

Equity Valuation: Thirty Years Later

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1 Introduction

This paper gives a guide to the valuation of equity claims consistent with classical asset pricing theory – yet it is simple and empirically implementable. The model recognizes that interest rates are stochastic and that equity premiums are time-varying, and to some degree predictable. Information in the currently observable term structure of interest rates is used explicitly in the valuation.

The analysis in this paper is not new. The seminal paper by Rubinstein (1976) includes most of the ideas (and in much fewer pages). However, the key lessons from that paper seem to have been largely forgotten in terms of practical equity valuation. At the time, the approach may have seemed impractical, but later advances in the theory of the term structure of interest rates and asset pricing in general may have changed this perception. Equities are long-term claims. Why is it that the information in a readily observable term structure of interest rates is not used in practical equity valuation? One of the key insights of the Rubinstein (1976) paper is that equity prices and the prices of fixed income claims are very closely related. In this paper, we explicitly reintroduce this link for the purpose of practical equity valuation.

The common practice in equity valuation is to use some kind of dividend, free cash flow, or residual operating income discount model. In the latter two models, the discount rate used is the firm's cost of capital, while an equity cost of capital is used in the former model. For example, using free cash flows, FCF_t , the value of the firm at date t , V_t , is determined as the discounted expected free cash flows conditional on information available at date t , i.e.,¹

$$V_t = \sum_{\tau=t+1}^T \frac{E_t[FCF_\tau]}{(1 + k_t)^{\tau-t}}, \quad (1)$$

where k_t is the firm's cost of capital. Of course, taking the value of the firm and the expected future free cash flows as given, the firm's cost of capital is merely an *internal rate of return*. This construct (or plug number) may be useful for strategy evaluation within the firm, but it is certainly not useful for security valuation or net present value calculations within the firm, since we in these cases are trying to determine the current value of the sequence of future uncertain cash flows. Instead, we must determine the firm's cost of capital directly. A common approach is to determine it as the weighted average cost of capital, i.e.,²

$$k_t \approx WACC_t = \frac{S_t}{S_t + D_t} k_t^e + \frac{D_t}{S_t + D_t} k_t^d, \quad (2)$$

¹We assume a finite horizon throughout the analysis to avoid subtle issues of equilibria in infinite horizon models.

²We don't want to be confused by taxes, so we assume throughout the analysis that the basic Modigliani/Miller assumptions are satisfied. Any tax advantage to debt can be found using the same principles as used in the paper.

where S_t and D_t are the market values of the firm's equity and net debt, respectively, with $S_t + D_t = V_t$, and k_t^e and k_t^d are the cost of equity and cost of debt, respectively. The market values of debt and equity are readily observable, while the cost of equity and the cost of debt must be estimated.³

We could determine these as the “implied” cost of capital for the two sources of capital, i.e.,

$$S_t = \sum_{\tau=t+1}^T \frac{E_t[U_\tau]}{(1 + k_t^e)^{\tau-t}}, \quad (3)$$

$$D_t = \sum_{\tau=t+1}^T \frac{E_t[Z_\tau]}{(1 + k_t^d)^{\tau-t}}, \quad (4)$$

where U_τ and Z_τ are the net payments to equity- and debtholders at date τ , respectively, with the sum being equal to the free cash flow by the firm's budget constraint, $FCF_\tau = U_\tau + Z_\tau$. If the expected payments are all infinite sequences growing at the same rate, then it is easy to show that the firm's cost of capital is indeed equal to the weighted average cost of capital.⁴ However, to compute the cost of capital for debt and equity in this way assumes we know both the market values of debt and equity and, thus, also the value of the firm. If the aim of the valuation exercise is to estimate the intrinsic value of equity, while we take the market value of debt to reflect its intrinsic value (as we will assume throughout the analysis), then the cost of debt can be found by using equation (4).

On the other hand, a different approach must be taken to estimate the cost of equity. The common approach is to use one of the standard asset pricing models such as the CAPM or the APT model. For example, using the CAPM, the first step is to estimate the *beta* of the equity as the slope of a time-series regression of equity returns on the returns of some stock index (as a proxy for the market portfolio). Of course, this assumes that the equity beta is constant over the estimation period. Secondly, the market price of risk, i.e., the expected excess return on the market portfolio relative to the riskless return, must be estimated. This can be done in a Fama-MacBeth type cross-section regression of average returns on beta-sorted portfolios on their estimated betas,

³Note the circularity here. We need the value of the firm and the values of debt and equity to find the WACC (which we, in turn, use to find these values). Of course, in an implementation of the model a search procedure can be used to find a fix-point for this relation, but this procedure is not very revealing in terms of what is really going on.

⁴Let both $E_t[U_\tau]$ and $E_t[Z_\tau]$ grow at a constant rate g . Then

$$S_t = E_t[U_{t+1}]/(k_t^e - g), \quad D_t = E_t[Z_{t+1}]/(k_t^d - g), \quad \text{and} \quad V_t = E_t[FCF_{t+1}]/(k_t - g).$$

Using the firm's budget constraint we get

$$k_t = [E_t[U_{t+1}] + E_t[Z_{t+1}]] / V_t + g = [S_t(k_t^e - g) + D_t(k_t^d - g)] / V_t + g = (S_t / V_t)k_t^e + (D_t / V_t)k_t^d$$

which is (2).

or in a Shanken-type direct time-series estimation of average excess return. Lastly, a choice has to be made regarding the maturity of the treasury yield used as the proxy for the riskless interest rate. As is well known, this approach has several problems.

Let us just mention a few in arbitrary order (and you can add your own favorites). (1) No consideration is given to the readily observable term structure of interest rates which we know is very important when we wish to value long-term bonds – for example, the yield on a bond does not only depend on its maturity, it also depends on its payment profile (see, e.g., Caks, 1977, and Christensen and Nielsen, 1987, for the so-called coupon- and bond-type effects on yield to maturity, respectively). (2) The standard CAPM is not valid in multi-period contexts unless investors have log-utility or the investment opportunity set is non-stochastic (or something close to it) – and even if it is valid, much care must be taken in discounting multi-period payments (see, e.g., Copeland and Weston, 1988). (3) The cost of equity is increasing in leverage – one of the curses of tests of asset pricing models. (4) Circularity in the beta estimates – return betas depend on the value of the security which we are trying to determine (see, e.g., Ang and Liu, 2004). (5) The future market prices of risk are uncertain although they may be predictable to some degree.

It is easy to criticize – the challenge is to come up with better alternatives which are also empirically implementable – or at least to identify the key issues. The paper by Ang and Liu (2004) is a good step in that direction. They recognize that one should not discount future expected payments with the same discount rate. They propose a model in which a *term structure of required rates of return* is used to discount future expected payments (see also Brennan, 1997, and Brennan and Xia, 2003). That is, the value of a sequence of uncertain future cash flows, say free cash flows, is determined as

$$V_t = \sum_{\tau=t+1}^T \frac{E_t[FCF_\tau]}{(1 + k_{\tau t})^{\tau-t}}, \quad (5)$$

where the risk-adjusted discount rates $k_{\tau t}$ are date-specific (like zero-coupon rates). They parameterize their model with specific processes for asset betas and market risk premia and find that significant pricing errors can occur if a constant discount rate is used. However, the approach requires that the conditional expected free cash flow at some future date, $E_t[FCF_\tau]$, and its current value have the same sign and, obviously, this need not be the case.⁵ Besides not always being applicable, the trouble with this kind of model is that everything is hidden in the date-specific risk-adjusted discount rates – and in a very convoluted way we might add. For example, the information in the current directly observable term structure of interest rates is not used – the date-specific discount rate is some complicated mixture of the zero-coupon rate for that date, the date-specific market

⁵This inconsistency can also be seen if (5) is used to value a forward contract. The forward price is set such that the current value of the contract is equal to zero. Hence, (5) can only be used if the forward price is equal to conditional expected value of the underlying security at maturity, i.e., there can be no risk-adjustments in forward prices!

price of risk, and the date-specific systematic risk of the cash flow where the latter is determined not only by the contemporaneous covariance with the market return but also by the time-series properties of cash flows and market returns.

Our approach aims at separating out these various effects on the current value of some future uncertain cash flow (or residual operating income). We take the classical path laid out in the seminal paper by Rubinstein (1976). In order to sketch the approach, we can use the well known result that no-arbitrage in the financial market implies that we can write the value of the firm as (see, for example, Christensen and Feltham, 2003, Chapter 6)

$$V_t = \sum_{\tau=t+1}^T \frac{E_t[FCF_\tau] + \text{Cov}_t[FCF_\tau, q_\tau]}{(1 + \iota_{\tau t})^{\tau-t}},$$

where $\iota_{\tau t}$ is the zero-coupon rate at date t for maturity τ , and q_τ is the ratio between the risk-neutral probabilities based on zero-coupon bonds (the so-called forward probabilities) and the true conditional probabilities. This latter object is also referred to as the valuation index or the Radon-Nikodym derivative for the two probability measures. The no-arbitrage approach is very useful to price *redundant* assets, but our aim is to price the *primitive* assets. Hence, we must know more than the fact that no arbitrage opportunities are available in the financial market. We go all the way and assume equilibrium prices in an effectively complete market (like in the CAPM, the representative agent model, or any other commonly used models to price primitive securities). Moreover, we assume homogeneous beliefs and time-additive preferences. In that case, it is known to some (see, for example, Christensen and Feltham, 2003, Chapter 6, or another favorite Christensen, Graversen, and Miltersen, 2000) that the valuation index for date τ is measurable with respect to aggregate consumption at that date, x_τ , i.e., we can write the valuation index as a function of aggregate consumption, i.e., $q_\tau = q_\tau(x_\tau)$. Now assume that aggregate (or log-aggregate) consumption and free cash flows are jointly normally distributed. Then we can use Stein's Lemma to separate the valuation index out of the covariance, i.e.,

$$V_t = \sum_{\tau=t+1}^T \frac{E_t[FCF_\tau] + E_t[q'_\tau(x_\tau)] \text{Cov}_t[FCF_\tau, x_\tau]}{(1 + \iota_{\tau t})^{\tau-t}}.$$

Lastly, the expected marginal valuation indices, $E_t[q'_\tau(x_\tau)]$, can be determined from prices of aggregate consumption claims, or by using a general equilibrium approach, i.e., by assuming a particular set of investor preferences.

The remainder of the paper is organized as follows. In Section 2 we review key results from multi-period asset pricing theory in discrete-time. Based on these results, we derive an accounting-based multi-period equity valuation model in Section 3 (a similar model can be derived based on

the free cash flows as above). Section 4 includes a general equilibrium analysis of a setting in which the investors have exponential utility, and aggregate consumption and residual operating income are jointly normally distributed. Appendix A shows how the analysis is adjusted if aggregate consumption is lognormally distributed and investors have power utility (average risk tolerance is replaced by the common risk cautiousness), and Appendix B extends the setting to preferences with external habit formation. For purpose of illustration, Section 5 examines a simple setting in which residual operating income and aggregate consumption are given by a simple first-order vector-autoregressive model with mean-reversion to a deterministic exponential trend. The analysis stresses the importance of both the contemporaneous correlation between residual operating income and aggregate consumption and the time-series properties of these processes. This analysis is the basis for the comparison of the general equilibrium analysis and the standard textbook approach of using a time-independent risk-adjustment to the required rates of returns in Section 6. We skip the concluding remarks. The reader is probably already sufficiently confused at this point – but hopefully at a higher level.

2 A Review of Multi-period Asset Pricing Theory

Our model closely follows the finite-horizon, discrete-time model in Christensen and Feltham, 2003, Chapters 6 and 9, except that we assume a continuous state space (in order to allow for continuous distributions). Let the standard probability space (Ω, Ξ, Φ) with homogeneous investor beliefs Φ be given, and denote probabilizable events observable at date t by $y_t \in \Xi_t$. Trading and consumption take place at dates $t = 0, 1, \dots, T$ conditional on public information Ξ_t . Dividends and ex-dividend prices of marketed securities are stochastic processes adapted to the filtration $\Xi = \{\Xi_t\}_{t=0, \dots, T}$. That is, the dividend d_{jt} and the ex-dividend price V_{jt} of security $j = 1, \dots, J$ at date t can be written as functions of the observable events at that date, i.e., $d_{jt} = d_{jt}(y_t)$ and $V_{jt} = V_{jt}(y_t)$, respectively. The fundamental theorem of asset pricing can now be stated as follows.

Theorem 1 *No-arbitrage in the securities market (and mild regularity conditions) implies that there exists a strictly positive event-price measure $P(y_\tau|y_t)$ such that the ex-dividend price of any marketed security can be written as*

$$V_{jt}(y_t) = \sum_{\tau=t+1}^T \int_{\Xi_\tau} d_{j\tau}(y_\tau) dP(y_\tau|y_t), \quad t = 0, 1, \dots, T-1; j = 1, \dots, J. \quad (6)$$

As is well known, this fundamental no-arbitrage asset pricing result can be given alternative representations. One set of representations normalizes the event-prices, and another set of representa-

tions re-expresses the dividends in terms of free cash flows or accounting numbers using accounting relations. The following theorem states alternative representations of the first type.

Theorem 2 *The fundamental asset pricing relation (6) is equivalent to each of the following:*

(a) *There exist forward measures $Q_{\tau t}$ for dates $\tau = t + 1, \dots, T$ such that*

$$V_{jt}(y_t) = \sum_{\tau=t+1}^T B_{\tau t}(y_t) E_{\tau t}^Q[d_{j\tau}|y_t], \quad (7)$$

where $B_{\tau t}(y_t)$ is the price at date t of a zero-coupon bond paying one unit of account at date τ , and $E_{\tau t}^Q[\cdot|y_t]$ is the conditional expectations operator under $Q_{\tau t}$.

(b) *There exist event-price deflators $\pi_\tau(y_\tau)$ for dates $\tau = t + 1, \dots, T$ such that*

$$V_{jt}(y_t) = \sum_{\tau=t+1}^T E[\pi_\tau d_{j\tau}|y_t]. \quad (8)$$

(c) *There exist valuation indices $q_{\tau t}(y_\tau|y_t)$ for dates $\tau = t + 1, \dots, T$ such that*

$$V_{jt}(y_t) = \sum_{\tau=t+1}^T B_{\tau t}(y_t) \{E[d_{j\tau}|y_t] + \text{Cov}[d_{j\tau}, q_{\tau t}|y_t]\}, \quad (9)$$

where the valuation indices are given by the Radon-Nicodym derivative between the probability measures $Q_{\tau t}$ and Φ , i.e.,

$$q_{\tau t}(y_\tau|y_t) = \frac{dQ_{\tau t}}{d\Phi}(y_\tau|y_t). \quad (10)$$

In (a) the event-prices are normalized by the riskless discount factor, $B_{\tau t}(y_t)$, for the period from t to τ (in order to transform the event-prices into probability densities), while in (b) the event-prices are normalized by the conditional densities, $d\Phi(y_\tau|y_t)$. In (c) the event-prices are normalized by both the discount factor and the probabilities and, furthermore, it follows that $E[q_{\tau t}|y_t] = 1$.

Accounting Assumptions:

As noted above, the set of alternative representations of no-arbitrage prices of equity claims of the second type builds on a set of accounting relations. For notational simplicity, we focus on a particular firm and, therefore, drop the subscript j on asset prices in the remainder of the paper.

Following Feltham and Ohlson (1999) we assume the following accounting relations (see also Christensen and Feltham, 2003, Chapter 9, for further discussion of these relations).

(A) Clean Surplus Relation (CSR):

All changes in the book value of equity except transactions with common equityholders go through the income statement, i.e.,

$$\begin{aligned}bv_t(y_t) &= bv_{t-1}(y_{t-1}) + ni_t(y_t) - d_t(y_t), \\bv_T(y_T) &= 0,\end{aligned}$$

where $bv_t(y_t)$ is the book value of equity at date t , $ni_t(y_t)$ is net income in period t , and $d_t(y_t)$ is the net-dividend paid to equityholders prior to closing the books at date t . Furthermore, the book value of equity and net income are separated into financial and operating activities, i.e.,

$$\begin{aligned}bv_t(y_t) &= fa_t(y_t) + oa_t(y_t), \\ni_t(y_t) &= fi_t(y_t) + oi_t(y_t).\end{aligned}$$

(B) Financial Asset Relation (FAR):

All transfers to common equityholders are made through the financial assets, and these assets are increased by financial income and free cash flows from operations denoted $oc_t(y_t)$, i.e.,

$$fa_t(y_t) = fa_{t-1}(y_{t-1}) + fi_t(y_t) + oc_t(y_t) - d_t(y_t).$$

(C) Financial Assets Marked-to-Market (FAM):⁶

The risk-adjusted expected financial income equals the riskless spot interest rate (denoted $\iota_{t-1}(y_{t-1})$) times the opening book value of the financial assets, i.e.,

$$E_{t,t-1}^Q[fi_t|y_{t-1}] = \iota_{t-1}(y_{t-1})fa_{t-1}(y_{t-1}).$$

⁶Note that we are using the same risk-adjusted probabilities for all claims. Hence, tax savings on interest paid or net financial debt is part of the operating income like any recognition of future tax benefits must be classified as an operating asset. This is different from the WACC-approach in which tax-savings on interest payments on net debt are part of the financial income – the tax benefits of debt is reflected in a lower cost of capital used to discount the free cash flows or residual operating income from operations (a somewhat strange idea). Our approach is similar to the so-called adjusted-present-value (APV) technique in which the tax advantage to debt is valued separately. Moreover, our analysis assumes that all claims are priced on a before investor-tax basis using the same set of event-prices. That is not restrictive as long as all claims are taxed equally in the hands of investors. If not, a separate set of event-prices must be used for claims of each tax class. Don't even think of letting everything being determined endogenously – infinite tax-arbitrage does not go well with no-arbitrage pricing!

(D) Operating Asset Relation (OAR):

The operating assets are increased by operating income and reduced by the free cash flows transferred to the financial assets, i.e.,

$$oa_t(y_t) = oa_{t-1}(y_{t-1}) + oi_t(y_t) - oc_t(y_t).$$

In addition to these accounting relations we define residual income for the financial and operating activities as income minus the riskless spot interest rate times the opening book value, i.e.,

$$\begin{aligned} ri_t(y_t) &= ni_t(y_t) - \iota_{t-1}(y_{t-1})bv_{t-1}(y_{t-1}), \\ rfi_t(y_t) &= fi_t(y_t) - \iota_{t-1}(y_{t-1})fa_{t-1}(y_{t-1}), \\ roi_t(y_t) &= oi_t(y_t) - \iota_{t-1}(y_{t-1})oa_{t-1}(y_{t-1}). \end{aligned}$$

Note that by the FAM relation, the risk-adjusted expected residual financial income is equal to zero. Feltham and Ohlson (1999) show the following theorem relating the current market value of equity $S_t(y_t)$ to future accounting numbers (see also Christensen and Feltham, 2003, Propositions 9.3 and 9.4).⁷

Theorem 3 *No-arbitrage, CSR, FAR, FAM, and OAR are sufficient for the following free operating cash flow and accounting-value relations*

$$S_t(y_t) = fa_t(y_t) + \sum_{\tau=t+1}^T B_{\tau t}(y_t) \{E[oc_{\tau}|y_t] + \text{Cov}[oc_{\tau}, q_{\tau t}|y_t]\}, \quad (11)$$

$$= fa_t(y_t) + oa_t(y_t) + \sum_{\tau=t+1}^T B_{\tau t}(y_t) \{E[roi_{\tau}|y_t] + \text{Cov}[roi_{\tau}, q_{\tau t}|y_t]\}, \quad (12)$$

where the valuation indices are the same as in the dividend-value relation (9).

The discount factors, $B_{\tau t}(y_t)$, at the valuation date t given current information y_t are easily estimated using current prices of traded treasury bonds and your favorite model of the term structure of interest rates.⁸ The book values of the financial and operating assets are readily observable from (reformulated) financial statements, and expected future free cash flows or residual operating

⁷Here we only report the value relations based on the valuation index since it is this version we will be using in subsequent analyses. Of course, equivalent relations can be established based on the event-prices, the risk-adjusted probabilities, and the event-price deflator.

⁸Note that, in principle, we do not need a fully dynamic model of the stochastic process for interest rates – we only need to calibrate to the current term structure of interest rates.

income numbers are outcomes of a strategic and financial statement analysis of the firm.⁹ The remaining item in the value relations is the risk-adjustment due to the covariance with the valuation index. Unfortunately, no-arbitrage alone does not tell us much about the valuation indices. All we know is that they are positive and have expected values of one – at this point they are merely mathematical constructs without economic content (except they are derived from an assumption of no-arbitrage).

There are various ways of getting more information about the valuation indices. We will assume that equilibrium prices are formed in an effectively dynamically complete market. By this we mean that there are sufficient trading opportunities for the investors to trade to a Pareto efficient risk sharing. Moreover, we will assume investors have time-additive utility functions and homogeneous beliefs. This leads to the following result (see, for example, Christensen and Feltham, 2003, Chapter 6, and the references therein).

Theorem 4 *Let an equilibrium in an effectively dynamically complete market be given. Assume investors have homogeneous beliefs, and differentiable, strictly increasing, and concave, time-additive utility functions of event-contingent consumption, $u_{it}(c_{it})$, defined on $C_i = [\underline{c}_i, \infty)$, where $u'_{it}(c_{it}) \rightarrow \infty$, for $c_{it} \downarrow \underline{c}_i$.*

(a) *Individual equilibrium consumption plans are measurable with respect to aggregate consumption, x_t , at each date, i.e., $c_{it} = c_{it}(x_t)$, and $c_{it}(x_t)$ is an increasing function of x_t , $i = 1, \dots, I$.*

(b) *The prices of zero-coupon bonds paying one unit of account at date τ are given by*

$$B_{\tau t}(y_t) = \frac{E[u'_{i\tau}(c_{i\tau})|y_t]}{u'_{it}(c_{it}(y_t))}, \quad \tau = t + 1, \dots, T; i = 1, \dots, I. \quad (13)$$

(c) *The valuation indices based on zero-coupon bonds are measurable with respect to aggregate consumption at each date and given by the “scarcity” of aggregate consumption as measured by the investors’ marginal utility of consumption, i.e.,*

$$q_{\tau t}(x_\tau | y_t) = \frac{u'_{i\tau}(c_{i\tau}(x_\tau))}{E[u'_{i\tau}(c_{i\tau})|y_t]}, \quad \tau = t + 1, \dots, T; i = 1, \dots, I. \quad (14)$$

This theorem is the key result in consumption-based asset pricing models. The first-order condition for the investors’ decision problems (with time-additive utility) implies that each investor’s

⁹In the calculation of expected residual operating income we need estimates of expected operating income, expected spot interest rates, and the covariance between spot interest rates and book values of operating assets. The latter two terms are not present in the WACC-approach with a constant cost of capital.

valuation index is determined by the investor’s marginal utility of consumption. The additional assumptions of homogeneous beliefs and efficient risk sharing imply that all investors’ valuation indices are perfectly aligned and measurable with respect to aggregate consumption. Hence, the risk-adjustments in the preceding value relations relate to the covariance of dividends, free cash flows, or residual operating income with the “scarcity” of aggregate consumption. But how can we empirically measure the “scarcity” of aggregate consumption?

3 An Accounting-based Multi-period Equity Valuation Model

Identification of the valuation index requires more assumptions on distributions and (or) preferences. Standard distributions in asset pricing models are normal distributions and lognormal distributions. First, we need a mathematical result for normal distributions known as Stein’s Lemma (see Rubinstein, 1976, for a proof).

Lemma 5 *If X and Y are jointly normally distributed random variables and $F(\cdot)$ is some differentiable real valued function with $E[|F'(Y)|] < \infty$, then*

$$\text{Cov}(X, F(Y)) = E[F'(Y)] \text{Cov}[X, Y].$$

If we assume that future residual operating income and aggregate consumption are jointly normally distributed, then Stein’s Lemma and the accounting-value relation (12) imply that¹⁰

$$S_t(y_t) = fa_t(y_t) + oa_t(y_t) + \sum_{\tau=t+1}^T B_{\tau t}(y_t) \left\{ E[roi_\tau | y_t] + E[q'_{\tau t}(x_\tau) | y_t] \text{Cov}[roi_\tau, x_\tau | y_t] \right\} \quad (15)$$

Recognizing that the valuation indices in the dividend-value relation (9) and the accounting-value relation (12) are the same, and that a claim on aggregate date τ consumption can be priced using

¹⁰Normally distributed aggregate consumption does not go well with standard utility functions defined only on positive consumption – really, only the negative exponential utility function works (see the following section). Instead we could assume that aggregate consumption is lognormally distributed and write the risk-adjustment in (12) as

$$\text{Cov}[roi_\tau, q_{\tau t}(x_\tau) | y_t] = \text{Cov}[roi_\tau, h_{\tau t}(z_\tau) | y_t],$$

where $h_{\tau t}(z_\tau) = q_{\tau t}(\exp(z_\tau))$ and $z_\tau = \ln(x_\tau)$, and then proceed in a similar manner from there. In this case, most finance researchers’ favorite utility function, the power utility function, would be the choice for similar general equilibrium analyses as in the subsequent sections. For simplicity, we stick with the normal distribution in the text. The general equilibrium analysis with lognormally distributed aggregate consumption and power utilities is worked out in Appendix A.

Note that a lognormal distribution would not be a good choice for future residual operating income, since these income numbers may very well be negative (which is precluded by the lognormal distribution).

(9), we get that the price of this claim is

$$V_{\tau t}^x(y_t) = B_{\tau t}(y_t) \left\{ E[x_\tau | y_t] + E[q'_{\tau t}(x_\tau) | y_t] \text{Cov}[x_\tau, x_\tau | y_t] \right\} \Leftrightarrow$$

$$E[q'_{\tau t}(x_\tau) | y_t] = \frac{V_{\tau t}^x(y_t) R_{\tau t}(y_t) - E[x_\tau | y_t]}{\text{Var}[x_\tau | y_t]},$$

where $R_{\tau t}(y_t) \equiv [B_{\tau t}(y_t)]^{-1}$ is the riskless return from date t to τ . If we define *aggregate consumption returns* and *expected aggregate consumption returns* as

$$R_{\tau t}^x \equiv \frac{x_\tau}{V_{\tau t}^x(y_t)}, \quad \bar{R}_{\tau t}^x(y_t) \equiv \frac{E[x_\tau | y_t]}{V_{\tau t}^x(y_t)},$$

then we can rewrite (15) as

$$S_t(y_t) = fa_t(y_t) + oa_t(y_t)$$

$$+ \sum_{\tau=t+1}^T B_{\tau t}(y_t) \left\{ E[roi_\tau | y_t] - [E[R_{\tau t}^x | y_t] - R_{\tau t}(y_t)] \frac{\text{Cov}[roi_\tau, R_{\tau t}^x | y_t]}{\text{Var}[R_{\tau t}^x | y_t]} \right\} \quad (16)$$

Similarly, we can define residual income returns on net operating assets by normalizing future residual operating income by the current book value of operating assets, i.e.,

$$ReNOA_{\tau t} \equiv \frac{roi_\tau}{oa_t(y_t)}, \quad \overline{ReNOA}_{\tau t}(y_t) \equiv \frac{E[roi_\tau | y_t]}{oa_t(y_t)}.$$

Subtracting the current financial assets from both sides and dividing by the current book value of operating assets in (16) yields the following result.

Proposition 6 *Assume the conditions in Theorem 4 hold and make the accounting assumptions (A)-(D). Furthermore, assume that future residual operating income and aggregate consumption are jointly normally distributed, and that $E[|q'_{\tau t}(x_\tau)| | y_t] < \infty$. Then the market-to-book ratio for the operating assets is given by*

$$\frac{S_t(y_t) - fa_t(y_t)}{oa_t(y_t)} = 1$$

$$+ \sum_{\tau=t+1}^T B_{\tau t}(y_t) \left\{ \overline{ReNOA}_{\tau t}(y_t) - [\bar{R}_{\tau t}^x(y_t) - R_{\tau t}(y_t)] \frac{\text{Cov}[ReNOA_{\tau t}, R_{\tau t}^x | y_t]}{\text{Var}[R_{\tau t}^x | y_t]} \right\}. \quad (17)$$

There are two unusual objects in this value relation. One is the *term structure of excess returns* at date t , i.e.,

$$\bar{R}_{\tau t}^x(y_t) - R_{\tau t}(y_t), \quad \tau = t + 1, \dots, T.$$

Of course, these are easily determined if one assumes that the expected return on a well diversified stock index is a good proxy for expected aggregate consumption returns, and that these excess returns are the same for all maturities (as in applications of the standard CAPM). However, there is significant empirical evidence that these excess returns are time-varying and predictable to some extent (see, for example, Campbell and Shiller, 1988a and 1988b, Lettau and Ludvigson, 2001, and Rangvid, 2006).

The second unusual object is the *term structure of systematic accounting risk*, i.e.,

$$\frac{\text{Cov}[ReNOA_{\tau t}, R_{\tau t}^x | y_t]}{\text{Var}[R_{\tau t}^x | y_t]}, \quad \tau = t + 1, \dots, T.$$

Except for the term structure aspect it is interesting that we have “accounting betas.” In the early seventies of the previous century there was a literature examining accounting betas (see, for example, Beaver, Kettler and Scholes, 1970).¹¹ The advantage of this approach is that we avoid the circularity in the estimation of betas using stock returns. However, the drawback is that we have much less data available to estimate the accounting betas. A possible solution might be to estimate “industry betas” and then use those. Note that the accounting betas we need are affected neither by leverage (which is one of the curses of tests of asset pricing models) nor by scale. The other interesting aspect is the term structure aspect. Here it may be useful to assume a particular stochastic process for residual operating income returns and aggregate consumption returns and then derive the term structure from there (see below).

Both of these unusual objects in equity valuation require new empirical work (which is not a comparative advantage of the authors’). It is an empirical issue whether Proposition 6 is a superior equity valuation model compared to a standard WACC-approach. In any case, Proposition 6 tells us what are the elements we should be looking for in order to have an equity valuation model consistent with classical asset pricing results.

4 An Accounting-based Multi-period Equity Valuation Model with Exponential Utility

In this section we make additional assumptions on investor preferences in order to get more insight into the determination of the term structure of excess returns, accounting betas, and interest rates. Hakansson (1970) and others (see also Christensen and Feltham, 2003, Proposition 6.6) show the following result.

¹¹In that literature, aggregate earnings on stock indices are used while we are using aggregate consumption returns defined by market values of aggregate consumption claims.

Theorem 7 *Let a Pareto efficient equilibrium be given and assume the investors have homogeneous beliefs and time-additive preferences represented by HARA utility functions with identical risk cautiousness. Then there are parameters v_{it} and f_{it} such that*

$$c_{it}(x_t) = f_{it} + v_{it}x_t, \quad i = 1, \dots, I,$$

$$\sum_{i=1}^I v_{it} = 1, \quad \text{and} \quad \sum_{i=1}^I f_{it} = 0$$

Each investor's fraction of aggregate consumption v_{it} is the same for all dates, i.e., $v_{it} = v_i$, if, and only if, one of the following conditions holds for all $i = 1, \dots, I$ and for all $t = 1, \dots, T$:

$$u_{it}(c_{it}) = -\beta_{it}^P \exp[-c_{it}/\rho_i], \quad \beta_{it}^P > 0, \rho_i > 0;$$

$$u_{it}(c_{it}) = \beta_i^P \ln(c_{it} - b_{it}), \quad \beta_i^P > 0, c_{it} - b_{it} > 0;$$

$$u_{it}(c_{it}) = \beta_i^P \frac{1}{\alpha - 1} [\alpha c_{it} - b_{it}]^{\frac{\alpha-1}{\alpha}}, \quad \beta_i^P > 0, \alpha c_{it} - b_{it} > 0.$$

The advantage of having constant fractions of aggregate consumption is that it is very easy to ensure an efficient equilibrium – essentially, a market portfolio and a complete set of zero-coupon bonds are sufficient if personal endowments are also spanned (see Christensen and Feltham, 2003, Proposition 6.7)

Assume investors have negative exponential utility and, for simplicity, that their risk aversion is time-independent with aggregate risk tolerance $\rho_o = \sum_i \rho_i$. Wilson (1968) shows that in this case $v_{it} = v_i = \rho_i/\rho_o$ (see also Christensen and Feltham, 2003, Proposition. 4.3). It then follows from (14) that the valuation index is given by

$$q_{\tau t}(x_{\tau}|y_t) = \frac{\exp[-x_{\tau}/\rho_o]}{\mathbb{E}[\exp[-x_{\tau}/\rho_o]|y_t]}, \quad y_{\tau} \subseteq y_t, \tau = t + 1, \dots, T.$$

Differentiating the valuation index with respect to aggregate consumption at date τ and taking conditional expectations yield

$$\mathbb{E} \left[q'_{\tau t}(x_{\tau})|y_{\tau} \right] = \mathbb{E} \left[-\frac{1}{\rho_o} \frac{\exp[-x_{\tau}/\rho_o]}{\mathbb{E}[\exp[-x_{\tau}/\rho_o]|y_t]} |y_{\tau} \right] = -\frac{1}{\rho_o}.$$

Inserting this into (15) yields

$$S_t(y_t) = fa_t(y_t) + oa_t(y_t) + \sum_{\tau=t+1}^T B_{\tau t}(y_t) \{ \mathbb{E}[roi_{\tau}|y_t] - \text{Cov}[roi_{\tau}, x_{\tau}/\rho_o|y_t] \}$$

If we define $\bar{\rho}$ as the investors' average risk tolerance (i.e., $\bar{\rho} = \rho_o/I$) and “aggregate consumption per capita” as $acc_{\tau} = x_{\tau}/I$, then the risk-adjusted aggregate consumption per capita is $racc_{\tau} =$

$acc_\tau/\bar{\rho} = x_\tau/\rho_o$.¹² Hence, normalizing by the book value of operating assets we get the following result.

Proposition 8 *Assume the conditions in Theorem 4 hold and make the accounting assumptions (A)-(D). Furthermore, assume that future residual operating income and aggregate consumption are jointly normally distributed, and investors have exponential utility with constant risk tolerances. Then the market-to-book ratio for the operating assets is given by*

$$\frac{S_t(y_t) - fa_t(y_t)}{oa_t(y_t)} = 1 + \sum_{\tau=t+1}^T B_{\tau t}(y_t) \{ \overline{ReNOA}_{\tau t}(y_t) - \text{Cov}[ReNOA_{\tau t}, racc_\tau | y_t] \}, \quad (18)$$

and the zero-coupon prices are given by

$$B_{\tau t}(y_t) = \frac{E[u'_{i\tau}(c_{i\tau})|y_t]}{u'_{it}(c_{it}(y_t))} = \beta_{\tau t}^o \exp[-\{E[racc_\tau | y_t] - racc_t - \frac{1}{2} \text{Var}[racc_\tau | y_t]\}], \quad (19)$$

where $\beta_{\tau t}^o$ is the investors' "average personal discount factor" from τ to t .¹³

5 A VAR Model with Exponential Utility

In order to get additional insights we now consider a simple vector-auto-regressive (VAR) model of the stochastic properties of residual operating income for a particular firm and risk-adjusted aggregate consumption per capita. More comprehensive models can be assumed, but we have chosen a simple one in order to more clearly focus on the central issues. Assume the residual operating income returns and the risk-adjusted aggregate consumption per capital are the only information available and that they follow a first-order vector auto-regressive process with mean-reversion to a deterministic exponential trend, i.e.,¹⁴

$$\begin{aligned} ReNOA_{\tau t} - ReNOA_t^o(1 + \alpha)^{\tau-t} \\ = \omega_r [ReNOA_{\tau-1,t} - ReNOA_t^o(1 + \alpha)^{\tau-1-t}] + \varepsilon_\tau, \quad \omega_r \in [0, 1), \end{aligned} \quad (20a)$$

$$racc_\tau - racc_t^o(1 + \gamma)^{\tau-t} = \omega_a [racc_{\tau-1,t} - racc_t^o(1 + \gamma)^{\tau-1-t}] + \delta_\tau, \quad \omega_a \in [0, 1), \quad (20b)$$

¹²Note that risk-adjusted aggregate consumption per capital is equal to the average relative risk aversion in the economy (which has an order of magnitude of 2-5, empirically).

¹³If we let the investors' personal discount factors be defined as $\beta_{it}^P \equiv \exp[-\theta_i t]$, then the average personal discount factor is given by

$$\beta_{\tau t}^o = \exp[-(\tau - t) \sum_i \theta_i \rho_i / \rho_o].$$

¹⁴That is, the trends for the two processes are $ReNOA_t^o(1 + \alpha)^{\tau-t}$ and $racc_t^o(1 + \gamma)^{\tau-t}$ with growth rate α and γ , respectively.

where ε_τ and δ_τ are zero-mean normally distributed and serially uncorrelated, and with conditional variances and contemporaneous conditional covariance: $Var[\varepsilon_\tau|y_t] = \sigma_r^2$, $Var[\delta_\tau|y_t] = \sigma_a^2$, $Cov[\varepsilon_\tau, \delta_\tau|y_t] = \sigma_{ra}$. Solving these equations recursively yields

$$ReNOA_{\tau t} = ReNOA_t^o(1 + \alpha)^{\tau-t} + \omega_r^{\tau-t}[ReNOA_{\tau t} - ReNOA_t^o] + \sum_{s=0}^{\tau-1-t} \omega_r^s \varepsilon_{\tau-s},$$

$$racc_\tau = racc_t^o(1 + \gamma)^{\tau-t} + \omega_a^{\tau-t}[racc_\tau - racc_t^o] + \sum_{s=0}^{\tau-1-t} \omega_a^s \delta_{\tau-s}.$$

This specification of the information dynamics allows us to calculate all the terms in the exponential utility accounting-value relation (18) explicitly.

Proposition 9 *The information dynamics in (20) implies that*

$$\overline{ReNOA}_{\tau t}(y_t) = ReNOA_t^o(1 + \alpha)^{\tau-t} + \omega_r^{\tau-t}[ReNOA_{\tau t} - ReNOA_t^o],$$

$$Var[ReNOA_{\tau t}|y_t] = \sigma_r^2 \frac{1 - \omega_r^{2[\tau-t]}}{1 - \omega_r^2},$$

$$E[racc_\tau|y_t] = racc_t^o(1 + \gamma)^{\tau-t} + \omega_a^{\tau-t}[racc_\tau - racc_t^o],$$

$$Var[racc_\tau|y_t] = \sigma_a^2 \frac{1 - \omega_a^{2[\tau-t]}}{1 - \omega_a^2},$$

and that the date τ risk-adjustment in (18) is given by

$$RA_{\tau t}(y_t) \equiv Cov[ReNOA_{\tau t}, racc_\tau|y_t] = \sigma_{ra} \frac{1 - (\omega_a \omega_r)^{\tau-t}}{1 - \omega_a \omega_r}. \quad (21)$$

Note that the risk-adjustment, $RA_{\tau t}(y_t)$, is an increasing function of τ (for $\omega_a \omega_r > 0$), but with an upper limit of

$$\sigma_{ra} \frac{1 - (\omega_a \omega_r)^{\tau-t}}{1 - \omega_a \omega_r} \rightarrow \sigma_{ra} \frac{1}{1 - \omega_a \omega_r}. \quad (22)$$

Moreover, if residual operating income is serially uncorrelated, i.e., $\omega_r = 0$, then the risk-adjustment is due solely to the contemporaneous correlation with risk-adjusted aggregate consumption, i.e., the risk-adjustment is σ_{ra} independently of maturity date τ . The reason for this latter result, of course, is that no new information about $ReNOA_{\tau t}$ is revealed until date τ – the claim to $ReNOA_{\tau t}$ is like a zero-coupon bond maturing at date $\tau - 1$ plus a one-period claim from $\tau - 1$ to τ with risk σ_{ra} . The important lesson from (21) is that, in general, the risk-adjustment depends not only on the contemporaneous conditional covariance between $ReNOA_{\tau t}$ and risk-adjusted aggregate consumption, σ_{ra} , but also on their time-series properties, i.e., ω_r and ω_a .

Note that the investors' average risk tolerance is not directly empirically observable, and it enters into the definition of risk-adjusted aggregate consumption per capita. There are various ways of estimating the average risk tolerance. For example, it can be estimated based on a time-series of average excess returns (probably giving rise to an equity premium puzzle and many other empirical problems). The obvious alternative is to determine it implicitly from a calibration to the readily observable current term structure of interest rates using the zero-coupon prices in (19) – in a general equilibrium, the risk tolerances determining zero-coupon prices are the same as those determining excess returns on equities! Once the implicit risk tolerance is determined, the parameters in the process for risk-adjusted aggregate consumption per capita can be estimated using a time-series of aggregate consumption per capita. Again, these parameters may alternatively be backed out from the current term structure of interest rates. We are not aware of empirical analyses determining the preference parameters and the parameters in the stochastic process for aggregate consumption from an observable term structure of interest rates. Of course, it is an empirical issue whether this approach works better than the standard time-series approaches. In Appendix B we consider a setting with preferences exhibiting habit formation. This allows for more flexibility in the types of equilibrium term structures of interest rates.

6 Comparison to the Standard Textbook Approach

The standard textbook approach uses a constant risk-adjusted cost of capital (WACC) to discount expected residual operating income, independently of the maturity. Of course, using a constant risk-adjusted discount rate which does not reflect the current term structure of interest rates is obviously a very bad choice. Therefore, in our comparison we will take the liberty to interpret the textbook approach as just assuming a constant risk premium. Hence, allowing for a general term structure of interest rates, the risk-adjusted discount factor has the form

$$\begin{aligned}\bar{B}_{\tau t}(y_t) &= [1 + \iota_{\tau t}(y_t) + \bar{r}\bar{p}(y_t)]^{-(\tau-t)} \\ &< B_{\tau t}(y_t) = [1 + \iota_{\tau t}(y_t)]^{-(\tau-t)}, \quad \text{for } \bar{r}\bar{p}(y_t) > 0 \text{ and } \tau > t,\end{aligned}$$

where $\iota_{\tau t}(y_t)$ is the zero-coupon riskless interest rate for maturity τ as of date t , and $\bar{r}\bar{p}(y_t)$ is a constant risk premium (i.e., excess return times beta) typically estimated on the basis of short-term stock returns. The present value at date t of the residual operating income return on operating assets at date τ then has the form $\bar{B}_{\tau t}(y_t)\overline{ReNOA}_{\tau t}(y_t)$. Hence, comparing to (18), the comparable *implicit* risk-adjustment for maturity date τ with this approach, $RA_{\tau t}^{TB}(y_t)$, is a solution to the equation

$$\bar{B}_{\tau t}(y_t)\overline{ReNOA}_{\tau t}(y_t) = B_{\tau t}(y_t)[\overline{ReNOA}_{\tau t}(y_t) - RA_{\tau t}^{TB}(y_t)].$$

Solving this equation for the implicit risk-adjustment yields

$$RA_{\tau t}^{TB}(y_t) = \overline{ReNOA}_{\tau t}(y_t) \left[1 - \left(\frac{1 + \iota_{\tau t}(y_t)}{1 + \iota_{\tau t}(y_t) + \overline{rP}(y_t)} \right)^{\tau-t} \right]. \quad (23)$$

Comparing the implicit risk-adjustment in (23) to the risk-adjustment of the exponential utility model in (21) shows a number of striking differences. First, note that the implicit risk-adjustment $RA_{\tau t}^{TB}(y_t)$ depends on the deterministic trend in residual operating income returns (through $\overline{ReNOA}_{\tau t}(y_t)$) and, obviously, it should not! For example, increasing (decreasing) the growth rate in residual operating income returns α , increases (decreases) the risk-adjustment and, thus, lowers (increases) the net present value of future residual operating income returns. Hence, high (low) growth firms will tend to be undervalued (overvalued) using the textbook approach. Note that our assumption that the variance of future residual operating income returns is independent of growth is important for this relation. Secondly, if there is low persistence in residual operating income returns, i.e., a high degree of mean reversion, then the risk-adjustment in (21) is almost independent of maturity τ , i.e., close to being equal to the contemporaneous covariance with risk-adjusted aggregate consumption, whereas the implicit risk-adjustment in (23) compounds the constant risk premium, $\overline{rP}(y_t)$, $(\tau - t)$ times. Hence, low persistence firms will tend to be undervalued using the textbook approach (compare Brennan and Xia, 2003). Note also that even if there is high persistence in residual operating income returns, the risk adjustment in (21) is bounded from above as τ goes to infinity (see (22)), while the implicit risk-adjustment in (23) approaches the expected residual operating income returns, $\overline{ReNOA}_{\tau t}(y_t)$. These latter relations highlight the fact that proper risk-adjustments should not only reflect the contemporaneous covariance with risk-adjusted aggregate consumption, but also the time-series properties of residual operating income returns and, in particular, the degree of persistence of these returns. The key problem with the textbook approach is that it does not reflect when *new* information about future residual operating income returns is revealed!

Will these differences between the textbook approach and the exponential utility model yield noticeable differences in equity values, or are they merely second-order effects? In the following, we try to answer this question by means of using a numerical example with reasonable parameters. The example is based on the set of parameters for risk-adjusted aggregate consumption shown in Table 1 and a personal discount rate of 2%.

Current $racc_t$	Trend $racc_t^o$	Growth rate γ	Persistence ω_a	Variance σ_a^2
2.25	2.00	2%	90%	1%

Table 1: Parameters of risk-adjusted aggregate consumption.

Using Proposition 8, these parameters result in the term structure of interest rates shown in Fig-

ure 1.

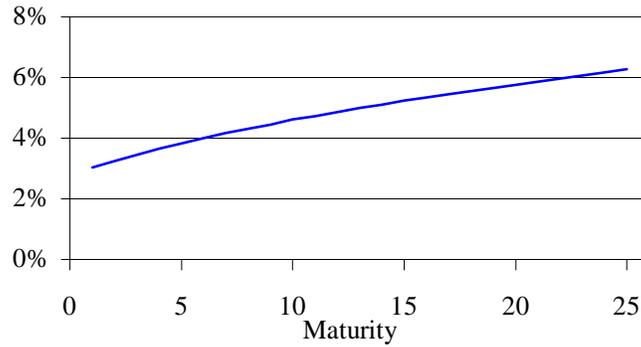


Figure 1: Term structure of interest rates.

We assume the firm has a horizon of 25 years, and that the residual operating income return process in our base case is characterized by the parameters shown in Table 2.

Current $ReNOA_t$	Trend $ReNOA_t^o$	Growth rate α	Persistence ω_r	Covariance σ_{ra}
15%	10%	0%	90%	1%

Table 2: Parameters of residual operating income return process for base case.

For the textbook approach we assume the firm has a beta of one, and that the excess return on the market portfolio is 5%. Using Propositions 8 and 9, this firm has a market-to-book of operating assets of 2.14, whether the textbook approach or the exponential utility model is used. The risk-adjustments for both approaches are shown in Figure 2.

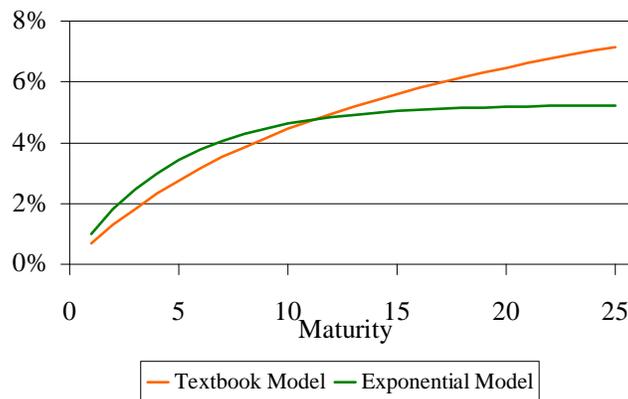


Figure 2: Risk-adjustments in base case.

Note that even though the market-to-book values are the same (indeed, we have chosen the parameters for this to be the case), the patterns across maturities are quite different. In particular, in

the textbook approach the implicit risk-adjustment is increasing without bounds, while the risk-adjustment in the exponential model quite quickly approaches its upper bound of (see (22))

$$\sigma_{ra} \frac{1}{1 - \omega_a \omega_r} = 0.01 \frac{1}{1 - 0.9 \times 0.9} = 5.26\%$$

despite the rather high persistence rates. Consider now two alternative cases: 5% growth, and 5% decline in residual operating income returns, i.e., $\alpha = 5\%$ and $\alpha = -5\%$, respectively. The market-to-book ratios are 2,64 and 3.15 in the 5% growth case, and 1.86 and 1,63 in the 5% decline case for the textbook and exponential utility model, respectively. The risk-adjustments are shown in Figures 3 and 4 for the $\alpha = 5\%$ and $\alpha = -5\%$ cases, respectively.

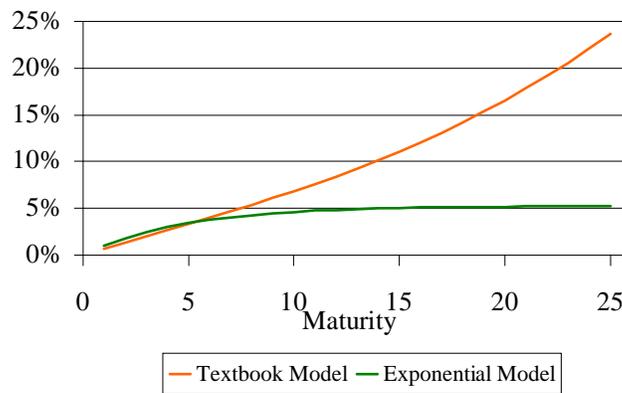


Figure 3: Risk-adjustments in 5% growth case.

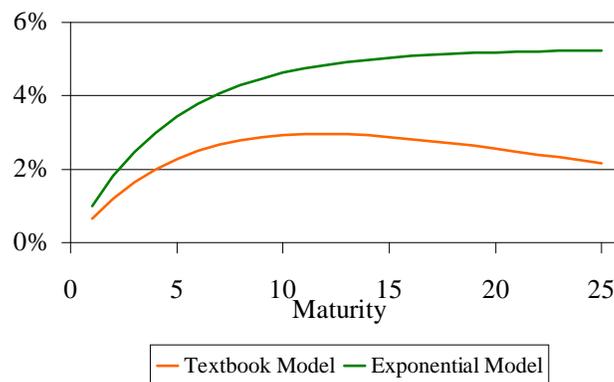


Figure 4: Risk-adjustments in 5% decline case.

As noted earlier the textbook approach leads to undervaluation of growth, while it overvalues declining expected residual operating income returns, and the differences are far from being trivial.

Finally, consider a case with low persistence in residual operating income returns (i.e., high mean reversion), $\omega_r = 10\%$. Compared to a market-to-book ratio of 2.14 for both approaches in the base case, the market-to-book ratio now becomes 1.91 and 2.25 for the textbook and exponential utility model, respectively. The risk-adjustments are shown in Figure 5.

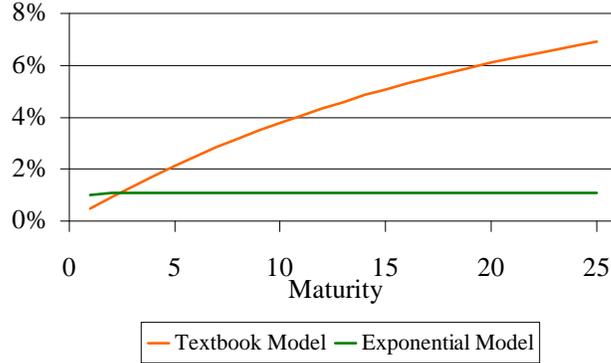


Figure 5: Risk-adjustments in 10% persistence case.

Even though there is now growth, i.e., $\alpha = 0$, the textbook approach risk-adjust the expected residual operating income returns far too much – due to the compounding of the constant risk premium. The exponential utility model risk-adjust with an almost constant amount equal to the upper bound of

$$\sigma_{ra} \frac{1}{1 - \omega_a \omega_r} = 0.01 \frac{1}{1 - 0.9 \times 0.1} = 1.10\%.$$

Above we have asked the question of what the implicit risk-adjustment is in the textbook approach assuming a constant risk premium. We can also ask the opposite question, i.e., what is the risk premium for maturity τ , $rp_{\tau t}(y_t)$, using a term structure of risk-adjusted discount rates (like Ang and Liu, 2004, and Brennan and Xia, 2003) which makes this approach equivalent to discounting risk-adjusted expected residual operating income returns with the zero-coupon riskless interest rates. That is, $rp_{\tau t}(y_t)$ is determined by the equation

$$\overline{ReNOA}_{\tau t}(y_t) (1 + \iota_{\tau t}(y_t) + rp_{\tau t}(y_t))^{-(\tau-t)} = [\overline{ReNOA}_{\tau t}(y_t) - RA_{\tau t}(y_t)] (1 + \iota_{\tau t}(y_t))^{-(\tau-t)} \quad (24)$$

Is there always a solution to this equation? Unfortunately, the answer is NO! To see why, assume $\overline{ReNOA}_{\tau t}(y_t) > 0$, and $(\tau - t)$ is even. Then the left-hand side of (24) is positive for any value of $rp_{\tau t}(y_t)$, i.e., the net present value of residual operating income returns at date τ is positive. However, the right-hand side is negative if $RA_{\tau t}(y_t) > \overline{ReNOA}_{\tau t}(y_t)$, i.e., the net present value of residual operating income returns at date τ is negative. Unfortunately, this latter condition is

likely to be satisfied for firms in which future residual operating income is expected to be small (i.e., low market-to-book firms). Hence, we can only use a term structure of risk-adjusted discount rates with great caution! In the following, assume $RA_{\tau t}(y_t) < \overline{ReNOA}_{\tau t}(y_t)$ such that there is a solution for $rp_{\tau t}(y_t)$. Solving (24) with respect to $rp_{\tau t}(y_t)$ (using Proposition 9) yields

$$\begin{aligned}
& \frac{rp_{\tau t}(y_t)}{1 + \iota_{\tau t}(y_t)} \\
&= \left[\frac{\overline{ReNOA}_{\tau t}(y_t)}{\overline{ReNOA}_{\tau t}(y_t) - RA_{\tau t}(y_t)} \right]^{1/(\tau-t)} - 1 \\
&= \left[\frac{ReNOA_t^o(1 + \alpha)^{\tau-t} + \omega_r^{\tau-t}[ReNOA_{tt} - ReNOA_t^o]}{ReNOA_t^o(1 + \alpha)^{\tau-t} + \omega_r^{\tau-t}[ReNOA_{tt} - ReNOA_t^o] - \sigma_{ra} \frac{1 - (\omega_a \omega_r)^{\tau-t}}{1 - \omega_a \omega_r}} \right]^{1/(\tau-t)} - 1 \\
&= \frac{1}{\left[1 - \frac{\sigma_{ra}}{ReNOA_t^o(1 + \alpha)^{\tau-t} + \omega_r^{\tau-t}[ReNOA_{tt} - ReNOA_t^o]} \frac{1 - (\omega_a \omega_r)^{\tau-t}}{1 - \omega_a \omega_r} \right]^{1/(\tau-t)}}} - 1.
\end{aligned}$$

Note that the risk premium is increasing in the contemporaneous covariance between $ReNOA_{\tau t}$ and $racc_{\tau}$, while it is decreasing in the growth rate in residual operating income returns.

For the base case of the numerical example examined above the risk premium is shown in Figure 6.

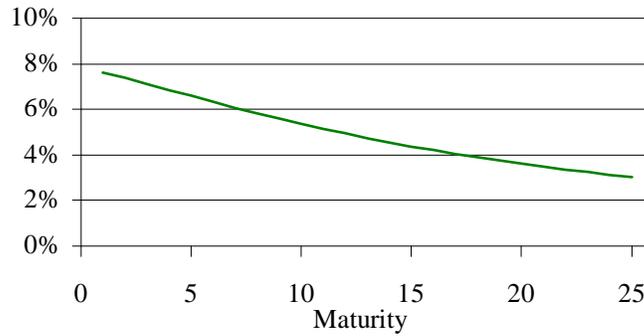


Figure 6: Risk-premium in risk-adjusted discount rate for base case.

Hence, there must be a quite substantial reduction in the risk premium for larger maturities to make this approach consistent with exponential utility model. Figure 7 shows the risk premium for the other cases examined above.

Note that the risk premium in the risk-adjusted discount rate is not defined for maturities larger than 16 years for the case with declining residual operating income returns – for these maturities $RA_{\tau t}(y_t) > \overline{ReNOA}_{\tau t}(y_t)$ and, thus, the risk premium is not defined.

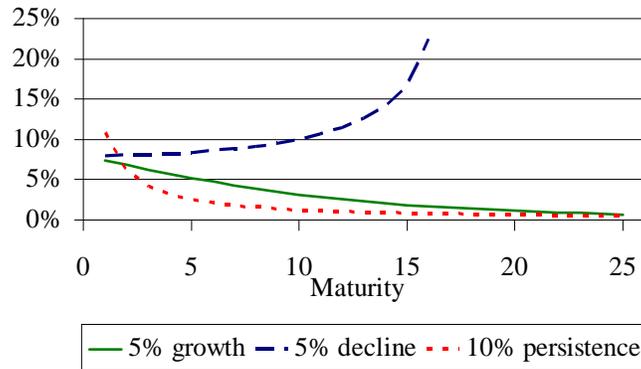


Figure 7: Risk premium in risk-adjusted discount rate for noted cases.

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Appendix A: Power Utility and Log-normal Distributions

In this appendix we examine the case in which investors have power utilities. In lieu of Theorem 7 we assume investors have the same risk cautiousness α (the slope of the linear risk tolerance function), identical personal discount factors β_t^P , and, for simplicity, that their personal minimum consumption levels b_{it} are time-independent, i.e.,

$$u_{it}(c_{it}) = \beta_t^P \frac{1}{\alpha - 1} [\alpha c_{it} - b_i]^{\frac{\alpha-1}{\alpha}}, \quad \beta_t^P > 0, \alpha c_{it} - b_i > 0.$$

The former two assumptions imply that the market is effectively dynamically complete if consumption endowments are spanned, and the investors can trade in a complete set of zero-coupon bonds

and the market portfolio. Note that the investors' relative risk aversion is increasing (decreasing) if $b_{it} < (>) 0$, while the constant relative risk aversion (CRRA) model has $b_i = 0$. Let $b_o = \sum_i b_i$ denote the aggregate minimum consumption level. In this case, Wilson (1968) shows that efficient consumption plans are on the form (see also Christensen and Feltham, 2003, Proposition 4.3)

$$c_{it}(x_t) = f_i + v_i x_t, \quad i = 1, \dots, I, \text{ where } f_i = (b_i - v_i b_o)/\alpha.$$

Scaling by the number of investors and defining the average minimum consumption level by $\bar{b} = b_o/I$, the marginal utility of consumption for any investor i is

$$u'_{it}(c_{it}) = \beta_i^P (v_i I)^{-1/\alpha} [\alpha acc_t + \bar{b}]^{-1/\alpha}.$$

Inserting this into (14) we get

$$q_{\tau t}(x_\tau | y_t) = \frac{[\alpha acc_\tau - \bar{b}]^{-1/\alpha}}{E[[\alpha acc_\tau - \bar{b}]^{-1/\alpha} | y_t]}, \quad y_\tau \subseteq y_t, \tau = t+1, \dots, T.$$

Define “log-aggregate consumption” per capita as $lacc_\tau \equiv \ln(\alpha acc_\tau - \bar{b})$, and assume this object is normally distributed. Then

$$h(lacc_\tau | y_t) \equiv q_{\tau t}(x_\tau | y_t) = \frac{\exp[-lacc_\tau/\alpha]}{E[\exp[-lacc_\tau/\alpha] | y_t]}, \quad y_\tau \subseteq y_t, \tau = t+1, \dots, T.$$

Differentiating the valuation index for log-aggregate consumption per capita and taking expectations yields

$$E \left[h'_{\tau t}(lacc_\tau) | y_\tau \right] = -\frac{1}{\alpha}.$$

Inserting this into the accounting-value relation (15) and defining risk-adjusted aggregate consumption per capita as $racc_\tau = lacc_\tau/\alpha$ yields the following result.

Proposition 10 *Assume the conditions in Theorem 4 hold and make the accounting assumptions (A)-(D). Furthermore, assume that future residual operating income and log-aggregate consumption per capita are jointly normally distributed, and investors have power utilities with identical and constant risk cautiousness and personal discount factors, and constant minimum consumption levels. Then the market-to-book ratio for the operating assets is given by*

$$\frac{S_t(y_t) - fa_t(y_t)}{oa_t(y_t)} = 1 + \sum_{\tau=t+1}^T B_{\tau t}(y_t) \{ \overline{ReNOA}_{\tau t}(y_t) - \text{Cov}[ReNOA_{\tau t}, racc_\tau | y_t] \}, \quad (25)$$

and the zero-coupon prices are given by

$$B_{\tau t}(y_t) = \frac{E[u'_{i\tau}(c_{i\tau})|y_t]}{u'_{it}(c_{it}(y_t))} = \beta_{\tau t}^P \exp[-\{E[racc_{\tau}|y_t] - racc_t - \frac{1}{2} \text{Var}[racc_{\tau}|y_t]\}], \quad (26)$$

where $\beta_{\tau t}^P$ is the investors' common personal discount factor from τ to t .

Comparing to the exponential utility model in the text, note that the only difference is a re-definition of risk-adjusted aggregate consumption per capita: average risk tolerance is substituted by the common risk cautiousness (= relative risk tolerance with $\bar{b} = 0$), and aggregate consumption per capital is substituted with log-aggregate consumption per capita. Note that if $\bar{b} = 0$ (i.e., the CRRA model), then $lacc_t \equiv \ln(\alpha) + \ln(acc_t)$ and, consequently, the VAR model for risk-adjusted aggregate consumption can be estimated directly based on log-aggregate consumption per capita without any preference-dependent parameters like in the exponential model (the risk cautiousness is just a time-independent mean adjustment). However, if $\bar{b} \neq 0$, then the VAR model for risk-adjusted aggregate consumption per capital must be estimated simultaneously with a calibration to the term structure of interest rates using (26).

Appendix B: Habit Formation

The analysis in Appendix A assumes that the investors' minimum consumption levels are constant across time. Allowing for time-dependent minimum consumption levels is straightforward. The next step is to allow for preferences exhibiting habit formation such that more variability in the types of equilibrium term structures of interest rates can be obtained. The literature distinguishes between *internal* and *external* habit formation. In the internal habit formation models, the investor's utility depends not only on his current consumption but also on his past consumption (the habit) and, in turn, his current choice of consumption affects his future habit level of consumption. In these models, the investors' utility functions are no longer time-additive, and general equilibrium analyses of this type are very rare (if they exist at all).

In the external habit formation models, the investor's date t utility depends on his current consumption as well as on some exogenous habit level of consumption, for example, on some weighted average of past aggregate consumption per capita (keeping-up-with-the-Joneses). That is, the marginal utility of consumption is increasing in the external habit level of consumption. Although several models of this type assuming a representative agent economy have been considered in the literature (see, for example, Wachter, 2006, and Campbell and Cochrane, 1999), we have not been able to locate a general equilibrium analysis with heterogeneous agents (see, however, Kraus and Sagi, 2006, for an extension to more general event-contingent preferences). The issue

is to establish aggregation, but as the following analysis demonstrates, this can be established in a similar fashion as with deterministic minimum consumption levels (see Christensen and Feltham, 2003, Chapters 4 and 6).

Consider date t utility functions of the type

$$u_{it}(c_{it}, y_t) = \beta_t^P \frac{1}{\alpha - 1} [\alpha c_{it} - b_{it}(y_t)]^{\frac{\alpha-1}{\alpha}}, \quad \beta_t^P > 0, \alpha c_{it} - b_{it}(y_t) > 0. \quad (27)$$

Note that this is the standard power utility function except that we allow the minimum consumption level to be date- and event-contingent, i.e., the utility function is event-dependent. In the following, we allow $b_{it}(y_t)$ to be some general date- and event-contingent habit formation function. Of course, in an application of the model, we could let this function be some weighted average of current and past aggregate consumption. First, we derive Pareto efficient consumption plans (assuming homogeneous beliefs as in the preceding analysis) and, secondly, we derive the equilibrium prices in an effectively dynamically complete market.

Note that the investors' utility functions are time-additive with external habit formation (as opposed to internal habit formation). Hence, the Borch first-order conditions characterizing necessary and sufficient conditions for Pareto efficient risk sharing are

$$\lambda_i u'_{it}(c_{it}(y_t), y_t) \varphi(y_t) = \mu_t(y_t), \quad \forall y_t, t = 1, \dots, T; i = 1, \dots, I; \quad (28)$$

$$\sum_{i=1}^I c_{it}(y_t) = x_t, \quad \forall y_t, t = 1, \dots, T; \quad (29)$$

where λ_i is the weight assigned to investor i , and $\mu_t(y_t)$ is the multiplier for the aggregate consumption constraint in a central planner's optimal risk sharing problem (see, for example, Christensen and Feltham, 2003, Chapter 4). Defining $\lambda_{it} \equiv \lambda_i \beta_t^P$ and inserting (27) in (28) yield

$$\lambda_{it} [\alpha c_{it}(y_t) - b_{it}(y_t)]^{-\frac{1}{\alpha}} = \frac{\mu_t(y_t)}{\varphi(y_t)} \Leftrightarrow \alpha c_{it}(y_t) - b_{it}(y_t) = \lambda_{it}^\alpha \left[\frac{\mu_t(y_t)}{\varphi(y_t)} \right]^\alpha. \quad (30)$$

Summation across investors using (29) yields

$$\alpha x_t - b_{ot}(y_t) = \lambda_{ot}^\alpha \left[\frac{\mu_t(y_t)}{\varphi(y_t)} \right]^\alpha,$$

where $b_{ot}(y_t) \equiv \sum_i b_{it}(y_t)$ and $\lambda_{ot}^\alpha \equiv \sum_i \lambda_{it}^\alpha$. Substitution back into (30) yields the investors' efficient consumption plans,

$$\alpha c_{it}(y_t) - b_{it}(y_t) = \frac{\lambda_{it}^\alpha}{\lambda_{ot}^\alpha} [\alpha x_t - b_{ot}(y_t)] = \frac{\lambda_{it}^\alpha}{\lambda_{ot}^\alpha} [\alpha x_t - b_{ot}(y_t)] \Leftrightarrow$$

$$c_{it}(y_t) = f_{it}(y_t) + v_i x_t, \quad \forall y_t, t = 1, \dots, T; i = 1, \dots, I;$$

where $f_{it}(y_t) = [b_{it}(y_t) - v_i b_{ot}(y_t)]/\alpha$ and $v_i = \lambda_i^\alpha/\lambda_o^\alpha$. Note that the efficient consumption plans have a similar form as in the standard power utility case in Appendix A. Each investor consumes a constant fraction of aggregate consumption (as a result of the assumed identical personal discount factors). However, the “fixed component” with habit formation is event-contingent such that efficient individual consumption is not necessarily measurable with respect to contemporaneous aggregate consumption. This also implies that trading in the market portfolio and a complete set of zero-coupon bonds is no longer sufficient to ensure an effectively dynamically complete market – there must be claims which allow implementation of the personal event-contingent “fixed components” associated with the habit formation. Of course, if the habit levels are generated by current and past aggregate consumption per capita, a sufficiently varied set of aggregate consumption claims will do.

Scaling by the number of investors and defining the average habit level by $\bar{b}_t(y_t) = b_{ot}(y_t)/I$, the marginal utility of consumption for any investor i is

$$u'_{it}(c_{it}(y_t)) = \beta_t^P (v_i I)^{-1/\alpha} [\alpha acc_t - \bar{b}_t(y_t)]^{-1/\alpha}.$$

Inserting this into (14) assuming an effectively dynamically complete market we get the equilibrium valuation index,

$$q_{\tau t}(y_\tau | y_t) = \frac{[\alpha acc_\tau - \bar{b}_\tau(y_\tau)]^{-1/\alpha}}{E[[\alpha acc_\tau - \bar{b}_\tau]^{-1/\alpha} | y_t]}, \quad y_\tau \subseteq y_t, \tau = t + 1, \dots, T.$$

Note that the valuation index for date τ may not be measurable with respect to aggregate consumption per capita as in the standard power utility case. Furthermore, the equilibrium prices of zero-coupon bonds are given by

$$B_{\tau t}(y_t) = \frac{E[u'_{i\tau}(c_{i\tau}) | y_t]}{u'_{it}(c_{it}(y_t))} = \beta_{\tau t}^P \frac{E[\alpha acc_\tau - \bar{b}_\tau(y_\tau)]^{-1/\alpha} | y_t]}{[\alpha acc_t - \bar{b}_t(y_t)]^{-1/\alpha}}.$$

Hence, both the equilibrium valuation index and the equilibrium prices of zero-coupon bonds are independent of both the distribution of initial wealth (as reflected in the equilibrium fractions of aggregate consumption, $v_i = \lambda_i^\alpha/\lambda_o^\alpha$) and the distribution of event-contingent habit levels, $b_{it}(y_t)$. That is, power utilities with identical risk cautiousness (and personal discount factors) and a general external habit formation allow aggregation. Using this result and assuming that “habit-adjusted log-aggregate consumption” per capita, $lacc_\tau(y_\tau) \equiv \ln(\alpha acc_\tau - \bar{b}_\tau(y_\tau))$, is normally distributed, the analysis proceeds as in the standard power utility case in Appendix A (see Wachter, 2006, and Campbell and Cochrane, 1999, for particular parameterizations of “habit-adjusted log-aggregate

consumption” per capita).