OPTIMALITY AND REPRESENTATION OF COMPETITIVE EQUILIBRIA WITH TIME-DEPENDENT PREFERENCES[1]

PAWEŁ DZIEWULSKI

JUNE 2013

ABSTRACT: This paper focuses on welfare properties of competitive equilibria of exchange economies with time-dependent preferences. We introduce a notion of recursive efficiency, and show that competitive equilibria are efficient in the sense defined. Moreover, we present a social welfare function with maximisers coinciding with recursively efficient allocations. We also show that every competitive equilibrium can be represented by a solution to a social welfare optimisation problem. Finally, we discuss the relevance of our results to allocations arising in sequential equilibria.

KEYWORDS: time-dependent preferences, Walrasian equilibrium, First Fundamental Welfare Theorem, social welfare.

1. INTRODUCTION

Consider an exchange economy consisting of consumers endowed with time-dependent preferences. Each period, an agent is represented by a different self, whose preferences are defined over paths of future consumption. Since preferences of subsequent incarnations may differ across periods and consumers have no access to any commitment technology, they need to take into account the behaviour of their future selves while determining their consumption. Assuming that the agents exhibit a sufficient level of sophistication and are able to correctly predict their future decisions, the demand is equivalent to a Subgame Perfect Nash Equilibrium (henceforth SPNE) path of a game between different selves of a consumer. In this paper we discuss welfare properties of equilibrium allocations arising in the framework.

Department of Economics, University of Oxford, Manor Road Building, Manor Road, Oxford OX1 3UQ, United Kingdom. Email: pawel.dziewulski@economics.ox.ac.uk. Website: http://users.ox.ac.uk/~shil3804/.

[1] I am especially grateful to John Quah for his invaluable help during writing this paper. In addition, I would like to thank Alan Beggs, Daniel Beary, Francis Dennig, Faruk Gul, Jakub Growiec, Jean-Jacques Herings, Sujoy Mukerji, Collin Raymond, Kevin Reffett, as well as participants of the XXII European Workshop on General Equilibrium Theory in Vienna (2013) for many insightful comments and suggestions. Finally, I thank Łukasz Woźny for his constant support over the years. This is a preliminary version of the paper. Errors may remain. Do not recirculate, rather write the author for the current version.

We introduce a notion of recursive efficiency, and present conditions under which every competitive equilibrium allocation is efficient in the defined sense. Moreover, we construct a social welfare function with maximisers coinciding with recursively efficient allocations. Eventually, the two results allow us to present a method of representing competitive equilibria by a solution to a social planner's optimisation problem.

The literature concerning welfare properties of economies with time-dependent preferences concentrates on conditions and mechanisms which allow to obtain equilibrium allocations that are efficient with respect to preferences of consumers in the initial period. We will say that such allocations are *Pareto efficient with respect to the initial selves*.[1] Laibson (1997) has shown that once agents have access to illiquid financial instruments, they are able to commit future selves to a plan which is optimal with respect to individual preferences of the initial self. Moreover, once agents can trade the long term assets, allocations resulting in equilibrium are Pareto efficient with respect to the initial selves.

Interestingly, in some special cases, even when individual agents are unable to commit, competitive equilibrium allocations can be efficient in the aforementioned sense. This property was first observed by Barro (1999) for production economies with consumers endowed with time-separable, logarithmic preferences, and hyperbolic discounting. The result is surprising, since it implies that even though every individual agent is bounded by a time-consistency constraint, there exists no other feasible allocation which could strictly improve upon the equilibrium allocation with respect to the initial agents. Unfortunately, apart from some very specific cases, such equilibria are non-generic. As shown by Luttmer and Mariotti (2007, Proposition 3), once preferences are not homothetic, the set of equilibria and the set of allocations which are Pareto efficient with respect to the initial selves intersect only at isolated points. Their negative result indicates, that this form of efficiency is rare in the discussed class of models.

In this paper we establish the general welfare properties of competitive economies with time-dependent preferences, rather than determine conditions under which equilibrium allocations satisfy a desired notion of optimality. This makes our results closely related to the ones obtained by Herings and Rohde (2006), who analysed a similar problem. However, there are several substantial differences between the two papers. First of all, we concentrate solely on economies with sophisticated agents who can correctly predict the behaviour of their future selves. Second of all, we do not discuss the existence of

[1] Herings and Rohde (2006, Definition 10) simply call such allocation *Pareto efficient*. On the other hand Luttmer and Mariotti (2007, Definition 1(i)) use the term *date-1 Pareto efficient* to describe the same notion of optimality.

competitive equilibrium. Moreover, we modify the definition of competitive equilibrium in a relevant way and present our results for a stronger definition of efficiency.

We consider a modified notion of optimality, which we call *recursive efficiency*. According to our definition, an allocation x is efficient if for any date t there exists no other feasible allocation which Pareto improves upon x with respect to preferences of all agents and their future selves following period t. The form of optimality is a stronger concept than the so called *time-consistent overall Pareto efficiency* introduced by Herings and Rohde (2006, Definition 27).[2]

Overall Pareto efficiency seems to be a natural way of understanding welfare when preferences are time-variant. According to the criterion, an allocation x is efficient if there exists no other feasible allocation which Pareto improves upon x with respect to preferences of all agents and their different selves. The notion implies that an improvement can be made only if it makes all the consumers and their different selves weakly better off, and some of them (i.e. at least one self of a consumer) strictly better off. Therefore, the definition of efficiency is equivalent to the standard Pareto criterion once we consider each self of every consumer to be a separate agent. As we discuss it in Section 3.1, overall Pareto efficient allocations exhibit a form of time-inconsistency. This is due to the fact, that as time progresses and the initial selves are gradually excluded from the economy, the remaining incarnations might have an incentive to reallocate the consumption in the following periods, and make themselves better off. Recursive efficiency excludes such cases.

Our notion of optimality is a weaker concept than the so called *renegotiation proofness* introduced by Luttmer and Mariotti (2007, Definition 1(ii)). According to their definition, an allocation x is renegotiation proof if it is recursively efficient according to our sense, and there exists no other recursively efficient allocation which Pareto dominates x with respect to preferences of the initial selves. In other words, once we constraint the set of feasible allocations to the set of recursively efficient consumption paths, renegotiation proofness coincides with Pareto efficiency with respect to the initial selves. We present a formal characterisation of the two notions in Definition 5.

Given the definition of efficiency, we show that any competitive equilibrium allocation is efficient in our sense. Therefore, we present a version of First Fundamental Welfare Theorem for competitive economies with time-variant preferences. Our result is positive, as we determine a general class of economies for which competitive equilibria are efficient in the defined sense.

The result differes from the one obtained by Herings and Rohde (2006, Theorem 30)

[2] Often the notion is called *weak efficiency* or *multiself Pareto criterion*.

in three aspects. First of all, our theorem concerns a stronger concept of optimality. In addition, we show that the efficiency of equilibria holds under weaker assumptions imposed on consumer preferences. Finally and most importantly, we apply a modified definition of competitive equilibrium.

In the equilibrium specification of Herings and Rohde (2006, Definition 11), agents are not allowed to transfer their wealth across time, as each period t they are bounded to consume only these consumption bundles which value does not exceed the value of their initial endowment of period t goods. This restricts consumers' possibility to save or borrow. Therefore, the assumption rules out an important channel of strategic interaction between different selves of an agent, which we consider to be the key feature of the discussed class of models. We relax the condition in this work.

Eventually, in the second part of our paper we present conditions under which every recursively efficient allocation can be represented by a solution to a social welfare maximisation problem. We consider this result to be important for two reasons. First of all, it allows to reduce a rather difficult problem of computing equilibrium allocations to a relatively simple maximisation program. Moreover, it establishes a form of a representative agent in the discussed class of economies.

The result refers to the characterization of competitive equilibria presented by Negishi (1960), who has shown that every competitive equilibrium can be represented as a solution to a weighted social welfare maximisation problem. We extend this idea to exchange economies with time-dependent preferences, and introduce a notion of *recursive social welfare allocation* obtained via a multi-stage optimisation problem. In our specification, at each stage the social planner maximises a weighted social welfare function of current selves, subject to him choosing amongst allocations that solve an analogous social welfare problem in all the subsequent periods. Therefore, the construction of recursive social welfare corresponds to the behaviour of every individual consumer, as it imposes a form of time-consistency on socially optimal allocations. The approach presented by Negishi have found a wide application to welfare economics, general equilibrium, as well as macroeconomics. For this reason, we believe that extending the idea to economies with time-dependent preferences will be useful for more applied studies of the discussed class of economies.

In Section 2 we introduce our framework and the necessary notation. Then, in Section 3 we characterise the notion of recursive efficiency and present our main result concerning efficiency of competitive equilibria. Section 4 concerns representation of recursively efficient allocations by a solution to a recursive social welfare optimisation problem. Finally in Section 5 we discuss several issues related to different notions of equilibrium, and how they affect results presented in the paper. Proofs as well as auxiliary results which are not included in the main body of the paper are presented in the Appendix.

2. ECONOMY WITH TIME-DEPENDENT PREFERENCES

Consider a multiple-period exchange economy with a finite set of consumers I. With a slight abuse of notation, I shall also denote the cardinality of the set. By T we denote a finite set of time indices. Dates are labelled in a decreasing manner, i.e. $t = 0$ denotes the final period in the economy, $t = 1$ the second to last, and so on. With a slight abuse of notation we label the initial date by T. Therefore, $T := \{T, T-1, \dots, 1, 0\}$.

Let $X_t = \mathbb{R}^{n_t}_+$ be a positive orthant of a n_t dimensional Euclidean space, $t \in T$. We shall refer to X_t as to the period t *commodity space*. Hence, n_t is the number of goods/markets available in the economy at date t. Denote elements of X_t by x^i_t, $i \in I$. That is, x^i_t is a consumption bundle of period t goods of consumer i.

Due to the dynamic nature of our framework, apart from consumption bundles in separate periods, it is useful to consider their paths. Let $\hat{X}_t := \times_{s=0}^{t} X_s$ be a *set of consumption paths* from date t to the final period 0. Therefore, an element $\hat{x}^i_t \in \hat{X}_t$ is a path/sequence of bundles $\hat{x}^i_t = (x^i_s)^t_{s=0}$, where for all s, $x^i_s \in X_s$. We shall refer to $\hat{x}^i_t \in \hat{X}_t$ as to a *consumption path* of consumer i following period t. In particular, $\hat{x}^i_T$ is a complete consumption path of consumer i from the initial date T till the final period 0. Moreover, by definition $\hat{x}^i_t = (x_t, \hat{x}^i_{t-1}) = (x^i_t, x^i_{t-1} \dots, \hat{x}^i_{t'})$, for any $t' \leq t$.

Apart from consumption bundles and their paths, we shall often refer to the notion of an *allocation*. A period t allocation is a vector $x_t \in X^I_t$, where $x_t := (x^i_t)_{i\in I}$. In addition, an *allocation path* following date t will be denoted by $\hat{x}_t \in \hat{X}^I_t$, $\hat{x}_t := (\hat{x}^i_t)_{i\in I}$. Similarly, $\hat{x}_T$ is a complete allocation path. Moreover, as previously $\hat{x}_t = (x_t, \hat{x}_{t-1}) = (x_t, x_{t-1} \dots, \hat{x}_{t'})$, for any $t' \leq t$.

As in Strotz (1955), we characterise agents by a sequence of preference relations $\{\succeq^i_t\}_{t\in T}$. We shall refer to $\succeq^i_t$ defined over $\hat{X}_t$, as to a preference relation of period t self of agent i.[3] For any $t' < t$ we allow for preference relations $\succeq^i_{t'}$ and $\succeq^i_t$ to differ over $\hat{X}_{t'}$. That is, we admit the case in which for some $\hat{x}^i_t \in \hat{X}_t$ and $\hat{x}'^i_{t'} \in \hat{X}_{t'}$, we have $\hat{x}'^i_{t'} \succeq^i_{t'} \hat{x}^i_{t'}$ and $\hat{x}^i_t \succeq^i_t (x^i_t, x^i_{t-1}, \dots, \hat{x}'^i_{t'})$ at the same time, which would suggest a preference reversal. In fact, the change of preferences in between the two periods is the source of time-inconsistency in our analysis. Moreover, we assume that preferences of period t selves are not directly affected, nor depend on consumption in the preceding periods.[4] We denote anti-symmetric, and

[3]Formally, we say that $\succeq^i_t \subset \hat{X}_t \times \hat{X}_t$.

[4]Nevertheless, as in the original formulation of the problem (see Strotz, 1955), period t self of agent i

symmetric elements of $\succeq_t^i$ in the standard fashion by $\succ_t^i$ and $\sim_t^i$, $t \in T$.

In order to make our presentation more transparent, we apply our framework to time-separable preferences with quasi-hyperbolic discounting.

Example 1 (Quasi-hyperbolic discounting) Let correspondence $v^i : \mathbb{R}_+^n \to \mathbb{R}$ denote an instantaneous utility function of agent $i \in I$, while $\delta_i, \gamma_i \in [0,1)$ his long-term and present-bias discount factor respectively. Let $X_t := \mathbb{R}_+^n$ for all $t \in T$. Hence, $\hat{X}_t = \mathbb{R}_+^{n(t+1)}$. Utility of period t self of consumer i is evaluated by function $u_t^i : \hat{X}_t \to \mathbb{R}$:

$$u_t^i(\hat{x}_t^i) := v^i(x_t^i) + \gamma_i \sum_{s=0}^{t-1} \delta_i^{t-s} v^i(x_s^i).$$

Therefore, whenever we define preferences $\{\succeq_t^i\}_{t\in T}$ such that for all $t \in T$ and any two $\hat{x}_t'^i$, $\hat{x}_t^i \in \hat{X}_t$ we have

$$\hat{x}_t'^i \succeq_t^i \hat{x}_t^i \quad \Leftrightarrow \quad u_t^i(\hat{x}_t'^i) \geq u_t^i(\hat{x}_t^i),$$

time-separable preferences with quasi-hyperbolic discounting are embedded in our framework.

In the initial period each consumer has an endowment $(e_t^i)_{t\in T} \in \times_{t\in T} X_t$, where e_t^i denotes rights to consumption in period t of consumer i. We find it useful to define paths of endowments following date t by $\hat{e}_t^i := (e_s^i)_{s=0}^t \in \hat{X}_t$. In particular, $\hat{e}_T^i$ is equivalent to the initial endowment of agent i.

In the remainder of the section we introduce the notion of a competitive equilibrium. To make our presentation more comprehensible, we first discuss in detail a two-period case. Then, we extend the framework to an arbitrary number of periods.

2.1. *Equilibrium in a two-period economy*

Consider a case where $T = \{1, 0\}$. Then, each consumer is characterized by a pair of preference relations $\{\succeq_1^i, \succeq_0^i\}$, where $\succeq_1^i$ is defined over $\hat{X}_1 := X_1 \times X_0$, and $\succeq_0^i$ over X_0. Moreover, in the initial period consumer i has a path of endowments following date 1 denoted by $\hat{e}_1^i := (e_1^i, e_0^i) \in \hat{X}_1$.

We consider the following structure of trade in the economy. At the initial date $t = 1$ agents may trade their initial endowments for the current consumption bundles $x_1^i \in X_1$,

is affected by the previous consumption in an indirect way, via the budget constraint. This condition is equivalent to *strong independence of past consumption* introduced by Herings and Rohde (2006, Definition 24).

as well as rights to consumption in the final period $t = 0$, denoted by $y_0^i \in X_0$. Since agents are are endowed with time-dependent preferences without the commitment technology, the vector of rights to consumption y_0^i acquired at date 1 may be different from the actual consumption x_0^i taking place in period 0.

Let $p_1 \in \mathbb{R}^{n_1}_{++}$ and $p_0 \in \mathbb{R}^{n_0}_{++}$ denote prices of date 1 and date 0 consumption goods/rights respectively. Again, we shall consider paths of prices $\hat{p}_1 = (p_1, p_0) \in \mathbb{R}^{n_1+n_0}_{++}$.

Given $\hat{p}_1$, the total wealth of consumer i is equal to the value of his initial endowment $\hat{p}_1 \cdot \hat{e}_1^i$. Hence, the budget set of agent i is determined by values of correspondence $B_1 : \mathbb{R}^{n_1+n_0}_{++} \times \hat{X}_1 \rightrightarrows \hat{X}_1$,

$$B_1(\hat{p}_1, \hat{e}_1^i) := \left\{ (x_1^i, y_0^i) \in \hat{X}_1 \;\middle|\; p_1 \cdot x_1^i + p_0 \cdot y_0^i \leq \hat{p}_1 \cdot \hat{e}_1^i \right\}. \tag{2.1}$$

Next, consider the budget set of agent i at date 0. At the beginning of the period, the agent inherits the rights to consumption y_0^i acquired at $t = 1$. Since x_1^i has already been consumed, it is no longer taken into account. Therefore, the disposable wealth of period 0 self is equal to the value of the inherited rights to consumption $p_0 \cdot y_0^i$, and so date 0 budget set is defined by values of correspondence $B_0 : \mathbb{R}^{n_0}_{++} \times X_0 \rightrightarrows X_0$,

$$B_0(p_0, y_0^i) := \left\{ x_0^i \in X_0 \mid p_0 \cdot x_0^i \leq p_0 \cdot y_0^i \right\}. \tag{2.2}$$

In the paper we analyse economies where sophisticated agents are endowed with time-dependent preferences and no commitment technology. This implies, that while determining their consumption paths consumers can correctly predict preferences and choices of their future selves, but cannot commit to any consumption plan. This implies, that date 1 selves determine their choices of (x_1^i, y_0^i) conditional on what will be chosen by their period 0 selves given y_0^i.

In order to formally define the demand, we first need to establish how the choice is made in the final period. Take a vector $y_0^i \in X_0$ of period 0 consumption rights and date 0 prices p_0. The set of choices of period 0 self is equivalent to the set of the greatest elements of $B_0(p_0, y_0^i)$ with respect to $\succeq_0^i$.[5] Hence, it is governed by values of correspondence $V_0^i : \mathbb{R}^{n_0}_{++} \times X_0 \rightrightarrows X_0$,

$$V_0^i(p_0, y_0^i) := \left\{ x_0^i \in X_0 \mid x_0^i \text{ is a } \succeq_0^i\text{-g.e. of } B_0(p_0, y_0^i) \right\}. \tag{2.3}$$

The sophistication of date 1 selves implies, that while acquiring (x_1^i, y_0^i) in the initial period, agents take into account that the actual consumption taking place at date 0 must

[5] For some binary relation $\succeq$, we consider x' to be a greatest element of set X with respect to $\succeq$, or a $\succeq$-g.e. of X, if $x' \in X$ and $\forall x \in X, x' \succeq x$.

belong to $V_0^i(p_0, y_0^i)$. This is to say, that since period 0 self is not committed to any plan, he will choose the most preferable bundle from his budget set given the inherited vector of consumption rights. Therefore, the problem of the consumer in the initial period is to maximise preferences over the set of affordable, time-consistent consumption paths, i.e. vectors $(x_1^i, x_0^i) \in B_1(\hat{p}_1, \hat{e}_1^i)$, where $x_0^i \in V_0^i(p_0, x_0^i)$. The set of all such consumption paths is determined by values of correspondence $F_1^i : \mathbb{R}_{++}^{n_1+n_0} \times \hat{X}_1 \rightrightarrows \hat{X}_1$,

$$(2.4) \qquad F_1^i(\hat{p}_1, \hat{e}_1^i) := \left\{ (x_1^i, x_0^i) \in B_1(\hat{p}_1, \hat{e}_1^i) \;\middle|\; x_0^i \in V_0^i(p_0, x_0^i) \right\},$$

Elements of $F_1^i(\hat{p}_1, \hat{e}_1^i)$ are time-consistent in the sense, that once period 1 self acquires $y_0^i = x_0^i$ rights to period 0 consumption, in the following period the date 0 self has no incentive to re-trade the inherited consumption rights. Hence, the consumption plan determined in period 1 will actually be implemented at the final date. This allows to define correspondence $V_1^i : \mathbb{R}_{++}^{n_1+n_0} \times \hat{X}_1 \rightrightarrows \hat{X}_1$,

$$(2.5) \qquad V_1^i(\hat{p}_1, \hat{e}_1^i) := \left\{ (x_1^i, x_0^i) \in \hat{X}_1 \;\middle|\; (x_1^i, x_0^i) \text{ is a } \succeq_1^i\text{-g.e. of } F_1^i(\hat{p}_1, \hat{e}_1^i) \right\},$$

which values determine the set of optimal, time-consistent choices of agent i.

By construction set $V_1^i(\hat{p}_1, \hat{e}_1^i)$ consists of consumption paths which emerge in a SPNE path of an intrapersonal game between different selves of agent i. Moreover, since the initial self is allowed to choose from the set of time consistent paths these elements $\hat{x}_1^i$ which maximise his current preferences, we assume that any ties that may arise in the choice of period 0 self are broken by the initial consumer.[6]

We proceed with a two-period definition of competitive equilibrium.

Definition 1 (Competitive equilibrium) *Given endowment distribution* $(\hat{e}_1^i)_{i\in I}$*, a* competitive equilibrium *of an economy starting at date 1 is a pair of allocation and price paths* $\{\hat{x}_1^*, \hat{p}_1^*\}$ *such that*

(i) given $\hat{p}_1^*$*, consumers maximise their preferences in a time-consistent manner, i.e. for all* $i \in I$*, we have* $\hat{x}_1^{*i} \in V_1^i(\hat{p}_1^*, \hat{e}_1^i)$*;*

(ii) markets clear, i.e. $\sum_{i\in I} \hat{x}_1^{*i} = \sum_{i\in I} \hat{e}_1^i$.

We discuss the relevance of the above definition in Section 2.3. In the following section we extend the above notion to an arbitrary number of periods.

[6] This specification is equivalent to the one introduced by Strotz (1955). However, it is a different formulation from the one investigated by Harris and Laibson (2001), or more recently Balbus, Reffett, and Woźny (2011) in the infinite dimensional framework.

2.2. *Equilibrium in a multi-period economy*

Let T be an arbitrary number in $\mathbb{N}$. Let $p_t \in \mathbb{R}^{n_t}_{++}$ denote prices of period t consumption goods, and $N_t := \sum_{s=0}^{t} n_s$. Similarly to the two-period case, vector $\hat{p}_t = (p_s)_{s=0}^{t} \in \mathbb{R}^{N_t}_{++}$ denotes a path of prices of consumption bundles consumed in periods following date t. By construction, we have $\hat{p}_t = (p_t, \hat{p}_{t-1}) = (p_t, p_{t-1} \dots, \hat{p}_{t'})$ for any t and $t' \leq t$.

We construct the optimisation problem of date t self of agent i as follows. In the final period 0, given prices p_0 as well as a vector of rights to consumption y^i_0, agent i determines the $\succeq^i_0$-greatest elements of $B_0(p_0, y^i_0)$, defined in (2.2). Hence, the set of his optimal choices is equal to $V^i_0(p_0, y^i_0)$, as in (2.3).

In period $t = 1$, the consumer determines the set of all affordable, time-consistent consumption paths and chooses the one which maximises his current preferences. The only difference with respect to the two-period case is that at the beginning of date 1 the current self is in possession of a path of consumption rights $\hat{y}^i_1 := (y^i_1, y^i_0) \in \hat{X}_1$ inherited from the preceding period, rather than $\hat{e}^i_1$. Therefore, the set of affordable, time-consistent consumption paths is $F^i_1(\hat{p}_1, \hat{y}^i_1)$, where correspondence F^i_1 is defined as in (2.4). Similarly, the set of choices is $V^i_1(\hat{p}_1, \hat{y}^i_1)$, where V^i_1 is defined as in (2.5).

By backward induction, it is possible to determine the set of all affordable and time-consistent consumption paths for any $t \in T$. At the beginning of date t, every consumer is in possession of a vector of rights to consumption $\hat{y}^i_t = (y^i_s)_{s=0}^{t} \in \hat{X}_t$. Budget set is then determined by values of correspondence $B_t : \mathbb{R}^{N_t}_{++} \times \hat{X}_t \rightrightarrows \hat{X}_t$,

$$B_t(\hat{p}_t, \hat{y}^i_t) := \left\{ \hat{x}^i_t \in \hat{X}_t \;\middle|\; \hat{p}_t \cdot \hat{x}^i_t \leq \hat{p}_t \cdot \hat{y}^i_t \right\}. \tag{2.6}$$

Hence, the set of affordable and time-consistent consumption paths is determined by $F^i_t : \mathbb{R}^{N_t}_{++} \times \hat{X}_t \rightrightarrows \hat{X}_t$,[7]

$$F^i_t(\hat{p}_t, \hat{y}^i_t) := \left\{ (x^i_t, \hat{x}^i_{t-1}) \in B_t(\hat{p}_t, \hat{y}^i_t) \mid \hat{x}^i_{t-1} \in V^i_{t-1}(\hat{p}_{t-1}, \hat{x}^i_{t-1}) \right\}, \tag{2.7}$$

where $V^i_{t-1}(\hat{p}_{t-1}, \hat{x}^i_{t-1})$ is the set of optimal, time-consistent choices of the following, date $(t-1)$ self. Being consistent with our recursive structure, the set is determined by values of correspondence $V^i_t : \mathbb{R}^{N_t}_{++} \times \hat{X}_t \rightrightarrows \hat{X}_t$,

$$V^i_t(\hat{p}_t, \hat{y}^i_t) := \left\{ \hat{x}^i_t \in \hat{X}_t \;\middle|\; \hat{x}^i_t \text{ is a } \succeq^i_t\text{-g.e. of } F^i_t(\hat{p}_t, \hat{y}^i_t) \right\}. \tag{2.8}$$

Correspondences F^i_t and V^i_t are constructed in the following way. Given date t, a path of prices – $\hat{p}_t$, and a sequence of rights to consumption following date t – $\hat{y}^i_t$, we determine

[7] Since we consider date 0 to be the final period, there is no consumption taking place beyond it. Therefore, every element of the set $B_0(p_0, y^i_0)$ is trivially time-consistent, and so $F^i_0(p_0, y^i_0) \equiv B_0(p_0, y^i_0)$.

the set of all affordable consumption paths following date t – $B_t(\hat{p}_t, \hat{y}_t^i)$. Every element $\hat{x}_t^i = (x_t^i, \hat{x}_{t-1}^i)$ of the set consists of the current consumption bundle x_t^i and a sequence of consumption rights/bundles following period $(t-1)$, $\hat{x}_{t-1}^i$. In order to make sure that $\hat{x}_t^i \in F_t^i(\hat{p}_t, \hat{y}_t^i)$, we need to guarantee that $\hat{x}_{t-1}^i$ is a solution to the optimisation problem of the subsequent incarnation of agent i, given that he inherits $\hat{x}_{t-1}^i$. Only then the future self has no incentive to choose a different consumption path when date $(t-1)$ arrives. In other words, as long as $\hat{x}_{t-1}^i \in V_{t-1}^i(\hat{p}_{t-1}, \hat{x}_{t-1}^i)$, date $(t-1)$ self cannot strictly benefit from re-trading $\hat{x}_{t-1}^i$. Finally, set $V_t^i(\hat{p}_t, \hat{y}_t^i)$ consists of $\succeq_t^i$-greatest elements of $F_t^i(\hat{p}_t, \hat{y}_t^i)$. Hence, it contains the most preferable, affordable, and time-consistent consumption bundles from the perspective of period t self.

At this point we define a generalised notion of competitive equilibrium introduced in Definition 1.

DEFINITION 1' (Competitive equilibrium) *Given endowment distribution $(\hat{e}_T^i)_{i\in I}$, a* competitive equilibrium *of an economy starting at date T is a pair of allocation and price paths $\{\hat{x}_T^*, \hat{p}_T^*\}$ such that*

(i) given $\hat{p}_T^*$, consumers maximise their preferences in a time-consistent manner, i.e. for all $i \in I$, we have $\hat{x}_T^{*i} \in V_T^i(\hat{p}_T^*, \hat{e}_T^i)$;

(ii) markets clear, i.e. $\sum_{i\in I} \hat{x}_T^{*i} = \sum_{i\in I} \hat{e}_T^i$.

Clearly, in the two-period case Definitions 1 and 1' are equivalent. The following result is directly implied by the above definition.

COROLLARY 1 *Let $\{\hat{x}_T^*, \hat{p}_T^*\}$ be a competitive equilibrium of an economy starting at date T. Then, for any $t \leq T$, $\{\hat{x}_t^*, \hat{p}_t^*\}$ is a competitive equilibrium of an economy starting at date t, for distribution of endowments $(\hat{x}_t^{*i})_{i\in I}$.*

PROOF: Let $\{\hat{x}_T^*, \hat{p}_T^*\}$ be a competitive equilibrium. Take any $t \leq T$, and adjust Definition 1' for $T = t$, with $(\hat{x}_t^{*i})_{i\in I}$ being the distribution of the initial endowment. Clearly, condition (ii) of Definition 1' is satisfied. By construction, $\forall i \in I$, $\hat{x}_T^{*i} \in V_T^i(\hat{p}_T^*, \hat{e}_T^i)$ implies $\hat{x}_t^{*i} \in V_t^i(\hat{p}_t^*, \hat{x}_t^{*i})$, for any $t < T$. Hence, condition (i) of Definition 1' also holds. *Q.E.D.*

In the next section we discuss several important issues concerning the nature of equilibria defined above.

2.3. *Comments on the notion of competitive equilibrium*

The definition of equilibrium introduced in the preceding sections requires some comment. First of all, the notion is very closely related to the standard concept of competitive equilibrium. As in the standard definition, trade in our economy takes place only once at the initial date T. This is implied by the requirement that regardless of the date, prices of the rights to consumption of period t goods – y_t^i, are equal to the prices of the actual consumption x_t^i taking place at date t. Therefore, there are no separate spot and futures markets where agent could exchange either goods or rights to future consumption.

The fact that markets do not reopen in the following periods does not imply that period T selves can commit their future incarnations to any consumption plan. Note, that the form of the optimisation problem of period T agents guarantees that all the future selves would not be willing to re-trade their consumption plans, and therefore captures the dynamic nature of the analysed problem. Since correspondence F_T^i imposes a time-consistency constraint on choices of period T selves, even though the above definition is static, equilibrium allocations are time-consistent.

Our notion is closely related to the one introduced by Luttmer and Mariotti (2003, 2006), who characterised the equilibrium for economies with time-separable preferences and hyperbolic discounting. Similarly to our definition, their economy is static in the sense that prices of rights to consumption in any period are constant regardless of the date when they are acquired. It is worth pointing out that the condition is also imposed by Herings and Rohde (2006) in their characterisation of competitive equilibrium.

What is interesting, is that even though the condition discussed above seems to be restrictive, it is not. In Section 5 we show that even if we allow for markets to reopen each period, and prices of rights to future consumption to vary from date to date, it does not affect our results. In fact, in equilibrium prices of the actual period t consumption and rights to consumption have to coincide at any date, which makes our requirement superfluous.

Regarding definition by Herings and Rohde (2006, Definition 11), our notion differs in one important aspect. Herings and Rohde characterise the optimisation problem of consumers in a way which does not allow them to spend on the current period consumption x_t^i more than the value of their initial endowment of period t goods – e_t^i. In other words, in every period t the budget set of agent i consists of only these consumption paths $\hat{x}_t^i$ for which $p_t \cdot x_t^i \leq p_t \cdot e_t^i$, for all $t \in T$.

The condition rules out the possibility of transferring wealth across periods strategically, which we consider to be the essence of the discussed class of models. Note that without commitment, the only channel that allows a sophisticated agent to influence his future consumption is through the value of consumption rights $\hat{y}_t^i$ which are inherited by the subsequent self. Only this way the current self can influence the decision of his future incarnations. Hence, by fixing the amount of wealth that an agent can spend each period, one looses an important feature of economies with time-dependent preferences.

As implied by Definition 1 and 1', this is not the case in our paper. In the discussed framework consumers may freely transfer their wealth across periods, as long as their total expenditure does not exceed the value of the initial endowment. Still, we are able to show that this does not affect the welfare properties of the discussed class of economies.

3. EFFICIENCY AND COMPETITIVE EQUILIBRIUM

In the following section we discuss welfare properties of competitive equilibria. We define our notion of efficiency and discuss its relevance with respect to the already existing concepts of optimality in the related literature. Moreover, we show that any competitive equilibrium allocation satisfies the efficiency condition.

3.1. *Efficient allocations and time-dependent preferences*

We say that period t allocation $x_t \in X_t$ is *feasible* if and only if $\sum_{i\in I} x_t^i \leq \sum_{i\in I} e_t^i$. An allocation path $\hat{x}_t \in \hat{X}_t^I$ is feasible if $\sum_{i\in I} \hat{x}_t^i \leq \sum_{i\in I} \hat{e}_t^i$, and we shall denote the set of all such allocation paths by

$$E_t := \left\{ \hat{x}_t \in \hat{X}_t^I \;\middle|\; \sum_{i\in I} \hat{x}_t^i \leq \sum_{i\in I} \hat{e}_t^i \right\}. \tag{3.1}$$

First, we introduce the notion of *post-t efficiency*, which shall become useful in the remainder of this section.[8]

DEFINITION 2 (Post-t efficiency) *For some $t \in T$, a path of allocations $\hat{x}_t \in E_t$ is* post-t efficient *if there exists no other $\hat{x}'_t \in E_t$ such that $\forall i \in I$ and $\forall t' \leq t$, we have $\hat{x}'^i_{t'} \succeq^i_{t'} \hat{x}^i_{t'}$, and for some $i \in I$ and some $t' \leq t$, $\hat{x}'^i_{t'} \succ^i_{t'} \hat{x}^i_{t'}$.*

According to Definition 2, we say that a feasible path of allocations following date t is post-t efficient if it is Pareto optimal with respect to preferences of all agents and their different selves following period t. In particular, given our framework, a post-T allocation is equivalent to the notion of time-consistent overall Pareto efficiency.

Building up on the previous definition, we characterize a notion of efficiency which will be of the central interest in the remainder of this paper.

DEFINITION 3 (Recursive efficiency) *A path of allocations $\hat{x}_T \in E_T$ is* recursively efficient*, if for any $t \in T$ path $\hat{x}_t$ is post-t efficient.*

[8]I am grateful to John Quah for inventing the term.

According to the definition, an allocation path is recursively efficient if it is Pareto optimal with respect to any subsequence of preference relations $\{\succeq^i_{t'}\}_{i\in I, t'\leq t}$ following any period t. Therefore, for $\hat{x}_T$ to be recursively efficient we require that allocation x_0 is Pareto optimal with respect to preference relations $\{\succeq^i_0\}_{i\in I}$, $\hat{x}_1$ is Pareto optimal with respect to $\{\succeq^i_1, \succeq^i_0\}_{i\in I}$, and so on. In particular we require, that an efficient allocation path is post-T efficient. Therefore, as mentioned earlier, it is overall Pareto efficient.

The main idea behind the definition of recursive efficiency concerns a form of time-consistency of optimal allocations. Consider a path $\hat{x}_T \in E_T$ which is post-T efficient, but not recursively efficient. By Definition 2, this implies that the allocation path is Pareto optimal with respect to preferences following period T, $\{\succeq^i_t\}_{(i,t)\in I\times T}$. Hence, there exists no other feasible sequence of allocations, which makes all the agents and their different selves weakly better off, and at least some of them (i.e. at least one self of any agent) strictly better off.

Now, assume that period T has passed, and consumers find themselves in period $T-1$. Since period T selves are no longer present in the economy, the remaining selves following date $T-1$ might be willing to alter the allocation of consumption in the remaining periods. As period T preferences are no longer taken into consideration, there might exist a distribution of goods which Pareto improves the previously determined allocation for the remaining selves. The idea of recursive efficiency is to exclude such cases.

As mentioned above, recursive efficiency is a stronger notion than overall Pareto efficiency. On the other hand, it is a weaker notion that renegotiation proofness. Consider a two-period case as in Luttmer and Mariotti (2007) (see Definition 5(ii) for an extension of the definition to multiple periods). Denote the set of period 0 Pareto efficient allocations by

$$R_0 := \{x_0 \in X_0 \mid \neg\exists x_0 \in E_0 \text{ that } \forall i \in I,\, x'^i_0 \succeq^i_0 x^i_0, \text{ and for some } i,\, x'^i_0 \succ^i_0 x^i_0\}.$$

Given the definition by Luttmer and Mariotti (2007, Definition 1(ii)), an allocation path $\hat{x}_1 := (x_1, x_0) \in E_1$ is renegotiation proof if: (a) $x_0 \in R_0$, and (b) there exist no other feasible allocation path $\hat{x}'_1 := (x'_1, x'_0)$, with $x'_0 \in R_0$, such that for all $i \in I$, $\hat{x}'^i_1 \succeq^i_1 \hat{x}^i_1$ and for some i, $\hat{x}'^i_1 \succ^i_1 \hat{x}^i_1$. Therefore, in a two-period economy renegotiation proofness is a special case of recursive efficiency. Clearly, since for any renegotiation proof allocation $\hat{x}_1$, x_0 belongs to R_0, it is post-0 efficient. In addition, there is no other consumption path $\hat{x}'_1 := (x'_1, x'_0)$, with $x'_0 \in R_0$, which could Pareto improve upon $\hat{x}_1$ with respect to period 1 preferences. Hence, any change in the welfare at the initial date would have to worsen off at least some agents in the final period. Therefore, it is recursively efficient. On the other hand, recursively efficient allocations are in general not renegotiation proof, as they fail to satisfy condition (b).

3.2. *Welfare properties of competitive equilibria*

In the remainder of this section we establish a result concerning recursive efficiency of equilibrium allocations. In order to prove our main theorem we impose the following assumption on the premises of the model.

ASSUMPTION 1 (Preferences) *For all $(i,t) \in I \times T$, preference relation $\succeq_t^i$ is*

(i) reflexive, complete, and transitive;

(ii) locally non-satiated on X_t, i.e. for all $\hat{x}_t^i \in \hat{X}_t$ and any $\varepsilon > 0$, there exists some $x_t^{i\prime} \in X_t$ such that $\|x_t^i - x_t^{i\prime}\|_{X_t} < \varepsilon$ and $(x_t^{i\prime}, \hat{x}_{t-1}^i) \succ_t^i \hat{x}_t^i$, where $\|\cdot\|_{X_t}$ is a norm on X_t.[9]

We state our first main result of the paper.

THEOREM 1 *Under Assumption 1, for any competitive equilibrium $\{\hat{x}_T^*, \hat{p}_T^*\}$, allocation path $\hat{x}_T^*$ is recursively efficient.*

The proof of the above theorem is rather extensive, hence we present it in the Appendix. In order to show the line of our argument we prove a two-period version of Theorem 1 below. Consider the following corollary.

COROLLARY 2 *Let $T := \{1, 0\}$. Under Assumption 1, for any competitive equilibrium $\{\hat{x}_1^*, \hat{p}_1^*\}$ allocation path $\hat{x}_1^*$ is recursively efficient.*

In order to make our argument transparent, we first present several claims which will be used in the proof. Throughout the following argument we shall assume that $\{\hat{x}_1^*, \hat{p}_1^*\}$ is a competitive equilibrium satisfying Definition 1.

CLAIM 1 *For all $i \in I$, $x_0^i \succeq_0^i x_0^{*i} \Rightarrow p_0^* \cdot x_0^i \geq p_0^* \cdot x_0^{*i}$, and $x_0^i \succ_0^i x_0^{*i} \Rightarrow p_0^* \cdot x_0^i > p_0^* \cdot x_0^{*i}$.*

PROOF: We prove the first part of the claim by contradiction. Assume that $x_0^i \succeq_0^i x_0^{*i}$ and $p_0^* \cdot x_0^i < p_0^* \cdot x_0^{*i}$. By Assumption 1(ii), there exists some $x_0^{\prime i} \in B_0(p_0^*, x_0^{*i})$ such that $x_0^{\prime i} \succ_0^i x_0^i \succeq_0^i x_0^{*i}$. This contradicts that x_0^{*i} is a $\succeq_0^i$-greatest element of $B_0(p_0^*, x_0^{*i})$. Therefore, $x_0^i \succeq_0^i x_0^{*i} \Rightarrow p_0^* \cdot x_0^i \geq p_0^* \cdot x_0^{*i}$.

[9] Observe that local non-satiation of $\succeq_t^i$ on X_t is a stronger condition than local non-satiation over the whole domain $\hat{X}_t$. Clearly the former implies the latter, but the opposite implication does not hold.

Now assume that $x_0^i \succ_0^i x_0^{*i}$. By the above argument, this implies that $p_0^* \cdot x_0^i \geq p_0^* \cdot x_0^{*i}$. If $p_0^* \cdot x_0^i = p_0^* \cdot x_0^{*i}$, then $x_0^i \in B_0(p_0^*, x_0^{*i})$, which contradicts that x_0^{*i} is a $\succeq_0^i$-greatest element of $B_0(p_0^*, x_0^{*i})$. Hence, $x_0^i \succ_0^i x_0^{*i} \Rightarrow p_0^* \cdot x_0^i > p_0^* \cdot x_0^{*i}$. *Q.E.D.*

In the next claim we simply restate the standard First Fundamental Welfare Theorem (see Mas-Colell, Whinston, and Green, 1995, Proposition 16.C.1).

CLAIM 2 *There exists no allocation $x_0 \in E_0$ such that for all $i \in I$, $x_0^i \succeq_0^i x_0^{*i}$, and for some i, $x_0^i \succ_0^i x_0^{*i}$.*

PROOF: We prove the result by contradiction. Assume that there exists some $x_0 \in E_0$ such that $\forall i \in I$, $x_0^i \succeq_0^i x_0^{*i}$, and for some i, $x_0^i \succ_0^i x_0^{*i}$. By the market clearing condition $p_0^* \cdot \sum_{i \in I} x_0^{*i} = p_0^* \cdot \sum_{i \in I} e_0^i$. By Claim 1, $\forall i \in I$, $p_0^* \cdot x_0^i \geq p_0^* \cdot x_0^{*i}$, and for some i, $p_0^* \cdot x_0^i > p_0^* \cdot x_0^{*i}$. Hence, $p_0^* \cdot \sum_{i \in I} x_0^i > p_0^* \cdot \sum_{i \in I} x_0^{*i} = p_0^* \cdot \sum_{i \in I} e_0^i$. Since $x_0 \in E_0$, we have $p_0^* \cdot \sum_{i \in I} x_0^i \leq p_0^* \cdot \sum_{i \in I} e_0^i$. Contradiction. *Q.E.D.*

CLAIM 3 *For any allocation $x_0 \in E_0$ such that for all $i \in I$, $x_0^i \sim_0^i x_0^{*i}$, we have $p_0^* \cdot x_0^i = p_0^* \cdot x_0^{*i}$, for all $i \in I$.*

PROOF: By Claim 2, $\forall i \in I$, $p_0^* \cdot x_0^i \geq p_0^* \cdot x_0^{*i}$. Assume that for some i, we have $p_0^* \cdot x_0^i > p_0^* \cdot x_0^{*i}$. By the market clearing condition this implies that $p_0^* \cdot \sum_{i \in I} x_0^i > p_0^* \cdot \sum_{i \in I} x_0^{*i} = p_0^* \cdot \sum_{i \in I} e_0^i$. However, we claim that $x_0 \in E_0$, that is $p_0^* \cdot \sum_{i \in I} x_0^i \leq p_0^* \cdot \sum_{i \in I} e_0^i$, which yields contradiction. *Q.E.D.*

CLAIM 4 *Take any $\hat{x}_1 := (x_1, x_0) \in E_1$ such that for all $i \in I$, we have $x_0^i \sim_0^i x_0^{*i}$. Then, $\hat{x}_1^i \succeq_1^i \hat{x}_1^{*i} \Rightarrow p_1^* \cdot x_1^i \geq p_1^* \cdot x_1^{*i}$, and $\hat{x}_1^i \succ_1^i \hat{x}_1^{*i} \Rightarrow p_1^* \cdot x_1^i > p_1^* \cdot x_1^{*i}$.*

PROOF: We prove the first part of the claim by contradiction. Assume that for some i we have $\hat{x}_1^i \succeq_1^i \hat{x}_1^{*i}$ and $p_1^* \cdot x_1^i < p_1^* \cdot x_1^{*i}$. By Assumption 1(ii), there exists some $x_1'^i \in X_1$ such that $p_1^* \cdot x_1'^i \leq p_1^* \cdot x_1^{*i}$ and $(x_1'^i, x_0^i) \succ_1^i \hat{x}_1^i$. Hence, $(x_1'^i, x_0^i) \succ_1^i \hat{x}_1^{*i}$.

Since $\hat{x}_1 \in E_1$, we have $x_0 \in E_0$. Moreover, $\forall i \in I$, $x_0^i \sim_0^i x_0^{*i}$, which by Claim 3 implies that $\forall i \in I$, $p_0^* \cdot x_0^i = p_0^* \cdot x_0^{*i}$. Therefore, $B_0(p_0^*, x_0^i) = B_0(p_0^*, x_0^{*i})$, and so $V_0^i(p_0^*, x_0^i) = V_0^i(p_0^*, x_0^{*i})$. Clearly, $(x_1'^i, x_0^i) \in F_1^i(\hat{p}_1^*, \hat{e}_1^i)$, which contradicts that $\hat{x}_1^{*i} \in V_1^i(\hat{p}_1^*, \hat{e}_1^i)$.

We proceed with the second part of the claim. By the above argument we know that for all i we have $p_1^* \cdot x_1^i \geq p_1^* \cdot x_1^{*i}$. Assume that $p_1^* \cdot x_1^i = p_1^* \cdot x_1^{*i}$. Since $\forall i \in I$, $x_0^i \sim_0^i x_0^{*i}$, we have $\forall i \in I$, $p_0^* \cdot x_0^i = p_0^* \cdot x_0^{*i}$ (by Claim 3). This implies that $B_0(p_0^*, x_0^i) = B_0(p_0^*, x_0^{*i})$ and $V_0^i(p_0^*, x_0^i) = V_0^i(p_0^*, x_0^{*i})$. Hence, we have $\hat{x}_1^i \in F_1^i(\hat{p}_1^*, \hat{e}_1^i)$, which contradicts that $\hat{x}_1^{*i} \in V_1^i(\hat{p}_1^*, \hat{e}_1^i)$. *Q.E.D.*

Having stated the necessary prerequisites, we may prove Corollary 2.

Proof of Corollary 2: We prove the result by contradiction. Assume that $\hat{x}_1^*$ is not recursively efficient. Therefore, there exists some feasible allocation path $\hat{x}_1$ such that (i) $\forall i \in I$, $\hat{x}_1^i \succeq_1^i \hat{x}_1^{*i}$, $x_0^i \succeq_0^i x_0^{*i}$, and for some i, $\hat{x}_1^i \succ_1^i \hat{x}_1^{*i}$ or $x_0^i \succ_0^i x_0^{*i}$; or (ii) $\forall i \in I$, $x_0^i \succeq_0^i x_0^{*i}$, and for some i, $x_0^i \succ_0^i x_0^{*i}$.

By Claim 2, we know that (ii) may never occur. Therefore, for (i) to hold it must be that $\forall i \in I$, $x_0^i \sim_0^i x_0^{*i}$. Moreover, we have $\forall i \in I$, $\hat{x}_1^i \succeq_1^i \hat{x}_1^{*i}$, and for some i, $\hat{x}_1^i \succ_1^i \hat{x}_1^{*i}$. Claim 4 implies that $\forall i \in I$, $p_1^* \cdot x_1^i \geq p_1^* \cdot x_1^{*i}$, and for some i, $p_1^* \cdot x_1^i > p_1^* \cdot x_1^{*i}$. By the market clearing condition $p_1^* \cdot \sum_{i \in I} x_1^i > p_1^* \cdot \sum_{i \in I} x_1^{*i} = p_1^* \cdot \sum_{i \in I} e_1^i$. However, by assumption $\hat{x}_1 \in E_1$, hence $p_1^* \cdot \sum_{i \in I} x_1^i \leq p_1^* \cdot \sum_{i \in I} e_1^i$. Contradiction. *Q.E.D.*

Theorem 1 requires some comment. First of all, it implies that there exists no other feasible allocation which can Pareto improve upon the equilibrium outcome, given that we consider each self of every consumer as a separate agent. Hence, every competitive equilibrium is time-consistently overall Pareto efficient. Therefore, the above proposition establishes a version of the First Fundamental Welfare Theorem for exchange economies with time-dependent preferences.

However, since the result implies that every equilibrium allocation is recursively efficient, it satisfies a stronger condition. By Theorem 1 and Definition 3, we know that additionally for any $t \in T$ allocation path $\hat{x}_t^*$ is post-t efficient. Therefore, competitive equilibrium allocations do not give much room for improvement when it comes to welfare. Clearly, equilibrium allocation is usually not Pareto efficient solely with respect to period T preferences, nor renegotiation proof, as shown by Luttmer and Mariotti (2007, Proposition 3). However, a strict improvement of the welfare of the initial consumers would worsen off at least some of the incarnations in the subsequent periods.

Theorem 1 differs from the result obtained by Herings and Rohde (2006, Theorem 30) in three ways. First of all, we show that competitive equilibria in economies with time-dependent preferences satisfy an optimality condition stronger than time-consistent overall Pareto efficiency. Equilibrium allocations are not only optimal with respect to preferences of all agents and their different selves, but also possess the recursive and time-consistent feature characterised in Definition 3, which preserves the efficiency of allocation paths as time progresses.

Second of all, our result refers to a modified definition of competitive equilibrium. As mentioned in Section 2.3, allowing agents to strategically interact with their future selves via budget constraints is an important component of the consumer choice when tastes change over time. Moreover, the behaviour does affect the resulting equilibrium allocations. Still, as suggested by Theorem 1, it does not change their general welfare properties.

Finally, we obtain our result under weaker conditions imposed on the preferences of consumers. As Herings and Rohde, we consider rational preferences which are independent of the past consumption.[10] However, we do not impose any assumption concerning continuity, monotonicity, nor convexity of the underlying tastes. In fact, our result requires only local non-satiation of preferences over the current period commodity space. Therefore, the property we have established in this section is a general feature of the discussed class of economies. Nevertheless, it is worth pointing out that our theorem is not a generalisation of the result by Herings and Rohde. Since their definition of equilibrium is substantially different from ours, the results need not apply to their framework.

In the remainder of the paper we use the implications of Theorem 1 to construct a social welfare function with maximisers coinciding with competitive equilibrium allocations. Finally, we discuss the issue of representation of economies in question.

4. REPRESENTATION OF RECURSIVELY EFFICIENT ALLOCATIONS

In the following section we concentrate on representation of recursively efficient allocations by solutions to a social welfare optimisation problem. We impose the following condition.

ASSUMPTION 2 (Utility representation) *For all* $(i,t) \in I \times T$, *preference relation* $\succeq_t^i$ *is represented by a utility function* $u_t^i : \hat{X}_t \to \mathbb{R}$. *That is, for any two* $\hat{x}_t^i$, $\hat{x}_t^{\prime i} \in \hat{X}_t$, *we have* $\hat{x}_t^{i\prime} \succeq_t^i \hat{x}_t^i \Leftrightarrow u_t^i(\hat{x}_t^{i\prime}) \geq u_t^i(\hat{x}_t^i)$.[11]

In the remainder of the section we characterize a notion of social welfare. Then, we discuss when the concept coincides with recursive efficiency presented in the preceding section.

4.1. *Social welfare function when preferences are time-dependent*

We construct our notion of social welfare function using backward induction. First, consider the social planner's problem in the final period $t = 0$. For any real, positive, non-zero weights $\alpha_0 := (\alpha_0^i)_{i \in I} \in \mathbb{R}_+^I$, define set

$$\Psi_0(\alpha_0) := \operatorname*{argmax}_{x_0 \in E_0} \sum_{i \in I} \alpha_0^i u_0^i(x_0^i). \tag{4.1}$$

[10] That is, they satisfy strong independence of past consumption. See Definition 6' in their paper.

[11] Sufficient conditions for utility representation of preferences are well-known (e.g. see Mas-Colell, Whinston, and Green, 1995, Chapter 3.C).

In other words, $\Psi_0(\alpha_0)$ contains all feasible period 0 consumption bundles which maximise the weighted social welfare function for a fixed vector of weights $\alpha_0 := (\alpha_0^i)_{i\in I}$. Since the form of the above functional is rather standard, we refrain ourselves from further discussion.

Next, consider the problem in period $t = 1$. Denote a path of real, positive, non-zero weights following period 1 by $\hat{\alpha}_1 := (\alpha_1, \alpha_0)$, where $\alpha_t = (\alpha_t^i)_{i\in I} \in \mathbb{R}_+^I$, $t \in \{1, 0\}$. Define set $\Psi_1(\hat{\alpha}_1)$ as

$$\Psi_1(\hat{\alpha}_1) := \underset{\hat{x}_1\in\Gamma_1(\alpha_0)}{\operatorname{argmax}} \sum_{i\in I} \alpha_1^i u_1^i(\hat{x}_1), \tag{4.2}$$

where

$$\Gamma_1(\alpha_0) := \{(x_1, x_0) \in E_1 \mid x_0 \in \Psi_0(\alpha_0)\},$$

and $\Psi_0(\alpha_0)$ is defined as in (4.1). Therefore, set $\Psi_1(\hat{\alpha}_1)$ contains all allocation paths following date 1 which maximise period 1 social welfare functional for weights α_1, given that period 0 allocation x_0 is a solution to the social planer's optimisation problem in the final period, given weights α_0. In other words, set $\Psi_1(\hat{\alpha}_1)$ contains time-consistent, welfare maximising allocations, in an environment where the social planner faces a similar time-inconsistency problem as individual consumers.

This allows us to define a simplified notion of a social welfare for a two-period case.

Definition 4 (Recursive social welfare) *For $T = \{1, 0\}$, an allocation path $\hat{x}_1^\circ \in \hat{X}_1^I$ is a* recursive social welfare allocation *if there exists a path of real, non-zero weights $\hat{\alpha}_1 := (\alpha_1, \alpha_0)$, where $\alpha_t \in \mathbb{R}_+^I$, $t \in T$, such that $\hat{x}_1^\circ \in \Psi_1(\hat{\alpha}_1)$.*

Using backward induction, one can determine corresponding sets $\Psi_t(\hat{\alpha}_t)$ and $\Gamma_t(\hat{\alpha}_{t-1})$ for any $t \in T$, and any non-zero path of weights $\hat{\alpha}_t := (\alpha_s)_{s=0}^t$, $\alpha_s \in \mathbb{R}_+^I$. Define

$$\Psi_t(\hat{\alpha}_t) := \underset{\hat{x}_t\in\Gamma_t(\hat{\alpha}_{t-1})}{\operatorname{argmax}} \sum_{i\in I} \alpha_t^i u_t^i(\hat{x}_t^i), \tag{4.3}$$

where

$$\Gamma_t(\hat{\alpha}_{t-1}) := \{(x_t, \hat{x}_{t-1}) \in E_t \mid \hat{x}_{t-1} \in \Psi_{t-1}(\hat{\alpha}_{t-1})\},$$

where $\Psi_{t-1}(\hat{\alpha}_{t-1})$ is defined as in (4.3) for the corresponding subsequence of weights $\hat{\alpha}_{t-1}$.

The construction of Ψ_t and Γ_t is similar to the construction of correspondences V_t^i and F_t^i for optimisation problems of individual agents in Section 2.2. Namely, for any t take the set of feasible allocations paths following date t, E_t. Let $\hat{x}_t \in E_t$. By definition, we have $\hat{x}_t = (x_t, \hat{x}_{t-1})$, where x_t is an allocation of period t consumption goods, and $\hat{x}_{t-1}$ is a path of allocations following period $(t-1)$. In order to make sure that $\hat{x}_t \in \Gamma_t(\hat{\alpha}_t)$, we need to guarantee that the subsequence $\hat{x}^i_{t-1}$ is a solution to the corresponding social welfare optimisation problem in the following period, given the path of weights $\hat{\alpha}_{t-1}$. This way, we obtain a form of time-consistency of socially optimal allocations. That is, given that the next period social planner is guided by a different social welfare function, he is not willing to change the allocation determined in the preceding period, as it could not strictly improve the welfare given his criterion. Finally, the social planner in period t chooses an element from the set of feasible, time-consistent sequences of allocations, which maximise his current welfare function.

We state the general definition or recursive efficiency.

Definition 4' (Recursive efficiency) *An allocations path* $\hat{x}^\circ_T \in \hat{X}^I_T$ *is a* recursive social welfare allocation *if there exists a path of real, positive, non-zero weights* $\hat{\alpha}_T := (\alpha_t)_{t\in T}$*, where* $\alpha_t \in \mathbb{R}^I_+$*, such that* $\hat{x}^\circ_T \in \Psi_T(\hat{\alpha}_T)$.

Clearly, Definition 4 is equivalent to 4' once $T = \{1, 0\}$. We can establish sufficient conditions under which for any path of weights there exists a recursive social welfare allocation.

Proposition 1 *Let Assumption 2 be satisfied and for all* $(i,t) \in I \times T$*,* u^i_t *be upper semi-continuous. Then, for any path of real, positive, non-zero weights* $\hat{\alpha}_T := (\alpha_t)_{t\in T}$*, where* $\alpha_t \in \mathbb{R}^I_+$*,* $t \in T$*, there exists a recursive social welfare allocation.*

Proof: Take any sequence of real, positive, non-zero weights $\hat{\alpha}_T$. We prove the result by induction. Take $t = 0$. Clearly, E_0 is non-empty and compact. Moreover, by upper semi-continuity of u^i_0, $\sum_{i\in I} \alpha^i_0 u^i_0$ is upper semi-continuous. Hence, due to Aliprantis and Border (2006, Theorem 2.43) $\Psi_0(\alpha_0)$ is non-empty and compact.

Next, take any $t \in T$ and the corresponding subsequence of weights $\hat{\alpha}_t$. Assume that set $\Psi_t(\hat{\alpha}_t)$ is non-empty and compact. Clearly, $\Gamma_{t+1}(\hat{\alpha}_t)$ is also non-empty and compact. By upper semi-continuity of u^i_{t+1}, $\sum_{i\in I} \alpha^i_{t+1} u^i_{t+1}$ is upper semi-continuous. Therefore, due to Aliprantis and Border (2006, Theorem 2.43) $\Psi_{t+1}(\hat{\alpha}_{t+1})$ is non-empty and compact. The proof is complete. *Q.E.D.*

A recursive social welfare allocation is a solution to a multi-stage maximisation problem, where at each stage t the social planner maximises the current period weighted social welfare function, given that the path of allocations following date t is a solution to an analogue problem in each of the following periods. Therefore, recursive social welfare is closely related to recursive efficiency, as it focuses on a form of time-consistency of optimal allocations. In fact, in the next section we present conditions under which the two notions coincide.

4.2. *Recursive efficiency and social welfare equivalence*

First, we show conditions under which every recursive social welfare allocation is recursively efficient.

PROPOSITION 2 *If $\hat{x}_T^{\circ}$ is a recursive social welfare allocation for some strictly positive path of weights $\hat{\alpha}_T := (\alpha_t)_{t\in T}$, $\alpha_t \in \mathbb{R}^I_{++}$, then it is recursively efficient.*

PROOF: Let $\hat{x}_T^{\circ}$ be a recursive social welfare allocation for some real, strictly positive path of weights $\hat{\alpha}_T$. We prove the result by induction. First, we show that x_0° is post-0 efficient. Assume the opposite. Then, there exists some $x_0 \in E_0$ such that $\forall i \in I$, $u_0^i(x_0^i) \geq u_0^i(x_0^{\circ i})$, and for some i, $u_0^i(x_0^i) > u_0^i(x_0^{\circ i})$. Since weights α_0 are strictly positive, this implies $\sum_{i\in I} \alpha_0^i u_0^i(x_0^i) > \sum_{i\in I} \alpha_0^i u_0^i(x_0^{\circ i})$, which contradicts that $x_0^{\circ} \in \Psi_0(\alpha_0)$ and $\hat{\alpha}_T \in \Psi_T(\hat{\alpha}_T)$.

Next, take any $t \in T$ and assume that $\forall t' \leq t$, $\hat{x}_{t'}^{\circ}$ is post-t' efficient. We claim that $\hat{x}_{t+1}^{\circ}$ is post-$(t+1)$ efficient. Assume the opposite. Therefore, there exists some $\hat{x}_{t+1} \in E_{t+1}$ such that $\forall i \in I$ and $\forall t' \leq (t+1)$, we have $u_{t'}^i(\hat{x}_{t'}^i) \geq u_{t'}^i(\hat{x}_{t'}^{\circ i})$, and for some i and some $t' \leq (t+1)$, $u_{t'}^i(\hat{x}_{t'}^i) > u_{t'}^i(\hat{x}_{t'}^{\circ i})$. By assumption $\forall t' \leq t$, $\hat{x}_{t'}^{\circ}$ is post-t' efficient, so it must be that $\forall i \in I$, and $\forall t' \leq t$, $u_{t'}^i(\hat{x}_{t'}^i) = u_{t'}^i(\hat{x}_{t'}^{\circ i})$. This implies that $\hat{x}_t \in \Psi_t(\hat{\alpha}_t)$, and so $\hat{x}_{t+1} \in \Gamma_{t+1}(\hat{\alpha}_t)$. Moreover, since the weights are strictly positive, $\sum_{i\in I} \alpha_{t+1}^i u_{t+1}^i(\hat{x}_{t+1}^i) > \sum_{i\in I} \alpha_{t+1}^i u_{t+1}^i(\hat{x}_{t+1}^{\circ i})$, which contradicts that $\hat{x}_{t+1}^{\circ} \in \Psi_{t+1}(\hat{\alpha}_{t+1})$ and $\hat{x}_T^{\circ} \in \Psi_T(\hat{\alpha}_T)$. *Q.E.D.*

Proposition 2 implies, that in general a set of recursively efficient allocations can be determined via a solution to the recursive social welfare maximisation problem, as long as weights corresponding to each self of every consumer are strictly positive. For the converse result to be true, we need to impose some convexity assumptions on preferences.

ASSUMPTION 3 (Concave utility) *For all $i \in I$, $t \in \{T-1, \dots, 0\}$, and $\hat{x}_{t-1}^i \in \hat{X}_{t-1}$, function $u_t^i(x_t^i, \hat{x}_{t-1}^i)$ is strictly concave with respect to x_t^i. Moreover, for all $i \in I$ and $\hat{x}_{T-1}^i \in \hat{X}_{t-1}^i$, $u_T^i(x_T^i, \hat{x}_{T-1}^i)$ is (weakly) concave with respect to x_T^i.*

Several times we shall refer to a slightly stronger version of the above assumption.

ASSUMPTION 3' (Strictly concave utility) *For all $(i,t) \in I \times T$ and $\hat{x}_{t-1}^i \in \hat{X}_{t-1}$, function $u_t^i(x_t^i, \hat{x}_{t-1}^i)$ is strictly concave with respect to x_t^i.*

We state the second main result of the paper.

THEOREM 2 *Let Assumptions 2, 3 be satisfied and $\hat{x}_T$ be a recursively efficient allocation. There exist some real, positive, non-zero weights $\hat{\alpha}_T := (\alpha_t)_{t\in T}$, where $\alpha_t \in \mathbb{R}^I_+$, such that $\hat{x}_T \in \Psi_T(\hat{\alpha}_T)$. If additionally Assumption 3' is satisfied, then $\{\hat{x}_T\} = \Psi_T(\hat{\alpha}_T)$.*

The proof of the theorem is presented in the Appendix. However, in order to show the intuition behind our main result, we provide the proof for the two-period case. Consider the following corollary

COROLLARY 3 *Assume $T = \{1, 0\}$. Let Assumptions 2, 3 be satisfied and $\hat{x}_1$ be a recursively efficient allocation. There exist some real, positive, non-zero weights $\hat{\alpha}_1 := (\alpha_1, \alpha_0)$, where $\alpha_t \in \mathbb{R}^I_+$, $t \in T$, such that $\hat{x}_1 \in \Psi_1(\hat{\alpha}_1)$. If additionally Assumption 3' is satisfied, then $\{\hat{x}_1\} = \Psi_1(\hat{\alpha}_1)$.*

PROOF: Assume that $\hat{x}_1 = (x_1, x_0)$ is a recursively efficient allocation. Let $u_0 : X^I_0 \to \mathbb{R}^I$, be defined as $u_0 := (u^i_0)_{i\in I}$, and $U'_0 := u_0(E_0)$. Since $\forall i \in I$, u^i_0 is convex, it is also continuous (see Rockafellar, 1970, Theorem 10.1). Therefore, by compactness of E_0, U'_0 is compact. Let $U_0 := \{u \in \mathbb{R}^I \mid \forall x'_0 \in E_0,\ \forall i \in I,\ u^i \leq u^i_0(x'^i_0)\}$. By Assumption 3, U_0 is convex. Moreover, by construction $U_0 = U'_0 - \mathbb{R}^I_+$. Hence, U_0 is closed and bounded above.

Denote $u^*_0 = u(x_0)$. By definition of $\hat{x}_1$, there exists no other $x'_0 \in E_0$ such that $\forall i \in I$, $u^i_0(x'^i_0) \geq u^i_0(x^i_0)$, and $u^i_0(x'^i_0) > u^i(x^i_0)$ for some i. Hence, it must be that $u^*_0 \in \partial U_0$. By the separating hyperplane theorem (see, e.g. Aliprantis and Border, 2006, Theorem 7.30), there exists some non-zero vector $\alpha_0 \in \mathbb{R}^I$ such that $\forall u \in U_0$, $\alpha_0 \cdot u^*_0 \geq \alpha_0 \cdot u$. Since $U_0 - \mathbb{R}^I_+ \subset U_0$, it must be that $\alpha_0 \in \mathbb{R}^I_+$. By construction, this implies that $x_0 \in \Psi_0(\alpha_0)$.

Strict concavity of u^i_0 implies that $\{x_0\} = \Psi_0(\alpha_0)$. Hence, it is compact and convex. Therefore, $\Gamma_1(\alpha_0)$ is also compact and convex. Define $u : X^I_1 \times X^I_0 \to \mathbb{R}^I$ as $u_1 := (u^i_1)_{i\in I}$, and $U'_1 := u_1(\Gamma_1(\alpha_0))$. By continuity of u^i_1 (implied by Rockafellar, 1970, Theorem 10.1), U'_1 is compact. Let $U_1 := \{u \in \mathbb{R}^I \mid \forall \hat{x}' \in \Gamma_1(\alpha_0),\ \forall i \in I,\ u^i \leq u^i_1(\hat{x}'^i)\}$, which by concavity of u^i_1 is convex. Moreover, by construction $U_1 := U'_1 - \mathbb{R}^I_+$, which is closed and bounded above.

To complete the proof, denote $u^*_1 = u_1(\hat{x}_1)$. By definition of $\hat{x}_1$, there exists no other $x'_1 \in X^I_1$ such that $\forall i \in I$, $u^i_1(x'^i_1, x^i_0) \geq u^i_1(\hat{x}^i)$, and $u^i_1(x'^i_1, x^i_2) > u^i_1(\hat{x}^i)$ for some i. Therefore, it must be that $u^*_1 \in \partial U_1$. By the separating hyperplane theorem (Aliprantis and Border, 2006, Theorem 7.30), there exists a non-zero vector $\alpha_1 \in \mathbb{R}^I$ such that $\forall u \in U_1$, $\alpha_1 \cdot u^*_1 \geq \alpha_1 \cdot u$. Since $U_1 - \mathbb{R}^I_+ \subset U_1$, it must be that $\alpha_1 \in \mathbb{R}^I_+$. Denote $\hat{\alpha}_1 = (\alpha_1, \alpha_0)$. By construction, $\hat{x}_1 \in \Psi_1(\hat{\alpha}_1)$.

To prove the second part of the proposition, recall that $\Gamma_1(\alpha_0)$ convex. Therefore, by strict concavity of u_1^i, we have $\{\hat{x}_1\} = \Psi_1(\hat{\alpha}_1)$, which completes the proof. *Q.E.D.*

Under some additional assumptions, it is possible to show that weights supporting a recursively efficient allocation are strictly positive.

PROPOSITION 3 *Let Assumptions 2 and 3 be satisfied. In addition, let for all $(i,t) \in I \times T$ and $\hat{x}_{t-1}^i \in \hat{X}_t$, $u_t^i(x_t^i, \hat{x}_{t-1}^i)$ be strictly increasing with respect to x_t^i. Then, for any recursively efficient allocation $\hat{x}_T$ such that for all $(i,t) \in I \times T$, x_t^i is non-zero, there exist some real, strictly positive weights $\hat{\alpha}_t := (\alpha_s)_{t\in T}$, where $\alpha_t \in \mathbb{R}_{++}^I$, such that $\hat{x}_T \in \Psi_T(\hat{\alpha}_T)$. If additionally Assumption 3' is satisfied, then $\{\hat{x}_T\} = \Psi_T(\hat{\alpha}_T)$.*

PROOF: Let $\hat{x}_T$ be a recursively efficient path of allocations such that $\forall (i,t) \in I \times T$, x_t^i is non-zero. By Theorem 2, there exist some real, positive, non-zero path of weights $\hat{\alpha}_T$, for which $\hat{x}_T \in \Psi_T(\hat{\alpha}_T)$.

Take any $t \in T$. Assume that for some $j \in I$, $\alpha_t^j = 0$. Let $x_t' = (x_t'^i)_{i\in I}$, where $\forall i \neq j$, $x_t'^i = x_t^i + 1/(I-1)x_t^j$, and $x_t'^j = 0$. Clearly, $(x_t', \hat{x}_{t-1}) \in \Gamma_t(\hat{\alpha}_{t-1})$. Since $\forall \hat{x}_{t-1}^i \in \hat{X}_{t-1}$, $u_t^i(x_t^i, \hat{x}_{t-1}^i)$ is strictly increasing in x_t^i, we have $\sum_{i\in I} \alpha_t^i u_t^i(x_t'^i, \hat{x}_{t-1}^i) > \sum_{i\in I} \alpha_t^i u_t^i(\hat{x}_t^i)$. This contradicts that $\hat{x}_t \in \Psi_t(\hat{\alpha}_t)$ and $\hat{x}_T \in \Psi_T(\hat{\alpha}_T)$. *Q.E.D.*

The results above require some comment. Proposition 2 implies that it is possible to determine a wide class of recursively efficient allocations by solving a social welfare optimisation problem. On the other hand, Theorem 2 provides conditions under which every recursively efficient allocation can be represented by a solution to the same optimisation problem.

The proof of Theorem 2 relies strongly on strict concavity of preferences following the initial period. Once we weaken the condition to *weak* concavity, there might exist a recursively efficient allocation which cannot be represented via a recursive social welfare function. For example, take $T = \{1, 0\}$ and assume that there exists some positive weights α_0 such that $x_0 \in \Psi_0(\alpha_0)$. Once u_0^i is (weakly) concave, set $\Psi_0(\alpha_0)$ is convex and contains set $\{x_0 \in E_0 \mid \forall i \in I,\ u_0^i(x_0'^i) = u_0^i(x_0^i)\}$, i.e. the set of period 0 allocations which are Pareto equivalent to x_0. However, in general the two sets are not equal. This implies, that there might exist some elements of set $\Gamma_0(\alpha_0)$ which are not Pareto ordered relatively to $\hat{x}_1$ with respect to preferences $(u_0^i)_{i\in I}$, but are Pareto dominant with respect to date 1 preferences. In such cases, $\hat{x}_1$ would never be a solution to recursive social welfare maximisation problem, like in the following example.

Example 2 Consider a pure exchange economy with two consumers and two goods $j = 1, 2$. Hence, $I = \{1, 2\}$ and $T = \{1, 0\}$. Let $X_0 = \mathbb{R}^2_+$, with its elements denoted by $x^i_0 = (x^{i1}_0, x^{i2}_0)$. Let $\forall i \in I$, period 0 preferences be defined by $u^i_0 : X_0 \to \mathbb{R}$,

$$u^i_0(x^i_0) := x^{i1}_0 + x^{i2}_0.$$

On the other hand, let period 1 preferences be defined by $u^i_1 : X_0 \to \mathbb{R}$,

$$u^i_1(x^i_0) := \sqrt{x^{i1}_0} + \gamma_i \sqrt{x^{i2}_0},$$

where $\gamma_1 = 1, \gamma_2 = 3$. Hence, we assume that period 1 preferences are defined solely over period 0 consumption bundles. Eventually, let the total endowment in the economy be $\sum_{i\in I} e^i_0 = (1, 1)$.

Observe, that allocation $x_0 = (x^1_0, x^2_0) = ((x^{11}_0, x^{12}_0), (x^{21}_0, x^{22}_0)) = ((0.8, 0.2), (0.2, 0.8))$ is recursively efficient. However, there exist no weights $\hat{\alpha}_1 := (\alpha_1, \alpha_0)$ such that the allocation is a solution to a recursive social welfare optimisation problem.

Clearly, $x_0 \in \Gamma_1(\alpha_0)$ only for these weights $\alpha_0 = (\alpha^1_0, \alpha^2_0)$ for which $\alpha^1_0 = \alpha^2_0$. However, then $\Gamma_1(\alpha_0) = \{x'_0 \in \mathbb{R}^4_+ \mid \sum_{i\in I} x'^{ij}_0 = 1,\ \forall j = 1, 2\}$, while the set of Pareto equivalent allocations to x_0 with respect to period 0 preferences is $\Lambda(\alpha_0) := \{x'_0 \in \mathbb{R}^4_+ \mid \sum_{i\in I} x'^{ij}_0 = 1,\ \text{and}\ \sum_{j=1,2} x'^{ij}_0 = 1,\ \forall i \in I,\ \forall j = 1, 2\}$. Therefore, $\Lambda(\alpha_0) \subset \Gamma_1(\alpha_0)$ (strictly).

Assume that there exist some period 1 weights $\alpha_1 = (\alpha^1_1, \alpha^2_1)$ such that x_0 is a recursive social welfare allocation. Given the weights, the allocation has to satisfy the following first order conditions:

$$\frac{\alpha^1_1}{\alpha^2_1} = \left(\frac{x^{11}_0}{x^{21}_0}\right)^{\frac{1}{2}}, \text{ and } \frac{\alpha^1_1}{\alpha^2_1} = 3\left(\frac{x^{12}_0}{x^{22}_0}\right)^{\frac{1}{2}}.$$

However, since $(x^{11}_0/x^{21}_0)^{\frac{1}{2}} = 2$ and $3\,(x^{12}_0/x^{22}_0)^{\frac{1}{2}} = 3/2$, there exists no α_1, for which the above conditions are met.

Once we restrict our attention to strictly concave utility functions, set $\Psi_0(\alpha_0)$ is a singleton and the case discussed above does not occur.

4.3. *Competitive equilibrium and social welfare*

Theorems 1 and 2 allow to characterize competitive equilibria via recursive social welfare function. Therefore, it is possible to define an optimisation problem with maximisers coinciding with any competitive equilibrium allocation. Consider the following proposition.

PROPOSITION 4 *Let Assumptions 1, 2, and 3 be satisfied. For any competitive equilibrium* $\{\hat{x}_T^*, \hat{p}_T^*\}$ *there exist some real, positive, non-zero weights* $\hat{\alpha}_T := (\alpha_t)_{t\in T}$*, where* $\alpha_t \in \mathbb{R}_+^I$*, such that* $\hat{x}_T^* \in \Psi_T(\hat{\alpha}_T)$*. In addition, if Assumption 3' is satisfied, then* $\{\hat{x}_T^*\} = \Psi_T(\hat{\alpha}_T)$*.*

PROOF: Theorem 1 implies, that for any competitive equilibrium $\{\hat{x}_T^*, \hat{p}_T^*\}$, allocation path $\hat{x}_T^*$ is recursively efficient. By Theorem 2, every recursively efficient allocation can be represented by a recursive social welfare allocation for some real, positive, non-zero weights $\hat{\alpha}_T$. In particular, this is true for $\hat{x}_T^*$. Moreover, once Assumption 3' holds, we have $\{\hat{x}_T^*\} = \Psi_T(\hat{\alpha}_T)$. *Q.E.D.*

The following corollary is implied by Proposition 3.

COROLLARY 4 *Let Assumptions 1, 2, and 3 be satisfied. In addition, let for all* $(i,t) \in I \times T$ *and* $\hat{x}_{t-1}^i \in \hat{X}_{t-1}$*,* $u_t^i(x_t^i, \hat{x}_{t-1}^i)$ *be strictly increasing with respect to* x_t^i*. Then, for any competitive equilibrium* $\{\hat{x}_T^*, \hat{p}_T^*\}$ *such that for any* $(i,t) \in I \times T$*,* x_t^{*i} *is non-zero, there exist some real, strictly positive weights* $\hat{\alpha}_T := (\alpha_t)_{t\in T}$*, where* $\alpha_t \in \mathbb{R}_{++}^I$*, such that* $\hat{x}_T^* \in \Psi_T(\hat{\alpha}_T)$*. In addition, if Assumption 3' is satisfied, then* $\{\hat{x}_T^*\} = \Psi_T(\hat{\alpha}_T)$*.*

PROOF: Let $\{\hat{x}_T^*, \hat{p}_T^*\}$ be a competitive equilibrium such that $\forall (i,t) \in I \times T$, x_t^{*i} is non-zero. Proposition 4 implies there exist some real, positive, non-zero weights $\hat{\alpha}_T$ such that $\hat{x}_T^* \in \Psi_T(\hat{\alpha}_T)$. Since $\forall (i,t) \in I \times T$ and $\forall \hat{x}_{t-1}^i \in \hat{X}_{t-1}$, $u_t^i(x_t^i, \hat{x}_{t-1}^i)$ is strictly increasing with respect to x_t^i, by Corollary 3 we conclude that path of weights $\hat{\alpha}_t$ is strictly positive. Finally, under Assumption 3' holds, we have $\{\hat{x}_T^*\} = \Psi_T(\hat{\alpha}_T)$. *Q.E.D.*

The above results state, that every allocation arising in a competitive equilibrium can be represented by a solution to a recursive social welfare optimisation problem, given the proper path of weights $\hat{\alpha}_T$. Therefore, instead of solving a competitive equilibrium problem, it is sufficient to determine an allocation maximising recursive social welfare function. In fact, in various applications finding a fixed point of some operator in order to determine equilibrium prices and allocations is much more difficult, if not impossible, than characterizing a solution to some optimisation problem (see Kehoe, 1991, for further discussion).

What is more, the corollary implies that there exists a method of aggregating preferences of agents with time-variant tastes and representing them by a single agent in the same class of preferences. Clearly, as it was mentioned before, social planner in our problem faces a very similar time-inconsistency issue as every individual agent in the economy.

Moreover, given the representation, we know that the resulting choice constitutes an allocation arising in some competitive equilibrium.

In the following section we apply our results to a class of time separable preferences with hyperbolic discounting.

4.4. *Efficiency, social welfare and hyperbolic discounting*

In general, our definition of efficiency does not coincide with Pareto efficiency with respect to the initial selves, nor renegotiation-proofness. However, in some special cases the three notions may be equivalent. We devote this section to one such example. In order to make the paper self-contained, we formally reintroduce the two efficiency notions.[12]

DEFINITION 5 *Let Assumption 2 be satisfied.*

(i) Allocation path $\hat{x}_T \in E_T$ is Pareto efficient with respect to the initial selves *if there exists no other $\hat{x}'_T \in E_T$ such that for all $i \in I$, $u_T(\hat{x}'_T) \geq u_T(\hat{x}_T)$, and for some i, $u_T(\hat{x}'_T) > u_T(\hat{x}_T)$.*

(ii) Let $R_T := \{\hat{x}_T \in E_T \mid \hat{x}_T$ is recursively efficient$\}$. Allocation path $\hat{x}_T \in R_T$ is renegotiation-proof *if there exists no other $\hat{x}'_T \in R_T$ such that for all $i \in I$, $u^i_T(\hat{x}'_T) \geq u^i_T(\hat{x}_T)$, and for some i, $u_T(\hat{x}'_T) > u_T(\hat{x}_T)$.*

We proceed with the following Proposition.

PROPOSITION 5 *Recall preferences in Example 1. Let for all $i \in I$, $v^i : \mathbb{R}^n_+ \to \mathbb{R}$ be strictly increasing, concave and once continuously differentiable, with $\lim_{z^k \to 0} \frac{\partial v^i}{\partial z^k}(z) = \infty$, for all $k = 1, \ldots, n$. In addition, for all $i \in I$, let $\delta_i = \delta$ and $\gamma_i = \gamma$. Then, any strictly positive allocation path $\hat{x}_T$ is Pareto efficient with respect to the initial selves if and only if there exist a vector $\alpha^* \in \mathbb{R}^I_{++}$ such that $\hat{x}_T$ is a recursive social welfare allocation for a path of weights $\hat{\alpha}_T := (\alpha_t)_{t \in T}$, where $\forall t \in T$, $\alpha_t = \alpha^*$. Moreover, $\hat{x}_T$ is recursively efficient and renegotiation-proof.*

PROOF: First, we prove ($\Rightarrow$). Let $\hat{x}_T$ be a strictly positive allocation path, Pareto efficient with respect to the initial selves. By a well-known result (see Mas-Colell, Whinston, and Green, 1995, Proposition 16.E.2), there exists some $\alpha^* \in \mathbb{R}^I_+$ such that $\hat{x}_T \in \operatorname{argmax}_{\hat{x}'_T \in E_T} \sum_{i \in I} \alpha^{*i} u^i_T(\hat{x}'^i_T)$. Since v^i is strictly increasing, $\forall i \in I$, $\alpha^{*i} > 0$. Moreover,

[12] Note, that renegotiation-proofness was defined by Luttmer and Mariotti (2007) only for two period economies. Therefore, we extend the definition in a way we think is accordant with the intuition of the authors.

$\hat{x}_T$ satisfies the following necessary and sufficient first order conditions, $\forall i, j \in I$, $t \in T$:

$$\begin{aligned} \alpha^{*i} \nabla v^i(x_t^i) &= \alpha^{*j} \nabla v^j(x_t^j), \\ \sum_{i \in I} x_t^i &= \sum_{i \in I} e_t^i. \end{aligned}$$

Define a path of weights $\hat{\alpha}_T := (\alpha_t)_{t \in T}$, $\alpha_t \in \mathbb{R}^I_+$, such that $\forall t \in T$, $\alpha_t = \alpha^*$. Note, that the unique recursive social welfare allocation for the weights has to satisfy the first order conditions as well. Therefore, it must be that $\hat{x}_T \in \Psi_T(\hat{\alpha}^i_T)$. Moreover, by Proposition 2 the allocation is recursively efficient.

Next, we show ($\Leftarrow$). Take any $\alpha^* \in \mathbb{R}^I_{++}$ and define $\hat{\alpha}_T := (\alpha_t)_{t \in T}$ such that $\forall t \in T$, $\alpha_t = \alpha^*$. Take some $\hat{x}_T \in \Psi_T(\hat{\alpha}_T)$. Clearly, it satisfies the above first order conditions, which implies that $\hat{x}_T \in \operatorname{argmax}_{\hat{x}'_T \in E_T} \sum_{i \in I} \alpha^{*i} u^i_T(\hat{x}'^i_T)$. Hence, by Mas-Colell, Whinston, and Green (1995, Proposition 16.E.2) it satisfies Definition 5(i). Again, by Proposition 2 the allocation is recursively efficient.

Finally, we show that $\hat{x}_T$ is renegotiation-proof. Let R_T be the set of all recursively efficient allocations. Clearly, $\hat{x}_T \in R_T$ and $R_T \subset E_T$. Since $\hat{x}_T$ is Pareto efficient with respect to the initial selves, there exists no other allocation $\hat{x}'_T$ in E_T (hence, in R_T) such that for all $i \in I$, $u^i_T(\hat{x}'^i_T) \geq u^i_T(\hat{x}^i_T)$, and for some i, $u^i_T(\hat{x}'^i_T) > u^i_T(\hat{x}^i_T)$. Therefore, Definition 5(ii) is satisfied. *Q.E.D.*

The above result crucially uses the assumption that discount factors are symmetric across consumers. Only then the first order conditions characterising all the three notions of efficiency are equivalent. Moreover, the proposition above does not require the quasi-hyperbolic specification of discounting. In fact, as long as values of discount factors in each period are equal across consumers, the claim of Proposition 5 remains true.

Finally, Proposition 5 does not imply that competitive equilibria in the discussed class of economies are efficient according to Definition 5(i). The result only states that allocations which are Pareto efficient with respect to the initial selves coincide with a class of recursively efficient ones, which can be represented by a solution to the recursive social welfare optimisation problem for some specific, time-invariant weights. In fact, by Luttmer and Mariotti (2007, Proposition 3) show that in general such allocations do not arise in a competitive equilibrium.

5. EFFICIENCY OF SEQUENTIAL EQUILIBRIA

As it was discussed in Section 2, our notion of equilibrium characterised in Definitions 1 and 1' might be considered restrictive for two reasons. First of all, we require that at each date $t \in T$, agents consume bundles x^i_t equal to the inherited consumption rights y^i_t.

Therefore, the consumption path is determined by the initial self and cannot be altered by the succeeding incarnations. Hence, even though our framework is dynamic, the notion of equilibrium seems to be static, as choices are made only once. Clearly, since agents are sophisticated, consumption plans determined in the initial period are time-consistent and none of the future selves would strictly benefit from changing the plan. Still, one might be willing to explore the implications of a more dynamic interaction.

Second of all, we require that for any date t prices of rights to period t consumption acquired prior to t, and prices of the actual period t goods traded at time t, have to be equal. The condition implies that the trade in the economy takes place only once, when the initial, sophisticated agents trade their optimal, time-consistent consumption plans. This once again imposes a static structure on an equilibrium.

In the following section we relax the two conditions, and claim that they do not affect the welfare properties of equilibrium allocations discussed in the previous sections.

5.1. *Sequential equilibrium in a two-period economy*

First we introduce our definition for a simplified, two-period case. Assume that $T = \{1, 0\}$. Each period we allow for the current selves to trade. In the initial period $t = 1$ we shall distinguish two types of markets: *spot markets*, where agent can trade the current period consumption goods (i.e. consumption bundles in X_1), and *futures markets*, where consumers exchange their rights to future consumption (i.e. consumption bundles in X_0). We denote consumption in the initial period by $x^i_1 \in X_1$, and rights to period 0 consumption traded at period 1 by $y^i_{1|0} \in X_0$. Moreover, let $y_{1|0} := (y^i_{1|0})_{i\in I} \in X^I_0$ denote an allocation of rights to period 0 consumption traded at date 1.

Let $p_1 \in \mathbb{R}^{n_1}_{++}$ denote prices of date 1 consumption goods evaluated at period 1 spot market. Let $q_{1|0} \in \mathbb{R}^{n_0}_{++}$ denote prices of rights to period 0 consumption, quoted at the futures market at date 1. Given period 1 prices $(p_1, q_{1|0})$, the total wealth of consumer i in period 1 is equal to the value of his total initial endowment $p_1 \cdot e^i_1 + q_{1|0} \cdot e^i_0$. Hence, the budget set of agent i at the initial date is determined by values of correspondence $\tilde{B}_1 : \mathbb{R}^{n_1}_{++} \times \mathbb{R}^{n_0}_{++} \times \hat{X}_1 \rightrightarrows \hat{X}_1$,

$$\tilde{B}_1(p_1, q_{1|0}, \hat{e}^i_1) := \left\{(x^i_1, y^i_{1|0}) \in \hat{X}_1 \mid p_1 \cdot x^i_1 + q_{1|0} \cdot y^i_{1|0} \leq p_1 \cdot e^i_1 + q_{1|0} \cdot e^i_0\right\}. \tag{5.1}$$

In the final period, given date 0 spot market prices $p_0 \in \mathbb{R}^{n_0}_{++}$, the total wealth of agent i is determined by the value of the rights to consumption $y^i_{1|0}$ inherited from the previous period. Therefore, the budget set is determined by values of correspondence

$\tilde{B}_0 : \mathbb{R}^{n_0}_{++} \times X_0 \rightrightarrows X_0$,

$$\tilde{B}_0(p_0, y^i_{1|0}) := \left\{ x^i_0 \in X_0 \mid p_0 \cdot x^i_0 \leq p_0 \cdot y^i_{1|0} \right\}. \tag{5.2}$$

Since no consumption takes place beyond date 0, there is no futures market in the final period.

Next, we establish how the choice is determined in the final period. Take a vector $y^i_{1|0} \in X_0$ of period 0 consumption rights, and period 0 spot market prices $p_0 \in \mathbb{R}^{n_0}_{++}$. The possible choices of period 0 self of agent i are determined by values of correspondence $\tilde{V}^i_0 : \mathbb{R}^{n_0}_{++} \times X_0 \rightrightarrows X_0$,

$$\tilde{V}^i_0(p_0, y^i_{1|0}) := \left\{ x^i_0 \in X_0 \mid x^i_0 \text{ is a } \succeq^i_0\text{-g.e. of } \tilde{B}^i_0(p_0, y^i_{1|0}) \right\}, \tag{5.3}$$

similarly to (2.3). Given that agents are sophisticated, while acquiring $(x^i_1, y^i_{1|0})$ in the initial period they take into account the actual consumption x^i_0 that will take place at date 0. In particular, that $x^i_0 \in \tilde{V}^i_0(p_0, y^i_{1|0})$. Hence, agents evaluate their preferences over vectors $(x^i_1, x^i_0) \in \hat{X}_1$, where $x^i_0 \in \tilde{V}^i_0(p_0, y^i_{1|0})$, rather than $(x^i_1, y^i_{1|0})$. Given path of prices $\hat{p}_1 := (p_1, p_0)$ and $q_{1|0}$, period 1 selves evaluate the set of all affordable, time-consistent consumption paths, i.e. values of correspondence $\tilde{F}^i_0 : \mathbb{R}^{n_1+n_0}_{++} \times \mathbb{R}^{n_0}_{++} \times \hat{X}_1 \rightrightarrows \hat{X}_1$,

$$\tilde{F}^i_1(\hat{p}_1, q_{1|0}, \hat{e}^i_1) := \left\{ (x^i_1, x^i_0) \in \hat{X}_1 \;\middle|\; x^i_0 \in \tilde{V}^i_0(p_0, y^i_{1|0}),\ \text{where } (x^i_1, y^i_{1|0}) \in \tilde{B}^i_1(p_1, q_{1|0}, \hat{e}^i_1) \right\}, \tag{5.4}$$

and choose an element of $\tilde{F}^i_1(\hat{p}_1, q_{1|0}, \hat{e}^i_1)$ which maximises their current preferences. Hence, the set of choices of date 1 self of agent i is determined by values of correspondence $\tilde{V}^i_1 : \mathbb{R}^{n_1+n_0}_{++} \times \mathbb{R}^{n_0}_{++} \times \hat{X}_1 \rightrightarrows \hat{X}_1$,

$$\tilde{V}^i_1(\hat{p}_1, q_{1|0}, \hat{e}^i_1) := \left\{ (x^i_1, x^i_0) \in \hat{X}_1 \;\middle|\; (x^i_1, x^i_0) \text{ is a } \succeq^i_1\text{-g.e. of } \tilde{F}^i_1(\hat{p}_1, q_{1|0}, \hat{e}^i_1) \right\}. \tag{5.5}$$

Note that the optimisation problem of sophisticated agents in the initial period is very similar to the one introduced in (2.5), apart from two differences. First of all, we do not require for prices of rights to period 0 consumption $q_{1|0}$ to be equal to prices p_0 of the actual consumption taking place at date 0. Second of all, we allow for the actual consumption taking place in period 0 – x^i_0, to differ from the inherited rights to consumption $y^i_{1|0}$. We proceed with our definition of sequential equilibrium.

Definition 6 (Sequential equilibrium) *Assume $T = \{1, 0\}$. A* sequential equilibrium *of an economy starting at date 1 is a tuple of an allocation path $\hat{x}^*_1$, an allocation of period 0 consumption rights $y^*_{1|0}$, a path of spot market prices $\hat{p}^*_1$, and futures market prices $q^*_{1|0}$, summarised by $\{\hat{x}^*_1, y^*_{1|0}, \hat{p}^*_1, q^*_{1|0}\}$, such that*

(i) *given* $\hat{p}_1^*$, $q_{1|0}^*$, *the initial selves choose an optimal, time-consistent consumption plan, i.e. for all* $i \in I$, $\hat{x}_1^{*i} \in \tilde{V}_1^i(\hat{p}_1^*, q_{1|0}^*, \hat{e}_1^i)$;

(ii) *given* $\hat{p}_1^*$, $q_{1|0}^*$, *the rights to consumption are chosen consistently, i.e. for all* $i \in I$, $(x_1^{*i}, y_{1|0}^{*i}) \in \tilde{B}_1(p_1^*, \hat{q}_{1|0}^*, \hat{e}_1^i)$ *and* $x_0^{*i} \in \tilde{V}_0^i(p_0^*, y_{1|0}^{*i})$;

(iii) *spot markets and futures markets clear, i.e.* $\sum_{i\in I} x_1^{*i} = \sum_{i\in I} e_1^i$, *and* $\sum_{i\in I} y_{1|0}^{*i} = \sum_{i\in I} x_0^{*i} = \sum_{i\in I} e_0^i$.

The above notion of sequential equilibrium is more general than the one characterised in Definition 1. In particular, it introduces a sequence of spot and futures markets where each period, the current selves can re-trade their consumption goods and claims.

5.2. *Sequential equilibrium in a multiple-period economy*

Let T be an arbitrary number in $\mathbb{N}$. As in the two-period case, at each date t we allow the current selves to trade on two types of markets: *spot markets*, where agents can trade the current period consumption goods (i.e. consumption bundles in X_t), and *futures markets*, where consumers may exchange their rights to future consumption (i.e. elements in $\hat{X}_{t-1}$). The current consumption bundle is denoted by $x_t^i \in X_t$. Let $y_{t|k} \in X_k$ be a vector of rights to period k consumption, which were acquired on futures market at date t. Therefore, a portfolio acquired at period t, of rights to consumption from date s till period 0, is represented by $\hat{y}_{t|s}^i := (y_{t|k}^i)_{k=0}^s$. By construction, we let $\hat{y}_{t|s}^i := (y_{t|s}^i, \hat{y}_{t|s-1}^i) = (y_{t|s}^i, y_{t|s-1}^i, \ldots, \hat{y}_{t|s'}^i)$, for any $s' \leq s$.

An allocation of rights to date s consumption acquired in period t is denoted by $y_{t|s} := (y_{t|s}^i)_{i\in I} \in X_s^I$. At the same time, an allocation of portfolios acquired at date t of rights to consumption following date s will be represented by $\hat{y}_{t|s} := (\hat{y}_{t|s}^i)_{i\in I} \in \hat{X}_s^I$. Similarly, we let $\hat{y}_{t|s} := (y_{t|s}, \hat{y}_{t|s-1}) = (y_{t|s}, y_{t|s-1}, \ldots, \hat{y}_{t|s'})$, for any $s' \leq s$.

Period t spot market prices are denoted by $p_t \in \mathbb{R}_{++}^{n_t}$. Let $N_t := \sum_{s=0}^t n_s$. Hence, $\hat{p}_t = (p_s)_{s=0}^t \in \mathbb{R}_{++}^{N_t}$ denotes a path of spot market prices following date t. As in the case of consumption paths, we have $\hat{p}_t = (p_t, \hat{p}_{t-1}) = (p_t, p_{t-1} \ldots, \hat{p}_{t'})$, for any $t' \leq t$. Finally, date t futures market prices to period k consumption goods are represented by $q_{t|k} \in \mathbb{R}_{++}^{n_k}$. Therefore, $\hat{q}_{t|s} := (q_{t|k})_{k=0}^s \in \mathbb{R}_{++}^{N_s}$ is a vector of prices quoted at the futures market in period t for rights to consumption bundles following date s.

We construct the optimisation problem of date t self of agent i as follows. In the final period 0, given spot market prices p_0 and an inherited vector of rights to consumption $y_{1|0}^i$, consumer i determines the set of $\succeq_0^i$-greatest elements of $\tilde{B}_0(p_0, y_{1|0}^i)$, defined in (5.2). The set of his optimal choices is then $\tilde{V}_0^i(p_0, y_{1|0}^i)$, as in (5.3).

In period $t = 1$, the consumer determines the set of all affordable, time-consistent consumption paths and chooses the one which maximises his preferences. The only difference with respect to the two-period case, is that at the beginning of period 1 the current self is in possession of a portfolio of rights to date 1 and date 0 consumption bundles inherited from period 2, $\hat{y}^i_{2|1} \in \hat{X}_1$, rather than $\hat{e}^i_1$. Therefore, the set of affordable, time-consistent consumption paths is $\tilde{F}^i_1(\hat{p}_1, q_{1|0}, \hat{y}^i_{2|1})$, where correspondence $\tilde{F}^i_1$ is defined as in (5.4). Similarly, the set of choices is $\tilde{V}^i_1(\hat{p}_1, q_{1|0}, \hat{y}^i_{2|1})$, where $\tilde{V}^i_1$ is defined as in (5.5).

By backward induction, we determine the set of all affordable and time-consistent consumption paths for any t. At the beginning of time t, every consumer has a portfolio of rights to consumption following date t, $\hat{y}^i_{t+1|t} := (y^i_{t+1|s})^t_{s=0} \in \hat{X}_t$, which was acquired on the futures market in the preceding period – $(t+1)$. The budget set is then determined by values of correspondence $\tilde{B}_t : \mathbb{R}^{n_t}_{++} \times \mathbb{R}^{N_{t-1}}_{++} \times \hat{X}_t \rightrightarrows \hat{X}_t$,

$$(5.6)\qquad \tilde{B}_t(p_t, \hat{q}_{t|t-1}, \hat{y}^i_{t+1|t}) := \left\{ (x^i_t, \hat{y}^i_{t|t-1}) \in \hat{X}_t \;\middle|\; p_t \cdot x^i_t + \hat{q}_{t|t-1} \cdot \hat{y}^i_{t|t-1} \leq p_t \cdot y^i_{t+1|t} + \hat{q}_{t|t-1} \cdot \hat{y}^i_{t+1|t-1} \right\}.$$

Hence, the set of affordable and time-consistent choices is determined by $\tilde{F}^i_t : \mathbb{R}^{N_t}_{++} \times \mathbb{R}^{\sum_{s=0}^{t-1} N_s}_{++} \times \hat{X}_t \rightrightarrows \hat{X}_t$,

$$(5.7)\qquad \tilde{F}^i_t\left(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}^i_{t+1|t}\right) := \Big\{ (x^i_t, \hat{x}^i_{t-1}) \in \hat{X}_t \;\Big|\; \hat{x}^i_{t-1} \in \tilde{V}^i_{t-1}(\hat{p}_{t-1}, (\hat{q}_{s|s-1})^{t-1}_{s=1}, \hat{y}^i_{t|t-1}), \text{ where } (x^i_t, \hat{y}^i_{t|t-1}) \in \tilde{B}_t(p_t, \hat{q}_{t|t-1}, \hat{y}^i_{t+1|t}) \Big\},$$

where $\tilde{V}^i_{t-1}(\hat{p}_{t-1}, (\hat{q}_{s|s-1})^{t-1}_{s=1}, \hat{y}^i_{t|t-1})$ is the set of optimal, time-consistent choices of the following self at date $(t-1)$. Being consistent with our recursive structure, the set is determined by values of correspondence $\tilde{V}^i_t : \mathbb{R}^{N_t}_{++} \times \mathbb{R}^{\sum_{s=0}^{t-1} N_s}_{++} \times \hat{X}_t \rightrightarrows \hat{X}_t$,

$$(5.8)\quad \tilde{V}^i_t\left(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}^i_{t+1|t}\right) := \left\{ \hat{x}^i_t \in \hat{X}_t \;\middle|\; \hat{x}^i_t \text{ is a } \succeq^i_t\text{-g.e. of } \tilde{F}^i_t\left(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}^i_{t+1|t}\right) \right\}.$$

Note, that correspondences $\tilde{F}^i_t$ and $\tilde{V}^i_t$ are constructed analogously to correspondences F^i_t and V^i_t in Section 2.2. However, there are two differences. First of all, prices of rights to consumption and the actual consumption may differ, which is implied by different notation for the spot market price of date t consumption – p_t, and rights to period t consumption acquired at date k – $q_{k|t}$. Moreover, note that prices of rights to consumption at date t, acquired in any two periods k and k', where $k, k' \geq t$, denoted respectively by $q_{k|t}$ and $q_{k'|t}$, may also be different, since they are quoted at different moments in time. For this reason, when making a choice at time t, the current period self needs to take into account not only the path of the spot market prices following date t, but also the prices of portfolios quoted at the futures markets in every period following date t.

Second of all, we do not require that the rights to consumption inherited from the previous period coincide with the actual consumption. Note that by definition of set $\tilde{F}^i_t(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}^i_{t+1|t})$, we consider consumption path $\hat{x}^i_t = (x^i_t, \hat{x}^i_{t-1})$ to be time consistent, if there exists a portfolio $\hat{y}^i_{t|t-1}$ which is affordable, i.e $(x^i_t, \hat{y}^i_{t|t-1}) \in \tilde{B}_t(p_t, \hat{q}_{t|t-1}, \hat{y}^i_{t+1|t})$, and once the following self inherits $\hat{y}^i_{t|t-1}$, consumption path $\hat{x}^i_{t-1}$ constitutes one of his optimal choices, i.e. $\hat{x}^i_{t-1} \in \tilde{V}^i_{t-1}(\hat{p}_{t-1}, (\hat{q}_{s|s-1})^{t-1}_{s=1}, \hat{y}^i_{t|t-1})$.

The above specification of the consumer optimisation problem admits a much more sophisticated behaviour of agents. In particular, it might be that case, that even though at time t the current self cannot afford $\hat{x}^i_{t-1}$, i.e. $(x^i_t, \hat{x}^i_{t-1}) \notin \tilde{B}_t(p_t, \hat{q}_{t|t-1}, \hat{y}^i_{t+1|t})$, he may acquire a portfolio $\hat{y}^i_{t|t-1}$ such that $\hat{x}^i_{t-1} \in \tilde{V}^i_{t-1}(\hat{p}_{t-1}, (\hat{q}_{s|s-1})^{t-1}_{s=1}, \hat{y}^i_{t|t-1})$. Therefore, even though some consumption bundles might not be affordable at the current set of prices, it is possible that once an agent finds himself in a different period, the bundle might become affordable under the new set of prices.

We define a generalised notion of competitive equilibrium introduced in Definition 6.

Definition 6' (Sequential equilibrium) *Given the endowment distribution* $(\hat{e}^i_T)_{i\in I}$*, a* sequential equilibrium *of an economy starting at date T is a tuple of an allocation path $\hat{x}^*_T$, a sequence of portfolio allocations $(\hat{y}^*_{t|t-1})^T_{t=1}$, a path of spot market prices $\hat{p}^*_T$, and a sequence of futures market prices $(\hat{q}^*_{t|t-1})^T_{t=1}$, summarised by $\{\hat{x}^*_T, (\hat{y}^*_{t|t-1})^T_{t=1}, \hat{p}^*_T, (\hat{q}^*_{t|t-1})^T_{t=1}\}$, such that*

- *(i) given $\hat{p}^*_T$ and $(\hat{q}^*_{t|t-1})^T_{t=1}$, the initial selves choose an optimal, time-consistent consumption plan, i.e. for all $i \in I$, $\hat{x}^{*i}_T \in \tilde{V}^i_T(\hat{p}^*_T, (\hat{q}^*_{t|t-1})^T_{t=1}, \hat{e}^i_T)$;*
- *(ii) given $\hat{p}^*_T$ and $(\hat{q}^*_{t|t-1})^T_{t=1}$, the rights to consumption are chosen consistently, i.e. for all $(i,t) \in I \times T/\{0\}$, we have $(x^{*i}_t, \hat{y}^{*i}_{t|t-1}) \in \tilde{B}_t(p^*_t, (\hat{q}^*_{s|s-1})^t_{s=1}, \hat{y}^i_{t+1|t})$ and $\hat{x}^{*i}_t \in \tilde{V}^i_t(\hat{p}^*_t, (\hat{q}^*_{s|s-1})^t_{s=1}, \hat{y}^{*i}_{t|t-1})$, where $\hat{y}^{*i}_{T+1|T} = \hat{e}^i_T$;*
- *(iii) spot and futures markets clear, i.e. for all $t \in T$, and any $s < t$, we have $\sum_{i\in I} y^{*i}_{t|s} = \sum_{i\in I} x^i_s = \sum_{i\in I} e^i_s$.*

Clearly, in the two-period case Definitions 6 and 6' are equivalent. The following result is directly implied by the above definition.

Corollary 5 *Let $\{\hat{x}^*_T, (\hat{y}^*_{t|t-1})^T_{t=1}, \hat{p}^*_T, (\hat{q}^*_{t|t-1})^T_{t=1}\}$ be a sequential equilibrium of an economy starting at date T. Then, for any $t \le T$, $\{\hat{x}^*_t, (\hat{y}^*_{s|s-1})^t_{s=1}, \hat{p}^*_t, (\hat{q}^*_{s|s-1})^t_{s=1}\}$ is a sequential equilibrium of an economy starting at date t, for distribution of endowments $(\hat{y}^{*i}_{t+1|t})_{i\in I}$.*

PROOF: Let $\{\hat{x}^*_T, (\hat{y}^*_{t|t-1})_{t=1}^T, \hat{p}^*_T, (\hat{q}^*_{t|t-1})_{t=1}^T\}$ be a sequential equilibrium. Take any $t \leq T$, and adjust Definition 6' for $T = t$, with $(\hat{y}^{*i}_{t+1|t})_{i\in I}$ being the distribution of the initial endowment. Clearly, condition (iii) of Definition 6' is satisfied. By construction, $\forall i \in I$ and $t \leq T$, $\hat{x}^{*i}_T \in \tilde{V}^i_T(\hat{p}^*_T, (\hat{q}^*_{t|t-1})_{t=1}^T, \hat{e}^i_T)$ implies $\hat{x}^{*i}_t \in \tilde{V}^i_t(\hat{p}^*_t, (\hat{q}^*_{s|s-1})_{s=1}^t, \hat{y}^{*i}_{t+1|t})$. Hence, conditions (i) and (ii) also hold. *Q.E.D.*

What is interesting, is that even though the above definition seems to be more general and admit more sophisticated interaction between agents and their different selves, the notions of competitive and sequential equilibrium are equivalent, as we show in the following section.

5.3. *Equilibrium equivalence*

Before we state our equivalence result, we show several properties of sequential equilibria which might be of a separate interest.

PROPOSITION 6 *Let Assumption 1 be satisfied and $\{\hat{x}^*_T, (\hat{y}^*_{t|t-1})_{t=1}^T, \hat{p}^*_T, (\hat{q}^*_{t|t-1})_{t=1}^T\}$ be a sequential equilibrium. Then, for all $(i,t) \in I \times T$, we have $\hat{q}^*_{t+1|t} \cdot \hat{x}^{*i}_t = \hat{q}^*_{t+1|t} \cdot \hat{y}^{*i}_{t+1|t}$, where $\hat{y}^{*i}_{T+1|T} = \hat{e}^i_T$, and $\hat{q}^*_{T+1|T} = (p^*_T, \hat{q}^*_{T|T-1})$. Moreover, $p^*_0 \cdot x^{*i}_0 = p^*_0 \cdot y^{*i}_{1|0}$.*

The argument supporting the above claim is rather extensive, hence we presented it in the Appendix. In the main body of the paper we prove a two-period version of the above proposition. Consider the following corollary.

COROLLARY 6 *Assume $T := \{1, 0\}$. Let Assumption 1 be satisfied and $\{\hat{x}^*_1, y^*_{1|0}, \hat{p}^*_1, q^*_{1|0}\}$ be a sequential equilibrium. Then, for all $i \in I$, we have $q^*_{1|0} \cdot x^{*i}_0 = q^*_{1|0} \cdot y^{*i}_{1|0}$ and $p^*_0 \cdot x^{*i}_0 = p^*_0 \cdot y^{*i}_{1|0}$.*

PROOF: By Assumption 1 we have $\forall i \in I$, $p^*_0 \cdot x^{*i}_0 = p^*_0 \cdot y^{*i}_{1|0}$. Hence, it suffices to prove that $q^*_{1|0} \cdot x^{*i}_0 = q^*_{1|0} \cdot y^{*i}_{1|0}$.

First, we claim that $q^*_{1|0} \cdot y^{*i}_{1|0} \leq q^*_{1|0} \cdot x^{*i}_0$. Assume the opposite, i.e. $q^*_{1|0} \cdot y^{*i}_{1|0} > q^*_{1|0} \cdot x^{*i}_0$. Then we have $p^*_1 \cdot x^{*i}_1 + q^*_{1|0} \cdot x^{*i}_0 < p^*_1 \cdot x^{*i}_1 + q^*_{1|0} \cdot y^{*i}_{1|0} \leq p^*_1 \cdot e^i_1 + q^*_{1|0} \cdot e^i_0$. By Assumption 1, there exists a $x'^i_1 \in X_1$ such that $(x'^i_1, x^{*i}_0) \in \tilde{B}_1(p^*_1, q^*_{1|0}, \hat{e}^i_1)$ and $(x'^i_1, x^{*i}_0) \succ^i_1 \hat{x}^{*i}_1$. By definition, we have $x^{*i}_0 \in \tilde{V}^i_0(p^*_0, y^{*i}_{1|0})$. Since $p^*_0 \cdot x^{*i}_0 = p^*_0 \cdot y^{*i}_{1|0}$, this implies that $x^{*i}_0 \in \tilde{V}^i_0(p^*_0, x^{*i}_0)$, which contradicts that $\hat{x}^{*i}_1 \in \tilde{V}^i_1(\hat{p}^*_1, q^*_{1|0}, \hat{e}^i_1)$.

Next, we show that $q^*_{1|0} \cdot y^{*i}_{1|0} \geq q^*_{1|0} \cdot x^{*i}_0$. By the previous argument we know that $\forall i \in I$, $q^*_{1|0} \cdot y^{*i}_{1|0} \leq q^*_{1|0} \cdot x^{*i}_0$. Assume that for some i we have $q^*_{1|0} \cdot y^{*i}_{1|0} < q^*_{1|0} \cdot x^{*i}_0$. Then $q^*_{1|0} \cdot \sum_{i\in I} y^{*i}_{1|0} < q^*_{1|0} \cdot \sum_{i\in I} x^{*i}_0$. However, by condition (iii) of Definition 6, we have

$\sum_{i\in I} y^{*i}_{1|0} = \sum_{i\in I} x^{*i}_0 = \sum_{i\in I} e^i_0$. Hence, it must be that $q^*_{1|0} \cdot \sum_{i\in I} y^{*i}_{1|0} \geq q^*_{1|0} \cdot \sum_{i\in I} x^{*i}_0$, which yields contradiction. *Q.E.D.*

Proposition 6 implies that in any sequential equilibrium, all agents can afford their actually consumed paths of bundles. Moreover, in any period the value of their portfolio equals the value of the equilibrium consumption path. This implies the following proposition.

PROPOSITION 7 *Let Assumption 1 be satisfied and* $\{\hat{x}^*_T, (\hat{y}^*_{t|t-1})^T_{t=1}, \hat{p}^*_T, (\hat{q}^*_{t|t-1})^T_{t=1}\}$ *be a sequential equilibrium. Then, tuple* $\{\hat{x}^*_T, (\hat{y}'_{t|t-1})^T_{t=1}, \hat{p}^*_T, (\hat{q}^*_{t|t-1})^T_{t=1}\}$*, where for all* $(i,t) \in I \times T/\{0\}$*, we have* $\hat{y}'^i_{t|t-1} = \hat{x}^{*i}_{t-1}$*, is also a sequential equilibrium.*

Once again, we move the proof of the result to the Appendix and present an argument supporting the two-period case of the above proposition.

COROLLARY 7 *Assume* $T = \{1, 0\}$*. Let Assumption 1 be satisfied, and* $\{\hat{x}^*_1, y^*_{1|0}, \hat{p}^*_1, q^*_{1|0}\}$ *be a sequential equilibrium. Then* $\{\hat{x}^*_1, x^*_0, \hat{p}^*_1, q^*_{1|0}\}$ *is also a sequential equilibrium.*

PROOF: Let $\{\hat{x}^*_1, y^*_{1|0}, \hat{p}^*_1, q^*_{1|0}\}$ be a sequential equilibrium. We need to verify, that tuple $\{\hat{x}^*_1, x^*_0, \hat{p}^*_1, q^*_{1|0}\}$ is also a sequential equilibrium. Clearly, condition (iii) of Definition 6 is satisfied. Therefore, it suffices to verify that conditions (i) and (ii) hold.

By Corollary 6, we have $\forall i \in I$, $q^*_{1|0} \cdot x^{*i}_0 = q^*_{1|0} \cdot y^{*i}_{1|0}$. Hence, $(x^{*i}_1, x^{*i}_0) \in \tilde{B}^i_1(p^*_1, q^*_{1|0}, \hat{e}^i_1)$. Moreover, $\forall i \in I$, $p^*_0 \cdot x^{*i}_0 = p^*_0 \cdot y^{*i}_{1|0}$. Therefore, $x^{*i}_0 \in \tilde{V}^i_0(p^*_0, x^{*i}_0)$, which implies that $\forall i \in I$, $\hat{x}^{*i}_1 \in \tilde{V}^i_1(\hat{p}^*_1, q^*_{1|0}, \hat{e}^i_1)$. *Q.E.D.*

In the following proposition we establish that at any period t, equilibrium spot prices of period t consumption p^*_t are equal to the futures market prices $q^*_{s|t}$, quoted at any date $s \geq t$.

PROPOSITION 8 *Let Assumption 1 be satisfied and* $\sum_{i\in I} \hat{e}^i_{T-1}$ *be strictly positive. Then, for any sequential equilibrium* $\{\hat{x}^*_T, (\hat{y}^*_{t|t-1})^T_{t=1}, \hat{p}^*_T, (\hat{q}^*_{t|t-1})^T_{t=1}\}$*, we have* $\hat{q}^*_{s|t} = (p^*_t, \hat{q}^*_{t|t-1})$ *and* $q^*_{1|0} = p^*_0$ *(up to a scalar), for any* $t \in T$ *and* $s \geq t$*.*

We prove a two-period version of the above corollary, while the general argument is presented in the Appendix.[13]

COROLLARY 8 *Assume* $T = \{1, 0\}$*. Let Assumption 1 be satisfied and* $\sum_{i\in I} e^i_0$ *be strictly positive. Then, for any sequential equilibrium* $\{\hat{x}^*_1, y^*_{1|0}, \hat{p}^*_1, q^*_{1|0}\}$*, we have* $q^*_{1|0} = p^*_0$ *(up to a scalar).*

[13] I am grateful to John Quah for useful comments which allowed me to simplify the proof.

PROOF: Let $\{\hat{x}_1^*, y_{1|0}^*, \hat{p}_1^*, q_{1|0}^*\}$ be a sequential equilibrium. For any $i \in I$ define sets $P(x_0^{*i}) := \{x_0^i \in X_0 \mid p_0^* \cdot x_0^i = p_0^* \cdot x_0^{*i}\}$ and $Q(x_0^{*i}) := \{x_0^i \in X_0 \mid q_{1|0}^* \cdot x_0^i = q_{1|0}^* \cdot x_0^{*i}\}$. By definition $x_0^{*i} \in P(x_0^{*i}) \cap Q(x_0^{*i})$.

First, we claim that for any $x_0^i \in P(x_0^{*i})$, we have $q_{1|0}^* \cdot x_0^i \geq q_{1|0}^* \cdot x_0^{*i}$. We show it by contradiction. Assume that there exists $i \in I$ and $x_0^i \in P(x_0^{*i})$ such that $q_{1|0}^* \cdot x_0^i < q_{1|0}^* \cdot x_0^{*i}$. Then, Assumption 1 and Corollary 6 imply that $p_1^* \cdot x_1^{*i} + q_{1|0}^* \cdot x_0^i < p_1^* \cdot x_1^{*i} + q_{1|0}^* \cdot x_0^{*i} = p_1^* \cdot x_1^{*i} + q_{1|0}^* \cdot y_{1|0}^{*i} \leq p_1^* \cdot e_1^i + q_{1|0}^* \cdot e_0^i$. In addition, Corollary 6 implies that $p_0^* \cdot x_0^{*i} = p_0^* \cdot y_{1|0}^{*i}$, and since $x_0^i \in P(x_0^{*i})$, we have $p_0^* \cdot x_0^i = p_0^* \cdot y_{1|0}^{*i}$. Therefore, $\tilde{B}_0(p_0^*, x_0^i) = \tilde{B}_0(p_0^*, y_{1|0}^{*i})$, which implies that $x_0^{*i} \in \tilde{V}_0^i(p_0^*, x_0^i)$. By Assumption **??**, there exists some $x_1'^i \in X_1$ such that $(x_1'^i, x_0^i) \in \tilde{B}_1(p_1^*, q_{1|0}^*, \hat{e}_1^i)$ and $(x_1'^i, x_0^{*i}) \succ_1^i \hat{x}_1^{*i}$, which contradicts that $\hat{x}_1^{*i} \in \tilde{V}_1^i(\hat{p}_1^*, q_{1|0}^*, \hat{e}_1^i)$. Hence, $q_{1|0}^* \cdot x_0^i \geq q_{1|0}^* \cdot x_0^{*i}$.

Next, assume that $q_{1|0}^* \neq p_0^*$ (up to a scalar). By Lemma A.5 and the above claim, this implies that $P(x_0^{*i}) \cap Q(x_0^{*i}) \subset \partial X_0$. For all $i \in I$, define $\nu^i = (p_0^* \cdot x_0^{*i})/(p_0^* \cdot \sum_{i\in I} e_0^i)$, and $x_\nu^i = \nu^i \sum_{i\in I} e_0^i$. By construction, $\forall i \in I$, we have $x_\nu^i \in P(x_0^{*i})$. Moreover, the market clearing condition implies $\sum_{i\in I} \nu^i = 1$.

Since $\sum_{i\in I} e_0^i$ is strictly positive, whenever x_0^{*i} is non-zero it must be that $x_\nu^i \notin P(x_0^{*i}) \cap Q(x_0^{*i})$. Hence, $q_{1|0}^* \cdot x_\nu^i > q_{1|0}^* \cdot x_0^{*i}$, as well as $q_{1|0}^* \cdot \sum_{i\in I} x_\nu^i > q_{1|0}^* \cdot \sum_{i\in I} x_0^{*i}$. However, then $q_{1|0}^* \cdot \sum_{i\in I} e_0^i = q_{1|0}^* \cdot \sum_{i\in I} x_\nu^i > q_{1|0}^* \cdot \sum_{i\in I} x_0^{*i} = q_{1|0}^* \cdot \sum_{i\in I} e_0^i$. Contradiction. *Q.E.D.*

Propositions 6 and 8 have an important implication, which we state in the following proposition.

PROPOSITION 9 (Equilibrium equivalence) *Let Assumption 1 be satisfied and* $\sum_{i\in I} \hat{e}_{T-1}^i$ *be strictly positive. Then* $\{\hat{x}_T^*, \hat{p}_T^*\}$ *is a competitive equilibrium if and only if there exist sequences* $(\hat{y}_{t|t-1}^*)_{t=1}^T$ *and* $(\hat{q}_{t|t-1}^*)_{t=1}^T$ *such that* $\{\hat{x}_T^*, (\hat{y}_{t|t-1}^*)_{t=1}^T, \hat{p}_T^*, (\hat{q}_{t|t-1}^*)_{t=1}^T\}$ *is sequential equilibrium.*

PROOF: First we prove ($\Rightarrow$). Let $\{\hat{x}_T^*, \hat{p}_T^*\}$ be a competitive equilibrium. We claim that $\{\hat{x}_T^*, (\hat{y}_{t|t-1}^*)_{t=1}^T, \hat{p}_T^*, (\hat{q}_{t|t-1}^*)_{t=1}^T\}$, where for all $t \in T$, $s > t$, $y_{s|t}^{*i} = x_t^{*i}$ and $q_{s|t}^* = p_t^*$, is a sequential equilibrium. Clearly, conditions (ii) and (iii) of Definition 6' are satisfied. Moreover, in this case $\tilde{V}_T^i(\hat{p}_T^*, (\hat{q}_{t|t-1}^*)_{t=1}^T, \hat{e}_T^i) = V_T^i(\hat{p}_T^*, \hat{e}_T^i)$. Hence condition (i) also holds.

To show ($\Leftarrow$), assume that $\{\hat{x}_T^*, (\hat{y}_{t|t-1}^*)_{t=1}^T, \hat{p}_T^*, (\hat{q}_{t|t-1}^*)_{t=1}^T\}$ is a sequential equilibrium. By Proposition 6, so is $\{\hat{x}_T^*, (\hat{y}_{t|t-1}')_{t=1}^T, \hat{p}_T^*, (\hat{q}_{t|t-1}^*)_{t=1}^T\}$, where $\forall t \in T$, $s > t$, $\hat{y}_{s|t}' = \hat{x}_t^*$. Moreover, by Proposition 8, $\forall t \in T$, $s > 0$, $\hat{q}_{s|t}^* = (p_t^*, \hat{q}_{t|t-1}^*)$ and $q_{1|0}^* = p_0^*$ (up to a scalar). Hence, it is always possible to normalise prices such that $q_{s|t}^* = p_t^*$, for all $t \in T$, $s > t$. Clearly, condition (ii) of Definition 1' is satisfied. Since $\tilde{V}_T^i(\hat{p}_T^*, (\hat{q}_{t|t-1}^*)_{t=1}^T, \hat{e}_T^i) = V_T^i(\hat{p}_T^*, \hat{e}_T^i)$, so is (i). *Q.E.D.*

Note, that in general there is a greater number of sequential equilibria than competitive equilibria, simply by the fact that there are many variations of portfolio structure which support the same equilibrium allocation $\hat{x}^*_T$. However, if one is interested solely in the actual consumption taking place in the equilibrium, then the two notions give equivalent implications.

APPENDIX

We begin this section with several lemmas used in the proofs of our main results. First we determine properties of optimal choices of consumers in a multi-period economy defined in Section 2.2. The notation used in the argument below refers to the one presented in that section.

LEMMA A.1 *For any $t \in T$, $\hat{p}_t \in \mathbb{R}^{N_t}_{++}$ and $\hat{y}'^i_t, \hat{y}^i_t \in \hat{X}_t$ such that $\hat{p}_t \cdot \hat{y}'^i_t = \hat{p}_t \cdot \hat{y}^i_t$, we have $F^i_t(\hat{p}_t, \hat{y}'^i_t) = F^i_t(\hat{p}_t, \hat{y}^i_t)$ and $V^i_t(\hat{p}_t, \hat{y}'^i_t) = V^i_t(\hat{p}_t, \hat{y}^i_t)$.*

PROOF: Take any $t \in T$ and any two $\hat{y}'^i_t, \hat{y}^i_t \in \tilde{X}_t$ such that $\hat{p}_t \cdot \hat{y}'^i_t = \hat{p}_t \cdot \hat{y}^i_t$. By definition, $F^i_t(\hat{p}_t, \hat{y}^i_t) := \left\{(x^i_t, \hat{x}^i_{t-1}) \in B_t(\hat{p}_t, \hat{y}^i_t) \mid \hat{x}^i_{t-1} \in V^i_{t-1}(\hat{p}_{t-1}, \hat{x}^i_{t-1})\right\}$. Clearly, $B_t(\hat{p}_t, \hat{y}^i_t) = B_t(\hat{p}_t, \hat{y}'^i_t)$, hence $F^i_t(\hat{p}_t, \hat{y}^i_t) = F^i_t(\hat{p}_t, \hat{y}'^i_t)$. Since $V^i_t(\hat{p}_t, \hat{y}^i_t)$ and $V^i_t(\hat{p}_t, \hat{y}'^i_t)$ contain the $\succeq^i_t$-greatest elements of $F^i_t(\hat{p}_t, \hat{y}^i_t)$ and $F^i_t(\hat{p}_t, \hat{y}'^i_t)$ respectively, the two sets are equal. *Q.E.D.*

LEMMA A.2 *Let $\hat{x}^i_t \in V^i_t(\hat{p}_t, \hat{x}^i_t)$. Then, for any $\hat{x}'^i_t \in \hat{X}_t$ such that for all $t' \leq t$, $\hat{x}'^i_{t'} \sim^i_{t'} \hat{x}^i_{t'}$ and $\hat{p}_{t'} \cdot \hat{x}'^i_{t'} = \hat{p}_{t'} \cdot \hat{x}^i_{t'}$, we have $\hat{x}'^i_t \in V^i_t(\hat{p}_t, \hat{x}^i_t)$.*

PROOF: We prove the result by induction. First consider the final period 0, and let $x^i_0 \in V^i_0(p_0, x^i_0)$. Take some $x'^i_0 \in X_0$ such that $x'^i_0 \sim^i_0 x^i_0$ and $p_0 \cdot x'^i_0 = p_0 \cdot x^i_0$. Clearly, $x'^i_0 \in B_0(p_0, x^i_0)$. Since $x'^i_0 \sim^i_0 x^i_0$, we have $x'^i_0 \in V^i_0(p_0, x^i_0)$.

Next, take any $t \in T$ and $\hat{x}^i_t \in V^i_t(\hat{p}_t, \hat{x}^i_t)$. Let $\hat{x}'^i_t \in \hat{X}_t$ be such that $\forall t' \leq t$, we have $\hat{x}'^i_{t'} \sim^i_{t'} \hat{x}^i_{t'}$ and $\hat{p}_{t'} \cdot \hat{x}'^i_{t'} = \hat{p}_{t'} \cdot \hat{x}^i_{t'}$. Assume that $\hat{x}'^i_{t-1} \in V^i_{t-1}(\hat{p}_{t-1}, \hat{x}^i_{t-1})$. Lemma A.1 implies that $\hat{x}'^i_{t-1} \in V^i_{t-1}(\hat{p}_{t-1}, \hat{x}'^i_{t-1})$. Moreover, by definition we have $F^i_t(\hat{p}_t, \hat{x}^i_t) := \left\{(z^i_t, \hat{z}^i_{t-1}) \in B_t(\hat{p}_t, \hat{x}^i_t) \mid \hat{z}^i_{t-1} \in V^i_{t-1}(\hat{p}_{t-1}, \hat{z}^i_{t-1})\right\}$. Clearly, $\hat{x}'^i_t \in B_t(\hat{p}_t, \hat{x}^i_t)$. Since $\hat{x}'^i_{t-1} \in V^i_{t-1}(\hat{p}_{t-1}, \hat{x}'^i_{t-1})$, we have $\hat{x}'^i_t \in F^i_t(\hat{p}_t, \hat{x}^i_t)$. Hence, $\hat{x}'^i_t \in V^i_t(\hat{p}_t, \hat{x}^i_t)$, which completes the proof. *Q.E.D.*

Now we can proceed with the proof of Theorem 1. We obtain the result by induction. However, since the proof is extensive, we present it through a sequence of claims. Throughout the following argument we assume that $\{\hat{x}^*_T, \hat{p}^*_T\}$ is a competitive equilibrium satisfying Definition 1'.

Claim A.1 *Allocation x_0^* is post-0 efficient. Moreover, for any $x_0 \in E_0$ such that for all $i \in I$, $x_0^i \sim_0^i x_0^{*i}$, we have $p_0^* \cdot x_0^i = p_0^* \cdot x_0^{*i}$, for all $i \in I$.*

Proof: Since $\{\hat{x}_T^*, \hat{p}_T^*\}$ is a competitive equilibrium, by Corollary 1 pair $\{\hat{x}_1^*, \hat{p}_1^*\}$ is a competitive equilibrium of an economy starting at date 1, given the distribution of endowments $(\hat{x}_1^{*i})_{i\in I}$. The rest follows from Claims 2 and 3 in Section 3.2. *Q.E.D.*

Claim A.2 *Take any $\hat{x}_{t+1}^i \in \hat{X}_{t+1}$ such that for all $t' \leq t$, $\hat{x}_{t'}^i \sim_{t'}^i \hat{x}_{t'}^{*i}$ and $\hat{p}_{t'}^* \cdot \hat{x}_{t'}^i = \hat{p}_{t'}^* \cdot \hat{x}_{t'}^{*i}$. Then $\hat{x}_{t+1}^i \succeq_{t+1}^i \hat{x}_{t+1}^{*i} \Rightarrow p_{t+1}^* \cdot x_{t+1}^i \geq p_{t+1}^* \cdot x_{t+1}^{*i}$ and $\hat{x}_{t+1}^i \succ_{t+1}^i \hat{x}_{t+1}^{*i} \Rightarrow p_{t+1}^* \cdot x_{t+1}^i > p_{t+1}^* \cdot x_{t+1}^{*i}$.*

Proof: Since $\{\hat{x}_T^*, \hat{p}_T^*\}$ is a competitive equilibrium, by Corollary 1 $\{\hat{x}_t^*, \hat{p}_t^*\}$ is a competitive equilibrium following date t, for the distribution of endowments $(\hat{x}_t^{*i})_{i\in I}$. In particular, $\forall i \in I$, $\hat{x}_t^{*i} \in V_t^i(\hat{p}_t^*, \hat{x}_t^{*i})$. Similarly, $\{\hat{x}_{t+1}^*, \hat{p}_{t+1}^*\}$ is a competitive equilibrium following date $t+1$, for distribution of endowments $(\hat{x}_{t+1}^{*i})_{i\in I}$. Hence $\hat{x}_{t+1}^{*i} \in V_{t+1}^i(\hat{p}_{t+1}^*, \hat{x}_{t+1}^{*i})$ for all i. Let $\hat{x}_{t+1}^i = (x_{t+1}^i, \hat{x}_t^i) \in \hat{X}_{t+1}$ be specified as in the thesis of the claim. Given the assumptions as well as Lemmas A.1 and A.2, we have $\hat{x}_t^i$, $\hat{x}_t^{*i} \in V_t^i(\hat{p}_t^*, \hat{x}_t^{*i})$, for any i.

We begin with proving the first part of the claim by contradiction. Assume that $\hat{x}_{t+1}^i \succeq_{t+1}^i \hat{x}_{t+1}^{*i}$, and $p_{t+1}^* \cdot x_{t+1}^i < p_{t+1}^* \cdot x_{t+1}^{*i}$. Since $\forall t' \leq t$, $\hat{p}_{t'}^* \cdot \hat{x}_{t'}^i = \hat{p}_{t'}^* \cdot \hat{x}_{t'}^{*i}$, we have $p_{t+1}^* \cdot x_{t+1}^i + \hat{p}_t^* \cdot \hat{x}_t^i < p_{t+1}^* \cdot x_{t+1}^{*i} + \hat{p}_t^* \cdot \hat{x}_t^{*i}$. By Assumption 1, there exists some $x_{t+1}'^i \in X_{t+1}$ such that $(x_{t+1}'^i, \hat{x}_t^i) \in B_{t+1}(\hat{p}_{t+1}^*, \hat{x}_{t+1}^{*i})$ and $(x_{t+1}'^i, \hat{x}_t^i) \succ_{t+1}^i \hat{x}_{t+1}^i$. Therefore, $(x_{t+1}'^i, \hat{x}_t^i) \succ_{t+1}^i \hat{x}_{t+1}^{*i}$.

Recall that by Lemma A.1, $\hat{x}_t^i \in V_t^i(\hat{p}_t^*, \hat{x}_t^{*i}) = V_t^i(\hat{p}_t^*, \hat{x}_t^i)$. Hence, $(x_{t+1}'^i, \hat{x}_t^i)$ belongs to $F_{t+1}^i(\hat{p}_{t+1}^*, \hat{x}_{t+1}^{*i})$. This contradicts that $\hat{x}_{t+1}^{*i} \in V_{t+1}^i(\hat{p}_{t+1}^*, \hat{x}_{t+1}^{*i})$, so it must be that $\hat{x}_{t+1}^i \succeq_{t+1}^i \hat{x}_{t+1}^{*i} \Rightarrow p_{t+1}^* \cdot x_{t+1}^i \geq p_{t+1}^* \cdot x_{t+1}^{*i}$.

Next, assume that $\hat{x}_{t+1}^i \succ_{t+1}^i \hat{x}_{t+1}^{*i}$. By the first part of the claim, we know that $p_{t+1}^* \cdot x_{t+1}^i \geq p_{t+1}^* \cdot x_{t+1}^{*i}$. Let $p_{t+1}^* \cdot x_{t+1}^i = p_{t+1}^* \cdot x_{t+1}^{*i}$. Then $\hat{x}_{t+1}^i \in B_{t+1}(\hat{p}_{t+1}^*, \hat{x}_{t+1}^{*i})$. Moreover, since $\hat{x}_t^i \in V_t^i(\hat{p}_t^*, \hat{x}_t^{*i}) = V_t^i(\hat{p}_t^*, \hat{x}_t^i)$ (by Lemma A.2), we have $\hat{x}_{t+1}^i \in F_{t+1}^i(\hat{p}_{t+1}^*, \hat{x}_{t+1}^{*i})$. This contradicts that $\hat{x}_{t+1}^{*i} \in V_{t+1}^i(\hat{p}_{t+1}^*, \hat{x}_{t+1}^{*i})$. Therefore, we have $\hat{x}_{t+1}^i \succ_{t+1}^i \hat{x}_{t+1}^{*i} \Rightarrow p_{t+1}^* \cdot x_{t+1}^i > p_{t+1}^* \cdot x_{t+1}^{*i}$. *Q.E.D.*

Claim A.3 *Let $\hat{x}_t^*$ be post-t efficient for some $t \in T$. Moreover, assume that for any feasible $\hat{x}_{t+1} \in E_{t+1}$ we have that if for all $i \in I$ and all $t' \leq t$, $\hat{x}_{t'}^i \sim_{t'}^i \hat{x}_{t'}^{*i}$, then for all $i \in I$ and all $t' \leq t$, $\hat{p}_{t'}^* \cdot \hat{x}_{t'}^i = \hat{p}_{t'}^* \cdot \hat{x}_{t'}^{*i}$. Then $\hat{x}_{t+1}^*$ is post-$(t+1)$ efficient.*

Proof: We prove the claim by contradiction. Assume, that $\hat{x}_{t+1}^*$ is not post-$(t+1)$ efficient. Therefore, there exists some other $\hat{x}_{t+1} \in E_{t+1}$ such that $\forall i \in I$ and $\forall t' \leq (t+1)$, we have $\hat{x}_{t'}^i \succeq_{t'}^i \hat{x}_{t'}^{*i}$, and for some i and $t' \leq (t+1)$, $\hat{x}_{t'}^i \succ_{t'}^i \hat{x}_{t'}^{*i}$.

By assumption, $\hat{x}_t^*$ is post-t efficient, so it must be that $\forall i \in I$ and $\forall t' \leq t$, $\hat{x}_{t'}^i \sim_{t'}^i \hat{x}_{t'}^{*i}$. Therefore, for the above statement to hold we must have $\forall i \in I$, $\hat{x}_{t+1}^i \succeq_{t+1}^i \hat{x}_{t+1}^{*i}$, and for some i, $\hat{x}^i_{t+1} \succ^i_{t+1} \hat{x}^{*i}_{t+1}$. Claim A.2 and the market clearing condition imply that $p^*_{t+1} \cdot \sum_{i\in I} x^i_{t+1} > p^*_{t+1} \cdot \sum_{i\in I} x^{*i}_{t+1} = p^*_{t+1} \cdot \sum_{i\in I} e^i_{t+1}$. But, $\hat{x}_{t+1} \in E_{t+1}$, so in particular $p^*_{t+1} \cdot \sum_{i\in I} x^i_{t+1} \leq p^*_{t+1} \cdot \sum_{i\in I} e^i_{t+1}$. Contradiction. *Q.E.D.*

CLAIM A.4 *Take any $\hat{x}_{t+1} \in E_{t+1}$ such that for all $i \in I$ and all $t' \leq t$, $\hat{x}^i_{t'} \sim^i_{t'} \hat{x}^{*i}_{t'}$ and $\hat{p}^*_{t'} \cdot \hat{x}^i_{t'} = \hat{p}^*_{t'} \cdot \hat{x}^{*i}_{t'}$. If for all $i \in I$, $\hat{x}^i_{t+1} \sim^i_{t+1} \hat{x}^{*i}_{t+1}$, then for all $i \in I$, $\hat{p}^*_{t+1} \cdot \hat{x}^i_{t+1} = \hat{p}^*_{t+1} \cdot \hat{x}^{*i}_{t+1}$.*

PROOF: Take any $\hat{x}_{t+1} \in E_{t+1}$ which satisfies the thesis of the claim. By Claim A.2, we know that $\forall i \in I$, whenever $\hat{x}^i_{t+1} \sim^i_{t+1} \hat{x}^{*i}_{t+1}$, then $p^*_{t+1} \cdot x^i_{t+1} \geq p^*_{t+1} \cdot x^{*i}_{t+1}$. Assume that for some i, $p^*_{t+1} \cdot x^i_{t+1} > p^*_{t+1} \cdot x^{*i}_{t+1}$. Then $p^*_{t+1} \cdot \sum_{i\in I} x^i_{t+1} > p^*_{t+1} \cdot \sum_{i\in I} x^{*i}_{t+1} = p^*_{t+1} \cdot \sum_{i\in I} e^i_{t+1}$. Since $\hat{x}_{t+1}$ is feasible, in particular $\sum_{i\in I} p^*_{t+1} \cdot x^i_{t+1} \leq \sum_{i\in I} p^*_{t+1} \cdot e^i_{t+1}$. Contradiction.

To complete the proof, recall that by assumption $\forall i \in I$ and $\forall t' \leq t$, we have $\hat{p}^*_{t'} \cdot \hat{x}^i_{t'} = \hat{p}^*_{t'} \cdot \hat{x}^{*i}_{t'}$. By the above argument $\forall i \in I$, $p^*_{t+1} \cdot x^i_{t+1} = p^*_{t+1} \cdot x^{*i}_{t+1}$. Hence, $\forall i \in I$, $\hat{p}^*_{t+1} \cdot \hat{x}^i_{t+1} = \hat{p}^*_{t+1} \cdot \hat{x}^{*i}_{t+1}$. *Q.E.D.*

Given the necessary prerequisites, we proceed with the proof of our first main theorem.

PROOF OF THEOREM 1: Let $\{\hat{x}^*_T, \hat{p}^*_T\}$ be a competitive equilibrium. We prove the result by induction. First, consider the final period 0. By Claim A.1, we know that x^*_0 is post-0 efficient. Moreover, for any $x_0 \in E_0$ such that $\forall i \in I$, $x^i_0 \sim^i_0 x^{*i}_0$, we have $\forall i \in I$, $p^*_0 \cdot x^i_0 = p^*_0 \cdot x^{*i}_0$.

By Claim A.3, we know that whenever $\hat{x}^*_t$ is post-t efficient, and for any $x_t \in E_t$ such that $\forall i \in I$ and $\forall t' \leq t$, $\hat{x}^i_{t'} \sim^i_{t'} \hat{x}^{*i}_{t'}$ implies that $\forall i \in I$ and $\forall t' \leq t$, $\hat{p}^*_{t'} \cdot \hat{x}^i_{t'} = \tilde{p}^*_{t'} \cdot \tilde{x}^{*i}_{t'}$, then $\hat{x}^*_{t+1}$ is post-$(t+1)$ efficient. In addition, Claim A.4 implies that for any $\hat{x}_{t+1} \in E_{t+1}$ such that $\forall i \in I$ and $\forall t' \leq (t+1)$, $\hat{x}^i_{t'} \sim^i_{t'} \hat{x}^{*i}_{t'}$, we have $\forall i \in I$ and $\forall t' \leq (t+1)$, $\hat{p}^*_{t'} \cdot \hat{x}^i_{t'} = \hat{p}^*_{t'} \cdot \hat{x}^{*i}_{t'}$.

Since the property is satisfied by the allocation in period 0, it also satisfied at any period $t \in T$. Therefore, for any $t \in T$, allocation $\hat{x}^*_t$ is post-t efficient, and so it is recursively efficient. *Q.E.D.*

PROOF OF THEOREM 2: Assume that $\hat{x}_T$ is a recursively efficient allocation. We prove the result by induction. First, take $t = 0$. Let $u_0 : X^I_0 \to \mathbb{R}^I$, be defined as $u_0 := (u^i_0)_{i\in I}$, and $U'_0 := u_0(E_0)$. Since $\forall i \in I$, u^i_0 is convex, it is also continuous (see Rockafellar, 1970, Theorem 10.1). Therefore, by compactness of E_0, U'_0 is compact. Let $U_0 := \{u \in \mathbb{R}^I \mid \forall x'_0 \in E_0,\ \forall i \in I,\ u^i \leq u^i_0(x'^i_0)\}$. By Assumption 3, U_0 is convex. Moreover, by construction $U_0 = U'_0 - \mathbb{R}^I_+$. Hence, U_0 is closed and bounded above.

Denote $u^*_0 = u(x_0)$. By definition of $\hat{x}_T$, there exists no other $x'_0 \in E_0$ such that $\forall i \in I$, $u^i_0(x'^i_0) \geq u^i_0(x^i_0)$, and $u^i_0(x'^i_0) > u^i(x^i_0)$ for some i. Hence, it must be that $u^*_0 \in \partial U_0$. By the separating hyperplane theorem (see, e.g. Aliprantis and Border, 2006, Theorem 7.30), there exists some non-zero vector $\alpha_0 \in \mathbb{R}^I$ such that $\forall u \in U_0$, $\alpha_0 \cdot u_0^* \geq \alpha_0 \cdot u$. Since $U_0 - \mathbb{R}^I_+ \subset U_0$, it must be that $\alpha_0 \in \mathbb{R}^I_+$. By construction, this implies that $x_0 \in \Psi_0(\alpha_0)$. Finally, by strict concavity of u_0^i and convexity of E_0, it must be that $\{x_0\} = \Psi_0(\alpha_0)$.

Next, take any $t \in T$. Assume that there exists a path of non-zero, positive vectors $\hat{\alpha}_t$ such that $\Psi_t(\hat{\alpha}_t) = \{\hat{x}_t\}$. Clearly, in this case $\Gamma_{t+1}(\hat{\alpha}_t)$ is compact and convex. Let $u_{t+1} : \tilde{X}^I_{t+1} \to \mathbb{R}^I$ be define by $u_{t+1} := (u^i_{t+1})_{i \in I}$, and $U'_{t+1} := u_{t+1}(\Gamma_{t+1})$. Since $\forall \hat{x}^i_t \in \hat{X}_t$, $u^i_{t+1}(x^i_{t+1}, \hat{x}^i_t)$ is concave with respect to x^i_{t+1}, it is also continuous with respect to x^i_{t+1} (by Rockafellar, 1970, Theorem 10.1). Therefore, U'_{t+1} is compact. Let $U_{t+1} := \{u \in \mathbb{R}^I \mid \forall \hat{x}'_{t+1} \in \Gamma_{t+1}(\hat{\alpha}_t),\ \forall i \in I,\ u^i \leq u^i_{t+1}(\hat{x}'^i_{t+1})\}$, which is convex. Moreover, $U_{t+1} := U'_{t+1} - \mathbb{R}^I_+$. Compactness of U'_{t+1} implies that U_{t+1} is closed and bounded above.

To complete the proof denote $u^*_{t+1} = u_{t+1}(\hat{x}_{t+1})$. By definition, $\hat{x}_{t+1}$ is post-$(t+1)$ efficient, so it must be that $u^*_{t+1} \in \partial U_{t+1}$. By the separating hyperplane theorem (see Aliprantis and Border, 2006, Theorem 7.30), there exists some non-zero vector $\alpha_{t+1} \in \mathbb{R}^I$ such that $\forall u \in U_{t+1}$, $\alpha_{t+1} \cdot u^*_{t+1} \geq \alpha_{t+1} \cdot u$. Since $U_{t+1} - \mathbb{R}^I_+ \subset U_{t+1}$, it must be that $\alpha_{t+1} \in \mathbb{R}^I_+$. Therefore, by construction $\hat{x}_{t+1} \in \Psi_{t+1}(\hat{\alpha}_{t+1})$, where $\hat{\alpha}_{t+1} := (\alpha_{t+1}, \hat{\alpha}_t)$. Eventually, as $\forall \hat{x}^i_t \in \hat{X}_t$, $u^i_{t+1}(x^i_{t+1}, \hat{x}^i_t)$ is strictly concave with respect to x^i_{t+1} and Γ_t convex, we have $\Psi_{t+1}(\hat{\alpha}_{t+1}) = \{\hat{x}_{t+1}\}$. *Q.E.D.*

Before we proceed with proofs of results presented in Section 5.3, we introduce the following lemmas.

LEMMA A.3 *For any $\hat{p}_t \in \mathbb{R}^{N_t}_{++}$, $(\hat{q}_{s|s-1})^t_{s=1} \in \mathbb{R}^{\sum_{s=0}^{t-1} N_s}_{++}$, and $\hat{y}^i_{t+1|t}$, $\hat{y}'^i_{t+1|t} \in \hat{X}^i_t$ such that $p_t \cdot y^i_{t+1|t} + \hat{q}_{t|t-1} \cdot \hat{y}^i_{t+1|t-1} = p_t \cdot y'^i_{t+1|t} + \hat{q}_{t|t-1} \cdot \hat{y}'^i_{t+1|t-1}$, we have $\tilde{V}^i_t(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}^i_{t+1|t}) = \tilde{V}^i_t(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}'^i_{t+1|t})$.*

PROOF: Recall that by (5.7) we have

$$\tilde{F}^i_t\left(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}^i_{t+1|t}\right) := \Big\{(x^i_t, \hat{x}^i_{t-1}) \in \hat{X}_t \;\Big|\; \hat{x}^i_{t-1} \in \tilde{V}^i_{t-1}(\hat{p}_{t-1}, (\hat{q}_{s|s-1})^{t-1}_{s=1}, \hat{y}^i_{t|t-1}),$$
$$\text{where } (x^i_t, \hat{y}^i_{t|t-1}) \in \tilde{B}_t(p_t, \hat{q}_{t|t-1}, \hat{y}^i_{t+1|t})\Big\}.$$

By assumption we have $p_t \cdot y^i_{t+1|t} + \hat{q}_{t|t-1} \cdot \hat{y}^i_{t+1|t-1} = p_t \cdot y'^i_{t+1|t} + \hat{q}_{t|t-1} \cdot \hat{y}'^i_{t+1|t-1}$, which implies that $\tilde{B}_t(p_t, \hat{q}_{t|t-1}, \hat{y}^i_{t+1|t}) = \tilde{B}_t(p_t, \hat{q}_{t|t-1}, \hat{y}'^i_{t+1|t})$, and so $\tilde{F}^i_t(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}^i_{t+1|t}) = \tilde{F}^i_t(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}'^i_{t+1|t})$. Since $\tilde{V}^i_t(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}^i_{t+1|t})$ and $\tilde{V}^i_t(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}'^i_{t+1|t})$ contain the $\succeq^i_t$-greatest elements of $\tilde{F}^i_t(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}^i_{t+1|t})$ and $\tilde{F}^i_t(\hat{p}_t, (\hat{q}_{s|s-1})^t_{s=1}, \hat{y}'^i_{t+1|t})$ respectively, they also must be equal. *Q.E.D.*

LEMMA A.4 *Let Assumption 1 be satisfied and $\{\hat{x}^*_T, (\hat{y}^*_{t|t-1})^T_{t=1}, \hat{p}^*_T, (\hat{q}^*_{t|t-1})^T_{t=1}\}$ be a sequential equilibrium. If $\hat{x}^{*i}_t \in \tilde{V}^i_t(\hat{p}^*_t, (\hat{q}^*_{s|s-1})^t_{s=1}, \hat{y}^{*i}_{t+1|t})$, then $\hat{x}^{*i}_t \in \tilde{V}^i_t(\hat{p}^*_t, (\hat{q}^*_{s|s-1})^t_{s=1}, \hat{x}^{*i}_t)$.*

PROOF: We prove the result by induction. Take $t = 0$, and $x_0^{*i} \in \tilde{V}_0^i(p_0^*, y_{1|0}^{*i})$. By by local non-satiation of preferences induced by Assumption 1, we have $p_0^* \cdot x_0^{*i} = p_0^* \cdot y_{1|0}^{*i}$, which implies that $\tilde{B}_0(p_0^*, y_{1|0}^{*i}) = \tilde{B}_0(p_0^*, x_0^{*i})$. Therefore, $\tilde{V}_0^i(p_0^*, x_0^{*i}) = \tilde{V}_0^i(p_0^*, y_{1|0}^{*i})$ and $x_0^{*i} \in \tilde{V}_0^i(p_0^*, x_0^{*i})$.

Next, take any $t \in T$ and assume that $\hat{x}_{t-1}^{*i} \in \tilde{V}_{t-1}^i(\hat{p}_{t-1}^*, (\hat{q}_{s|s-1}^*)_{s=1}^{t-1}, \hat{x}_{t-1}^{*i})$. We show that the same property holds at period t. By (5.7)

$$\tilde{F}_t^i\left(\hat{p}_t^*, (\hat{q}_{s|s-1}^*)_{s=1}^t, \hat{y}_{t+1|t}^{*i}\right) := \Big\{(x_t^i, \hat{x}_{t-1}^i) \in \hat{X}_t \;\Big|\; \hat{x}_{t-1}^i \in \tilde{V}_{t-1}^i(\hat{p}_{t-1}^*, (\hat{q}_{s|s-1}^*)_{s=1}^{t-1}, \hat{y}_{t|t-1}^i), \text{ where } (x_t^i, \hat{y}_{t|t-1}^i) \in \tilde{B}_t(p_t^*, \hat{q}_{t|t-1}^*, \hat{y}_{t+1|t}^{*i})\Big\}.$$

We begin by showing that $\hat{x}_t^{*i} \in \tilde{B}_t(p_t^*, \hat{q}_{t|t-1}^*, \hat{y}_{t+1|t}^{*i})$, with $p_t^* \cdot x_t^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{x}_{t-1}^{*i} = p_t^* \cdot y_{t+1|t}^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{y}_{t+1|t-1}^{*i}$.

First, we argue that $p_t^* \cdot x_t^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{x}_{t-1}^{*i} \geq p_t^* \cdot y_{t+1|t}^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{y}_{t+1|t-1}^{*i}$. Assume the opposite. By Assumption 1, there exists some $x_t'^i \in X_t$ such that $(x_t'^i, \hat{x}_{t-1}^{*i}) \in \tilde{B}_t(p_t^*, \hat{q}_{t|t-1}^*, \hat{y}_{t+1|t}^{*i})$ and $(x_t'^i, \hat{x}_{t-1}^{*i}) \succ_t^i \hat{x}_t^{*i}$. Moreover, by assumption $\hat{x}_{t-1}^{*i} \in \tilde{V}_{t-1}^i(\hat{p}_{t-1}^*, (\hat{q}_{s|s-1}^*)_{s=1}^{t-1}, \hat{x}_{t-1}^{*i})$. Hence, $(x_t'^i, \hat{x}_{t-1}^{*i}) \in \tilde{F}_t^i(\hat{p}_t^*, (\hat{q}_{s|s-1}^*)_{s=1}^t, \hat{y}_{t+1|t}^{*i})$, which contradicts that $\hat{x}_t^{*i} \in \tilde{V}_t^i(\hat{p}_t^*, (\hat{q}_{s|s-1}^*)_{s=1}^t, \hat{y}_{t+1|t}^{*i})$.

Next we argue that $p_t^* \cdot x_t^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{x}_{t-1}^{*i} \leq p_t^* \cdot y_{t+1|t}^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{y}_{t+1|t-1}^{*i}$. By the previous argument we know that for all $i \in I$, $p_t^* \cdot x_t^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{x}_{t-1}^{*i} \geq p_t^* \cdot y_{t+1|t}^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{y}_{t+1|t-1}^{*i}$. Assume that for some i, we have $p_t^* \cdot x_t^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{x}_{t-1}^{*i} > p_t^* \cdot y_{t+1|t}^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{y}_{t+1|t-1}^{*i}$. Then, $p_t^* \cdot \sum_{i\in I} x_t^{*i} + \hat{q}_{t|t-1}^* \cdot \sum_{i\in I} \hat{x}_{t-1}^{*i} > p_t^* \cdot \sum_{i\in I} y_{t+1|t}^{*i} + \hat{q}_{t|t-1}^* \cdot \sum_{i\in I} \hat{y}_{t+1|t-1}^{*i}$. However, by the market clearing condition we have $p_t^* \cdot \sum_{i\in I} e_t^i + \hat{q}_{t|t-1}^* \cdot \sum_{i\in I} \hat{e}_{t-1}^i = p_t^* \cdot \sum_{i\in I} x_t^{*i} + \hat{q}_{t|t-1}^* \cdot \sum_{i\in I} \hat{x}_{t-1}^{*i} > p_t^* \cdot \sum_{i\in I} y_{t+1|t}^{*i} + \hat{q}_{t|t-1}^* \cdot \sum_{i\in I} \hat{y}_{t+1|t-1}^{*i} = p_t^* \cdot \sum_{i\in I} e_t^i + \hat{q}_{t|t-1}^* \cdot \sum_{i\in I} \hat{e}_{t-1}^i$, which yields contradiction.

By the above claim as well as Lemma A.3, we conclude that $\tilde{V}_t^i(\hat{p}_t^*, (\hat{q}_{s|s-1}^*)_{s=1}^t, \hat{x}_t^{*i}) = \tilde{V}_t^i(\hat{p}_t^*, (\hat{q}_{s|s-1}^*)_{s=1}^t, \hat{y}_{t+1|t}^{*i})$. Hence, $\hat{x}_t^{*i} \in \tilde{V}_t^i(\hat{p}_t^*, (\hat{q}_{s|s-1}^*)_{s=1}^t, \hat{x}_t^{*i})$. *Q.E.D.*

PROOF OF PROPOSITION 6: Let $\{\hat{x}_T^*, (\hat{y}_{t|t-1}^*)_{t=1}^T, \hat{p}_T^*, (\hat{q}_{t|t-1}^*)_{t=1}^T\}$ be a sequential equilibrium. By Corollary 5, $\{\hat{x}_1^*, y_{1|0}^*, \hat{p}_1^*, q_{1|0}^*\}$ constitutes a sequential equilibrium starting at date 1, with $(\hat{y}_{2|1}^*)_{i\in I}$ being the distribution of initial endowments. Hence, by Corollary 6 we have $p_0^* \cdot y_{1|0}^{*i} = p_0^* \cdot x_0^{*i}$.

Take any $t \in T$. First we show that $\hat{q}_{t|t-1}^* \cdot \hat{y}_{t-1|t}^{*i} \leq \hat{q}_{t|t-1}^* \cdot \hat{x}_{t-1}^{*i}$. Assume the opposite, i.e. $\hat{q}_{t|t-1}^* \cdot \hat{x}_{t-1}^{*i} < \hat{q}_{t|t-1}^* \cdot \hat{y}_{t|t-1}^{*i}$. Then, $p_t^* \cdot x_t^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{x}_{t-1}^{*i} < p_t^* \cdot x_t^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{y}_{t|t-1}^{*i} \leq p_t^* \cdot y_{t+1|t}^{*i} + \hat{q}_{t|t-1}^* \cdot \hat{y}_{t+1|t-1}^{*i}$. By Assumption 1, there exists a $x_t'^i \in X_t$ such that $(x_t'^i, \hat{x}_{t-1}^{*i}) \in \tilde{B}_t(p_t^*, \hat{q}_{t|t-1}^*, \hat{y}_{t+1|t}^{*i})$ and $(x_t'^i, \hat{x}_{t-1}^{*i}) \succ_t^i \hat{x}_t^{*i}$. By Lemma A.4, $\hat{x}_{t-1}^{*i} \in \tilde{V}_{t-1}^i(\hat{p}_{t-1}^*, (\hat{q}_{s|s-1}^*)_{s=1}^{t-1}, \hat{x}_{t-1}^{*i})$. Hence, we have $(x_t'^i, \hat{x}_{t-1}^{*i}) \in \tilde{F}_t^i(\hat{p}_t^*, (\hat{q}_{s|s-1}^*)_{s=1}^t, \hat{y}_{t+1|t}^{*i})$, which contradicts that $\hat{x}_t^{*i} \in \tilde{V}_t^i(\hat{p}_t^*, (\hat{q}_{s|s-1}^*)_{s=1}^t, \hat{y}_{t+1|t}^{*i})$.

Next, we show that $\hat{q}_{t|t-1}^* \cdot \hat{y}_{t|t-1}^{*i} \geq \hat{q}_{t|t-1}^* \cdot \hat{x}_{t-1}^{*i}$. By the previous argument we have $\hat{q}_{t|t-1}^* \cdot \hat{y}_{t|t-1}^{*i} \leq \hat{q}_{t|t-1}^* \cdot \hat{x}_{t-1}^{*i}$ for all i. Assume that for some i we have $\hat{q}_{t|t-1}^* \cdot \sum_{i\in I} \hat{y}_{t|t-1}^{*i} <$

$\hat{q}^*_{t|t-1} \cdot \hat{x}^{*i}_{t-1}$. Then $\hat{q}^*_{t|t-1} \cdot \sum_{i\in I} \hat{y}^{*i}_{t|t-1} < \hat{q}^*_{t|t-1} \cdot \sum_{i\in I} \hat{x}^{*i}_{t-1}$. However, by the market clearing condition we have $\hat{q}^*_{t|t-1} \cdot \sum_{i\in I} \hat{e}^i_{t-1} = \hat{q}^*_{t|t-1} \cdot \sum_{i\in I} \hat{y}^{*i}_{t|t-1} < \hat{q}^*_{t|t-1} \cdot \sum_{i\in I} \hat{x}^{*i}_{t-1} = \hat{q}^*_{t|t-1} \cdot \sum_{i\in I} \hat{e}^i_{t-1}$. Contradiction. *Q.E.D.*

Proof of Proposition 7: Let $\{\hat{x}^*_T, (\hat{y}^*_{t|t-1})^T_{t=1}, \hat{p}^*_T, (\hat{q}^*_{t|t-1})^T_{t=1}\}$ be a sequential equilibrium. Let $\forall (i,t) \in I \times T/\{0\}$, $\hat{y}'^i_{t|t-1} = \hat{x}^{*i}_{t-1}$. We show that $\{\hat{x}^*_T, (\hat{y}'_{t|t-1})^T_{t=1}, \hat{p}^*_T, (\hat{q}^*_{t|t-1})^T_{t=1}\}$ is also a sequential equilibrium.

Clearly, the market clearing condition holds. Moreover, by Lemma A.4, $\forall (i,t) \in I \times T$, $\hat{x}^{*i}_t \in \tilde{V}^i_t(\hat{p}^*_t, (\hat{q}^*_{s|s-1})^t_{s=1}, \hat{x}^{*i}_t)$. On the other hand, by Proposition 6 we have $\hat{x}^{*i}_t \in \tilde{B}_t(p^*_t, \hat{q}^*_{t|t-1}, \hat{y}^{*i}_{t+1|t})$. The proof is complete. *Q.E.D.*

Lemma A.5 *Let $X = \mathbb{R}^n_+$, and $y \in X$ be a non-zero vector. For any $p, q \in \mathbb{R}^n_{++}$, where $p \neq q$, define $P(y) := \{x \in X \mid p \cdot x = p \cdot y\}$ and $Q(y) := \{x \in X \mid q \cdot x = q \cdot y\}$ such that for any $x \in P(y)$, we have $q \cdot x \geq q \cdot y$. Then $x' \in P(y) \cap Q(y)$ implies $x' \in \partial X$.*

Proof: Let $x' \in P(y) \cap Q(y)$. Since $p \neq q$ and for any $x \in P(y)$ we have $q \cdot x \geq q \cdot y$, there exists a $x'' \in P(y)$ such that $q \cdot x'' > q \cdot y$. As $x' \in Q(y)$, we have $q \cdot x'' > q \cdot x'$. Assume that $x' \in X/\partial X$. Then there exists $\alpha \in (0,1)$ and $x \in P(y)$ such that $\alpha x'' + (1-\alpha)x = x'$. Hence, $x = (1-\alpha)^{-1}(x' - \alpha x'')$. However, this implies that $q \cdot x = (1-\alpha)^{-1}(q \cdot x' - \alpha q \cdot x'') < q \cdot y$, which yields contradiction since for any $x \in P(y)$ we have $q \cdot x \geq q \cdot y$. *Q.E.D.*

Proof of Proposition 8: Let $\{\hat{x}^*_T, (\hat{y}^*_{t|t-1})^T_{t=1}, \hat{p}^*_T, (\hat{q}^*_{t|t-1})^T_{t=1}\}$ be a sequential equilibrium. We prove the result by induction.

By Corollary 5, for any sequential equilibrium $\{\hat{x}^*_T, (\hat{y}^*_{t|t-1})^T_{t=1}, \hat{p}^*_T, (\hat{q}^*_{t|t-1})^T_{t=1}\}$, tuple $\{\hat{x}^*_1, y^*_{1|0}, \hat{p}^*_1, q^*_{1|0}\}$ is a sequential equilibrium starting at date 1, given the initial distribution of endowments $(\hat{y}^{*i}_{2|1})_{i\in I}$. By Corollary 8, this implies that $p^*_0 = q^*_{1|0}$ (up to a scalar).

Take any $t \in T$ and assume that $\hat{p}^*_{t-1} = \hat{q}^*_{t|t-1}$ (up to a scalar). We claim that this implies $\hat{p}^*_t = \hat{q}^*_{t+1|t}$ (up to a scalar). For any $i \in I$ define sets $P(\hat{x}^{*i}_t) := \{\hat{x}^i_t \in \hat{X}_t \mid \hat{p}^*_t \cdot \hat{x}^i_t = \hat{p}^*_t \cdot \hat{x}^{*i}_t\}$ and $Q(\hat{x}^{*i}_t) := \{\hat{x}^i_t \in \hat{X}_t \mid \hat{q}^*_{t+1|t} \cdot \hat{x}^i_t = \hat{q}^*_{t+1|t} \cdot \hat{x}^{*i}_t\}$. By definition $\hat{x}^{*i}_t \in P(\hat{x}^{*i}_t) \cap Q(\hat{x}^{*i}_t)$.

First, we claim that for any $\hat{x}^i_t \in P(\hat{x}^{*i}_t)$, we have $\hat{q}^*_{t+1|t} \cdot \hat{x}^i_t \geq \hat{q}^*_{t+1|t} \cdot \hat{x}^{*i}_t$. We prove it by contradiction. Assume that there exists $\hat{x}^i_t \in P(\hat{x}^{*i}_t)$ such that $\hat{q}^*_{t+1|t} \cdot \hat{x}^i_t < \hat{q}^*_{t+1|t} \cdot \hat{x}^{*i}_t$. Assumption 1 and Proposition 6 imply that $p^*_{t+1} \cdot x^{*i}_{t+1} + \hat{q}^*_{t+1|t} \cdot \hat{x}^i_t < p^*_{t+1} \cdot x^{*i}_{t+1} + \hat{q}^*_{t+1|t} \cdot \hat{x}^{*i}_t = p^*_{t+1} \cdot x^{*i}_{t+1} + \hat{q}^*_{t+1|t} \cdot \hat{y}^{*i}_{t+1|t}$. By Assumption 1 there exists a $x'^i_{t+1} \in X_{t+1}$ such that $(x'^i_{t+1}, \hat{x}^i_t) \in \tilde{B}_{t+1}(p^*_{t+1}, \hat{q}^*_{t+1|t}, \hat{y}^{*i}_{t+2|t+1})$ and $(x'^i_{t+1}, \hat{x}^{*i}_t) \succ^i_{t+1} \hat{x}^{*i}_{t+1}$. In addition, since $\hat{x}^i_t \in P(\hat{x}^{*i}_t)$ and $\hat{p}^*_{t-1} = \hat{q}^*_{t|t-1}$, we have $p^*_t \cdot x^i_t + \hat{q}^*_{t|t-1} \cdot \hat{x}^i_{t-1} = \hat{p}^*_t \cdot \hat{x}^i_t = \hat{p}^*_t \cdot \hat{x}^{*i}_t = p^*_t \cdot x^{*i}_t + \hat{q}^*_{t|t-1} \cdot \hat{x}^{*i}_{t-1}$. By Lemmas A.3 and A.4, this implies that $\hat{x}^{*i}_t \in \tilde{V}^i_t(\hat{p}^*_t, (\hat{q}^*_{s|s-1})^t_{s=1}, \hat{x}^i_t)$.

Therefore, it must be that $(x'^i_{t+1}, \hat{x}^{*i}_t) \in \tilde{F}^i_{t+1}(\hat{p}^*_{t+1}, (\hat{q}^*_{s|s-1})^{t+1}_{s=1}, \hat{y}^{*i}_{t+2|t+1})$, which contradicts that $\hat{x}^{*i}_{t+1} \in \tilde{V}^i_{t+1}(\hat{p}^*_{t+1}, (\hat{q}^*_{s|s-1})^{t+1}_{s=1}, \hat{y}^{*i}_{t+2|t+1})$.

Next, assume that $\hat{q}^*_{t+1|t} \neq \hat{p}^*_t$ (up to a scalar). By Lemma A.5 and the above claim, this implies that $P(\hat{x}^{*i}_t) \cap Q(\hat{x}^{*i}_t) \subset \partial \hat{X}_t$. For all $i \in I$ define $\nu^i = (\hat{p}^*_t \cdot \hat{x}^{*i}_t)/(\hat{p}^*_t \cdot \sum_{i\in I} \hat{e}^i_t)$, and $\hat{x}^i_\nu = \nu^i \sum_{i\in I} \hat{e}^i_t$. By construction, $\forall i \in I$, we have $\hat{x}^i_\nu \in P(\hat{x}^{*i}_t)$. Moreover, the market clearing condition implies $\sum_{i\in I} \nu^i = 1$.

Since $\sum_{i\in I} \hat{e}^i_t$ is strictly positive, whenever $\hat{x}^{*i}_t$ is non-zero it must be that $\hat{x}^i_\nu \notin P(\hat{x}^{*i}_t) \cap Q(\hat{x}^{*i}_t)$. Hence, $\hat{q}^*_{t+1|t} \cdot \hat{x}^i_\nu > \hat{q}^*_{t+1|t} \cdot \hat{x}^{*i}_t$, as well as $\hat{q}^*_{t+1|t} \cdot \sum_{i\in I} \hat{x}^i_\nu > \hat{q}^*_{t+1|t} \cdot \sum_{i\in I} \hat{x}^{*i}_t$. This leads to contradiction, since by construction $\hat{q}^*_{t+1|t} \cdot \sum_{i\in I} \hat{e}^i_t = \hat{q}^*_{t+1|t} \cdot \sum_{i\in I} \hat{x}^i_\nu > \hat{q}^*_{t+1|t} \cdot \sum_{i\in I} \hat{x}^{*i}_t = \hat{q}^*_{t+1|t} \cdot \sum_{i\in I} \hat{e}^i_t$. *Q.E.D.*

REFERENCES

ALIPRANTIS, C. D., AND K. C. BORDER (2006): *Infinite dimentional analysis. A hitchhiker's guide*. Springer-Verlag, Berlin Heilbelberg, 3rd edn.

BALBUS, Ł., K. REFFETT, AND Ł. WOŹNY (2011): "Computing time-consistent Markov policies for quasi-hyperbolic consumers under uncertainty," Mimeo, Arizona State University.

BARRO, R. J. (1999): "Ramsey Meets Laibson In The Neoclassical Growth Model," *The Quarterly Journal of Economics*, 114(4), 1125–1152.

HARRIS, C., AND D. LAIBSON (2001): "Dynamic choices of hyperbolic consumers," *Econometrica*, 69(4), 935–57.

HERINGS, P. J., AND K. ROHDE (2006): "Time-inconsistent preferences in a general equilibrium model," *Economic Theory*, 29(3), 591–619.

KEHOE, T. J. (1991): "Computation and multiplicity of equilibria," in *Handbook of Mathematical Economics*, ed. by W. Hildenbrand, and H. Sonnenschein, vol. 4 of *Handbook of Mathematical Economics*, chap. 38, pp. 2049–2144. Elsevier.

LAIBSON, D. (1997): "Golden eggs and hyperbolic discounting," *Quarterly Journal of Economics*, 112(2), 443–77.

LUTTMER, E. G. J., AND T. MARIOTTI (2003): "Subjective Discounting in an Exchange Economy," *Journal of Political Economy*, 111(5), 959–989.

——— (2006): "Competitive equilibrium when preferences change over time," *Economic Theory*, 27(3), 679–690.

——— (2007): "Efficiency and equilibrium when preferences are time-inconsistent," *Journal of Economic Theory*, 132(1), 493–506.

MAS-COLELL, A., M. D. WHINSTON, AND J. R. GREEN (1995): *Microeconomic theory*. Oxford University Press.

NEGISHI, T. (1960): "Welfare economics and existence of equilibrium for a competitive economy," *Metroeconomica*, 12, 92–97.

Rockafellar, T. R. (1970): *Convex analysis*. Princeton University Press.

Strotz, R. H. (1955): "Myopia and inconsistency in dynamic utility maximization," *Review of Economic Studies*, 23(3), 165–180.