

# Monetary Policy and Quantitative Easing in an Open Economy: Prices, Exchange Rates and Risk Premia \*

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August 16, 2013

## Abstract

Under Quantitative Easing, Open Market Operations involve arbitrary portfolios of assets and not exclusively nominally risk free bonds held with a specific target composition. In a simple stochastic cash-in-advance model of a large open economy, quantitative easing inhibits the ability of the central bank to control the path of prices and exchange rates. This is the case even with non-Ricardian fiscal policy.

Alternative modes of conduct of monetary policy have measurable implications. A financial stability target, where the central bank trades only in nominally risk free bonds, implies that the risk premium is positively correlated with future interest rates. A price stability, or inflation, target induces the same correlation, while a monetary stability target reverses the sign of the correlation. Naïve estimations of aggregate risk premia may be misleading if monetary policy is not accounted for.

**Key words:** monetary policy; uncertainty; indeterminacy; fiscal policy; open economy.

**JEL classification numbers:** D50; E31; E40; E50; F41.

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\*First version: December 2010. Acknowledgements: Andrew Chernih, Li Lin.

# 1 Introduction

How monetary policy transmits inflation expectations to other countries is a question of theoretical interest and practical importance. The failure to control inflation domestically can be the cause of suboptimal domestic fluctuations, if indeterminacy is real, and can de-stabilise trading partners via current account changes. Optimal fiscal-monetary policy supports an optimal allocation of resources; if such a policy is also consistent with other, suboptimal, equilibrium allocations, then, it does not “implement” the targeted allocation.<sup>1</sup> Under normal conditions, monetary policy sets a target for the short-term (here one period) interest rates, and conducts open market operations or repo transactions, using as collateral Treasury securities, with various maturities, but to conform to an ex-ante determined overall portfolio composition which has an exclusive focus on Treasuries of short maturity. Unconventional monetary policy expands the balance sheet by increasing the maturity range (and possibly range of assets) of the monetary authority portfolio. As under conventional monetary policy, under the recent US experience of Credit Easing it is the explicit target for the composition of the balance sheet that allows the monetary authority to target the stochastic path of inflation: the target for the composition of the portfolio guarantees the necessary restrictions to obtain determinacy. The absence of such restrictions under the UK and Japanese versions of QE manifests nominal (and possibly) real indeterminacy. Here we show that, non-traditional methods of conducting monetary policy such as quantitative easing affect the path of prices and furthermore, the interaction with interest rate rules generate specific risk premia associated with the correlation between interest rates and the martingale measure in an open economy.

To address these issues, we consider an open economy extension of McMahon et al (2012) and Nakajima and Polemarchakis (2005), similar in spirit to Lucas (1982) and Geanakoplos and Tsomocos (2002). Specifically, we consider large open stochastic cash-in-advance economies, and first show that indeterminacy is pervasive: monetary policy does not suffice to determine the stochastic path of inflation. This indeterminacy may affect real allocations even with flexible prices, depending on the conduct of monetary policy, the completeness of asset markets, and the timing of transactions in goods and asset markets.

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<sup>1</sup>Chari and Kehoe (1999) and Bloise et al. (2005) survey the literature.

In an open economy, this indeterminacy proliferates. The stochastic distribution of prices is now independently indeterminate in each country. If all countries coordinate on to an interest rate monetary policy rule, the indeterminacy is purely nominal, while if even one country runs a money supply rule, then via current account changes, the indeterminacy becomes (globally) indeterminate. This result is beyond that of [Dupor \(2000\)](#), where like us, they explore exchange rate determination in a multi country/ currency model under a nominal interest rate peg. They too restrict the substitutability of currency as a method of payment across borders and maintained the possibility that the exchange rate is not unique for a conventional monetary/fiscal policy. Our result is stronger however. Their result resets on agents being indifferent as to the currency in which they hold their money balances, ours does not. Although the non-Ricardian fiscal policy pins down initial price levels and hence the initial exchange rate, the stochastic distribution of prices and exchange rates depends on asset demands. As the monetary authority is willing to supply state-contingent bonds, maintaining only the interest rate, individual asset prices are left undetermined. Furthermore, as agents are indifferent between purchasing assets in any country, the indeterminacy in one country proliferates globally.

The fact that the initial price level and the nominal equivalent martingale measure are indeterminate implies that monetary policy leaves indeterminacy of degree equal to the number of unique martingale probabilities in a finite-period model (1 less than the number of terminal nodes)<sup>2</sup>.

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<sup>2</sup> There is a vast and important literature on indeterminacy of monetary equilibria. [Sargent and Wallace \(1975\)](#) discussed the indeterminacy of the initial price level under interest rate policy; [Lucas and Stokey \(1987\)](#) derived the condition for the uniqueness of a recursive equilibrium with money supply policy; [Woodford \(1994\)](#) analyzed the dynamic paths of equilibria associated with the indeterminacy of the initial price level under money supply policy. In this paper, we give the exact characterization of the indeterminacy in stochastic economies in terms of the initial price level and the nominal equivalent martingale measure and extend the argument to the sticky-price case. Also, we show that there is a continuum of recursive equilibria with interest rate policy. In closely related models, [Dubey and Geanakoplos \(1992, 2003\)](#) considered non-Ricardian fiscal policy with no transfers and [Geanakoplos and Tsomocos \(2002\)](#) and [Tsomocos \(2008\)](#) extended their model to an open economy. [Dreze and Polemarchakis \(2000\)](#) and [Bloise et al. \(2005\)](#) studied the existence and indeterminacy of monetary equilibria with a particular Ricardian fiscal policy, seigniorage distributed contemporaneously as dividend to the private sector. The literature on incomplete markets shows the degree of real indeterminacy which proliferates when contracts are in nominal terms. [Geanakoplos and Mas-Colell \(1989\)](#) showed that there are generically  $S - 1$  degrees of indeterminacy, where  $S$  is the number of states. In an

The mainstream competitive model has locally unique equilibria with respect to the real side of the economy; however, it manifests nominal indeterminacy. [Kareken and Wallace \(1981\)](#) extend the O.L.G. indeterminacy result to a monetary model of the international economy. [Tsomocos \(2008\)](#) show that under non-ricardian fiscal policy, international monetary equilibria are locally unique<sup>3</sup>.

The necessity of analysis of the determinacy of any model and specifically any monetary model is the question of money non-neutrality or lack thereof. In other words, a model as the traditional competitive model that produces real determinacy but nominal indeterminacy manifests neutrality of monetary policy. Changes of the money supply affect nominal variables without influencing the determination of the real allocations of an economy. Therefore, the study of the number of equilibria in an economic model lies at the heart of the neutrality debate in macroeconomics.

We then study determinate equilibria and argue that the correlation between monetary costs and real asset payoffs in monetary models creates risk-premia in expected exchange rates. Monetary costs generate a wedge between cash and credit goods, and consequently affect marginal utilities and equilibrium prices. This premium causes the term structure to lie above levels predicted by the pure expectation hypothesis. In equilibrium models where monetary policy is neutral, as in [Lucas \(1982\)](#), as risk premia are constant, interest rate differentials move one-for-one with the expected change in the exchange rate. Empirically, however, the expected change in the exchange rate is roughly constant and interest differentials move approximately one-for-one with risk premia. Furthermore, the forward premium anomaly, as documented by [Fama \(1984\)](#), [Hodrick \(1987\)](#), and [Backus et al. \(1995\)](#) among others, states that when a currency's interest rate is high, that currency is expected to appreciate. Here we show that not only does the stochastic distribution of prices and interest rates domestically matter, but also the correlation of monetary policy across countries, in determining risk premia. We do this by considering the general equilibrium model of [Lucas \(1982\)](#) who considered only “cash goods”, to include “credit goods”. The

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abstract open economy, [Polemarchakis \(1988\)](#) allow  $A$  assets to be dominated in  $N$  distinct units of account or currencies. In addition to the purchasing power of one currency, the rates of exchange across currencies may now vary. In this setting [Polemarchakis \(1988\)](#) shows that, generically, the economy displays  $NS - A(N - 1) - N$  degrees of indeterminacy.

<sup>3</sup>This is in the model of [Geanakoplos and Tsomocos \(2002\)](#), which has qualitatively a similar structure to [Lucas \(1982\)](#)

International Finance models of Geanakoplos and Tsomocos (2002), Tsomocos (2008), Peiris and Tsomocos (2010) and Peiris (2010) study the effects of this and monetary policy becomes non-neutral since monetary changes affect nominal variables which in turn determine different real allocations. In a closed economy Espinoza et al. (2009) show that the risk-premia generated by the non-neutrality of a monetary policy exist in addition to the ones derived from the stochastic distribution of endowments as presented in Lucas (1978) and Breeden (1979). They provide a potential explanation for the Term Premium Puzzle. In such a setting there is a role for monetary policy to determine the equilibrium allocation, as presented in Tsomocos (2003) and Goodhart et al. (2006)<sup>4</sup>.

## 2 Monetary World Economy

In this section, we describe the benchmark economy with flexible prices and characterize the set of equilibria. All markets are perfectly competitive. Money is valued through a cash-in-advance constraint, as in Lucas and Stokey (1987). We consider non-ricardian fiscal policy which determines the initial price level but leaves the probability measure associated with nominal state prices, which is referred to as the nominal equivalent martingale measure, indeterminate.

Consider an economy with an infinite time horizon. In each discrete period  $t \geq 0$ , one of  $S$  possible shocks  $s \in \mathcal{S}$  is realized. Denote the shock occurring in period  $t$  as  $s_t$ . We represent the resolution of uncertainty by an event tree  $\Sigma$ , with a given date-event  $\sigma \in \Sigma$ . Each date-event  $\sigma$  is characterized by the history of shocks up to and including the current period  $s^t = (s^0, \dots, s_t)$ .<sup>5</sup> The root of  $\Sigma$  is the date-event  $\sigma^0$  with realization  $s^0$ , where  $s^0 \in \mathcal{S}$  is a fixed state of the economy. Each  $\sigma \in \Sigma$  has  $S$  immediate successors that are randomly drawn from  $\mathcal{S}$  according to a Markov process with transition matrix  $\Pi$ . Each  $\sigma \in \Sigma$  has a unique predecessor, where the unique predecessor of the date-event  $s^t$  is  $s^{t-1}$ . An (immediate) successor of

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<sup>4</sup>In these models, the demand for money is supported by cash-in-advance constraints and financial frictions are explicitly introduced through endogenous default on nominal obligations. Shubik and Yao (1990), Shubik and Tsomocos (1992) and Shubik and Tsomocos (2002) present the importance of monetary transaction costs and nominal wealth within a strategic market game framework.

<sup>5</sup>The vector  $s^t$  is equivalently interpreted as an ordered set, so that  $s \in s^t$  refers to a particular shock in the history of shocks up to period  $t$ .

a date-event  $s^t = (\bar{s}_0, \dots, \bar{s}_t)$  is  $s^{t+1} = (\bar{s}_0, \dots, \bar{s}_t, s_{t+1}) = (s^t, s_{t+1})$ , and, inductively,  $s^{t+k} = (\bar{s}_0, \dots, \bar{s}_t, s_{t+1}, \dots, s_{t+k}) = (s^t, s_{t+1}, \dots, s_{t+k})$ . Conditional on a date-event, probabilities of successors are

$$f(s^{t+1}|s^t)$$

and, inductively,

$$f(s^{t+k}|s^t) = f(s^{t+k}|s^{t+k-1})f(s^{t+k-1}|s^t).$$

At a date-event, a perishable input, labor,  $l(s^t)$ , is employed to produce a perishable output in each country, consumption,  $y(s^t)$ , according to a linear technology:

$$y(s^t) = a(s^t)l(s^t), \quad a(s^t) > 0.$$

The price level is  $p(s^t)$ , and the wage rate is

$$w(s^t) = a(s_t)p(s^t),$$

as profit maximization requires. Labour is immobile and so only domestic residents can supply labour to domestic labour markets.

A representative individual is endowed with 1 unit of leisure.

He supplies labor and demands the consumption good and derives utility according to the cardinal utility index

$$u(c(s^t), 1 - l(s^t)).$$

that satisfies standard monotonicity, curvature and boundary conditions.

**Assumption 1.** *The flow utility function,  $u : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}$ , is continuously differentiable, strictly increasing, and strictly concave. Both goods are normal:*

$$u_{11}u_2 - u_{12}u_1 < 0, \quad \text{and} \quad u_{22}u_1 - u_{12}u_2 < 0.$$

*The Inada conditions hold:*

$$\lim_{c \rightarrow 0} u_1 = \lim_{l \rightarrow 0} u_2 = \infty.$$

In particular, this guarantees that  $u_1(c, y-c)/u_2(c, y-c)$  is strictly decreasing in  $c$ .

The preferences of the individual over consumption-employment paths commencing then are described by the separable, von Neumann-Morgenstern utility function

$$u(c(s^t), 1 - l(s^t)) + \mathbf{E}_{s^t} \sum_{k>0} \beta^k u(c(s^{t+k}), 1 - l(s^{t+k})), \quad 0 < \beta < 1. \quad (1)$$

## 2.1 Monetary Structure

We follow the monetary cash-in-advance structure of ? and ?.<sup>6</sup> Our timing convention is such that transactions occur after uncertainty is realized so that there is only a transactions demand for money. We assume a unitary velocity of money.

In each country there exists a complete set of one-period state-contingent bonds (ie Arrow securities), so that markets are complete. A home bond at date event  $s^t$  maturing at state  $s_{t+1}|s^t$  is denoted  $b(s_{t+1}|s^t)$ . The price of these securities are denoted by  $q(s_{t+1}|s^t)$  and  $q^*(s_{t+1}|s^t)$  in the home and foreign country respectively. These fundamental securities can then be used to price a term structure of (untraded) bonds in each country. The nominal, risk-free rate of interest is  $r(s^t)$  and  $r^*(s^t)$  at home and abroad respectively. The price of elementary securities are

$$q(s^{t+1}|s^t) = \frac{\mu(s^{t+1}|s^t)}{1 + r(s^t)},$$

with  $\mu(\cdot|s^t)$  a “nominal pricing measure,” which guarantees the non-arbitrage relation

$$\sum_{s^{t+1}} q(s^{t+1}|s^t) = \frac{1}{1 + r(s^t)}.$$

for some  $\mu(s_{t+1}|s^t)$ ,  $s \in \mathcal{S}$ , satisfying

$$\sum_{s^{t+1}} \mu(s^{t+1}|s^t) = 1.$$

It follows that  $\mu$  is viewed as a probability measure over  $\mathcal{S}$ , and called the *nominal equivalent martingale measure*. We shall see that there are no equilibrium conditions that determine  $\mu$ , regardless of whether monetary policy sets interest rates or money supplies nor if exchange rates are managed. Note that there is a martingale measure in each country.

Inductively,

$$\mu(s^{t+k}|s^t) = \mu(s^{t+k}|s^{t+k-1})\mu(s^{t+k-1}|s^t),$$

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<sup>6</sup>These models are closely related to the open economy models of [Lucas \(1982\)](#) and [Geanakoplos and Tsomocos \(2002\)](#) and the open economy model with incomplete markets of [Peiris and Tsomocos \(2010\)](#).

and the implicit price of revenue at successor date-events is

$$q(s^{t+k}|s^t) = \frac{\mu(s^{t+k}|s^t)}{1 + r(s^{t+k-1})} q(s^{t+k-1}|s^t).$$

As the goods in each country are perfect substitutes, in equilibrium the Law of One Price must hold for goods,

$$p(s^t) = e^*(s^t)p^*(s^t) \quad (2)$$

and (redundant) assets,

$$q(s^{t+1}|s^t) = \frac{e^*(s^t)q^*(s^{t+1}|s^t)}{e^*(s^{t+1}|s^t)}. \quad (3)$$

The uncovered interest parity condition can be derived by summing across states as follows:

$$\begin{aligned} q^*(s^{t+1}|s^t)e^*(s^t) &= q(s^{t+1}|s^t)e^*(s^{t+1}|s^t) \\ e^*(s^t) \sum_s q^*(s^{t+1}|s^t) &= \sum_s q(s^{t+1}|s^t)e^*(s^{t+1}|s^t) \\ e^*(s^t) \sum_s \frac{\mu^*(s^{t+1}|s^t)}{1 + r^*(s^t)} &= \sum_{s^{t+1}|s^t} \frac{\mu(s^{t+1}|s^t)}{1 + r(s^t)} e^*(s^{t+1}|s^t) \\ e^*(s^t) \frac{1 + r(s^t)}{1 + r^*(s^t)} &= \sum_s \mu(s^{t+1}|s^t)e^*(s^{t+1}|s^t). \end{aligned} \quad (4)$$

Consider the initial date-event  $s^t$ . The home household and the foreign household begin this date-event with nominal assets  $w(s^t)$  and  $w^*(s^t)$ , respectively, where each is valued in terms of the local currency.

## 2.2 Timing of Markets

The timing proceeds as follows. First, the asset market opens, in which cash and the bonds, one from each country, are traded. Additionally, the currency market opens, in which cash denominated in one currency is traded for cash denominated in another currency.

Let  $e^*(s^t)$  be the nominal exchange rate for the foreign country (number of units of home currency for each unit of foreign currency) and  $e(s^t)$  is

the exchange rate for the home country, where  $e(s^t) = \frac{1}{e^*(s^t)}$ . Let  $r(s^t)$  and  $r^*(s^t)$  denote the nominal interest rates for the home and foreign country, respectively, implying that  $\frac{1}{1+r(s^t)}$  is the price of a nominal bond which pays 1 identically in every proceeding state in the home currency and  $\frac{1}{1+r^*(s^t)}$  is the price of such a nominal bond in the foreign currency.

Accounting for the asset and foreign exchange markets, the budget constraint for the home household in terms of the home currency (and similarly for the foreign household) is given by:

$$\hat{m}_h(s^t) + e^*(s^t)\hat{m}_f(s^t) + \sum_{s^{t+1}} b_h(s^{t+1}|s^t)q(s^{t+1}|s^t) + e^*(s^t) \sum_{s^{t+1}} b_f(s^{t+1}|s^t)q^*(s^{t+1}|s^t) \leq \tau(s^t). \quad (5)$$

The variables  $\hat{m}_h(s^t)$  and  $\hat{m}_f(s^t)$  are the amounts of the home and foreign currency held, while  $b_h(s^{t+1}|s^t)$  and  $b_f(s^{t+1}|s^t)$  are the home and foreign bond positions (net savings).  $\tau(s^t)$  is the nominal wealth agents bring into each date-event. At date 0, initial wealth constitutes a claim against a monetary-fiscal authority. Alternatively, it can be interpreted as outside money.

Cash amounts are nonnegative variables, while the bond holdings can take any values.

The market for goods opens next. Denote  $p(s^t)$  and  $p^*(s^t)$  as the commodity prices in the home and foreign country, respectively. The purchase of consumption goods at home and abroad, respectively, is subject to the cash-in-advance constraints:

$$\begin{aligned} p(s^t)c_h(s^t) &\leq \hat{m}_h(s^t), \\ p^*(s^t)c_f(s^t) &\leq \hat{m}_f(s^t). \end{aligned} \quad (6)$$

The home household also receives cash by selling its labour receiving real income of,  $a_h(s_t)l_h(s^t)$ . Hence, the amount of cash that it carries over to the next period is

$$\begin{aligned} m_h(s^t) &= p(s^t)a_h(s_t)l_h(s^t) + \hat{m}_h(s^t) - p(s^t)c_h(s^t). \\ m_f(s^t) &= \hat{m}_f(s^t) - p^*(s^t)c_f(s^t). \end{aligned} \quad (7)$$

Given (7), the cash-in-advance constraints (6) are equivalent to

$$\begin{aligned} m_h(s^t) &\geq p(s^t)a(s_t)l(s^t), \\ m_f(s^t) &\geq 0. \end{aligned} \quad (8)$$

In equilibrium, the Law of One Price must hold, meaning that  $p(s^t) = e^*(s^t)p^*(s^t)$ . Furthermore, as the goods are perfect substitutes, agents only care about the total consumption of the two goods  $c(s^t) = c_h(s^t) + c_f(s^t)$ . Using this fact and substituting for  $m_h(s^t)$  and  $m_f(s^t)$  from (7) into (5) yields the budget constraint in date-event  $s^t$  :

$$p(s^t)z(s^t) + m_h(s^t) + \sum_{s^{t+1}} b_h(s^{t+1}|s^t)q(s^{t+1}|s^t) + e^*(s^t) \sum_{s^{t+1}} b_f(s^{t+1}|s^t)q^*(s^{t+1}|s^t) \leq \tau(s^t), \quad (9)$$

where

$$z(s^t) = c(s^t) - a(s^t)l(s^t) = c(s^t) - y(s^t)$$

is the effective excess demand for consumption.

Debt limit constraints are

$$-\tau(s^t) \leq - \sum_k \sum_{s^{t+k}} q(s^{t+k}|s^t) \frac{1}{1+r(s^t)} a(s^{t+k})$$

or, equivalently

$$\lim_{k \rightarrow \infty} \sum_{s^{t+k}} q(s^{t+k}|s^t) \tau(s^{t+k}|s^t) \leq 0.$$

Wealth at successor date-events is

$$\tau(s^{t+1}|s^t) = b_h(s^{t+1}|s^t) + e^*(s^{t+1}|s^t)b_f(s^{t+1}|s^t) + m_h(s^t),$$

and the flow budget constraint reduces to

$$p(s^t)z(s^t) + \frac{r(s^t)}{1+r(s^t)}p(s^t)a(s^t)l(s^t) + \sum_{s^{t+1}} q(s^{t+1}|s^t)\tau(s^{t+1}|s^t) \leq \tau(s^t).$$

The life-time or present value budget constraint is

$$\begin{aligned} \frac{\tau(0)}{p(0)} &= \sum_{t=0}^{\infty} \sum_{s^t} \frac{q(s^t|0)p(s^t)}{p(0)} \left\{ z(s^t) + \frac{r(s^t)}{1+r(s^t)} a(s^t)l(s^t) \right\} \\ &= \sum_{t=0}^{\infty} \sum_{s^t} \frac{\beta^t u_1[c(s^t), 1-y(s^t)]f(s^t)}{u_1[c(0), 1-y(0)]} \left\{ z(s^t) + \frac{r(s^t)}{1+r(s^t)} a(s^t)l(s^t) \right\} \end{aligned} \quad (10)$$

### 2.2.1 The Monetary-Fiscal Authority

Each country contains a monetary-fiscal authority whose responsibilities include monetary (interest rate) policy and exchange rate policy.

The parameters  $W(s_0)$  and  $W^*(s_0)$  are the nominal payments owed to the household in the home and foreign country, respectively, where the debt is owed by the monetary-fiscal authority in each country. In the initial date-event  $s_0$ , the monetary-fiscal authority in the home country chooses the domestic money supply  $M(s_0)$ , the domestic debt obligations  $B_h(s_0)$ , and the foreign debt obligations  $B_f(s_0)$ . The money supplies are nonnegative, while the debt obligations can be either positive (net borrow) or negative (net save). The similar choices for the monetary-fiscal authority in the foreign country are  $M^*(s_0)$ ,  $B_h^*(s_0)$ , and  $B_f^*(s_0)$ . The constraint in  $s_0$  for the monetary-fiscal authority in the home country (and similarly for the foreign country) is given by:

$$M(s_0) + \sum_{s^1} B_h(s^1|s_0)q(s^1|s_0) + e^*(s_0) \sum_{s^1} B_f(s^1|s_0)q^*(s^1|s_0) = T(s_0). \quad (11)$$

Similarly, the constraint in date-event  $s^t$  for any  $t > 0$  is given by:

$$\begin{aligned} & M(s^t) + \sum_{s^{t+1}} B_h(s^{t+1}|s^t)q(s^{t+1}|s^t) + e^*(s^t) \sum_{s^{t+1}} B_f(s^{t+1}|s^t)q^*(s^{t+1}|s^t) \\ &= M(s^{t-1}) + B_h(s^{t-1}) + e^*(s^t) B_f(s^{t-1}). \end{aligned} \quad (12)$$

where  $T(s^{t-1}) = M(s^{t-1}) + B_h(s^{t-1}) + e^*(s^t) B_f(s^{t-1})$ .

The flow budget constraint reduces to

$$M(s^t) \frac{r(s^t)}{1+r(s^t)} + \sum_{s^{t+1}} T(s^{t+1}|s^t)q(s^{t+1}|s^t) = T(s^{t-1}). \quad (13)$$

Define the choice vectors as  $M \in \ell_+^\infty$  and  $B_h, B_f \in \ell^\infty$  for the home household (and  $M^* \in \ell_+^\infty$  and  $B_h^*, B_f^* \in \ell^\infty$  for the foreign household), where  $M = (M(s^t))_{t \geq 0, s^t}$  is the infinite sequence of money supplies for all date-events, with similar definitions for all other choice vectors.

## 2.3 Sequential Competitive Equilibria

The market clearing conditions are such that  $\tau(s_0) = T(s_0)$  and  $\tau^*(s_0) = T^*(s_0)$  hold in the initial date-event and for all date-events  $s^t$  :

$$\begin{aligned} c_h(s^t) + c_h^*(s^t) &= y_h(s_t), \\ c_f(s^t) + c_f^*(s^t) &= y_f(s_t), \\ m_h(s^t) + m_h^*(s^t) &= M(s^t), \\ m_f(s^t) + m_f^*(s^t) &= M^*(s^t) \\ b_h(s^t) + b_h^*(s^t) &= B_h(s^t) + B_h^*(s^t), \\ b_f(s^t) + b_f^*(s^t) &= B_f(s^t) + B_f^*(s^t). \end{aligned}$$

A sequential competitive equilibrium is defined as follows.

**Definition 1.** *Given initial nominal obligations  $W(s_0)$  and  $W^*(s_0)$ , a sequential competitive equilibrium consists of an allocation  $(c, c^*, l, l^*)$ , household money holdings  $(m_h, m_f, m_h^*, m_f^*)$ , household portfolios  $(b_h, b_f, b_h^*, b_f^*)$ , money supplies  $(M, M^*)$ , monetary-fiscal authority debt positions  $(B_h, B_f, B_h^*, B_f^*)$ , interest rates  $(r, r^*)$ , commodity prices  $(p, p^*)$ , and exchange rates  $(e, e^*)$  such that:*

1. *the monetary-fiscal authorities satisfy their constraints (11) and (13);*
2. *given interest rates  $(r, r^*)$ , commodity prices  $(p, p^*)$ , and exchange rates  $(e, e^*)$ , households solve the problem (1) subject to their budget constraints (10) and cash-in-advance constraints (8);*
3. *all markets clear.*

## 2.4 Equilibria with interest rate policy

We will show that with a Ricardian fiscal policy, the initial price level and nominal equivalent martingale measure in each country is indeterminate.

First, consider the real variables of this economy. Define,

- $\tilde{m}(s^t) = \frac{1}{p(s^t)}m(s^t)$  are real balances,
- $\tilde{\tau}(s^t) = \frac{1}{p(s^t)}\tau(s^t)$  is real wealth,

- $\pi(s^{t+1}|s^t) = \frac{p(s^{t+1})}{p(s^t)} - 1$  is the rate of inflation, and
- $\tilde{q}(s^{t+1}|s^t) = \frac{\mu(s^{t+1}|s^t)(1+\pi(s^{t+1}|s^t))}{1+r(s^t)}$  are prices of indexed securities.

Real wealth at successor date-events is

$$\tilde{\tau}(s^{t+1}|s^t) = \left( \frac{b_h(s^{t+1}|s^t) + e^*(s^{t+1}|s^t)b_f(s^{t+1}|s^t) + m(s^t)}{p(s^t)} \right) \frac{1}{1 + \pi(s^{t+1}|s^t)},$$

and the flow budget constraint reduces to

$$z(s^t) + \frac{r(s^t)}{1+r(s^t)}a(s^t)l(s^t) + \sum_{s^{t+1}} \tilde{q}(s_{t+1}|s^t)\tilde{\tau}(s_{t+1}|s^t) \leq \tilde{\tau}(s^t).$$

We can obtain a single life-time present-value budget constraint

$$z(s^t) + \frac{a(s^t)l(s^t)}{1+r(s^t)} + \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^t} \tilde{q}(s^{t+j}|s^t) \left( z(s^{t+j}) + \frac{r(s^{t+j})}{1+r(s^{t+j})}a(s^{t+j})l(s^{t+j}) \right) \leq 0,$$

or

$$c(s^t) + \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^t} \tilde{q}(s^{t+j}|s^t)c(s^{t+j}) \leq \frac{a(s^t)l(s^t)}{1+r(s^t)} + \sum_{j=1}^{\infty} \sum_{s^{t+j}|s^t} \tilde{q}(s^{t+j}|s^t) \frac{a(s^{t+j})l(s^{t+j})}{1+r(s^{t+j})}. \quad (14)$$

First order conditions for an optimum (for each home and foreign agent) are

$$\frac{\partial u(c(s^t), 1-l(s^t))}{\partial c(s^t)} = \frac{\partial u(c(s^t), 1-l(s^t))}{\partial l(s^t)} \left( \frac{a(s^t)}{1+r(s^t)} \right)^{-1}, \quad (15)$$

$$\beta f(s^{t+1}|s^t) \frac{\partial u(c(s^{t+1}), 1-l(s^{t+1}))}{\partial c(s^{t+1})} \tilde{q}(s^{t+1}|s^t)^{-1} = \frac{\partial u(c(s^t), 1-l(s^t))}{\partial c(s^t)}, \quad (16)$$

and the transversality condition is

$$\lim_{k \rightarrow \infty} \sum_{s^{t+k}} q(s^{t+k}|s^t) \tau(s^{t+k}|s^t) = 0.$$

The monetary-fiscal authority in each country sets rates of interest and accommodates the demand for balances.

**Proposition 1.** *Given initial real wealth,  $\tilde{\tau}(s_0) = \tilde{T}(s_0)$  and interest rate policy,  $\{r(s^t)\}$ , all prices,  $p(s^t)$  in each country and exchange rates  $e(s^t)$  are indeterminate;*

**Proof**  
**Part 1**

Equilibrium requires that the excess demand for output vanishes:

$$z(s^t) + z^*(s^t) = c(s^t) - a(s^t)l(s^t) + c^*(s^t) - a^*(s^t)l^*(s^t) = 0,$$

which, together with the real (normalized) budget constraints of the agents, the consumption-labour first order conditions

$$\frac{\partial u(c(s^t), 1 - l(s^t))}{\partial c(s^t)} = \frac{\partial u(c(s^t), 1 - l(s^t))}{\partial l(s^t)} \left( \frac{a(s^t)}{1 + r(s^t)} \right)^{-1}$$

and the transversality condition determines the path of employment and consumption. In turn, this determines the prices of indexed elementary securities:

$$\beta f(s^{t+1}|s^t) \frac{\partial u(c(s^{t+1}), 1 - l(s^{t+1}))}{\partial c(s^{t+1})} \tilde{q}(s^{t+1}|s^t)^{-1} = \frac{\partial u(c(s^t), 1 - l(s^t))}{\partial c(s^t)}.$$

As we have solved the real side of the economy, and have made no claims on the nature of fiscal policy (we have assumed it is Ricardian), it can be shown that the initial price level remains indeterminate as well. More importantly, the decomposition of equilibrium asset prices into an inflation process,  $\pi(\cdot|s^t)$ , and a nominal pricing measure,  $\mu(\cdot|s^t)$ , remains indeterminate:

$$\tilde{q}(s^{t+1}|s^t) = \frac{\mu(s^{t+1}|s^t)(1 + \pi(s^{t+1}|s^t))}{1 + r(s^t)}.$$

To see this, assume each of the representative households commence, at date 0, with real wealth  $\tilde{\tau}(0)$  and  $\tilde{\tau}^*(0)$ . Choosing an arbitrary  $p(0)$  and  $p^*(0)$ , we find the nominal budget constraints at date 0. From the cash-in-advance specification, we obtain the aggregate money supply  $M(0) = p(0)y(0)$ . The monetary-fiscal authority will accommodate any demand for assets at the given interest rate, so the difference between the money supply and the initial liabilities of each monetary authority will give the nominal value of its portfolio. Now, choose an arbitrary martingale measure in each country. Using the real price of the bonds from [16](#), we can then solve for the stochastic rates of inflation. As there is no restriction on the portfolio of assets that each monetary-fiscal authority purchases, then the market clearing in the nominal state-contingent bond market follows trivially. As we have chosen an

arbitrary initial price and have found arbitrary stochastic rates of inflation, given a martingale measure, from the law of one price the nominal exchange rate is also indeterminate.

**Remark**

A non-Ricardian fiscal policy which sets initial nominal wealth, rather than real wealth, determines only the initial prices and exchange rates irrespective of whether there is an explicit exchange rate target. To see this, consider the present-value budget constraint of each monetary-fiscal authority:

$$\begin{aligned}
 W(0) &= \sum_{t=0}^{\infty} \sum_{s^t} q(s^t|0) \frac{r(s^t)}{1+r(s^t)} M(s^t) \\
 \frac{W(0)}{p(0)} &= \sum_{t=0}^{\infty} \sum_{s^t} \frac{q(s^t|0)p(s^t)}{p(0)} \left\{ \frac{r(s^t)}{1+r(s^t)} a(s^t)l(s^t) \right\} \\
 &= \sum_{s^t} \frac{\beta^t u_1[c(s^t), 1-y(s^t)]f(s^t)}{u_1[c(0), 1-y(0)]} \left\{ \frac{r(s^t)}{1+r(s^t)} a(s^t)l(s^t) \right\} \quad (17)
 \end{aligned}$$

The foreign bond holdings and hence exchange rates do not enter as the only revenue is the seigniorage revenue. There are now two additional variables to be determined, namely the initial price levels in each country. Hence, with nominal initial wealth given to agents, the two present-value budget constraints of the monetary-fiscal authorities provide the necessary equations to determine them in addition to the allocation.

## 2.5 A stationary economy

We now consider an economy where the resolution of uncertainty follows a stationary stochastic process. We show that consigning ourselves to such economies does not remove the indeterminacy. That is, there exists a continuum of stationary markov equilibria.

**Proposition 2.** *Given initial real wealth,  $\tilde{\tau}(s_0) = \tilde{T}(s_0)$  and interest rate policy,  $\{r(s^t)\}$ , equilibria with stationary allocations, prices,  $p(s^t)$  and exchange rates  $e(s^t)$  are indeterminate;*

**Proof** Suppose that shocks follow a Markov chain with transition probabilities. That is, conditional on an elementary state of the world, transition probabilities are  $f(s'|s)$ .  $F$  is the  $S \times S$  matrix of all transition probabilities. As markets are complete, we can define  $\tilde{p}(s) = \frac{\partial u(c(s^t), 1-l(s^t))}{\partial c(s^t)}$  as the real price of goods at a state.  $I_s$  is the  $S \times S$  identity matrix. Furthermore, we will use  $\circ$  to denote the element-by-element multiplication of vectors. For two  $S$  dimensional vectors  $x$  and  $y$ ,  $x \circ y = (x_1y_1, x_2y_2, \dots, x_Sy_S)'$ . Let the present value of consumption for the home agent in state  $s$  be

$$V(s) = \tilde{p}(s)c(s) + \sum_{s'} \beta f(s'|s)V(s').$$

In matrix terms this is  $V = \tilde{p} \circ c = \beta FV$  and has unique solution

$$V = [I_s - \beta F]^{-1}(\tilde{p} \circ c). \quad (18)$$

Similarly the present value of income can be denoted by

$$W(s) = \tilde{p}(s)a(s)l(s)R(s)^{-1} + \sum_{s'} \beta f(s'|s)W(s')$$

where  $R(s) = 1 + r(s)$  are the state-contingent interest rates. The solution to  $W$  in matrix terms is

$$W = [I_s - \beta F]^{-1}(\tilde{p} \circ a \circ l \circ R), \quad (19)$$

where  $R = [R(1), \dots, R(S)]'$

If the economy starts in the state  $s_0$  at period  $t = 0$ , then the present value budget constraint requires that  $V_{s_0} = W_{s_0}$  for each of the representative households, though due to Walras' Law only one is required. That is, we require

$$[I_s - \beta F]^{-1}(\tilde{p} \circ (c - a \circ l \circ R))_{s_0} = 0. \quad (20)$$

Market clearing requires that

$$\begin{aligned} c(s) + c^*(s) &= y(s) + y^*(s) \\ &= a(s)l(s) + a^*(s)l^*(s) \end{aligned} \quad (21)$$

Finally the labour supply decisions are given, in matrix form, by

$$\tilde{p} \circ a \circ R = Du_2, \quad (22)$$

and

$$\tilde{p} \circ a^* \circ R^* = Du_2^*, \quad (23)$$

where  $Du_2 = \left[ \frac{\partial u(c(1), 1-l(1))}{\partial l(1)}, \dots, \frac{\partial u(c(S), 1-l(S))}{\partial l(S)} \right]$ .

For  $H = 2$  representative households, we have  $2HS + S$  unknowns,  $HS$  consumption and labour supplies and  $S$  real prices. Any solution to (20, 21, 22, 23) is an equilibrium state-contingent consumption and labour for each agent in state  $s$ .

The path of consumption,  $c(s)$ , and employment,  $l(s)$ , at equilibrium, in turn, determine the prices of indexed elementary securities:

$$\beta f(s'|s) \frac{\partial u(c(s'), 1-l(s'))}{\partial c(s')} \tilde{q}(s'|s)^{-1} = \frac{\partial u(c(s), 1-l(s))}{\partial c(s)}$$

or

$$\tilde{Q} = \beta \bar{D} u^{-1} F \bar{D} u.$$

Note that the real Arrow price is independent of the country. The nominal Arrow prices, and hence martingale measures, across countries differ in their stochastic rates of inflation (and consequently the no-arbitrage condition).

Here,

$$\bar{D} u = \text{diag}(\dots, \frac{\partial u(c(s), 1-l(s))}{\partial c(s)}, \dots)$$

is the diagonal matrix of marginal utilities of consumption, and

$$F = (f(s'|s)) \quad \text{and} \quad \tilde{Q} = (\tilde{q}(s'|s))$$

are, respectively, the matrices of transition probabilities and of prices of indexed elementary securities.

For the home household,

$$\tilde{m} = (\dots \frac{r(s)}{1+r(s)} a(s) l(s) \dots)$$

is the vector of real balances at equilibrium,

$$\tilde{z} = (\dots z(s) \dots)$$

is the vector of excess demands and the real wealth at the steady state is given by

$$\tilde{\tau} = (\dots \tau(s), \dots).$$

$\tilde{\tau}$  is determined by the equations

$$\tilde{z} + \tilde{m} + \tilde{Q}\tilde{\tau} = \tilde{\tau} \quad \text{or} \quad \tilde{\tau} = (I - \tilde{Q})^{-1} [\tilde{z} + \tilde{m}].$$

$$\tilde{z}^* + \tilde{m}^* + \tilde{Q}\tilde{\tau}^* = \tilde{\tau}^* \quad \text{or} \quad \tilde{\tau}^* = (I - \tilde{Q})^{-1} [\tilde{z}^* + \tilde{m}^*].$$

As we have solved the entire real economy without nominal variables, the initial price level in each country remains indeterminate. More importantly, the decomposition of equilibrium asset prices into an inflation process,  $\pi(\cdot|s)$ , and a nominal pricing measure,  $\mu(\cdot|s)$ , remain indeterminate in each country. For the home country:

$$\tilde{Q} = \bar{R}^{-1}\bar{M} \otimes \Pi.$$

Here,

$$\bar{R} = \text{diag}(\dots, (1 + r(s)), \dots)$$

is the diagonal matrix of interest factors, and

$$\bar{M} = (\mu(s'|s)) \quad \text{and} \quad \Pi = ((1 + \pi(s'|s)))$$

are, respectively, the matrices of “nominal pricing transition probabilities” and inflation factors.

Suppose the inflation process, which is endogenous, is restricted to take the form

$$(1 + \pi(s'|s)) = \hat{a}(s)\hat{h}(s)\hat{b}(s'),$$

where  $\hat{a}(s)$  and  $\hat{b}(s)$  are known function of the fundamentals of the economy or of the economy, and, as a consequence,

$$\bar{M} \otimes \Pi = AH\bar{M}B.$$

Here,

$$a = (\dots, \hat{a}(s), \dots), \quad B = (\dots, \hat{b}(s) \dots), \quad \text{and} \quad h = (\dots, \hat{h}(s), \dots),$$

and  $A, B$  and  $H$  are the associated diagonal matrices.

Then,

$$\tilde{Q} = R^{-1}\bar{M} \otimes \Pi \quad \Leftrightarrow \quad A^{-1}R\tilde{Q}B^{-1} = H\bar{M},$$

which determines the inflation process as well as nominal pricing probabilities, since

$$\bar{M}\mathbf{1}_S = \mathbf{1}_S \quad \Leftrightarrow \quad H = A^{-1}R\tilde{Q}B^{-1}\mathbf{1}_S,$$

$$\bar{M}^* \mathbf{1}_S = \mathbf{1}_S \quad \Leftrightarrow \quad H^* = (A^*)^{-1} R^* \tilde{Q} (B^*)^{-1} \mathbf{1}_S.$$

This is indeed the case under conventional monetary policy.

Let  $\tilde{T}$  be the real wealth at successor date-events of the home monetary-fiscal authority

$$\tilde{T}(s') = \left( \frac{B(s'|s) + M(s)}{p(s)} \right) \frac{1}{1 + \pi(s'|s)},$$

and conventional monetary policy requires that

$$B(s'|s) = B(s).$$

The argument fails if, alternatively,

$$(1 + \pi(s'|s)) = \tilde{h}(s') \tilde{a}(s) \tilde{b}(s').$$

In this case,

$$\tilde{Q} = R^{-1} \bar{M} \otimes P \quad \Leftrightarrow \quad A^{-1} R \tilde{Q} B^{-1} H^{-1} = \bar{M},$$

and

$$\bar{M} \mathbf{1}_S = \mathbf{1}_S \quad \Leftrightarrow \quad H^{-1} \mathbf{1}_S = B Q^{-1} R^{-1} A \mathbf{1}_S$$

that need not be positive.

Non-ricardian monetary-fiscal policy determines the initial price level by setting exogenously the level of initial nominal claims.

□

The indeterminacy of  $\mu$  implies that the inflation rate, is indeterminate. Thus, interest rate policy alone does not determine the stochastic path of inflation. The reason that  $\mu$  is indeterminate is simple, and closely related to the well known fact that only relative prices are determined in equilibrium. In an open economy, the indeterminacy proliferates. Even with perfect substitutes, as we have here, only the relative prices across countries are determined. As the stochastic path of prices in each country is indeterminate, then so is the path of exchange rates. Furthermore, fixing the path of exchange rates fixes only the ratio of prices in countries but not price levels globally.

### 3 Risk Premia in a Monetary Open Economy

Typically in cash-in-advance economies with complete markets the optimal rate of interest is zero and agents are indifferent between holding bonds and money. A positive interest rate on the other hand, causes money holdings to incur the cost of foregone interest and are economised by agents. If the demand for money is purely for transactions (which obtains if markets open after uncertainty is realised), then positive interest rates reduces aggregate demand and, when supply is endogeneous, consequently aggregate supply. Furthermore, independent of the real economy, an environment with stochastic rates of interests will then generate aggregate risk premia which are both nominal and real. More precisely, a correlation is generated between the nominal martingale measure and nominal interest rates which results in risk-neutral pricing being systematically biased (from subjective pricing alone). [Espinoza et al. \(2009\)](#) characterise this risk premia in a closed economy and consider the implication for the term structure of interest rates. In an open economy, the terms of trade effects means that aggregate demand is determined by the choice of interest rates in all trade partners: it is the correlation between interest rates across countries and the martingale measure in an open economy that determines the direction of the bias in asset pricing.

We study determinate equilibria with non-Ricardian fiscal policy, which determines the initial price level, and also portfolio restrictions on the monetary-fiscal authority which also fix the distribution of prices across states. The restrictions on the monetary-fiscal authority determine the path of prices within each country. We consider three alternative objectives. The first is choosing a stable growth rate in inflation: we call this price stability. In a world of stochastic outputs, the portfolio choice alters the money supplies inversely with the output to maintain the same price across states of nature. Nominal GDP Targeting results in money supplies to grow in a non-stochastic manner, and is consistent with the Friedman  $k\%$  rule. Finally we consider Traditional Monetary Policy which is the result of the monetary-fiscal authority holding a portfolio composed of a nominally riskless bond. Its implications are a combination of that under price and Nominal GDP Targeting and allows a positive role for interest rates to target the price level in order to maintain a stable growth rate in prices. All proofs in this section are in the Appendix. The section proceeds as follows...

### 3.1 Primitives

There are two periods. In the second period uncertainty is resolved. We fix a complete probability space  $(\Omega, \mathcal{F}, P)$  for period 1. Here,  $\Omega$  is a complete description of the exogenous uncertain environment at Period 1, the  $\sigma$ -algebra  $\mathcal{F}$  is the collection of events distinguishable at period 1, and  $P$  is a probability measure over  $(\Omega, \mathcal{F})$ .

There are three periods:  $t = \{0, 1, 2\}$ . There is no uncertainty in the first or third period. In the second period, a single state  $\omega \in \Omega$  realizes<sup>7</sup>. In what follows we assume that  $\omega$  lies on the real segment between 0 and 1 ( $[0; 1]$ ). Furthermore the uniform probability density  $f \sim U[0; 1]$  is defined on  $\Omega$ .

Production and consumption occur in the first two periods. The last period is added for an accounting purpose, where households and the fiscal authority redeem their debt.

In general, for some variable  $x$ ,  $x(0)$  denotes the value at date 0,  $x(1, \omega)$  at date 1, state  $\omega$  and  $x(2, \omega)$  the value at date 2 of the date-event immediately proceeding  $(1, \omega)$ .

### 3.2 Households

The world is inhabited by a continuum of individual producer-consumers in each of 2 countries (home and foreign), each of unit mass, and producing a single homogeneous good. Agents are identically endowed with  $\bar{y}$  of labour at each date-event and we assume that agents supply an amount of labour  $y$  which produces  $y$  consumption goods. Otherwise, we use the same notational convention as the previous section.

Individuals everywhere in the world have the same preferences, which are defined over consumption and effort expended in production. The preferences of the home agent is

$$\frac{c(0)^{1-\rho} - 1}{1-\rho} + \frac{(\bar{y} - y(0))^{1-\rho} - 1}{1-\rho} + \beta \int_{\omega} f(\omega) \left\{ \frac{c(1, \omega)^{1-\rho} - 1}{1-\rho} + \frac{(\bar{y} - y(1, \omega))^{1-\rho} - 1}{1-\rho} \right\} d\omega. \quad (24)$$

---

<sup>7</sup>In the following, all the uncertainty will be due to the path of interest rates in one country.

Note that the endowment of leisure is state and agent independent while the probability measure and rate of discount factor is also agent independent for simplicity.

As before, there are no impediments or costs to trade between the countries. The budget constraints for the representative home household is

$$\begin{aligned}
p(0)c_h(0) + \int_{\omega} q(1, \omega)b_h(1, \omega)d\omega \\
+ e^*(0) \left[ p^*(0)c_f(0) + \int_{\omega} q^*(1, \omega)b_f(1, \omega)d\omega \right] + m_h(0) \\
\leq w_h(0) + p(0)y(0).
\end{aligned} \tag{25}$$

$$\begin{aligned}
p(1, \omega)c_h(1, \omega) - b_h(1, \omega) + \frac{b_h(2, \omega)}{1 + r(1, \omega)} \\
+ e^*(1, \omega) \left[ p^*(1, \omega)c_f(1, \omega) - b_f(1, \omega) + \frac{b_f(2, \omega)}{1 + r^*(1, \omega)} \right] + m_h(1, \omega) \\
\leq m_h(0) + p(1, \omega)y(1, \omega).
\end{aligned} \tag{26}$$

and in the final period

$$0 \leq m_h(1, \omega) + b_f(2, \omega) + e^*(2, \omega)b_f(2, \omega).$$

### 3.3 Individual Maximization

The first order conditions for the representative households in period 1 gives us:

$$y(1, \omega) = \bar{y} - c(1, \omega)(1 + r(1, \omega))^{1/\rho}, \tag{27}$$

and

$$y^*(1, \omega) = \bar{y} - c^*(1, \omega)(1 + r^*(1, \omega))^{1/\rho}. \tag{28}$$

The marginal rates of substitution

$$q(1, \omega) = \beta f(\omega) \frac{p(0)}{p(1, \omega)} \left\{ \frac{c(0)}{c(1, \omega)} \right\}^{\rho}. \tag{29}$$

Equating 29 for the home and foreign agent gives

$$c^*(1, \omega) = \frac{c^*(0)}{c(0)} c(1, \omega). \quad (30)$$

Market clearing condition is

$$c(1, \omega) + c^*(1, \omega) = y(1, \omega) + y^*(1, \omega). \quad (31)$$

### 3.4 Monetary-Fiscal Authority

In the first part of the paper we showed that in the absence of a restriction on the composition of the portfolio of the monetary-fiscal authority, indeterminacy proliferates. Conversely, placing restrictions on the relative quantities of state-contingent bonds traded will obtain a determinate stochastic path of prices. We characterise the the monetary-fiscal authority portfolio restriction for Country 1 as:

$$B(1, \omega) = B(1)\pi(\omega) \quad (32)$$

where  $\int_{\omega} \pi(\omega) d\omega = 1$ . Hence

$$M(0) = B(1) \int_{\omega} q(1, \omega) \pi(\omega) d\omega + W(0) \quad (33)$$

$$B(1) = \frac{M(0) - W(0)}{\int_{\omega} q(1, \omega) \pi(\omega) d\omega}$$

$$B(1)\pi(\omega) = M(0) - \frac{r(1, \omega)}{1 + r(1, \omega)} M(1, \omega) \quad (34)$$

where  $B(1)$  is the gross value of debt purchased by the monetary-fiscal authority of country 1. These restrictions correspond to a particular stochastic distribution of prices. We consider three possible targets which can be obtained by the portfolio restriction in the next section.

#### 3.4.1 Monetary Policy Options

We now define the various monetary policy regimes available. As we are in a stochastic world, the monetary-fiscal authority is required to choose a path

of interest rates and a choice of its portfolio to target a stable growth rate in prices or money supplies. In addition it can choose a portfolio of state contingent bonds in equal proportion producing the payoff of a nominally riskless bond. We now define these policy targets formally.

**Definition 2.** *Nominal GDP Targeting is the outcome of monetary policy that sets interest rates and money supplies in the second period which are state independent. Formally,  $r(0), r(\omega) \geq 0$  and a choice of  $\pi(\omega)$  such that  $\int_{\omega} \pi(\omega) d\omega = 1$  and  $M(1, \omega) = M(1, \omega') \forall \omega, \omega' \in \Omega$ .*

**Definition 3.** *Price stability is the outcome of monetary policy that sets interest rates and prices in the second period which are state independent. Formally,  $r(0), r(\omega) \geq 0$  and a choice of  $\pi(\omega)$  such that  $\int_{\omega} \pi(\omega) d\omega = 1$  and  $p(1, \omega) = p(1, \omega') \forall \omega, \omega' \in \Omega$ .*

**Definition 4.** *Traditional Monetary Policy occurs when the Central Bank purchases equal quantities of state-contingent bonds. Formally,  $r(0), r(\omega) \geq 0$  and  $B(1, \omega) = B(1, \omega') \forall \omega \in \Omega$ , and where  $\bar{B}$  is the state-independent value of debt.*

### 3.5 The Aggregate Risk Premium

The state space is continuous, and will be indexed by the interest rates of country 1, between two bounds. Formally  $\Omega = [\omega_0, \dots, \omega]$ , where interest rates in country 1 are  $r(1, \omega_0) = \underline{r}$  and  $r(1, \omega) = \bar{r}$ , while for country 2,  $r^*(1, \omega_i) = r^*(1, \omega_0) \forall i \in [0, 1]$ . That is, the only uncertainty is the date 1 interest rate in country 1.

The nominal stochastic discount factor, or the nominal state-contingent bond price here, is given by

$$q(1, \omega) = \beta f(\omega) \frac{p(0)}{p(1, \omega)} \left\{ \frac{c(0)}{c(1, \omega)} \right\}^{\rho}.$$

The following two lemmas decompose this to nominal and real terms and determine the correlation with expected nominal interest rates. We show that nominal interest rates affect both the real stochastic discount factor as well as the expected rates of inflation reflecting both real and nominal risk

premia. The particular policy target then determines the overall correlation of the nominal stochastic discount factor with nominal interest rates.

### **Real Risk Premium**

The real risk premium caused by monetary policy is determined by change in  $\frac{u(c(1,\omega))}{u(c(0))}$ . In the following we characterise how the direction of the risk premia in response to higher interest rates.

**Lemma 1.** *The real stochastic discount factor is positively correlated with expected interest rates.*

This shows that the path of interest rates in one country effects the allocation globally: the non-neutrality of monetary policy results in the global real risk premium being determined by the combination of interest rates globally.

### **Inflation Risk**

The (stochastic) rate of inflation depends on the choice of nominal targets. Clearly a policy of price stability denies the presence of a nominal risk premium. However a policy of Monetary or Traditional Monetary Policy has clear implications for the nominal risk premium.

**Lemma 2.** *A global policy of*

- 1. Nominal GDP Targeting results in expected interest rates being positively related to the expected price levels.*
- 2. Traditional Monetary Policy results in expected interest rates being negatively related to expected money supplies and price levels.*

### **Nominal Stochastic Discount Factor and Risk Premia**

We now combine the previous two lemmas to obtain the overall direction of the correlation between expected nominal interest rates and the Nominal Stochastic Discount Factor (NSDF).

**Lemma 3.** *A global policy of*

- 1. Price Stability results in a positive correlation between expected nominal interest rates and the NSDF.*
- 2. Nominal GDP Targeting results in a negative correlation between expected nominal interest rates and the NSDF.*

3. *Traditional Monetary Policy results in a positive correlation between expected nominal interest rates and the NSDF.*

We can now examine the bias in the term structure of interest rates using the previous lemmas.

### **The Term Structure of Interest Rates**

Here we examine the implications of the choice of monetary policy on the risk premium in the interest rate market. Under a policy of price stability we find the same result as in [Espinoza et al. \(2009\)](#) and [Espinoza and Tsomocos \(2008\)](#) the forward interest rate is an upwardly biased indicator of future interest rates.. However, under Nominal GDP Targeting we get the opposite result reflecting the importance of considering the choice of monetary policy in determining the informational content in observed risk premia in the market.

**Proposition 3.** *Given a distribution of future interest rates,*

1. *Price stability and Traditional Monetary Policy result in the forward interest rate being an upwardly biased indicator of expected interest rates.*
2. *Nominal GDP Targeting result in the forward interest rate being an downwardly biased indicator of expected interest rates.*

### **The Stochastic Path of Exchange Rates**

Here we characterise the path of exchange rates under alternative monetary policy regimes globally. The exchange rate is given by

$$e(1, \omega) = \frac{p(1, \omega)}{p^*(1, \omega)} \tag{35}$$

$$= \frac{M(1, \omega)}{M^*(1, \omega)} \frac{y^*(1, \omega)}{y(1, \omega)}. \tag{36}$$

To determine the correlation between the nominal exchange rate and the NSDF we need to determine the effect that expected interest rates have on both relative money supplies and relative outputs across countries. Under different policy objectives either or both money supply and output may change, hence we need to consider them individually and then conclude the overall direction of the correlation. We first consider the effect of expected interest rates on output, independent of monetary policy objectives, in the following lemma.

**Lemma 4.** *At each date-event, output in the country with the relatively higher interest rate will be relatively lower.*

Money supplies on the other hand will depend on the monetary policy choice. We now consider the overall effect on the exchange rate of interest rates. Note that under Price stability, the exchange rate is unchanged across states by definition.

**Proposition 4.** *Under a global policy of*

- 1. Traditional Monetary Policy, the exchange rate in the country with the relatively higher interest rate across states, will be relatively more appreciated across countries.*
- 2. Nominal GDP Targeting, the exchange rate in the country with the relatively higher interest rate across states, will be relatively more depreciated across countries.*

We can now use the results of the previous lemmas on the correlation between the NSDF and interest rates and the correlation between interest rates and exchange rates to determine the direction of the bias in Uncovered Interest Parity, or the Forward Exchange Rate premium.

### **Monetary Policy and Forward Exchange Rate Premium**

**Proposition 5.** *Monetary Policy results in the Forward Exchange Rate being*

- 1. downwardly biased under Traditional Monetary Policy,*
- 2. unbiased under Price Stability,*
- 3. downwardly biased under Nominal GDP Targeting.*

If home interest rates are higher and more volatile, then it may seem economically profitable for foreign investors to take advantage of this difference.

### 3.6 Numerical Analysis

The lifetime budget constraint for the home household in domestic currency is

$$p(0) \left[ c(0) - \frac{y(0)}{1+r(0)} \right] + \int_{\omega} q(1, \omega) p(1, \omega) \left[ c(1, \omega) - \frac{y(1, \omega)}{1+r(1, \omega)} \right] d\omega \leq w(0). \quad (37)$$

Recall that the state price gives us  $q(1, \omega) = \beta f(\omega) \frac{p(0)}{p(1, \omega)} \left\{ \frac{c(0)}{c(1, \omega)} \right\}^{\rho}$ . Substituting this in and rearranging gives

$$c(0)^{-\rho} \left[ c(0) - \frac{y(0)}{1+r(0)} \right] + \int_{\omega} f(\omega) c(1, \omega)^{-\rho} \left[ c(1, \omega) - \frac{y(1, \omega)}{1+r(1, \omega)} \right] d\omega \leq w(0) \frac{c(0)^{-\rho}}{p(0)}. \quad (38)$$

From the first order conditions and market clearing, we get<sup>8</sup>  $c(1, \omega)$  and  $c^*(1, \omega)$  as functions of constants, endogenous variables  $\{c(0), c^*(0)\}$  and state variables  $\{r(1, \omega), r^*(1, \omega)\}$ . Using this, and the first order equations 27 and 28 we get expressions for  $y(1, \omega)$  and  $y^*(1, \omega)$  also as functions of constants, endogenous variables  $\{c(0), c^*(0)\}$  and state variables  $\{r(1, \omega), r^*(1, \omega)\}$ .

What remains is to determine the initial price level in each country. This can be obtained using the present value budget constraint for each country by combining equations 33 and 34.

$$M(0) \frac{r(0)}{1+r(0)} + \int_{\omega} q(1, \omega) \frac{r(1, \omega)}{1+r(1, \omega)} M(1, \omega) = W(0).$$

Using the definition of the state price, and the cash-in-advance constraint:  $p(1, \omega)y(1, \omega) = M(1, \omega)$ ,

$$M(0) \frac{r(0)}{1+r(0)} + \int_{\omega} p(0)y(1, \omega) \left\{ \frac{c(0)}{c(1, \omega)} \right\}^{\rho} \frac{r(1, \omega)}{1+r(1, \omega)} = W(0).$$

Finally using the cash-in-advance constraint:  $p(0)y(0) = M(0)$ , and rearranging

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<sup>8</sup>see proof for proposition ?? in Appendix.

$$p(0) = \frac{W(0)}{y(0) \frac{r(0)}{1+r(0)} + \int_{\omega} y(1, \omega) \left\{ \frac{c(0)}{c(1, \omega)} \right\}^{\rho} \frac{r(1, \omega)}{1+r(1, \omega)}} \quad (39)$$

which, using the arguments used earlier is a function of constants, endogenous variables  $\{c(0), c^*(0)\}$  and state variables  $\{r(1, \omega), r^*(1, \omega)\}$ .

Substituting equation 39 into the two budget constraints represented by equation 38, we have a system of two equations that solve  $\{c(0), c^*(0)\}$  as a function of state variables  $\{r(1, \omega), r^*(1, \omega)\}$ .

### 3.6.1 Simulation

The parameters of the initial allocation are given as follows.

	<i>Country1</i>	<i>Country2</i>
<i>Initial Wealth, <math>w(h, 1, i)</math></i>	1.0	1.0
<i>Endowment of Leisure <math>\bar{l}</math></i>	1.0	1.0
<i>Risk Aversion, <math>\rho(h)</math></i>	0.9	0.9
<i>Preference for Leisure, <math>\kappa(i)</math></i>	1.0	1.0
<i>Period 1 Interest Rate 0, <math>r(1, i)</math></i>	3.0%	3.0%
<i>Discount Factor, <math>\beta</math></i>	1.0	1.0

Table 1: Parameters of Initial Equilibrium

In the second period the interest rates in the two countries follow a bivariate log-normal distribution. The mean of the interest rates in each country are given by  $e^{\mu+\sigma^2/2}$ , the variance is given by  $(e^{\sigma^2} - 1)(e^{2\mu+\sigma^2})$  and the correlation by  $\rho$  where  $\mu$ ,  $\sigma$  and  $\rho$  are the parameters from bivariate normal distribution. We fix the Country 2 parameters to be  $\mu_2 = -4.5$  and  $\sigma_2 = 1.5$  which translate into a mean of 0.0514 and a standard deviation of 0.2319 in the log normal distribution. For country 1 we assume the same mean but solve the economy for 100 values of  $\sigma$  between 1.25 and 1.75 and 100 values of  $\rho$  between .059 and .095. The plot of these are presented below.

**Bias in Expected Exchange Rates, Correlation of Interest Rates and Standard Deviation of Interest Rates**

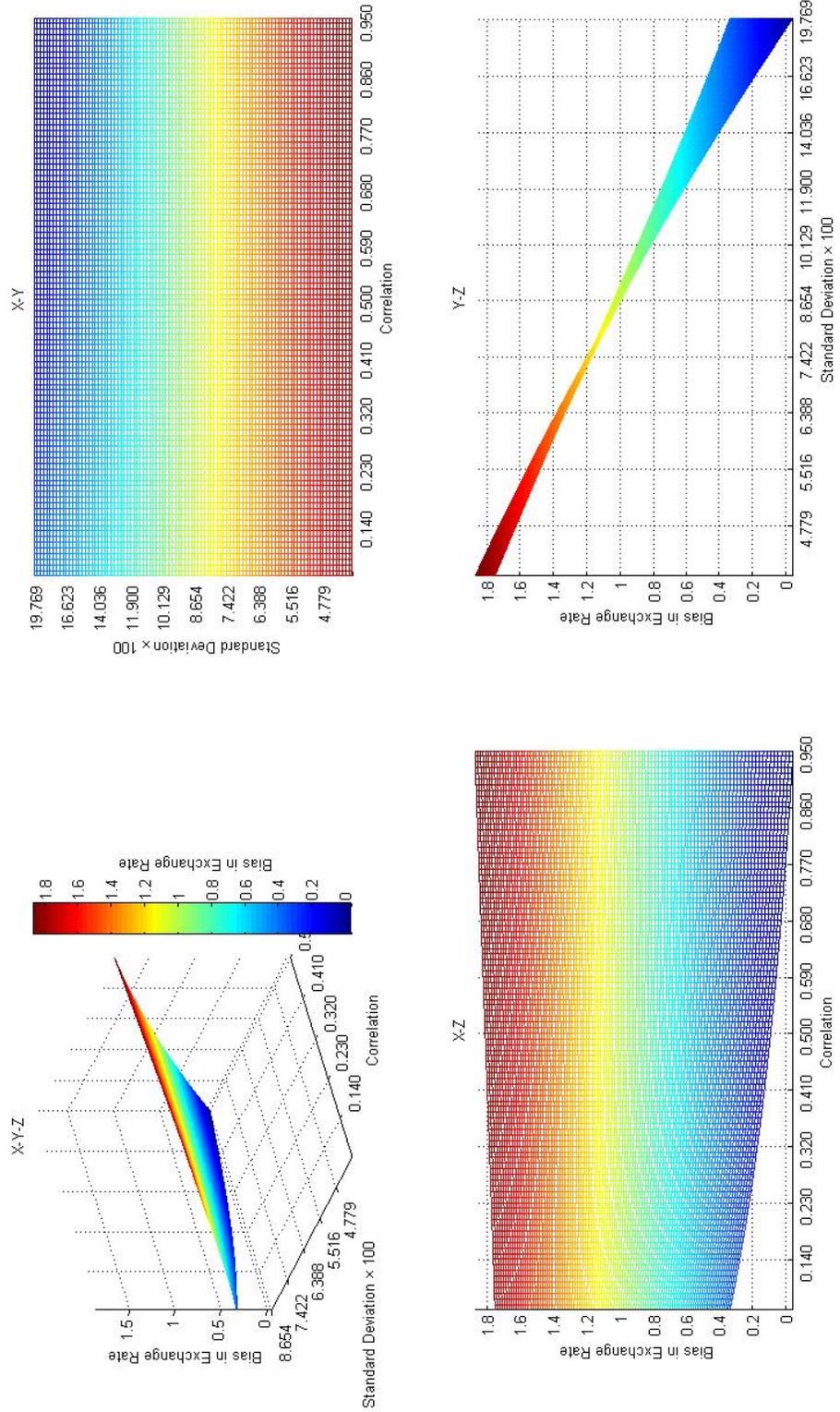


Figure 1: Log Difference between Objective and Risk Neutral Expected Exchange Rate

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## 4 Appendix

### Proof of Lemma 1

*Proof.* **Consumption** Substituting 27 and 28 into market clearing equation 31

$$c^*(1, \omega) = \frac{2\bar{y} - c(1, \omega)(1 + (1 + r(1, \omega))^{1/\rho})}{(1 + (1 + r^*(1, \omega))^{1/\rho})}. \quad (40)$$

Finally substitute 30 into 40

$$c(1, \omega) = \frac{2\bar{y}}{\frac{c^*(0)}{c(0)}(1 + (1 + r^*(1, \omega))^{1/\rho}) + (1 + (1 + r(1, \omega))^{1/\rho})}. \quad (41)$$

Now two states  $\omega$  and  $\omega'$  where monetary policy sets interest rates such that  $r(1, \omega) > r(1, \omega')$  but  $r^*(1, \omega) = r^*(1, \omega')$  we get

$$c(1, \omega) < c(1, \omega').$$

From 30

$$c^*(1, \omega) < c^*(1, \omega').$$

**Production** From 27 and 28

$$y(1, \omega) < y(1, \omega')$$

and

$$y^*(1, \omega) < y^*(1, \omega').$$

□

### Proof of Lemma 2

#### Part (i)

*Proof.* By construction,  $M(1, \omega) = M(1, \omega')$ . In section ?? we showed that  $y(1, \omega) < y(1, \omega')$  whenever  $r(1, \omega) > r(1, \omega')$ . As the cash-in-advance holds, then it must be that  $p(1, \omega) > p(1, \omega')$ . □

#### Part (ii)

*Proof.* The period 0 budget constraint

$$\frac{r(0)}{1+r(0)}M(0) + \bar{B} \sum_{\omega} q(1, \omega) = W(0) \quad (42)$$

$$\frac{r(0)}{1+r(0)}M(0) + \bar{B} \frac{1}{1+r(0)} = W(0) \quad (43)$$

The period 1 money supply is then

$$\frac{r(1, \omega)}{1+r(1, \omega)}M(1, \omega) + \bar{B} = M(0) \quad (44)$$

hence

$$M(1, \omega) = \frac{1+r(1, \omega)}{r(1, \omega)} [M(0) - \bar{B}]. \quad (45)$$

This gives us that taking two states  $\omega$  and  $\omega'$  where monetary policy sets interest rates such that  $r(1, \omega) > r(1, \omega')$ ,  $M(1, \omega) < M(1, \omega')$ .

Take states  $\omega, \omega' \in S$  such that  $r(1, \omega) > r(1, \omega')$  and  $r^*(1, \omega) = r^*(1, \omega')$ . From the cash-in-advance constraint

$$\begin{aligned} p(1, \omega) &= \frac{M(1, \omega)}{y(1, \omega)} \\ &= \frac{(M(0) - \bar{B}) \frac{1+r(1, \omega)}{r(1, \omega)}}{\bar{y} - \frac{c^*(0)}{c(0)} \frac{2(1+r(1, \omega))^{1/\rho}}{(1+(1+r^*(1, \omega))^{1/\rho}) + (1+(1+r(1, \omega))^{1/\rho})}}. \end{aligned}$$

The relative price levels are

$$\frac{p(1, \omega)}{p(1, \omega')} = \frac{\frac{r(\omega')}{1+r(\omega')} \left[ 1 - \frac{2(1+r(1, \omega))^{1/\rho}}{\frac{c^*(0)}{c(0)} \left( (1+(1+r^*(1, \omega))^{1/\rho}) + (1+(1+r(1, \omega))^{1/\rho}) \right)} \right]}{\frac{r(\omega)}{1+r(\omega)} \left[ 1 - \frac{2(1+r(1, \omega'))^{1/\rho}}{\frac{c^*(0)}{c(0)} \left( (1+(1+r^*(1, \omega'))^{1/\rho}) + (1+(1+r(1, \omega'))^{1/\rho}) \right)} \right]}.$$

Given our assumptions about interest rates, the first part of the expres-

sion,  $\frac{\frac{r(\omega')}{1+r(\omega')}}{\frac{r(\omega)}{1+r(\omega)}}$ , is less than 1, as is the second,  $\frac{1 - \frac{2(1+r(1, \omega))^{1/\rho}}{\frac{c^*(0)}{c(0)} \left( (1+(1+r^*(1, \omega))^{1/\rho}) + (1+(1+r(1, \omega))^{1/\rho}) \right)}}{1 - \frac{2(1+r(1, \omega'))^{1/\rho}}{\frac{c^*(0)}{c(0)} \left( (1+(1+r^*(1, \omega'))^{1/\rho}) + (1+(1+r(1, \omega'))^{1/\rho}) \right)}}$ .

Hence  $p(1, \omega) < p(1, \omega')$ .  $\square$

### Proof of Lemma 3

#### Part (i)

*Proof.* As prices are non-stochastic under this policy regime, all the variation in the state price is derived from how consumption changes. In Proposition ?? we showed that  $c(1, \omega) < c(1, \omega')$  whenever  $r(1, \omega) > r(1, \omega')$ , hence it must be that

$$\begin{aligned} r(1, \omega) > r(1, \omega') &\Leftrightarrow q(1, \omega) > q(1, \omega') \\ &\Leftrightarrow q_2(1, \omega) > q_2(1, \omega') \\ &\Leftrightarrow \mu(1, \omega) > \mu(1, \omega') \\ &\Leftrightarrow \mu_2(1, \omega) > \mu_2(1, \omega'). \end{aligned}$$

□

#### Part (ii)

*Proof.*

$$\begin{aligned} q(1, \omega) &= \beta f(\omega) \frac{p(0)}{p(1, \omega)} \left\{ \frac{c(0)}{c(1, \omega)} \right\}^\rho \\ &= \beta f(\omega) \frac{p(0)c(0)^\rho}{M(1, \omega)} \left\{ \frac{y(1, \omega)}{c(1, \omega)} \right\}^\rho y(1, \omega)^{1-\rho}. \end{aligned}$$

Note that from 40 and 27, the ratio  $\frac{y(1, \omega)}{c(1, \omega)} = \frac{c^*(0)}{c(0)} (1 + (1 + r^*(1, \omega))^{1/\rho}) - \frac{1}{2} (1 + (1 + r(1, \omega))^{1/\rho})$  and comparing states, is negatively correlated with  $r(1, \omega)$ . The additional relevant ratio to determining the risk premium is  $\frac{y(1, \omega)^{1-\rho}}{M(1, \omega)}$ . As under Nominal GDP Targeting money supplies are non-stochastic, then this ratio will move in the same direction as output. As we have shown in Proposition ?? that output falls, then the state price must be negatively correlated with interest rates. It follows that:

$$\begin{aligned} r(1, \omega) > r(1, \omega') &\Leftrightarrow q(1, \omega) < q(1, \omega') \\ &\Leftrightarrow q_2(1, \omega) < q_2(1, \omega') \\ &\Leftrightarrow \mu(1, \omega) < \mu(1, \omega') \\ &\Leftrightarrow \mu_2(1, \omega) < \mu_2(1, \omega'). \end{aligned}$$

This result shows that the effect of inflation outweighs the effect of the real allocation in determining the risk premium.  $\square$

**Part (iii)**

*Proof.* From Proposition ??, we found that under Traditional Monetary Policy, the price level is negatively correlated with interest rates. From Proposition ?? we found that consumption is also negatively correlated with interest rates. Hence it follows that:

$$\begin{aligned} r(1, \omega) > r(1, \omega') &\Leftrightarrow q(1, \omega) > q(1, \omega') \\ &\Leftrightarrow q_2(1, \omega) > q_2(1, \omega') \\ &\Leftrightarrow \mu(1, \omega) > \mu(1, \omega') \\ &\Leftrightarrow \mu_2(1, \omega) > \mu_2(1, \omega'). \end{aligned}$$

$\square$

**Proof of Proposition 3**

**Part (i)**

*Proof.* We can calculate the price of a bond which pays a unit of currency at the end of the second period through no arbitrage and is given by  $q(0 : 2) = \int_{\omega} \frac{q(1, \omega)}{1+r(1, \omega)} d\omega$  or equivalently  $\frac{1}{1+r(0)} \int_{\omega} \frac{\mu(1, \omega)}{1+r(1, \omega)} d\omega$ . The positive correlation between the martingale measure and the nominal interest rate implies that  $\int_{\omega} \frac{\mu(1, \omega)}{1+r(1, \omega)} < \int_{\omega} \frac{f(\omega)}{1+r(1, \omega)} d\omega$  or in other words  $q(0 : 2) < \frac{1}{1+r(0)} \int_{\omega} f(\omega) \frac{f(\omega)}{1+r(1, \omega)} d\omega$ . Let the (per period) interest rate on the long term bond be  $r(0 : 2)$  and the forward interest rate between period 1 and 2 be  $r^f(1 : 2)$ . Therefore the price of the two period bond is

$$q(0 : 2) = \frac{1}{(1 + r(0 : 2))^2} = \frac{1}{1 + r(0)} \frac{1}{1 + r^f(1 : 2)}.$$

It follows that

$$\frac{1}{1 + r^f(1 : 2)} < \int_{\omega} \frac{f(\omega)}{1 + r(1, \omega)} d\omega$$

and  $f(1, \omega) > \int_{\omega} f(\omega) r(1, \omega) d\omega$ : the forward interest rate is an upwardly biased indicator of future interest rates.  $\square$

**Part (ii)**

*Proof.* The proof is the same as above, with inequalities reversed.  $\square$

**Proof of Lemma 4**

*Proof.*

$$\begin{aligned}
\frac{y(1, \omega)}{y^*(1, \omega)} &= \frac{\bar{y} - c(1, \omega)(1 + r(1, \omega))^{1/\rho}}{\bar{y} - c^*(1, \omega)(1 + r^*(1, \omega))^{1/\rho}} \\
&= \frac{\frac{\bar{y}}{c(1, \omega)} - (1 + r(1, \omega))^{1/\rho}}{\frac{\bar{y}}{c(1, \omega)} - \frac{c^*(1, \omega)}{c(1, \omega)}(1 + r^*(1, \omega))^{1/\rho}} \\
&= \frac{.5 \frac{c^*(0)}{c(0)}(1 + (1 + r^*(1, \omega))^{1/\rho}) + .5(1 + (1 + r(1, \omega))^{1/\rho}) - (1 + r(1, \omega))^{1/\rho}}{.5 \frac{c^*(0)}{c(0)}(1 + (1 + r^*(1, \omega))^{1/\rho}) + .5(1 + (1 + r(1, \omega))^{1/\rho}) - \frac{c^*(0)}{c(0)}(1 + r^*(1, \omega))^{1/\rho}} \\
&= \frac{\frac{c^*(0)}{c(0)}(1 + (1 + r^*(1, \omega))^{1/\rho}) + 1 - (1 + r(1, \omega))^{1/\rho}}{\frac{c^*(0)}{c(0)}(1 - (1 + r^*(1, \omega))^{1/\rho}) + 1 + (1 + r(1, \omega))^{1/\rho}}.
\end{aligned}$$

Taking two states, such that  $r(\omega^*) > r(\omega')$  and  $r^*(\omega^*) = r^*(\omega')$ , it must be the case that  $\frac{y(1, \omega^*)}{y^*(1, \omega^*)} < \frac{y(1, \omega')}{y^*(1, \omega')}$ .  $\square$

**Proof of Proposition 4**

*Proof.*

$$\frac{M(1, \omega)}{M^*(1, \omega)} = \frac{M(0) - \bar{B}(1)}{M^*(0) - \bar{B}_2(0)} \left\{ \frac{\frac{1+r(1, \omega)}{r(1, \omega)}}{\frac{1+r^*(1, \omega)}{r^*(1, \omega)}} \right\} \quad (46)$$

Taking two states, such that  $r(\omega^*) > r(\omega')$  and  $r^*(\omega^*) = r^*(\omega')$ , it must be the case that  $\frac{M(1, \omega^*)}{M^*(1, \omega^*)} < \frac{M(1, \omega')}{M^*(1, \omega')}$ .

**Part (i)**

Propositions ?? and ?? tell us that the nominal and real effects on the exchange rates move in opposite directions. Therefore we will determine the

net effect by considering them jointly:

$$\begin{aligned}
e(1, \omega) &= \frac{p(1, \omega)}{p^*(1, \omega)} \\
&= \frac{M(1, \omega)}{M^*(1, \omega)} \frac{y^*(1, \omega)}{y(1, \omega)} \\
&= \frac{M(0) - \bar{B}(1)}{M^*(0) - \bar{B}_2(0)} \left\{ \frac{1+r(1, \omega)}{r(1, \omega)} \right\} \frac{\frac{c^*(0)}{c(0)}(1 - (1 + r^*(1, \omega))^{1/\rho}) + 1 + (1 + r(1, \omega))^{1/\rho}}{\frac{c^*(0)}{c(0)}(1 + (1 + r^*(1, \omega))^{1/\rho}) + 1 - (1 + r(1, \omega))^{1/\rho}}
\end{aligned}$$

This function is clearly non-monotonic. However taking two states such that  $r(\omega^*) > r(\omega')$  and  $r^*(\omega^*) = r^*(\omega')$ . Define  $g(\omega^*) = \log(e(\omega^*))$ . This gives us that

$$\begin{aligned}
\frac{\partial g(\omega^*)}{\partial s^*} \Big|_{r(\omega^*)=0} &= \frac{1}{1 + r(\omega^*)} - \frac{1}{r(\omega^*)^2} \\
&+ \frac{1/\rho(1 + r(1, \omega^*))^{1/\rho-1}}{\frac{c^*(0)}{c(0)}(1 - (1 + r^*(1, \omega^*))^{1/\rho}) + 1 + (1 + r(1, \omega^*))^{1/\rho}} \\
&+ \frac{1/\rho(1 + r(1, \omega^*))^{1/\rho-1}}{\frac{c^*(0)}{c(0)}(1 + (1 + r^*(1, \omega^*))^{1/\rho}) + 1 - (1 + r(1, \omega^*))^{1/\rho}} \\
&\rightarrow -\infty.
\end{aligned}$$

Setting this to zero we can find the interest rate at which the inequality is reversed, though for equilibrium values of  $\frac{c^*(0)}{c(0)} \approx 1$  and  $r^*(1, \omega^*) \approx 0$ , for reasonable values of  $r(1, \omega^*)$ , the effect on the exchange rate is negative (appreciation): exchange rates are stronger or more appreciated in states where interest rates are higher. That is  $r(1, \omega^*) > r(1, \omega') \Leftrightarrow e(1, \omega^*) < e(1, \omega')$ .

**Part (ii)**

As  $e(1, \omega) = \frac{M(1, \omega)}{M^*(1, \omega)} \frac{y^*(1, \omega)}{y(1, \omega)}$ , and using Proposition ??, it follows that  $r(1, \omega^*) > r(1, \omega') \Leftrightarrow e(1, \omega^*) > e(1, \omega')$ .  $\square$

**Proof of Proposition 5**

*Proof.* The propositions above tell us that for Price Stability and Traditional Monetary Policy, the risk premium or state price is negatively correlated with the exchange rate. This means that, given an objective probability measure

$f(\omega), \mu(\omega^*) < f(\omega^*)$  whenever  $e(\omega^*) < \int_{\omega} f(\omega)e(\omega)d\omega$  Let the forward Exchange Rate be  $F(1)$  or:

$$e_{(0)} \frac{1 + r(0)}{1 + r^*(0)} = \int_{\omega} \mu(\omega)e(1, \omega)d\omega \quad (47)$$

$$= e^f(1) \quad (48)$$

$$< \int_{\omega} f(\omega)e(1, \omega)d\omega. \quad (49)$$

The expected exchange rate under the subjective measure is  $\int_{\omega} f(\omega)e(1, \omega)$ . Now as we have shown that as Arrow prices, and hence the Martingale measure are positively correlated with the exchange rate, then the Forward exchange rate, or the expected exchange rate under the Martingale Measure, is biased upwards (more depreciated). That is,  $e^f(1) < \int_{\omega} f(\omega)e(1, \omega)d\omega$ .  $\square$