

# Pricing Large Financial Products

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## Abstract

This paper develops a method to price risky assets that are in non-negligible supply. The introduction of large financial products affects consumption; hence, Arrow prices and the original pricing kernel become obsolete in the counterfactual experiment. Thus, the standard Arrow pricing approach cannot be applied in this context. This paper contributes to the literature on the differentiated product markets. Contrary to the commonly used discrete choice approach, our model gives predictions in settings in which agents are allowed to simultaneously trade arbitrary quantities of many financial products.

**Key words:** Asset pricing, large IPOs, differentiated products

**JEL classification numbers:** D43, D53, G11, G12, L13

## 1 Introduction

The classic approach to the pricing of new assets, which underlies many important results (e.g., the Black-Scholes formula or the Modigliani-Miller theorem) relies on the assumption that investors' consumption is constant in the counterfactual experiment. From the prices of existing assets, the method derives marginal rates of substitution for each state—which correspond to Arrow prices—and then uses the latter to infer prices of newly created assets. This approach does not allow the determination of market value of an asset that is in non-negligible supply, e.g., large Initial Public Offerings. Introducing large financial products affects the equilibrium consumption of the investors and hence the prices of existing assets. It follows that the original pricing kernel cannot be used in the counterfactual experiment.

An alternative approach to counterfactual pricing, developed by the IO literature in the context of product differentiation, relies on the characteristic (or addresses) model. From the

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market-level data, the method estimates the distribution of preferences over characteristics in the population of buyers. Then, the estimated demographic structure is used to predict the effects of newly created products. Even though financial markets fit well the assumptions of differentiated product markets,<sup>1</sup> the IO methodology is not applicable to markets for shares. The method relies heavily on the discrete choice assumption, i.e., that each buyer makes mutually exclusive choices from the collection of all available products.<sup>2</sup> In financial markets, investors simultaneously trade arbitrary, possibly negative, quantities of shares of many different assets. Moreover, the preferences of financial investors exhibit risk aversion i.e., their marginal utility is decreasing in share holdings, a feature that cannot be captured by the discrete choice settings.

This paper develops a method to predict prices of newly issued assets that are in non-negligible supply from the prices of existing assets and aggregate wealth of the investors. Investors are allowed to trade different quantities of risky shares, and can sell assets short. The approach gives price and welfare predictions under general assumptions about heterogeneous quasilinear Von Neumann-Morgenstern preferences. The method uses the fact that, in the model of financial monopoly, the distribution of investors' preferences can be summarized by an aggregate Bernoulli index, a function of aggregate wealth that is the same across the state of the world. Observed prices of existing assets, through Arrow prices, impose restrictions on such an index, which can be used to derive the demand system and the change of the surplus in an arbitrary counterfactual experiment.

## 2 Financial Monopoly

### 2.1 Motivating Example

We first explain the proposed methodology in the context of a large IPO. Consider the following example of a financial economy.

**Example 1.** *In a two-period economy, there is a continuum of identical investors with mass normalized to one and a monopoly. Competitive investors have identical quasilinear utility  $U(c_0, c_1) = c_0 + E u(c_1)$ , where the Bernoulli function  $u(c_1) = 4c_1 - c_1^2$  is quadratic, and endowments are normalized to zero. In the second period, there are two equally likely states. Existing assets consist of two contingent claims with payments*

$$\hat{a} = (\hat{a}_1, \hat{a}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

*The market position of the monopoly in two assets is  $\hat{q} = (1, 0)^T$ . Given symmetric utilities, in the unique competitive equilibrium, the consumption of each investor coincides with aggregate*

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<sup>1</sup>Hedonic models have a structure similar to a financial model. Financial products are described by  $S$  characteristics, i.e., asset payoffs in each state. Agents derive utility from the total quantity of each characteristic.

<sup>2</sup>Generalizations i.e., by Igal Hendel RFS.

wealth  $\hat{w} = \hat{q} \cdot \hat{a} = (1, 0)^T$ , and the corresponding prices of the contingent claims, which are equal to the marginal utilities, are given by  $\hat{p} = (1, 2)^T$ . By assumption, the monopolist understands the structure of the model and knows that the investors' utility functions belong to a quadratic family,

$$u^i(c_1) = \bar{\alpha}_0 + \bar{\alpha}_1 c_1 - \frac{1}{2} \bar{\alpha}_2 c_1^2,$$

but he lacks information about the particular values of the parameters  $(\bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2)$ .

The monopoly contemplates issuing a riskless IPO that pays  $a^* = (1, 1)^T$ , with the (non-negligible) supply of shares normalized to one. We want to study the problem of what the issuer can infer about the *post-issuance* prices of the three assets, from the market-level data available *before* the issuance takes place. The standard asset pricing method assumes that the Arrow prices  $\hat{p}$  are not affected by the introduction of the asset and the predicted price of the IPO is  $\hat{p}^* = a^* \cdot \hat{p} = 3$ . However, since shares of the IPO have positive supply, the investors' consumption in the counterfactual equilibrium is going to change to  $(2, 1)$ , so and marginal utilities and Arrow prices will go down to  $p = (0, 1)$ . Thus, the price of the IPO is smaller than that predicted by the standard Arrow pricing method,  $p^* = a^* \cdot p = 1 < \hat{p}^*$ .

In order to correctly predict the effects of the IPO, one first needs to determine the demand schedules for contingent claims faced by the monopoly. In single product markets, one equilibrium trade-price pair does not pin down the entire demand function: there are many downward-sloping functions that pass through the observed point. In differentiated product markets, the determination of a demand system is further complicated by the large dimensionality of cross-product elasticities; however, the structure of the model of financial monopoly with expected utility functions makes it possible to infer counterfactual demands for existing and newly issued assets, from the observed prices  $\hat{p}$  and monopoly's position  $\hat{q}$ .

The prices of contingent claims are equal to investors' marginal utility at the consumption that coincides with *per capita* wealth  $\hat{c} = \hat{w} \equiv \hat{a} \cdot \hat{q} = (1, 0)^T$ , namely

$$\hat{p}_s = 0.5u'(\hat{w}_s) = 0.5\bar{\alpha}_1 - 0.5\bar{\alpha}_2\hat{w}_s. \quad (2)$$

Thus, even if *a priori* the monopolist does not know the values of preference parameters, he can infer  $\bar{\alpha}_1, \bar{\alpha}_2$  (but not  $\bar{\alpha}_0$ ) from the observed prices of contingent claims  $\hat{p}$  and trade  $\hat{q}$ . Given that the IPO increases the aggregate wealth to

$$c_1 = w \equiv \hat{a} \cdot \hat{q} + a^* = (2, 1)^T,$$

the two parameters allow the prediction of the counterfactual Arrow prices  $p$ , which can be extrapolated to the prices of newly created assets.

The standard counterfactual methodology rules out experiments that have non-negligible welfare effects: there, by assumption, the consumption and hence the equilibrium levels of utility are the same before and after the issuance of new assets. Assets in non-negligible supply affect

the utility of the investors, and the method developed in this paper facilitates the determination of the associated welfare effects. In the example, the change in the overall level of utility (i.e., the utility of the investors and the monopoly's revenue) is geometrically represented by the area below marginal utility. With known values of parameters  $\bar{\alpha}_1$  and  $\bar{\alpha}_2$ , the change of total surplus can be calculated as  $\Delta S = 4$ .

In the following sections, we demonstrate how one can obtain predictions regarding counterfactual prices and surplus effects based solely on the prices of existing assets and aggregate wealth under general assumptions. The method is applicable to markets with an arbitrary number of states, with heterogenous (but quasilinear) Von Neumann-Morgenstern utility functions, and with arbitrary distributions of investors' endowments.

## 2.2 Assumptions and Definitions

We consider two-period financial markets with one large trader, *a monopoly*, and a continuum of mass 1 of competitive investors, indexed by  $i$ . Period 1 is uncertain, and the set of states,  $\mathcal{S}$ , has finite cardinality  $\equiv S \geq 2$ . A typical state is denoted by  $s = 1, \dots, S$ , and the probability distribution over states is denoted by  $\pi = (\pi_1, \dots, \pi_S)^T$ . Investor  $i$  has quasilinear, Von Neumann-Morgenstern preferences defined over current and future consumption,

$$U^i(c_0^i, c_1^i) = c_0^i + \text{Eu}^i(c_1^i), \quad (3)$$

and the Bernoulli utility function of investor  $i$  satisfies standard assumptions:

**Assumption 1.** *Function  $u^i$  is twice-continuously differentiable and strictly concave.*

In period 1, investor  $i$  is endowed with  $e^i$  units of numéraire, which is a random variable over  $\mathcal{S}$ . The distribution of investor types is  $F$ , the endowment *per capita* in the economy is  $\bar{e} \equiv \int e^i dF$ .<sup>3</sup>

To insure against idiosyncratic risk, in period 0 investors trade  $K$  assets. Each of these assets is a random variable with respect to  $\mathcal{S}$ : asset  $k$  promises to pay an amount of numéraire,  $a_k$ , which is random.

The collection of all asset payoffs, which we call an *asset structure*, is given by a random column vector  $a = (a_1, \dots, a_K)^T$ . We focus on the case of complete financial markets, so we assume that for any stochastic consumption  $c_1^i$ , there exists a portfolio of assets,  $t^i \in \mathbb{R}^K$ , such that  $c_1^i = a \cdot t^i$ .

By  $q \in \mathbb{R}^K$ , we denote monopoly's position in assets  $a$ , and hence  $w \equiv a \cdot q + \bar{e}$  is the corresponding aggregate wealth *per capita*. Importantly, for our results we assume that the shares of risky assets are perfectly divisible, so that investors can trade arbitrary portfolios

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<sup>3</sup>Throughout the paper, we assume that  $e^i$  and all other functions that depend on  $i$  are integrable with respect to  $F$ .

assets, and that they can sale assets short. Formally, investor  $i$  can chose his trades from the unrestricted set  $\mathbb{R}^K$ .

Given the economy  $(\{u^i, e^i\}_i, F)$ , function  $\bar{U}$  assigns the sum of the investors' utilities in a Pareto efficient allocation, for any level of aggregate wealth  $w$ :

$$\bar{U}(w) := \max_{\{c_1^i\}_i} \int \mathbb{E} u^i(c_1^i) dF : \int c_1^i dF \leq w. \quad (4)$$

Define  $\mathcal{W}$  as the set of random variables  $w > 0$  for which the solution to program (4) is interior and, for each investor, preferences are strictly increasing and in all states.

Suppose that Assumption 1 holds for  $F$ -almost all investors and let the set

$$\mathcal{D} = \{\bar{w} \in \mathbb{R} \mid \exists (s, w) \in \mathcal{S} \times \mathcal{W} : w(s) = \bar{w}\}$$

be defined. The following lemma constructs a *Bernoulli index* for the economy  $(\{u^i, e^i\}_i, F)$ .

**Lemma 2.** *Given function  $\bar{U} : \mathcal{W} \rightarrow \mathbb{R}$ , there exists a unique function  $\bar{u} : \mathbb{R}_{++} \rightarrow \mathbb{R}$  for which  $\bar{U}(w) = \mathbb{E} \bar{u}(w)$ . Moreover, function  $\bar{u}$  is once-continuously differentiable, strictly increasing and strictly concave.*

### 2.3 Demand System and Total Surplus

In economy  $(\{u^i, e^i\}_i, F)$  with assets  $a$ , and monopoly's position,  $q$ , a competitive equilibrium in financial markets is defined as a profile of investors' trades and asset prices  $(\{t^{i*}\}_i, p^*)$ , where  $t^{i*} \in \mathcal{T}$ ,  $p^* \in \mathbb{R}^K$ , for which trades are optimal,  $t^{i*} = \operatorname{argmax}_{t^i \in \mathcal{T}} U^i(-p^* \cdot t^i, a \cdot t^i + e^i)$ , and all financial markets clear,  $\int t^{i*} dF = q$ . The competitive equilibrium is uniquely defined for all trading positions  $q \in \mathcal{Q}^a \equiv \{q \in \mathbb{R}^K \mid a \cdot q + \bar{e} \in \mathcal{W}\}$ .<sup>4</sup>

A *demand system*<sup>5</sup> for assets  $a$  is a mapping that assigns the vector of competitive prices for all, possibly short, positions of the monopoly  $q$ .

**Definition 3.** *Map  $p^a : \mathcal{Q}^a \rightarrow \mathbb{R}^K$  gives a vector of asset prices in competitive equilibrium for all  $q \in \mathcal{Q}^a$ , i.e.,  $p^a(q) = p^*$ .*

With quasilinear preferences, demand system is a function. The next proposition characterizes the system in terms of index  $\bar{u}$ .

<sup>4</sup>For example, with Inada conditions,  $\lim_{c_s \rightarrow 0} u^i(c_s) = \infty$  and  $\lim_{c_s \rightarrow \infty} u^i(c_s) = 0$ , the Pareto-efficient allocation satisfies the non-satiation condition for all  $w \in \mathcal{W} = \{w \mid w \geq 0\}$ , and demand system is well-defined for monopoly's positions in  $\mathcal{Q}^a = \{q \in \mathbb{R}^K \mid a \cdot q > -\bar{e}\}$ .

<sup>5</sup>As in a characteristics models for some range asset prices one asset can strictly dominate the other asset in terms of its payoff. In the discrete choice model the dominated asset would get zero demand. With unrestricted trade choice set, for such price vector in optimum traders take unbounded positions and aggregate demand is not a well-defined function. This is why in the context of financial markets it is more convenient to work with an inverse demand function. Since from inverse function one can unambiguously derive demand correspondence, we refer to  $p^a(\cdot)$  as a *demand system*.

**Proposition 4.** *In economy  $(\{u^i, e^i\}_i, F)$ , with complete financial structure, a demand system is given by*

$$p^a(q) = E[a\bar{u}'(a \cdot q + \bar{e})]. \quad (5)$$

The method developed in this paper also allows for welfare analysis. The total surplus in the economy is defined as a sum of the utilities of all agents, including monopoly.

**Definition 5.** *Map  $S^a : \mathcal{Q}^a \rightarrow \mathbb{R}$  gives a sum of investors utilities and monopoly's revenue in a competitive equilibrium for all  $q \in \mathcal{Q}^a$ , i.e.,*

$$S^a(q) \equiv \int U^i(-p^* \cdot t^{i*}, a \cdot t^{i*} + e^i) dF + p^* \cdot q. \quad (6)$$

As in the case of demand system, by the uniqueness of competitive equilibrium, surplus  $S^a(\cdot)$  is a function. Conditional on position  $q$ , equilibrium allocation is Pareto-efficient; therefore, with transferable utilities, the total surplus coincides with  $\bar{U}(a \cdot q + \bar{e})$ . Thus, in terms of Bernoulli index the surplus is given by

$$S^a(q) = E[\bar{u}(\bar{e} + a \cdot q)]. \quad (7)$$

## 2.4 Equilibrium Observation

In financial markets, large issuers often lack precise information about the distribution of investors' preferences or endowments. Their knowledge of markets is restricted to existing asset structure  $\hat{a}$ , own market positions  $\hat{q}$ , equilibrium asset prices  $\hat{p}$  and *per capita* level endowment  $\bar{e}$ . The central question of this paper is what monopoly can infer about counterfactual prices of new assets and the effects of their issuance on welfare, given such limited information.

Let  $(\hat{q}, \hat{p}, \hat{a})$  be the *market-level data* available in financial markets. Price taking assumption by the investors implies no arbitrage opportunities; from the prices of assets  $\hat{p}$ , one can derive a unique vector of Arrow prices  $\hat{p}^{Arrow} \in \mathbb{R}^S$ , for which  $\hat{p}_k = \sum_s [a_k]_s \hat{p}_s^{Arrow}$  for all  $k$ , where  $[x]_s$  denotes realisation of random variable  $x$  in state  $s$ . In the model of financial monopoly, by equation (4) in terms of aggregate wealth, Arrow prices are equal to

$$\hat{p}_s^{Arrow} = \pi_s \bar{u}'(\hat{w}_s). \quad (8)$$

Let  $\hat{\mathcal{S}} \subset \mathcal{S}$  be a maximal subset of states for which  $\hat{w}_s \neq \hat{w}_{s'}$  for  $s \neq s'$  (within each class of states with identical aggregate wealth  $\hat{w}_s = \hat{w}_{s'}$ , drop all but one states). Without loss of generality, the states in  $\hat{\mathcal{S}}$  are reordered so that aggregate wealth  $\hat{w} = \hat{a} \cdot \hat{q} + \bar{e}$  is increasing in  $s$ , i.e.,  $\hat{w}_{s+1} > \hat{w}_s$ . Equilibrium observation is defined as an array

$$\hat{o} \equiv \{\hat{o}_s\}_{s \in \hat{\mathcal{S}}} \equiv \{\hat{w}_s, \hat{p}_s^{Arrow} / \pi_s\}_{s \in \hat{\mathcal{S}}}. \quad (9)$$

In terms of primitives that characterize economy, observation  $\hat{o}$  is informationally equivalent to corresponding data  $(\hat{q}, \hat{p}, \hat{a})$ .<sup>6</sup> It is also useful to define a vector of slopes of line segments connecting observation points  $\hat{o}_s$  and  $\hat{o}_{s+1}$  for all  $s \in \hat{S}_- \equiv \hat{S}/\{s = \hat{S}\}$ ,

$$\Delta\hat{o} \equiv \{\Delta\hat{o}_s\}_{s \in \hat{S}_-} = \left\{ \frac{\hat{p}_s^{Arrow}/\pi_s - \hat{p}_{s+1}^{Arrow}/\pi_{s+1}}{\hat{w}_s - \hat{w}_{s+1}} \right\}_{s \in \hat{S}_-}. \quad (10)$$

By Lemma 2, the marginal index  $\bar{u}'$  is strictly positive and decreasing. Thus, from equation (8), it follows that market-level data can be rationalized by a model of financial monopoly if and only if corresponding equilibrium observation satisfies  $\hat{o} \in \mathbb{R}_{++}^{2\hat{S}}$  and the slopes of line segments connecting observation points are negative  $\Delta\hat{o} < 0$ , which is assumed in this paper.

Throughout the paper, we illustrate the counterfactual methods using the following example of an economy.

**Example 6.** Consider economy with  $S$ , equally likely states,  $s = 1, \dots, S$  and the deterministic endowment per capita,  $\bar{e} = 1$ . Existing financial structure,  $\hat{a}$ , consists of a complete set of Arrow securities. Position of a monopoly in  $s$ -contingent claim is  $\hat{q}_s = (s-1)/(S-1)$  so that equilibrium wealth is increasing in  $s$ ,  $\hat{w}_s = 1 + (s-1)/(S-1)$ . We consider two scenarios. In scenario (a), equilibrium price of contingent claim is

$$\hat{p}_s = \frac{S-1}{S(S+s-2)}, \quad (11)$$

while in scenario (b), it is given by

$$\hat{p}_s = \frac{1}{S} \left( \frac{S-s}{S-1} \right). \quad (12)$$

In the example, wealth has uniform distribution on the support  $[1, 2]$  for all values of  $S$ . The inverse of  $S$  measures the coarseness of the state space *ceteris paribus*. The corresponding equilibrium observations for the example are depicted in Figure 1. Observe that in none of the two scenarios do observations reject the hypothesis of the model of financial monopoly.

### 3 Linear Financial Monopoly

We first consider markets in which Bernoulli utility functions of the investors are quadratic.

**Assumption 2.** Bernoulli utility function  $u^i$  satisfies

$$u^i(c_1) = \alpha_0^i + \alpha_1^i c_1 - 0.5\alpha_2^i (c_1)^2, \quad (13)$$

for some parameters  $(\alpha_0^i, \alpha_1^i, \alpha_2^i)$ , where  $\alpha_1^i, \alpha_2^i > 0$ .

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<sup>6</sup>An economy rationalizes data  $(\hat{q}, \hat{p}, \hat{a})$  if a demand system  $p^{\hat{a}}(\cdot)$  corresponding to this economy satisfies  $\hat{p} = p^{\hat{a}}(\hat{q})$ . An economy rationalizes observation  $\hat{o}$  if Bernoulli index satisfies  $\hat{p}_s^{Arrow}/\hat{\pi}_s = \bar{u}'(\hat{w}_s)$  for all  $s$ . It can be shown that economy  $(\{u^i, e^i\}_i, F)$  rationalizes  $(\hat{q}, \hat{p}, \hat{a})$  if and only if it rationalizes  $\hat{o}$ .

Parameters  $(\alpha_0^i, \alpha_1^i, \alpha_2^i)$  can differ across agents. Scalar  $\alpha_2^i$  gives the slope of marginal utility from contingent consumption and it measures investor's risk aversion. Lemma 7 characterizes Bernoulli index for an economy with quadratic Bernoulli utility functions for almost all investors.

**Lemma 7.** *Bernoulli index  $\bar{u}$  is quadratic*

$$\bar{u}(w) = \bar{\alpha}_0 + \bar{\alpha}_1 w - 0.5\bar{\alpha}_2 w^2, \quad (14)$$

with coefficient of aggregate risk aversion  $\bar{\alpha}_2 = \left( \int (\alpha_1^i)^{-1} dF \right)^{-1} > 0$ .

Not all market level data can be explained by a linear model. By condition (8) and Lemma 7, data  $(\hat{q}, \hat{p}, \hat{a})$  can be rationalized by a model of financial monopoly with preferences satisfying Assumptions 1 and 2 if and only if corresponding observation is consistent with the general model and in addition  $\Delta\hat{o}_s = \Delta\hat{o}_{s'}$  for all  $s, s' \in \hat{\mathcal{S}}_-$ . Geometrically, the necessary and sufficient condition for rationalizability of data  $(\hat{q}, \hat{p}, \hat{a})$  requires that there exists a line with negative slope in  $\mathbb{R}_{++}^2$  whose graph contains observation  $\hat{o}$ . In Example 6, the hypothesis of demand linearity is rejected in scenario (a) but not (b) (see Figure 1).

### 3.1 Counterfactual Analysis

Consider an experiment in which existing assets  $\hat{a}$  are replaced by an arbitrary complete asset structure  $a$  with  $K \geq S$  assets.

**Definition 8.** *Counterfactual demand system  $p^a : \mathbb{R}^K \rightarrow \mathbb{R}^K$  is consistent with market-level data  $(\hat{q}, \hat{p}, \hat{a})$  if there exists economy  $(\{u^i, e^i\}_i, F)$  that rationalizes  $(\hat{q}, \hat{p}, \hat{a})$ , and which gives rise to demand  $p^a(\cdot)$  when assets  $a$  are traded.*

The assumption of expected quadratic utility typically allows the identification of demand system  $p^a(\cdot)$  from data  $(\hat{q}, \hat{p}, \hat{a})$ . To see this, recast economy with financial structure  $\hat{a}$  as an economy with contingent claims. In such an economy, the demand system consists of  $S$  independent schedules, one for each contingent claim, determined by *the same* marginal index  $\bar{u}'$ . Since, generically in  $\hat{q}$ ,  $\hat{\mathcal{S}} = S \geq 2$ , observation  $\hat{o}$  gives two or more points on the linear index  $\bar{u}'$  one can infer parameters  $(\bar{\alpha}_1, \bar{\alpha}_2)$ , from the following equations

$$\bar{\alpha}_2 = \Delta\hat{o}_s, \text{ and } \bar{\alpha}_1 = \hat{p}_s^{Arrow} / \pi_s + \bar{\alpha}_2 \hat{w}_s, \quad (15)$$

where  $s \in \hat{\mathcal{S}}_-$ . Since  $(\hat{q}, \hat{p}, \hat{a})$  is generated by a linear model,  $\Delta\hat{o}_s$  attains the same value for all choices of  $s \in \hat{\mathcal{S}}_-$ . Once derived, marginal index  $\bar{u}'$  determines a consistent demand system for an arbitrary counterfactual financial structure  $a$ .

**Proposition 9.** *Generically in  $\hat{q}$ , demand system  $p^a : \mathcal{Q}^a \rightarrow \mathbb{R}^K$  consistent with  $(\hat{q}, \hat{p}, \hat{a})$  is unique. The demand system is given by*

$$p^a(q) = \bar{\alpha}_1 E[a] - \bar{\alpha}_2 E[\bar{e}a] - \bar{\alpha}_2 E[(aa^T)q], \quad (16)$$

A straightforward implication of the proposition is that equilibrium information makes it possible to infer a  $K \times K$  matrix of counterfactual cross-asset price impacts,  $D_q p^a(\cdot) = -\bar{\alpha}_2 E[aa^T]$ , which is negative semi-definite and symmetric.

Equilibrium prices are derived from, and hence provide information about the slope but not the level of index  $\bar{u}$ . Thus, parameter  $\alpha_0$  and hence, by equation (7), the level of total surplus in the economy is not pinned down by  $(\hat{q}, \hat{p}, \hat{a})$ ; however, as in Example 1, within a given state  $s$ , the change in total surplus relative to the observed equilibrium is geometrically represented by the area below marginal index  $\bar{u}'$  and it can be calculated from the market-level data. In terms of the change of monopoly promised payment,  $\delta \equiv a \cdot q - \hat{a} \cdot \hat{q}$ , the impact of experiment  $(a, q)$  on the total surplus,  $\Delta S^a(q) = S^a(q) - S^{\hat{a}}(\hat{q})$ , is given by

$$\Delta S^a(q) = \bar{\alpha}_1 E[\delta] - \bar{\alpha}_2 E[\delta \bar{e}] - \bar{\alpha}_2 E[\delta \hat{a} \cdot \hat{q}] - \bar{\alpha}_2 \frac{1}{2} E[\delta^2]. \quad (17)$$

Data  $(\hat{q}, \hat{p}, \hat{a})$  also allow the decomposition of  $\Delta S^a(\cdot)$  into investors' surplus and the revenue of the monopoly.<sup>7</sup>

### 3.2 Large IPOs

An important instance of a counterfactual experiment considered in the previous section is an issuance of a large IPO. At which price should the issuer offer newly created shares? Pricing new financial products is associated with two risks. If shares are underpriced, i.e., if the chosen price is below their market value, then the offering generates capital losses for the issuer. If, on the other hand, the stock is offered at a higher price than the markets are willing to pay, then the underwriters will be unable to sell shares, which is also a suboptimal outcome. Thus, ideally, the price should coincide with the maximal price for which the demand for shares is no smaller than its supply. The counterfactual method developed in this paper allows the determination of such a price from the data available in financial markets.

Suppose monopoly contemplates the issuance of an IPO so that the counterfactual asset structure is  $a \equiv (\hat{a}^T, a_{IPO})^T$  and the supply of shares is normalized to one. Let  $\hat{p}_{IPO}$  be the IPO price predicted assuming constant Arrow prices. The actual market price of an IPO that is going to be observed in equilibrium after issuance is

$$p_{IPO} = \hat{p}_{IPO} - \underbrace{\bar{\alpha}_2 (E^2[a_{IPO}] + Var[a_{IPO}])}_{\text{large IPO discount}}. \quad (18)$$

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<sup>7</sup>Investors' surplus changes by  $\Delta IS^a(q) =$ , while the revenue of the monopoly changes by  $\Delta R^a(q) =$ . Both values can be inferred from the market-level data.

Thus, for all  $a_{IPO} \neq 0$ , price is below the one predicted in the standard approach. The discount is directly proportional to the market risk aversion  $\bar{\alpha}_1$  and can be determined from the market-level data.

The issuance of new assets affects the prices of existing assets. For asset  $k$  with non-negative expected payoff the price effect is

$$p_k = \hat{p}_k - \underbrace{\bar{\alpha}_2(E[\hat{a}_k]E[a_{IPO}] + Cov[a_{IPO}, \hat{a}_k])}_{\text{effect of IPO}}, \quad (19)$$

where  $\hat{p}_k$  is the price before issuance. If the existing asset and the IPO are substitutes or at most weak complements, (i.e.,  $Cov[a_{IPO}, \hat{a}_k] > -E[\hat{a}_k]E[a_{IPO}]$ ), then asset experiences a negative price shock. This pattern is observed empirically. Large IPOs have a significant and negative effect on prices of existing assets, and the magnitudes of the effects depends on payoff correlation.<sup>8</sup>

In Example 6, the price effects of an IPO that pays

$$a_{IPO,s} = \left\{ \begin{array}{ll} 1 & \text{if } \hat{w}_s \leq E(\hat{w}) = 1.5 \\ 0 & \text{otherwise} \end{array} \right\}, \quad (20)$$

is depicted in Figure 2. Even though preferences of the investors are heterogenous, monopoly can precisely infer counterfactual prices and the change in total surplus resulting from introduction of IPO.

## 4 Non-parametric Method

In this section, we consider a model with arbitrary Bernoulli utility functions that satisfy Assumption 1. Analogously to a linear model, counterfactual demand system is consistent with the market-level data if there exists an economy with general preferences that generates data  $(\hat{q}, \hat{p}, \hat{a})$  and which gives rise to demand system  $p^a(\cdot)$ .

For equilibrium observation  $\hat{o}$ , define  $\bar{u}'_+ : \bar{\mathbb{R}}_+ \rightarrow \bar{\mathbb{R}}_+$  and  $\bar{u}'_- : \bar{\mathbb{R}}_+ \rightarrow \bar{\mathbb{R}}_+$  as the maximal and minimal, right- and left-continuous step functions, respectively, whose graphs contain corresponding  $\hat{o}$ . (For  $w \in (0, \hat{w}_1)$  let  $\bar{u}'_+(w) \equiv \infty$  and for  $w \in (\hat{w}_S, \infty)$  let  $\bar{u}'_-(w) \equiv 0$ .) In Example 6, the two functions are depicted in Figure 3. By condition (8), the graph of the true marginal Bernoulli index  $\bar{u}'$  contains observation  $\hat{o}$ . Since in addition  $\bar{u}'$  is non-increasing, functions  $\bar{u}'_+$  and  $\bar{u}'_-$  give the highest and the lowest marginal utility that can be explained by a model given observation  $\hat{o}$ , respectively.

The bounds, derived from the equilibrium with existing assets  $\hat{a}$  can be used to predict the prices and changes of surplus in arbitrary counterfactual experiment  $(q, a)$ . Let  $a_+ \equiv (\max(a_1, 0), \dots, \max(a_K, 0))$ , and  $a_- \equiv a - a_+$ . Since Arrow prices are proportional to marginal Bernoulli index, the bounds on counterfactual prices are given by

<sup>8</sup>Here give empirical papers on large IPOs.

**Corollary 10.** *Counterfactual demand system  $p^a(\cdot)$  is consistent with the market-level data only if*

$$p_-^a(\cdot) \leq p^a(\cdot) \leq p_+^a(\cdot), \quad (21)$$

where

$$p_+^a(q) \equiv E [a_+ \bar{u}'_+(a \cdot q + \bar{e}) + a_- \bar{u}'_-(a \cdot q + \bar{e})], \quad (22)$$

and

$$p_-^a(q) \equiv E [a_- \bar{u}'_+(a \cdot q + \bar{e}) + a_+ \bar{u}'_-(a \cdot q + \bar{e})]. \quad (23)$$

The cross-asset price impacts have a structure similar to that of the linear model. Define  $\bar{\alpha}_2(w) \equiv E[\bar{u}''(w)]$  as the expected market risk aversion given wealth  $w$ , and let  $\bar{\pi}(w) = \{\bar{\pi}_1(w), \dots, \bar{\pi}_S(w)\}$  be a modified probability measure over states  $\mathcal{S}$ , where  $\bar{\pi}_s(w) \equiv \pi_s \bar{u}''(w_s) / \bar{\alpha}_2(w)$ . Measure  $\bar{\pi}(w)$  alters the objective distribution  $\pi$  by increasing mass probability for the states with above average convexity of Bernoulli index. Given the two definitions, the cross-asset price impact is given by

$$D_q p^a(q) = -\bar{\alpha}_2(w) E_{\bar{\pi}(w)}[aa^T], \quad (24)$$

where  $w$  is wealth corresponding to position  $q$ .<sup>9</sup>

The sign and the magnitude of cross-asset price impact depends on how close the substitutes of any two assets are to each other. Under general preferences, the substitutability/complementarity is determined by the covariance of asset payoffs given altered probability measure  $\bar{\pi}(w)$ . Since prices of assets are determined by marginal index and not the second derivative of  $\bar{u}$ , observation  $\hat{o}$  does not impose any restrictions on  $\bar{\alpha}_2(w)$  or  $\bar{\pi}(w)$ .<sup>10</sup> Thus, contrary to a linear model, market-level data do not allow the inference of cross-asset elasticities from the prices of existing assets.

The maximal change in total surplus resulting from counterfactual experiment  $(q, a)$ , that is consistent with equilibrium data, is given by

$$\Delta S_+^a(q) = E \left[ \int_{\hat{w}}^{\hat{w}+\delta_+} \bar{u}'_+(w) dw + \int_{\hat{w}+\delta_-}^{\hat{w}} \bar{u}'_-(w) dw \right], \quad (25)$$

and the change is bounded from below by

$$\Delta S_-^a(q) = E \left[ \int_{\bar{w}}^{\bar{w}+\delta_+} \bar{u}'_-(w) dw + \int_{\bar{w}+\delta_-}^{\bar{w}} \bar{u}'_+(w) dw \right], \quad (26)$$

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<sup>9</sup>From the characteristics discrete choice model it is known that  $K^2$  cross-asset price impacts can be determined from  $S^2$  cross elasticities of contingent claims (which correspond to attributes, see Section X). Given separability of the expected utility function, the dimensionality of cross asset elasticities is further reduced to  $S$ , i.e., cross asset price impacts are uniquely determined by a vector of convexity of Bernoulli utility function in all state. In the linear model, such convexity is constant across all states and the matrix of cross asset price impacts can be derived from one number,  $\bar{\alpha}$ .

<sup>10</sup>This holds generically in  $w$ . Identical Arrow prices normalized by state probabilities indicate that aggregate wealth and hence convexity of utility function are the same in the two states. This in turn implies restriction  $\bar{\pi}_s(w) / \bar{\pi}_{s'}(w) = \pi_s / \pi_{s'}$ .

where  $\delta_+ \equiv \max(\delta, 0)$ , and  $\delta_- \equiv \delta - \delta_+$ . Figure X depicts the predicted prices and the surplus change in Example 6, resulting from the introduction of IPO given by (20). The price of a new asset is decreasing, and surplus is increasing in supply of shares,  $q_{IPO}$ .

## 5 Precision of the Predictions

In this section, we discuss two factors that affect the precision of the predictions of the non-parametric method: information structure and preference structure.

### 5.1 Information Structure

In Example 6, when the number of distinct realizations of wealth within interval  $[1, 2]$ , equal to  $S$ , is large, then step functions  $\bar{u}'_+$  and  $\bar{u}'_-$  closely approximate marginal index  $\bar{u}'$ . The predictions become exact for  $S$  approaching infinity. We now formalize the observation that the richer the state space, the greater the accuracy of the predictions of the nonparametric method. For this, we construct a sequence of economies with finer and finer state space *ceteris paribus*.

Let the limit state space be given by a cube,  $\mathcal{S}^\infty = [0, 1]^X \subset \mathbb{R}^X$ , where  $X \geq 1$  is a number of factors that generate uncertainty in the economy. The probability distribution over  $\mathcal{S}^\infty$  has density given by a density function  $\pi^\infty$ . In a *limit economy*,  $(\{u^i, e^i(\cdot)\}_i, F)$  investors' utility functions satisfy Assumption 1 and endowments are continuous functions  $e^i : \mathcal{S}^\infty \rightarrow \mathbb{R}_{++}$ . Financial structure consists of  $K$  assets,  $a(\cdot) = \{a_1(\cdot), \dots, a_K(\cdot)\}$ , where individual asset is a continuous function,  $a_k : \mathcal{S}^\infty \rightarrow \mathbb{R}$ . Given position  $\hat{q} \in \mathbb{R}^K$ , let  $\hat{w}_1 \equiv \min_{x \in \mathcal{S}^\infty} a(x) \cdot \hat{q} + \int e^i(x) dF$  and  $\hat{w}_\hat{s} \equiv \max_{x \in \mathcal{S}^\infty} a(x) \cdot \hat{q} + \int e^i(x) dF$  be the maximal and the minimal realization of *per capita* wealth, respectively.

The sequence of economies with an increasing number of states that corresponds to a limit economy is constructed as follows. For a natural number,  $n = 1, 2, \dots$ , the limit state space  $\mathcal{S}^\infty$  is partitioned into  $n^X$  sub-cubes with equal Lebesgue measures (the boundary points of the sub-cubes can be assigned to adjacent sub-cubes in arbitrary manner). In the  $n^{th}$  economy, each sub-cube  $C_s^n \subset \mathcal{S}^\infty$  corresponds to state  $s$ , and thus the cardinality of the state space of the  $n^{th}$  economy is  $S^n = n^X$ . The probability of state  $s \in \mathcal{S}^n$  is equal to the probability of the corresponding sub-cube given limit density, i.e.,  $\pi_s^n \equiv \int_{x \in C_s^n} \pi^\infty(x) dx$ . For any function  $h : \mathcal{S}^\infty \rightarrow \mathbb{R}$ , the discretized random variable  $h^n$  in state  $s$  gives realizations equal to the expected value of function  $h$ , conditional on sub-cube  $C_s$ , i.e.,  $h_s^n \equiv \int_{x \in C_s^n} h(x) \pi^\infty(x) dx$ . The  $n^{th}$  economy is given by  $(\{u^i, e^{i,n}\}_i, F)$ , where  $e^{i,n}$  is a discretization of function  $e^i(\cdot)$ . Asset structure  $a^n$  for such an economy is given by discretization of assets  $a(\cdot)$  and a complete set of Arrow securities in zero net supply.<sup>11</sup> Finally,  $\bar{u}'_+^n$  and  $\bar{u}'_-^n$  are bounds on marginal utility

<sup>11</sup>This assumption assures the completeness of financial structure for each  $n$ , without affecting expected wealth

derived in equilibrium of the  $n^{\text{th}}$  economy.

**Proposition 11.** *In the sequence of economies, the bounds on marginal utility pointwise converge*

$$\lim_{n \rightarrow \infty} \bar{u}'_+{}^n = \lim_{n \rightarrow \infty} \bar{u}'_-{}^n = \bar{u}', \quad (27)$$

on interval  $(\hat{w}_1, \hat{w}_{\hat{S}})$ , where  $\bar{u}$  is a Bernoulli index of the limit economy.

Thus, for all experiments in which wealth remains within range  $[\hat{w}_1, \hat{w}_{\hat{S}}]$  in all states, the non-parametric method allows the prediction of prices and changes in total surplus with arbitrarily precision when state space is sufficiently fine.

## 5.2 Preference structure

The second factor that affects the predictive power of the non-parametric method is the strength of assumptions on investors' preferences. A natural property of investors' preferences observed in financial markets is a convexity of investors' marginal utility.<sup>12</sup> We now consider economies in which utility functions  $u^i$  of all  $i$ , apart from Assumption 1 satisfy the following assumption

**Assumption 3.** *Bernoulli function  $u^i$  is thrice-continuously differentiable, and marginal utility is convex,  $u^{i'''} \geq 0$ .*

Among many others, functions that satisfy Assumption 3 include quadratic, CARA or CRRA utility functions. The assumption of convex marginal utility has a natural economic interpretation; it more than proportionally increases the value of consumption in states with small wealth and thus it captures fear of catastrophe. The next lemma shows that if preferences of almost all investors are characterized by a convex marginal utility, the Bernoulli index exhibits this property as well.

**Lemma 12.** *Marginal Bernoulli index is convex,  $\bar{u}''' \geq 0$ .*

One implication of the lemma is that market-level data can be rationalized by an economy with convex marginal utilities whenever the slopes,  $\Delta\hat{o}$ , are non-decreasing in wealth, i.e.,  $\Delta\hat{o}_{s+1} \geq \Delta\hat{o}_s$ . Note that in Example 6, in both scenarios, market-level data are consistent with convex marginal utility.

Assumption of convex marginal utilities significantly reduces the range of values of the marginal Bernoulli index consistent with equilibrium. Since  $\bar{u}'$  is convex, for any  $s \in \hat{\mathcal{S}}$ , the line that passes through points  $\hat{o}_s$  and  $\hat{o}_{s+1}$  is above marginal index  $\bar{u}'$  for all  $w \in [\hat{w}_s, \hat{w}_{s+1}]$  and

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*per capita.*

<sup>12</sup>Evidence for convex marginal utility here. By the symmetric argument, one can increase the precision of the predictions by *a priori* assuming concave marginal utility. Concavity, however, is not an economically plausible assumption in financial markets.

is below the index otherwise. (Moreover vertical line passing through point  $\hat{o}_1$  and horizontal line passing through point  $\hat{o}_{\hat{S}}$  give bounds for values outside of the interval  $[\hat{w}_1, \hat{w}_{\hat{S}}]$ ). Thus the union of line segments connecting points  $\{\hat{o}_s, \hat{o}_{s+1}\}$  for all  $s \in \hat{S}$ , constant function  $\hat{p}_{\hat{S}}^{Arrow}/\pi_{\hat{S}}$  for  $w > \hat{w}_{\hat{S}}$ , and infinity for  $w < \hat{w}_1$ , jointly define the upper bound  $\bar{u}'_+$  on marginal index  $\bar{u}'$ . The upper envelope of the parts of lines that are outside of each interval  $[\hat{w}_s, \hat{w}_{s+1}]$  for all  $s$  give the lower bound  $\bar{u}'_-$  (there are  $2 \times \hat{S}$  such line segments). The construction of the two bounds is depicted in Figure 5.

Equations (22) and (23), with original bounds replaced by  $\bar{u}'_+$  and  $\bar{u}'_-$  give the range of counterfactual prices that are consistent with  $(\hat{q}, \hat{p}, \hat{a})$  under Assumption 3.<sup>13</sup> Figure 6 depicts price predictions for the IPO (6), and contrasts it with the ones from the general model. In both scenarios, the method that relies on convexity of marginal utility functions significantly improves accuracy. Under scenario (b), the non-parametric approach from this section exactly uncovers the counterfactual demand system. It can be shown that within the class of economies satisfying Assumption 3, equilibrium observation for which  $\Delta\hat{o}_s = \Delta\hat{o}_{s'}$  for all  $s, s'$  implies that in the underlying economy in the interval  $(\hat{w}_1, \hat{w}_{\hat{S}})$ , utility functions are quadratic for almost all investors. Contrary to the method that is based solely on Assumption 1, in quadratic economies the non-parametric method from this section has the same predictive power as the linear one from Section 3.

Under Assumption 3, market-level data  $(\hat{q}, \hat{p}, \hat{a})$  contains information about the convexity of Bernoulli index and hence facilitate inference about cross-asset elasticities (24). Lemma 14 (in the Appendix) demonstrates that for all  $s \in \hat{S}_-$  and  $w \in (\hat{w}_s, \hat{w}_{s+1})$ , the convexity of the Bernoulli index cannot be smaller than

$$\bar{u}''_- \equiv \frac{\bar{u}'_-(w) - \hat{p}_s^{Arrow}/\pi_s}{w - \hat{w}_s}, \quad (28)$$

and larger than

$$\bar{u}''_+ \equiv \frac{\hat{p}_{s+1}^{Arrow}/\pi_{s+1} - \bar{u}'_-(w)}{\hat{w}_{s+1} - w}. \quad (29)$$

Functions (28) and (29) in Example 6 are depicted in Figure 7. Observe that in scenario (b) for  $w \in (\hat{w}_1, \hat{w}_{\hat{S}})$  the bounds exactly uncover the second derivative of the Bernoulli index.

Equations (28) and (29) give restrictions on the expected risk aversion

$$E[\bar{u}''_-(w)] \leq \bar{\alpha}_2(w) \leq E[\bar{u}''_+(w)], \quad (30)$$

and altered measure  $\bar{\pi}(w)$ . Relative to objective distribution  $\pi$ , the states characterized by

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<sup>13</sup>Observe that  $\bar{u}'_-$  is not convex; for various values of  $w$ , the bound is approached by different convex marginal utilities. Thus, for a given experiment  $(a, q)$ , the precision can be increased by minimizing counterfactual price over the set of all convex marginal utilities bounded from below by  $\bar{u}'_-$ . The upper bound can be arbitrarily closely approximated by one smooth convex marginal utility and hence the upper bound gives tight predictions regarding prices and surplus.

smaller wealth get probability premium. For any  $s > s'$ , probability distortion satisfies

$$\frac{\bar{u}''_-(w_{s'})}{\bar{u}''_+(w_s)} \frac{\pi_s}{\pi_{s'}} \leq \frac{\bar{\pi}_s(w)}{\bar{\pi}_{s'}(w)} \leq \frac{\bar{u}''_-(w_{s'})}{\bar{u}''_+(w_s)} \frac{\pi_s}{\pi_{s'}}, \quad (31)$$

Equations (30) and (31) determine the ranges of consistent cross-asset price elasticities. As with the price levels, in quadratic economies, the bounds coincide and elasticities are determined with perfect accuracy. The price impacts inferred from market level data in Example 6 are depicted in Figure 8.

## 6 Discussion

This paper develops a method to determine price and welfare effects resulting from introducing new assets that are in non-negligible supply. The paper contributes to two strands of literature; 1) the literature on asset pricing in perfectly competitive (complete) markets, and, the literature on product differentiation. We now discuss each of the two contributions to the corresponding literatures.

### 6.1 Arrow Pricing Approach

In the paper we depart from the classic Arrow-Debreu methodology to price assets, by allowing the investors' consumption and hence Arrow prices to change in a counterfactual experiment. For such experiments, the values of marginal utility (and hence Arrow prices) state-by-state are insufficient for the post-experiment prices and surplus. In order to make predictions, one must to develop a model of how Arrow prices will be affected by new financial products. The pricing method from this paper not only utilizes the information contained in levels but also in the differences of marginal utility across states, which, jointly with the information about the distribution of aggregate wealth  $\hat{w}$ , makes it possible to uncover a shape of a marginal Bernoulli index  $\bar{u}'$  for an entire economy. How closely the index can be approximated depends on the coarsnes of the state space and the strenght of *a priori* assumptions on preferences. The Bernoulli index then can be used to derive demand system for existing assets,  $\hat{a}$ , which gives prices for arbitrary positions of the issuer.

The extension of the demand system  $p^{\hat{a}}(\cdot)$  to include non-existing assets relies on the following logic. Since, by assumption, original financial structure  $\hat{a}$  is complete, for any supply of a new assets  $q^{IPO}$ , there exists a change in portfolio of original assets,  $\Delta q$ , that gives rise to precisely the same promised payment as new IPO, i.e., it satisfies  $a^{New} \cdot q^{New} = a \cdot \Delta q$ . Thus, new asset has the same impact on prices and surplus as a change in supply of existing assets  $\Delta q$ . The prices of existing assets  $a$  in counterfactual experiment are  $p^{\hat{a}}(\bar{q} + \Delta q)$ . From the latter prices, one can infer counterfactual Arrow prices as well as prices of newly created assets.

The method developed in this paper is less universal than the standard Arrow pricing approach. The method requires that an economy with heterogenous traders can be represented by a fictitious agent whose preferences over aggregate wealth are von-Neumann-Morgenstern. The existence of such representative agent rules out financial markets with non-negligible income effects. Thus contrary to asset pricing approach based on constant Arrow prices the method developed in this paper is restricted to settings with quasilinear utilities.

## 6.2 Product Differentiation

The paper also contributes to the theoretical literature on product differentiation, that dates back to a seminal work by Hotelling (1929) and that has been further developed by Lancaster (1971) and McFadden (1974).<sup>14</sup> This strand of literature relies on a characteristics model (also called a hedonic or an addresses model), in which buyers' preferences are defined over *bundles of product characteristics* rather than directly over a product space.<sup>15</sup> The advantage of the characteristic over the traditional product-space approach is that it facilitates analysis of the price and welfare effects for products that are not yet marketed. Observe that in the model of financial monopoly each asset  $a_k$  is completely described by  $S$  attributes,  $[a_k] \equiv ([a_k]_1, \dots, [a_k]_S)$ , its payoffs in  $S$  states. (The results from this paper hold under the assumption that the asset payoffs are non-negative.) Investors' utility is derived from consumption of the attributes given by sum of payoff of all assets and initial endowment within each  $s$ . Thus the model of financial monopoly has a structure of a characteristic model and financial markets can be interpreted as differentiated product markets.

The literature on product differentiation has predominantly focused on markets for consumption goods. In such markets individual buyers typically buy one unit of a selected variant of a product. In terms of notation from this paper, the literature restricts the choice of individual trades  $t^i$  to a finite set  $\mathcal{T}^{DC} = \{0\} \cup \{1_k\}_{k=1}^K \subset \mathbb{R}^K$ , where  $1_k = (0, \dots, 1, \dots, 0)$  are unit vectors with one in the  $k^{th}$  position.<sup>16</sup> The *discrete choice* assumption is clearly too restrictive in the context of financial markets. Financial investors can simultaneously hold arbitrary quantities of various assets and they can sell assets short (i.e., components of  $t^i$  can be negative). This paper contributes to the literature on product differentiation by characterizing demand systems

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<sup>14</sup>For the review of the theoretical literature on product differentiation see e.g., Anderson Palma and Thisse (1992).

<sup>15</sup>Despite the theoretical advantages, a pure characteristics model is less popular in applied work. The latter relies on framework that combines characteristics and tastes-for-product components in preferences. This is due to practical considerations. Contrary to a pure characteristics framework, models with logistically distributed product-specific shocks define non-zero purchase probabilities for all values of parameters. Moreover, the inverse functions of market shares are linear functions of unobservable characteristics, which allows to solve the problem of endogeneity. For the estimation technics of pure characteristics models see Berry and Pakes (2007).

<sup>16</sup>In some discrete choice models, buyers are allowed to chose arbitrary quantity of a selected type of a product, i.e.,  $D = \{t^i \in \mathbb{R}^K | t^i = \lambda e_k \text{ for some } \lambda \geq 0 \text{ and } k = 1, \dots, K\}$ .

in a characteristics model, in which choice sets of individual traders are maximal, i.e.,  $\mathcal{T} = \mathbb{R}^K$ . The results from this critically rely on the assumption of unrestricted choice set.

The literature based on a characteristic model assumes that buyers initial endowments are zero  $e^i = 0$ , and it captures heterogeneity of preferences over bundles of attributes  $[c_1^i] \equiv ([c_1^i]_1, \dots, [c_1^i]_S)$  through random buyer-specific ideal points. In particular, utility from any bundle of characteristics  $[c_1^i]$  is decreasing in the distance between vector of characteristics  $[c_1^i]$  and the buyers most preferred location  $[\bar{c}_1^i]$ . The functional form most commonly used in the literature is given by

$$U^i(c_0^i, c_1^i) = c_0^i - 0.5\tau \|[c_1^i] - [\bar{c}_1^i]\|^2 \quad (32)$$

where  $\|\cdot\|$  is the Euclidean distance and  $\tau > 0$ .<sup>17</sup> The ideal points  $[\bar{c}_1^i]$  are distributed over  $\mathbb{R}^K$  according to some non-negative continuous density function. In the model of financial monopoly preferences (32) can be obtained by assuming quadratic utility function (see Section 3) with parameters set to  $\alpha_0^i = \alpha_1^i = 0$ ,  $\alpha_2^i = \tau S$ , ideal locations  $\bar{c}_1^i = -e^i$  and uniform probability distribution  $\pi$ . The model of financial monopoly with choice set  $\mathcal{T}$  remains tractable under general assumptions. The primitives of the economy are summarized by a Bernoulli index and the model gives sharp predictions in economies with arbitrary heterogenous, non-linear (Bernoulli) utility functions and with arbitrary empirical distribution  $F$ , possibly with atoms.

The model of financial monopoly is not directly applicable to markets of consumption goods: buyers of such commodities cannot “sell” the products “short”. We now introduce additional structure on the product space, under which short-selling constraints are not binding, and the model will characterize the demand system for divisible consumption goods, in which agents can simultaneously buy several types of products.

Consider markets in which buyers’ choice sets of  $K$  products are given by  $\mathcal{T}_+ = \mathbb{R}_+^K$ . As in the case of financial assets, each product is a bundle of  $S$  characteristics, with some abuse of notation characterized by a vector  $a_k \in \mathbb{R}_+^S$ . Trading position of a monopoly is strictly positive  $q \in \mathbb{R}_{++}^K$ . The demand side of the market is modeled as follows. For any bundle of characteristics  $c_1^i \in \mathbb{R}_{++}^S$  and money  $c_0^i$  their utility functions are quasilinear

$$U^i(c_0^i, c_1^i) = c_0^i + \sum_{s \in S} \pi_s u^i(c_{1,s}^i).$$

In the model of differentiated products vector  $\pi = (\pi_1, \dots, \pi_S)$  gives weights assigned to each attribute. Buyer-specific utilities  $u^i(\cdot)$  satisfy Assumption 1 and, in order to assure non-negative consumption, two Inada conditions  $\lim_{c_s \rightarrow 0} u^i(c_s) = \infty$  and  $\lim_{c_s \rightarrow \infty} u^i(c_s) = 0$ . Each buyer’s choice of  $K$  products is restricted to non-negative trades  $\mathcal{T}_+ = \mathbb{R}_+^K$ . Finally endowments of the agents are normalized to zero, so that consumption of all characteristics by agent  $i$  is given by  $c_1^i = a \cdot t^i$ .

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<sup>17</sup>Give examples.

We now provide sufficient conditions on the existing product structure  $\hat{a}$  under which short-selling constraints are not binding. We assume that  $K$  existing products are divided into  $S$  categories indexed by  $s$ . If product  $k$  is of type  $s$ , its attributes are given by  $\hat{a}_k = 1_s + \gamma \tilde{a}_k$ , where  $\tilde{a}_k \in \mathbb{R}_+^S$  is a non-negative column vector and  $\gamma > 0$  is a small positive scalar. By construction type- $s$  products are key sources of attribute  $s$  in the market. Fix  $\tilde{a}_k$ . For any position of a monopoly  $q \in \mathbb{R}_{++}^K$  there exists  $\gamma$

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## 7 Appendix

*Proof of Lemma 2:* Problem  $\bar{U}(w) \equiv \max_{\{c_1^i\}_i} \int E(u^i(c_1^i)dF) : \int_i c_1^i dF \leq w$  can be written as  $\bar{U}(w) \equiv \max_{\{c_{1,s}^i\}_{i,s}} \sum_{s \in \mathcal{S}} \pi_s \int u^i(c_{1,s}^i)dF$  subject to  $\int c_{1,s}^i dF \leq w_s$  for all  $s$ . Since inequality constraints are independent across states,  $\bar{U}(w) \equiv E\bar{u}(w)$ , where

$$\bar{u}(y) \equiv \max_{\{x^i\}_i} \int u^i(x^i)dF : \int x^i dF \leq y. \quad (33)$$

Observe that function  $\bar{u}$  does not depend on state and is uniquely defined.

Since by assumption, for any  $w \in \mathcal{W}$  investors’ preferences in optimum are non-satiated, consumption is implicitly defined by  $u^i(x^i) = \lambda$  and hence  $\lambda > 0$ . Let  $x^i(\lambda)$  be the optimal solution for any value of multiplier  $\lambda$ . From the Implicit Function Theorem,  $x^{i'}(\lambda) = 1/u^{i''} < 0$  and  $x^i(\lambda)$  is once continuously differentiable. By non-satiation, the inequality constraint is

binding and the Lagrangian multiplier is implicitly defined as a function of  $y$  by condition  $\int x^i(\lambda)dF = y$ . It follows that  $\lambda(y)$  is once-continuously differentiable and

$$\lambda'(y) = \left( \int (u'''(x^i))^{-1}dF \right)^{-1}. \quad (34)$$

Since  $\bar{u}$  is a value function of program (33), it follows that  $\bar{u}'(y) = \lambda(y) > 0$ , and hence index  $\bar{u}$  is increasing. Moreover

$$\bar{u}''(y) = \lambda'(y) = \left( \int (u'''(x^i))^{-1}dF \right)^{-1} = \mathcal{H}(u'''(x^i)) < 0, \quad (35)$$

where  $\mathcal{H}(\cdot)$  is a harmonic average of  $u'''$  across all  $i$ , and thus  $\bar{u}$  is strictly concave. Finally,  $\bar{u}$  is a composition of once-continuously differentiable functions  $u^i$ ,  $x^i(\cdot)$  and  $\lambda(\cdot)$ , and hence it is once-continuously differentiable as well. *Q.E.D.*

*Proof of Proposition 4:* Fix complete financial structure  $a$ , and monopoly's position,  $q$ . Let  $(\{t^{i*}\}_i, p^*)$ , denote a competitive equilibrium in economy  $(\{u^i, e^i\}_i, F)$ . Since markets are complete, the allocation  $\{c_1^{i*} \equiv a \cdot t^{i*} + e_i\}_i$  is Pareto-efficient. By the transferability of utility, the competitive allocation in the second period solves

$$\max_{\{c_1^i\}_i} \int E(u^i(c_1^i)dF) : \int c_1^i dF \leq a \cdot q + \bar{e}. \quad (36)$$

By strict concavity of the objective function, allocation is unique. By the standard argument, Lagrangian multiplier for each  $s$  (that correspond to Arrow price in state  $s$  is given by the slope of value function  $\bar{U}(\cdot)$  at  $w_s$ , which, by Lemma (2) is given by

$$\lambda_s = \pi_s \bar{u}'((a \cdot q + \bar{e})_s). \quad (37)$$

The (necessary and sufficient) first-order condition for program (36) requires that for each investor  $i$ ,

$$\pi_s u^i(c_1^{i*}) = \pi_s \bar{u}'((a \cdot q + \bar{e})_s). \quad (38)$$

The optimality of trade  $t^{i*} = \operatorname{argmax}_{t^i \in \mathbb{R}^K} U^i(-p^* \cdot t^i, e^i + a \cdot t^i)$  requires  $p_k^* = E(a_k u^i(c_1^{i*}))$ . Replacing marginal utility from equation (38) it follows that  $p_k^* = E(a_k \bar{u}'(a \cdot q + \bar{e}))$ , which gives demand system (36). *Q.E.D.*

*Proof of Lemma 7:* We derive  $\bar{u}$  in closed form. From the definition of Bernoulli utility function (33) the first-order (necessary and sufficient) condition requires that  $\lambda = \alpha_1^i - \alpha_2^i x^i$ , where  $\lambda$  is a Lagrangian multiplier of program 33. It follows that

$$x^i = \frac{\alpha_1^i}{\alpha_2^i} - \frac{1}{\alpha_2^i} \lambda. \quad (39)$$

Integrating consumption over investors gives

$$\lambda = \left( \int (\alpha_2^i)^{-1} dF \right)^{-1} \int \frac{\alpha_1^i}{\alpha_2^i} dF - \left( \int (\alpha_2^i)^{-1} dF \right)^{-1} y. \quad (40)$$

Optimal consumption is given by  $x^i = \gamma_0^i + \frac{\bar{\alpha}_2}{\alpha_2^i} y$  where

$$\gamma_0^i \equiv \frac{\alpha_1^i}{\alpha_2^i} - \frac{1}{\alpha_2^i} \left[ \int (\alpha_2^i)^{-1} dF \right]^{-1} \int \alpha_1^i (\alpha_2^i)^{-1} dF, \quad (41)$$

and

$$\bar{\alpha}_2 \equiv \left[ \int (\alpha_2^i)^{-1} dF \right]^{-1}. \quad (42)$$

By plugging efficient consumption into definition (33) and using the fact that  $\int \gamma_0^i dF = 0$  one obtains

$$\begin{aligned} \bar{u}(y) &= \int \left( \alpha_0^i - 0.5 \alpha_2^i (\gamma_0^i)^2 \right) dF + \bar{\alpha}_2 \int \frac{\alpha_1^i}{\alpha_2^i} dF w_s - 0.5 \bar{\alpha}_2 w_s^2 \\ &= \bar{\alpha}_0 + \bar{\alpha}_1 y - 0.5 \bar{\alpha}_2 y^2, \end{aligned} \quad (43)$$

where  $\bar{\alpha}_2$  is defined by equation (42) and two other coefficients are given by  $\bar{\alpha}_1 \equiv \bar{\alpha}_2 \int \frac{\alpha_1^i}{\alpha_2^i} dF$  and  $\bar{\alpha}_0 \equiv \int (\alpha_0^i + \alpha_1^i \gamma_0^i - 0.5 \alpha_2^i \gamma_0^i{}^2) dF$ . *Q.E.D.*

*Proof of Proposition 9:* In terms of primitives, the functional form of a counterfactual demand (16) straightforwardly follows from Lemmas 4 and 7. To pin down counterfactual demand, it suffices to recover parameters  $(\bar{\alpha}_1, \bar{\alpha}_2)$  from equilibrium data  $(\hat{q}, \hat{p}, \hat{a})$ . By no arbitrage, equilibrium prices  $\hat{p}$  make it possible to derive Arrow prices that take the form

$$\hat{p}_s^{Arrow} = \pi_s \bar{u}'(w_s). \quad (44)$$

Given functional form  $\bar{u}$ , condition (4) can be written as

$$\hat{p}_s^{Arrow} / \pi_s = \bar{\alpha}_1 - \bar{\alpha}_2 \hat{w}_s. \quad (45)$$

Generically in  $\hat{q}$ ,  $\hat{w} \equiv a \cdot \hat{q} + \bar{e}$  is random, i.e., there exist two or more states,  $s, s'$ , such that  $\hat{w}_s \neq \hat{w}_{s'}$  and condition (45) give two independent equations that determine  $(\bar{\alpha}_1, \bar{\alpha}_2)$ . *Q.E.D.*

*Proof of Corollary 10:* Given equilibrium data  $(\hat{q}, \hat{p}, \hat{a})$ , the Arrow price for state  $s$  is  $\pi_s \bar{u}'_+(w_s)$  and the minimal one is  $\pi_s \bar{u}'_-(w_s)$ . Thus, the maximal price of asset  $a_k$  cannot exceed the one for which maximal Arrow price is assumed for states in which  $a_{k,s} > 0$  and the minimal for  $a_{k,s} > 0$ . The lower bound is determined in analogous way. *Q.E.D.*

*Proof of Lemma 12:* Consider program (33). By steps of Lemma 12,

$$\bar{u}''(y) = \lambda'(y) = \left( \int (u^{i''}(x^i(\cdot)))^{-1} dF \right)^{-1} = \mathcal{H}(u^{i''}(x^i)), \quad (46)$$

where  $\mathcal{H}(\cdot)$  is a harmonic average of  $u^{i''}$  across all  $i$  at consumption  $x^i(\lambda(y))$ . Since  $\lambda'(y) < 0$  and  $x^{i''}(\lambda) < 0$ , it follows that the consumption of each individual is increasing in  $y$ . Moreover, harmonic average is increasing in all its arguments and, by assumption  $u^{i''} \geq 0$ , it follows that  $\bar{u}'' \geq 0$ . *Q.E.D.*

*Proof of Proposition 11:* Divide  $(\hat{w}_1, \hat{w}_{\hat{s}}) \subset R_+$  into  $G > 1$  disjoint right-open intervals of equal length  $(\hat{w}_1 - \hat{w}_{\hat{s}})/G$ . Each interval  $I_g$  is indexed by  $g = 1, 2, \dots, G$ , and the midpoint of  $I_g$  is denoted by  $i_g$ .

**Lemma 13.** *For any grid  $G > 0$ , there exists  $\bar{n} \geq 1$  such that in all discretized economies  $n \geq \bar{n}$ , for each interval  $\{I_g\}_{g=1}^G$ , there exists state for which realization of wealth  $w_s^n \in I_g$ .*

*Proof of Lemma 13:* Let  $\hat{w}(x) \equiv \hat{q} \cdot a(x) + \int e^i(x) dF$ . Since  $a(x)$ ,  $e^i(x)$  are continuous, wealth function  $\hat{w}(x)$  is continuous in  $x$  as well. The image of a closed cube  $\mathcal{S}^\infty = [0, 1]^X$  by any continuous function is a closed interval (by Intermediate Value Theorem), and hence  $w(\mathcal{S}^\infty) = [\hat{w}_1, \hat{w}_{\hat{s}}]$ . It follows that for any  $g = 1, \dots, G$ , there exists  $x_g \in \mathcal{S}^\infty$  for which  $\hat{w}(x_g) = i_g$ . By continuity of  $\hat{w}(\cdot)$ , for each  $x_g$  there exists  $\delta_g > 0$  such that for all  $x \in B(x_g, \delta)$ , the value of a function is in the interval  $\hat{w}(x) \in I_g$ . For partition  $n$ , let  $C_s^n(x_g)$  denote the sub-cube in the partition that contains  $x_g$ . Since the maximal distance between any two points within each cube is bounded by  $X/n$ , there exists  $\bar{n}_g$  so that  $C_s^n(x_g) \subset B(x_g, \delta)$  for all  $n \geq \bar{n}_g$ . Define  $\bar{n} \equiv \max \{\bar{n}_g\}_{g=1}^G$ . Since  $\hat{w}(x) \in I_g$  for all  $x \in C_s^n(x_g)$ , conditional expectations  $w_s^n = \int_{x \in C_s^n(x_g)} \hat{w}(x) \pi^\infty(x) dx \in I_g$  for all  $g = 1, \dots, G$ . *Q.E.D.*

Let  $\bar{n}(G)$  denote the smallest number that satisfies premise of Lemma 13. Consider any  $w \in (\hat{w}_1, \hat{w}_{\hat{s}})$ . By construction, the value of the upper bound on marginal utility at  $w$  cannot be greater than  $\bar{u}_+^n(w) \leq \hat{p}_s^{Arrow} / \pi_s$ , where the state  $s$  corresponds to the maximal realization of wealth  $\hat{w}_s^n$  that is smaller than  $w$ . Given  $n \geq \bar{n}(G)$ , the distance  $|w - \hat{w}_s^n|$  is bounded by  $2(\hat{w}_1 - \hat{w}_{\hat{s}})/G$ . Since  $\bar{u}(\hat{w}_s^n) = \hat{p}_s^{Arrow} / \pi_s$ , the gap between the bound and the actual value of marginal utility at  $w$  is

$$\bar{u}_+^n(w) - \bar{u}'(w) = \bar{u}'(\hat{w}_s^n) - \bar{u}'(w). \quad (47)$$

By continuity of  $\bar{u}$ , the right-hand side can be made arbitrarily small by properly choosing  $\bar{n}$  (and hence corresponding grid  $G$ .) Thus, on the open interval  $(\hat{w}_1, \hat{w}_{\hat{s}})$ , the upper bound converges to the true value  $\bar{u}_+^n \rightarrow \bar{u}'$ . By the symmetric argument, the lower bound pointwise converges to true value  $\bar{u}_-^n \rightarrow \bar{u}'$ . *Q.E.D.*

**Lemma 14.** *Equations (28) and (29) respectively provide lower and upper bound on convexity of Bernoulli index,  $\bar{u}''$ , .*

*Proof of Lemma 13:* The proof uses the following property (Property 1). Consider  $w' > w$  and convex marginal index  $\bar{u}'$ . Let  $L$  be a line segment connecting points  $(w, \bar{u}'(w))$  and  $(w', \bar{u}'(w'))$ . The slope of line segment is denote by  $\Delta L \equiv (\bar{u}'(w') - \bar{u}'(w)) / (w' - w)$ . By mean-value theorem, there exists  $w'' \in [w, w']$  for which  $\bar{u}''(w'') = \Delta L$ . By convexity of  $\bar{u}'$  it follows that  $\bar{u}''(w') \geq \Delta L$  and  $\bar{u}''(w) \leq \Delta L$ . Observe that one implication of Property 1 is that for  $s \in \hat{\mathcal{S}} / \{1, S\}$  and for any  $w \in [\hat{w}_s, \hat{w}_{s+1}]$ ,

$$\bar{u}''(w) \in [\Delta \hat{\delta}_{s-1}, \Delta \hat{\delta}_{s+1}]. \quad (48)$$

Equation (28): For any  $s \in \hat{\mathcal{S}}_-$  and  $w \in (\hat{w}_s, \hat{w}_{s+1}]$ , equation (28) gives the slope of line segment,  $L$ , connecting  $\hat{\delta}_s$  with the corresponding point on the lower bound of marginal Bernoulli index  $(w, \bar{u}'_-(w))$ .

$$\Delta L = \frac{\bar{u}'_-(w) - \hat{p}_s^{Arrow} / \pi_s}{w - \hat{w}_s} \quad (49)$$

On the interval  $[\hat{w}_s, \hat{w}_{s+1}]$  function  $\bar{u}'_-$  is composed of two line segments: the extension of the segment passing through points  $\{\hat{\delta}_{s-1}, \hat{\delta}_s\}$  (the *left part*) and the extension the segment passing through points  $\{\hat{\delta}_{s+1}, \hat{\delta}_{s+2}\}$  (the *right part*). The two parts are depicted in Figure 9. By  $\bar{w} \in (\hat{w}_s, \hat{w}_{s+1})$  denote the level of wealth that separates bound  $\bar{u}'_-$  into the two parts.

We consider three cases that correspond to Figures 9 b,c, and d.

1. For  $w \in (\hat{w}_s, \bar{w}]$  line segment  $L$  is located on the left part of bound and thus its slope coincides with  $\Delta \hat{\delta}_{s-1}$ . By equation (48) threshold  $\Delta \hat{\delta}_{s-1}$  is a lower bound for the convexity of Bernoulli index for all  $w \in (\hat{w}_s, \bar{w}]$ .

2. Given  $w \in (\bar{w}, \hat{w}_{s+1}]$ , let  $w' \in (\bar{w}, w)$ . We show that the slope of  $L$  is the lower bound for  $u''(w)$ . Consider consistent function  $\bar{u}'$  for which point  $(w', u'(w'))$  is located weakly above line segment  $L$ . Let  $L'$  be a segment connecting  $\hat{\delta}_s$  with  $(w', u'(w'))$ . Since  $L$  and  $L'$  share end-point  $\hat{\delta}_s$ , the slope  $\Delta L$ , gives the lower bound for the slope  $\Delta L'$ . By construction  $\bar{u}'$  passes through two end-points of  $L'$ , and by Property 1 slope  $\Delta L'$  gives lower bound for  $u''$  for all levels of wealth above  $w'$ , in particular for  $w$ . It follows that

$$\frac{\bar{u}'_-(w) - \hat{p}_s^{Arrow} / \pi_s}{w - \hat{w}_s} = \Delta L \leq \Delta L' \leq u''(w') \leq u''(w), \quad (50)$$

and equation (28) gives lower bound for  $u''(w)$ .

3. Consider  $w \in (\bar{w}, \hat{w}_{s+1}]$  and a function for which there exists  $w' \in (\bar{w}, w)$  such that  $(w', u'(w'))$  is located below  $L$ . Consider line segment  $L''$  that connects points  $(w', u'(w'))$  and  $(w, u'(w))$ . Since  $u'(w) \geq u'(w')$ , the slope  $\Delta L''$  is bounded from below by the slope  $\Delta L$ . By Property 1, slope  $\Delta L''$  gives lower bound for  $u''(w)$ . Thus

$$\frac{\bar{u}'_-(w) - \hat{p}_s^{Arrow} / \pi_s}{w - \hat{w}_s} = \Delta L \leq \Delta L'' \leq u''(w), \quad (51)$$

and again equation (28) gives lower bound for  $u''(w)$ .

Equation (29): For any  $s \in \hat{\mathcal{S}}_-$  and  $w \in [\hat{w}_s, \hat{w}_{s+1})$ , equation (28) gives the slope of line segment,  $L$ , connecting  $(w, \bar{u}'_-(w))$  with  $\hat{\delta}_{s+1}$ .

$$\Delta L = \frac{\hat{p}_{s+1}^{Arrow}/\pi_{s+1} - \bar{u}'_-(w)}{\hat{w}_{s+1} - w} \quad (52)$$

1. For  $w \in [\bar{w}, \hat{w}_{s+1})$ , line segment  $L$  is located on the right part of bound and thus its slope coincides with  $\Delta\hat{\delta}_{s+1}$ . By equation (48) threshold  $\Delta\hat{\delta}_{s+1}$  is the upper bound for the convexity of Bernoulli index for all  $w \in [\bar{w}, \hat{w}_{s+1})$ .

2. Given  $w \in [\hat{w}_s, \bar{w})$ , let  $w' \in (\bar{w}, w)$ . We show that the slope of segment  $L$  is the lower bound for  $u''(w)$ . Consider consistent function  $\bar{u}'$  for which point  $(w', u'(w'))$  is located weakly above line segment  $L$ . Let  $L'$  be a segment connecting  $\hat{\delta}_{s+1}$  with  $(w', u'(w'))$ . Since  $L$  and  $L'$  share end-point  $\hat{\delta}_{s+1}$ , the slope  $\Delta L$ , gives the upper bound for the slope  $\Delta L'$ . By construction  $\bar{u}'$  passes through two end-points of  $L'$ , and by Property 1 slope  $\Delta L'$  gives upper bound for  $u''$  for all levels of wealth below  $w'$ , in particular for  $w$ . It follows that

$$\frac{\hat{p}_{s+1}^{Arrow}/\pi_{s+1} - \bar{u}'_-(w)}{\hat{w}_{s+1} - w} = \Delta L \geq \Delta L' \geq u''(w') \geq u''(w), \quad (53)$$

and equation (28) gives the lower bound for  $u''(w)$ .

3. Consider  $w \in (\bar{w}, \hat{w}_{s+1}]$  and a function  $\bar{u}'$  for which there exists  $w' \in (\bar{w}, w)$  such that  $(w', u'(w'))$  is located below  $L$ . Consider line segment  $L''$  that connects points  $(w', u'(w'))$  and  $(w, u'(w))$ . Since  $u'(w) \geq u'(w')$ , the slope  $\Delta L''$  is bounded from above by the slope  $\Delta L$ . By Property 1, slope  $\Delta L''$  gives upper bound for  $u''(w)$ . Thus

$$\frac{\hat{p}_{s+1}^{Arrow}/\pi_{s+1} - \bar{u}'_-(w)}{\hat{w}_{s+1} - w} = \Delta L \frac{\hat{p}_{s+1}^{Arrow}/\pi_{s+1} - \bar{u}'_-(w)}{\hat{w}_{s+1} - w} \geq \Delta L'' \geq u''(w) \quad (54)$$

and equation (28) gives the upper bound for  $u''(w)$ .

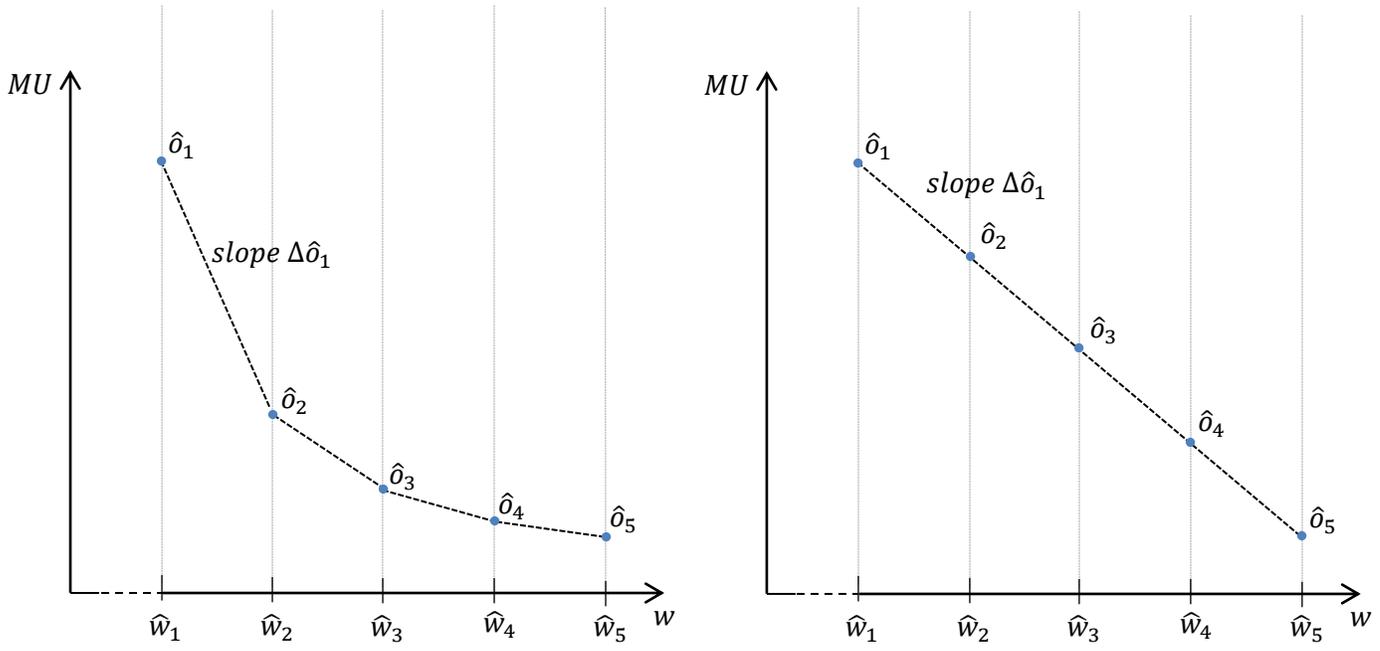


Figure 1. Equilibrium observations in example. Scenario (a) and (b).

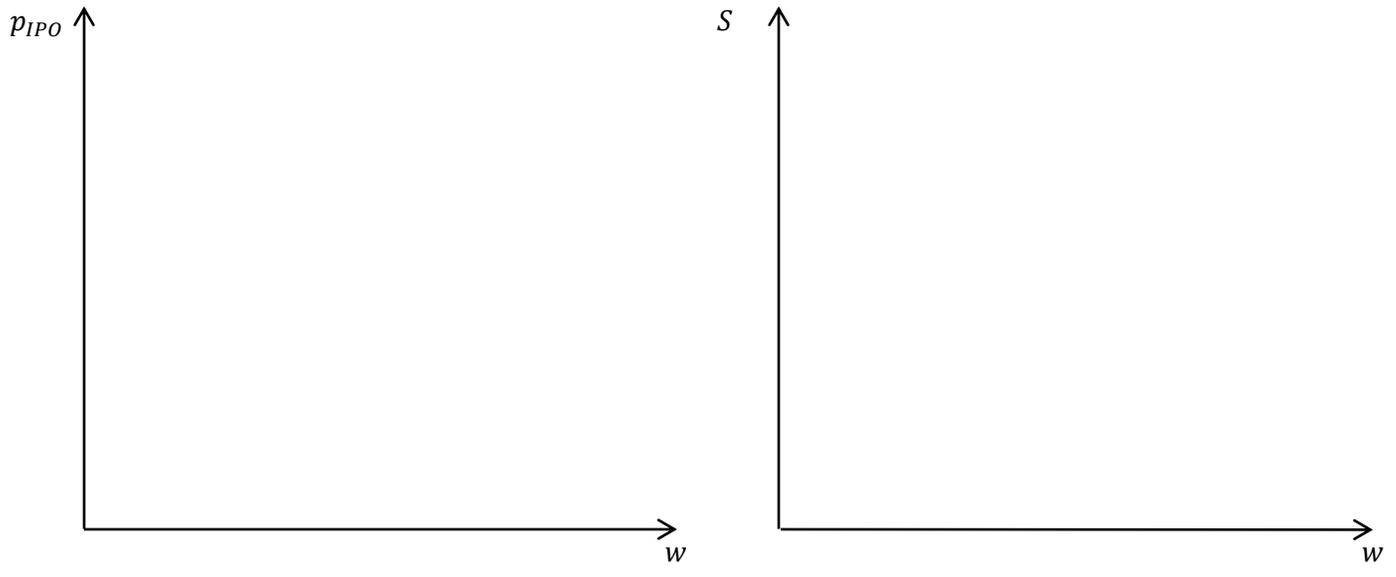


Figure 4. Predictions for prices and surplus

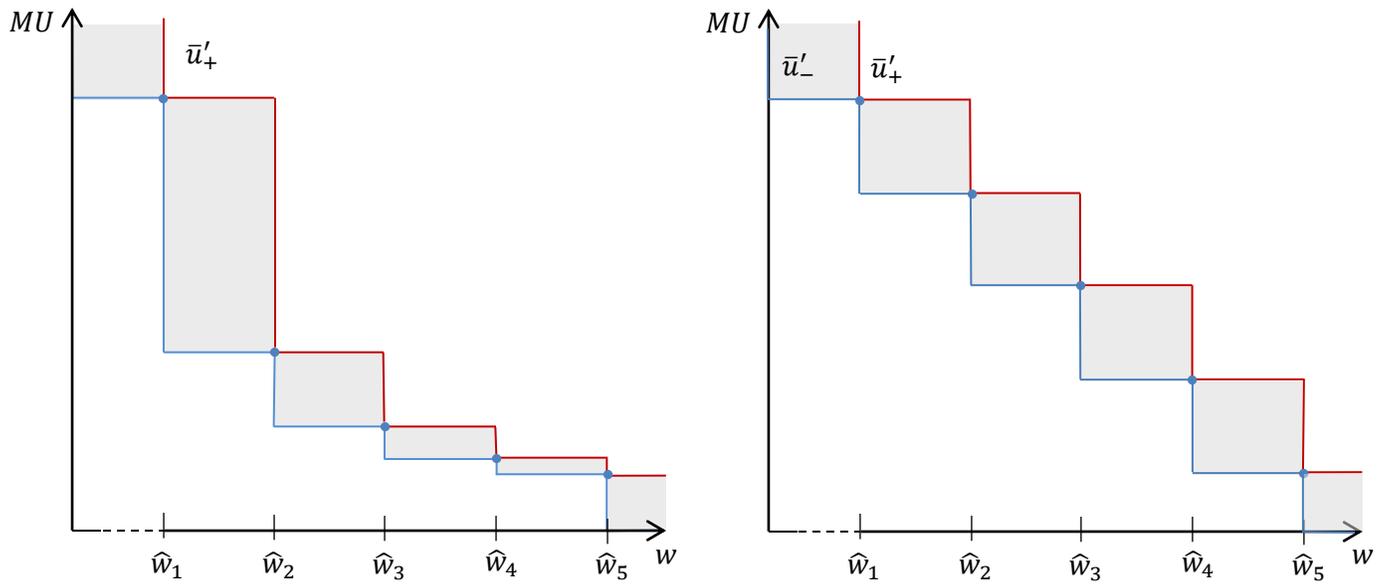


Figure 3. Bounds on marginal utility. Scenario (a) and (b).

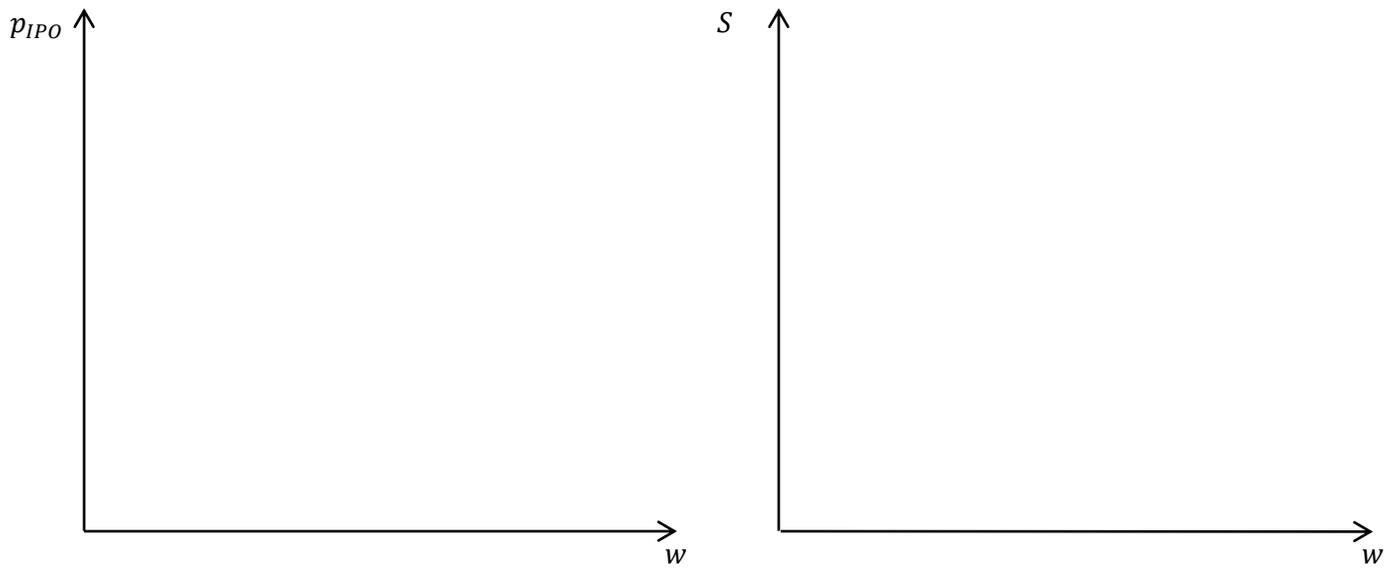


Figure 4. Predictions for prices and surplus

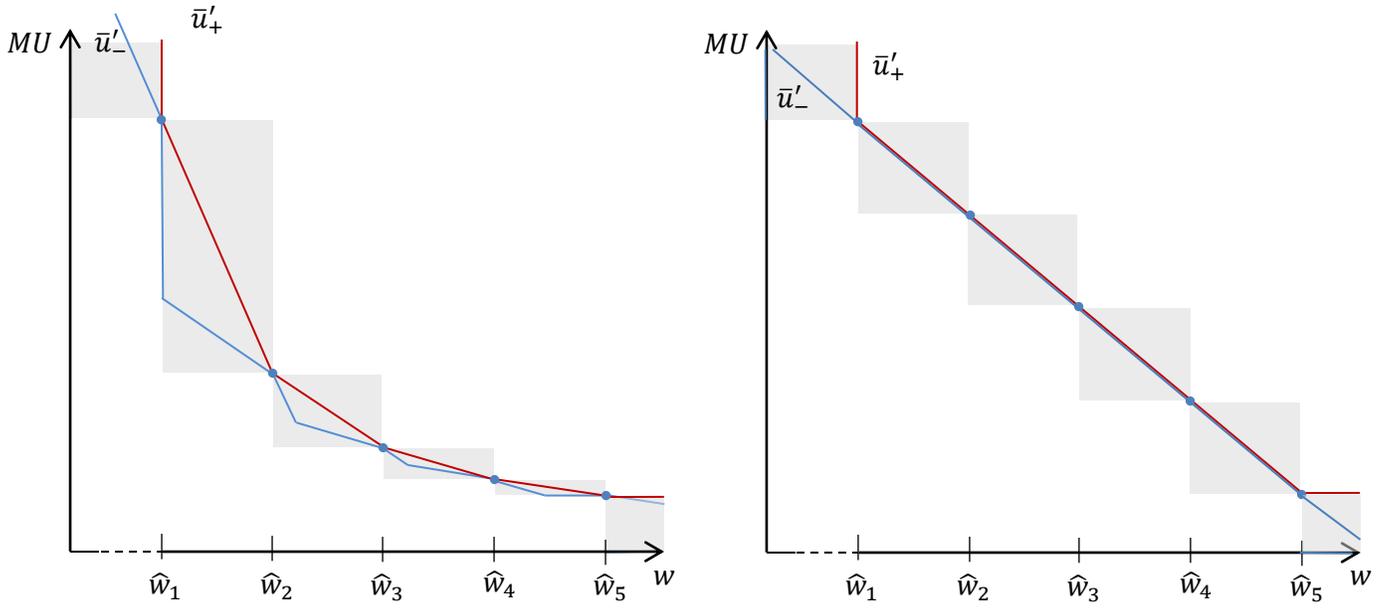


Figure 5. Bounds on marginal utility with convex marginal utility. Scenario (a) and (b).

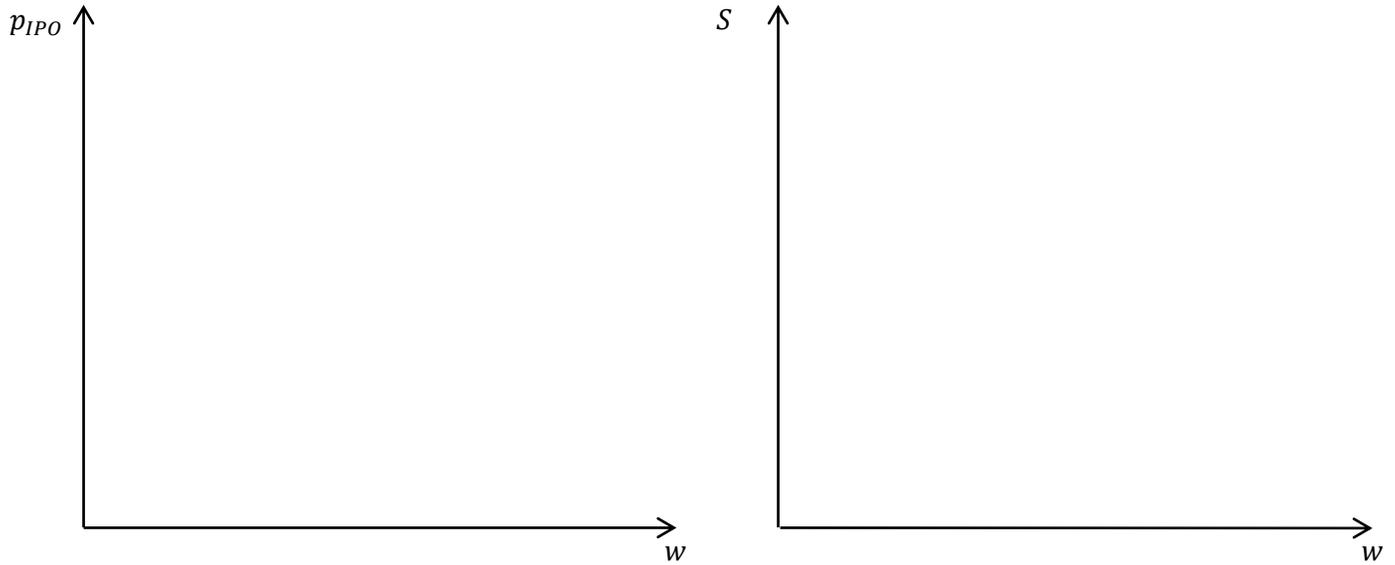


Figure 6. Predictions for prices and surplus

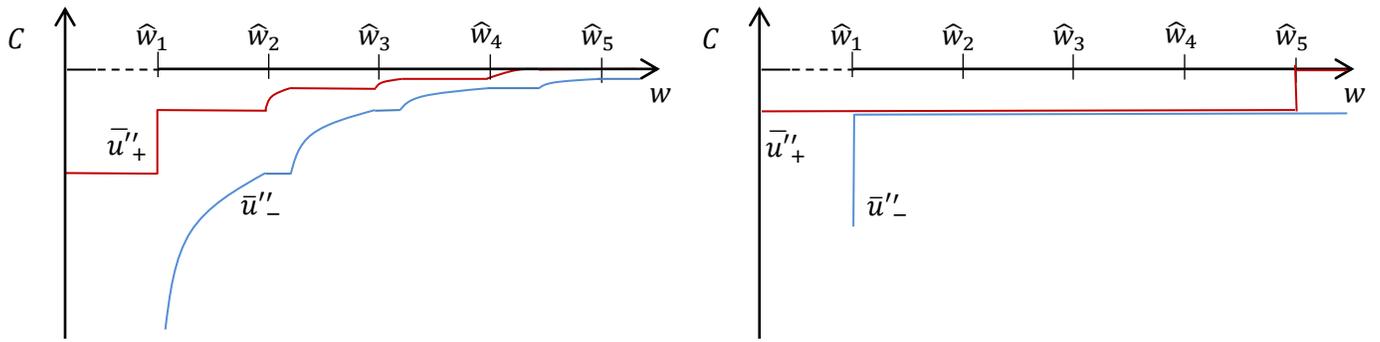


Figure 7. Bounds on convexity. Scenario (a) and (b).

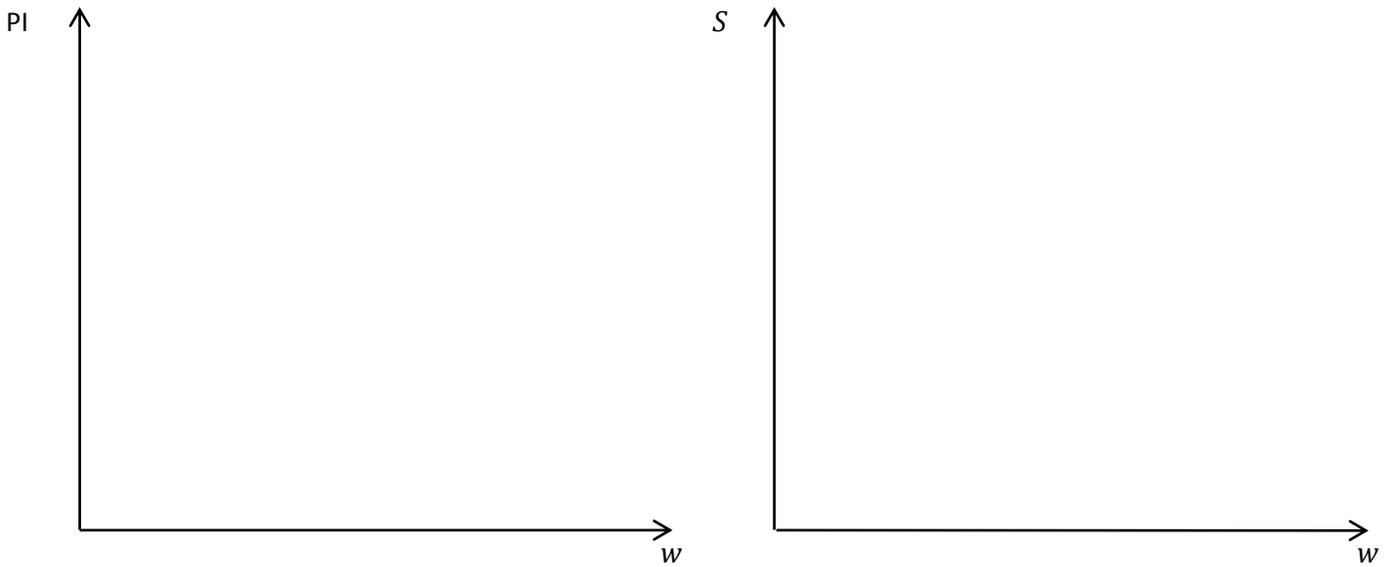


Figure 8. Predictions regarding price impacts

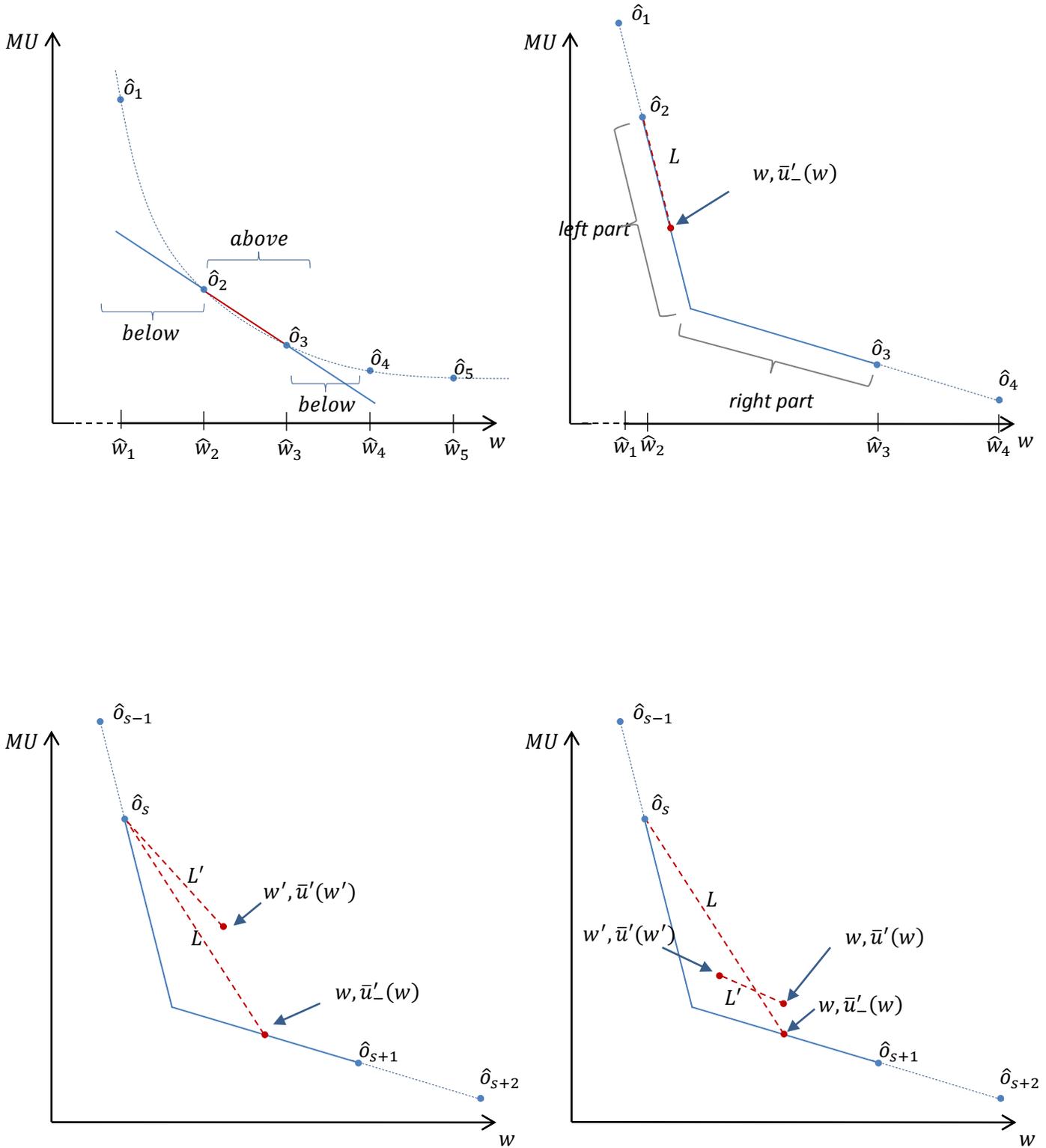


Figure 9. Proof of Lemma

Figure 2 Simulations Linear model

Figure 3: Construction of index bounds in a non-parametric Model

Figure 4 Simulations in a general non-parametric model.

Figure 5: Construction of index bounds in a non-parametric convex Model

Figure 6 Simulations in a general non-parametric model.

Figure 7 Bounds on convexity.

Figure 8 Simulations of price impacts.

Figure 9 a,b,c Construction of bounds.