Framework Agreements in Procurement: an Auction Model and Design Recommendations

Yonatan Gur, Lijian Lu, and Gabriel Y. Weintraub*
Columbia Business School
August 27, 2013

Abstract

Framework agreements (FAs) are procurement mechanisms commonly used by buying agencies around the world to satisfy demand that arises randomly over a certain time horizon. This paper is one of the first in the literature that provides a formal understanding of FAs, with a particular focus on the cost uncertainty faced by bidders over the FA time horizon. More specifically, we introduce a novel model that generalizes standard auction models to incorporate the complexities that arise in FAs. We analyze the model through a combination of theoretical and numerical results. Based on these results, we provide concrete design recommendations to decrease the buying prices in FAs, highlighting the importance of (i) investing in implementing price indexes for the random part of suppliers’ costs; and (ii) allowing the right form of flexibility in suppliers’ pricing over the time horizon. These prescriptions are already applied by the Chilean government procurement agency that buys US$1.9 billion worth of contracts yearly using FAs.

Keywords: Procurement, mechanism design, auctions

1 Introduction

Governments around the world spend billions of dollars every year buying a wide range of products and services from private firms. Auctions are often used to allocate contracts in these procurement processes. Another class of mechanisms that are a common alternative to standard auctions in public procurement settings, are the so-called framework agreements (FAs), also called indefinite-delivery/indefinite-quantity (IDIQ) contracts in the United States. FAs award tens of billions of dollars worth of contracts annually around the globe and constitute a steadily increasing fraction of governments’ procurement processes. For example, FAs awarded €85 billion in 2010 in the

*This paper is the result of a collaboration with the Chilean government procurement agency Dirección ChileCompra and was partially funded by this institution. We are grateful to its executives, specially to David Escobar and Guillermo Burr, for introducing us to this problem and for the valuable insights they shared with us. The opinions expressed here are those of the authors and do not necessarily reflect the positions of Dirección ChileCompra nor of its executives. This work was also supported by the Chazen Institute of International Business at Columbia Business School. We thank Ciamac Moallemi and seminar participants at Columbia Business School, Informs, MSOM, and POMS for useful comments. Correspondence: ygur14@gsb.columbia.edu, llu16@gsb.columbia.edu, gweintraub@columbia.edu.
European Union only, accounting for 17% of the total value of all contracts awarded, and their use has increased in the EU at an average rate of 18% since 2006.\footnote{Data from “Public Procurement in Europe,” http://ec.europa.eu/internal_market/publicprocurement/docs/modernising_rules/cost-effectiveness_en.pdf.}

Broadly speaking, FAs are anticipated arrangements for the delivery of goods and services over a certain period of time that provide more flexibility to the auctioneer regarding the exact mix of products and services that will be bought, compared to standard auctions.\footnote{The European Parliament defined FAs as “an agreement between one or more contracting authorities and one or more economic operators, the purpose of which is to establish the terms governing contracts to be awarded during a given period, in particular with regard to price and, where appropriate, the quantity envisaged”. (The Directive 2004/18/EC of the European Parliament and of the council of March 31, 2004.)} There are important characteristics of FAs that intrigue practitioners and this paper is one of the first in the literature that provides a more formal understanding of them. More specifically, in this work we introduce a novel auction model for FAs and use it to propose concrete design recommendations. This work is the result of a collaboration with the Chilean government procurement agency Dirección ChileCompra (ChileCompra for short) that buys around 9.1 billion dollars worth of products and services every year, of which 1.9 billion is bought using FAs. (The left Table in Figure 1 provides some aggregate figures by buying sector.)

<table>
<thead>
<tr>
<th>Buying Sector</th>
<th>Volume (M$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central Government</td>
<td>224.1</td>
</tr>
<tr>
<td>Health System</td>
<td>249.2</td>
</tr>
<tr>
<td>Ministry of Education</td>
<td>214.7</td>
</tr>
<tr>
<td>Armed Forces</td>
<td>101.4</td>
</tr>
<tr>
<td>Municipalities</td>
<td>85.7</td>
</tr>
<tr>
<td>Public Works</td>
<td>24.1</td>
</tr>
<tr>
<td>Others</td>
<td>22.8</td>
</tr>
<tr>
<td>Overall</td>
<td>921.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Product Category</th>
<th>Volume (M$)</th>
<th>Number of Orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>Printers - Supplies and Services</td>
<td>35.4</td>
<td>46167</td>
</tr>
<tr>
<td>Office Supplies</td>
<td>55.3</td>
<td>130523</td>
</tr>
<tr>
<td>Communication and Media</td>
<td>46</td>
<td>16441</td>
</tr>
<tr>
<td>Gas and Liquefied Petroleum</td>
<td>24</td>
<td>25492</td>
</tr>
<tr>
<td>Courier and Transportation Services</td>
<td>15.1</td>
<td>8860</td>
</tr>
<tr>
<td>Accommodation and Hospitality</td>
<td>15.3</td>
<td>9728</td>
</tr>
<tr>
<td>Furniture, Mattresses, and Bedding</td>
<td>27.8</td>
<td>14256</td>
</tr>
<tr>
<td>Management and Consulting</td>
<td>136.4</td>
<td>3984</td>
</tr>
</tbody>
</table>

Figure 1: **Demand from FAs.** *(Left)* The overall order volume (in millions of dollars) by buying sector transacted through ChileCompra using FAs in the year 2010. *(Right)* The number of orders and overall order volume (in millions of dollars) transacted through ChileCompra using FAs in some selected product categories for the year 2010. *Source: Dirección ChileCompra.*

More specifically, suppose a government wants to buy computers for all of its public agencies for the next few years. At the beginning of the time horizon, the government does not know what the specific requests and demands from its agencies will be. On one hand, running one auction whenever a request arises is too expensive administratively, especially considering that these requests arise frequently and can be quite small (e.g., few laptop computers for a rural school); see also the right Table of Figure 1. On the other hand, letting each agency run its own procurement process does not exploit the central government’s bargaining power and buying know-how. FAs are meant to deal with these issues and strike a balance between a decentralized procurement process and the central government’s bargaining power, where providers typically have incentives to participate in
FAs due to the large demand they can potentially have access to.

A typical FA is composed of two stages: in the first stage, an auction-type mechanism is run to decide the potential providers of a set of products or services (e.g., computers) to the government. This mechanism results in FA winners, a set of providers required to sell over the time horizon of the FA (typically few years) at the prices determined at this first stage. Each FA winner can be assigned to provide one or more of the products/services in the FA. In the second stage, the different government agencies can buy products from the FA winners as needs arise over the established time horizon. It is common that government agencies have the obligation to buy from FA winners unless they can provide evidence of a more convenient outside option. Albano and Sparro (2010) provide a more detailed description of FAs.

As a procurement process, FAs exhibit distinctive attributes. Notably, providers face significant uncertainty when submitting their bids in the first stage: they do not know when, what, and how much they will sell over the time horizon specified. In particular, while typically the price of a product or service in an FA is locked at the beginning of the time horizon, the suppliers’ costs may change over that period. Anecdotal and empirical evidence suggests that providers “charge” for this uncertainty through higher bids. Guillermo Burr, head of the Research Department at ChileCompra, says “we wanted to enhance our understand of why in some categories standard auctions resulted in lower prices relative to FAs. In addition, in some cases, prices observed in the ‘general open market’ were more competitive than those in the FAs.”

Thus motivated, in this paper we develop a model for FAs that considers the cost uncertainty faced by suppliers and the resulting bidding incentives. Using the results of our model we provide concrete design recommendations to decrease the buying prices for governments; in particular, these prescriptions are already applied by ChileCompra to improve the design of their FAs. Moreover, despite their practical importance, there is surprisingly very little academic research on FAs. In fact, this is one of the very few academic papers that study FAs and the first that provides a formal understanding on how the uncertainty suppliers face affects market outcomes. Overall, we believe this work provides a contribution to the literature on procurement mechanisms in the interface between operations and economics, and at the same time, can have a concrete practical impact.

**Basic FA model.** The buyer proposes an FA mechanism that induces a game of incomplete information between sellers and we use Bayes Nash equilibrium (BNE) as our solution concept. In this way, we generalize the treatment of standard auction models to incorporate the complexities that arise in FAs. While our FA model is stylized, we think it captures the key features needed to understand the issues alluded in the motivation above.

In our model the buyer wants to buy one unit of a good at the end of a given time period. There is a set of risk neutral bidders or sellers that at the beginning of the time period participate in a first price auction to obtain the right of selling the good. There is one FA winner in this auction. Each bidder has a cost that is composed by the sum of two components. First, an i.i.d. private cost
that captures idiosyncratic differences among firms and relates to managerial ability, logistics, etc.

Second, a random common cost component that is realized only at the end of the time horizon. The common cost represents costs of inputs that all bidders need to buy to provide the good or service. It is random from the bidders’ perspective at the bidding stage to capture that this cost may change until the good is provided. At the end of the time period, the buyer has the option to buy from the FA at the agreed price given by the auction, or if it is more convenient, she can buy from an outside option given by the spot market. The spot market price has a similar structure to the bidders’ costs. Notably, the spot market price also incorporates the common cost component, as an outside provider may also need to pay this cost to provide the good.

**Summary of main results.** Our first result derives an integral equation that characterizes the BNE bids in the FA. The equation makes explicit the fact that bidders “charge” for the cost uncertainty, because the event in which the FA winner defeats the spot market is positively correlated with a large realization of the common cost component. This effect, that we call the FA catch, formalizes practitioners’ intuitions. The following results study how to alleviate the FA catch and, therefore, reduce buying prices.

A natural benchmark against which to compare the extent of the FA catch is a FA with a perfect price index (PPI) in which the auctioneer perfectly observes the realization of the common cost and indexes the bid of the FA winner to its changes. In this way, the buyer completely removes the cost uncertainty from bidders. First, we show that under broad conditions a PPI FA with an appropriately set reserve price is the optimal FA mechanism, so it minimizes the buyer’s expected buying price among all feasible mechanisms. In practice, governments generally do not optimize reserve prices to decrease buying prices. We also show that the PPI FA induces lower equilibrium bids and also a lower expected buying price compared to the basic FA in many settings of practical interest, even without a reserve price. The previous results suggest that using PPIs is useful; such price indexes are available in practice when the common cost is a commodity such as gas. However, for many of the goods and services procured with FAs (e.g., computers, office equipment, services) such perfect price indexes do not exist or are hard to build. Thus motivated, we next study how different practically relevant operational changes to the design of FAs impact and potentially reduce the expected buying price relative to the basic FA.

First, we study a variation of the basic FA, the flexible FA (FLE), in which an FA winner with an initial bid larger than the realized spot market price is allowed to match it and sell the product. We show that the flexible FA achieves the same expected buying price than the PPI FA, and therefore, it induces a smaller expected buying price compared to the basic FA in many settings of practical interest. Second, we study a variation of the FLE FA, the flexible out-of-stock FA (OOS), in which the FA winner declares himself out-of-stock to avoid loosing money in scenarios where he is supposed to sell the product for a large common cost realization. We show that even though BNE bids of the OOS FA could be lower than those of the basic FA, the expected buying price is
generally larger, because the FA winner declares himself out-of-stock and the buyer ends up buying from the spot market too often.

We compare between the various BNE bids by analyzing the integral equations or the ordinary differential equations that characterize the equilibria of the different models. Notably, we are able to provide sharp comparison results even though the equilibrium bids for the basic FA model and the variants we study (expect for PPI) do not admit closed-form solutions. We do the comparisons of the expected buying prices, that not only consider the equilibrium bids but also the outside option of the spot market and the additional payments in PPI, using an envelope theorem approach. In addition, we complement our theoretical results with numerical experiments.

**Design recommendations.** Our results suggest the following prescriptions for the design of FAs. First, if possible, governments should make an effort to invest in finding and implementing price indexes for the random common part of costs. Second, allowing flexibility in the design of FAs, for example, by letting FA winners match the spot market price can reduce buying prices. These conclusions are valid for many settings of practical interest. However, they need to be taken with some caution because in some instances, in particular when the number of bidders and/or the variance of the common cost are small, introducing a perfect price index or a flexible FA can actually increase buying prices. In addition, too much flexibility can also systematically hurt: not enforcing suppliers’ commitments generally increases buying prices and this should be avoided. These prescriptions are currently being considered by the Chilean government to improve their procurement processes. Mr. Burr says “These results have provided important insights regarding the design of our FAs. They have encourage our FA department to make larger efforts to build adequate price indexes. They have also showed us the types of flexibility that we should encourage and discourage in FAs.” We hope that these prescriptions will also be considered by other procurement agencies to improve their buying mechanisms.

**Related literature.** Our work relates to the growing stream of work in operations that studies procurement mechanisms. In particular, Chen (2007) and Duenyas et al. (2013) study optimal procurement mechanisms in a newsvendor-like setting where a buyer facing an uncertain demand finds both the quantity and purchasing price through interactions with suppliers. Li and Scheller-Wolf (2011) considers the problem of a buyer facing uncertain demand that sources from suppliers that need to invest in capacity before the demand uncertainty is resolved, and studies whether the buyer should offer a pull or push contract. Related to these papers is Zhang (2010) that also studies a procurement mechanism in a supply chain setting, but includes supplier delivery performance and price-sensitive market demand. In addition, Schummer and Vohra (2003) study the mechanism design problem of a buyer that can procure purchase options to satisfy an unknown future demand. In the economics literature, Klemperer and Meyer (1989) study how supply functions can aid firms in an oligopolistic setting with demand uncertainty.

All the previous papers are related in some way or another to the problem of dealing with the
demand uncertainty suppliers face over the time horizon in an FA. In our work we instead focus on
the cost uncertainty faced by suppliers an we found much less literature related to this problem. An
exception is Elmaghraby and Oh (2013) that studies, in a very different setting, how to structure
two sequential auctions in the presence of learning-by-doing and whether the buyer is better-off by
limiting competition and contracting with a single supplier in the hope of extracting a better future
price. These questions are at some level related to the issue of how to structure the competition
with the spot market price in the second stage of our FA model (e.g., whether to allow the flexibility
to match it or not). On another related work, Tunca and Zenios (2006) provides conditions under
which supply auctions are preferred to relational contracts, because similarly to FAs, they identify
the most cost efficient supplier.

To the best of our knowledge the only other paper that have studied a model of FAs is Albano
and Sparro (2008). However, this paper does not study the uncertainties supplier face, which is
the focus of our work. Instead, they consider a stylized model of complete information in which
suppliers are horizontally differentiated and study the following trade-off: contracting with more
suppliers decreases competition in prices to enter into the FA, but increases the chances that a
buying agency will find the right “variety” of supplier in the buying stage. Our paper contributes
to this nascent literature in FAs.

Finally, we note that we have presented preliminary results of this work in our own conference
paper in a practitioners’ outlet (Anonymous, 2012). However, that paper does not contain the
theoretical results that support the managerial insights and design prescriptions of this work. In
particular, it does not include the analysis regarding the expected buying prices from the different
FAs, which is a central part of this study. Moreover, the managerial insights themselves presented
in the conference paper were incomplete; for example, they did not discuss the optimality of the
PPI FA nor the precise conditions under which certain FA design are preferred over others.

Rest of the paper. §2 introduces the basic FA model, while §3 provides the analysis of its
BNE bidding strategies together with the analysis of the perfect price index FA. §4 provides results
to compare the expected buying prices of the basic FA, the perfect price index FA, and other FA
models, and it characterizes the optimal FA mechanism. §5 studies the operational variations of
the basic FA, namely the flexible FA, and the flexible out-of-stock FA. §6 provide numerical results
that complement the analytical results of the previous sections. §7 concludes, summarizes the main
design prescriptions, and discusses directions for future research.

2 Basic FA model

We model a framework agreement (FA) as a game of incomplete information between sellers,
similarly to the classic modeling approach in auction theory (see, e.g., Milgrom 2004 and Krishna
2002). The buyer (or auctioneer) wants to procure one unit of a product or service at time period
At time period $t = 0$, there are $N$ competing bidders or sellers (also referred to as suppliers or providers). As is common in auction models we assume bidders are risk neutral. This assumption implies that bidders will generally not be interested in hedging with financial instruments such as options. However, as we will see, cost uncertainty will play a key role in bidding incentives even with risk neutral bidders. We briefly discuss the case of risk averse bidders in §7. We next describe the main elements of the basic FA model.

**Cost Structure.** A bidder’s cost of providing the good at $t = 1$ is given by the sum of two components: $c_i + X$. The cost component $c_i$ corresponds to an independent private cost for bidder $i$, known to himself at $t = 0$, but not to his competitors. The costs $\{c_i : i = 1, ..., N\}$ are independent and identically distributed (i.i.d.) across bidders with distribution function $F$, continuous density function $f$, finite mean $\mu_c$, and support $[c, \bar{c}]$. The cost component $X$ is common to all bidders and its realization is unknown to all bidders and the buyer at $t = 0$. The realization of $X$ at $t = 1$ is drawn from a distribution function $G$, with continuous density function $g$, finite mean $\mu_x$, and support $[x, \bar{x}]$. The random variables $\{c_i : i = 1, ..., N\}$ and $X$ are independent.

The private cost component $c_i$ represents idiosyncratic characteristics of the firm, such as its managerial ability, logistics and production costs, or technology, that for the most part do not change over time. On the other hand, the cost component $X$ is common, and we interpret it as being related to the price of inputs that all firms may require to provide the product or service. It is random at $t = 0$ because these prices may change over time until $t = 1$. To illustrate this cost structure, consider for example the provision of a transportation service at $t = 1$. Costs associated with the logistics of the firm and its transportation network are private and represented as part of $c_i$. On the other hand, the costs of some inputs such as gas are common to all firms and subject to random fluctuations between $t = 0$ and $t = 1$; these costs are represented by $X$. While in reality, a fraction of the private costs may also be subject to uncertainty between $t = 0$ and $t = 1$, we abstract away from this effect to be able to focus on the impact of the common cost uncertainty.

**First stage.** At time $t = 0$, the sellers participate in a first price auction to determine the winner of the first stage. Each supplier bids a selling price, and the lowest submitted bid wins; we call this firm the FA winner. In the basic FA model, the winner is committed to sell at the winning bid price if required at $t = 1$.

**Second stage.** At $t = 1$, after the realization of the common cost, the buyer has the option to buy from the FA winner at the winning bid price of the first price auction at $t = 0$. As an alternative, she can buy from the spot market, that represents an outside option for an organization wanting to buy the product. As discussed in the §1, these outside alternatives appear in many FA settings.

We assume that the spot market price is given by $z + x$, where $x$ is the realization of $X$ at $t = 1$, and $z$ is the realization of $Z$, a random variable that also gets realized at $t = 1$.

---

3For this and all distributions defined below, the lower and upper ends of the support are non-negative finite numbers, unless otherwise noted.
random variable $Z$ has a distribution function $H$, continuous density function $h$, finite mean $\mu_Z$, and support $[\underline{z}, \bar{z}]$, and is independent of $\{c_i : i = 1, ..., N\}$ and $X$.\footnote{We assume $\underline{c} < \underline{z}$. Also, to simplify some arguments we assume that $\bar{z} \leq \bar{c}$.} At $t = 1$, the buyer buys from the FA winner if and only if the bid submitted by the FA winner is lower than the realized spot market price.

Importantly, we assume that the spot market price shares the cost component $X$ as any provider would need to pay for the cost of inputs. The random variable $Z$ represents the private cost component of the outside provider plus, potentially, a markup this provider may charge on top of his costs. Organizations wanting to buy from the FA typically do not have the ability to screen the most efficient providers in the outside market. Therefore, it may be reasonable to assume, for example, that $H$ is the same distribution as $F$. Under such assumption the organization can “sample” one provider from the private cost distribution $F$ in the outside market, and pay a spot market price equal to $c + X$, where $c$ is the sampled cost. Another convenient specification that we will use to gain intuition and simplify the analysis regarding the common cost uncertainty and the way it affects the bids and the selling price, is to assume that $Z$ is deterministic and equal to $z_0$. In this case, it may be reasonable to assume, for example, that $z_0 = \mu_c$, so that the organization has access to an “average supplier” in the spot market.

We note that in our model the spot market price is independent of the bidders’ private costs that participate in the FA. Moreover, we assume that when bidders participate in the FA at $t = 0$, they ignore the possibility of selling as the outside option at $t = 1$. This assumption significantly simplifies the analysis. Moreover, anecdotal evidence from ChileCompra suggests that these are reasonable simplifications in diffused markets with many suppliers, in which FA winners may not necessarily be the outside option for a typical organization. Indeed, most FA markets have ten or more suppliers (see Figure 2).

![Figure 2: Competitiveness in FAs. The histogram summarizes the number of competitors in 83 different product categories of FAs taken place between 2007 and 2011 in Chile. We note that in some of these FAs, in particular those with tens of suppliers, it may be the case that some of them only submit bids for a subset of products/services auctioned in the FA. Source: Dirección ChileCompra.](image-url)
Information structure. The structure and elements of the model are assumed to be common knowledge, and in particular, the distributions $F$, $G$, and $H$, and the number of bidders $N$.

Preliminary analysis and the “FA Catch”. We now present the key uncertainties that govern the decision-making process of bidders in an FA. For the sake of simplicity, we assume in this discussion that $Z$ is deterministic and takes the value $z_0$. Note that in the case of deterministic $Z$, the FA winner must have a private cost $c_i \leq z_0$; if not, he could not be profitable in the face of the spot market competition and would not participate in the FA in the first place. Note that the spot market price $z_0 + x$ is an upper bound on the bidder’s revenues and his costs are given by $c_i + x$.

Assuming he participates and wins, the FA winner faces two positively correlated uncertainties at time period $t = 1$: (1) his cost component $X$ is random; and (2) the competition against the spot market price $z_0 + X$ is also random. We describe the different scenarios the FA winner faces depending on the outcome of $X$.

First, if the realization of $X$ is low, the spot market price may be lower than the FA winning bid, and the auctioneer buys from the spot market. Second, for a moderate realization of $X$, the FA winning bid may be lower than the spot market price $z_0 + X$, but larger than his realized costs $c_i + X$; hence, the FA winner provides the product making positive profits. Finally, for a large realization of $X$, the FA winner still provides the product, but he makes negative profits because of the high realized costs. The different scenarios are described in Figure 3. We refer to the above

<table>
<thead>
<tr>
<th>$X$ Realization Scenario</th>
<th>Graphic Description</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low $X$ realization</td>
<td>$c_i + x$ $z_0 + x$</td>
<td>$b_i$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Moderate $X$ realization</td>
<td>$c_i + x$ $z_0 + x$</td>
<td>$i$ wins with positive margin</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>High $X$ realization</td>
<td>$b_i$ $c_i + x$ $z_0 + x$</td>
<td>$i$ wins with negative margin</td>
</tr>
</tbody>
</table>

Figure 3: The “FA catch”. (For simplicity we assume that $Z$ is deterministic and takes the value $z_0$.) At $t = 0$ bidder $i$ wins the FA with bid $b_i$. At time period $t = 1$ the common cost is realized to be $x$. Then, the realized cost of $i$ at $t = 1$ is $c_i + x$, and the realized market spot price at $t = 1$ is $z_0 + x$. The three different realization scenarios lead to different outcomes for the FA winner.

situation as the FA catch: winning may be bad news as this event is positively correlated with a large realization of the common cost $X$. This general observation is carried over (through the probability of possible events) to the general case in which $Z$ is random. In equilibrium, one may expect bidders to anticipate this possible outcome and “charge” for it in their bids. Such a bidding behavior, by which bids in FAs tend to be higher than bids in a procurement standard first price auction for supplying the same item, is observed by procurement operators in practice as discussed
in the Introduction. In the next sections we formalize the FA catch and describe ways to alleviate it by eliminating or reducing the uncertainty associated to the common cost, with the objective of reducing the expected buying price paid by the auctioneer.

**Variations.** In the rest of the paper we discuss several variations of the basic FA model. In particular, we will discuss different allocation rules between the FA winner and the spot market. Moreover, we study models for which the payment to the FA winner not only depends on the winning bid, but also on the realizations of \( Z \) and \( X \).

**Notation.** Throughout the paper, unless stated otherwise, we denote by \( \mathbb{E}[\cdot] \) and \( \mathbb{P}[\cdot] \) the expectation and probability with respect to both random variables \( X \) and \( Z \). As an exception, in §4 in which we study the expected buying price paid by the auctioneer, these are taken with respect to \( X \), \( Z \), and also over the random private costs vector \( c \). We use boldfaces to denote vectors throughout the paper.

**Final comments.** We note that our model has abstracted away from various complexities that arise in real-world FAs, such as the existence of many products, having more than one winner at \( t = 0 \), and uncertainty on the quantity demanded. However, as we argue below this model allows us to capture important insights regarding the design of FAs, and in particular regarding the cost uncertainty faced by bidders, which is the focus of this paper. We discuss some extensions and future directions in §7.

## 3 Analysis of Basic FA and Perfect Price Index FA

In this section we provide a formal analysis of the basic FA model introduced in the previous section. §3.1 revisits the notion of Bayes Nash equilibrium, which is the solution concept we use. §3.2 provides the equilibrium analysis of the basic FA model. In §3.3 we introduce a model in which the common cost uncertainty is completely removed through a perfect price index (PPI). This model serves as a useful benchmark to compare the basic FA model and quantify the FA catch.

### 3.1 Bayes-Nash equilibrium

Following the standard auction framework, we use pure strategy Bayes Nash equilibrium (BNE) as our solution concept of the FA game of incomplete information between sellers. For completeness, we revisit the definition of BNE (see, e.g., Mas-Colell et al. (1995)). After observing his own private cost \( c_i \), seller \( i \) decides on a bid \( b_i \in \mathcal{A} \subset \mathbb{R} \), where \( \mathcal{A} \) is a compact interval. The bidding strategy of \( i \) is a mapping \( \beta_i : [c_i, \bar{c}_i] \to \mathcal{A} \). Given the bidding strategy \( \beta_{-i} = \{ \beta_j, j \neq i \} \) of all other sellers, the expected profit of bidder \( i \) is denoted by \( \pi_i(b_i, c_i, \beta_{-i}) \). The specific structure of the expected profit depends on the specific terms of the different FA models we consider. A bidding strategy profile \( \beta = \{ \beta_i, \beta_{-i} \} \) is a BNE if it satisfies the following:
\[ \pi_i(\beta_i(c_i), c_i, \beta_{-i}) \geq \pi_i(b, c_i, \beta_{-i}) \quad \forall b \in A, \forall c_i \in [\underline{c}, \overline{c}], \forall i \in \{1, \ldots, N\}. \]

We further say that \( \beta \) is (strictly) increasing if, for all sellers \( i \), \( \beta_i(\cdot) \) is a (strictly) increasing function of the private cost, and that \( \beta \) is symmetric if \( \beta_i = \beta_j \) for any sellers \( i, j \). In this case, we refer to \( \beta \) as the symmetric equilibrium strategy.

### 3.2 Equilibrium strategy for Basic FA

In this section we study symmetric, continuous, and strictly increasing BNE strategies under the basic FA framework. Throughout this section we assume the existence of such equilibria, and at the end of the section we provide conditions under which such equilibria actually exist.

A seller \( i \) with bid \( b_i \) wins the basic FA auction at \( t = 0 \) if and only if all other bids are larger than \( b_i \), that is \( b_i \leq b_j \), for all \( j \neq i \).\(^5\) Moreover, the FA winner makes the sell at \( t = 1 \) at his bid price \( b_i \) if and only if his winning bid is smaller than the realized spot market price, that is, \( b_i \leq z + x \). Note that if a seller has a private cost of \( c_i \geq \overline{z} \), he cannot be profitable in face of the spot market competition, because his revenues are upper bounded by \( \overline{z} + x \) and his costs are given by \( c_i + x \). We assume that such a seller submits a large enough bid with zero chances of winning, for example, \( b_i = \overline{z} + \overline{x} \). The latter observation and assumption regarding participation will be valid for all the different FA designs we study in the paper.

Given that his competitors play a strictly increasing strategy profile \( \beta_{-i} \), the expected profit of seller \( i \) in the basic FA, with private cost \( c_i \leq \overline{z} \) and bid price \( b_i \) is given by:

\[
\pi^\text{FA}_i(b_i, c_i, \beta_{-i}) = \prod_{j \neq i} \overline{F} \left( \beta_j^{-1}(b_i) \right) \cdot \mathbb{E} \left[ (b_i - c_i - X) \cdot I \{ b_i \leq Z + X \} \right]. \quad (1)
\]

where \( \overline{F}(x) = 1 - F(x) \) and \( I \{ \cdot \} \) denotes the indicator function. The above reflects that bidder \( i \) sells only when he is the winner in the first stage and when he offers a better price than the spot market at the post-auction competition. Note that at the time the bid is submitted at \( t = 0 \), \( X \) and \( Z \) are random from the bidder’s perspective.

To study the equilibrium strategies we could solve the ordinary differential equation (ODE) derived from the first-order condition associated to the maximization of (1). However, differently to standard first price auctions, this ODE does not have a closed-form solution, because of the presence of the random spot market price. Instead, we derive the integral equation that describes the BNE strategy, by using the envelope theorem (see, e.g., chapter 4 in Milgrom 2004 for a treatment of this approach). This equation provides more intuition regarding the equilibrium relative to the ODE. We have the following result.

\(^5\)We assume throughout the paper that if more than one bidder makes the same lowest bid, ties are broken randomly where all such bidders have the same chance of being selected. Note that if bidders use strictly increasing strategies ties occur with zero probability, because the distribution of private costs is atomless.
Proposition 1. Suppose that a symmetric BNE $\beta^{FA}(\cdot)$ in continuous and strictly increasing strategies exists for the basic FA model. Then, the equilibrium strategy $\beta^{FA}(\cdot)$ must satisfy the following integral equation for all $c \leq \bar{z}$:

$$\beta^{FA}(c) = c + \mathbb{E} \left[ X \mid \beta^{FA}(c) \leq Z + X \right] + \int_{c}^{\bar{z}} \bar{F}^{N-1}(t) \cdot \mathbb{P} \left[ \beta^{FA}(t) \leq Z + X \right] dt. \quad (2)$$

Proof. To simplify notation, we ignore the superscript FA in the proof. Considering that $\beta(\cdot)$ is a symmetric and strictly increasing BNE, equilibrium profits for bidder $i$ with cost $c$ are given by:

$$\pi_i(\beta(c), c, \beta) = \bar{F}^{N-1}(c) \cdot \mathbb{E} \left[ (\beta(c) - c - X) \cdot \mathbb{I} \{ \beta(c) \leq Z + X \} \right], \quad (3)$$

where we abuse notation to denote that all competitors use the common strategy $\beta$. On the other hand,

$$\pi_i(\beta(c), c, \beta) = \max_{b \in A} \bar{F}^{N-1} \left( \beta^{-1}(b) \right) \cdot \mathbb{E} \left[ (b - c - X) \cdot \mathbb{I} \{ b \leq Z + X \} \right]. \quad (4)$$

It is simple to verify that this setting satisfies all assumptions needed to apply the envelope theorem (see, e.g., Theorem 3.1 in Milgrom (2004)). Therefore,

$$\pi_i(\beta(\bar{z}), \bar{z}, \beta) - \pi_i(\beta(c), c, \beta) = \int_{c}^{\bar{z}} \frac{\partial \pi_i(b, t, \beta)}{\partial t} \bigg|_{b=\beta(t)} dt = -\int_{c}^{\bar{z}} \bar{F}^{N-1}(t) \mathbb{P} \left[ \beta(t) \leq Z + X \right] dt. \quad (5)$$

Recall that a bidder with cost $c_i = \bar{z}$ has no chance of defeating the spot market, and therefore, $\pi_i(\beta(\bar{z}), \bar{z}, \beta) = 0$. Hence, replacing in (5) and equating with (3), we obtain

$$\int_{c}^{\bar{z}} \bar{F}^{N-1}(t) \mathbb{P} \left[ \beta(t) \leq Z + X \right] dt = \bar{F}^{N-1}(c) \cdot \mathbb{E} \left[ (\beta(c) - c - X) \cdot \mathbb{I} \{ \beta(c) \leq Z + X \} \right]$$

$$= (\beta(c) - c) \cdot \bar{F}^{N-1}(c) \mathbb{P} \left[ \beta(c) \leq Z + X \right] - \bar{F}^{N-1}(c) \mathbb{E} \left[ X \mathbb{I} \{ \beta(c) \leq Z + X \} \right].$$

Therefore, we get:

$$\beta(c) = c + \mathbb{E} \left[ X \mathbb{I} \{ \beta(c) \leq Z + X \} \right] + \int_{c}^{\bar{z}} \bar{F}^{N-1}(t) \cdot \mathbb{P} \left[ \beta(t) \leq Z + X \right] dt \frac{\mathbb{P} \left[ \beta(c) \leq Z + X \right]}{\bar{F}^{N-1}(c) \cdot \mathbb{P} \left[ \beta(c) \leq Z + X \right]},$$

proving the result.

We refer to the first two components of the right-hand-side of equation (2) as the implied cost: the expected cost of the seller, conditional on offering a better price than the spot market. We refer to the third term of the right hand side as the markup the bidder charges on top of the implied cost for having a private cost lower than $\bar{z}$. The last term is similar to the “information rent” term in a standard first price auction; however, in our model the bidder needs to defeat not only all the
other bidders but also the spot market. One can show that:

\[ E\left[X\mid \beta^{FA}(c) \leq Z + X\right] \geq E[X] = \mu_x. \tag{6} \]

Equation (6) captures the essence of the FA catch: conditional on the event in which the FA winner defeats the spot market, the expected common cost is larger than its unconditional average. This part of the equilibrium bid quantifies the way the bidder “charges” for this effect. It is simple to observe that as a consequence bidders always make positive profits in equilibrium.

### 3.2.1 Existence of equilibrium

We finish this section by proving the existence of symmetric BNE for the basic FA model. Moreover, we show that all symmetric BNE must be continuous and strictly increasing. We note that these results do not follow by standard existence results for first price auctions, because of the presence of the random common cost component and its correlation with the random spot market price through \( X \).

**Proposition 2.** Assume \( Z \) is deterministic equal to \( z_0 \). Then, the basic FA model admits a symmetric BNE in increasing strategies.

We prove this result by applying and specializing to our setting the result of Athey (2001) that establishes the existence of BNE for a large class of games of incomplete information in two steps. First, for the corresponding game with a finite action space, the so-called single-crossing condition (SCC) is shown to be sufficient for the existence of an increasing and symmetric BNE. Second, for games with continuous and compact action spaces, she establishes the existence of a symmetric BNE by taking a limit of a sequence of games with finite action space as the granularity of the action space increases. The essence of our existence proof, provided in Appendix B, is to first establish that the basic FA model satisfies the SCC property, and then to show that the regularity conditions required for the limiting argument are valid in our setting. In that appendix, we also provide an extension of the result to cases in which \( Z \) is random.

The following condition is used to establish that all symmetric BNE must be continuous.

**Assumption 1.** Assume that \( \arg\max_{b \in A} E[(b - c - X) \mathbb{I}\{b \leq X + Z\}] \) is unique for all \( c \in [\underline{c}, \bar{c}] \).

We have the following result:

**Proposition 3.** Any symmetric and increasing BNE strategy of the basic FA model must be strictly increasing for \( c \in [\underline{c}, \bar{z}] \). Moreover, if Assumption 1 holds, any symmetric and increasing BNE strategy must be continuous in \( c \in [\underline{c}, \bar{z}] \).

---

\( ^6 \)This inequality is easy to prove by generalizing the result \( E[X \mid X \geq a] \geq E[X] \) for any constant \( a \) (see Lemma B1). We also note that for \( c < \bar{z} \), it must be that \( P[\beta^{FA}(c) \leq Z + X] > 0 \). If not the bidder would make zero expected profits in equilibrium. It is simple to observe that this strategy is dominated by a strategy in which the bidder makes strictly positive profits if his private cost is \( c < \bar{z} \).
A sufficient condition for Assumption 1 to hold is that \( E[(b - c - X) \mathbb{I}\{b \leq X + Z\}] \) is strictly quasi-concave in \( b \), for all \( c \in [c, \tilde{z}] \) and all \( b \leq \tilde{z} + \tilde{x} \). As an example, one can show this is the case when the common cost \( X \) has a uniform distribution and \( \mu_x - \tilde{z} > \tilde{x} \). In addition, when \( Z \) is deterministic, one can verify that if \( X \) has a strictly increasing failure rate (IFR) distribution, then Assumption 1 holds. Examples of strictly IFR distributions include uniform, normal or truncated normal, Weibull, beta, and gamma among others (see Bagnoli and Bergstrom (2005)). We also note that we can derive Proposition 1 and related results without assuming that the BNE strategy is continuous (hence, dispensing of Assumption 1), if we instead assume that for all strictly increasing symmetric candidate equilibrium strategies, optimization problem (4) always admits a solution for all private costs \( c \).

The same approach to existence of symmetric, strictly increasing, and continuous BNE described here applies to the other FA models studied below, with the only exception of the flexible out-of-stock FA presented in §5.2; this model does not fit Athey’s framework. For brevity, we only provide the existence proof for the basic FA in the appendix; the analysis is similar for the other models.\(^7\)

### 3.3 The Perfect Price Index FA

In this section we introduce an alternative FA model, in which the auctioneer can perfectly observe the outcome of the common cost \( X \) and perfectly indexes the winning bid to the changes in \( X \), completely removing the common cost uncertainty for the FA winner. We call this model the **Perfect Price Index (PPI) FA** and we use it as a benchmark against which to compare the extent of the FA catch.

In the PPI FA the auctioneer offers a PPI to the bidders, so that the payment to the FA winner with bid \( b \) is equal to \( b + (x - \mu_x) \) at \( t = 1 \) if he sells the unit. Hence, the costs of the PPI FA winner are effectively given by \( c + \mu_x \). Moreover, note that in this case, the auctioneer will buy from the winner of the PPI FA if and only if the payment to the bidder is smaller than the spot market price, that is \( b + (x - \mu_x) \leq z + x \), which is equivalent to \( b \leq z + \mu_x \). Hence, given that his competitors play a strictly increasing strategy profile \( \beta_{-i} \), the expected profit of seller \( i \) in a PPI FA is given by:

\[
\pi_i^{\text{PPI}}(b_i, c_i, \beta_{-i}) = \prod_{j \neq i} \tilde{F}(\beta_j^{-1}(b_i)) \cdot (b_i - c_i - \mu_x) \cdot \mathbb{P}[b_i \leq Z + \mu_x].
\]

We note that the PPI FA is equivalent to a first price auction with private costs \( c_i + \mu_x \) and a random reserve price \( r = Z + \mu_x \). Therefore, we have the following result.

**Proposition 4.** Suppose that a symmetric BNE \( \beta^{\text{PPI}}(\cdot) \) in strictly increasing and continuous strategies exists for the PPI FA model. Then, the equilibrium strategy \( \beta^{\text{PPI}}(\cdot) \) must satisfy the following

\(^7\)In fact, Proposition 2 holds for the perfect price index and the flexible FA and similar results to Proposition 3 can also be obtained for those models.
Then, BNE bids in the basic FA are larger than those in the PPI FA, that is
\[
\beta^{PPI}(c) = c + \mu_x + \frac{\int_c^{\bar{z}} \bar{F}^{N-1}(t) \mathbb{P}[\beta^{PPI}(t) \leq Z + \mu_x] \, dt}{F^{N-1}(c) \mathbb{P}[\beta^{PPI}(c) \leq Z + \mu_x]}.
\] (8)

The proof is similar to the one of Proposition 1 and is omitted. We note that the implied cost in the PPI FA is simply the actual (and known) cost the seller is about to face at \( t = 1 \), and that the markup is the same as in a first price auction with a random reserve price \( \tau = Z + \mu_x \). The implied cost under the basic FA is always larger than the implied cost under the PPI FA (recall equation (6)). However, the markup term could sometimes be larger in the PPI FA compared to the basic FA, because in the latter there is additional randomness with regard to the competition with the spot market price, that is relaxed in the former. Formally, the difference of the markup terms is equal to:

\[
\frac{\int_c^{\bar{z}} \bar{F}^{N-1}(t) \mathbb{P}[\beta^{PPI}(t) \leq Z + \mu_x] \, dt}{F^{N-1}(c) \mathbb{P}[\beta^{PPI}(c) \leq Z + \mu_x]} - \frac{\int_c^{\bar{z}} \bar{F}^{N-1}(t) \cdot \mathbb{P}[\beta^{FA}(t) \leq Z + X] \, dt}{F^{N-1}(c) \cdot \mathbb{P}[\beta^{FA}(c) \leq Z + X]}. \]

In the case where \( Z \) is deterministic and equal to \( z_0 \), it is simple to observe that the condition \( \beta^{PPI}(c) \leq z_0 + \mu_x \) is always satisfied if \( c \leq z_0 \), because \( \beta^{PPI}(z_0) = z_0 + \mu_x \) and the equilibrium strategy is strictly increasing. Moreover, in this case, bidders with a private cost \( c \geq z_0 \) cannot be profitable in face of the spot market competition. Therefore, for \( c \leq z_0 \), the previous expression becomes:

\[
\int_c^{z_0} \frac{\bar{F}^{N-1}(t)dt}{F^{N-1}(c)} - \int_c^{z_0} \frac{\bar{F}^{N-1}(t) \bar{G}(\beta^{FA}(t) - z_0) \, dt}{F^{N-1}(c) \bar{G}(\beta^{FA}(c) - z_0)} = \frac{1}{F^{N-1}(c)} \int_c^{z_0} \bar{F}^{N-1}(t) \left[ 1 - \frac{\bar{G}(\beta^{FA}(t) - z_0)}{\bar{G}(\beta^{FA}(c) - z_0)} \right] \, dt \geq 0,
\]

where the inequality follows because \( \bar{G}(\cdot) \) is a decreasing function. Hence, there are potentially two countering effects when comparing the basic FA with PPI FA bidding strategies. In the next result, we provide instances in which the implied cost effect dominates the markup effect, and therefore, equilibrium bids are larger in the basic FA model compared to the PPI FA model. Throughout the rest of this section we denote by \( \beta^{FA}(\cdot) \) and \( \beta^{PPI}(\cdot) \) as symmetric, strictly increasing, and continuous BNE strategies for the basic FA and PPI FA models, respectively.

**Lemma 1.** Assume that \( Z \) is deterministic equal to \( z_0 \). Suppose the following condition holds:

\[
\mathbb{E}[X \mid X \geq \mu_x - z_0] - \mu_x \geq z_0. \quad (9)
\]

Then, BNE bids in the basic FA are larger than those in the PPI FA, that is \( \beta^{FA}(c) \geq \beta^{PPI}(c) \), \( \forall c \in [c, z_0] \).\(^8\) When the inequality in (9) is strict, so is the comparison of bid functions. Further, assume that \( X \sim N(\mu, \sigma^2) \), then the sufficient condition (9) takes the form \( \sigma \geq z_0/t_0 \), where \( t_0 \approx 0.506 \) is

\(\footnotesize{\text{\(^8\)Recall that bidders with private costs } c > z_0 \text{ do not participate in the FA.}}\)
the positive, unique solution to the equation $\frac{\phi(x)}{x\Phi(x)} = 1$, where $\phi$ is the density of a standard normal random variable, and $\Phi$ is the corresponding cumulative distribution function.\footnote{Throughout our formulation we assumed that $X$ has bounded support to insure the existence of equilibrium. We note that a similar, yet less simple, sufficient condition can be obtained under the assumption that $X$ has a truncated normal distribution.}

The above sufficient condition has a clear interpretation: If the implied cost difference is large enough, then the equilibrium bid prices in the basic FA are larger than those in the PPI FA. When $X$ is normally distributed this happens when its variance is large enough, which coincides with the intuition that the FA catch is more pronounced when there is more common cost uncertainty.

The proof of Lemma 1 is relegated to the appendix; we next provide the main ideas of the proof. First, we define two functional mappings, $T_{FA}$ and $T_{PPI}$, that correspond to the right hand sides of the integral equations (2) and (8), respectively, when $Z$ is deterministic. We show that under condition (9), $T_{FA}$ is larger than $T_{PPI}$, in the sense that $T_{FA}(\beta) \geq T_{PPI}(\beta)$ for any symmetric equilibrium candidate bid function $\beta$. Finally, we show that this implies that any fixed point of $T_{FA}$ is larger than any fixed point of $T_{PPI}$, and therefore, for any equilibrium bid functions $\beta^{FA}(\cdot)$, $\beta^{PPI}(\cdot)$, we have that $\beta^{FA}(c) \geq \beta^{PPI}(c)$, for all $c$.

The next result provides conditions for which BNE bids under the basic FA are higher than BNE bids for the PPI FA, even if condition (9) does not hold. For this result, we show that in the presence of enough competition, the implied cost effect dominates the markup effect. To emphasize the way competition affects the equilibrium bids, we denote by $\beta_N(\cdot)$ the symmetric equilibrium bid function in the presence of $N$ bidders. We also introduce the following assumption.

**Assumption 2.** Assume that there exist $\varepsilon > 0$ and $\alpha > 0$, such that $\mu_x \geq z_0 + \varepsilon$, and $1 > G(\varepsilon) \geq \alpha$.

We have the following result.

**Lemma 2.** Assume that $Z$ is deterministic equal to $z_0$ and that Assumption 2 holds. Then, there exists $N_0$, such that for all $N > N_0$, one has $\beta^{FA}_N(c) > \beta^{PPI}_N(c)$, $\forall c \in [\underline{c}, z_0]$.

The proof of Lemma 2 follows a similar argument to the proof of Lemma 1, showing that for $N$ large enough $T_{FA}^N(\beta) > T_{PPI}^N(\beta)$, for any symmetric equilibrium candidate bid function $\beta$, and therefore $\beta^{FA}_N(c) > \beta^{PPI}_N(c)$, for all $c \in [\underline{c}, z_0]$. Roughly speaking, Lemma 2 implies that when the expected common cost is higher than the deterministic market cost $z_0$, but the distribution of the common cost $X$ assigns a large enough probability to low realizations, bids under the basic FA are higher than bids under the PPI FA if there is enough competition. Note that the condition $G(\varepsilon) \geq \alpha$ in Assumption 2 is more likely to be satisfied if the variance of $X$ is large (e.g., if $X$ is normally distributed).

The conditions used in the previous results to obtain the ordering of bids are sufficient, but not necessary. In §6 we complement these results with numerical experiments and show that the
number of bidders required so that bids under the basic FA are higher than those under the PPI FA is generally small. Moreover, we show that this ordering of bids can be obtained even for values of the variance of $X$ that are much smaller than what condition (9) suggests.

The comparison of bid prices is useful, since it provides intuition about bidding behavior. However, from the perspective of the auctioneer the more relevant quantity is the buying price, which also takes into account the spot market price in the FA and the perfect price index in the PPI FA. We assume the buyer, such as a government, is risk neutral and cares about expected buying prices. In particular, the expected buying prices for the basic FA and PPI FA under the respective BNE strategies are given by the following expressions:

$$\mathbb{E}[P^\text{FA}] = \mathbb{E}\left[\beta^\text{FA}(c(1)) \mathbb{I}\{\beta^\text{FA}(c(1)) \leq Z + X\} + (Z + X)\mathbb{I}\{\beta^\text{FA}(c(1)) > Z + X\}\right],$$ (10)

$$\mathbb{E}[P^\text{PPI}] = \mathbb{E}\left[(\beta^\text{PPI}(c(1)) + X - \mu_x) \mathbb{I}\{\beta^\text{PPI}(c(1)) \leq Z + \mu_x\} + (Z + X)\mathbb{I}\{\beta^\text{PPI}(c(1)) > Z + \mu_x\}\right],$$

where $c(1)$ is the lowest order statistic among the private costs. In both cases, recall that the FA winner must have a private cost smaller than $\bar{z}$. Moreover, the auctioneer buys from the (PPI) FA winner if it offers a smaller (effective) price compared to the spot market price.

Comparing the expected buying prices using the previous expressions is challenging because they have different terms; hence, an ordering on the equilibrium strategies does not directly imply an ordering in the expected buying prices. In the next section, we introduce a unified result that allows to more easily compare the expected buying prices between different FAs.

4 Comparison of expected buying prices

In this section we compare the expected buying prices of different FAs using an envelope theorem approach similar to the one used in typical proofs of the revenue equivalence theorem and in the derivation of the optimal auction (Myerson 1981). In fact, we use this result to derive the optimal auction in our setting. Our exposition and analysis closely follows Milgrom (2004).

To develop a general treatment, it is useful to define an auction or FA mechanism as an allocation and a payment rule. Formally, let us define the allocation function $p_i : A^N \times [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}] \to [0, 1]$, where $p_i(b_i, b_{-i}, x, z)$ is the probability that bidder $i$ sells the product when $i$ submit a bid $b_i$, $i$’s competitors submit bids $b_{-i}$, the common cost realization is $x$ and the realization of $Z$ is equal to $z$. We let $p = (p_1, ..., p_N)$. We only consider allocation rules that satisfy $\sum_{i=1}^N p_i(b, x, z) \leq 1$, for all $b, x, z$.

Similarly, define the payment function $t_i : A^N \times [\underline{x}, \bar{x}] \times [\underline{z}, \bar{z}] \to \mathbb{R}$, where $t_i(b_i, b_{-i}, x, z)$ is the expected payment bidder $i$ receives for given bids $(b_i, b_{-i})$ and given realization of common cost $x$ and realization $z$. We let $t = (t_1, ..., t_N)$. An FA mechanism is given by the functions $w = (p, t)$ and we denote by $\mathcal{W}$ the set of all such mechanisms.

For concreteness, we define the allocation and payment functions for different FAs. To simplify
the exposition we ignore ties; however, all our results in this section are valid even if ties are allowed.

1. Perfect Price Index FA (PPI):

\[ p_i(b_i, b_{-i}, x, z) = \mathbb{I}\{b_i \leq b_j, \forall j \neq i\} \mathbb{I}\{b_i \leq z + \mu_x\} \]

\[ t_i(b_i, b_{-i}, x, z) = p_i(b_i, b_{-i}, x, z)(b_i + (x - \mu_x)) \]

2. Basic FA (FA):

\[ p_i(b_i, b_{-i}, x, z) = \mathbb{I}\{b_i \leq b_j, \forall j \neq i\} \mathbb{I}\{b_i \leq z + x\} \]

\[ t_i(b_i, b_{-i}, x, z) = p_i(b_i, b_{-i}, x, z)b_i. \]

Note that different FA mechanisms have different allocation and payment functions. For example, while the allocation function of PPI depends on \( z \) and its payment function also depends on \( x \), the allocation and payment functions of the basic FA depend on \( z + x \) only. The following result characterizes the expected buying price in the BNE of a given FA mechanism. The proof is provided in the appendix.

**Proposition 5.** Let \( \beta(\cdot) \) be a BNE strategy profile induced by a mechanism \( w = (p, t) \), such that equilibrium expected profits satisfy

\[ \pi_i(\beta_i(c), c, \beta_{-i}) = 0, \text{ for all } i. \]

Then, the expected buying price for the auctioneer is given by:

\[
E[P] = \mu_z + \mu_x + \mathbb{E}\left[ \sum_{i=1}^{N} p_i(\beta(c), X, Z)(v(c_i) - Z) \right],
\]

(11)

where the “virtual cost” function is \( v(c) = c + F(c)/f(c). \)

Equation (11) shows that the expected buying price for a given FA mechanism is equal to the expected spot market price plus a term related to the savings an FA can potentially achieve relative to the spot market price. An effective FA should indeed induce a negative last term in (11) and the smaller its value the larger the savings from using the FA.

Note that the expected buying price is determined only by the functions \( p_i(\beta(c), X, Z) \). Hence, two FAs that have the same allocation rules in equilibrium (and in which the largest cost supplier receives zero profits) achieve the same expected buying price. For FAs with equilibrium in strictly increasing strategies, one has that \( p_i(\beta(c), x, z) = \mathbb{I}\{c_i \leq c_j, \forall j \neq i\} \mathbb{I}\{c_i \leq z\} \tilde{p}_i(\beta_i(c_i), x, z) \), where \( \tilde{p}_i(\beta_i(c_i), x, z) \) is the allocation rule when comparing the winning bid with quantities that depend on \( x \) and \( z \) only. That is, for the PPI FA we have \( \tilde{p}_i(\beta_i(c_i), x, z) = \mathbb{I}\{\beta_i(c_i) \leq z + \mu_x\} \), and for the basic FA we have \( \tilde{p}_i(\beta_i(c_i), x, z) = \mathbb{I}\{\beta_i(c_i) \leq z + x\} \). Recall that when \( Z \) is deterministic and equal to \( z_0 \), the condition \( \beta^{PPI}(c) \leq z_0 + \mu_x \) is always satisfied if \( c \leq z_0 \). Therefore, in this case we get \( p_i(\beta^{PPI}(c), X, Z) = \mathbb{I}\{c_i \leq c_j, \forall j \neq i\} \mathbb{I}\{c_i \leq z_0\} \).
Recall that in Lemma 2 we provided conditions under which the BNE bids in the basic FA are larger than those of the PPI FA, when the number of competitors is large enough. In the next result, we use Proposition 5 to show that under similar conditions the same holds for the expected buying prices. We denote by $P_N$ the expected buying price in presence of $N$ bidders.

**Proposition 6.** Assume that $Z$ is deterministic equal to $z_0$ and that Assumption 2 holds. In addition, assume that $\min_{c \in [\underline{c}, \bar{c}]} f(c) > 0$. Then, there exists a number of bidders $N_0$, such that $E[P_{N_{FA}}^N - P_{N_{PPI}}^N] > 0$, for all $N \geq N_0$.

**Proof.** In the proof we denote by $\beta_N$ the equilibrium bid function in presence of $N$ bidders, and by $c_{(1:N)}$ the smallest order statistics of the realized private costs when there are $N$ bidders. Note that for any basic FA candidate equilibrium strategy and any $N$, we must have (see equations (2), (6), and also (A-1)):

$$\beta_{N_{FA}}(c) \geq c + \mu_x \geq c + \varepsilon + z_0 \geq \varepsilon + z_0, \quad \forall c \in [\underline{c}, \bar{c}] .$$

(12)

Define the random variables $Y_N = \mathbb{I}\{c_{(1:N)} \leq z_0\} (z_0 - v(c_{(1:N)}))$ and the events $B_N = \{\beta_{N_{FA}}(c_{(1:N)}) > z_0 + X\}$. We note that using (12), one has that for all $N$:

$$1 \geq \mathbb{P}[B_N] \geq \alpha .$$

(13)

Then, we have (recall expectations are taken with respect to both $X$ and the random vector $c$):

$$E[P_{N_{FA}}^N - P_{N_{PPI}}^N] = E \left[ \sum_{i=1}^{N} \mathbb{I}\{c_i \leq \epsilon, \forall j \neq i\} \mathbb{I}\{c_i \leq z_0\} (\hat{\beta}_{i_{FA}}^N(\beta_{N_{FA}}(c_i), X) - \hat{\beta}_{i_{PPI}}^N(\beta_{N_{PPI}}(c_i), X))(v(c_i) - z_0) \right]$$

$$= E \left[ \mathbb{I}\{c_{(1:N)} \leq z_0\} (\mathbb{I}\{\beta_{N_{FA}}(c_{(1:N)}) \leq z_0 + X\} - 1)(v(c_{(1:N)}) - z_0) \right]$$

$$= E [Y_N \mathbb{I}\{B_N\}] .$$

(14)

We note that $c_{(1:N)}$, the lowest private cost, converges to $\underline{c} < z_0$ in probability as $N \to \infty$, and therefore, the random variable $\mathbb{I}\{c_{(1:N)} \leq z_0\}$ converges to 1 in probability. In addition, $v(c_{(1:N)}) = c_{(1:N)} + \frac{f(c_{(1:N)})}{\beta_{N_{FA}}(c_{(1:N)}, X)}$ converges in probability to $\underline{c}$ (recall that $f(\underline{c}) > 0$). Therefore, the random variables $Y_N$ converge in probability to $z_0 - \underline{c}$. Hence, because the indicator function is bounded by 1, $|Y_N \mathbb{I}\{B_N\} - (z_0 - \underline{c})| \mathbb{I}\{B_N\}$ converge in probability to zero. In addition, since $\min_{c \in [\underline{c}, \bar{c}]} f(c) > 0$ we get that $Y_N$ is uniformly bounded, and therefore $|Y_N \mathbb{I}\{B_N\} - (z_0 - \underline{c})| \mathbb{I}\{B_N\}$ converge in expectation to zero, by the Bounded Convergence Theorem. Hence, for all $\delta > 0$ there exists $N_N$, such that for all $N > N_N$,

$$E [Y_N \mathbb{I}\{B_N\}] \geq (z_0 - \underline{c}) \mathbb{P}[B_N] - \delta$$

$$\geq (z_0 - \underline{c}) \alpha - \delta ,$$

(15)

where the last inequality follows by equation (13). The result follows by choosing $\delta = \alpha (z_0 - \underline{c}) / 2,$
Another interesting consequence of Proposition 5 is that we can characterize the optimal mechanism, the one that minimizes the expected buying price for the auctioneer over all feasible auction mechanisms. More formally, let us define an augmented mechanism as a pair \((w, \beta)\), where \(\beta\) is a BNE induced by mechanism \(w \in W\). We say that an augmented mechanism is feasible if individual rationality (IR) is satisfied, i.e., \(\pi_i(\beta(c), c, \beta_{-i}) \geq 0\), for all players \(i\) and cost realizations \(c_i\). That is, all bidders receive non-negative equilibrium profits, and therefore, they are willing to voluntarily participate in the mechanism (where the profits of non-participating have been normalized to zero).

To prove this and the next result in this section, we introduce the standard regularity assumption used in mechanism design.

**Assumption 3.** The virtual cost function \(v(c) = c + F(c)/f(c)\) is strictly increasing in \(c\), for all \(c \in [\bar{c}, \bar{c}]\).

In the following result we provide a characterization of mechanisms that minimize the expected buying price of the auctioneer.

**Proposition 7.** Suppose Assumption 3 holds. An augmented mechanism \((w, \beta)\) minimizes the expected buying price for the auctioneer among all feasible augmented mechanisms if it satisfies
\[
\pi_i(\beta_{\bar{c}}(\bar{c}), \bar{c}, \beta_{-i}) = 0, \quad \text{for all } i,
\]
and its allocation rule under the BNE strategy profile \(\beta\) satisfies the following: (1) if \(v(c_{(1)}) \leq z\), then buy from the lowest cost FA supplier; and (2) if \(v(c_{(1)}) > z\), then buy from the spot market. Moreover, there exists at least one such augmented mechanism that achieves the optimum.

**Proof.** Following the same argument as Proposition 5, the expected buying price for an augmented feasible mechanism \((w, \beta)\) satisfies:
\[
E[P] = \mu_z + \mu_x + \mathbb{E}\left[\sum_{i=1}^{N} p_i(\beta(c), X, Z)(v(c_i) - Z)\right] + \sum_{i=1}^{N} \pi_i(\beta_{\bar{c}}(\bar{c}), \bar{c}, \beta_{-i})
\geq \mu_z + \mu_x + \mathbb{E}\left[\min_{0 \leq \min_{i=1,...,N} (v(c_i) - Z)}\right],
\]
where the inequality follows because for a feasible mechanism \(\pi_i(\beta_i(c_i), c_i, \beta_{-i}) \geq 0\), for all \(i\) and \(c_i\), and because \(\sum_{i=1}^{N} p_i(b, x, z) \leq 1\) and \(p_i(b, x, z) \geq 0\), for all \(b, x, z\). The right hand side of (16) provides a lower bound on the expected buying price for any feasible mechanism, therefore, a feasible mechanism that achieves it must be optimal. Hence, using the fact that \(v(\cdot)\) is strictly increasing, a feasible augmented mechanism with an allocation rule in equilibrium like the one proposed in the statement of the proposition and that satisfies \(\pi_i(\beta_{\bar{c}}(\bar{c}), \bar{c}, \beta_{-i}) = 0\), for all \(i\), must be optimal.

To prove the second part of the proposition we construct a mechanism that achieves the optimum. Consider a “modified” second-price auction in which bidders submit bids \(b_i\) and the spot market
“submits” a bid equal to \( b_0 = v^{-1}(z) \) after observing the realization of \( Z \). The lowest bid among \( b_0, b_1, \ldots, b_N \) wins and sells the object. If one of the bidders \( 1, \ldots, N \) wins, after observing the realization of \( X \), the auctioneer pays him \( b(2) + x \), where \( b(2) \) is the second lower order statistics among \( b_0, b_1, \ldots, b_N \). Loosing bidders do not receive payments. Therefore, the actual payoff for a winning bidder \( i \) is given by \( b(2) + x - (c_i + x) = b(2) - c_i \), which is the same as the payoff in a standard second price auction. Hence, truthful bidding is a dominant strategy, so that bidder \( i \) submitting a bid \( b_i = c_i \) is a BNE. Clearly, a bidder with cost \( \bar{c} \) has no chance of winning and \( \pi_i(\beta(\bar{c}), \bar{c}, \beta_{-i}) = 0 \). Moreover, the winning bidder is determined by the minimum between \( c_{(1)} \) and \( v^{-1}(z) \). Because \( v(\cdot) \) is strictly increasing, it follows that the allocation rule satisfies: (1) if \( v(c_{(1)}) \leq z \), then buy from the lowest cost FA supplier; and (2) if \( v(c_{(1)}) > z \), then buy from the spot market. These facts prove the result.

The previous result characterizes the optimal mechanism. The proof shows that a “modified” second price auction with an appropriately chosen (random) reserve price is optimal. In this auction a winning bidder gets payed \( b(2) + x \), where \( b(2) \) is the second lower order statistics among submitted bids. Hence, this auction is like a second price implementation of the PPI FA. In the next result, we show that in the case in which \( Z \) is deterministic, then the optimal mechanism can also be implemented as the first price PPI FA (introduced in §3.3) with an appropriately chosen reserve price.

**Proposition 8.** Suppose Assumption 3 holds and that \( Z \) is deterministic equal to \( z_0 \). Consider the augmented mechanism induced by the PPI FA with its symmetric BNE, in which the winning bidder with bid \( b^* \) receives a payment equal to \( b^* + (x - \mu_x) \). Moreover, assume there is a reserve price equal to \( v^{-1}(z_0) + \mu_x \). Then, this augmented mechanism minimizes the expected buying price for the auctioneer among all feasible augmented mechanisms.

**Proof.** When \( Z \) is deterministic, this PPI FA is equivalent to a first price auction with a reserve price \( r = v^{-1}(z_0) + \mu_x \), in which bidders have costs \( c_i + \mu_x \). A symmetric BNE \( \beta(\cdot) \) exists by a result similar to Proposition 2 for the PPI FA. Moreover, it must be that \( \beta(v^{-1}(z_0)) = r \). We know that \( v^{-1}(z_0) \leq z_0 \leq \bar{c} \), and therefore, \( \pi_i(\beta(\bar{c}), \bar{c}, \beta) = 0 \), for all \( i \). Moreover, because the equilibrium must be in strictly increasing strategies by a result similar to Proposition 3 for the PPI FA, we get that in equilibrium: (1) if \( v(c_{(1)}) \leq z_0 \), then the auctioneer buys from the lowest cost FA supplier; and (2) if \( v(c_{(1)}) > z_0 \), then it buys from the spot market. The result follows by Proposition 7.

The previous analysis suggests that using PPIs is useful, because in many situations of interest the PPI FA achieves a lower expected buying price than the basic FA. To complement our theoretical results, in §6 we provide numerical experiments that show that in fact the number of bidders required to obtain the latter result is generally small. Moreover, with an appropriately chosen reserve price, PPI FA is optimal as it minimizes the expected buying price. It is worth noting that in practice typically public procurement agencies do not use reserve prices to optimize their
buying processes. In addition, it is important to observe that a PPI FA is implementable only if such price indexes are available, for example, if $X$ represents the cost of a commodity such as gas. However, for many of the goods and services procured with FAs (e.g., computers, office equipment, services) such perfect price indexes do not exist. Thus motivated, in the next sections we study how different operational changes to the design of FAs impact and potentially reduce the expected buying price relative to the basic FA. Studying and implementing these modifications may be particularly relevant in the absence of PPIs.

5 Operational variations of the Basic FA

Recall from the discussion in §2 summarized in Figure 3 that in the basic FA there are two negative scenarios for the supplier that affect bidding incentives and may increase equilibrium bids. First, for a low realization of the common cost $X$, the auctioneer may prefer to buy from the spot market. Second, for high realizations of $X$, the FA winner may be forced to sell the product at a loss. In this section we study the impact on the expected buying price of two variants of the basic FA that address these two cases.

Regarding the first case, we study a variation of the basic FA in which an FA winner with an initial bid larger than the realized spot market price is allowed to match it and sell the product. We call this variation the flexible FA (FLE). Regarding the second case, we study a variation of the flexible FA in which the FA winner can declare himself out-of-stock to avoid losses. We refer to this variation as flexible out-of-stock FA (OOS). These two FA designs are practically important; in fact, ChileCompra has experienced with flexible FAs and they are concerned about the out-of-stock providers behavior that is occasionally observed. We study them in the next subsections.

5.1 Flexible FA

In the flexible FA (FLE), the FA winner is allowed to match the spot market price if his initial bid is larger than the realized spot market price. The FA winner will always match if it is profitable to do so. Given that his competitors play a strictly increasing strategy profile $\beta_{-i}$, the expected profit of seller $i$ in a FLE FA is given by:

$$\pi_i^{\text{FLE}}(b_i, c_i, \beta_{-i}) = \prod_{j \neq i} F_j^{-1}(b_i) \mathbb{E} \left[ (b_i - c_i - X) \cdot \mathbb{I} \{b_i \leq Z + X\} + (Z - c_i) \cdot \mathbb{I} \{b_i > Z + X \geq c_i + X\} \right].$$

Note that the profit function captures that bidder $i$ wins and makes the sell if his bid is lower than all competitors’ bids and also if one of two events occur: (1) his initial bid is lower than the realized spot market price (in which case he makes a profit $b_i - c_i - X$); or (2) his initial bid is larger than the realized spot market price, but he matches the latter because it is profitable to do so (in the case $Z + X \geq c_i + X$, making a profit of $Z - c_i$). We have the following characterization.
Proposition 9. Suppose that a symmetric BNE $\beta^{F\text{LE}}(\cdot)$ in strictly increasing and continuous strategies exists for the FLE FA model. Then, the equilibrium strategy $\beta^{F\text{LE}}(\cdot)$ must satisfy the following integral equation for all $c \leq \bar{z}$:

$$
\beta^{F\text{LE}}(c) = c + \mathbb{E} \left[ X \mathbb{I} \{ \beta^{F\text{LE}}(c) \leq Z + X \} \right] - \frac{\mathbb{E} \left[ (Z - c) \cdot \mathbb{I} \{ \beta^{F\text{LE}}(c) > Z + X, Z \geq c \} \right]}{\mathbb{P} \{ \beta^{F\text{LE}}(c) \leq Z + X \}}
+ \int^{\bar{z}} c \cdot F_{N-1}(t) \cdot \mathbb{P} \left[ \beta^{F\text{LE}}(t) \leq Z + X \right] + \mathbb{P} \left[ \beta^{F\text{LE}}(t) > Z + X, Z \geq t \right] dt.
$$

The proof can be found in the appendix. We can make several observations by comparing the right-hand sides of equations (2) and (17) for the same bid function. First, the implied costs coincide for the basic FA and the FLE FA. Second, the last markup term is larger for the FLE FA relative to the basic FA. This is because in the flexible FA the winner does not need to overcome the spot market price to make the sell, he has the option to match it ex-post. On the other hand, FA winners face a more convenient environment given by the flexibility of matching the spot market price, introducing an additional negative term in equation (17) that we call the benefit of flexibility. In summary, allowing the bidders to match the spot market price introduces two opposing effects: (1) it relaxes the competition with the spot market price potentially pushing equilibrium bids upwards; and (2) it intensifies the competition with other bidders potentially pushing equilibrium bids downwards. In the next result we show that when $N$ is large so that the effect of the difference in the last markup term vanishes, the expected buying price of the basic FA becomes larger than the one for the FLE FA.

To prove the result we will assume that $Z$ is deterministic and equal to $z_0$. In this case, it is simple to observe that an FA winner with a cost $c \leq z_0$ will always match the spot market price, because it is profitable to do so. Hence, the allocation rule for the FLE FA with an equilibrium in strictly increasing strategies is given by $p_i(\beta^{F\text{LE}}(c), X, Z) = \mathbb{I} \{ c_i \leq c_j, \forall j \neq i \} \mathbb{I} \{ c_i \leq z_0 \}$. Therefore, the allocation rule in equilibrium coincides with that of the PPI FA. We obtain the following result that directly follows from this observation and Propositions 5 and 6.

Proposition 10. Assume that $Z$ is deterministic and equals to $z_0$. Then, the expected buying price for the FLE FA is equal to that of the PPI FA. Moreover, if Assumption 2 holds and $\min_{c \in [c, \bar{c}]} f(c) > 0$, then there exists a number of bidders $N_0$, such that $E[P_N^{FA} - P_N^{F\text{LE}}] > 0$, for all $N \geq N_0$.

The equivalence result between the FLE FA and the PPI FA in terms of the expected buying price does not directly follow from comparing their equilibrium strategies; indeed, BNE bid strategies for FLE FA and PPI FA are generally different. In addition, the result shows that when there is enough competition, introducing the flexibility of matching the spot market price into the FA may reduce the expected buying price compared to the basic FA. In the numerical experiments we will show that in fact the number of bidders required to obtain the latter result is generally small.
5.2 Flexible Out-of-Stock FA

In the flexible out-of-stock FA (OOS), similarly to the FLE FA, the FA winner is allowed to match the spot market price if his initial bid is larger than the realized spot market price. However, in addition, the FA winner can also declare himself out-of-stock to avoid selling at a loss. Given that his competitors play a strictly increasing strategy profile $\beta_{-i}$, the expected profit of seller $i$ in an OOS FA is given by:

$$\pi_{i}^{OOS}(b_i, c_i, \beta_{-i}) = \prod_{j \neq i} F \left( \beta_{j}^{-1}(b_i) \right) \cdot E \left[ (b_i - c_i - X) \cdot I \{ c_i + X \leq b_i \leq Z + X \} \right] + (Z - c_i) \cdot I \{ b_i > Z + X \geq c_i + X \}.$$

Note that the profit function is similar to that of the FLE FA, with the only difference that the bidder sells at his initial bid only if it is higher than his realized cost (and lower than the realized spot market price). We have the following characterization.

**Proposition 11.** Suppose that a symmetric BNE $\beta^{OOS}(\cdot)$ in strictly increasing and continuous strategies exists for the OOS FA model. Then, the equilibrium strategy $\beta^{OOS}(\cdot)$ must satisfy the following integral equation for all $c \leq \bar{z}$:

$$\beta^{OOS}(c) = c + E \left[ X \Big| c + X \leq \beta^{OOS}(c) \leq Z + X \right] - \frac{E \left[ (Z - c) \cdot I \{ \beta^{OOS}(c) > Z + X, Z \geq c \} \right]}{P \{ c + X \leq \beta^{OOS}(c) \leq Z + X \}}$$

$$+ \left[ \frac{\bar{z}}{c} F_{N}^{-1}(t) \cdot \left( \mathbb{P} [ t + X \leq \beta^{OOS}(t) \leq Z + X ] + \mathbb{P} [ \beta^{OOS}(t) > Z + X, Z \geq t ] \right) \right] dt.$$

The proof is similar to the one of Proposition 9 and is omitted. We make several observations by comparing the right-hand sides of equations (2) and (18) for the same bid function. First, the implied costs may be smaller for the OOS FA relative to the basic FA, since the FA winner under OOS FA will not sell when the common cost is very large (see also Lemma B1 in the appendix). In addition, the benefit of flexibility for the FLE FA also applies to FA winners for the OOS FA. One can provide conditions for which the bids under the OOS FA are smaller than those under the basic FA. However, the bids do not consider the chance that the FA winner declares himself out-of-stock. As above, we consider the expected buying price instead. The expected buying price for the OOS FA is given by:

$$E[P^{OOS}] = E \left[ \beta^{OOS}(c(1)) \cdot I \{ c(1) + X \leq \beta^{OOS}(c(1)) \leq Z + X \} \right] + (Z + X) (1 - I \{ c(1) + X \leq \beta^{OOS}(c(1)) \leq Z + X \}).$$

The allocation function for the OOS FA depends not only on the vector of bids, the realization of common cost $x$ and spot market price $z$, but also on the bidder’s private cost $c_i$. In particular,
the latter affects whether an FA winner declares himself out-of-stock. Hence, the OOS FA is not included in the set of mechanisms studied in §4, because of the dependence of the allocation function on each bidder’s private cost c_i. As a consequence, the comparison of the expected buying prices between OOS and other FAs requires a different methodology to that used in §4, and follows two steps. In the first step, we derive the ordinary differential equation (ODE) that any symmetric BNE strategy for the basic FA model should satisfy using first order conditions. We use the ODE to derive a range in which the equilibrium bids must fall (for this we assume a uniformly distributed common cost). In the second step, we use this range to show that the expected buying price is higher for the OOS FA relative to the basic FA by comparing equations (10) and (19). The proofs of the next two propositions are presented in the appendix.

**Proposition 12.** Assume that Z is deterministic equal to z_0. A symmetric and differentiable BNE strategy for the basic FA satisfies the following ODE for any c ∈ [c, z_0):

\[
\frac{d\beta^{FA}(c)}{dc} = (N - 1) f(c) \cdot \frac{\mathbb{E}_X \left[ \left( \beta^{FA}(c) - c - X \right) \cdot \mathbb{I} \left\{ \beta^{FA}(c) \leq z_0 + X \right\} \right]}{\left[ G(\beta^{FA}(c) - z_0) - (z_0 - c) g(\beta^{FA}(c) - z_0) \right]}. \tag{20}
\]

In addition, when X is uniformly distributed [0, \bar{x}] and \mu_x \geq z_0, we have \beta^{FA}(c) \leq c + \bar{x} for any c ∈ [c, z_0].

Using the previous result, we prove the following proposition.

**Proposition 13.** Assume that Z is deterministic equal to z_0 and that X is uniformly distributed in [0, \bar{x}]. Moreover, assume that \mu_x \geq z_0 and that \beta^{OOS}(c) \leq z_0 + \bar{x}, for all c ∈ [c, z_0]. Then, the expected buying price is higher for OOS FA relative to the basic FA, that is \mathbb{E}[P^{OOS}] \geq \mathbb{E}[P^{FA}].

Note that the assumption \beta^{OOS}(c) \leq z_0 + \bar{x}, for all c ∈ [c, z_0] is essentially done with out loss of generality, because a bid b > z_0 + \bar{x} has no chance of winning and is payoff equivalent to submitting a bid equal to z_0 + \bar{x}. Note that Propositions 10 and 13 together imply that the expected buying price is higher for the OOS FA relative to the FLE FA under the conditions of both propositions.

It is interesting to note that even though equilibrium bids may be larger for the basic FA relative to the OOS FA, this ordering is reversed for the expected buying price under the conditions of the proposition. This is because in the OOS FA the FA winner declares himself out-of-stock too often. Moreover, the FA winner tends to declare out-of-stock when the realized cost is high (and therefore prices in the spot market are high) and provide the product when the realized cost is low (and therefore prices in the spot market are low anyway). In §6 we show with numerical experiments that this result is quite general and is also obtained for common cost distributions other than uniform.
6 Numerical results

In this section we perform numerical experiments that illustrate and complement the theoretical results obtained in the previous sections regarding the relations between the bids and the expected buying prices under the various FA mechanisms. We start by providing the setup.

Setup. We consider a wide range of model instances. We assume there are $N$ bidders, where bidder $i$ has a private cost $c_i \sim U [0, \frac{1}{2}]$. The uniform assumption simplifies the ODEs and we fix this distribution throughout the numerical experiments, and we vary the other parameters. The common cost $X$ is distributed according to a truncated normal distribution with mean $\mu_x$, standard deviation $\sigma_x$, and support $[0, 2\mu_x]$. We assume $Z$ is deterministic equal to $z_0 \in [0, \frac{1}{2}]$. We consider all possible combinations of a large range of parameter values that capture different settings: $N \in \{2, 4, 6, 8, 10\}$, $z_0 \in \{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}\}$, $\mu_x \in \{z_0, 1.1z_0, 1.5z_0, 2z_0\}$, and $\sigma_x \in \{\frac{3\sigma_{max}}{4}, \frac{2\sigma_{max}}{3}, \sigma_{max}\}$, where $\sigma_{max} = \frac{2\mu_x}{\sqrt{12}}$. In total, there are 180 model instances.

Algorithm. For each model instance, we first solve the ordinary differential equations (ODEs) for the different FA models. The ODE for the basic FA was provided in Proposition 12. The ODEs for the rest of the models can be similarly derived and are provided in the following result. To be able to numerically solve the ODEs, we also derive their boundary conditions.

Proposition 14. Assume that $Z$ is deterministic equal to $z_0$. A symmetric and differentiable BNE strategy for the PPI FA, FLE FA, and OOS FA satisfy the following ODEs, for any $c \in [c_0, z_0)$:

\[
\frac{d\beta^{PPI}(c)}{dc} = \frac{(N-1)f(c)}{F(c)} \cdot (\beta^{PPI}(c) - c - E(X)),
\]

\[
\frac{d\beta^{FLE}(c)}{dc} = \frac{(N-1)f(c)}{F(c)} \cdot \frac{\mathbb{E}_X [(b - c - X) \cdot \mathbb{1} \{b \leq z_0 + X\} + (z_0 - c) \cdot \mathbb{1} \{b > z_0 + X\}]}{\mathbb{P}[b \leq z_0 + X]} \bigg|_{b = \beta^{FLE}(c)},
\]

\[
\frac{d\beta^{OOS}(c)}{dc} = \frac{(N-1)f(c)}{F(c)} \cdot \frac{\mathbb{E}_X [(b - c - X) \cdot \mathbb{1} \{c + X \leq b \leq z_0 + X\} + (z_0 - c) \cdot \mathbb{1} \{b > z_0 + X\}]}{\mathbb{P}[c + X \leq b \leq z_0 + X]} \bigg|_{b = \beta^{OOS}(c)}.
\]

Moreover, assume that the pdf for $X$ satisfies $g(\bar{x}) > 0$.\(^{11}\) Then, the boundary conditions for the basic FA and the FLE FA must be given by $\beta^{FA}(z_0) = \beta^{FLE}(z_0) = z_0 + \bar{x}$, while it is $\beta^{PPI}(z_0) = z_0 + \mu_x$ for the PPI FA. In addition, it must be that $\beta'(z_0) = 0$ for these three models.

In the appendix we provide the proof of the derivation of the boundary conditions. The derivation of the ODEs is similar to Proposition 12 and is omitted. In contrast to the other models, it is not possible to analytically uniquely determine the boundary conditions for the OOS FA; for the sake of comparison, we assume they coincide with those of the basic FA and the FLE FA.

We note that similarly to asymmetric first-price auctions, our ODEs are ill-behaved at the boundary $z_0$, because at the right-hand-side of the ODE we obtain $\frac{d}{dc}$. Hubbard and Paarsch (2011)

\(^{10}\)When $\sigma_x = \sigma_{max} = \frac{2\mu_x}{\sqrt{12}}$, $X$ becomes a uniform distribution with support $[0, 2\mu_x]$.

\(^{11}\)The latter condition is satisfied for truncated normal distributions.
and Fibich and Gavish (2011) provide useful summaries of the challenges involved in numerically solving for BNE strategies using ODEs in asymmetric first-price auctions and of ways to overcome them. To avoid the singularity at the boundary, we make the approximation $\beta(z_0 - \epsilon) = \beta(z_0) - \epsilon$ for a small value of $\epsilon = 10^{-5}$. We solve the ODEs with a Runge-Kutta method in Matlab to obtain the BNE bid functions for FA, PPI, FLE, and OOS mechanisms.

Having computed the equilibrium bid functions, we use simulation to determine the expected buying prices for each model instance and each FA mechanism. In particular, we randomly generate the private costs $c_i$ for all suppliers to obtain, using the BNE bid functions, the winning bid $\beta(c_{(1)})$, where $c_{(1)}$ is the lowest realized private cost. Then, we randomly generate a common cost $X$, the realized cost of the winning bidder $c_{(1)} + X$, and the realized spot market price $z_0 + X$. Given these quantities, the buying price is determined according to the different mechanisms. To simulate the expected buying price, we repeat the above procedure 50,000 times for each parametric combination and each FA mechanism (all relative errors are smaller than 0.5% with 98% confidence intervals).

**Results.** We obtain the results for all 180 model instances and we summarize the main insights. The plot in the left side of Figure 4 shows the BNE bid functions under the different mechanisms for one specific instance. The bid functions in the plot are ordered in “the typical order”, which commonly repeats itself over different instances: the basic FA has the highest equilibrium bids, followed by the flexible FA, and then by the PPI FA. Equilibrium bids under the OOS FA are typically lowest (recall that under the OOS sellers are not committed to their bids). The exceptions to this ordering are cases with small $N$ (typically 2 sellers) and small $\sigma_x$, in which for small private costs bids under the basic FA may "cross" the FLE and PPI bids, and be lower than

A first-order approximation yields $\beta(z_0 - \epsilon) = \beta(z_0)$, because $\beta'(z_0) = 0$. In addition, because we are looking for BNE in strictly increasing strategies and to avoid a flat curve at the boundary we subtract $\epsilon$ from $\beta(z_0)$.

All numerical results can be obtained from the authors upon request.

---

Figure 4: **Representative results.** (Left) BNE bid function under the FA, PPI, FLE, and OOS mechanisms. In this instance there are $N = 10$ suppliers, $\mu_x = 0.5$, $\sigma_x = \frac{\mu_x}{\sqrt{12}} = \frac{\sigma_{\text{max}}}{2}$, and $z_0 = 0.25 = \frac{\mu_x}{2}$. (Right) Expected savings (relative to the expected spot market price) when $\mu_x = 0.5$, $\sigma_x = \frac{\mu_x}{\sqrt{12}}$, and $z_0 = 0.25$ for the different mechanisms as a function of the number of participating suppliers.

---

12 A first-order approximation yields $\beta(z_0 - \epsilon) = \beta(z_0)$, because $\beta'(z_0) = 0$. In addition, because we are looking for BNE in strictly increasing strategies and to avoid a flat curve at the boundary we subtract $\epsilon$ from $\beta(z_0)$.

13 All numerical results can be obtained from the authors upon request.
them. While these exceptional cases indeed fail to satisfy the conditions stated in Lemma 1 ($\sigma_x$ is too small) and Lemma 2 ($N$ is too small), it is notable that our results show that the typical order of bids is obtained in many cases in which $\sigma_x$ is much lower than the one required for Lemma 1. For instance, the plot in the left side of Figure 4 shows a typical order of bids even though $\sigma_x < z_0$.

The plot in the right side of Figure 4 shows the expected savings relative to the spot market price as measured by,

$\text{Expected Savings} = \frac{z_0 + \mu_x - \mathbb{E}[P]}{z_0 + \mu_x} \times 100\%,$

as a function of $N$ under the different mechanisms, for one model instance. Note that a larger expected savings is equivalent to a smaller expected buying price. The expected savings order that appears in the plot is “the typical order”. The expected savings under the flexible FA coincide with those of the PPI FA, because as shown in Proposition 10, the expected buying prices are the same. Moreover, their savings are the highest (implying the lowest expected buying price). The savings under the OOS are the lowest, implying the highest expected buying price (even though BNE bids can be the lowest). Finally, savings under the basic FA are in between. This typical order is consistent with the theoretic results obtained in §4 and §5 and is obtained in most instances, except in some scenarios in which the expected buying price under PPI is larger than that of the basic FA. Consistent with our theoretical results, most of these scenarios correspond to either a small number of firms ($N = 2, 4$) or a small value of $\sigma_x$ (corresponding to the smallest value of the coefficient of variation of $X$, 0.29). There are only 5 model instances that do not satisfy these requirements for which $E[P_{PPI}] > E[P_{FA}]$, but the differences are all less than 2.5%.

**Discussion.** In the plot in the right side of Figure 4 we used the letters A, B, and C to illustrate some important insights. The gap marked by “A” represents the savings generated by the basic FA compared to the spot market price. These savings show the importance of FAs as procurement mechanisms, illustrating the buyer may save up to 20% by using FAs instead of buying directly from the market and these savings increase with the number of bidders. This is because the FA “screens” the most efficient providers through competition. Typical savings across model instances vary between 5 and 30% with an average of 15%.

The gap marked by “B” represents the benefit of flexibility or, equivalently in the case of deterministic $Z$, the benefit of implementing a PPI, relative to the savings of the basic FA. The difference between the expected buying prices of PPI FA and the basic FA is usually between 5 and 10% of the expected spot market price under the typical order.

Finally, it is notable that the savings under the OOS are minimal relative to the spot market, with an average of 5% across instances. Even though OOS equilibrium bids are low, the expected buying price is large because of the lack of commitment. The gap marked by “C” represents the cost of lack of commitment: the savings gap between the FLE and the OOS. This gap typically varies between 10 and 35% of the expected spot market price.
7 Concluding remarks

Conclusions and design recommendations. In this paper we introduced a novel auction model to analyze FAs, a commonly used procurement mechanism around the world. Our paper focused on the cost uncertainty faced by bidders when participating in these agreements.

Based on our theoretical and numerical results we suggest the following prescriptions for the design of FAs. First, building and implementing perfect price indexes for the random part of costs may reduce buying prices. Second, in the absence of a perfect price index, allowing FA winners to lower and match the spot market price may also be an effective way of reducing buying prices. These conclusions are valid for many settings of practical interest. However, they need to be taken with some caution because in some instances, in particular when the number of bidders and/or the variance of the common cost are small, introducing a perfect price index or a flexible FA can actually increase buying prices. Finally, not enforcing suppliers’ commitments generally increases buying prices and this should be avoided. These prescriptions are already being considered by the Chilean government to improve their procurement processes and we hope they are also consider in the future by other buying agencies.

Future research directions. Our work is the first that provides a formal understanding on how the uncertainty suppliers face affects outcomes in FAs. In this paper we made several modeling choices; these were generally driven by practical concerns expressed by our partner ChileCompra. However, several interesting directions were left unexplored. In particular, throughout the paper we assumed risk neutral bidders and showed that even in this case the discussion regarding cost uncertainty is rich and relevant. An interesting extension of our model would be to consider risk averse bidders. This would introduce additional complications in the analysis; for example, under risk aversion revenue equivalence-style arguments may no longer hold. On the other hand, introducing risk averse bidders would allow a meaningful discussion on bidders’ use of financial instruments such as options to hedge the risk associated to the common cost. We conjecture that the use of options may play a similar role to the perfect price index in the case of risk neutral bidders to alleviate the extent of the FA catch.

FAs are rich and complex buying mechanisms. In this paper we focused on one particular important aspect of FAs, namely, the cost uncertainty bidders face. In future research, we plan to study other important dimensions of FAs that would require different analyses. In particular, an interesting direction for future research may be to study how the demand uncertainty suppliers face over the time horizon affects outcomes in FAs. Overall, we hope that this paper together with the follow up work it may generate, improve how FAs and buying processes are designed in practice.
References


Online Appendix

A Proofs

A.1 Proof of Lemma 1

The proof is similar to the approach presented in Villas-Boas (1997) to compare fixed points of different mappings. It is suffices to compare the bids when \( c \leq z_0 \). The structure of the proof is as follows: In the first part of the proof, we define two mappings, \( T_{FA}(\beta) \) and \( T_{PPI}(\beta) \) that correspond to the right-hand side of the integral equations (2) and (8), in the case of deterministic \( Z \). In the second part of the proof, we show that under condition (9), \( T_{FA} \) is larger than \( T_{PPI} \), in the sense that \( T_{FA}(\beta) \geq T_{PPI}(\beta) \) for any equilibrium candidate function \( \beta \). In the third part, we show that the latter implies that any (candidate equilibrium) fixed point of \( T_{FA} \) is larger than any (candidate equilibrium) fixed point of \( T_{PPI} \), and therefore for any BNE bid strategies \( \beta_{FA}(\cdot) \), \( \beta_{PPI}(\cdot) \), it must be that \( \beta_{PPI}(c) \leq \beta_{FA}(c) \) for all private costs \( c \). Finally, we adapt the established sufficient condition to the special case of \( X \sim N(\mu, \sigma^2) \).

Part 1. Fix \( c \in [c, z_0] \). Let \( S \subset C^1 [c, z_0] \) be the set of functions that are candidates BNE bid functions for FA or PPI FA. In particular, we define \( S \) to be the set of continuous, strictly increasing functions \( \beta(\cdot) \) on \([c, z_0]\), such that

\[
\beta(c) \geq c + \mu_x \quad \forall c \in [c, z_0]. \tag{A-1}
\]

The last inequality follows because equilibrium bids must be at least the expected cost given the private cost \( c \) so that firms make positive profits conditional on winning (see also equations (2), (6) and (8)). In addition, we define the following two mappings:

\[
T_{FA}(\beta) = c + \mathbb{E}[X | X \geq \beta(c) - z_0] + \frac{1}{F_{N-1}(c)G(\beta(c) - z_0)} \int_c^{z_0} \hat{F}_{N-1}(t)\hat{G}(\beta(t) - z_0) \, dt
\]

and

\[
T_{PPI}(\beta) = c + \mu_x + \frac{1}{F_{N-1}(c)} \int_c^{z_0} \hat{F}_{N-1}(t) \, dt
\]

We note that \( T_{PPI}(\beta) \) is a constant mapping, in the sense that it does not depend on the function \( \beta(\cdot) \).

Part 2. Fix some \( \beta \in S \). We note that if \( c \in [c, z_0] \):

\[
\frac{1}{F_{N-1}(c)G(\beta(c) - z_0)} \int_c^{z_0} \hat{F}_{N-1}(t)\hat{G}(\beta(t) - z_0) \, dt - \frac{1}{F_{N-1}(c)} \int_c^{z_0} \hat{F}_{N-1}(t) \, dt \\
\geq \frac{1}{F_{N-1}(c)G(\beta(c) - z_0)} \int_c^{z_0} \hat{F}_{N-1}(z_0)\hat{G}(\beta(z_0) - z_0) \, dt - \frac{1}{F_{N-1}(c)} \int_c^{z_0} \hat{F}_{N-1}(c) \, dt \\
= \frac{F_{N-1}(z_0)G(\beta(z_0) - z_0)}{F_{N-1}(c)G(\beta(c) - z_0)} (z_0 - c) - \frac{F_{N-1}(c)}{F_{N-1}(c)} (z_0 - c) \\
= \left( \frac{F_{N-1}(z_0)\hat{G}(\beta(z_0) - z_0)}{F_{N-1}(c)G(\beta(c) - z_0)} - 1 \right) (z_0 - c) \\
\geq z_0 \left( \frac{F_{N-1}(z_0)\hat{G}(\beta(z_0) - z_0)}{F_{N-1}(c)G(\beta(c) - z_0)} - 1 \right) \geq -z_0, \tag{A-2}
\]
since \( \beta(\cdot) \) is increasing, and therefore both \( \bar{F}(c) \) and \( \bar{G}(\beta(c) - z_0) \) are decreasing functions of \( c \), and \( c \leq z_0 \). Then, for all \( c \in [\underline{c}, z_0] \):
\[
T^{FA}(\beta) - T^{PPI}(\beta) \geq \mathbb{E}[X | X \geq \beta(c) - z_0] - \mu_x - z_0 \\
\geq \mathbb{E}[X | X \geq \mu_x - z_0] - \mu_x - z_0 \geq 0,
\]
where the first inequality follows by (A-2); the second because \( \beta(c) \geq \mu_x + c \geq \mu_x \), for all \( c \), and by Lemma B1; and the last by condition (9).

**Part 3.** Let \( \beta^{FA} \) be a fixed point of \( T^{FA} \), and let \( \beta^{PPI} \) be a fixed point of \( T^{PPI} \). Then,
\[
\beta^{PPI} = T^{PPI}(\beta^{PPI}) \overset{(a)}{=} T^{PPI}(\beta^{FA}) \overset{(b)}{\leq} T^{FA}(\beta^{FA}) = \beta^{FA},
\]
where (a) holds since \( T^{PPI} \) is independent of \( \beta \), and where (b) follows from applying (A-3) with \( \beta = \beta^{FA} \). Finally, let \( \beta^{PPI}(c) \) be an increasing and symmetric equilibrium bid function under the PPI FA, and let \( \beta^{FA}(c) \) be an increasing and symmetric equilibrium bid function under the basic FA. Then, \( \beta^{PPI}(c) \) is a solution of integral equation (8) and therefore is a fixed point of \( T^{PPI} \) when \( Z \) is deterministic (Proposition 4). In addition, \( \beta^{FA}(c) \) is a solution of integral equation (2) and therefore is a fixed point of \( T^{FA} \) when \( Z \) is deterministic (Proposition 1). By (A-4) we have that \( \beta^{PPI}(c) \leq \beta^{FA}(c) \) for all \( c \in [\underline{c}, z_0] \), and overall the result holds for all \( c \in [\underline{c}, \bar{c}] \). Note that the latter inequality is strict if condition (9) is strict.

**Part 4.** We consider the special case of normally distributed common cost and assume \( X \sim N(\mu, \sigma^2) \). Then,
\[
\mathbb{E}[X | X \geq \mu - z_0] - \mu = \frac{\sigma \cdot \phi \left( \frac{\mu - z_0 - \mu}{\sigma} \right)}{1 - \Phi \left( \frac{\mu - z_0 - \mu}{\sigma} \right)} = \frac{\sigma \cdot \phi \left( \frac{-z_0}{\sigma} \right)}{1 - \Phi \left( \frac{-z_0}{\sigma} \right)} = \frac{\sigma \cdot \phi \left( \frac{z_0}{\sigma} \right)}{\Phi \left( \frac{z_0}{\sigma} \right)},
\]
where \( \phi \) is the density of a standardized normal random variable, and \( \Phi \) is the corresponding cumulative distribution function. Therefore, the sufficient condition (9) is equivalent to the inequality
\[
\frac{\phi \left( \frac{z_0}{\sigma} \right)}{\frac{z_0}{\sigma} \cdot \Phi \left( \frac{z_0}{\sigma} \right)} \geq 1.
\]
The function \( f(t) = \frac{\phi(t)}{t \cdot \Phi(t)} \) is decreasing in \( t \), and the equation \( f(t) = 1 \) admits the unique and positive solution \( t_0 \approx 0.506 \). Therefore, the inequality (A-5) holds if and only if \( \frac{z_0}{\sigma} \leq t_0 \), or \( \sigma \geq z_0/t_0 \approx 1.976 z_0 \).

\[\square\]

**A.2 Proof of Lemma 2**

In the proof we use the mappings \( T^{FA}_N(\beta) \) and \( T^{PPI}_N(\beta) \) as defined in the proof of Lemma 1, where we explicitly index the number of bidders \( N \). We know that for any equilibrium bid function (see equations (2), (6) and (8)):
\[
\beta(c) \geq c + \mu_x \geq c + \varepsilon + z_0 \geq \varepsilon + z_0, \quad \forall c \in [\underline{c}, \bar{c}] .
\]
\[\text{(A-6)}\]
Therefore, comparing the implied cost terms of the mappings, one has:

\[
\mathbb{E}[X \mid X > \beta(c) - z_0] - \mu_x \overset{(a)}{\geq} \mathbb{E}[X \mid X > \epsilon] - \mu_x \overset{(b)}{\geq} \frac{\mu_x - \epsilon G(\epsilon)}{G(\epsilon)} - \mu_x
\]

\[
= \frac{(\mu_x - \epsilon) G(\epsilon)}{G(\epsilon)} \geq \frac{\alpha z_0}{1 - \alpha} > 0, \tag{A-7}
\]

where (a) follows from (A-6) and (b) holds by

\[
\mathbb{E}[X] = \mathbb{E}[X \mid X > \epsilon] G(\epsilon) + \mathbb{E}[X \mid X \leq \epsilon] G(\epsilon) \leq \mathbb{E}[X \mid X > \epsilon] G(\epsilon) + \epsilon G(\epsilon).
\]

In addition, comparing the markup terms of the mappings, and taking the limit as \( N \to \infty \), one has:

\[
\frac{1}{F N^{-1}(\epsilon) G(\beta(c) - z_0)} \int_c^{z_0} F N^{-1}(t) \bar{G}(\beta(t) - z_0) dt - \frac{1}{F N^{-1}(\epsilon)} \int_c^{z_0} \bar{F} N^{-1}(t) dt
\]

\[
= \frac{1}{G(\beta(c) - z_0)} \int_c^{z_0} \left( \frac{\bar{F}(t)}{F(c)} \right)^{N-1} \bar{G}(\beta(t) - z_0) dt - \int_c^{z_0} \left( \frac{\bar{F}(t)}{F(c)} \right)^{N-1} dt \to 0,
\]

since \( \frac{\bar{F}(t)}{F(c)} < 1 \) for all \( t \in (c, z_0) \) and the dominated convergence theorem. Also note that the difference is negative, because any candidate equilibrium bidding function is increasing, and therefore \( \bar{G}(\beta(t) - z_0) / \bar{G}(\beta(c) - z_0) \leq 1, \forall t \in [c, z_0] \). Therefore, there exists \( N_0 \), such that for all \( N > N_0 \), one has

\[
-\frac{1}{F N^{-1}(\epsilon) G(\beta(c) - z_0)} \int_c^{z_0} F N^{-1}(t) \bar{G}(\beta(t) - z_0) dt + \frac{1}{F N^{-1}(\epsilon)} \int_c^{z_0} \bar{F} N^{-1}(t) dt < \frac{\alpha z_0}{1 - \alpha},
\]

and therefore by the previous inequality and (A-7) we get that for all \( N > N_0 \), \( T_N^{FA}(\beta) - T_N^{PPI}(\beta) > 0 \). Following (A-4), if \( \beta_N^{PPI}(c) \) is a solution of integral equation (8) and therefore is a fixed point of \( T_N^{PPI} \), and \( \beta_N^{FA}(c) \) is a solution of integral equation (2) and therefore is a fixed point of \( T_N^{FA} \), we have that \( \beta_N^{PPI}(c) < \beta_N^{FA}(c) \) for all \( c \in [c, z_0] \) and \( N > N_0 \), proving the result. \( \square \)

### A.3 Proof of Proposition 5

Recall that if the auctioneer does not buy from one of the FA bidders, she buys from the spot market. Therefore, the expected buying price for an FA given its equilibrium strategy can be expressed as:

\[
\mathbb{E}[P] = \mathbb{E} \left[ \sum_{i=1}^{N} t_i(\beta(c), X, Z) + (1 - \sum_{i=1}^{N} p_i(\beta(c), X, Z)) (Z + X) \right], \tag{A-8}
\]

where the expectation is taken with respect to the random variables \( X \) and \( Z \), and the random vector \( c \). In addition, throughout this proof we use the notation \( \mathbb{E}_{-i} \) to denote expectation with respect to \( X \), \( Z \), and the random vector \( c_{-i} \). Consider the equilibrium payoff for bidder \( i \):

\[
\pi_i(\beta_i(c_i), c_i, \beta_{-i}) = \mathbb{E}_{-i} \left[ t_i(\beta_i(c_i), \beta_{-i}(c_{-i}), X, Z) - (c_i + X)p_i(\beta_i(c_i), \beta_{-i}(c_{-i}), X, Z) \right]. \tag{A-9}
\]
Using the envelope theorem, similarly to equation (5), and using the fact that $\pi_i(\beta_i(\bar{c}), \bar{c}, \beta_{-i}) = 0$, we obtain:

$$\pi_i(\beta_i(c_i), c_i, \beta_{-i}) = \int_{c_i}^\infty \mathbb{E}_{-i} \left[ p_i(\beta_i(y), \beta_{-i}(c_{-i}), X, Z) \right] dy. \tag{A-10}$$

Equating (A-9) and (A-10) we obtain:

$$\mathbb{E}_{-i} \left[ i_i(\beta_i(c_i), \beta_{-i}(c_{-i}), X, Z) \right] = \mathbb{E}_{-i} \left[ (c_i + X) p_i(\beta_i(c_i), \beta_{-i}(c_{-i}), X, Z) \right] + \int_{c_i}^\infty \mathbb{E}_{-i} \left[ p_i(\beta_i(y), \beta_{-i}(c_{-i}), X, Z) \right] dy. \tag{A-11}$$

Replacing in equation (A-8) and using the fact that the private costs $c_i$ are independent across firms, we get:

$$\mathbb{E}[P] = \mathbb{E} \left[ \sum_{i=1}^N \left( (c_i + X) p_i(\beta(c), X, Z) + \int_{c_i}^\infty p_i(\beta_i(y), \beta_{-i}(c_{-i}), X, Z) dy \right) \right] + \mathbb{E} \left[ \left( 1 - \sum_{i=1}^N p_i(\beta(c), X, Z) \right) (Z + X) \right] = \mu_x + \mu_z + \mathbb{E} \left[ \sum_{i=1}^N \left( p_i(\beta(c), X, Z)(c_i - Z) + \int_{c_i}^\infty p_i(\beta_i(y), \beta_{-i}(c_{-i}), X, Z) dy \right) \right]. \tag{A-12}$$

Next, note that

$$\mathbb{E} \left[ \int_{c_i}^\infty p_i(\beta_i(y), \beta_{-i}(c_{-i}), X, Z) dy \right] = \mathbb{E}_{-i} \left[ \int_{c_i}^\infty \int_{y}^{\infty} p_i(\beta_i(y), \beta_{-i}(c_{-i}), X, Z) f_c(c_i) dc_i dy \right] = \mathbb{E}_{-i} \left[ \int_{c_i}^\infty \int_{y}^{\infty} p_i(\beta_i(y), \beta_{-i}(c_{-i}), X, Z) f_c(c_i) dc_i dy \right] = \mathbb{E} \left[ p_i(\beta(c), X, Z) F_c(c_i)/f_c(c_i) \right], \tag{A-12}$$

where the first equation follows by the independence of the private costs and the second by changing the order of integration. Replacing (A-12) in (A-11), we obtain:

$$\mathbb{E}[P] = \mu_x + \mu_z + \mathbb{E} \left[ \sum_{i=1}^N p_i(\beta(c), X, Z)(c_i + F_c(c_i)/f_c(c_i) - Z) \right],$$

proving the result.

### A.4 Proof of Proposition 9

The proof is similar to that of Proposition 1. We omit the superscript FLE to simplify notation. Recall that

$$\pi_i(b, c, \beta) = F_b^{-1}(\beta^{-1}(b)) \mathbb{E} \left[ (b - c - X) \cdot 1 \{b \leq Z + X\} + (Z - c) \cdot 1 \{b > Z + X \geq c + X\} \right].$$
Note that
\[
\mathbb{E}[(Z - c) \cdot \mathbb{I}\{b > Z + X \geq c + X\}] = \mathbb{E}_Z \left[ \mathbb{E}_X \left[ (Z - c) \cdot \mathbb{I}\{b > Z + X \geq c + X\} \left| Z \right. \right] \right] \\
= \int_c^\bar{z} (z - c) \mathbb{E}_X [\mathbb{I}\{b > z + X\}] h(z) \, dz.
\]

Taking derivatives w.r.t. \(c\) in the above equation and applying Leibniz rule, one obtains:
\[
\frac{\partial}{\partial c} \left( \mathbb{E}[(Z - c) \cdot \mathbb{I}\{b > Z + X \geq c + X\}] \right) = \frac{\partial}{\partial c} \left( \int_c^\bar{z} (z - c) \mathbb{E}_X [\mathbb{I}\{b > z + X\}] h(z) \, dz \right) \\
= -\int_c^\bar{z} \mathbb{E}_X [\mathbb{I}\{b > z + X\}] h(z) \, dz \\
= -\mathbb{E} [\mathbb{I}\{b > Z + X, Z \geq c\}] \\
\quad \text{ (A-13)}
\]

Similarly to equation (5), we apply the envelope theorem to obtain:
\[
-\pi_i(\beta(c), c, \beta) = \int_c^\bar{z} \left. \frac{\partial \pi_i(b, t, \beta)}{\partial t} \right|_{b=\beta(t)} \, dt \\
= -\int_c^\bar{z} \tilde{F}^{N-1}(t) \cdot \mathbb{E} [\mathbb{I}\{\beta(t) \leq Z + X\} + \mathbb{I}\{\beta(t) > Z + X, Z \geq c\}] \, dt.
\]

The reminder of the proof can essentially be followed line-by-line to the proof of Proposition 1. \(\square\)

### A.5 Proof of Proposition 12

If the action set \(\mathcal{A}\) is appropriately chosen, any symmetric BNE strategy \(\beta\) must be interior and satisfy the first order conditions:
\[
\frac{\partial \pi_i(\beta(c), c, \beta)}{\partial b} = 0. \quad \text{ (A-14)}
\]

For the basic FA, by Eq. (1) and the fact that \(\frac{\partial \beta^{-1}(b)}{\partial b}|_{b=\beta(c)} = \frac{1}{\beta'(c)}\), one has
\[
\frac{\partial \pi_i^{FA}(b, c, \beta)}{\partial b} \bigg|_{b=\beta(c)} = -\frac{(N - 1) f(c) \tilde{F}^{N-2}(c)}{\beta'(c)} \cdot \int_{\beta(c)-z_0}^{\bar{x}} \left( \tilde{G} (\beta(c) - z_0) - (z_0 - c) g(\beta(c) - z_0) \right) \, dx \\
+ \tilde{F}^{N-1}(c) \cdot \left[ \tilde{G} (\beta(c) - z_0) - (z_0 - c) g(\beta(c) - z_0) \right].
\]

Thus, by (A-14), we have shown that any symmetric equilibrium satisfies ODE (20).

Now, we show the second part. Recall that the bid for the basic FA satisfies: \(\beta(c) \geq c + \mu_x \geq z_0\), for e.g., by (A-1). Also note that, by Eq. (1), \(0 \leq \pi_i(\beta(c), c, \beta) = \tilde{F}^{N-1}(c) \cdot \mathbb{E}_X \left[ (\beta(c) - c - x) \cdot \mathbb{I}\{\beta(c) \leq z_0 + X\} \right] \). Therefore, by the increasing property of the symmetric equilibrium and (20), we obtain
\[
0 \leq \tilde{G} (\beta(c) - z_0) - (z_0 - c) g(\beta(c) - z_0) = \frac{\bar{x} - \beta(c) + c}{\bar{x}},
\]

This implies \(\beta(c) \leq c + \bar{x}\) for any \(c \in [\underline{c}, z_0]\). \(\square\)
A.6 Proof of Proposition 13

By (10) and (19), one has

\[
\mathbb{E}[P_{FA}] - \mathbb{E}[P_{OOS}] = \mathbb{E}_{c(1),X}\left[-(z_0 + X - \beta_{FA}(c(1))) \mathbb{1}\{\beta_{FA}(c(1)) \leq z_0 + X\} + (z_0 + X - \beta_{OOS}(c(1))) \mathbb{1}\{c(1) + X \leq \beta_{OOS}(c(1)) \leq z_0 + X\}\right] \\
\stackrel{(a)}{=} \int_\xi \mathbb{E}_X\left[-(z_0 + X - \beta_{FA}(t)) \mathbb{1}\{\beta_{FA}(t) \leq z_0 + X\} + (z_0 + X - \beta_{OOS}(t)) \mathbb{1}\{t + X \leq \beta_{OOS}(t) \leq z_0 + X\}\right] dF_{c(1)}(t) \\
\stackrel{(b)}{=} \int_\xi \left[-\int_{\beta_{FA}(t)-z_0}^{\bar{x}} \frac{(z_0 + x - \beta_{FA}(t))}{\bar{x}} dx + \int_{\min\{z_0, \beta_{OOS}(t)-z_0\}}^{\max\{0, \beta_{OOS}(t)-z_0\}} \frac{(z_0 + x - \beta_{OOS}(t))}{\bar{x}} dx\right] dF_{c(1)}(t) \\
\stackrel{(c)}{\leq} \int_\xi \left[-\int_{\beta_{FA}(t)-z_0}^{\bar{x}} \frac{(z_0 + x - \beta_{FA}(t))}{\bar{x}} dx + \int_{\beta_{OOS}(t)-z_0}^{\beta_{OOS}(t)-t} \frac{(z_0 + x - \beta_{OOS}(t))}{\bar{x}} dx\right] dF_{c(1)}(t) \\
\stackrel{(d)}{=} \int_\xi \left[-\int_0^{\bar{x}+z_0-\beta_{FA}(t)} \frac{y}{\bar{x}} dy + \int_0^{z_0-t} \frac{y}{\bar{x}} dy\right] dF_{c(1)}(t) \\
\stackrel{(e)}{=} \int_\xi \left[-\frac{(z_0 + \bar{x} - \beta_{FA}(t))^2}{2\bar{x}} + \frac{(z_0 - t)^2}{2\bar{x}}\right] dF_{c(1)}(t) \\
\leq 0,
\]

where (a) follows from the independence of \(c(1)\) and \(X\); (b) holds by \(X \sim [0, \bar{x}]\); (c) holds by combining the following two cases: (c.1) If \(\beta_{OOS}(t) - z_0 < 0\), obviously, \(\int_{\beta_{OOS}(t)-z_0}^{\beta_{OOS}(t)-t} \frac{(z_0 + x - \beta_{OOS}(t))}{\bar{x}} dx \geq 0\); (c.2) If \(\beta_{OOS}(t) - t \geq \bar{x}\), \(\int_{\beta_{OOS}(t)-z_0}^{\beta_{OOS}(t)-t} \frac{(z_0 + x - \beta_{OOS}(t))}{\bar{x}} dx \geq 0\), since \(\beta_{OOS}(t) \leq z_0 + \bar{x}\), for all \(t \leq z_0\); (d) holds by the change of variables \(y = z_0 + x - \beta_{FA}(t)\) and \(y = z_0 + x - \beta_{OOS}(t)\); (e) holds since, by Lemma 12, \(\beta_{FA}(t) \leq t + \bar{x}\) and \(2z_0 + \bar{x} - t \geq t + \bar{x} \geq \beta_{FA}(t)\) for all \(t \leq z_0\). Hence, \(\mathbb{E}[P_{FA}] \leq \mathbb{E}[P_{OOS}]\).

A.7 Proof of Proposition 14

We derive the boundary conditions for the basic FA; the boundary conditions for the FLE FA and the PPI FA can be characterized in a similar way and the proofs are thus omitted. In the proof, we will omit the superscript “FA”. Taking limit \(c \uparrow z_0\) in (2), one obtains

\[
\beta(z_0) = z_0 + \mathbb{E}\left[X | \beta(z_0) \leq z_0 + X\right].
\]

It is easy to verify that the only solution to the above equation is \(\beta(z_0) = z_0 + \bar{x}\) by applying L’Hôpital’s rule. Next, we show \(\beta'(z_0) = 0\).
Thus, we have proved the lemma.

For any $y$, let $y = \bar{y}$, and/or $\bar{y} = \infty$ for $\beta_1(y)$ with pdf $f$, cdf $F$ and support $[y, \bar{y}]$ (possibly $y = -\infty$ and/or $\bar{y} = \infty$), let

$$B(a,b) = \mathbb{E}[Y|a < Y < b] = \frac{\int_a^b t dH(t)}{H(b) - H(a)}, \quad y \leq a < b \leq \bar{y}.$$ 

Then, $B(a,b)$ is increasing in $a$ and $b$.

**Proof.** For any $y \leq a < b \leq \bar{y}$, we have

\[
\frac{\partial B}{\partial a}(a,b) = -ah(a)[H(b) - H(a)] + h(a) \int_a^b tdH(t) \\
\frac{\partial B}{\partial b}(a,b) = bh(b)[H(b) - H(a)] - h(b) \int_a^b tdH(t)
\]


Thus, we have proved the lemma.
B.2 Proof of Proposition 2

First, we introduce the following definition. A twice-differentiable function \( h : \mathbb{R}^2 \rightarrow \mathbb{R} \) is called supermodular or log-supermodular, respectively, if for all \( x \) and \( \theta \):

\[
\frac{\partial^2 h(x, \theta)}{\partial x \partial \theta} \geq 0, \quad \text{or if } h > 0 \quad \frac{\partial^2 \ln(h(x, \theta))}{\partial x \partial \theta} \geq 0.
\]

Note that these are sufficient conditions for supermodularity. There are weaker related conditions that do not require differentiability and use function differences.

**Definition B1.** (Athey 2001) The Single Crossing Condition (SCC) for games of incomplete information is satisfied if for each \( i = 1, 2, \ldots, N \), whenever every opponent \( j \neq i \) uses a strategy \( \beta_j \) that is increasing, player \( i \)'s profit function, \( \pi_i(b_i, c_i, \beta_{-i}) \) is supermodular or log-supermodular in \((b_i, c_i)\).

We have the following result.

**Proposition B1.** The basic FA game satisfies the SCC property. Hence, it admits a symmetric BNE in increasing strategies if the action space \( A \) is restricted to be finite.

**Proof.** Recall that the private costs \( \{c_j\} \) are i.i.d. and independent with the common cost \( X \). For any increasing profile \( \beta_{-i} \), by (1), we have

\[
\pi_i(b, c, \beta_{-i}) = \mathbb{P}[i \text{ wins with } b] \cdot \mathbb{E}[(b - c - X)I\{b \leq X + Z\}].
\]

(B-1)

where \( \mathbb{P}[i \text{ wins with } b] \) is the probability bidder \( i \) defeats its competitors’ with a bid \( b \) and is given by:

\[
\mathbb{P}[i \text{ wins with } b] = \mathbb{P}(\beta_j(c_j) > b, \text{ for all } j \neq i) + \sum_{k=1}^{N-1} \mathbb{P}(\text{exactly } k \text{ bidders other than } i \text{ bid } b \text{ and the rest higher than } b) \frac{k + 1}{k + 1} \mathbb{P}[\text{winning probability with ties}].
\]

It can be easily shown that the winning probability \( \mathbb{P}[i \text{ wins with } b] \) is decreasing in \( b \), i.e., a higher bid induces a lower winning probability. Thus, taking the partial derivatives with respect to \( c \), we have

\[
\frac{\partial \pi_i(b, c, \beta_{-i})}{\partial c} = -\mathbb{P}[i \text{ wins with } b] \cdot \mathbb{P}(b \leq X + Z).
\]

Since both \( \mathbb{P}(b \leq X + Z) \) and \( \mathbb{P}[i \text{ wins with } b] \) are decreasing in \( b \), the partial derivative \( \frac{\partial \pi_i(b, c, \beta_{-i})}{\partial c} \) is increasing in \( b \). Therefore, by Definition B1, SCC is satisfied. The existence of a symmetric BNE in increasing strategies follows by Theorem 1 in Athey (2001).

To complete the proof of Proposition 2 we need to verify that our basic FA game satisfies some technical conditions required for a limiting argument used by Athey (2001) to pass from games with finite action spaces to games with continuous action space. We postpone this argument to §B.4. We also note that a similar analysis can be used to establish the SCC property for the other FA models studied in the paper.
B.3 Proof of Proposition 3

Part 1. Equilibrium is strictly increasing. We need the following Lemma for the proof.

Lemma B2. Let $\beta(\cdot)$ be a symmetric and increasing BNE strategy of the basic FA model. Then, $\beta(c) < \bar{z} + \bar{x}$, for all $c < \bar{z}$.

Proof. We argue by contradiction. Suppose that $\beta(c) = \bar{z} + \bar{x}$, for some $c < \bar{z}$. Note that in this case, $\pi_i(\beta(c), c, \beta) = 0$, because bidder $i$ with private cost $c$ has no chance of defeating the spot market. We show that $\beta'(c) = \bar{z} + \bar{x} - \epsilon$, for small enough $\epsilon > 0$ is a profitable unilateral deviation, so the initially proposed strategy cannot be a BNE. Let $\{i \text{ wins}\}$ be the event in which $\beta_i(c_i) \leq \beta_j(c_j)$, $\forall j$ and bidder $i$ is selected in case of a tie. Then,

$$
\pi_i(\beta'(c), c, \beta) = \mathbb{P}[i \text{ wins}] \cdot \mathbb{E} \left[ (\beta'(c) - c - X) \cdot 1 \{ \beta'(c) \leq Z + X \} \right] \\
= \mathbb{P}[i \text{ wins}] \cdot \mathbb{E} \left[ (\bar{z} - c - \epsilon + (\bar{x} - X)) \cdot 1 \{ \beta'(c) \leq Z + X \} \right] \\
= \mathbb{P}[i \text{ wins}] \cdot [ (\bar{z} - c - \epsilon) \mathbb{P}[\beta'(c) \leq Z + X] + \mathbb{E} [(\bar{x} - X) \cdot 1 \{ \beta'(c) \leq Z + X \}] ] .
$$

Clearly, $\mathbb{P}[i \text{ wins}] > 0$, $\mathbb{P}[\beta'(c) \leq Z + X] > 0$, and $(\bar{x} - x) \geq 0$, for all realizations $x$. Moreover, for small enough $\epsilon$, $\bar{z} - c - \epsilon > 0$. The result follows.

Now, we show that the equilibrium is strictly increasing by contradiction. Let us write:

$$
\pi_i(\beta(c), c, \beta) = \mathbb{P}[i \text{ wins}] \cdot \left( \beta(c) - c - \mathbb{E} \left[ X \middle| \beta(c) \leq Z + X \right] \right) \mathbb{P}[\beta(c) \leq Z + X].
$$

Assume that there is an interval with positive length $[\hat{c}_1, \hat{c}_2]$, with $\hat{c}_2 < \hat{z}$, such that $\beta(c) = \hat{b}$ for all $c \in [\hat{c}_1, \hat{c}_2]$. We consider two cases. First, suppose that $\pi_i(\beta(\hat{c}_2), \hat{c}_2, \beta) > 0$. In this case,

$$
\left( \beta(\hat{c}_2) - \hat{c}_2 - \mathbb{E} \left[ X \middle| \beta(\hat{c}_2) \leq Z + X \right] \right) \mathbb{P}[\beta(\hat{c}_2) \leq Z + X] > 0. $$

It is simple to observe that the bidder with private cost $\hat{c}_2$ is strictly better off by unilaterally deviating from $\hat{b}$ to $\hat{b} - \delta$ for small enough $\delta > 0$. To see this, note that with this deviation, $\mathbb{P}[i \text{ wins}]$ increases by a strictly positive discrete amount and the other terms in $\pi_i(\beta(\hat{c}_2), \hat{c}_2, \beta)$ remain essentially unchanged for small enough $\delta > 0$ by continuity.

The second case we consider is $\pi_i(\beta(\hat{c}_2), \hat{c}_2, \beta) = 0$. Because $\hat{c}_2 < \hat{z}$, $\beta(\cdot)$ is increasing, and Lemma B2, it must be that $\mathbb{P}[i \text{ wins}] \cdot \mathbb{P}[\beta(\hat{c}_2) \leq Z + X] > 0$. Hence, it must be that $\beta(\hat{c}_2) - \hat{c}_2 - \mathbb{E} \left[ X \middle| \beta(\hat{c}_2) \leq Z + X \right] = 0$. Take small enough $\epsilon$, for which $\beta(\hat{c}_2 - \epsilon) = \beta(\hat{c}_2)$. We have that $\pi_i(\beta(\hat{c}_2 - \epsilon), \hat{c}_2 - \epsilon, \beta) > 0$, and we can use the same argument regarding a unilateral deviation like in the first case. The result follows.

Part 2. Equilibrium is continuous. We show it by contradiction using the first part of the proof. Assume there is a symmetric and strictly increasing equilibrium $\beta(\cdot)$ and $\hat{c}_1 \in [\hat{z}, \bar{z}]$ such that $\beta(\cdot)$ has a jump at $\hat{c}_1$. Let the left-limit and right-limit of $\beta$ at $\hat{c}_1$ be $b_- = \lim_{c \searrow \hat{c}_1} \beta(c)$ and $b_+ = \lim_{c \nearrow \hat{c}_1} \beta(c)$, respectively. Then, $b_- < b_+$. By (B-1) and the fact that the ties happens with probability zero, one has

$$
\pi_i(b, \hat{c}_1, \beta) = \mathbb{P}(b < \beta(c_j), j \neq i) \cdot \mathbb{E} \left[ (b - \hat{c}_1 - X) \cdot 1 \{ b \leq X + Z \} \right] \mathbb{P}[\beta(c) \leq Z + X] , \text{ for any } b \in [b_-, b^+] . \tag{B-2}
$$

There are two cases to consider. Suppose $\beta(\hat{c}_1) = b_-$. Then, $b_-$ must be the maximum of $\mathbb{E} \left[ (b - \hat{c}_1 - X) \cdot 1 \{ b \leq X + Z \} \right]$ by the previous equation. Moreover, by continuity and taking the limit $\lim_{c \searrow \hat{c}_1}$, $b_+$ must also be the maximum of $\mathbb{E} \left[ (b - \hat{c}_1 - X) \cdot 1 \{ b \leq X + Z \} \right]$. This contradicts
our assumption of the unique maximum of $\mathbb{E}[(b - c - X)\mathbb{I}\{b \leq X + Z\}]$ for any $c$. The second case is analogous, proving the result. 

\section*{B.4 Existence of a Symmetric BNE for Games with Compact Action Space}

In this section, we show that an increasing symmetric pure strategy BNE exists for the basic FA model, by applying Theorem 6 in Athey (2001). For self-completeness, we briefly summarize notation and assumptions made in Theorem 6 of Athey (2001). After introducing the theorem, we then show that all conditions are satisfied for the basic FA model.

\textbf{Part 1. Restatement of results in Athey (2001).} Consider a game of incomplete information between $I$ players, $i = 1, \ldots, I$, where each player first observes his own type $t_i \in T_i = [\underline{t}_i, \bar{t}_i]$ and then takes an action $a_i$ from a compact set $\mathcal{A}_i \subseteq \mathbb{R}$. Let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_I$, $T = T_1 \times \cdots \times T_I$, $\underline{a} = \min \mathcal{A}$, and $\bar{a} = \max \mathcal{A}$. The joint density over player types is $f(\cdot)$, with the conditional density of $t_{-i}$ given $t_i$ denoted $f(t_{-i}|t_i)$. Player $i$’s payoff function is $u_i : \mathcal{A} \times T \to \mathbb{R}$. Given any set of strategies for the opponents, $\alpha_j : T_j \to \mathcal{A}_j, j \neq i$, player $i$’s objective function is defined as follows (using the notation $(a_i, \alpha_{-i}(t_{-i})) = (\cdots, \alpha_{i-1}(t_{i-1}), a_i, \alpha_{i+1}(t_{i+1}), \cdots)$):

$$U_i(a_i, t_i, \alpha_{-i}(t_{-i})) = \int_{t_{-i}} u_i((a_i, \alpha_{-i}(t_{-i})), t) f(t_{-i}|t_i) dt_{-i}.$$  

\textbf{Assumption B1.} The types have joint density with respect to Lebesgue measure, $f(\cdot)$, which is bounded and atomless. Further, $\int_{t_{-i} \in S} u_i((a_i, \alpha_{-i}(t_{-i})), t) f(t_{-i}|t_i) dt_{-i}$ exists and is finite for all convex $S$ and all increasing functions $\alpha_j : T_j \to \mathcal{A}_j, j \neq i$.

For games with finite action spaces, say $\mathcal{A}_i = \{A_0, A_1, \ldots, A_M\}$, she shows that the Kakutani’s fixed point theorem is applicable when SCC is satisfied. Thus, a pure strategy BNE exists for games with finite action spaces. For games with compact action spaces, she assumes that player $i$’s payoff, given a realization of types and actions, has the following form

$$u_i(a, t) = \varphi_i(a) \cdot \bar{v}_i(a_i, t) + (1 - \varphi_i(a)) \cdot \underline{v}_i(a_i, t) = \underline{v}_i(a_i, t) + \varphi_i(a) \cdot \Delta v_i(a_i, t), \quad (B-3)$$

where $\Delta v_i(a_i, t) = \bar{v}_i(a_i, t) - \underline{v}_i(a_i, t)$. Intuitively, the winners receive payoffs $\bar{v}_i(a_i, t)$ with probability $\varphi_i(a)$, while losers receive payoffs $\underline{v}_i(a_i, t)$ with probability $1 - \varphi_i(a)$. In most auction models, participation is voluntary: there is some outside option such as not placing a bid that provides a fixed certain utility to the agent, typically normalized to zero. We refer to this action as $Q$. We introduce the following assumption.

\textbf{Assumption B2.} There exists $\lambda > 0$ such that, for all $i = 1, \ldots, I$, all $a_i \in [\underline{a}_i, \bar{a}_i]$ and all $t \in T$:

(i) the types have support $T_1 \times \cdots \times T_I$; (ii) $\bar{v}_i(a_i, t)$ and $\underline{v}_i(a_i, t)$ are bounded and continuous in $(a_i, t)$; (iii) $\bar{v}_i(Q, t) = 0, \underline{v}_i(Q, t) = 0, \underline{v}_i(a_i, t) \leq 0$, and $\Delta v_i(a_i, t) < 0$; (iv) $\Delta v_i(a_i, t)$ is strictly increasing in $(-a_i, t_i)$; (v) for all $\varepsilon > 0$, $\Delta v_i(a_i, t_{-i}, t_i + \varepsilon) - \Delta v_i(a_i, t_{-i}, t_i) \geq \lambda \varepsilon$.

Let $W_i(a_i, \alpha_{-i})$ denote the event that the realization of $t_{-i}$ and the outcome of the tie-breaking mechanism are such that player $i$ wins with $a_i$, when opponents use strategies $\alpha_{-i}$ with the realization of $t_{-i}$. Thus,

$$\mathbb{P}(W_i(a_i, \alpha_{-i})|t_i) = \int \varphi_i(a_i, \alpha_{-i}(t_{-i})) f(t_{-i}|t_i) dt_{-i}. \quad (B-4)$$
Assumption B3. For all $i = 1, \ldots, I$, all $a_i, a_i' \in [a_i, \bar{a}_i]$, and whenever every opponent $j \neq i$ uses a strategy $\alpha_j$ that is increasing, $E_{t_{i-1}} \left[ \Delta v_i(a_i, t) \bigg| t_i, W_i(a_i', \alpha_{-i}) \right]$ is strictly increasing in $t_i$ and increasing in $a_i'$.

Theorem B1. (Athey 2001) For all $i$, let $A_i = Q \cup [a_i, \bar{a}_i]$. Suppose Assumptions B1, B2, and B3 hold, and that the game satisfies the SCC. Then, there exists a pure strategy BNE in increasing strategies.

It is simple to use the previous result to establish the existence of a symmetric BNE for symmetric games with incomplete information, which is our case of interest.

Part 2. Verifications of the Assumptions B1–B3. Now, we are ready to show the existence of a BNE in the basic FA model by verifying the conditions in Assumptions B1–B3. This together with the verification of SCC guarantee the existence of increasing BNE. Our proof is presented for the general case of random $Z$. For this we need one additional assumption:

Assumption B4. Assume that the random variables $X$ and $Z$ satisfy: $E[|X| X+Z > b]$ is increasing in $b$, for all $b \geq 0$.

The above assumption is used to guarantee Assumption B3. It can be shown that the condition in the assumption is satisfied in the following cases: 1) if $Z$ is deterministic, which is the case stated in Proposition 2; 2) if $Z$ and $X$ are independent and identically distributed; and 3) if $Z$ and $X$ are both uniformly distributed (with potentially different supports).

Recall that a bidder in a basic FA will participate only if his private cost is less than $\bar{z}$. Since the bid has to be at least the private cost, thus the lowest possible rational bid is $\bar{c}$. For technical reasons, however, we define $\bar{b} = \bar{c} - \Delta$, $\Delta > 0$. Also, the bid will not be higher than $\bar{z} + \bar{x}$, namely, $\bar{b} = \bar{z} + \bar{x}$, which is the highest possible price in the spot market. To handle the two random components in the spot market price, we decompose the spot market into two “virtual” bidders: one has private cost $c_0^1 = x$ and the other has private cost $c_0^2 = z$, and they bid their true cost, i.e., $b_0^1 = x$ and $b_0^2 = z$. The FA winner competes with the “aggregate” price of these two virtual bidders, with bid $b_0^i = b_0^i + b_0^j$. To be consistent with notations in Athey (2001), we make the following transformation of the private cost and the bids: for all bidders $i = 1, 2, \ldots, N$ with private cost $c_i$ and bid $b_i$, let

\[
\begin{align*}
    a_i &= \bar{c} + \bar{x} - b_i, \quad a_i' = \bar{c} + \bar{x} - b = \bar{c} + \bar{x} - c + \Delta, \\
    t_i &= \bar{c} + \bar{x} - c_i, \quad t_i' = \bar{c} + \bar{x} - c, \\
    a_0^1 &= t_0^1 = \bar{x} - x, \quad a_0^1 = t_0^1 = 0, \quad a_0^1 = t_0^1 = \bar{x} - \bar{x}, \\
    a_0^2 &= t_0^2 = \bar{c} - z, \quad a_0^2 = t_0^2 = \bar{c} - z, \quad a_0^2 = t_0^2 = \bar{c} - \bar{z}.
\end{align*}
\]

For any given $a = (a_0^0, a_0^1, a_1, \ldots, a_N), t = (t_0^0, t_0^1, t_1, \ldots, t_N)$, corresponding to $\alpha_{-i}$, our basic FA model can be specified as follows: For any $i = 1, 2, \ldots, N$,

\[
\begin{align*}
    \varphi_i(a) &= \begin{cases} 
    1, & \text{if } b_i < b_j, \forall j \neq i \\
    0, & \text{o.w.}
    \end{cases} = \mathbb{I} \{ b_i < b_j, \forall j \neq i \} = \mathbb{I} \{ a_i > a_j, \forall j \neq i \}, \\
    \varphi_i(a_i, t) &= 0, \quad \nu_i(a_i, t) = b_i - c_i - x = t_i - a_i - x = t_i - a_i - \bar{x} + t_i^0.
\end{align*}
\]

For simplification, we ignore ties in the winning probability in $\varphi_i$; a similar analysis applies if we consider them.
Proposition B2. Assume that Assumption B4 holds. Then, Assumptions B1-B3 hold for the basic FAs.

Proof. We will check all conditions in Assumptions B1-B3 hold for basic FAs.

- Assumption B1: Assumption B1 is trivially true since $u_i(a, t)$ is bounded for any $a, t$ by (B-3)-(B-7) and the fact that $a_i, t_i$ is bounded for any $i$.

- Assumption B2:
  - (i) and (ii) are trivial by (B-7) and the fact that $a_i, t_i$ is bounded for any $i$.
  - (iii). Let $Q = \bar{c} - \bar{z}$, i.e., the bid equals to the highest possible spot market price $\bar{z} + \bar{x}$. Thus, by bidding $Q$, the bidder will never win against spot market and $u_i(a, t|a_i=Q) = 0$, thus, $a_i \geq Q$. $\Delta v_i(a_i, t_i) = \bar{v}_i(a_i, t_i) - \bar{z} = -\bar{x} - \Delta < 0$ by (B-5).
  - (iv). By (B-7), $\Delta v_i(a_i, t) = \bar{v}_i(a_i, t) = t_i - a_i + t_0^1 - \bar{x}$. Obviously, $\Delta v_i(a_i, t)$ is strictly increasing in $(-a_i, t_i)$.
  - (v). For all $\varepsilon > 0$, by (B-7), $\Delta v_i(a_i, t, t_i + \varepsilon) = \Delta v_i(a_i, t, t_i) = \varepsilon$. Thus, (v) in Assumption B2 is true for any $\lambda \in (0, 1]$.

- Assumption B3:

\[
\mathbb{E}_{t_{i-1}} \left[ \Delta v_i(a_i, t_i, W_i(a'_i, \alpha_{i-1}) \right| t_i, W_i(a'_i, \alpha_{i-1})] = \frac{\mathbb{E}_{t_{i-1}} \left[ \Delta v_i(a_i, t_i) \cdot I \{W_i(a'_i, \alpha_{i-1}) \} \right| t_i]}{\mathbb{P} \left( W_i(a'_i, \alpha_{i-1}) \right| t_i)}
\]
\[
= \frac{\mathbb{E}_{t_{i-1}} \left[ (t_i - a_i + t_0 - \bar{x}) \cdot I \{W_i(a'_i, \alpha_{i-1}) \} \right| t_i]}{\mathbb{P} \left( W_i(a'_i, \alpha_{i-1}) \right| t_i)}
\]
\[
= \frac{\mathbb{E}_{t_{i-1}} \left[ t_0 \cdot I \{W_i(a'_i, \alpha_{i-1}) \} \right| t_i]}{\mathbb{P} \left( W_i(a'_i, \alpha_{i-1}) \right| t_i)}
\]

\[
\overset{(a2)}{=} t_i - a_i - \bar{x} + \frac{\mathbb{E}_{t_{i-1}} \left[ t_0 \cdot I \{W_i(a'_i, \alpha_{i-1}) \} \right| t_i]}{\mathbb{P} \left( W_i(a'_i, \alpha_{i-1}) \right| t_i)}
\]
Therefore, we have shown conditions in Assumptions B1-B3 hold for the basic FAs. except SCC. This together with the SCC property established in Proposition B1 imply that a game is symmetric. The above analysis establishes the conditions required to use Theorem B1. Recall that Athey’s method is applicable to establishing the existence of a symmetric BNE when E B4, one has

\[ \mathbb{E}_{t_{-i}} \left[ t_0^1 \cdot \mathbb{I} \{ W_i(a_i', \alpha_{-i}) \} \right] = \frac{\int_{t_{-i}} t_0^1 \cdot \mathbb{I} \{ a_i' > \alpha_j(t_j), \forall j \neq i \} f(t_{-i}) dt_{-i}}{\int_{t_{-i}} \mathbb{I} \{ a_i' > \alpha_j(t_j), \forall j \neq i \} f(t_{-i}) dt_{-i}} \]

\[ \equiv \mathbb{E}_{t_{01}^2, t_{02}^2} \left[ t_0^1 \mathbb{I} \{ a_i' > a_0^1 + a_0^2 \} \cdot \left( \prod_{j \neq i} f_j(t_j) dt_j \right) \right] \]

\[ \equiv \mathbb{E}_{t_{01}^2, t_{02}^2} \left[ t_0^1 \mathbb{I} \{ a_i' > a_0^1 + a_0^2 \} \cdot \left( \prod_{j \neq i} f_j(t_j) dt_j \right) \right] \]

where (a3) follows from the fact that t_{-i} are independent and the last equality from the facts that a_0^1 = t_0^1 and a_0^2 = t_0^2. Thus,

\[ \mathbb{E}_{t_{-i}} \left[ \Delta v_i(a_i, t) \mid t_i, W_i(a_i', \alpha_{-i}) \right] = t_i - a_i - \bar{x} + \mathbb{E} \left[ t_0^1 a_i' > t_0^1 + t_0^2 \right]. \]

Obviously, \( \mathbb{E}_{t_{-i}} \left[ \Delta v_i(a_i, t) \mid t_i, W_i(a_i', \alpha_{-i}) \right] \) is strictly increasing in t_i. Next, we show that \( \mathbb{E} \left[ t_0^1 a_i' > t_0^1 + t_0^2 \right] \) is increasing in a_i'. By tower property of conditional expectation and independence of t_0^1 and t_0^2, one has

\[ \mathbb{E}_{t_{01}^2, t_{02}^2} \left[ t_0^1 a_i' > t_0^1 + t_0^2 \right] = \mathbb{E} \left[ \bar{x} - X \mid a_i' > \bar{x} - X + \bar{c} - Z \right] = \bar{x} - \mathbb{E} \left[ X \mid X + Z > \bar{x} + \bar{c} - a_i' \right]. \]

The first equation follows by the definition t_0^1 = \bar{x} - X and t_0^2 = \bar{c} - Z. Thus, by Assumption B4, one has \( \mathbb{E}_{t_{01}^2, t_{02}^2} \left[ t_0^1 a_i' > t_0^1 + t_0^2 \right] \) is increasing in a_i'. Thus, Assumption B3 is true, namely,

\[ \mathbb{E}_{t_{-i}} \left[ \Delta v_i(a_i, t) \mid t_i, W_i(a_i', \alpha_{-i}) \right] \] is strictly increasing in t_i and increasing in a_i'.

Therefore, we have shown conditions in Assumptions B1-B3 hold for the basic FAs.

Recall that Athey’s method is applicable to establishing the existence of a symmetric BNE when the game is symmetric. The above analysis establishes the conditions required to use Theorem B1 except SCC. This together with the SCC property established in Proposition B1 imply that a increasing symmetric BNE for basic FAs exists.