Decomposition of Optimal Dynamic Portfolio Choice with Wealth-Dependent Utilities in Incomplete Markets

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Abstract

This paper establishes a new decomposition of optimal dynamic portfolio choice under general incomplete-market diffusion models by disentangling the fundamental impacts on optimal policy from market incompleteness and flexible wealth-dependent utilities. We derive explicit dynamics of the components for the optimal policy, and obtain an equation system for solving the shadow price of market incompleteness, which is found to be dependent on both market state and wealth level. We identify a new important hedge component for non-myopic investors to hedge the uncertainty in shadow price due to variation in wealth level. As an application, we establish and compare the decompositions of optimal policy under general models with the prevalent HARA and CRRA utilities. Under nonrandom but possibly time-varying interest rate, we solve in closed-form the HARA policy as a combination of a bond holding scheme and a corresponding CRRA strategy. Finally, we develop a simulation method to implement the decomposition of optimal policy under the general incomplete market setting, whereas existing approaches remain elusive.

Keywords: optimal portfolio choice, decomposition, incomplete market, wealth-dependent utility, closed-form.

JEL Codes: C61, C63, G11.

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1 Introduction

Optimal portfolio choice is a central topic in modern financial economics, drawing continual attention from both industry and academia. Hedge funds, asset management firms, and pension funds, which manage large positions of portfolios, as well as individual investors, are confronted with this type of decision frequently. Due to its own importance or as an indispensable tool amply applied in both theoretical and empirical literature, the optimal portfolio choice problem has also drawn long-standing interest in academia. The celebrated static mean-variance framework of Markowitz (1952) laid a foundation. Following the seminal work by Samuelson (1969) and Merton (1969, 1971), various studies have been developed for the optimal dynamic portfolio choice; see the comprehensive surveys in, e.g., Detemple (2014), Brandt (2010), and Wachter (2010), as well as the references therein. In continuous-time setting, it is an optimal stochastic control problem, that combines stochastic modeling and optimization techniques. In early works, a large number of relevant contributions relied on the dynamic programming approach, which employs the highly nonlinear Hamilton-Jacobi-Bellman (HJB hereafter) equation to characterize the optimal policy. An alternative notable approach is the martingale method pioneered and developed by, e.g., Pliska (1986), Karatzas et al. (1987), Cox and Huang (1989), Ocone and Karatzas (1991), Cvitanic and Karatzas (1992), and Detemple et al. (2003). Koijen (2014) explains how we can also use the martingale approach to estimate continuous-time optimization models.

For the purpose of understanding and analyzing the behavior of optimal portfolios, existing works largely focus on some specific affine models (see, e.g., Duffie et al. (2000)) and constant relative risk aversion (CRRA hereafter) utilities that yield closed-form optimal portfolio policies, though such analytically tractable cases are rare and limited. As an effective method for applying flexible models without closed-form optimal portfolio policies, Detemple et al. (2003) further developed the aforementioned martingale approach and derived, at the theoretical level, an explicit decomposition of the optimal policy under general diffusion models as well as flexible utilities, and consequently pioneered a flexible Monte Carlo simulation approach for implementation. See also Cvitanic et al. (2003) for an alternative simulation approach. However, this milestone of methods is by far limited to the complete market setting. Besides simulation, other numerical methods were proposed; see, e.g.,

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the early attempts based on the dynamic programming approach.\textsuperscript{2} We refer to the recent book of Dumas and Luciano (2017) for a survey of different numerical methods available for optimal portfolio choice. It is a common consensus that the notable challenge lies in the incomplete market settings at not only the theoretical but also the numerical and even empirical levels.

In this paper, we develop and implement a new decomposition for the optimal policy in general incomplete market models, under which we cannot fully hedge the risk by investing in the risky assets. Our contribution to the literature holds for general diffusion models of assets prices and state variables, with flexible utilities (rather than limited to those of the CRRA type) over both intermediate consumption and terminal wealth. The optimal policy is decomposed to the mean-variance component, the interest rate hedge component, and the price of risk hedge component, which are all functions of current market state variable and investor wealth level. This type of decomposition reconciles the seminal work in Merton (1971). In the decomposition, we explicitly express each component of the optimal policy as conditional expectation of suitable random variables underlaid by sophisticated but explicit dynamics, with the necessary aid of Malliavin calculus (see an accessible survey of Malliavin calculus for finance in Appendix D of Detemple et al. (2003)). Our decomposition fundamentally and substantially extends the representation results under the complete market setting in Detemple et al. (2003) to general incomplete market models, and thus lays an important foundation for developing subsequent theoretical analysis, numerical methods, and empirical studies.

To handle the market incompleteness, we apply and explore the “least favorable completion” principle developed by Karatzas et al. (1991) under general diffusion models. It completes the market by introducing suitable fictitious assets. Then, the equivalence between the optimal policy in the completed market and that in the original market is established via choosing the appropriate price of risk associated with those fictitious assets. Such a price of risk is endogenously determined by the investor utility function and the investment horizon, and thus is referred to as the investor-specific price of risk under market incompleteness. It is also known as the “shadow price” of market incompleteness in the literature; see, e.g., Detemple and Rindisbacher (2010). We begin by showing and applying the following structure. Under general incomplete market models with wealth-dependent utilities, the appropriate investor-specific price of risk not only depends on current market state, but also implicitly depends on investor’s wealth level. The latter dependence is completely absent in the market price of risk associated with the real assets. Consequently, a new important hedge component emerges in the optimal policy for hedging the uncertainty in investor-specific price of risk due to the variation in investor wealth level. This reveals an additional type of hedging demand when investors allocate their portfolio in incomplete markets, which essentially arises from the market incomplete-

ness and wealth-dependent utilities, more precisely, from the structure of investor-specific price of risk. Finally, as an indispensable part of the decomposition, we establish an equation system for directly characterizing the deterministic functional form of investor-specific price of risk. Comparing with the other types of differential equations employed for characterizing optimal portfolios in incomplete markets\(^3\), our equation system offers more generality, explicitness, and analytical convenience for further analysis.

Equipped with the decomposition results for general incomplete market models, we study how the optimal policy is fundamentally impacted by market incompleteness and wealth-dependent utilities. Specifically, we derive the corresponding decomposition results for complete market models and incomplete market models with the wealth-independent CRRA utility, and compare them with the result for the general incomplete market models. This new analytical contribution allows us to analyse the impacts, not yet addressed in the literature, of market incompleteness and wealth-dependence utility. In two special cases, we show that the dynamics of components underlying the optimal policy are fundamentally different compared with the general case. Besides, the new component in the optimal policy, which hedges the fluctuation in investor-specific price of risk due to variation in wealth level, vanishes in these two special cases. These two comparative studies show the fundamental differences between our decomposition and the existing ones in, e.g., Detemple et al. (2003), Detemple and Rindisbacher (2005, 2010), and Detemple (2014). Obviously, the departure from the complete market setting and wealth-independent CRRA utility has both technical and economic implications on the resulting optimal policy, and we cannot simply use the results currently available in the literature for this investigation.

Our decomposition of optimal policy under general incomplete market models with flexible utility functions advances the frontier of related theoretical studies and offers a broader foundation for conducting relevant analysis. As the first application, among many others, of our representation results, we apply our new decomposition to disentangle the optimal policy under general incomplete market models with hyperbolic absolute risk aversion (HARA) utility, which, compared with the CRRA utility, is more flexible and effective in reflecting investor preference due to its wealth-dependent property. While solving the optimal policy under HARA utility is commonly believed to be difficult and is thus much less studied compared with the CRRA case in the literature, we apply our decomposition results to explicitly reveal how the optimal policy under HARA utility is impacted by investor’s wealth level and the minimum requirements for terminal wealth and intermediate consumption. Furthermore, under the special case with nonrandom but possibly time-varying interest rate, we apply our decomposition to solve and interpret the optimal policy under HARA utility. The

\(^3\)See, e.g., the forward-backward stochastic differential equations in in Detemple and Rindisbacher (2005) and Detemple and Rindisbacher (2010) under a model featuring partially hedgeable Gaussian interest rate with CRRA utility, as well as the quasi-linear partial differential equation employed in He and Pearson (1991) for indirectly relating to the optimal policy.
policy is surprisingly decomposed as a product of its counterpart under the CRRA utility and a multiplier related to current wealth level and a continuum of bond prices. This new interpretation indicates that the HARA optimal policy is constructed by first buying a series of zero-coupon bonds to exactly satisfy the minimum requirements for terminal wealth and intermediate consumption in the entire future investment horizon, and then allocating the remaining wealth just as an investor with CRRA utility.

As an illustrative example of our decomposition result under HARA utility, we solve the optimal policy in closed-form for HARA investors under the celebrated incomplete market stochastic volatility model of Heston (1993). We then use the closed-form formulae to explicitly analyze how the optimal policies under CRRA and HARA utilities are differently impacted by wealth level, interest rate, and investment horizon. We show that the wealth-dependent property of HARA utility should not be taken only literally. The HARA utility impacts the optimal policy via not only the wealth level, but also the interest rate and investment horizon. This is because the latter two factors impact the prices of bonds held by HARA investors for the purpose of satisfying their minimum requirements for terminal wealth, as revealed in the portfolio construction for HARA utility with nonrandom interest rate. Furthermore, we show that the optimal policies for HARA investors with different (i.e., high and low) initial wealth levels become more (resp. less) resembling to each other, when the market experiences the bull (resp. bear) regime; such cycle impact is entirely absent for CRRA investors.

These results illustrate the importance of our theoretical findings in analyzing the behavior of optimal policies under market incompleteness and wealth-dependent utility. They also corroborate empirical evidence in the literature that the investment in risky assets increases concavely in investors' financial wealth; see, e.g., Roussanov (2010), Wachter and Yogo (2010), and Calvet and Sodini (2014). Besides, they imply that investment recommendations based on a CRRA utility are incorrect for a HARA investor except if her current wealth is sufficiently high. In the numerical experiments of Section 3.2 (see Figure 2), we observe a relative (resp. absolute) increase up to 24% (resp. 10%) in the optimal allocation in the risky asset when shifting from a low-wealth investor to a high-wealth investor.

As demonstrated in the aforementioned application, our decomposition of optimal policy leads to potential success of solving optimal policy in closed-form under some analytically tractable models. Nevertheless, if no closed-form solutions are available in nature, as for most of the flexible models, its direct implementation obviously encounters significant challenges. We therefore propose and implement a Monte Carlo simulation method for optimal policy in general incomplete market models, as the second application based on our decomposition results. Such a method provides a solution to the open problem of extending the simulation approach proposed in Detemple et al. (2003) for the complete market case under HARA utility (see Basak and Chabakauri (2010) for use in dynamic

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4While the optimal policy under Heston’s model has been solved in closed-form under CRRA utility as in Liu (2007), its counterpart under HARA utility, to our best knowledge, remains elusive in the existing literature.
mean-variance asset allocation in incomplete markets). By exploiting the structure of optimal policy developed in our decomposition, especially the equation system characterizing the investor-specific price of risk, our simulation method successfully circumvents the essential difficulty stemming from market incompleteness, and thus renders a useful tool for studying optimal policies under general incomplete market models, where optimal policies are difficult or impossible to solve in closed-form and/or existing numerical approaches do not efficiently apply.

As an illustrative example of our decomposition and the subsequent simulation method, we perform novel analysis of the impacts on optimal portfolios from specifications of stochastic volatility dynamics. Indeed, little is known about the behavior of optimal policies under this typical issue of market incompleteness, although there exists a large amount of discussions on what is the proper volatility dynamics for fitting prices of underlying asset or/and its derivative securities; see, e.g., Jones (2003), Ait-Sahalia and Kimmel (2007), Medvedev and Scaillet (2007), and Christoffersen et al. (2010). We thus follow these literature to consider a flexible CEV-type stochastic volatility model, which nests some prevalent volatility dynamics such as the stochastic volatility models of Heston (1993) type, of the GARCH diffusion type proposed in Nelson (1990), and of the 3/2 type investigated in, e.g., Christoffersen et al. (2010). Due to its incomplete-market and nonaffine nature, the optimal policy under this model is analytically intractable. Applying our simulation method, we explore and explain various impacts on optimal portfolio policies from model parameters that control the specification of volatility dynamics. This study substantially complements and extends the aforementioned closed-form impact analysis for the Heston model based on our decomposition results under HARA utility. Our findings bring comprehensive economic insights of the optimal policy under stochastic volatility. For instance, the price of risk hedge component increases with the degree of elasticity of the variance process, but decreases as market becomes more incomplete, i.e., the leverage effect parameter approaches zero. Besides, it exhibits a hump-shape with respect to the level of risk aversion.

The rest of this paper is organized as follows. In Section 2, we establish the decomposition for general incomplete market models with flexible utilities and analyze the impacts from market incompleteness and/or wealth-dependent utilities. In Section 3, we apply our decomposition to general models under HARA utility, and reveal the fundamental connection between CRRA and HARA policies under nonrandom but possibly time-varying interest rate. In Section 4, we propose and implement a Monte Carlo simulation method based on our decomposition. Section 5 concludes and provides discussions. We collect proofs for the decompositions of optimal policy in Appendix A.
2 A new decomposition of optimal dynamic portfolio choice in incomplete markets

We begin by setting up the model, utility function, and optimal dynamic portfolio choice problem in Section 2.1. Then, we establish our new decomposition of the optimal policy in Section 2.2. In Sections 2.3 and 2.4, we analyze the impacts of market incompleteness and wealth-dependent utilities, respectively, which illustrate the fundamental differences between our decomposition and existing results.

2.1 The optimal dynamic portfolio choice problem

We assume that the market consists of $m$ stocks and one savings account. The price of the stock $S_{it}$, for $i = 1, 2, \ldots, m$, follows the generic SDE:

$$\frac{dS_{it}}{S_{it}} = (\mu_i(t, Y_t) - \delta_i(t, Y_t)) dt + \sigma_i(t, Y_t) dW_t,$$

where $Y_t$ is an $n$-dimensional state variable driven by the following generic SDE:

$$dY_t = \alpha(t, Y_t) dt + \beta(t, Y_t) dW_t.$$

In (1), $W_t$ is a standard $d$-dimensional Brownian motion; $\mu_i(t, y)$ is a scalar function for modeling the mean rate of return; $\delta_i(t, y)$ is a scalar function for modeling the dividend rate; $\sigma_i(t, y)$ is a $d$-dimensional vector-valued function for modeling the volatility. In (2), $\alpha(t, y)$ is an $n$-dimensional vector-valued function for modeling the drift of the state variable $Y_t$; $\beta(t, y)$ is an $n \times d$ matrix-valued function for modeling the diffusion of the state variable $Y_t$. Besides, we assume that the savings account appreciates at the instantaneous interest rate $r_t = r(t, Y_t)$ for some scalar-valued function $r(t, y)$. The state variable $Y_t$ governs all the investment opportunities in the market through the rate of return, the dividend rate, the volatility, and the instantaneous interest rate. We mainly focus on the incomplete market case where the number of independent Brownian motions is strictly larger than the number of tradable risky assets, i.e., $d > m$. In this case, we can not fully hedge the risk introduced by the Brownian motion. Owing to the market incompleteness, the model and the subsequent portfolio choice problem enjoy their multidimensional nature, even for one-asset cases.

Denote by $X_t$ the investor wealth process. Then, it satisfies the following wealth equation:

$$dX_t = (r_t X_t - c_t) dt + X_t \pi_t^T [(\mu_t - r_t 1_m) dt + \sigma_t dW_t].$$

In (3), $\mu_t$ and $\sigma_t$ represent the mean rate of return and volatility of the risky assets respectively, which satisfy $\mu_t = \mu(t, Y_t)$ and $\sigma_t = \sigma(t, Y_t)$ with the $m$-dimensional vector $\mu(t, y)$ and the $m \times d$-dimensional matrix $\sigma(t, y)$ defined by $\mu(t, y) := (\mu_1(t, y), \mu_2(t, y), \ldots, \mu_m(t, y))^\top$ and $\sigma(t, y) := (\sigma_1(t, y), \sigma_2(t, y), \ldots, \sigma_m(t, y))^\top$. We assume the volatility function $\sigma(t, y)$ has rank $m$, i.e., its rows
are linearly independent. Besides, \( c_t \) is the instantaneous consumption rate; \( \pi_t \) is an \( m \)-dimensional vector representing the weights of the risky assets; \( 1_m \) denotes an \( m \)-dimensional column vector with all elements equal to 1. The investor maximizes her expected utility over both intermediate consumptions and terminal wealth by dynamically allocating her wealth among the risky assets and the risk-free asset, subject to the non-bankruptcy condition. We can formulate this optimization problem as

\[
\sup_{(\pi_t, c_t)} E \left[ \int_0^T u(t, c_t) dt + U(T, X_T) \right], \quad \text{with } X_t \geq 0 \text{ for all } t \in [0, T],
\]

where \( u(t, \cdot) \) and \( U(T, \cdot) \) are the utility functions of the intermediate consumptions and the terminal wealth, respectively.

In (4), both utility functions \( u(t, \cdot) \) and \( U(T, \cdot) \) are time-varying, in order to reflect the time value, e.g., the discount effect. Furthermore, we assume them to be strictly increasing and concave with \( \lim_{x \to \infty} \partial u(t, x)/\partial x = 0 \) and \( \lim_{x \to \infty} \partial U(T, x)/\partial x = 0 \). One important specification is the constant relative risk aversion (CRRA) utility. Following the convention (see, e.g., Pratt (1964)), the CRRA utility function is defined by

\[
u(t, c) = we^{-\rho t} \frac{c^{1-\gamma}}{1-\gamma} \quad \text{and} \quad U(T, x) = (1-w)e^{-\rho T} \frac{x^{1-\gamma}}{1-\gamma},
\]

(5a)

where \( \gamma \in (0, +\infty) \) is the constant relative risk aversion coefficient, \( w \in [0, 1] \) is a weight for balancing the intermediate consumption and the terminal wealth, and \( \rho \) is the discount rate. The CRRA utility is wealth independent, as the investor relative risk aversion coefficient is a constant \( \gamma \) and does not vary with the wealth level. This property brings mathematical convenience that leads to closed-form formulae of optimal policy or simplifications of the optimization problem under some specific models; see, e.g., Wachter (2002), Kim and Omberg (1996), and Liu (2007) for closed-form optimal policies, as well as Detemple et al. (2003) for developing a Monte Carlo simulation approach.

Another prevalent example is the hyperbolic absolute risk aversion (HARA) utility. Following the convention (see, e.g., Pratt (1964)), other things being equal, the HARA utility function is defined by

\[
u(t, c) = we^{-\rho t} \frac{(c - \bar{c})^{1-\gamma}}{1-\gamma} \quad \text{and} \quad U(T, x) = (1-w)e^{-\rho T} \frac{(x - \bar{x})^{1-\gamma}}{1-\gamma},
\]

(5b)

for \( c > \bar{c} \) and \( x > \bar{x} \), where \( \bar{c} \) and \( \bar{x} \) are set as the minimum allowable levels for the intermediate consumption and terminal wealth, respectively. The HARA utility allows for imposing lower bound constraints on the intermediate consumption and/or terminal wealth. This feature is particularly suitable for incorporating, e.g., portfolio insurance, investment goal constraints, and subsistence level constraints. Unlike the CRRA utility in (5a), the HARA utility is wealth dependent. Besides, closed-form optimal policies under the HARA utility are generally rare; see Kim and Omberg (1996) for one such case with the stochastic market price of risk modelled by an Ornstein-Uhlenbeck process. The Monte Carlo simulation approaches of Detemple et al. (2003) and Cvitanic et al. (2003) offer remedies for such mathematical inconvenience.
2.2 Decomposition of optimal policy

To establish a decomposition of the optimal policy under the incomplete market setting \((d > m)\) for models with flexible dynamics (1) – (2) and general utility functions, we begin by applying and further exploring the least favorable completion\(^5\) principle introduced in Karatzas et al. (1991) under the general diffusion model (1) – (2). First, we complete the market by introducing \(d - m\) candidate fictitious assets. Then, we solve the optimal portfolio choice problem in this completed market via the martingale method for complete market case. Finally, by letting the optimal weights for the fictitious assets be zero, we characterize the price of risk of these suitable fictitious assets and pin down the desired optimal policy for the real assets. In what follows, we develop the procedures described above to circumvent the challenge in explicitly decomposing and implementing the optimal portfolio policy for the general incomplete market model.

We introduce \(d - m\) fictitious assets to complete the market as discussed in Karatzas et al. (1991). Their prices \(F_{it}\), for \(i = 1, 2, \ldots, d - m\), satisfy the following SDE:

\[
dF_{it} = \frac{\mu^F_i}{F_{it}} dt + \sigma^F_i(t, Y_t) dW_t,
\]

where the mean rates of returns \(\mu^F_i\) are stochastic processes adaptive to the filtration generated by the Brownian motion \(W_t\). According to Karatzas et al. (1991), we can choose the volatility function \(\sigma^F(t, y) := (\sigma^F_1(t, y), \ldots, \sigma^F_{d-m}(t, y))^\top\) arbitrarily, as long as it has rank \(d - m\) and satisfies the following orthogonal condition with respect to the volatility function \(\sigma(t, y)\) of the real risky assets \(S_t\):

\[
\sigma(t, y)\sigma^F(t, y)^\top \equiv 0_{m \times (d-m)}.
\]

This condition guarantees that the fictitious assets are driven by different Brownian shocks, and thus leads to the success of market completion. Moreover, we assume that the fictitious assets do not pay dividend.

Combining the \(m\) real risky assets with prices \(S_t\) in (1) and the \(d - m\) fictitious risky assets with prices \(F_t\) in (6), we construct a completed market consisting of \(d\) risky assets and driven by \(d\) independent Brownian motions. In this completed market, we represent the prices of the risky assets, including both the real and the fictitious ones, by a \(d\)-dimensional column vector \(S^S_t = (S^\top_t, F^\top_t)^\top\). According to (1) and (6), \(S^S_t\) is driven by the SDE:

\[
ds^S_t = \text{diag}(S^S_t) \left[ \mu^S_t dt + \sigma^S(t, Y_t) dW_t \right],
\]

\(^5\)As documented in the literature (see, e.g., Karatzas et al. (1991)), we can interpret the terminology “least favorable completion” as follows: Consider all the possible fictitious completions and their associated optimal policies, we naturally say that a completion is more (resp. less) favorable if its corresponding optimal policy results in higher (resp. lower) expected utility. The completion (23) below, which leads to an optimal portfolio with zero weight on the fictitious assets, must be the least favorable one. Indeed, in any other fictitious completion, since this portfolio without the fictitious assets is admissible (i.e., a candidate portfolio strategy), the optimal one must result in a higher expected utility and thus becomes more favorable.
with the diagonal matrix \( \text{diag}(S_t) = \text{diag}(S_{1t}, S_{2t}, \cdots, S_{mt}, F_{1t}, F_{2t}, \cdots, F_{(d-m)t}) \), the \( d \)-dimensional column vector \( \mu_t^S = \left( (\mu(t, Y_t) - \delta(t, Y_t))^\top, (\mu_t^F)^\top \right)^\top \), and the \( d \times d \) dimensional matrix \( \sigma^S(t, Y_t) = (\sigma(t, Y_t)^\top, \sigma^F(t, Y_t)^\top)^\top \). By linear algebra, the orthogonal condition (7) implies that \( \sigma^S(t, y) \) must be nonsingular. Thus, we are now in a complete market, where we can fully hedge the uncertainty from all Brownian motions by investing in the real and fictitious assets. The completed market allows for investing in both the real assets \( S_t \) and fictitious assets \( F_t \). We denote by \( \pi_t \) and \( \pi_t^F \) their corresponding weights, which are \( m \) and \( (d-m) \)-dimensional vectors, respectively. Similar to (4), we consider the utility maximization problem in this completed market, with the non-bankruptcy constraint \( X_t \geq 0 \) still holds.

To represent the policy, we introduce the following necessary notations. First, in the completed market, we define the total price of risk as

\[
\theta_t^S := \sigma^S(t, Y_t)^{-1}(\mu_t^S - r(t, Y_t)1_d). \tag{9}
\]

By the orthogonal condition (7), it follows from matrix calculations that

\[
(\sigma^S(t, y))^{-1} = (\sigma(t, y)^+ \sigma^F(t, y)^+) \tag{10}
\]

where the notation

\[
A^+ := A^\top(AA^\top)^{-1} \tag{11}
\]

denotes the Moore–Penrose inverse (see, e.g., Penrose (1955)) of a general matrix \( A \) with linearly independent rows. Thus, \( \sigma(t, y)^+ \) (resp. \( \sigma^F(t, y)^+ \)) is a \( d \times m \) dimensional (resp. \( d \times (d-m) \) dimensional) matrix satisfying \( \sigma(t, y)\sigma(t, y)^+ = I_m \) (resp. \( \sigma^F(t, y)\sigma^F(t, y)^+ = I_{d-m} \)), with \( I_m \) being the \( m \)-dimensional identity matrix. By (10), the total price of risk \( \theta_t^S \) allows the following decomposition:

\[
\theta_t^S = \theta^h(t, Y_t) + \theta^u_t, \tag{12}
\]

where \( \theta^h(t, Y_t) \) and \( \theta^u_t \) are the prices of risk associated with the real and fictitious assets respectively. They are defined by \( d \)-dimensional column vectors

\[
\theta^h(t, Y_t) := \sigma(t, Y_t)^+ (\mu(t, Y_t) - r(t, Y_t)1_m), \tag{13a}
\]

and

\[
\theta^u_t := \sigma^F(t, Y_t)^+ (\mu^F_t - r(t, Y_t)1_{d-m}), \tag{13b}
\]

respectively. The term \( \theta^h(t, Y_t) \) in (13a) is referred to as the market price of risk, as it is fully determined by the real assets shared by all investors in the market. The term \( \theta^u_t \) in (13b), however, is purely associated with the fictitious assets, which are introduced for solving the optimal portfolio choice problem (18) in the incomplete market. As we will show momentarily, \( \theta^u_t \) is endogenously determined by the investor utility function and the investment horizon. Thus, we refer to \( \theta^u_t \) as the investor-specific price of risk, since it varies from one investor to another.
By definitions (11), (13a), and (13b), we can translate the orthogonal condition (7) as

\[ \sigma^F(t, Y_t) \theta^b(t, Y_t) \equiv 0_{d-m} \quad \text{and} \quad \sigma(t, Y_t) \theta^u_t \equiv 0_m. \tag{14a} \]

As we assume matrix \( \sigma(t, Y_t) \) has linear independent rows, the second condition imposes \( m \) linear constraints for the \( d \)-dimensional vector \( \theta^u_t \). Besides, definition (13a) and the second condition imply that

\[ \theta^b(t, Y_t) \top \theta^u_t \equiv 0, \tag{14b} \]

i.e., the market and investor-specific price of risk are orthogonal. According to Karatzas et al. (1991), we can fully determine the optimal policy \( \pi_t \) in the real market by the choice of \( \theta^u_s \).

Next, we introduce the state price density as

\[ \xi^S_t := \exp \left( -\int_0^t r(v, Y_v) dv - \int_0^t (\theta^S_v) \top dW_v - \frac{1}{2} \int_0^t (\theta^S_v) \top \theta^S_v dv \right). \tag{15} \]

For any \( s \geq t \geq 0 \), we define the relative state price density as

\[ \xi^S_{t,s} = \xi^S_s / \xi^S_t, \tag{16} \]

which satisfies

\[ d\xi^S_{t,s} = -\xi^S_s [r(s, Y_s) ds + (\theta^S_s) \top dW_s], \tag{17} \]

with initial value \( \xi^S_{t,t} = 1 \), according to a straightforward application of Ito formula. The above dynamics of \( \xi^S_{t,s} \) clearly hinges on the undetermined investor-specific price of risk \( \theta^u_s \).

The martingale approach pioneered by Karatzas et al. (1987) and Cox and Huang (1989) starts by formulating the dynamic problem (4) with information up to time \( t \) as an equivalent static optimization problem:

\[ \sup_{(c_t, X_T)} E_t \left[ \int_t^T u(s, c_s) ds + U(T, X_T) \right] \quad \text{subject to} \quad E_t \left[ \int_t^T \xi^S_{t,s} c_s ds + \xi^S_{t,T} X_T \right] \leq X_t, \tag{18} \]

where, throughout the paper, \( E_t \) denotes the expectation condition on the information up to time \( t \) and \( X_t \) is the wealth level assuming that the investor always follows the optimal policy. Then, we can solve this problem via the method of Lagrangian multiplier. The optimal intermediate consumption and terminal wealth satisfy

\[ c_t = I^u(t, \lambda^*_t) \quad \text{and} \quad X_T = I^U(T, \lambda^*_T), \tag{19} \]

To guarantee the martingale property of \( \xi^S_t \exp(\int_0^T r(v, Y_v) dv) \), we assume that the total price of risk \( \theta^u_s \) satisfies the Novikov condition:

\[ E \left[ \exp \left( \frac{1}{2} \int_0^T (\theta^u_v) \top \theta^u_v dv \right) \right] < \infty. \]

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6To guarantee the martingale property of \( \xi^S_t \exp(\int_0^T r(v, Y_v) dv) \), we assume that the total price of risk \( \theta^u_v \) satisfies the Novikov condition: \( E \left[ \exp \left( \frac{1}{2} \int_0^T (\theta^u_v) \top \theta^u_v dv \right) \right] < \infty. \)
respectively, with $I^u(t, \cdot)$ and $I^U(t, \cdot)$ being the inverse marginal utility functions of $u(t, \cdot)$ and $U(t, \cdot)$, i.e., the functions satisfying $\partial u/\partial x(t, I^u(t, y)) = y$ and $\partial U/\partial x(t, I^U(t, y)) = y$. In (19), we employ $\lambda^*_t$ to denote the Lagrangian multiplier for the wealth constraint in (18). It is uniquely characterized by following wealth equation:

$$X_t = E_t[G_{t,T}(\lambda^*_t)],$$

(20)

where $G_{t,T}(\lambda^*_t)$ is defined as

$$G_{t,T}(\lambda^*_t) := \Gamma^U_{t,T}(\lambda^*_t) + \int_t^T \Gamma^u_{t,s}(\lambda^*_t) ds.$$  

(21)

Here, $\Gamma^U_{t,T}(\lambda^*_t)$ and $\Gamma^u_{t,s}(\lambda^*_t)$ are given by

$$\Gamma^U_{t,T}(\lambda^*_t) = \xi^S_{t,T} I^U_{t,T}(T, \lambda^*_t \xi^S_{t,T})$$

and

$$\Gamma^u_{t,s}(\lambda^*_t) = \xi^S_{t,s} I^u_{t,s}(s, \lambda^*_t \xi^S_{t,s}).$$  

(22a)

By (20), we can determine the multiplier $\lambda^*_t$ with information up to time $t$. For representing the portfolio decomposition momentarily, we also introduce following quantities:

$$\Upsilon^U_{t,T}(\lambda^*_t) = \lambda^*_t \left( \xi^S_{t,T} \right)^2 \frac{\partial I^U_{t,T}}{\partial y} (T, \lambda^*_t \xi^S_{t,T})$$

and

$$\Upsilon^u_{t,s}(\lambda^*_t) = \lambda^*_t \left( \xi^S_{t,s} \right)^2 \frac{\partial I^u_{t,s}}{\partial y} (s, \lambda^*_t \xi^S_{t,s}).$$  

(22b)

Consequently, we can represent the optimal policy $(\pi_t, \pi^F_t)$ for the completed market via the martingale representation theorem (see, e.g., Section 3.4 in Karatzas and Shreve (1991)). With the Clark-Ocone formula (see, e.g., the survey provided in Detemple et al. (2003)), we can further represent the optimal policy in the form of conditional expectations of suitable random variables (see Ocone and Karatzas (1991)). Under a general and flexible diffusion model, Detemple et al. (2003) propose an explicit conditional expectation form for the optimal policy, and develop a Monte Carlo simulation method for its implementation; see also, e.g., Detemple and Rindisbacher (2010) along this line of contributions and Detemple (2014) for a comprehensive survey of the related developments. We aim at explicitly developing such results for the incomplete market case.

By the least favorable completion principle, the optimal policy $\pi_t$ for the real assets in the original incomplete market coincides with its counterpart in the completed market, as long as we properly choose the investor-specific price of risk $\theta^u_v$ such that the optimal weights for the fictitious assets are always identically zero, i.e.,

$$\pi^F_v \equiv 0_{d-m}, \text{ for any } 0 \leq v \leq T.$$  

(23)

Given an arbitrary choice of the volatility function $\sigma^F(v, y)$, the least favorable constraint (23) together with the second orthogonal condition in (14a) determines the desired $\theta^u_v$ for $0 \leq v \leq T$. Then, the corresponding optimal policy $\pi_t$ of the real assets for the completed market is also optimal for the original incomplete market. In particular, according to Karatzas et al. (1991), the desired $\theta^u_v$
satisfying (23) and the resulting optimal policy \( \pi_t \) are independent of the specific choice of \( \sigma^F(v, y) \). Thus, in what follows, we focus on characterizing \( \theta_v^u \).

Now, we notice and apply the following useful representation of the unknown investor-specific price of risk \( \theta_v^u \) that satisfies the least favorable completion constraint (23):

\[ \theta_v^u = \theta^u(v, Y_v, \lambda_v^*; T) \]  

(24)

for some function \( \theta^u(v, y, \lambda; T) \) endogenously determined by investor’s utility function and investment horizon. We can derive such a result by combining the fictitious completion approach in Karatzas et al. (1991) and the minimax local martingale approach in He and Pearson (1991); we document a brief verification in Appendix A.1. Representation (24) reveals the fundamental structure of the investor-specific price of risk \( \theta^u(v, Y_v, \lambda_v^*; T) \), which is strikingly different from that of the market price of risk \( \theta^h(v, Y_v) \) defined in (13a), and thus leads to fundamental difference between incomplete and complete market cases. First, \( \theta^u(v, Y_v, \lambda_v^*; T) \) depends on the constraint multiplier \( \lambda_v^* \), which solves \( X_v = E_v[G_v,T(\lambda_v^*)] \) by (20). Thus, \( \theta^u(v, Y_v, \lambda_v^*; T) \) implicitly depends on the wealth level \( X_v \). Second, \( \theta^u(v, Y_v, \lambda_v^*; T) \) also depends on the investment horizon \( T \), which is economically meaningful and technically indispensable at the level of implementation. However, neither of these two types of dependence exists in the market price of risk \( \theta^h(v, Y_v) \). Accordingly, we get the clear insight that, when completing the market following the least favorable principle (23), the introduced fictitious assets ought to depend on both the current wealth level and the investment horizon. In addition to this economic message, representation (24) is key not only for establishing the decomposition of optimal policy in Theorems 1 and 2 below, but also for developing the simulation method for implementation purposes in Section 4.

**Theorem 1.** Under the incomplete market model (1) and (2), the optimal policy \( \pi_t \) for the real assets with prices \( S_t \) admits the following decomposition

\[ \pi_t = \pi^{mv}(t, X_t, Y_t) + \pi^r(t, X_t, Y_t) + \pi^\theta(t, X_t, Y_t). \]  

(25)

Here, the mean-variance component \( \pi^{mv} \), the interest rate hedge component \( \pi^r \), and the price of risk hedge component \( \pi^\theta \) satisfy

\[ \pi^{mv}(t, X_t, Y_t) = -\frac{1}{X_t}(\sigma(t, Y_t)^+)^\top \theta^h(t, Y_t) E_t[Q_{t,T}(\lambda_v^*)], \]  

(26a)

\[ \pi^r(t, X_t, Y_t) = -\frac{1}{X_t}(\sigma(t, Y_t)^+)^\top E_t[H^r_{t,T}(\lambda_v^*)], \]  

(26b)

\[ \pi^\theta(t, X_t, Y_t) = -\frac{1}{X_t}(\sigma(t, Y_t)^+)^\top E_t[H^\theta_{t,T}(\lambda_v^*)], \]  

(26c)

where, throughout the paper, \( E_t \) denotes the expectation condition on the information up to time \( t \); \( X_t \) is the wealth level assuming that the investor always follows the optimal policy; the quantities

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\[ Q_{t,T}(\lambda^*_t), \mathcal{H}^u_{t,T}(\lambda^*_t), \text{ and } \mathcal{H}^\theta_{t,T}(\lambda^*_t) \text{ are given by} \]
\[ Q_{t,T}(\lambda^*_t) := \mathcal{Y}^U_{t,T}(\lambda^*_t) + \int_t^T \mathcal{Y}^u_{t,s}(\lambda^*_s)ds, \]  
(27a)  
\[ \mathcal{H}^u_{t,T}(\lambda^*_t) := (\Gamma^U_{t,T}(\lambda^*_t) + \mathcal{Y}^U_{t,T}(\lambda^*_t))H^u_{t,T} + \int_t^T (\Gamma^u_{t,s}(\lambda^*_t) + \mathcal{Y}^u_{t,s}(\lambda^*_t))H^u_{t,s}ds, \]  
(27b)  
\[ \mathcal{H}^\theta_{t,T}(\lambda^*_t) := (\Gamma^U_{t,T}(\lambda^*_t) + \mathcal{Y}^U_{t,T}(\lambda^*_t))H^\theta_{t,T} + \int_t^T (\Gamma^\theta_{t,s}(\lambda^*_t) + \mathcal{Y}^\theta_{t,s}(\lambda^*_t))H^\theta_{t,s}ds. \]  
(27c)  

Hereafter, \( \lambda^*_t \) is the multiplier uniquely determined by (20), i.e., \( X_t = E_t[\mathcal{G}_{t,T}(\lambda^*_t)] \), and thus it depends on \( X_t \) and satisfies the relation

\[ \lambda^*_t = \lambda^*_0 \xi^S_t, \]  
(28)  

\[ \Gamma^U_{t,T}(\lambda^*_t), \Gamma^u_{t,s}(\lambda^*_t), \mathcal{Y}^U_{t,T}(\lambda^*_t), \text{ and } \mathcal{Y}^u_{t,s}(\lambda^*_t) \] are defined in (22a) and (22b) except for replacing the relative state price density \( \xi^S_{t,s} \) by a \( \lambda^*_t \)-dependent version \( \xi^S_{t,s}(\lambda^*_t) \), for \( t \leq s \leq T \), which solves the SDE:

\[ d\xi^S_{t,s}(\lambda^*_t) = -\xi^S_{t,s}(\lambda^*_t)[v(s,Y_s)ds + \theta^S_{s}(\lambda^*_t)\top dW_s], \]  
(29)  

with

\[ \theta^S_{s}(\lambda^*_t) = \theta^h(s,Y_s) + \theta^u(s,Y_s,\lambda^*_t \xi^S_{t,s}(\lambda^*_t)) T \]  
(30)

being the \( \lambda^*_t \)-parameterized version of the total price of risk (12). Herein and thereafter, as an indispensable foundation, \( \theta^u(v,y,\lambda; T) \) is the function for representing the desired investor-specific price of risk \( \theta^u \), which satisfies the least favorable completion constraint (23), via \( \theta^u = \theta^u(v,Y_v,\lambda^*_v; T) \) as introduced in (24). Besides, \( H^u_{t,s} \) and \( H^\theta_{t,s}(\lambda^*_t) \) in (27b) and (27c) are both \( d \)-dimensional vector-valued processes evolving according to SDEs:

\[ dH^u_{t,s} = (D_t Y_s) \nabla v(s,Y_s)ds, \]  
(31)

and

\[ dH^\theta_{t,s}(\lambda^*_t) = [(D_t Y_s)(\nabla \theta^h(s,Y_s) + \nabla \theta^u(s,Y_s,\lambda^*_t \xi^S_{t,s}(\lambda^*_t))) T) - \lambda^*_t \xi^S_{t,s}(\lambda^*_t)(\theta^S_{t,s}(\lambda^*_t) + H^u_{t,s} + H^\theta_{t,s}(\lambda^*_t)) \partial \theta^u / \partial \lambda(s,Y_s,\lambda^*_t \xi^S_{t,s}(\lambda^*_t)) T)] \theta^S_{t,s}(\lambda^*_t) ds + dW_s, \]  
(32)

for \( t \leq s \leq T \), with initial values \( H^u_{t,t} = H^\theta_{t,t}(\lambda^*_t) = 0_d \) and \( \theta^S_{t,s}(\lambda^*_t) \) given in (30). Here and throughout this paper, \( \nabla \) denotes the gradient of functions with respect to the arguments in the place of \( Y_s \).

In (32), the random variable \( D_t Y_s \) is the time-\( t \) Malliavin derivative for the time-\( s \) state variable \( Y_s \), i.e., a \( d \times n \) matrix with \( D_t Y_s = [(D_{t1} Y_s)^\top, (D_{t2} Y_s)^\top, \cdots, (D_{tn} Y_s)^\top]^\top \), where each \( D_{ti} Y_s \) is an \( n \)-dimensional column vector satisfying SDE:

\[ dD_{ti} Y_s = (\nabla \alpha(s,Y_s))^\top D_{ti} Y_s ds + \sum_{j=1}^d (\nabla \beta_{ij}(s,Y_s))^\top D_{ti} Y_s dW_{js}, \lim_{s \to t} D_{ti} Y_s = \beta_i(t,Y_t), \]  
(33)

For an \( m \)-dimensional vector-valued function \( f(t,y) = (f_1(t,y), f_2(t,y), \cdots, f_m(t,y)) \), its gradient is an \( n \times m \) matrix with each element given by \( [\nabla f(t,y)]_{ij} = \partial f_j / \partial y_i(t,y) \), for \( i = 1,2, \ldots, n \) and \( j = 1,2, \ldots, m \).
for $t \leq s \leq T$, where $\beta_i(s, y)$ denotes the $i$th column of $\beta(s, y)$; $W_{js}$ denotes the $j$th dimension of Brownian motion $W_s$. Finally, the optimal intermediate consumption $c_t$ and terminal wealth $X_T$ are given by (19), i.e., $c_t = I^u(t, \lambda_t^*)$ and $X_T = I^U(T, \lambda_T^*)$.

Proof. See Appendix A.1.

The contribution of Theorem 1, relative to the previous literature, lies in that it reveals the structure of optimal policy under general incomplete market models with flexible utilities. The first component $\pi^{mv}(t, X_t, Y_t)$ is the mean-variance component, as reflected in (26a) through the market price of risk $\theta^h(t, Y_t)$ defined in (13a) – a mean-variance trade-off. The component $\pi^r(t, X_t, Y_t)$ (resp. $\pi^{\theta}(t, X_t, Y_t)$) is for hedging the uncertainty in interest rate (resp. price of risk), as reflected by the interest rate sensitive term $\nabla r$ involved in (26b) via $H_{t,T}^r(\lambda_t^*)$ in (27b) and $H_{t,s}^r(\lambda_t^*)$ in (31) (resp. the price of risk sensitive terms $\nabla \theta$, $\nabla \theta^u$, and $\partial \theta^u / \partial \lambda$ involved in (26c) via $H_{t,T}^\theta(\lambda_t^*)$ in (26c) and $H_{t,s}^\theta(\lambda_t^*)$ in (32)) as well as the Malliavin derivative $D_t Y_s$ appearing in (31) (resp. (32)). In particular, naturally analogous to classical derivatives, we can intuitively understand the Malliavin derivative $D_t Y_s$ as the sensitivity of the state variable $Y_s$ to the underlying Brownian motion $W_t$; see an accessible survey of Malliavin calculus for finance in Appendix D of Detemple et al. (2003).

This type of decomposition of optimal policy was first proposed in Merton (1971) for a complete market model. By (26a) – (26c), we represent each component as an explicit conditional expectation, which substantially extends the complete-market results in Detemple et al. (2003). In contrast to the complete market counterpart, the hedge component $\pi^\theta(t, X_t, Y_t)$ in (26c) is designed to hedge the uncertainty in both market and investor-specific price of risk. Indeed, this is because, as shown in (26c), $\pi^\theta(t, X_t, Y_t)$ depends on the term $H_{t,T}^\theta(\lambda_t^*)$ defined in (27c) via $H_{t,s}^\theta(\lambda_t^*)$, of which the dynamics involves both $\theta^h$ and $\theta^u$ according to (32).

In line with the time–$t$ formulation of the optimization problem (18), we express these components by the time–$t$ state variable $Y_t$ and the current wealth $X_t$, rather than the time–0 wealth $X_0$ as employed in most of the existing literature, e.g., Detemple et al. (2003). We can easily see that, by solving constraint (20), $\lambda_t^*$ is a function of $t$, $X_t$, and $Y_t$. In addition to the consistent expressions based on the time–$t$ information, this setup leads to convenience for further developing numerical methods for implementing our decomposition of optimal policy, e.g., the Monte Carlo simulation approach to propose momentarily in Section 4. Technically, the expressions based on time–$t$ and time–0 variables are related according to relation (28) linking the time–$t$ multiplier $\lambda_t^*$ and the time–0 multiplier $\lambda_0^*$ via state price density $\xi_t^S$, which by (15) depends on the entire path of interest rate

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We can view Malliavin calculus as the stochastic calculus of variation in the space of sample paths. Malliavin calculus has proven its important role in financial economics through its merit in solving portfolio choices problems, see, e.g., Ocone and Karatzas (1991), Detemple et al. (2003), Detemple and Rindisbacher (2005), and Detemple and Rindisbacher (2010). We can find a book-length discussion of the theory of Malliavin calculus in, for example, Nualart (2006).
$r(v,Y_t)$ and total price of risk $\theta^S_v$ for $0 \leq v \leq t$.

Moreover, we establish the explicit dynamics for $\xi^S_s(\lambda^*_t)$ and $H^\theta_{t,s}(\lambda^*_t)$ as in (29) and (32), respectively, which are key for both decomposition and implementation of the optimal policy under the incomplete market models. The representation of individual-specific price of risk $\theta^u_s$ in (24) plays a crucial role in deriving these dynamics and then further establishing the representations in Theorem 1. In particular, to naturally separate the given information up to time $t$, we follow (24) to express $\theta^u_s$ as the $\lambda^*_t$-dependent version:

$$\theta^u_s(\lambda^*_t) = \theta^u(s,Y_s;\lambda^*_t;T) = \theta^u(s,Y_s;\lambda^*_t\xi^S_{t,s}(\lambda^*_t);T), \quad (34)$$

where the second equality follows relations (16) and (28), i.e., $\lambda^*_s = \lambda^*_s\xi^S_s = \lambda^*_s\xi^S_{t,s}$. This leads to (30) – the parameterized version of the total price of risk (12). By (29) and (32), we see the separation of $\lambda^*_s$ and $\xi^S_s(\lambda^*_t)$ is fully reflected in the dynamics of $\xi^S_{t,s}(\lambda^*_t)$ and $H^\theta_{t,s}(\lambda^*_t)$. Besides, these $\lambda^*_t$-dependent versions naturally isolate the information up to time $t$, by which we can fully determine the multiplier $\lambda^*_t$. As we will show in Section 4, such a separation not only clearly reveals the structure regarding information, but also provides much convenience for our implementation of the decomposition by simulation method. Besides, in Sections 2.3 and 2.4, we employ the dynamics of $\xi^S_{t,s}(\lambda^*_t)$ and $H^\theta_{t,s}(\lambda^*_t)$ to analyze the impact on optimal policy from market incompleteness and wealth dependent properties of utility functions.

As an extension and enhancement of Theorem 1, we can further decompose the price of risk hedge component $\pi^\theta_t$ into three parts according to the economic nature of the uncertainties embedded in the market and investor-specific price of risk. Proposition 1 below shows that our decomposition not only brings insights to the structure of price of risk hedge component, but also disentangles the fundamental impacts on optimal policy from market incompleteness and wealth-dependent utilities. In particular, we identify a new important hedge component for non-myopic investors to hedge the uncertainty in investor-specific price of risk due to variation in wealth level.

**Proposition 1.** The price of risk hedge component $\pi^\theta(t,X_t,Y_t)$ in (26c) can be further decomposed as

$$\pi^\theta(t,X_t,Y_t) = \pi^{h,Y}(t,X_t,Y_t) + \pi^{u,Y}(t,X_t,Y_t) + \pi^{u,\lambda}(t,X_t,Y_t), \quad (35)$$

where components $\pi^{h,Y}(t,X_t,Y_t)$, $\pi^{u,Y}(t,X_t,Y_t)$, and $\pi^{u,\lambda}(t,X_t,Y_t)$ are given by:

$$\pi^{h,Y}(t,X_t,Y_t) = -\frac{1}{X_t}(\sigma(t,Y_t^+))^\top E_t[\mathcal{H}^{h,Y}_{t,t^*}], \quad (36a)$$

$$\pi^{u,Y}(t,X_t,Y_t) = -\frac{1}{X_t}(\sigma(t,Y_t^+))^\top E_t[\mathcal{H}^{u,Y}_{t,t^*}(\lambda^*_t)], \quad (36b)$$

$$\pi^{u,\lambda}(t,X_t,Y_t) = -\frac{1}{X_t}(\sigma(t,Y_t^+))^\top E_t[\mathcal{H}^{u,\lambda}_{t,t^*}(\lambda^*_t)]. \quad (36c)$$

Here, the terms $\mathcal{H}^{h,Y}_{t,t^*}$, $\mathcal{H}^{u,Y}_{t,t^*}(\lambda^*_t)$, and $\mathcal{H}^{u,\lambda}_{t,t^*}(\lambda^*_t)$ in (36a)–(36c) are defined in the same way as that for $\mathcal{H}^\theta_{t,s}(\lambda^*_t)$ in (27c) except for replacing $H^\theta_{t,s}$ by $H^{h,Y}_{t,s}$, $H^{u,Y}_{t,s}(\lambda^*_t)$, and $H^{u,\lambda}_{t,s}(\lambda^*_t)$ for $t \leq s \leq T$. 

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respectively, which follow the SDEs:

\[ dH_{t,s}^{h,Y} = (\mathcal{D}_t Y_s) \nabla \theta^h(s, Y_s)(\theta^h(s, Y_s)ds + dW_s), \quad (37a) \]
\[ dH_{t,s}^{u,Y}(\lambda^*_t) = (\mathcal{D}_t Y_s) \nabla \theta^u(s, Y_s, \lambda^*_t \xi^S_{t,s}(\lambda^*_t); T)(\theta^u_s(\lambda^*_t)ds + dW_s), \quad (37b) \]

and

\[ dH_{t,s}^{u,\lambda}(\lambda^*_t) = -\lambda^*_t \xi^S_{t,s}(\lambda^*_t)(\theta^S_s(\lambda^*_t) + H_{t,s}^r + H_{t,s}^\phi(\lambda^*_t)) \]
\[ \cdot \partial \theta^u/\partial \lambda(s, Y_s, \lambda^*_t \xi^S_{t,s}(\lambda^*_t); T)(\theta^u_s(\lambda^*_t)ds + dW_s), \quad (37c) \]

with initial values \( H_{t,t}^{h,Y} = H_{t,t}^{u,Y}(\lambda^*_t) = H_{t,t}^{u,\lambda}(\lambda^*_t) = 0_d \). Here, the \( \lambda^*_t \)-parameterized variables \( \xi^S_{t,s}(\lambda^*_t), \theta^S_s(\lambda^*_t), H_{t,s}^r(\lambda^*_t) \) and \( \theta^u_s(\lambda^*_t) \) are given in (29), (30), (32), and (34), respectively.

**Proof.** It follows by straightforward algebraic calculations based on the orthogonal condition (14b), representation (30), and the dynamics (32). \( \square \)

Decomposition (35) of the price of risk hedge component \( \pi^\theta(t, X_t, Y_t) \) has the following interpretations that reveal the fundamental structure of market incompleteness. The component \( \pi^{h,Y}(t, X_t, Y_t) \) hedges the uncertainty in the market price of risk \( \theta^h \) associated with the real assets. This because it hinges on (37a), in which the gradient \( \nabla \theta^h(s, Y_s) \) reflects the sensitivity of market price of risk with respect to the random state variable \( Y_s \). The other two components \( \pi^{u,Y}(t, X_t, Y_t) \) and \( \pi^{u,\lambda}(t, X_t, Y_t) \) are both hedges for the investor-specific price of risk associated with the fictitious assets. However, their causes are fundamentally different. The first component \( \pi^{u,Y}(t, X_t, Y_t) \) hedges the fluctuation in investor-specific price of risk that arises from the state variable, as explicitly reflected from the gradient \( \nabla \theta^u(s, Y_s, \lambda^*_t \xi^S_{t,s}(\lambda^*_t); T) \) in (37b). Its structure resembles that of \( \pi^{h,Y}(t, X_t, Y_t) \), except for replacing \( \theta^h \) by \( \theta^u \) in (37a). The second component \( \pi^{u,\lambda}(t, X_t, Y_t) \), however, is introduced by the sensitivity with respect to the multiplier \( \lambda^*_t = \lambda^*_t \xi^S_{t,s}(\lambda^*_t) \), i.e., \( \partial \theta^u/\partial \lambda(s, Y_s, \lambda^*_t \xi^S_{t,s}(\lambda^*_t); T) \) in (37c). Recall that, by (20), the multiplier \( \lambda^*_t \) is directly related to wealth level \( X_s \) via \( X_s = E_s[G_{s,t}(\lambda^*_t)] \). Thus, as a new important hedge component, \( \pi^{u,\lambda}(t, X_t, Y_t) \) essentially hedges the fluctuation in investor-specific price of risk due to the variation in wealth level, which is completely absent from the market price of risk. This structure reconciles and further develops the discussion following (24): under general incomplete market models, the suitable investor-specific price of risk depends on current market state and wealth level, and thus the uncertainties from both channels need to be hedged.

With the decomposition in Theorem 1 and Proposition 1, we face only one remaining difficulty in solving the optimal policy – the investor-specific price of risk function \( \theta^u(v, y, \lambda; T) \) is still undetermined. Indeed, by checking the function definitions for the four conditional expectations \( E_t[G_{t,T}(\lambda^*_t)], E_t[Q_{t,T}(\lambda^*_t)], E_t[H^r_{t,T}(\lambda^*_t)], \) and \( E_t[H^\phi_{t,T}(\lambda^*_t)] \) in (21) and (27a) – (27c) with \( \xi^S_{t,s}(\lambda^*_t) \) and \( H_{t,s}^r(\lambda^*_t) \) given in (29) and (32), we can verify that we can fully determine all of them after
we choose an investor-specific price of risk $\theta^u_s$. This reconciles the previous claim, in Karatzas et al. (1991) and He and Pearson (1991), that the characterization of $\theta^u_s$ plays a key role in solving the desired optimal policy for the real assets via fictitious completion. Meanwhile, by inspecting, e.g., (29) and (34) jointly, we find that the desired investor-specific price of risk function $\theta^u_s(s, y; \lambda; T)$ is entangled in a complex stochastic system. To circumvent this difficulty, our next task is to establish a novel equation system for solving this function in Theorem 2 below, which then suffices to explicitly express the optimal policy. The equation technically follows representation (24), the second orthogonal constraint in (14a), and the least favorable completion principle (23).

**Theorem 2.** The function $\theta^u(v, y; \lambda; T)$ for representing the investor-specific price of risk $\theta^u_s$ via representation (24) satisfies the orthogonal condition

$$\sigma(v, y)\theta^u(v, y; \lambda; T) \equiv 0_m$$

and the least favorable completion constraint (23). Or equivalently, $\theta^u(v, y; \lambda; T)$ satisfies the following $d$-dimensional equation

$$\theta^u(v, y, \lambda; T) = \frac{\sigma(v, y)^+\sigma(v, y) - I_d}{E[Q_{v,T}(\lambda)|Y_v = y]} \times [E[H_{v,T}^r(\lambda)|Y_v = y] + E[H_{v,T}^\theta(\lambda)|Y_v = y]],$$

(39)

for $0 \leq v \leq T$, where $I_d$ denotes the $d$-dimensional identity matrix; $\sigma(v, y)^+$ is given by (11). The function $\theta^u(s, y, \lambda; T)$ is fully characterized by a multidimensional coupled equation system consisting of equation (39), as well as the SDEs of $Y_s$, $\xi^{S}_{t,s}(\lambda)$, $H^{r}_{t,s}$, $H^{\theta}_{t,s}(\lambda)$, and $D_tY_s$ given in (2), (29), (31), (32), and (33), respectively, except for replacing $\lambda^*_s$ by $\lambda$. Consequently, given the function $\theta^u(v, y, \lambda; T)$, the optimal portfolio is determined by decomposition (25) based on (26a) – (26c).

**Proof.** See the derivation of equation (39) in Appendix A.2. \qed

We now analyze the structure of above-mentioned equation system that characterizes function $\theta^u(v, y, \lambda; T)$. It suffices to disentangle the structure of equation (39) by revealing how the unknown function $\theta^u$ gets involved in the conditional expectations $E[Q_{v,T}(\lambda)|Y_v = y]$, $E[H_{v,T}^r(\lambda)|Y_v = y]$, and $E[H_{v,T}^\theta(\lambda)|Y_v = y]$ therein. Without loss of generality, we take $E[Q_{v,T}(\lambda)|Y_v = y]$ as an example. By definitions in (27a) and (22b), it can be expressed as:

$$E[Q_{v,T}(\lambda)|Y_v = y] = E\left[\lambda \xi^{S}_{v,T}(\lambda) \frac{\partial I^U}{\partial y} (T, \lambda \xi^{S}_{v,T}(\lambda)) + \int_v^T \lambda \xi^{S}_{v,s}(\lambda)^2 \frac{\partial I^u}{\partial y} (s, \lambda \xi^{S}_{v,s}(\lambda)) ds | Y_v = y \right].$$

Here, we see that the conditional expectation is taken over the entire path of $\xi^{S}_{v,s}(\lambda)$ for $v \leq s \leq T$. Meanwhile, the dynamics of $\xi^{S}_{v,s}(\lambda)$ follows $d\xi^{S}_{v,s}(\lambda) = -\xi^{S}_{v,s}(\lambda)[r(s, Y_s)ds + (\theta^u s, Y_s, \lambda \xi^{S}_{v,s}(\lambda); T)]^\top dW_s$, by (29) and (30). It clearly depends on the unknown investor-specific price of risk function $\theta^u(s, y', \lambda'; T)$ for $v \leq s \leq T$ and all possible values of $y'$ and $\lambda'$. A similar structure holds for the conditional expectations $E[H_{v,T}^r(\lambda)|Y_v = y]$ and $E[H_{v,T}^\theta(\lambda)|Y_v = y]$. From
the above analysis, we verify that the function \( \theta^u(s, y, \lambda; T) \) is indeed fully characterized by the multidimensional equation system consisting of equation (39), as well as SDEs of \( Y_s, \xi^S_t(s), H^t(s), \) and \( D_t Y_s \) given in (2), (29), (31), (32), and (33), respectively, except for replacing \( \lambda^*_t \) by \( \lambda \). Finally, by checking the definitions in (27a) – (27c), as well as the SDEs of \( Y_s, \xi^S_t(s), H^t(s), H^\theta_t(s), \) and \( D_t Y_s \), we verify that all the components involved in this equation system are independent of the functional form of \( \sigma^F(v, y) \). Thus, \( \theta^u(v, y, \lambda; T) \), as the solution to the aforementioned equation system, is obviously also independent of the choice of \( \sigma^F(v, y) \). Such an independence lines up with the claim in Karatzas et al. (1991) that the desired price of risk \( \theta^u \) for fictitious assets is independent of \( \sigma^F(v, y) \).

Second, equation (39) obviously implies its terminal condition as

\[
\theta^u(T, y, \lambda; T) = 0_d,
\]

which corresponds to the investor-specific price of risk with the investment horizon shrinking to zero. An immediate consequence of this condition is that, as the investment horizon shrinks to zero, the return of the fictitious assets \( \mu^F_t \) employed in the least favorable completion converges to the risk-free return \( r(t, Y_t) \). This indeed directly follows from \( \theta^u(v, y, \lambda^*_t; T) = \sigma^F(v, y)^+ (\mu^F_v - r(v, y)1_{d-m}) \), which combines (13b) and (24). The terminal condition (40) plays an important role in potential numerical methods for solving function \( \theta^u(v, y, \lambda; T) \). For example, when employing a discretization and simulation approach, it serves as the terminal condition for numerical backward induction.

This demonstrates the importance of highlighting the investment horizon \( T \) as an argument of the investor-specific price of risk function \( \theta^u(v, y, \lambda; T) \) in (24).

In the literature, other types of differential equations were employed for characterizing optimal portfolios in incomplete markets; see, e.g., Detemple and Rindisbacher (2005) for deriving and solving a forward-backward SDE (FBSDE) governing the shadow price under a model featuring partially hedgeable Gaussian interest rate and CRRA utility, also see Detemple and Rindisbacher (2010) for generalizing such results, as well as He and Pearson (1991) for indirectly relating the optimal policy and the equivalent local martingale measure to the solution of a quasi-linear PDE. Thanks to the application of representation (24), when compared with, e.g., the forward-backward SDE, our equation system with (39), as given in Theorem 2 for solving the deterministic function \( \theta^u(v, y, \lambda; T) \), appears more explicit in terms of revealing the fundamental structure of the investor-specific price of risk for our general incomplete market model with flexible utilities, and it is relatively easier to handle technically. Besides, compared with the quasi-linear PDE based approach in He and Pearson (1991), our Theorem 1 together with Theorem 2 jointly provide an explicit decomposition of the

---

9If seeking for a purely analytical characterization, we can obtain an integral-partial differential equation system consisting of an integral-form expression of equation (39) and a Kolmogorov forward or backward PDE for governing the transition density of the Markov process \( (Y_s, \xi^S_t(s), D_t Y_s, H^t(s), H^\theta_t(s)) \) underlying the stochastic system. For the sake of space limitation, we omit such routine and technical details.
optimal policy.

Correct implementation of our decomposition obviously hinges on the system of equations for $\theta^u(s, y, \lambda; T), Y_s, \xi^S_t(s, y, \lambda), H^r_{t,s}, H^\theta_{t,s}(\lambda), \text{and } D_t Y_s$ as established in Theorem 2. Owing to its intricate structure and high-dimensional nature even for the one-asset case, it is a significant challenge to solve the policy analytically or even numerically via existing methods. We will tackle this difficulty in Section 4 via a Monte Carlo simulation method, which provides a practical method for numerically solving optimal policies under incomplete market models without closed-form solutions.

2.3 Impact from market incompleteness

In this section, we demonstrate the role of market incompleteness in the decomposition of optimal policy. For this purpose, we compare the optimal policy in general incomplete market models, as established in the previous section, with its complete market counterpart. More precisely, the complete market model for this comparison is set under the general model (1) and (2) by requiring the number of risky assets and the number of driving Brownian motions are equal, i.e., $m = d$. In such a complete market, we can fully hedge the risk by investing in the risky assets, and we do not need the above completion procedure with fictitious assets. We can decompose the optimal policy and implement it following the framework of Detemple et al. (2003).

In this complete market model, we assume that the square matrix $\sigma(t, y) := (\sigma_1(t, y), \ldots, \sigma_m(t, y))^\top$ is non-singular; see, for example, the related discussion in Ocone and Karatzas (1991). Then, we define the market price of risk as $\theta(t, Y_t) := \sigma(t, Y_t)^{-1} (\mu(t, Y_t) - r(t, Y_t) 1_m)$, which is the complete-market counterpart of the total price of risk $\theta^S_t$ in the completed market given in (9). Next, the state price density follows

$$
\xi_t := \exp \left( - \int_0^t r(v, Y_v) dv - \int_0^t \theta(v, Y_v)^\top dW_v - \frac{1}{2} \int_0^t \theta(v, Y_v)^\top \theta(v, Y_v) dv \right). \tag{41}
$$

For any $s \geq t \geq 0$, the relative state price density $\xi_{t,s} = \xi_s / \xi_t$ satisfies

$$
d\xi_{t,s} = -\xi_{t,s}[r(s, Y_s) ds + \theta(s, Y_s)^\top dW_s], \tag{42}
$$

with initial value $\xi_{t,t} = 1$, according to an application of Ito formula. Similarly, the dynamics of $H^\theta_{t,s}$ under the complete market setting specifies to

$$
dH^\theta_{t,s} = (D_t Y_s) \nabla \theta(s, Y_s)[\theta(s, Y_s) ds + dW_s], \tag{43}
$$

with initial value $H^\theta_{t,t} = 0_d$.

Comparing the above dynamics with their incomplete market counterparts (29) and (32), we observe that the dynamics (42) and (43) here are now explicitly given, as opposed to involving the undetermined investor-specific price of risk process $\theta^u_s$. Besides, we see an important feature that differentiates the complete and incomplete market settings: the dynamics (42) and (43) for the
The above structure of investor-specific price of risk is crucial for deriving the optimal policy. If we fail to recognize such structure that originates from market incompleteness, we might incorrectly write the dynamics of $\xi^{S}_{t,s}(\lambda^{*}_t)$ and $H^{\theta}_{t,s}(\lambda^{*}_t)$ for incomplete market models via a mechanical imitation of the complete-market dynamics (42) and (43), i.e., by simply changing the notation $\theta$ to $\theta^{S}$. This leads to not only wrong dynamics of $\xi^{S}_{t,s}(\lambda^{*}_t)$ and $H^{\theta}_{t,s}(\lambda^{*}_t)$, e.g., missing the term related to $\partial \theta^{u}/\partial \lambda$ in the second line of (32), but also the failure to capture the hedge term $\pi^{u,\lambda}(t, X_t, Y_t)$ in (35) that stems from the uncertainty in investor-specific price of risk due to variation in wealth level. This highlights again the importance of applying the representation (24) and the equation system in Theorem 2 for the investor-specific price of risk function $\theta^{u}(v, y, \lambda; T)$.

With the building blocks in (42) and (43), the decomposition of optimal policy in the complete market follows as a special case of the incomplete market results in Theorem 1, via replacing the components by their counterparts in the complete market. Specifically, the optimal policy $\pi_t$ still admits decomposition (25). The three components are given by (26a)–(26c), except for replacing $(\sigma(t, Y_t)^{+})^{\top}$ and $\theta^{h}(t, Y_t)$ by $(\sigma(t, Y_t)^{+})^{-1}$ and $\theta(t, Y_t)$, respectively. The time–t multiplier $\lambda^{*}_t$ is still characterized by the wealth equation (20). The functions $G_{t,T}(\lambda^{*}_t)$, $Q_{t,T}(\lambda^{*}_t)$, $H_{t,T}(\lambda^{*}_t)$, and $H_{t,T}^{\theta}(\lambda^{*}_t)$ appearing therein still satisfy (21), (27a), (27b), and (27c), respectively, except for replacing the relative state price density $\xi^{S}_{t,s}(\lambda^{*}_t)$ by $\xi_{t,s}$. Obviously, unlike the incomplete market case, the optimal policy $\pi_t$ for complete market models does not involve the investor-specific price of risk $\theta^{u}$ or the function $\theta^{u}(v, y, \lambda; T)$ for representing it via (24). We omit the details of these adaptation and reductions from incomplete market to complete market cases; see, e.g., Detemple et al. (2003), for the complete market policy decomposition.

2.4 Impact from wealth-dependent utilities

The previous literature on optimal portfolio choice in incomplete markets largely assumes that investors have the wealth-independent CRRA utility (see, e.g., Detemple and Rindisbacher (2005), Liu (2007), and Detemple and Rindisbacher (2010)). Hence, little is known about the cases with wealth-dependent utilities, e.g., the HARA utility (5b). We hereby start from discussing explicitly how the wealth-dependent utility impacts the structure of optimal policy, by comparing the policy decomposition results in Theorems 1 and 2 under the general utility case and those given in Corollary
where the mean-variance component 

\[ \tilde{\varphi}_{t,s} = -\tilde{\varphi}_{t,s} [r(s, Y_s) ds + (\theta^h(s, Y_s) + \theta^n(s, Y_s; T))^\top dW_s], \]  

and 

\[ dH^\theta_{t,s} = (D_t Y_s) (\nabla \theta^h(s, Y_s) + \nabla \theta^n(s, Y_s; T)) [\theta^S(s, Y_s; T) ds + dW_s], \]  

with \( \theta^S(s, Y_s; T) = \theta^h(s, Y_s) + \theta^n(s, Y_s; T) \). Then, the optimal portfolio policy \( \pi_t \) in (25) is independent of the current wealth level \( X_t \), and admits the following decomposition:

\[ \pi_t = \pi^{mv}(t, Y_t) + \pi^r(t, Y_t) + \pi^\theta(t, Y_t), \]  

where the mean-variance component \( \pi^{mv}(t, Y_t) \) follows

\[ \pi^{mv}(t, Y_t) = \frac{1}{\gamma} (\sigma(t, Y_t)^+) \Theta^h(t, Y_t) = \frac{1}{\gamma} (\sigma(t, Y_t) \sigma(t, Y_t)^\top)^{-1} (\mu(t, Y_t) - r(t, Y_t) 1_m); \]  

the interest rate hedge and price of risk hedge components are given by

\[ \pi^r(t, Y_t) = - (\sigma(t, Y_t)^+) \Theta^r(t, Y_t) \]  

and \( \pi^\theta(t, Y_t) = - (\sigma(t, Y_t)^+) \Theta^\theta(t, Y_t) \).

The functions \( \tilde{\varphi}_{t,T}, \tilde{\varphi}_{t,T}^\theta, \) and \( \tilde{\varphi}_{t,T}^\gamma \) in above are defined as

\[ \tilde{\varphi}_{t,T} := \left(1 - \frac{1}{\gamma} \right) \left[ (1 - w) \frac{1}{\gamma} e^{-\frac{\rho T}{\gamma}} (\Xi_{t,T}^S)_{1/\gamma}^\top H_{t,T}^{r} + w \frac{1}{\gamma} \int_t^T e^{-\frac{\rho s}{\gamma}} (\Xi_{s,T}^S)_{1/\gamma}^\top H_{s,T}^{r} ds \right], \]  

\[ \tilde{\varphi}_{t,T}^\theta := \left(1 - \frac{1}{\gamma} \right) \left[ (1 - w) \frac{1}{\gamma} e^{-\frac{\rho T}{\gamma}} (\Xi_{t,T}^S)_{1/\gamma}^\top H_{t,T}^{\theta} + w \frac{1}{\gamma} \int_t^T e^{-\frac{\rho s}{\gamma}} (\Xi_{s,T}^S)_{1/\gamma}^\top H_{s,T}^{\theta} ds \right], \]  

and

\[ \tilde{\varphi}_{t,T}^\gamma := (1 - w) \frac{1}{\gamma} e^{-\frac{\rho T}{\gamma}} (\Xi_{t,T}^S)_{1/\gamma}^\top H_{t,T}^{\gamma} + w \frac{1}{\gamma} \int_t^T e^{-\frac{\rho s}{\gamma}} (\Xi_{s,T}^S)_{1/\gamma}^\top H_{s,T}^{\gamma} ds, \]

with \( \Xi_{t,T}^S, H_{t,T}^{r}, \) and \( H_{t,T}^{\theta} \) evolving according to (44a), (31), and (44b), respectively. The investorspecific price of risk function \( \sigma^u(v, y; T) \) satisfies the following \( d \)–dimensional equation

\[ \sigma^u(v, y; T) = \frac{\sigma(v, y)^+ \sigma(v, y) - I_d}{E[Q_{v,T} | Y_v = y]} \times (E[\tilde{\varphi}_{v,T}^{\gamma} | Y_v = y] + E[\tilde{\varphi}_{v,T}^{\theta} | Y_v = y]), \]  

where \( \tilde{Q}_{v,T} = - \tilde{\varphi}_{v,T} / \gamma \), defined by

\[ \tilde{Q}_{v,T} := -\frac{1}{\gamma} \left[ (1 - w) \frac{1}{\gamma} e^{-\frac{\rho T}{\gamma}} (\Xi_{v,T}^S)_{1/\gamma}^\top + w \frac{1}{\gamma} \int_v^T e^{-\frac{\rho s}{\gamma}} (\Xi_{s,v}^S)_{1/\gamma}^\top ds \right]. \]  

Corollary 1. Under the incomplete market model (1) – (2) with the CRRA utility function given in (5a), the investor-specific price of risk function \( \sigma^u(v, y; T) \) introduced through (24) does not depend on the parameter \( \lambda \), and thus can be written as \( \sigma^u(v, y; T) \). Consequently, the relative state price density \( \xi^S_{t,s}(\lambda^*_t) \) and Malliavin term \( H^\theta_{t,s}(\lambda^*_t) \) characterized in general by dynamics (29) and (32), respectively, become independent of \( \lambda^*_t \). So we write them simply as \( \xi^S_{t,s} \) and \( H^\theta_{t,s} \), and spell their dynamics as
The function \( \theta^u(s, y; T) \) is fully characterized by a multidimensional equation system consisting of equation (48), as well as the SDEs of \( Y_s, \xi^S_{t,s}, H^\theta_{t,s}, H^r_{t,s}, \) and \( D_t Y_s \) given in (2), (44a), (31), (44b), and (33), respectively, which are all independent of the parameter \( \lambda \).

Proof. See Appendix A.3.

Though as a special case, the explicit decomposition in Corollary 1 is new relative to the existing analysis on the structure of optimal policy under CRRA utility by means of, e.g., HJB equations (see, e.g., Liu (2007).) By comparing the decomposition in Corollary 1 and that in Theorems 1 and 2, we explicitly observe how the structure of optimal policy under wealth-independent CRRA utility differs from that under the general wealth-dependent utilities, as well as how the specific structure of CRRA utility allows for significant simplification of the decomposition. This comparative study demonstrates again the importance of our explicit decomposition of optimal policy in Theorems 1 and 2.

First, the building blocks employed in these two decompositions obviously have different structures. Since the function \( \theta^u(v, y, \lambda; T) \) does not depend on the parameter \( \lambda \), the investor-specific price of risk for the CRRA case takes the form \( \theta^u_s = \theta^u(s, Y_s; T) \), which is independent of the multiplier \( \lambda^*_s \). By the analysis similar to those immediately prior to Theorem 1, \( \theta^u_s \) here is also independent of the wealth level \( X_s \). Thus, the market completion under the CRRA utility enjoys a simpler mechanism.

We now compare the dynamics of \( \xi^S_{t,s}(\lambda^*_t) \) in (29) with that of \( \xi^S_{t,s} \) in (44a), as well as the dynamics of \( H^\theta_{t,s}(\lambda^*_t) \) in (32) with that of \( H^\theta_{t,s} \) in (44b). Owing to the absence of parameter \( \lambda \) from function \( \theta^u(v, y, \lambda; T) \), dynamics (44a) and (44b) are obviously simpler than (29) and (32). In particular, the term related to \( \partial \theta^u/\partial \lambda \) in the second line of (32) vanishes in the corresponding equation (44b) for the CRRA utility case.

Next, our decomposition results illustrate how current wealth level impacts the optimal policy under general wealth-dependent utilities, but not under the CRRA utility. By Theorems 1 and 2 for general wealth-dependent utilities, the current wealth level \( X_t \) impacts the optimal policy through two channels. First, it directly appears in the optimal policy as the denominator in (26a) – (26c). Second, due to the wealth equation \( X_t = E_t[G_{t,T}(\lambda^*_t)] \), \( X_t \) is implicitly involved in the optimal policy through the time–t multiplier \( \lambda^*_t \) in the functions \( Q_{t,T}(\lambda^*_t), H^r_{t,T}(\lambda^*_t), \) and \( H^\theta_{t,T}(\lambda^*_t) \) with the building blocks \( \xi^S_{t,s}(\lambda^*_t) \) and \( H^\theta_{t,s}(\lambda^*_t) \). However, both channels are absent under the CRRA utility, thanks to its special structure. As shown in (46a) and (46b), both \( X_t \) and \( \lambda^*_t \) vanish in the components of the optimal policy. Furthermore, by (44a) and (44b), the building blocks \( \xi^S_{t,s} \) and \( H^\theta_{t,s} \) are also independent of \( \lambda^*_t \). Such independence guarantees that \( X_t \) is not implicitly involved in the optimal policy through \( \lambda^*_t \) as in the case with wealth-dependent utility. In essence, this is again because the investor-specific price of risk \( \theta^u(s, Y_s; T) \) does not depend on \( \lambda^*_s \) under the CRRA utility.

\[10\] This wealth-independent property reconciles the conclusions in Detemple et al. (2003) and Ocone and Karatzas (1991) for complete market models.
Beside the wealth-independent property, the simplified structure under CRRA utility has a fundamental impact on the resulting optimal policy. First, by (46a), the mean-variance component \(\pi_{mv}(t,Y_t)\) under CRRA utility is independent of investment horizon \(T\) and only depends on current time \(t\) and state variable \(Y_t\). Besides, it reflects the classic mean-variance trade-off structure, as it is proportional to the excess return of the risky assets \(\mu(t,Y_t) - r(t,Y_t)1_m\) and is inversely proportional to the covariance matrix \(\sigma(t,Y_t)\sigma(t,Y_t)^\top\) as well as the risk aversion level \(\gamma\).

As a direct implication of Proposition 1, the price of risk hedge component \(\pi_{\theta}(t,Y_t)\) has the following decomposition under CRRA utility:

\[
\pi_{\theta}(t,Y_t) = \pi_{h,Y}(t,Y_t) + \pi_{u,Y}(t,Y_t),
\]

with \(\pi_{h,Y}(t,Y_t) = -(\sigma(t,Y_t)^+)^\top E_t[\tilde{H}_{t,T}^{h,Y}/E_t[\tilde{G}_{t,T}]]\) and \(\pi_{u,Y}(t,Y_t) = -(\sigma(t,Y_t)^+)^\top E_t[\tilde{H}_{t,T}^{u,Y}/E_t[\tilde{G}_{t,T}]].\)

The functions \(\tilde{H}_{t,T}^{h,Y}\) and \(\tilde{H}_{t,T}^{u,Y}\) are defined in the same way as \(\tilde{H}_{t,T}^{\theta}\) in (47b), except for replacing \(H_{t,T}^{\theta}\) by \(H_{t,T}^{h,Y}\) and \(H_{t,T}^{u,Y}\), respectively, which follow (37a) and (37b) with \(\theta^u(s,Y_s,\lambda^*_{S_t}(\lambda^*_t);T)\) replaced by the CRRA–utility counterpart \(\theta^u(s,Y_s;T)\). It is straightforward to verify that \(\pi_{h,Y}(t,Y_t)\) and \(\pi_{u,Y}(t,Y_t)\) are both independent of the wealth level \(X_t\). The last component \(\pi_{u,\lambda}(t,X_t,Y_t)\) in (35), however, vanishes under the CRRA utility, as the investor-specific price of risk \(\theta^u_u = \theta^u(s,Y_s;T)\) is independent of multiplier \(\lambda^*_s\). As discussed after Proposition 1, the term \(\pi_{u,\lambda}(t,X_t,Y_t)\) hedges the uncertainty in investor-specific price of risk that arises from the variation in wealth level. However, as the investor-specific price of risk does not depend on wealth level under CRRA utility, this hedge component naturally disappears.

### 3 Application 1: Decomposing optimal policy under HARA utility

With the wealth-dependent property and minimum requirements for terminal wealth and/or intermediate consumption, the HARA utility offers more flexibility in capturing investor risk preference than the CRRA utility. However, though desired, solving for optimal policy under HARA utility is usually regarded as a notoriously difficult and even prohibitive task. To the best of our knowledge, only a few analytical results on optimal portfolio under HARA utility exist in the literature; see, e.g., Kim and Omberg (1996). Instead, previous literatures largely assume CRRA utility for simplicity, because of the technical inconvenience with HARA utility.

In the first part of this section, we apply our decomposition results in Theorems 1 and 2 to explicitly characterize optimal policy under HARA utility, illustrating the potential role of our decomposition for solving optimal policies in closed form, which facilitate subsequent economic analysis. We first develop the results under general incomplete market models with HARA utility. Then, more surprisingly, we further show that under the special case with nonrandom but possibly time-varying interest rate, we can fundamentally connect the HARA optimal policy to its CRRA–utility counterpart. We can explicitly solve the HARA policy as a product of its counterpart under CRRA utility and an explicit multiplier involving current wealth level and bond prices, further revealing how the
optimal policy behaves. In the second part of this section, we specialize our decomposition to a representative example of incomplete market – the celebrated stochastic volatility model of Heston (1993), leading to closed-form optimal policy under the wealth-dependent HARA utility and the subsequent explicitly analysis of the behavior of optimal policies under the complex settings of market incompleteness and wealth-dependent utility.

3.1 Decomposition under HARA utility in incomplete markets

We start our discussions from the general case under the incomplete market model (1) and (2).

**Corollary 2.** Under the HARA utility (5b) with \( w > 0 \), the investor-specific price of risk function \( \theta^u(v, y; \lambda; T) \) for representing the investor-specific price of risk \( \theta_v^u \) via representation (24) satisfies the following \( d \)-dimensional equation

\[
\theta^u(v, y; \lambda; T) = \frac{\sigma(v, y)^+ \sigma(v, y) - I_d}{E[\hat{Q}_{v,T}(\lambda)|Y_v = y]} (E[\hat{H}_{r,v,T}(\lambda)|Y_v = y] + E[\hat{H}_{\theta,v,T}(\lambda)|Y_v = y] + \lambda^\frac{1}{\gamma} E[\zeta_{v,T}(\lambda)|Y_v = y]) .
\]

(50)

Here, \( \hat{H}_{r,v,T}(\lambda) \), \( \hat{H}_{\theta,v,T}(\lambda) \), and \( \hat{Q}_{v,T}(\lambda) \) are defined as

\[
\hat{H}_{r,v,T}(\lambda) := \left(1 - \frac{1}{\gamma}\right) \left[(1 - w)^{\frac{1}{\gamma}} e^{-\frac{\rho_T}{\gamma} (\xi_{v,T}(\lambda))^{1-\frac{1}{\gamma}}} H_{r,v,T} + w^{\frac{1}{\gamma}} \int_v^T e^{-\frac{\rho_s}{\gamma} (\xi_{v,s}(\lambda))^{1-\frac{1}{\gamma}}} H_{r,v,s} ds\right], \quad (51a)
\]

\[
\hat{H}_{\theta,v,T}(\lambda) := \left(1 - \frac{1}{\gamma}\right) \left[(1 - w)^{\frac{1}{\gamma}} e^{-\frac{\rho_T}{\gamma} (\xi_{v,T}(\lambda))^{1-\frac{1}{\gamma}}} H_{\theta,v,T}(\lambda) + w^{\frac{1}{\gamma}} \int_v^T e^{-\frac{\rho_s}{\gamma} (\xi_{v,s}(\lambda))^{1-\frac{1}{\gamma}}} H_{v,s}(\lambda)ds\right], \quad (51b)
\]

and

\[
\hat{Q}_{v,T}(\lambda) := -\frac{1}{\gamma} \left[(1 - w)^{\frac{1}{\gamma}} e^{-\frac{\rho_T}{\gamma} (\xi_{v,T}(\lambda))^{1-\frac{1}{\gamma}}} + w^{\frac{1}{\gamma}} \int_v^T e^{-\frac{\rho_s}{\gamma} (\xi_{v,s}(\lambda))^{1-\frac{1}{\gamma}}} ds\right], \quad (51c)
\]

with \( \xi_{v,s}(\lambda) \), \( H_{r,v,T}(\lambda) \), and \( H_{\theta,v,T}(\lambda) \) evolving according to SDEs (29), (31), and (32), respectively, except for replacing \( \lambda_t^* \) by \( \lambda \). Besides, \( \zeta_{v,T}(\lambda) \) is a \( d \)-dimensional column vector given by

\[
\zeta_{v,T}(\lambda) = \zeta^r_{v,T}(\lambda) + \zeta^\theta_{v,T}(\lambda),
\]

(52a)

where

\[
\zeta^r_{v,T}(\lambda) := \tilde{x}_v \xi^S_{v,T}(\lambda) H_{r,v,T} + \tilde{c} \int_v^T \xi^S_{v,s}(\lambda) H_{v,s}(\lambda) ds,
\]

(52b)

\[
\zeta^\theta_{v,T}(\lambda) := \tilde{x}_v \xi^S_{v,T}(\lambda) H_{\theta,v,T}(\lambda) + \tilde{c} \int_v^T \xi^S_{v,s}(\lambda) H_{v,s}(\lambda) ds,
\]

(52c)

with \( \tilde{x} \) and \( \tilde{c} \) being the minimum allowable levels for terminal wealth and intermediate consumption under the HARA utility (5b). The optimal policy under HARA utility follows by \( \pi_t = \pi^{mv}(t, X_t, Y_t) + \pi^r(t, X_t, Y_t) + \pi^\theta(t, X_t, Y_t) \), where

\[
\pi^{mv}(t, X_t, Y_t) = -\frac{1}{X_t} (\sigma(t, Y_t)^+) \theta^h(t, Y_t) (\lambda_t^*)^{-\frac{1}{\gamma}} E_t[\hat{Q}_{v,T}(\lambda_t^*)],
\]

(53a)
and the hedging components given by:

\[ \pi^r(t, X_t, Y_t) = -\frac{1}{X_t} \sigma(t, Y_t)^+ (\lambda_t^*)^{-\frac{1}{2}} E_t[\tilde{H}^r_{t,T}(\lambda_t^*)] + E_t [\tilde{\theta}^r_{t,T}(\lambda_t^*)], \]

\[ \pi^\theta(t, X_t, Y_t) = -\frac{1}{X_t} \sigma(t, Y_t)^+ (\lambda_t^*)^{-\frac{1}{2}} E_t[\tilde{H}^\theta_{t,T}(\lambda_t^*)] + E_t [\tilde{\theta}^\theta_{t,T}(\lambda_t^*)]. \]

The multiplier \( \lambda_t^* \) is characterized as the unique solution for the wealth constraint:

\[ (\lambda_t^*)^{-\frac{1}{2}} E_t[\tilde{G}_{t,T}(\lambda_t^*)] + \pi E_t [\tilde{\theta}^S_{t,T}(\lambda_t^*)] + \tau E_t \left[ \int_t^T \tilde{\theta}^S_{t,s}(\lambda_t^*) \, ds \right] = X_t, \]

with \( \tilde{G}_{t,T}(\lambda) = -\gamma \tilde{Q}_{v,T}(\lambda) \), defined by

\[ \tilde{G}_{t,T}(\lambda) := (1 - w)^{\frac{1}{2}} e^{-\frac{v}{2}} (\xi_{t,T}^S(\lambda))^{-\frac{1}{2}} + w^{\frac{1}{2}} \tau \int_t^T e^{-\frac{v}{2}} (\xi_{t,s}^S(\lambda))^{-\frac{1}{2}} \, ds. \]

For case of \( w = 0 \) in utility (5b), the above representation still holds except for dropping the terms related to \( \tau \) in (52b), (52c), and (54).

We now analyze the structure of equation (50) and the fundamental difference from its CRRA counterpart. Comparing equation (50) under the HARA utility with its CRRA utility counterpart (48), a striking difference lies in that equation (50) contains the additional term \( \lambda^2 E[\xi_{v,T}(\lambda)|Y_v = y] \) on its right-hand side. This explicitly introduces parameter \( \lambda \) to the equation and then to the solution of \( \theta^u(v, y, \lambda; T) \). Then, it follows from (51a), (51b), and (51c) that the components \( \tilde{H}^r_{v,T}(\lambda) \), \( \tilde{H}^\theta_{v,T}(\lambda) \), and \( \tilde{G}_{v,T}(\lambda) \) also depend on \( \lambda \). Besides, by (52a) – (52c), \( \lambda^2 E[\xi_{v,T}(\lambda)|Y_v = y] \) appears as a linear combination of the minimum allowable levels \( \bar{x} \) and \( \bar{c} \) for terminal wealth and intermediate consumption under the HARA utility, and is the only term explicitly involving these two constants that feature the HARA utility. From this aspect, equation (50) provides a further decomposition for equation (39) under the HARA utility by isolating the parts involving \( \bar{c} \) and \( \bar{c} \). As a result, the multiplier \( \lambda_t^* \) and thus the current wealth level \( X_t \) also get involved in the optimal policy (53a) – (53c) under HARA utility.

Based on the general HARA results in Corollary 2, we further show in Proposition 2 below that under nonrandom but possibly time-varying interest rate, investor-specific price of risk under HARA utility is indeed identical to that under the corresponding CRRA utility, and furthermore, the optimal policies under HARA and CRRA utilities are connected to each other by a simple multiplier related to current wealth level and bond prices, which shed lights on the construction of the optimal policy under HARA utility.

**Proposition 2.** With nonrandom but possibly time-varying interest rate, the investor-specific price of risk \( \theta^u_v \) under HARA utility (5b) coincides with its counterpart under the CRRA utility (5a). That is, it does not depend on multiplier \( \lambda_t^* \) and indeed allows the following representation \( \theta^u_v = \theta^u(v, Y_v; T) \), with the function \( \theta^u(v, y; T) \) characterized by equation

\[ \theta^u(v, y; T) = \frac{\sigma(v, y)^+ \sigma(v, y) - I_d}{E[\tilde{Q}_{v,T}|Y_v = y]} \times E[\tilde{H}^\theta_{v,T}|Y_v = y], \]
with $\tilde{H}^v_{t,T}$ and $\tilde{Q}_{v,T}$ given in (47b) and (49), respectively. The optimal policy under HARA utility satisfies the following simple ratio relationship with its counterpart under CRRA utility:

$$
\pi^m_H(t, X_t, Y_t) = \pi^m_C(t, Y_t) \frac{\bar{X}_t}{X_t} \quad \text{and} \quad \pi^\theta_H(t, X_t, Y_t) = \pi^\theta_C(t, Y_t) \frac{\bar{X}_t}{X_t},
$$

as well as $\pi^r_H(t, X_t, Y_t) = \pi^r_C(t, Y_t) \bar{X}_t / X_t \equiv 0$ due to deterministic nature of interest rate. Here, the subscripts $H$ and $C$ represent for the HARA and CRRA utilities, respectively. The CRRA components $\pi^m_C(t, Y_t)$ and $\pi^\theta_C(t, Y_t)$ are given in (46a) and (46b). Besides, $\bar{X}_t$ in (57) is given by

$$
\bar{X}_t = X_t - \pi B_{t,T} - \bar{v} \int_t^T B_{t,s} ds, \text{ for } w > 0,
$$

(58a)

and

$$
\bar{X}_t = X_t - \pi B_{t,T}, \text{ for } w = 0,
$$

(58b)

where $B_{t,s} := \exp(-\int_t^s \bar{r}_v dv)$ is the price at time $t$ for a zero-coupon bond maturing at time $s$.

**Proof.** See Appendix A.4. \qed

Let us further discuss the fundamental and explicit connection stated in Proposition 2. First, the investor-specific price of risk under the two utilities indeed agree with each other. That is, when there is no uncertainty in interest rate, the investor completes market in exactly the same way under the two utilities, and the impact of current wealth level entirely vanishes in the investor-specific price of risk, which is different from the situation under otherwise more general cases as analyzed in Corollary 2 and its follow-up discussions. Second, the ratio relationship (57) bridges the gap between HARA and CRRA policies, and thus renders a convenient way to compute or approximate the optimal policy under HARA utility based on its CRRA counterpart, which is more advantageous to solve in closed-form or implement via, e.g., simulation methods due to the wealth independent nature.

Although deterministic interest rate is assumed in Proposition 2, no assumptions are imposed on the state variable. Thus, we can apply the relationship (57) to various models with sophisticated state variable evolving according to complex dynamics. As long as the investor-specific price of risk and optimal policy under CRRA utility can be solved, so do those under the HARA utility. For example, Liu (2007) assumes constant interest rate in his incomplete market with stochastic volatility, and obtains the closed-form optimal policy under CRRA utility. With relationship (57), we can now solve the policy in closed form under HARA utility for this typical incomplete market example. Extension of Proposition 2 to the case with random interest rate can be regarded as an open research topic, for which the change of numeraire techniques in Detemple and Rindisbacher (2010) may render a useful tool.

We can view relationship (57) as a decomposition of the optimal policy under HARA utility by disentangling how the state variable and current wealth level get involved in the optimal policy: the
state variable $Y_t$ impacts the optimal policy only through the CRRA counterparts $\pi_{C}^{mv}(t,Y_t)$ and $\pi_{C}^{\theta}(t,Y_t)$, while the current wealth level $X_t$ impacts optimal policy only through the ratio $\bar{X}_t/X_t$.

Such a structure leads to an explicit explanation of the puzzle on how HARA investor optimally allocates her/his wealth, while fulfilling the minimum terminal wealth and intermediate consumption required by the utility specification. With deterministic interest rate, $B_{t,s}$ represents the time–$t$ price of a zero-coupon bond with face value one maturing at time $s$. Thus, $\bar{X}_t$ given in (58a), i.e., $\bar{X}_t = X_t - \bar{x} B_{t,T} - \bar{c} \int_{t}^{T} B_{t,s} ds$ is the remaining wealth, after the investor buys $\bar{x}$ zero-coupon bonds maturing at $T$, and a continuum of $\bar{c} ds$ zero-coupon bond maturing at $s$ for all $s \in [t,T]$. The continuum payments from this bonds holding position exactly render the minimum terminal wealth $x$ and intermediate consumption $c$ required by the HARA utility (5b), i.e., $x$ at time $T$ and $c$ at each $s \in [t,T]$. Immediately after the bonds purchasing, the investor allocates the remaining wealth following the optimal policy under CRRA utility, i.e., $\pi_{C}^{mv}(t,Y_t)\bar{X}_t$ and $\pi_{C}^{\theta}(t,Y_t)\bar{X}_t$ amount of wealth for the mean-variance and price of risk hedge components respectively, thus leading to the optimal policy under HARA utility given in (57). We can summarize the insights out of relationship (57) as follows. With deterministic interest rate, the investor first buys a series of zero-coupon bonds to satisfy his minimum requirements for terminal wealth and intermediate consumption in the entire future investment horizon, then allocates his remaining wealth just as a pure CRRA investor. The reason why the neat and seemingly simple relationship (57) was not discovered, to our best knowledge, in the literature though much desired, is probably because the investment scheme (58a) and the corresponding mathematical formulation are not easy to conjecture and handle.

Moreover, the decomposition in Proposition 2 leads to explicit evidences for the following economic common consensus. First, the HARA investor allocates more on risky assets as his wealth level increases, since the ratio $\bar{X}_t/X_t$ monotonically increases in current wealth level $X_t$. Second, the optimal components $\pi_{H}^{mv}(t,X_t,Y_t)$ and $\pi_{H}^{\theta}(t,X_t,Y_t)$ converge to their CRRA counterparts as $X_t$ approaches to infinity, since it follows from (57) and (58a) that

$$\lim_{X_t \to \infty} \pi_{H}^{mv}(t,X_t,Y_t) = \pi_{C}^{mv}(t,Y_t) \lim_{X_t \to \infty} \frac{1}{X_t} \left( X_t - \bar{x} B_{t,T} - \bar{c} \int_{t}^{T} B_{t,s} ds \right) = \pi_{C}^{mv}(t,Y_t),$$

and

$$\lim_{X_t \to \infty} \pi_{H}^{\theta}(t,X_t,Y_t) = \pi_{C}^{\theta}(t,Y_t) \lim_{X_t \to \infty} \frac{1}{X_t} \left( X_t - \bar{x} B_{t,T} - \bar{c} \int_{t}^{T} B_{t,s} ds \right) = \pi_{C}^{\theta}(t,Y_t).$$

These findings reconcile the fact implied by the Arrow-Pratt relative risk aversion coefficient

$$\gamma_{U}(x) := -\left( \frac{\partial U}{\partial x} (t,x) \right)^{-1} \frac{\partial^{2} U}{\partial x^{2}} (t,x).$$

Under the HARA utility for terminal wealth, it is given by $\gamma_{U}(X_t) = \gamma X_t/(X_t - \bar{x})$. It clearly decreases as $X_t$ increases, and approaches its CRRA counterpart $\gamma$ as $X_t$ goes to infinity. Thus, the decrease in risk aversion degree motivates HARA investor to invest more in risky assets.
3.2 Illustrations by the incomplete-market stochastic volatility model of Heston (1993)

In this section, we apply the decomposition developed in the previous section to a representative example of incomplete market – the celebrated stochastic volatility model of Heston (1993). Our decomposition allows for solving the optimal policy under the wealth-dependent HARA utility in closed form. We then use the closed-form formulae to explicitly analyze the behavior of optimal policies by comparing it with the CRRA counterpart. In particular, we will investigate how they are impacted by, e.g., wealth level, interest rate, and investment horizon, that play important roles in the wealth-dependent HARA utility. This example illustrates how we can further apply our theoretical findings on the decompositions established in the previous sections to explicitly analyze the behavior of optimal policies under complex settings such as market incompleteness and wealth-dependent utility. We now begin by setting the model:

**Example 1** (The incomplete-market stochastic volatility model of Heston (1993)). The asset price $S_t$ follows

$$
\frac{dS_t}{S_t} = (r + \lambda V_t)dt + \sqrt{(1 - \rho^2)V_t}dW_{1t} + \rho \sqrt{V_t}dW_{2t},
$$

(60a)

and the variance $V_t$ follows

$$
dV_t = \kappa(\theta - V_t)dt + \sigma \sqrt{V_t}dW_{2t},
$$

(60b)

where $W_{1t}$ and $W_{2t}$ are two independent standard one-dimensional Brownian motions. Here, the parameter $r$ denotes the risk-free interest rate; the parameter $\lambda$ controls the market price of risk; the positive parameters $\kappa, \theta,$ and $\sigma$ represent the speed of mean-reversion, the mean-reverting level, and the proportional volatility of the variance process $V_t$, respectively. We assume the Feller’s condition holds: $2\kappa \theta > \sigma^2$. The leverage effect parameter $\rho \in [-1, 1]$ measures the instantaneous correlation between the asset return and the change in its variance.

As shown in Example 1, the stochastic volatility model of Heston (1993) (Heston SV thereafter) is an incomplete market model featuring stochastic volatility, and it belongs to the class of affine models (Duffie et al. (2000)). We study the optimal policy under the Heston SV model for risk averse investors with HARA or CRRA utilities over the terminal wealth, i.e., the following maximization problem

$$
sup_{\pi_t} E(U(T, X_T)),
$$

with the HARA utility function

$$
U(T, x) = \frac{(x - \bar{x})^{1-\gamma}}{1-\gamma},
$$

(61a)

for some risk aversion coefficient $\gamma > 1$ as well as some minimum allowable level $\bar{x}$, and for comparison purposes, the corresponding CRRA utility function

$$
U(T, x) = \frac{x^{1-\gamma}}{1-\gamma},
$$

(61b)
By applying Proposition 2 on decomposing optimal policy for HARA investors, we immediately obtain the closed-form formulae for the mean-variance component, the price of risk hedge component, and the interest rate hedge component under the HARA utility as

$$\pi_{H}^{mv}(t, X_t, V_t) = \pi_{C}^{mv}(t, V_t) \frac{\bar{X}_t}{X_t}, \quad \pi_{H}^{\theta}(t, X_t, V_t) = \pi_{C}^{\theta}(t, V_t) \frac{\bar{X}_t}{X_t},$$

(62)

and

$$\pi_{H}^{r}(t, X_t, V_t) = \pi_{C}^{r}(t, V_t) \frac{\bar{X}_t}{X_t},$$

where $$\pi_{C}^{mv}(t, V_t), \pi_{C}^{\theta}(t, V_t),$$ and $$\pi_{C}^{r}(t, V_t)$$ are the counterparts under the corresponding CRRA utility, and $$\bar{X}_t$$ follows

$$\bar{X}_t = X_t - x \exp(-r(T-t)),\quad (63)$$

according to (58b). In particular, for the interest rate hedge components, we have

$$\pi_{H}^{r}(t, X_t, V_t) = \pi_{C}^{r}(t, V_t) \frac{\bar{X}_t}{X_t} \equiv 0.$$ \quad (64)

This follows from the representation of interest rate hedge component in (46b) and the constant nature of risk-free interest rate $$r$$. From an obvious economics perspective, it leads to no hedging demand for interest rate; from a technical perspective, it leads to a zero for the Malliavin derivative term $$H_{r,t,s}$$ defined in (31), and, according to (47a), further leads to a zero of the component $$\tilde{H}_{r,T}$$, which determines $$\pi_{C}^{r}(t, V_t)$$ according to (46b). Thus, the optimal policy under HARA utility is given by

$$\pi_{H}(t, X_t, V_t) = \pi_{C}^{mv}(t, X_t, V_t) + \pi_{C}^{\theta}(t, V_t) = \bar{X}_t \pi_{C}(t, V_t),$$

(65)

where $$\pi_{C}(t, V_t)$$ is the optimal policy under the CRRA utility, given by:

$$\pi_{C}(t, V_t) = \pi_{C}^{mv}(t, V_t) + \pi_{C}^{\theta}(t, V_t).$$ \quad (66)

Formulae (62), (65), and (66) are all in closed-form, since the corresponding CRRA components $$\pi_{C}^{mv}(t, V_t)$$ and $$\pi_{C}^{\theta}(t, V_t)$$ can be obtained in closed form based on our decomposition (45) in Corollary 1 and the closed-form CRRA optimal policy solved in Liu (2007). While the optimal policy for CRRA investors under the Heston SV model is available in closed-form, its counterpart for HARA investors, to our best knowledge, was absent in the literature. This is probably due to the common consensus that HARA utility causes mathematical inconvenience. We now briefly state the closed-form results for the CRRA components $$\pi_{C}^{mv}(t, V_t)$$ and $$\pi_{C}^{\theta}(t, V_t)$$ for purposes of representing their HARA counterparts and our comparative studies: The mean-variance component follows

$$\pi_{C}^{mv}(t, V_t) = \frac{\lambda}{\gamma},$$ \quad (67a)

which we can obtain by (46a). For an investor with risk aversion coefficient $$\gamma > 1$$, the price of risk hedge component is given by

$$\pi_{C}^{\theta}(t, V_t) = -\rho \sigma \delta \frac{2[\exp(\varsigma(T-t)) - 1]}{(\bar{k} + \varsigma)[\exp(\varsigma(T-t)) - 1] + 2\varsigma},$$ \quad (67b)
with \( \tilde{\kappa} = \kappa - (1 - \gamma)\lambda\rho\sigma/\gamma \), \( \delta = -(1 - \gamma)\lambda^2/(2\gamma^2) \), and \( \varsigma = \sqrt{\tilde{\kappa}^2 + 2\delta\sigma^2(\rho^2 + \gamma(1 - \rho^2))} \).

These closed-form formulae obviously illustrate again the potential role of our decomposition for solving optimal policies in closed form. For our subsequent impact analysis, it is helpful to notice the obvious fact that both the mean-variance component (67a) and the price of risk hedge component (67b) are independent of the wealth level \( X_t \) and the interest rate \( r \); so, the wealth level \( X_t \) and the interest rate \( r \) impact the HARA policy only through \( \bar{X}_t \) in (63).

We now apply the above closed-form formulae to explicitly analyze the behavior of optimal policies under HARA utility by comparing with the corresponding CRRA case. We focus on the impacts from wealth level \( X_t \), interest rate \( r \), and investment horizon \( T - t \), that significantly distinguish the HARA and CRRA cases. In particular, the next comparative study provides us with an explanation for empirical evidence that the investment in risky assets increases concavely in investors’ financial wealth; see, e.g., Roussanov (2010), Wachter and Yogo (2010), and Calvet and Sodini (2014). For our numerical experiments, we set the model parameters at the following representative annualized values \( \kappa = 5.07 \), \( \rho = -0.767 \), \( \lambda = 1.1 \), \( \theta = 0.0457 \), and \( \sigma = 0.48 \) according to the Maximum Likelihood estimation results in Aït-Sahalia and Kimmel (2007), while choosing \( \gamma = 2 \) and \( V_t = 0.0457 \). Their Table 6 provides estimates on daily data from January 2, 1990 until September 30, 2003 for the dynamics of Heston’s model under the historical measure. For more complicated dynamics, we can rely on the Simulated Method of Moments approach of Duffie and Singleton (1993) as in Moreira and Muir (2019) (see also Bazdresch et al. (2017)).

Impact of wealth level \( X_t \): According to the closed-form formulae (62), the optimal policy \( \pi_H(t,X_t,V_t) \) under HARA utility is impacted by current wealth level \( X_t \) only via the ratio \( \bar{X}_t/X_t \), which is exactly the ratio between the optimal policies (resp. the corresponding components) under HARA and CRRA utilities according to (65) (resp. (62)). By (63), we express it as

\[
\bar{X}_t/X_t = 1 - \left(\frac{X_t}{\bar{X}}\right)^{-1} \exp(-r(T-t)), \tag{68}
\]

where \( X_t/\bar{X} \) measures the current wealth \( X_t \) relative to the minimum requirement \( \bar{X} \). This formula explicitly shows that \( \bar{X}_t/X_t \) increases with \( X_t/\bar{X} \), and it approaches 1 as \( X_t/\bar{X} \) approaches infinity. In the upper left panel of Figure 1, we follow formulae (67a), (67b), (66), (65) and (68) to show how the optimal policies under CRRA and HARA utilities according to (65) (resp. (62)). By (63), we express it as

\[
\bar{X}_t/X_t = 1 - \left(\frac{X_t}{\bar{X}}\right)^{-1} \exp(-r(T-t)), \tag{68}
\]

In addition to the closed-form optimal policies, we can also solve equation (56) to obtain the investor-specific price of risk in closed-form as

\[
\theta^g(s,V_s;T) = \theta^g_1(s,V_s;T)/\theta^g_2(s,V_s;T) = \frac{\rho \sqrt{(1 - \rho^2)}}{(\kappa + \varsigma) \exp(\varsigma (T - t)) - 1 + 2\varsigma \sqrt{V_s}}.
\]

We document this result here as a by-product an illustration for the solution of equation (56) without further analysis.
\( X_t/\pi \) and approaches its asymptote at level of the optimal policy under CRRA utility, i.e., \( \pi_C(t, V_t) \), showing the shape of a hyperbola, as revealed by formulae (65) and (68). Besides, we observe that the impact of wealth level can be substantial. As \( X_t/\pi \) increases from 1 to 10, the allocation on risky asset from HARA investors increases by approximately three times from less than 20% to more than 50%. But such an impact decreases as the wealth level becomes higher.

**Impact of interest rate \( r \):** The optimal policy \( \pi_H(t, X_t, V_t) \) under HARA utility is impacted by \( r \) only via \( \bar{X}_t/X_t \), according to (65) as well as the fact that the corresponding CRRA optimal policy \( \pi_C(t, V_t) \) given in (66) is independent of interest rate \( r \). Formulae (68) explicitly shows that \( \bar{X}_t/X_t \) increases with \( r \). In the upper right panel of Figure 1, we follow formulae (67a), (67b), (66), (65) and (68) to show how the optimal policies under CRRA and HARA utilities depend on interest rate \( r \). As exhibited in the figure, the optimal policy under CRRA utility (red dotdash line) is independent of \( r \). However, the optimal policy under HARA utility (blue solid line) increases concavely with \( r \), showing the shape of an exponential function, as revealed by formulae (65) and (68). An economic interpretation based on the optimal investment strategy designed following Proposition 2 proceeds as follows. While higher risk-free rate does not impact the optimal policy under CRRA utility, it lowers the bond price \( B_{t,T} \). So, it costs less for HARA investors to satisfy their minimum allowable level \( x \) by investing in bonds, and thus increases their remaining wealth \( \bar{X}_t \) for investing in the risky asset according to the corresponding CRRA policies.

**Impact of investment horizon \( T - t \):** The mean-variance component \( \pi_{mv}^H(t, X_t, V_t) \) under HARA utility is impacted by the investment horizon \( T - t \) only via \( \bar{X}_t/X_t \), according to (62) as well as the fact that the corresponding CRRA mean-variance component \( \pi_{mv}^C(t, V_t) \) given in (67a) is independent of \( T - t \). Formulae (68) explicitly shows that \( \bar{X}_t/X_t \) increases with \( T - t \), and it approaches 1 as \( T - t \) approaches infinity. On the other hand, according to (62), the price of risk hedge component \( \pi_\theta^H(t, X_t, V_t) \) under HARA utility depends on \( T - t \) via two channels: the ratio \( \bar{X}_t/X_t \) given in (68) and the corresponding CRRA component \( \pi_\theta^C(t, V_t) \) given in (67b). We now follow formulae (67a), (67b), (62), and (68) to show the different behaviors of the mean-variance and price of risk hedge components with respect to \( T - t \) in the lower left and lower right panels in Figure 1 respectively. As exhibited by the lower left panel, the mean-variance component under CRRA utility (red dotdash line) is independent of \( T - t \). However, the mean-variance component under HARA utility (blue solid line) increases concavely with \( T - t \). Similar to our previous economic interpretation for the impact of interest rate \( r \), the observed behavior of \( \pi_{mv}^H(t, X_t, V_t) \) with respect to \( T - t \) is because, given the interest rate \( r \), longer investment horizon lowers the bond price \( B_{t,T} \), and thus increases the remaining wealth \( \bar{X}_t \) for investing in risky asset according to the corresponding CRRA policies. Next, as shown in the lower right panel, both the price of risk hedge components under the CRRA and HARA utilities increase with investment horizon \( T - t \). For the CRRA case, the sharp increase of \( \pi_\theta^C(t, V_t) \) mainly occurs when investment horizon is short; however, under longer investment horizons,
\(\pi^\theta_C(t, V_t)\) becomes almost insensitive to the increase in \(T - t\). For the HARA case, besides the similar sharp increase for short horizons, \(\pi^\theta_H(t, X_t, V_t)\) increases faster relatively than its CRRA counterpart \(\pi^\theta_C(t, V_t)\) for longer horizons. An economic interpretation proceeds as follows: The increase in \(\pi^\theta_C(t, V_t)\) under the CRRA utility is because longer investment horizon increases the uncertainty in the price of risk, thus leading to larger hedging demand. On the other hand, the increase in the HARA component \(\pi^\theta_H(t, X_t, V_t)\) is generated by a combination of two, as analyzed above based on the closed-form formula: first, the decrease of bond price and thus increase of remaining wealth \(\bar{X}_t\) to be invested in the risky asset according to the corresponding CRRA policies, and second, the increase of the CRRA hedging demand \(\pi^\theta_C(t, V_t)\) as mentioned above. In particular, the first effect is significant even under long investment horizons. The combination of these two effects leads to a faster and more lasting increase under the HARA utility compared with its CRRA counterpart.

The above comparative analysis illustrates how our theoretical decompositions allow for understanding behaviors of optimal policy under incomplete market models with wealth-dependent utilities. In particular, it reveals that the wealth-dependent property of HARA utility should not be taken only literally. We observe that the HARA utility impacts the optimal policy via other channels beyond current wealth level, i.e., the interest rate and investment horizon, as shown in the upper right and the lower two panels of Figure 1. This is supported by the fact that the remaining wealth \(\bar{X}_t\), as a whole, plays the key role in determining how HARA policy fundamentally differs from its CRRA counterpart, as established in Proposition 2 regarding the decomposition of optimal policy under HARA utility; in particular, under the Heston example illustrated above, the remaining wealth \(\bar{X}_t\) depends on current wealth level \(X_t\), minimum requirement \(\bar{x}\), interest rate \(r\), and investment horizon \(T - t\) according to the closed-form formula (63).

**Dynamic impact of HARA utility:** In addition to the above static analysis of optimal policies at any arbitrary fixed time \(t\), we now examine the wealth impact of HARA utility from a dynamic perspective. Consider a market with the stock price \(S_t\) and its variance \(V_t\) following the Heston SV model. Without loss of generality, the initial price is set as \(S_0 = 100\) and the initial variance is set as \(V_0 = \theta = 0.0457\). We consider two investors with HARA utilities over terminal wealth for an investment horizon of \(T = 10\) years. Their risk aversion coefficient and minimum allowable level for terminal wealth are set as \(\gamma = 2\) and \(\bar{x} = 10^6\), i.e., one million, respectively. The two investors only differ in their initial wealth levels: the high-wealth investor has an initial wealth of \(X^H_0 = 10^7\), i.e., ten million, while the low-wealth investor has an initial wealth of \(X^L_0 = 3 \times 10^6\), i.e., three million. Thus, their ratios of initial wealth over the minimum allowable level are given by \(X^H_0/\bar{x} = 10\) and \(X^L_0/\bar{x} = 3\), respectively.

Denote by \(\pi^H_t\) and \(\pi^L_t\) the optimal policies of the two investors, respectively. Then, the ratio \(\pi^H_t/\pi^L_t\) measures how the optimal policy of the high-wealth investor differs from that of the low-
wealth investor. We simulate the market as well as the dynamics of optimal policies \( \pi_t^H \) and \( \pi_t^L \) during the entire investment horizon, and thus the dynamics of \( \pi_t^H / \pi_t^L \) simply by ratio calculation. The simulation is conducted based on a standard Euler scheme on the Heston model (60a) – (60b). Along the simulated path, the optimal policies \( \pi_t^H \) and \( \pi_t^L \) are evaluated via the closed-form solution, and the investors’ wealth evolve according to equation (3) with \( c_t \equiv 0 \) and \( \pi_t \) being the optimal policy given in (65).

Figure 2 shows a representative simulated path of ratio \( \pi_t^H / \pi_t^L \) (red solid line with the right y-axis) and the corresponding path of stock price \( S_t \) (blue dotdash line with the left y-axis). The optimal policy under CRRA utility, as given in (67a) and (67b), only depends on \( t \) via the investment horizon \( T - t \), and is independent of variance level \( \nu_t \), stock price \( S_t \), and wealth level \( X_t \). Thus we have \( \pi_t^H / \pi_t^L \equiv 1 \) for all \( t \) under the CRRA utility, i.e., the optimal policies from the high-wealth and low-wealth investors always coincide. However, under HARA utility, we observe an interesting pattern that the ratio \( \pi_t^H / \pi_t^L \) and the stock price \( S_t \) are negatively correlated, i.e., \( \pi_t^H / \pi_t^L \) tends to be high (resp. low) when \( S_t \) is low (resp. high). Actually, this is not an incidental result out of a specific path. With 1000 trials of simulations, we calculate the average correlation between \( \pi_t^H / \pi_t^L \) and \( S_t \) as approximately \(-0.95\). We can explain this negative correlation in the following way. As shown in the lower two panels of Figure 1 under the Heston SV model with the representative parameters, the HARA investors always hold positive positions in the stock. Thus, when the stock price \( S_t \) increases, the wealth levels \( X_t \) of the investors increase, making both of them wealthier. We now recall that, as shown in the upper left panel of Figure 1, under HARA utility, the impacts of wealth level on optimal policies decrease and the HARA optimal policy approaches its CRRA counterpart, as wealth level becomes higher. So, when the stock price increases and thus both investors become wealthier, the difference in their optimal policies becomes smaller, in particular, the ratio \( \pi_t^H / \pi_t^L \) approaches 1, as in the case under CRRA utility. We can also interpret this as follows. As investors become wealthier, the minimum constraint \( \bar{x} \) is less binding for them, and thus they behave more like CRRA investors. This example demonstrates the wealth-dependent property of HARA utility from a dynamic perspective. As opposed to being fully determined by their different initial wealth levels, the investors’ dynamic optimal investment decisions depend on the historical performance of the market, e.g., the time spent in the bull or bear regimes. Obviously, cycles matter for HARA investors. This important path dependence is totally absent under the wealth-independent CRRA utility, as discussed above. Besides, Figure 2 shows that the policy ratio \( \pi_t^H / \pi_t^L \) can reach levels as large as 1.24, namely a 24% relative increase when shifting from the low-wealth type to the high-wealth type. The corresponding values for \( \pi_t^L \) and \( \pi_t^H \) at the peak of the red solid line in year 10 are 41% and 51%, which yields an absolute increase of 10%. On the 1000 trials of simulations, we have observed a relative difference as large as 80% corresponding to an absolute difference of 19%. This implies that delegated portfolio management assuming a CRRA utility can be completely erroneous...
for a HARA investor and her investment profile can be suboptimal to a large extent except if her current wealth $X_t$ is sufficiently high.

4 Application 2: Implementation by Monte Carlo simulation

The decomposition developed in Section 2 provides an indispensable foundation for revealing the structure of optimal policy and conducting relevant analysis under incomplete market models. As shown in Section 3, it facilitates the potential success in solving optimal policy in closed-form under some settings. Now, a more important question is of course how to implement the decomposition under the general setting as proposed in Section 2.1. Monte Carlo simulation is obviously one of the most natural methods. For the complete market counterpart, Detemple et al. (2003) and Cvitanic et al. (2003) proposed simulation approaches under general diffusion models. However, due to the essential challenge stemming from the market incompleteness, their extension or even alternative approach, that is applicable to incomplete market model, remains an important open problem so far.

In this section, we propose and implement a Monte Carlo simulation method for optimal dynamic portfolio choice in the general incomplete market model, as an application of and based on the decomposition developed in Section 2. It potentially allows for not only computing the optimal policies, but also conducting various comparative studies regarding their behavior and economic implications. Our Monte Carlo method is an extension of the simulation approach for complete market models developed in Detemple et al. (2003). By fully exploiting the explicit structure of our decomposition, including, e.g., the equation system characterizing the investor-specific price of risk $\theta^u(v, y, \lambda; T)$, it successfully handles the essential difficulty due to market incompleteness and the generality of utility specification. It is thus of special importance for settings under which optimal policies are difficult or impossible to explicitly solve and/or existing numerical approaches do not efficiently apply. As an illustrative example, we apply the simulation method in analyzing the behavior of optimal policies under a flexible class of stochastic volatility models that in general lack analytical solutions.

4.1 Simulation method and its numerical performance

We now outline the major technical aspects for our simulation method. First, it is natural to follow Theorem 1 to simulate for estimating the conditional expectations $E_{t}[Q_{v, T}(\lambda^*_t)]$, $E_{t}[H_{v, T}(\lambda^*_t)]$, and $E_{t}[H_{v, T}^{\theta}(\lambda^*_t)]$ of the optimal policy. Nevertheless, the difficulty stems from the unknown form of the investor-specific price of risk function $\theta^u(v, y, \lambda; T)$, which is involved in the dynamics (29) and (32) for simulation. Second, we numerically solve the unknown function $\theta^u(v, y, \lambda; T)$ based on the equation system established in Theorem 2, in which the involved conditional expectations $E[Q_{v, T}(\lambda)|Y_v = y]$, $E[H_{v, T}^{\theta}(\lambda)|Y_v = y]$, and $E[H_{v, T}^{\theta}(\lambda)|Y_v = y]$ need to be simulated at the same
time. In other words, equations (2), (29), (31), (32), (33), and (39) in Theorems 1 and 2 form together an equation systems to solve simultaneously. For the expectations to be simulated, we resort to, e.g., the standard discretization techniques for simulation, e.g., the Euler scheme (see, e.g., Chapter 6 of Glasserman (2004)); for numerically solving function \( \theta^u \), we resort to, e.g., the standard finite difference techniques for numerical solution of initial value problems of differential and/or integral equations; see, e.g., Section 2.4 of Morton and Mayers (2005). Third, in principle, we need to solve the wealth constraint multiplier \( \lambda^*_t \) involved in the dynamics (29) and (32). It can be numerically solved from equation (20) via standard root-finding methods, where we can simulate the conditional expectation \( E_t[G_{t,T}(\lambda^*_t)] \) by the same procedure as proposed above for a given candidate value of \( \lambda^*_t \). This demonstrates the benefit of employing the time-\( t \) multiplier \( \lambda^*_t \) in the decomposition instead of the initial version \( \lambda^*_0 \). Since we have clarified the major steps of our numerical approach, we omit the routine details for the sake of space.

We now examine the numerical performance of our simulation method for optimal portfolio choice. By employing two representative incomplete-market examples with closed-form formulae for optimal policy, we demonstrate the accuracy of simulated policies by comparing with the corresponding explicitly known benchmarks. The first example is the stochastic volatility model of Heston (1993) as given in Example 1. The second example is the mean-reverting return model of Kim and Omberg (1996), which is given in Example 2 below. These two examples cover two main features of the dynamics in real market: the stochastic volatility and stochastic drift, which feature market incompleteness.

Example 2 (The incomplete-market mean-reverting return model of Kim and Omberg (1996)). The asset price \( S_t \) follows \( dS_t/S_t = (r + \sigma \theta_t)dt + \sigma dW_{1t} \), where \( r, \sigma, \) and \( \theta_t \) represent the constant interest rate, market price of risk, and volatility, respectively; \( W_t \) is a standard one dimensional Brownian motion. Assume that \( \theta_t \) is governed by the following Ornstein-Uhlenbeck process \( d\theta_t = \lambda (\theta - \theta_t)dt - \sigma \theta dW_{2t} \). Assume that the instantaneous correlation between \( W_{1t} \) and \( W_{2t} \) is given by \( \rho \in [-1, 1] \), i.e., \( dW_{1t}dW_{2t} = \rho dt \).

Both Examples 1 and 2 are affine models (Duffie et al. (2000)), and closed-form formulae for the optimal policies exist for both HARA utility (61a) and CRRA utility (61b) over terminal wealth with risk aversion coefficient \( \gamma > 1 \). We can find the closed-form policies in Kim and Omberg (1996) for the mean-reverting return model under the HARA utility, in Liu (2007) for the Heston SV model under the CRRA utility, and (62) – (65) derived in this paper for the Heston SV model under the HARA utility. Therefore, we are allowed to use these closed-form formulae as benchmarks to examine the accuracy of our simulation method, which is generally applicable but of course not confined to affine models. Due to the relation between optimal policies under CRRA and HARA utilities with deterministic interest rate, as established in Proposition 2, it suffices to consider the CRRA case in our numerical experiments. The two models can be nested into our general framework (1)–(2) by
setting \( m = 1, n = 1, \) and \( d = 2. \) The state variable \( Y_t \) is specialized as \( V_t \) in Example 1, and \( \theta_t \) in Example 2, respectively.

As shown in Corollary 1, the decomposition of the optimal policy enjoys the special structure under the CRRA utility, while maintaining the impact from market incompleteness. The mean-variance component can be directly obtained in closed-form by (46a), which is given by \( \pi^{mv}(t, V_t) = \lambda / \gamma \) for Example 1 and \( \pi^{mv}(t, \theta_t) = \theta_t / \gamma \) for Example 2. Besides, as both the two examples assume constant interest rate, we have zero interest rate hedge components for them by its representation in (46b), i.e., \( \pi^r(t, V_t) = \pi^r(t, \theta_t) \equiv 0. \) Thus, we only need to simulate the price of risk hedge component \( \pi^\theta(t, V_t) \) or \( \pi^\theta(t, \theta_t) \) and then compare results with the closed-form benchmark formulae. In particular, as shown in Corollary 1, the investor-specific price of risk function \( \theta^u(v, y, \lambda; T) \) is independent of parameter \( \lambda \) and we can represent the hedge components of the optimal policy as in (46b). These structures lead to slight simplification of our simulation method outlined in the beginning of this section, but without reducing the essential challenges, e.g., we still need to follow the general method to deal with the equation (48) for the unknown investor-specific price of risk function \( \theta^u(v, y; T) \) as well as the complex dynamics of variables \( \xi_{t,s}^S \) and \( H_{t,s}^\theta \) in (44a) and (44b), respectively.

In our numerical experiments under Examples 1 and 2, we set for illustration purposes the risk aversion level at \( \gamma = 2 \) and consider three representative investment horizons with \( T = 0.5, T = 1.0, \) and \( T = 3.0 \) years, respectively. Then, for each value of \( T, \) we compare the simulated optimal hedge components \( \pi^\theta(t, V_t) \) (resp. \( \pi^\theta(t, \theta_t) \)) with the benchmark for five representative values of the state variable \( V_t \) (resp. \( \theta_t \)). In the implementation, we simulate for \( M = 10^6 \) trials, and for each, we set the time increment for discretization as \( \Delta = 0.001. \) For verification purposes, we employ two methods to compute the standard error of the estimator and then accordingly construct confidence intervals: First, since the components take the form of ratios as shown in (46b), we follow the formula given in Section 4.3.3 of Glasserman (2004) for asymptotic standard deviation of ratio estimator derived based on the Delta method. Second, we follow a standard bootstrap procedure to approximate for the small-sample standard deviation of the estimator; see, e.g., Efron (1979).

The simulation results for Examples 1 and 2 are summarized in Tables 1 and 2. The first column shows our choices of investment horizon and the value of state variable with which the simulation is implemented. The second and third columns report the simulated value \( \hat{\pi}^\theta \) and the true value \( \pi^\theta_{\text{true}} \) calculated from benchmark formulae. The fourth column shows the relative error of simulation \( e_{\text{Rel}} \), which is computed by \( e_{\text{Rel}} = |\hat{\pi}^\theta - \pi^\theta_{\text{true}}| / |\pi^\theta_{\text{true}}|. \) The fifth and sixth columns, i.e., Std and CI_{95}, report the standard error of estimator \( \hat{\pi}^\theta \) calculated from the asymptotic standard deviation of ratio estimator (the first method mentioned above) and the corresponding 95% level confidence interval for \( \pi^\theta_{\text{true}}. \) The next two columns, i.e., Std_{B} and CI_{95,B}, report bootstrap standard error of estimator \( \hat{\pi}^\theta \) calculated based on \( J = 1,000 \) bootstrap samples (the second method mentioned above) and
the corresponding 95% level confidence interval for $\pi_{\text{true}}^\theta$. The last column reports the relative root mean square error (RMSE) estimated based on the bootstrap estimators, i.e., $\text{RMSE}_B/|\pi_{\text{true}}^\theta|$, where $\text{RMSE}_B$ is calculated according to $\text{RMSE}_B = \sqrt{\frac{\sum_{j=1}^J (\hat{\pi}_{\theta,j} - \pi_{\text{true}}^\theta)^2}{J}}$, with $\hat{\pi}_{\theta,1}, \hat{\pi}_{\theta,2}, ..., \hat{\pi}_{\theta,J}$ being the bootstrap estimators of $\pi_{\text{true}}^\theta$ based on $J$ bootstrap samples.

As shown in the results from Tables 1 and 2, our simulation method performs accurately across all the representative choices of investment horizons and state variables for both Examples 1 and 2. For all entries, the relative errors and the relative RMSE are at the magnitude of $10^{-2}$ or less; the standard errors are relatively small compared with the corresponding simulated values $\hat{\pi}^\theta$ and the true values $\pi_{\text{true}}^\theta$ all lie in the corresponding confidence intervals. Moreover, by comparing the columns of Std and Std$_B$ (resp. CI$_{95}$ and CI$_{95,B}$), we observe that the standard errors (resp. confidence intervals) calculated by the asymptotic standard deviation and bootstrap are close to each other. This further demonstrates the numerical validity and accuracy of our simulation method.

4.2 Illustration by analyzing impacts of stochastic volatility dynamics

In this section, we illustrate by an example how to apply our decomposition and the subsequent simulation method in analyzing the behavior of optimal policies under incomplete market models. Under a flexible class of incomplete market stochastic volatility models, which is more general and flexible than the Heston-SV model, we reveal and explain various impacts on optimal portfolio from model parameters that control the volatility dynamics, i.e., to analyze how the optimal policy behaves with respect to different model specifications. This is as a substantial complement and extension to the impact analysis conducted in Section 3.2 under the Heston SV model with our closed-form HARA optimal policy (67a) and (67b). Nevertheless, as the closed-form solution for optimal policy is no longer available for now, our simulation method plays an important role. The analysis in this section further demonstrates the importance and application potential of our decomposition developed in Section 2.2.

We begin by introducing the model. Consider the following incomplete-market bivariate CEV-type stochastic volatility (CEV-SV hereafter) model of Jones (2003), which generalizes the volatility dynamics of Heston SV model in Example 1.

**Example 3** (The incomplete-market CEV-type stochastic volatility model of Jones (2003)). The asset price $S_t$ follows

$$dS_t/S_t = (r + \lambda V_t)dt + \sqrt{1-\rho^2} V_t dW_{1t} + \rho \sqrt{V_t} dW_{2t}, \quad (69a)$$

and the variance $V_t$ follows

$$dV_t = \kappa (\theta - V_t) dt + \sigma V_t^\nu dW_{2t}, \quad (69b)$$

where $W_{1t}$ and $W_{2t}$ are two independent standard Brownian motions. Here, the parameters $r, \lambda, \kappa$,
\( \kappa, \theta, \sigma, \) and \( \rho \) have the same interpretations as those for the stochastic volatility model of Heston (1993) given in Example 1. The constant elasticity parameter \( \nu \) controls the elasticity of volatility.

The CEV-SV model (69a) – (69b) belongs to the nonaffine class (see, e.g., Duffie et al. (2000)). It was proposed to offer great flexibility in modeling volatility dynamics; see, e.g., Jones (2003), Aït-Sahalia and Kimmel (2007), Medvedev and Scaillet (2007), and Christoffersen et al. (2010) for empirical evidence in favour of this model. Owing to its generality, the CEV-SV model reduces, by letting \( \nu = 1/2, 1, \) and \( 3/2, \) to the stochastic volatility model of Heston (1993) as in Example 1, the GARCH diffusion type (GARCH-SV hereafter) proposed in Nelson (1990), and the \( 3/2-\) type (\( 3/2-SV \) hereafter) investigated in, e.g., Christoffersen et al. (2010). While the literature has witnessed a large amount of enthusiastic discussions on what is the proper dynamics of stochastic volatility for fitting prices of underlying asset or/and its derivative securities, little is known about the impact of such dynamics on optimal portfolios.\(^{12}\) As an attractive issue featuring market incompleteness, this now becomes our task.

For these flexible models, there is no general closed-form formulae for the optimal policies, except for the affine case with \( \nu = 1/2 \) (i.e., the model of Heston (1993) in Example 1); see Liu (2007) for the CRRA utility case and (62) – (65) derived in this paper for the HARA utility case. Thus, we apply our decomposition and the subsequent Monte Carlo simulation method proposed in Section 4.1 to implement the optimal policy and then perform the subsequent impact analysis on the optimal portfolios from various model parameters, other than those already considered in Section 3.2. We assume, as before, the investors maximize the expected HARA utility (61a) and/or CRRA utility (61b) over terminal wealth. Due to the closed-form relation between HARA and CRRA optimal policies in Proposition 2, it suffices to focus on the CRRA case in our impact analysis below.

First of all, by the similar reasons leading to (64) and (67a) for the Heston-SV case, we obtain for CEV-SV model under the CRRA utility that the interest rate hedge component is given by \( \pi^r(t,Y_t) = 0 \) as the interest rate is constant, and the mean-variance component is given by

\[
\pi^{mv}(t,V_t) = \frac{\lambda}{\gamma}
\]

(70) according to (46a). Both components are independent of the specification of volatility dynamics. In particular, the behaviors of \( \pi^{mv}(t,V_t) \) are clear: \( \pi^{mv}(t,V_t) \) increases (resp. decreases) with the market price of risk parameter \( \lambda \) (resp. the risk aversion level \( \gamma \)). The explanation is that investors are more willing (resp. reluctant) to hold the risky asset as the market price of risk increases (resp. as their risk aversion increases.)

In what follows, we focus on understanding the price of risk hedge component \( \pi^\theta(t,V_t) \) in the CEV-SV model. We simulate \( \pi^\theta(t,V_t) \) by the method introduced in Section 4.1. For illustration purpose,

\(^{12}\)See, e.g., Chacko and Viceira (2005), Liu (2007), and Moreira and Muir (2019) for investigations on optimal portfolio choice under stochastic volatility models.
we consider a representative medium investment horizon \( T - t = 1 \) year. The three panels of Figure 3 illustrate how the optimal hedge component \( \pi^\theta(t, V_t) \) depends on the current level of variance \( V_t \), the correlation \( \rho \), and the risk aversion \( \gamma \), under the CEV-SV model in Example 3, complementing the impact with respect to wealth level, interest rate, and investment horizon analyzed in Section 3.2 under the Heston-SV model as a representative illustration. To take various volatility dynamics into consideration, instead of only an arbitrary one, we consider three choices of the elasticity parameter: \( \nu = 1/2, 1, \) and \( 3/2 \), which correspond to the Heston-SV, GARCH-SV, and \( 3/2 \)-SV models; see, e.g., the discussions in Jones (2003) and Christoffersen et al. (2010). Thus, as analyzed in what follows, the general patterns of impacts are robust with respect to the specification of volatility dynamics. Such an analysis is made possible, owing to our our decomposition and its subsequent use in our simulation method.

To facilitate comparisons across different choices of \( \nu \) while controlling the effect of volatility, we keep \( \sigma \theta^\nu \), the volatility of the variance process \( V_t \) evaluated at the mean-reverting level \( \theta \), at a constant and realistic level \( \phi_{V_t} := \sigma \theta^\nu \) by setting \( \sigma \) accordingly as \( \sigma = \phi_{V_t}/\theta^\nu \) for each value of \( \nu \). Otherwise, if we merely change the value of \( \nu \), the volatility \( \phi_{V_t} = \sigma \theta^\nu \) would change accordingly, even to some unrealistic levels, and thus simultaneously impacts \( \pi^\theta(t, V_t) \). By controlling the volatility \( \phi_{V_t} \), we are able to expose and analyze the neat impact from the elasticity of the variance process, without changing the volatility \( \phi_{V_t} \) simultaneously.

**Impact of variance level \( V_t \):** The left panel demonstrates how the optimal hedge component \( \pi^\theta(t, V_t) \) varies with the current wealth level \( V_t \). The optimal hedge component \( \pi^\theta(t, V_t) \) remains constant under the Heston-SV model. This observation reconciles the corresponding result in Liu (2007) that the optimal policy in the Heston model is independent of current variance level, as shown in (67b). However, this mathematical property does not necessarily hold under other specifications of volatility dynamics, and instead optimal policies may depend on variance level. In the GARCH-SV model and \( 3/2 \)-SV models, the hedge component \( \pi^\theta(t, V_t) \) increases with the current variance \( V_t \). Moreover, the increment in the \( 3/2 \)-SV model is more significant than that in the GARCH-SV model.

We can explain such an observed behavior of optimal hedge component \( \pi^\theta(t, V_t) \) with respect to variance \( V_t \) as follows. By (69a), the market price of risk in the CEV-SV model is calculated as \( \lambda \sqrt{V_t} \), which represents the premium in expected return per a unit of volatility. Obviously, the uncertainty in \( \lambda \sqrt{V_t} \) depends on that of \( V_t \). As \( \lambda \sqrt{V_t} \) is a concave increasing function in \( V_t \), the impact of \( V_t \) on market price of risk \( \lambda \sqrt{V_t} \) decreases as \( V_t \) increases\(^\text{13}\). On the other hand, the volatility of the variance process, as given by \( \sigma V_t^{\nu} \) in (69b), increases in \( V_t \) given the elasticity parameter \( \nu > 0 \), and it increases faster in \( V_t \) when \( \nu \) is larger. Thus, according these helpful calculus facts, the increase of

\(^{13}\)This can also be seen from the derivative of market price of risk with respect to \( V_t \), since this derivative is given by \( \lambda/(2 \sqrt{V_t}) \) and clearly decreases in \( V_t \).
\( V_t \) has two effects that oppositely impact the hedge component \( \pi^\theta(t, V_t) \). First, as the market price of risk \( \lambda \sqrt{V_t} \) becomes less sensitive to \( V_t \), its uncertainty decreases and leads to a smaller hedging demand. Second, as the volatility of the variance process increases, increase in \( V_t \) leads to more uncertainty in the variance process, which translates to a larger hedging demand. For Heston-SV model with \( \nu = 1/2 \), the independence of \( \pi^\theta(t, V_t) \) with respect to \( V_t \) can be possibly attributed to the offset of these two opposite effects. For the GARCH-SV and 3/2-SV models however, as they have larger elasticity parameters \( \nu \), the volatility of the variance process \( \sigma V_t^\nu \) increases faster in \( V_t \) compared with the case under the Heston-SV model. Thus, the second effect of increasing \( V_t \) (i.e., more volatility in the variance process) outweighs the first one (i.e., less sensitivity of the market price of risk) in the GARCH-SV and 3/2-SV models. Furthermore, as the volatility in the variance process \( \sigma V_t^\nu \) increases faster in \( V_t \) in the 3/2-SV model with \( \nu = 3/2 \) than that in the GARCH-SV model with \( \nu = 1 \), the hedging demand also increases faster in \( V_t \) in the 3/2-SV model, as shown the left panel.

**Impact of leverage effect parameter \( \rho \):** In the middle panel, we analyze how the optimal hedge component \( \pi^\theta(t, V_t) \) depends on the correlation parameter \( \rho \), namely on the leverage effect when \( \rho < 0 \). As a by-product, this analysis provides an angle for understanding the impact of market incompleteness. This is because the absolute value of \( \rho \) is related to the degree of market incompleteness: the smaller is \( |\rho| \), the more incomplete is the market. In particular, the market is incomplete (resp. complete) for \( \rho \in (-1, 1) \) (resp. \( \rho = -1 \) or \( \rho = 1 \)). For \( \rho = -1 \) and \( \nu = 1/2 \), the CEV-SV model reduces to the Heston-Nandi model in Heston and Nandi (2000). As exhibited in the middle panel, the hedge component \( \pi^\theta(t, V_t) \) interestingly changes its sign at \( \rho = 0 \), and it decreases as \( \rho \) increases. This pattern is robust across all the three representative model specifications.

An economic interpretation of the above relationship between the hedge component \( \pi^\theta(t, V_t) \) and correlation \( \rho \) proceeds as follows. Recall the market price of risk is calculated as \( \lambda \sqrt{V_t} \) by (69a). According to (70), investors take a long position in the mean-variance component given the condition \( \lambda > 0 \). Thus, they clearly benefit from a higher market price of risk, therefore regard its decrease, i.e., the decrease of variance \( V_t \), as the downside risk. To hedge against this adverse situation, investors ought to take a position via \( \pi^\theta(t, V_t) \) so that the decrease of \( V_t \) is favorable, i.e., leading to potentially larger terminal wealth and thus the utility. To achieve this, we can buy (resp. short sell) the asset, as its price \( S_t \) tends to increase (resp. decrease) as \( V_t \) decreases when \( \rho < 0 \) (resp. \( \rho > 0 \)). This explains the positive (resp. negative) sign of hedge component \( \pi^\theta(t, V_t) \) when \( \rho < 0 \) (resp. \( \rho > 0 \)). In particular, for \( \rho = -1 \) or \( \rho = 1 \), the instantaneous changes of price \( S_t \) and its variance \( V_t \) are perfectly correlated and thus result in a complete market. In this situation, the risky asset serves as a perfect hedge for the uncertainty in market price of risk \( \lambda \sqrt{V_t} \). This naturally leads to the largest possible absolute value of the hedge component \( \pi^\theta(t, V_t) \) across all values of \( \rho \), as shown in the middle panel. However, when \( \rho = 0 \), the instantaneous changes of price \( S_t \) and its variance \( V_t \) are totally
uncorrelated and accordingly lead to the “most incomplete” market. Thus, investors are not able to use the asset to hedge the uncertainty in the market price risk \( \lambda \sqrt{V_t} \) as under \( \rho \neq 0 \). This naturally results in the zero hedge component \( \pi^\theta(t, V_t) = 0 \) when \( \rho = 0 \).

**Impact of risk aversion level \( \gamma \):** The right panel demonstrates that the optimal hedge component \( \pi^\theta(t, V_t) \) exhibits a hump shape with respect to the risk aversion \( \gamma \) for all the three representative model specifications. The aforementioned hump shape is also discussed in Detemple et al. (2003) under the complete-market non-linear mean-reverting elastic volatility (NMREV) model. We provide the following economic interpretation in our incomplete-market environment. When \( \gamma \) approaches 1, the CRRA utility reduces to the logarithm utility. As discussed in Merton (1971), the investors with the logarithm utility behave myopically and do not invest to hedge the risk in their investment opportunities. Thus, their hedge component \( \pi^\theta(t, V_t) \) is identically zero. As the risk aversion \( \gamma \) increases above 1, investors become more risk averse than myopic and start to hedge the risk by using the risky asset. This explains the increase in the absolute value of the hedge component \( \pi^\theta(t, V_t) \). However, as \( \gamma \) further increases significantly, the investors become much more risk averse so that they do not want to bear risk associated with the risky asset. This can be seen from the mean-variance component (70), which obviously decreases with \( \gamma \). As investors reduce their holdings in the risky asset, the hedging demand for market price of risk also decreases. These two effects together yield the hump shape. The documentation in our incomplete-market CEV-SV model and in the complete-market NMREV model studied in Detemple et al. (2003) suggest that the hump shape effect for the optimal price of risk hedge component with respect to risk aversion may hold in a robust fashion insensitive to model specifications.

**Impact of elasticity parameter \( \nu \):** Finally, we analyze the impact from the elasticity parameter \( \nu \) for different values of \( V_t \) and \( \rho \), as shown in the first two panels, respectively. As shown in the left panel, if \( V_t > \theta \) (resp. \( V_t < \theta \)), as \( \nu \) increases from 1/2 to 1 and further to 3/2, the curves turn anti-clockwise and thus the positive value of the hedge component \( \pi^\theta(t, V_t) \) increases (resp. decreases). We can interpret the aforementioned behavior of \( \pi^\theta(t, V_t) \) with respect to elasticity \( \nu \) as follows. Plugging \( \sigma = \phi_V / \theta^\nu \) into (69b), i.e., fixing the volatility of \( V_t \) evaluated at its mean-reverting level \( \theta \) as the constant \( \phi_V \), the dynamics of \( V_t \) specifies to \( dV_t = \kappa (\theta - V_t) dt + \phi_V (V_t / \theta)^\nu dW_{2t} \). We see from this dynamics that the instantaneous volatility of \( V_t \), as given by \( \phi_V (V_t / \theta)^\nu \), clearly increases (resp. decreases) with elasticity \( \nu \) when \( V_t > \theta \) (resp. \( V_t < \theta \)). So, when \( V_t > \theta \), a higher \( \nu \) yields more volatility in the variance process \( V_t \), and thus more uncertainty in the market price of risk \( \lambda \sqrt{V_t} \) that needs to be hedged. This demand naturally leads to a larger absolute value of the hedge component \( \pi^\theta(t, V_t) \). By similar argument, when \( V_t < \theta \), a higher \( \nu \) leads to a smaller absolute value of the hedge component \( \pi^\theta(t, V_t) \). Such interpretations reconcile the behavior of \( \pi^\theta(t, V_t) \) with respect to \( \nu \) observed in our precedent analysis.
5 Conclusions and discussions

This paper establishes and implements a new decomposition of the optimal dynamic portfolio choice under general incomplete-market diffusion models with flexible wealth-dependent utilities. By noticing and applying the functional form of the investor-specific price of risk in a suitable market completion procedure, we derive explicit dynamics of the components underlying the optimal policy and obtain an equation system for characterizing the investor-specific price of risk, which is shown to depend on both market state and wealth level, and further for expressing the optimal policy. Our decomposition substantially extends the representation results under complete market setting in, e.g., Detemple et al. (2003) to general incomplete market models. The decomposition reveals the impacts on the optimal policy from market incompleteness and wealth-dependent utilities. In particular, we report a new important hedge component for non-myopic investors with wealth-dependent utilities. This new component hedges the uncertainty in investor-specific price of risk due to variation in wealth level.

As the first application, we establish and compare the decompositions of optimal policy under general models with the prevalent HARA and CRRA utilities. Moreover, under nonrandom but possibly time-varying interest rate, we explicitly solve the HARA policy as a combination of a bond holding scheme and the corresponding CRRA strategy. As a representative illustration, we apply the decomposition results to solve the optimal policy for HARA investors under the incomplete market stochastic volatility model of Heston (1993) in closed-form and then conduct in-depth comparative studies for understanding the nature of wealth dependency of optimal policies.

In addition to the theoretical findings and the potential applications in explicitly solving optimal policies, our decomposition renders an indispensable and flexible foundation for implementing the optimal policy by appropriate numerical methods. As the second application, we propose a Monte Carlo simulation approach for computing the optimal policies in general incomplete market models, where existing numerical approaches do not efficiently apply. Such a simulation method is made possible, owing to the indispensable characterization of market incompleteness through our decomposition. As a representative illustration, we apply this simulation approach in a novel analysis of the behavior of optimal portfolio policies under a flexible class of stochastic volatility models.

We can adapt or generalize our decomposition for optimal portfolio policies to other settings, e.g., the forward measure based representation of optimal portfolios considered in Detemple and Rindisbacher (2010). Moreover, it is interesting, among other possible extensions, to consider other (exotic) types of market incompleteness, e.g., the short-selling constraint or the “rectangular” constraint considered in Cvitanic and Karatzas (1992) and/or Detemple and Rindisbacher (2005), as well as the presence of jumps considered in, e.g., Aït-Sahalia et al. (2009) and Jin and Zhang (2012). Regarding implementation by simulation, it is beneficial to investigate the convergence rate of our approach and enhance its efficiency via various techniques; see, e.g., the relevant analysis in
Detemple et al. (2006) for the simulation method under complete market models. Besides, it is also interesting to develop alternative numerical methods that can be efficiently applied to incomplete market models, based on our theoretical decomposition results. We defer these investigations, among others, to future projects.
References


Appendix A  Proofs

In this appendix, we document the detailed proofs for Theorem 1, Theorem 2, and Corollary 1.

Appendix A.1  Proof of Theorem 1

We first provide the following lemma that represents the optimal policy under the completed market with both real and fictitious assets, assuming the investor-specific price of risk process $\theta^u_s$ were known.

**Lemma 1.** In the completed market with dynamics (8) and (2), the optimal policy $(\pi_t, \pi^F_t)\top$ for both the real and fictitious assets admits the following representation

$$
(\pi_t, \pi^F_t)\top = -\frac{1}{X_t}(\sigma(t,Y_t)\top)^{-1}\left(\theta^S_t E_t[Q_{t,T}(\lambda^*_0\xi^S_t)] + E_t[H^r_{t,T}(\lambda^*_0\xi^S_t)] + E_t[H^\theta_{t,T}(\lambda^*_0\xi^S_t)]\right), 
$$

(A.1)

where $\theta^S_t$ is the total price of risk defined in (12); $E_t$ denotes the expectation conditional on the information up to time $t$; $\xi^S_t$ is the state price density defined in (15); $\lambda^*_0$ is the multiplier uniquely determined by the wealth equation

$$
X_0 = E[\mathcal{G}_{0,T}(\lambda^*_0)], 
$$

(A.2)

where $X_0$ is the initial wealth and the function $\mathcal{G}_{0,T}(\cdot)$ is defined in (21); the components $Q_{t,T}(\lambda^*_0\xi^S_t)$, $H^r_{t,T}(\lambda^*_0\xi^S_t)$, and $H^\theta_{t,T}(\lambda^*_0\xi^S_t)$ are given by

$$
Q_{t,T}(\lambda^*_0\xi^S_t) = \Upsilon^U_{t,T}(\lambda^*_0\xi^S_t) + \int_t^T \Upsilon^u_{t,s}(\lambda^*_0\xi^S_s) ds, 
$$

(A.3a)

$$
H^r_{t,T}(\lambda^*_0\xi^S_t) = (\Gamma^U_{t,T}(\lambda^*_0\xi^S_t) + \Upsilon^U_{t,T}(\lambda^*_0\xi^S_t))H^r_{t,T} + \int_t^T (\Gamma^u_{t,s}(\lambda^*_0\xi^S_s) + \Upsilon^u_{t,s}(\lambda^*_0\xi^S_s))H^r_{t,s} ds, 
$$

(A.3b)

$$
H^\theta_{t,T}(\lambda^*_0\xi^S_t) = (\Gamma^U_{t,T}(\lambda^*_0\xi^S_t) + \Upsilon^U_{t,T}(\lambda^*_0\xi^S_t))H^\theta_{t,T} + \int_t^T (\Gamma^u_{t,s}(\lambda^*_0\xi^S_s) + \Upsilon^u_{t,s}(\lambda^*_0\xi^S_s))H^\theta_{t,s} ds, 
$$

(A.3c)

with functions $\Gamma^U_{t,T}(\cdot)$, $\Gamma^u_{t,s}(\cdot)$, $\Upsilon^U_{t,T}(\cdot)$, and $\Upsilon^u_{t,s}(\cdot)$ defined in (22a) and (22b). Here, the terms $H^r_{t,s}$ and $H^\theta_{t,s}$ in (A.3b) and (A.3c) satisfy

$$
dH^r_{t,s} = D_t r(s,Y_s) ds \quad \text{and} \quad dH^\theta_{t,s} = D_t \theta^S_o[\theta^S_s ds + dW_s], 
$$

(A.4)

with initial values $H^r_{t,t} = H^\theta_{t,t}(\lambda^*_t) = 0_d$, where $D_t r(s,Y_s)$ and $D_t \theta^S_o$ denote the Malliavin derivatives of the interest rate $r(s,Y_s)$ and total price of risk $\theta^S_o$, respectively.

**Proof.** The statement follows from the martingale approach arguments that lead to Theorem 1 in Detemple et al. (2003) (see also, e.g., Karatzas et al. (1987) and Cox and Huang (1989)). \qed

In what follows, we prove Theorem 1.
Proof. This proof consists of three parts consecutively. In the first part, we prove the relationship (28) and apply it to verify the representation of the investor-specific price of risk in (24). In the second part, we start to apply Lemma 1 and focus on deriving the explicit dynamics of $\xi_t^S(\lambda^*_t)$, $H_t^r$, and $H_t^s(\lambda'_t)$ as (29), (31), and (32), respectively, based on the representation (24) of $\theta_v^*$ and the dynamics in (15) and (A.4). In the third part, we consequently establish the decomposition of the optimal policy given in (26a) – (26c).

Part 1: We first briefly prove the relationship in (28). As a foundation, the existence and uniqueness of $\lambda^*_t$, as the solution to equation (20), follow from standard calculus: the utilities $u(t, \cdot)$ and $U(t, \cdot)$ are strictly increasing and concave with \( \lim_{x \to \infty} \partial u(t, x) / \partial x = 0 \) and \( \lim_{x \to \infty} \partial U(T, x) / \partial x = 0 \) (see similar discussions in Cox and Huang (1989)). We now proceed to show (28), i.e., $\lambda^*_t = \lambda_0^* \xi_t^S$.

Assuming the investor follows the optimal policy in the completed market, we follow Karatzas et al. (1991) and Cox and Huang (1989) to derive that the time–$t$ optimal wealth satisfies

$$
\xi_t^S X_t = E_t[\xi_T^S I^U(T, \lambda_0^* \xi_T^S) + \int_t^T \xi_s^S I^u(s, \lambda_0^* \xi_s^S) ds],
$$

where $\lambda_0^*$ is characterized by (A.2). By dividing $\xi_t^S$ on both sides of the above equation and using the relation $\xi_s^S = \xi_t^S \xi_s^S$ for any $s \geq t$, we obtain

$$
X_t = E_t[\xi_{t,T}^S I^U(T, \lambda_0^* \xi_T^S) + \int_t^T \xi_s^S I^u(s, \lambda_0^* \xi_s^S) ds].
$$

By the definition of $G_{t,T}(\cdot)$ in (21), the above equation is equivalent to $X_t = E_t[G_{t,T}(\lambda_0^* \xi_T^S)]$. By the uniqueness of solution to equation (20), we establish the relationship (28).

Then, we verify the representation of the investor-specific price of risk in (24), i.e., $\theta^*_v = \theta^u(v, Y_v, \lambda_v^*; T)$ for some function $\theta^u(v, y, \lambda; T)$. This verification hinges on linking the least favorable completion approach of Karatzas et al. (1991) and the minimax local martingale approach of He and Pearson (1991), two independently developed martingale approaches for solving optimal portfolios under incomplete market settings.

By Theorem 9.3 of Karatzas et al. (1991), the investor-specific price of risk $\theta^*_v$ satisfying (23) must lead to the smallest utility among all possible completions, i.e., the least favorable completion. More precisely, the desired $\theta^*_v$ satisfying (23) serves as the optimizer for the following dual problem

$$
\inf_{\theta^u \in \text{Ker}(\sigma)} \left\{ \sup_{(c_t, X_T) \in A_{\theta^u}} E \left[ \int_0^T u(t, c_t) dt + U(T, X_T) \right] \right\}, \quad (A.5)
$$

where $A_{\theta^u} = \{(c_t, X_T) : E[\int_0^T \xi_t^S c_t dt + \xi_T^S X_T] \leq X_0 \text{ and } X_t \geq 0 \text{ for all } t \in [0, T]\}$. Here, corresponding to the second orthogonal condition in (14a), we use $\theta^u \in \text{Ker}(\sigma)$ to abbreviate $\theta^u \in \text{Ker}(\sigma(v, Y_v))$ for any $0 \leq v \leq T$, with $\text{Ker}(\sigma(v, Y_v)) := \{w \in R^d : \sigma(v, Y_v)w \equiv 0_m\}$ denoting the kernel of $\sigma(v, Y_v)$. Problem (A.5) is also discussed in He and Pearson (1991) for the same goal of characterizing the optimal portfolio in the incomplete market case, though the language of He and Pearson (1991) hinges on the class of arbitrage-free state prices, which indeed correspond to the state price density $\xi_t^S$ of the completed market defined by (15). According to Theorem 2 and the discussion prior to Theorem 7 of He and Pearson (1991), the solution of problem (A.5) also solves the following optimization
problem:

\[
\inf_{\theta^u \in \text{Ker}(\sigma)} E \left[ \int_0^T \tilde{u}(v, \lambda_v^*)dv + \tilde{U}(T, \lambda_T^*) \right],
\]

where \( \lambda_v^* \) is the time–v multiplier characterized by the equation \( X_v = E_v[G_{v,T}(\lambda_v^*)] \). By (28), i.e., \( \lambda_v^* = \lambda^S_{\theta^u} \), an application of Ito formula leads to \( d\lambda_v^* = -\lambda_v^*[r(v, Y_v)dv + (\theta_v^u(v, Y_v) + \theta_v^u)^\top dW_v] \), in which \( \theta_v^u \) serves as a control process. Besides, \( \tilde{u}(t, x) \) and \( \tilde{U}(t, x) \) in (A.6) denote the conjugates of utility functions \( u(t, x) \) and \( U(t, x) \), defined by \( \tilde{u}(t, y) := \sup_{x \geq 0}(u(t, x) - xy) \) and \( \tilde{U}(t, y) := \sup_{x \geq 0}(U(t, x) - xy) \), respectively. We can check that \( \sup_{x \geq 0}(u(t, x) - xy) \) and \( \sup_{x \geq 0}(u(t, x) - xy) \) take their maximum at \( x = I^u(t, y) \) and \( x = I^U(t, y) \) respectively.

To temporarily summarize, by linking the problems (A.5) and (A.6), we verify that the desired investor-specific price of risk \( \theta_v^u \) satisfying the least favorable completion (23) is also the solution of the optimization problem (A.6). Next, we proceed to obtain the functional representation of \( \theta_v^u \) by looking into the optimization problem (A.6). Since \( (Y_v, \lambda_v^*) = (Y_v, \lambda^S_{\theta^u}) \) forms a Markov process, an application of the feedback law (see, e.g., Theorem 9.1 of Touzi (2012)) to the control problem (A.6) implies that the control process \( \theta_v^u \) must be a measurable function of the time \( v \), the state variable \( Y_v \), and the multiplier \( \lambda_v^* \). Besides, \( \theta_v^u \) also depends on the investment horizon \( T \), since it is involved in the objective function of the optimization problem (A.6). Although \( T \) is a fixed parameter, it is economically and technically important for revealing the structural differences of optimal policies between complete and incomplete market settings. Thus, we arrive to the representation (24) of the control process \( \theta_v^u \), i.e., \( \theta_v^u = \theta^u(v, Y_v, \lambda_v^*; T) \) for some investor-specific price of risk function \( \theta^u(v, y, \lambda; T) \).

Part 2: In this part, we prove the dynamics of \( \xi^S_{t,s}(\lambda_v^*) \), \( H^r_{t,s} \), and \( H^\theta_{t,s}(\lambda_v^*) \) given in (29), (27b), and (32), respectively. To begin, by applying the representation of \( \theta_v^u \) in (24) and the relationship (28), we obtain the following \( \lambda_v^* \)-dependent version of the total price of risk \( \theta^S_s \) introduced in (12):

\[
\theta^S_s(\lambda_v^*) = \theta^R(s, Y_s) + \theta^u(s, Y_s, \lambda_v^* \xi^S_{t,s}(\lambda_v^*); T),
\]

which plays an important role in explicitly deriving the desired dynamics in what follows. We now apply Lemma 1 to the completed market (8) with the total price of risk \( \theta^S_s \) taking the specific form given in (A.7).

Accordingly, by the generic dynamics of \( \xi^S_{t,s} \) in (17) and that of \( H^R_{t,s} \) in (A.4), it is easy to verify that \( \lambda_v^* \) gets involved in them through the term \( \theta^S_s(\lambda_v^*) \). Thus, similar to the spirit of creating the \( \lambda_v^* \)-dependent notation \( \theta^S_s(\lambda_v^*) \), we express \( \xi^S_{t,s} \) and \( H^R_{t,s} \) by their \( \lambda_v^* \)-dependent versions \( \xi^S_{t,s}(\lambda_v^*) \) and \( H^R_{t,s}(\lambda_v^*) \), respectively, for emphasizing their dependences on \( \lambda_v^* \). Applying the representation (A.7) to the generic dynamics of \( \xi^S_{t,s}(\lambda_v^*) \) given in (17), we obtain the explicit dynamics of \( \xi^S_{t,s}(\lambda_v^*) \) in (29), i.e., \( d\xi^S_{t,s}(\lambda_v^*) = -\xi^S_{t,s}(\lambda_v^*)[r(s, Y_s)ds + \theta^S_s(\lambda_v^*)^\top dW_s] \).

For \( H^R_{t,s} \) from the first SDE in (A.4), it is straightforward to apply the chain rule of Malliavin
derivative to obtain (31), i.e., \( dH_{t,s}^\theta = (D_t Y_s) \nabla r(s, Y_s) ds \).

Here, \( D_t Y_s \) denotes the Malliavin derivative of the state variable, which satisfies SDE (33). Next, to derive the explicit dynamics of \( H_{t,s}^\theta (\lambda_t^\ast) \), we plug the representation of \( \theta^S (\lambda_t^\ast) \) in (A.7) into the second SDE in (A.4) to obtain

\[
dH_{t,s}^\theta = D_t \theta^S (\lambda_t^\ast) [\theta^S (\lambda_t^\ast) \, ds + dW_s].
\]  

(A.8)

By the chain rule of Malliavin derivative, we express the term \( D_t \theta^S (\lambda_t^\ast) \) as

\[
D_t \theta^S (\lambda_t^\ast) = (D_t Y_s) (\nabla \theta^b(s, Y_s) + \nabla \theta^u(s, Y_s, \lambda_t^\ast \xi^S_{t,s}(\lambda_t^\ast); T)) + \lambda_t^\ast (D_t \xi^S_{t,s}(\lambda_t^\ast)) \partial \theta^u / \partial \lambda(s, Y_s, \lambda_t^\ast \xi^S_{t,s}(\lambda_t^\ast); T),
\]

(A.9)

where we must take into account the dependence of \( \theta^S (\lambda_t^\ast) \) on both state variable \( Y_s \) and relative state price density \( \xi^S_{t,s}(\lambda_t^\ast) \). Besides, by the property of Malliavin derivative on Ito integrals (see, e.g., the survey in Appendix D of Detemple et al. (2003)), we have

\[
\frac{d}{dt} \theta^S (\lambda_t^\ast) = -\xi^S_{t,s}(\lambda_t^\ast) (\theta^S (\lambda_t^\ast) + H_{t,s}^\theta (\lambda_t^\ast)).
\]

Further applying this to (A.9), we have

\[
D_t \theta^S (\lambda_t^\ast) = (D_t Y_s) (\nabla \theta^b(s, Y_s) + \nabla \theta^u(s, Y_s, \lambda_t^\ast \xi^S_{t,s}(\lambda_t^\ast); T)) - \lambda_t^\ast \xi^S_{t,s}(\lambda_t^\ast) (\theta^S (\lambda_t^\ast) + H_{t,s}^\theta (\lambda_t^\ast)) + H_{t,s}^\theta (\lambda_t^\ast)
\]

(A.10)

Then, the explicit dynamics of \( H_{t,s}^\theta (\lambda_t^\ast) \) in (32) follows by plugging (A.7) and (A.10) into SDE (A.8).

Part 3: We now proceed to prove the decomposition of optimal policy given in (26a) – (26c). Since we apply Lemma 1 to the completed market (8) with the total price of risk \( \theta^\ast \) taking the specific form given in (A.7), the components \( Q_{t,T}(\lambda_0^\ast \xi^S_0), H_{t,T}^\theta(\lambda_0^\ast \xi^S_0), \) and \( H_{t,T}^\theta(\lambda_0^\ast \xi^S_0) \) in (A.3a) – (A.3c) of Lemma 1 exactly coincide with the components \( Q_{t,T}(\lambda_t^\ast), H_{t,T}^\theta(\lambda_t^\ast), \) and \( H_{t,T}^\theta(\lambda_t^\ast) \) in (27a) – (27c) of Theorem 1. Indeed, this correspondence hinges on the following two reasons. First, by the relationship (28), i.e., \( \lambda_t^\ast = \lambda_0^\ast \xi^S_0 \), we can substitute \( \lambda_0^\ast \xi^S_0 \) in \( Q_{t,T}(\lambda_0^\ast \xi^S_0), H_{t,T}^\theta(\lambda_0^\ast \xi^S_0), \) and \( H_{t,T}^\theta(\lambda_0^\ast \xi^S_0) \) by the time–\( t \) multiplier \( \lambda_t^\ast \). Second, as proved in Part 2 above, the building blocks \( \xi^S_{t,s}(\lambda_t^\ast) \) and \( H_{t,s}^\theta(\lambda_t^\ast) \) in Lemma 1 are realized by their \( \lambda_t^\ast \)-dependent versions \( \xi^S_{t,s}(\lambda_t^\ast) \) and \( H_{t,s}^\theta(\lambda_t^\ast) \), with explicit dynamics (29) and (32), while the dynamics of \( H_{t,s}^\theta(\lambda_t^\ast) \) is explicitly computed as (31).

Following the above discussions, we can represent the optimal policy \((\pi_t, \pi^F_t)\) in (A.1) for the completed market as

\[
(\pi_t, \pi^F_t)^\top = -\frac{1}{\lambda_t} (\sigma^S(t, Y_t)^\top)^{-1} \left( \theta^S (\lambda_t^\ast) E_t[Q_{t,T}(\lambda_t^\ast)] + E_t[H_{t,T}^\theta(\lambda_t^\ast)] + E_t[H_{t,T}^\theta(\lambda_t^\ast)] \right),
\]

(A.11)

where \( \theta^S (\lambda_t^\ast) = \theta^b(t, Y_t) + \theta^u(t, Y_t, \lambda_t^\ast; T) \) according to the \( \lambda_t^\ast \)-dependent representation in (A.7) and the fact that \( \xi^S_{t,t}(\lambda_t^\ast) = 1 \). Here, in (A.11), the components \( Q_{t,T}(\lambda_t^\ast), H_{t,T}^\theta(\lambda_t^\ast), \) and \( H_{t,T}^\theta(\lambda_t^\ast) \) are

\[\text{This dynamics coincides with the complete market counterpart derived in Detemple et al. (2003). However, as shown in what follows, the dynamics of } H_{t,s}^\theta(\lambda_t^\ast) \text{ is further sophisticated and fundamentally different from its complete market counterpart.}\]
given by (27a), (27b), and (27c), respectively; besides, the building blocks $\xi^{S}_{v,s}(\lambda^{*}_{v})$, $H^{s}_{v,s}$, and $H^{\theta}_{v,s}(\lambda^{*}_{v})$ now follow the dynamics in (29), (31), and (32), respectively.

Next, combining (A.11) with the following algebraic fact:

$$
(\sigma^{S}(t,Y_{t})^{\top})^{-1} = (\sigma^{S}(t,Y_{t})^{-1})^{\top} = ((\sigma(t,Y_{t})^{+})^{\top}, (\sigma^{F}(t,Y_{t})^{+})^{\top})^{\top},
$$

(A.12)

where the second equality follows (10), we explicitly represent the optimal policy for real assets as

$$
\pi_{t} = -\frac{1}{X_{t}}(\sigma(t,Y_{t})^{+})^{\top}\left(\theta^{S}_{t}(\lambda^{*}_{t})E_{t}[Q_{t,T}(\lambda^{*}_{t})] + E_{t}[H^{s}_{t,T}(\lambda^{*}_{t})] + E_{t}[H^{\theta}_{t,T}(\lambda^{*}_{t})]\right).
$$

(A.13)

We can further simplify this expression using the following algebraic fact

$$
(\sigma(t,Y_{t})^{+})^{\top}\theta^{S}_{t}(\lambda^{*}_{t}) = (\sigma(t,Y_{t})^{+})^{\top}\theta^{h}(t,Y_{t})
$$

(A.14)

with $\theta^{h}(t,Y_{t})$ defined in (13a). To verify this, we use definition (11) for $(\sigma(t,Y_{t})^{+})^{\top}$, the second orthogonal condition in (14a), as well as representation (24) to deduce that $(\sigma(t,Y_{t})^{+})^{\top}\theta^{h}(t,Y_{t},\lambda^{*}_{t};T) = (\sigma(t,Y_{t})\sigma(t,Y_{t})^{\top})^{-1}\sigma(t,Y_{t})\theta^{h}(t,Y_{t},\lambda^{*}_{t};T) = 0_{m}$. By (12), we can compute the terms $(\sigma(t,Y_{t})^{+})^{\top}\theta^{S}_{t}(\lambda^{*}_{t})$ in (A.13) as $(\sigma(t,Y_{t})^{+})^{\top}\theta^{S}_{t}(\lambda^{*}_{t}) = (\sigma(t,Y_{t})^{+})^{\top}(\theta^{h}(t,Y_{t}) + \theta^{a}(t,Y_{t},\lambda^{*}_{t};T)) = (\sigma(t,Y_{t})^{+})^{\top}\theta^{h}(t,Y_{t})$. Hence, by (A.14), we can further simplify the representations (A.13) as

$$
\pi_{t} = -\frac{1}{X_{t}}(\sigma(t,Y_{t})^{+})^{\top}\left(\theta^{h}(t,Y_{t})E_{t}[Q_{t,T}(\lambda^{*}_{t})] + E_{t}[H^{s}_{t,T}(\lambda^{*}_{t})] + E_{t}[H^{\theta}_{t,T}(\lambda^{*}_{t})]\right).
$$

(A.15)

Finally, the decomposition in (26a), (26b), and (26c) of the optimal policy $\pi_{t}$ for real assets directly follows the representation (A.15).

\[ \square \]

**Appendix A.2 Proof of Theorem 2**

*Proof.* We begin by verifying the simple fact that

$$
E_{v}[Q_{v,T}(\lambda^{*}_{v})] = E[Q_{v,T}(\lambda^{*}_{v})|Y_{v},\lambda^{*}_{v}], \quad E_{v}[H^{s}_{v,T}(\lambda^{*}_{v})] = E[H^{s}_{v,T}(\lambda^{*}_{v})|Y_{v},\lambda^{*}_{v}], \quad E_{v}[H^{\theta}_{v,T}(\lambda^{*}_{v})] = E[H^{\theta}_{v,T}(\lambda^{*}_{v})|Y_{v},\lambda^{*}_{v}].
$$

(A.16)

Without loss of generality, we take $E_{v}[Q_{v,T}(\lambda^{*}_{v})]$ as an example to verify this fact. Indeed, it follows from (2), (29), and (27a) that the joint process $(Y_{s},\xi^{S}_{v,s}(\lambda^{*}_{v}),Q_{v,s}(\lambda^{*}_{v}))$ in the time variable $s \geq v$ is Markovian with the starting point given by $(Y_{v},\xi^{S}_{v,v}(\lambda^{*}_{v}),Q_{v,v}(\lambda^{*}_{v})) \equiv (Y_{v},1,\lambda^{v}_{v}\partial U/\partial y(v,\lambda^{*}_{v}))$. Thus, the conditioning in $E_{v}[Q_{v,T}(\lambda^{*}_{v})]$ is reduced to $Y_{v}$ and $\lambda^{v}_{v}$, i.e., $E_{v}[Q_{v,T}(\lambda^{*}_{v})] = E[Q_{v,T}(\lambda^{*}_{v})|Y_{v},\lambda^{*}_{v}]$.

The orthogonal condition (38), i.e., $\sigma(v,y)\theta^{a}(v,y,\lambda;T) \equiv 0_{m}$, easily follows from the second condition in (14a). Next, we establish equation (39) for governing the $d-$dimensional column vector-valued function $\theta^{a}(v,y,\lambda;T)$. For this purpose, we explicitly deduce the least favorable completion constraint (23), i.e., $\pi_{v}^{F} \equiv 0_{d-m}$, for any $0 \leq v \leq T$. To begin, we first explicitly represent the optimal policy for fictitious assets as

$$
\pi_{v}^{F} = -\frac{1}{X_{v}}(\sigma^{F}(v,Y_{v})^{+})^{\top}\left(\theta^{a}(v,Y_{v},\lambda^{*}_{v};T)E_{v}[Q_{v,T}(\lambda^{*}_{v})] + E_{v}[H^{s}_{v,T}(\lambda^{*}_{v})] + E_{v}[H^{\theta}_{v,T}(\lambda^{*}_{v})]\right).
$$

(A.17)
We establish this representation following a similar argument for proving (A.15). By combining (A.11) with the algebraic fact (A.12), we can represent the optimal policy for fictitious assets as

$$\pi^F_v = -\frac{1}{X_v} (\sigma^F(v, Y_v)^+)^\top \left( \theta^S_v(\lambda^*_v) E_v[Q_{v,T}(\lambda^*_v)] + E_v[H^\theta_{v,T}(\lambda^*_v)] \right).$$  \hspace{1cm} (A.18)

We can further simplify this representation using the following algebraic fact

$$ (\sigma^F(v, Y_v)^+)^\top \theta^S_v(\lambda^*_v) = (\sigma^F(v, Y_v)^+)^\top \theta^a(v, Y_v, \lambda^*_v; T),$$ \hspace{1cm} (A.19)

with $\theta^a(v, Y_v, \lambda^*_v; T)$ introduced in (24) for representing $\theta^a_v$. To verify this, we use definition (11) for $(\sigma^F(v, Y_v)^+)^\top$ and the first orthogonal condition in (14a) to deduce that $(\sigma^F(v, Y_v)^+)^\top \theta^h(v, Y_v) = (\sigma^F(v, Y_v)\sigma^F(v, Y_v)^\top)^{-1}\sigma^F(v, Y_v)\theta^h(v, Y_v) = 0_{d-m}$. Then, by (12), we can compute the term

$$ (\sigma^F(v, Y_v)^+)^\top \theta^S_v(\lambda^*_v) = (\sigma^F(v, Y_v)^+)^\top (\theta^h(v, Y_v) + \theta^a(v, Y_v, \lambda^*_v; T)) = (\sigma^F(v, Y_v)^+)^\top \theta^a(v, Y_v, \lambda^*_v; T)$$

in (A.18). By (A.19), we can further simplify representation (A.18) to obtain (A.17).

Thus, by plugging (A.17) into the least favorable completion constraint (23), we have

$$ (\sigma^F(v, Y_v)^+)^\top \left( \theta^a(v, Y_v, \lambda^*_v; T) E_v[Q_{v,T}(\lambda^*_v)] + E_v[H^\theta_{v,T}(\lambda^*_v)] \right) \equiv 0_{d-m}, \hspace{1cm} (A.20)$$

for any $0 \leq v \leq T$. By representation (A.16), equation (A.20) is equivalent to

$$ (\sigma^F(v, Y_v)^+)^\top \left( \theta^a(v, Y_v, \lambda^*_v; T) E[Q_{v,T}(\lambda^*_v)]|Y_v = y] + E[H^\theta_{v,T}(\lambda^*_v)]|Y_v = y] + E[H^\theta_{v,T}(\lambda^*_v)]|Y_v, \lambda^*_v] \right) \equiv 0_{d-m}. $$

Since this equation holds for any value of $Y_v$ and $\lambda^*_v$, we replace them with arbitrary deterministic arguments $y$ and $\lambda$ to obtain

$$ (\sigma^F(v, y)^+)^\top \left( \theta^a(v, y, \lambda; T) E[Q_{v,T}(\lambda)]|Y_v = y] + E[H^\theta_{v,T}(\lambda)]|Y_v = y] + E[H^\theta_{v,T}(\lambda)]|Y_v, \lambda^*_v] \right) \equiv 0_{d-m}. $$

It is straightforward to obtain from (A.21) that

$$ (\sigma^F(v, y)^+)^\top \theta^a(v, y, \lambda; T) = - (\sigma^F(v, y)^+)^\top E[H^\theta_{v,T}(\lambda)]|Y_v = y] + E[H^\theta_{v,T}(\lambda)]|Y_v = y] \right) E[Q_{v,T}(\lambda)]|Y_v = y]. $$ \hspace{1cm} (A.22a)

Since $(\sigma^F(v, y)^+)^\top$ is a $(d-m) \times d$ matrix, (A.22a) provides $(d-m)$ equations governing the $d$-dimensional column vector $\theta^a(v, y, \lambda, T)$. We get the other $m$ equations for governing $\theta^a(v, y, \lambda; T)$ out of the second orthogonal condition in (38), i.e., $\sigma(v, y)\theta^a(v, y, \lambda; T) = 0_m$. Thus, it follows that

$$ (\sigma(v, y)^+)^\top \theta^a(v, y, \lambda, T) = (\sigma(v, y)\sigma(v, y)^+)^{-1}\sigma(v, y)\theta^a(v, y, \lambda; T) = 0_m. $$ \hspace{1cm} (A.22b)

By combining (A.22a) and (A.22b), the function $\theta^a(v, y, \lambda; T)$ solves

$$ \theta^a(v, y, \lambda; T) = - \left( \frac{(\sigma(v, y)^+)^\top}{(\sigma^F(v, y)^+)^\top} \right)^{-1} \left( \begin{array}{c} 0_{m \times d} \\ (\sigma^F(v, y)^+)^\top \end{array} \right)^T \frac{E[H^\theta_{v,T}(\lambda)]|Y_v = y] + E[H^\theta_{v,T}(\lambda)]|Y_v = y]}{E[Q_{v,T}(\lambda)]|Y_v = y]}. $$ \hspace{1cm} (A.23)
We now further simplify the above equation. By (A.12), we have
\[
\left( \begin{array}{c} (\sigma(v,y)^\dagger)^\top \\ (\sigma^F(v,y)^\dagger)^\top \end{array} \right)^{-1} = \sigma^S(v,y)^\top = (\sigma(v,y)^\top \sigma^F(v,y)^\top).
\] (A.24)

Thus, equation (A.23) can be further deduced as
\[
\theta^u(v,y,\lambda;T) = -(\sigma(v,y)^\top \sigma^F(v,y)^\top) \left( \begin{array}{cc} 0_{m\times d} \end{array} \right) E[\mathcal{H}^r_{v,T}(\lambda)|Y_v = y] + E[\mathcal{H}^\theta_{v,T}(\lambda)|Y_v = y]
\]
\[
\equiv -\sigma^F(v,y)^\top (\sigma^F(v,y)^\dagger)^\top E[\mathcal{H}^r_{v,T}(\lambda)|Y_v = y] + E[\mathcal{H}^\theta_{v,T}(\lambda)|Y_v = y]
\]
\[
\frac{E[Q_{v,T}(\lambda)|Y_v = y]}{E[Q_{v,T}(\lambda)|Y_v = y]}.
\] (A.25)

By (11), we can simplify the coefficient in the above equation as
\[
\sigma^F(v,y)^\top (\sigma^F(v,y)^\dagger)^\top = \sigma^F(v,y)^\top (\sigma^F(v,y)^\top)^{-1} \sigma^F(v,y) = \sigma^F(v,y)^+ \sigma^F(v,y).
\] (A.26)

Besides, by (10), we note that
\[
I_d = (\sigma(v,y)^+ \sigma^F(v,y)^+) \left( \begin{array}{c} (\sigma(v,y)^\top \\ (\sigma^F(v,y)^\top) \end{array} \right) \equiv \sigma(v,y)^+ \sigma(v,y) + \sigma^F(v,y)^+ \sigma^F(v,y).
\] (A.27)

Combining (A.27) with (A.26), we get
\[
\sigma^F(v,y)^\top (\sigma^F(v,y)^\dagger)^\top = \sigma^F(v,y)^+ \sigma^F(v,y) = I_d - \sigma(v,y)^+ \sigma(v,y).
\] (A.28)

Then, (39) follows by plugging (A.28) into (A.25).

\[\Box\]

**Appendix A.3 Proof of Corollary 1**

**Proof.** For the incomplete market model with CRRA utility (5a), we follow the general decomposition established in Theorems 1 and 2, and then develop substantial structural simplifications of the results based on the special properties of CRRA utility.

First, we prove that the investor-specific price of risk function \(\theta^u(v,y,\lambda;T)\) is independent of the parameter \(\lambda\) for any \(0 \leq v \leq T\) under the CRRA utility. Equivalently, this leads to that the investor-specific price of risk \(\theta^u_s\), which ought to be \(\theta^u(v,Y_v,\lambda^*_v;T)\) under general utilities according to representation (24), is independent of the multiplier \(\lambda^*_v\). To begin, with the explicit forms of functions \(I^u(t,y)\) and \(I^U(t,y)\) under the CRRA utility (5a), we can specify the dual problem in (A.6) for characterizing the investor-specific price of risk function as:

\[
\inf_{\theta^u \in \text{Ker}(\sigma)} E \left[ (1 - w)^\frac{1}{\gamma} e^{-\frac{\theta^u T}{\gamma}} (\lambda^*_T)^{1-\frac{1}{\gamma}} + w^\frac{1}{\gamma} \int_0^T e^{-\frac{\theta^u s}{\gamma}} (\lambda^*_s)^{1-\frac{1}{\gamma}} ds \right].
\] (A.29)

According to the principle of dynamic programming, we can solve the optimal \(\theta^u\) at arbitrary time \(v\) from the following time-v version of problem (A.29):

\[
\inf_{\theta^u \in \text{Ker}(\sigma)} E_v \left[ (1 - w)^\frac{1}{\gamma} e^{-\frac{\theta^u Y_v}{\gamma}} (\lambda^*_T)^{1-\frac{1}{\gamma}} + w^\frac{1}{\gamma} \int_v^T e^{-\frac{\theta^u s}{\gamma}} (\lambda^*_s)^{1-\frac{1}{\gamma}} ds \right].
\] (A.30)
Using the relationship $\lambda^*_v = \lambda^*_0 \xi^S_{s,t} = \lambda^*_s \xi^S_{s,t} + \eta^s_{s,t}$, as well as the fact that the multiplier $\lambda^*_v$ is known with information available up to time $v$, we can extract the factor $(\lambda^*_v)^{1 - \frac{1}{\gamma}}$ from the conditional expectation in (A.30) to get

$$\inf_{\theta \in \text{Ker} (\sigma)} (\lambda^*_v)^{1 - \frac{1}{\gamma}} E_v \left[ (1 - w)^{-\frac{1}{\gamma}} \right] (t, Y_s) \xi^S_{s,T} (1 - \frac{1}{\gamma}) + w^\gamma \int_0^T e^{-w^\gamma} (\xi^S_{t,s})^{-\frac{1}{\gamma}} \gamma \, ds \right].$$

According to He and Pearson (1991), as the Lagrangian multiplier of the static optimization problem (18), $\lambda^*_0$ must be positive. This implies that $\lambda^*_v = \lambda^*_0 \xi^S_{s,t}$ is also positive. Thus we can drop the factor $(\lambda^*_v)^{1 - \frac{1}{\gamma}}$ in this optimization problem. Besides, the process $(Y_s, \xi^S_{s,t})$ for $v \leq s \leq T$ is Markovian with the initial value $(Y_v, 1)$. Then, we use the feedback control law to conclude that $\theta^u_v$ admits the representation $\theta^u_v = \theta^u (v, Y_s; T)$ for some function $\theta^u (v, y; T)$. In other words, the function $\theta^u (v, y, \lambda; T)$ introduced in (24) is independent of the parameter $\lambda$ under the CRRA utility.

Then, by (12), the total price of risk under the CRRA utility can be parameterized and represented as $\theta^S (s, Y_s; T) = \theta^h (s, Y_s) + \theta^u (s, Y_s; T)$. Plugging this representation into (29) and (32), we can prove that $\xi^S_{s,t}$ and $H^\theta_{s,t}$, under the CRRA utility, satisfy the dynamics in (44a) and (44b), respectively. In particular, as both $\xi^S_{s,t}$ and $H^\theta_{s,t}$ are independent of the time–$t$ multiplier $\lambda^*_t$ under the CRRA utility, we drop $\lambda^*_t$ as opposed to writing their general expressions $\xi^S_{s,t}(\lambda^*_t)$ and $H^\theta_{s,t}(\lambda^*_t)$.

We now establish the representations of optimal policy in (46a) and (46b) as well as the equation governing function $\theta^u (v, y, T)$ in (48). First, we note the following algebraic fact: with the specification of the CRRA utility function given in (5a), the functions $Q_{t,T}(\lambda^*_t)$, $H^\theta_{t,T}(\lambda^*_t)$, $H^\theta_{t,T}(\lambda^*_t)$, and $G_{t,T}(\lambda^*_t)$ defined in (27a) – (27c) and (21) are simplified to the following separable forms:

$$Q_{t,T}(\lambda^*_t) = (\lambda^*_t)^{-\frac{1}{\gamma}} \hat{Q}_{t,T}, \quad H^\theta_{t,T}(\lambda^*_t) = (\lambda^*_t)^{-\frac{1}{\gamma}} \hat{H}_{t,T}, \quad H^\theta_{t,T}(\lambda^*_t) = (\lambda^*_t)^{-\frac{1}{\gamma}} \tilde{H}_{t,T},$$

(A31)

and $G_{t,T}(\lambda^*_t) = (\lambda^*_t)^{-\frac{1}{\gamma}} \hat{G}_{t,T}$, where $\hat{H}_{t,T}, \tilde{H}_{t,T},$ and $\hat{G}_{t,T}$ are introduced in (47b), (47a), and (47c), respectively, and the function $\hat{Q}_{t,T}$ is given by

$$\hat{Q}_{t,T} = -\frac{1}{\gamma} \hat{G}_{t,T}.$$  

(A32)

With the separable forms, the wealth equation in (20), i.e., $X_t = E_t [\hat{G}_{t,T}(\lambda^*_t)]$, is equivalent to

$$X_t = (\lambda^*_t)^{-\frac{1}{\gamma}} E_t [\hat{G}_{t,T}].$$  

(A33)

For the mean–variance component $\pi^{mv}(t, X_t, Y_t) = - (\sigma(t, Y_t)^+)^T \theta^h (t, Y_t) E_t [Q_{t,T}(\lambda^*_t)] / X_t$ in (26a), we use the relationships (A31) and (A32) to get

$$E_t [Q_{t,T}(\lambda^*_t)] = (\lambda^*_t)^{-\frac{1}{\gamma}} E_t [\hat{Q}_{t,T}] = - (\lambda^*_t)^{-\frac{1}{\gamma}} E_t [\hat{G}_{t,T}] / \gamma.$$  

Then, plugging it into (26a) yields $\pi^{mv}(t, X_t, Y_t) = (\lambda^*_t)^{-\frac{1}{\gamma}} E_t [\hat{G}_{t,T}] (\sigma(t, Y_t)^+)^T \theta^h (t, Y_t) / \gamma$. Then, by the definition of $\theta^h (t, Y_t)$ in (13a), we obtain the representation in (46a) as $\pi^{mv}(t, X_t, Y_t) = (\sigma(t, Y_t)^+)^T \theta^h (t, Y_t) / \gamma = (\sigma(t, Y_t) \sigma(t, Y_t)^T)^{-1} (\mu(t, Y_t) - r(t, Y_t) 1_m) / \gamma$. Similarly, for the interest rate and price of risk hedge components given by (26b) and (26c), their representations under CRRA utility (46b) follow by combining the separable forms in (A31) and the constraint (A33).

Finally, we derive equation (48) for the investor-specific price of risk function $\theta^u (v, y, T)$ under the CRRA utility. With the separable forms given in (A31) and the relationship (A32), we can express
the conditional expectations in the general equation (39) as $E[\mathcal{H}_{v,T}(\lambda)|Y_v = y] = \lambda^{-\frac{1}{2}} E[\tilde{\mathcal{H}}_{v,T}|Y_v = y]$, $E[\mathcal{H}_{v,T}^\theta(\lambda)|Y_v = y] = \lambda^{-\frac{1}{2}} E[\tilde{\mathcal{H}}_{v,T}^\theta|Y_v = y]$, and $E[\mathcal{Q}_{v,T}(\lambda)|Y_v = y] = \lambda^{-\frac{1}{2}} E[\tilde{\mathcal{Q}}_{v,T}|Y_v = y] = -\lambda^{-\frac{1}{2}} E[\tilde{\mathcal{G}}_{v,T}|Y_v = y]/\gamma$. Then, (48) follows directly by plugging them in (39). The term $\lambda^{-\frac{1}{2}}$ cancels out in both the nominator and denominator, which reconciles with the investor-specific price of risk function $\theta^u(v, y; T)$ being indeed independent of parameter $\lambda$. 

\[ \blacksquare \]

**Appendix A.4 Proof of Proposition 2**

We first prove the following lemma that plays a crucial role in proving Proposition 2.

**Lemma 2.** With deterministic interest rate $r_s$, the following relationship holds:

\[ E_t[\xi_t^S(\lambda) H_{t,s}^\theta(\lambda)] \equiv 0_d, \text{ for any } s \geq t. \]  

(A.34)

Here, $\xi_t^S(\lambda)$ is the relative state price density, and $H_{t,s}^\theta(\lambda)$ is the Malliavin term related to the uncertainty in the total price of risk, with their dynamics given explicitly in (29) and (27c), respectively.

**Proof.** By (15), we get $\xi^S_t \equiv \exp\left(-\int_0^t r_s ds - \int_0^t (\theta^S_s)^\top dW_s - \frac{1}{2} \int_0^t (\theta^S_s)^\top \theta^S_s ds\right)$ for the state price density in incomplete markets. We can decompose it to two parts related to interest rate and total price of risk respectively, i.e., $\xi_t^S = B_t \eta_t$, where

\[ B_t = \exp\left(-\int_0^t r_s ds\right) \text{ and } \eta_t = \exp\left(-\int_0^t (\theta^S_s)^\top dW_s - \frac{1}{2} \int_0^t (\theta^S_s)^\top \theta^S_s ds\right). \]  

(A.35)

With deterministic interest rate $r_s$, the discount term $B_t$ is also deterministic. A straightforward application of Ito formula leads to the SDEs $\eta_t$ as

\[ d\eta_t = -\eta_t (\theta^S_t)^\top dW_t. \]  

(A.36)

The martingale property of $\eta_t$ leads to

\[ E_t[\eta_s] = \eta_t, \text{ for any } s \geq t. \]  

(A.37)

Next, we prove that

\[ E_t[\eta_s H_{t,s}^\theta(\lambda)] \equiv 0_d. \]  

(A.38)

On one hand, by computing Malliavin derivative (see, e.g., the tutorial in Appendix D of Detemple et al. (2003)), we obtain the time–$t$ Malliavin derivative of $\eta_s$ as $\mathcal{D}_t \eta_s = -\eta_s (\theta^S_t + H_{t,s}^\theta(\lambda))$. Taking conditional expectation on the both sides, we have

\[ E_t[\mathcal{D}_t \eta_s] = -E_t[\eta_s \theta^S_t] - E_t[\eta_s H_{t,s}^\theta(\lambda)] = -E_t[\eta_s] \theta^S_t - E_t[\eta_s H_{t,s}^\theta(\lambda)] = -\eta_t \theta^S_t - E_t[\eta_s H_{t,s}^\theta(\lambda)], \]  

(A.39)

where the last equality follows from the martingale property of $\eta_s$ in (A.37). On the other hand, we compute the Malliavin derivative $\mathcal{D}_t \eta_s$ again, using the SDE of $\eta_s$. By (A.36), we have $\eta_s =
We express the conditional expectation and (A.39). which follows the martingale property of Itô integrals. Thus, (A.38) follows by comparing (A.40) to see this, we use the relationship \( \lambda \). Taking time–

Finally, relationship (A.34) comes from \( E_t[\xi^S_{t,s}(\lambda) H^0_{t,s}(\lambda)] = E_t[\xi^S_s H^0_{t,s}(\lambda)]/\xi^S_t = B_s E_t[\eta_s H^0_{t,s}(\lambda)]/\xi^S_t = 0_d \), where the second equality follows from the deterministic nature of the discount term \( B_s \) as well as (A.38).

Now, we are ready to prove Proposition 2 for the optimal policy under HARA utility with deterministic interest rate. Without loss of generality, we assume \( w > 0 \) in utility (5b), as the case of \( w = 0 \) follows in a similar fashion.\(^{15}\)

**Proof.** Part 1: First, we show that with deterministic interest rate, the investor-specific price of risk function \( \theta^u(v,y,\lambda; T) \) under HARA utility coincides with its counterpart under CRRA utility, and thus is independent of parameter \( \lambda \). To begin, like in other proofs, we employ the dual problem (A.6) as a tool. Using the explicit forms of functions \( I^u(t,y) \) and \( I^U(t,y) \) under HARA utility, we can explicitly specify the dual problem in (A.6) as

\[
\inf_{\theta^u \in \text{Ker}(\sigma)} E_v \left[ (1 - w)^{\frac{1}{2}} e^{-\frac{\theta^u T}{2}} (\lambda^*_T)^{1 - \frac{1}{\gamma}} + w^\frac{1}{2} \int_v^T e^{-\frac{\theta^u s}{2}} (\lambda^*_s)^{1 - \frac{1}{\gamma}} ds + \frac{\gamma - 1}{\gamma} A_v[T] \right],
\]

where \( A_v[T] = \bar{x} \lambda^*_T + \bar{c} \int_v^T \lambda^*_s ds \). Here, \( \bar{x} \) and \( \bar{c} \) are the minimum requirements for terminal wealth and intermediate consumption, respectively. Comparing (A.41) and (A.29), we see that the term \( A_v[T] \) distinguish the dual problem under HARA utility from that under CRRA utility.

With deterministic interest rate, we then verify that \( E_v[A_v,T] \) does not depend on the control process \( \theta^u \) for \( v \in [v, T] \) and thus can be dropped from the dual problem (A.41) to simplify it as

\[
\inf_{\theta^u \in \text{Ker}(\sigma)} E_v \left[ (1 - w)^{\frac{1}{2}} e^{-\frac{\theta^u T}{2}} (\lambda^*_T)^{1 - \frac{1}{\gamma}} + w^\frac{1}{2} \int_v^T e^{-\frac{\theta^u s}{2}} (\lambda^*_s)^{1 - \frac{1}{\gamma}} ds \right].
\]

To see this, we use the relationship \( \lambda^*_u = \lambda^*_0 \xi^S_s = \lambda^*_v \xi^S_{v,s} \) to derive that

\[
E_v[A_v,T] = \bar{x} E_v[\lambda^*_T] + \bar{c} \int_v^T E_v[\lambda^*_s] ds = \lambda^*_v \left[ \bar{x} E_v[\xi^S_{v,T}] + \bar{c} \int_v^T E_v[\xi^S_{v,s}] ds \right].
\]

We express the conditional expectation \( E_v[\xi^S_{v,s,T}] \) as \( E_v[\xi^S_{v,s,T}] = E_v[B_{v,s} \eta_{v,s}] \), where \( B_{v,s} = B_s/B_v \) and \( \eta_{v,s} = \eta_s/\eta_v \) following (A.35). A straightforward application of Itô formula leads to the SDE of \( \eta_{v,s} \)

\(^{15}\)For the proof under the case of \( w = 0 \), we just need to drop all the terms related to \( \bar{c} \).
as \( d\eta_{v,s} = -\eta_{v,s} (\theta_v^S)\top dW_s \). As we assume a deterministic interest rate, \( B_{v,s} \) is also deterministic. Thus, we have

\[
E_v[\xi_{v,s}^S] = E_v[B_{v,s} \eta_{v,s}] = B_{v,s} E_v[\eta_{v,s}] = B_{v,s},
\]

(A.44)

where the last equality follows from the martingale property of \( \eta_{v,s} \) as a process in \( s \) and the fact that \( \eta_{v,v} = 1 \). Plugging \( E_v[\xi_{v,s}^S] \) into (A.43), we obtain that 

\[
E_v[A_{v,T}] = \lambda_v^u [\bar{x} B_{v,T} + \bar{c} \int_v^T B_{v,s} ds],
\]

which obviously does not depend on the control process \( \theta_v^u \) for \( v \in [v, T] \). Thus, we can drop the term \( A_{v,T} \) from (A.41).

By the above arguments, we show that with deterministic interest rate, the investor-specific price of risk \( \theta_v^u \) under HARA utility is uniquely characterized as the control process for the dual problem (A.42), with the underlying Markov process \( (Y_s, \xi_{v,s}^S) \) for \( v \leq s \leq T \). Comparing the dual problem (A.42) with its counterpart (A.30) under CRRA utility, we can verify that the two dual problems, as well as the underlying Markov process, are actually the same under the two utility specifications. Thus, the unique optimal control process \( \theta_v^u \) is also the same for the two dual problems. This proves that with deterministic interest rate, the investor-specific price of risk function \( \theta^u (v, y, \lambda; T) \) under HARA utility coincides with its counterpart under CRRA utility, and thus is independent of the parameter \( \lambda \). So, we can express it as 

\[
\theta_v^u = \theta^u (v, Y_v; T)
\]

with the same function \( \theta^u (v, y; T) \) that satisfies the equation (50) under the CRRA utility. Consequently, with deterministic interest rate, the quantities \( \hat{H}_{v,T}(\lambda), \hat{H}_v(\lambda), \hat{Q}_{v,T}(\lambda), \) and \( \hat{G}_{v,T}(\lambda) \) in (51a), (51b), (51c), and (55) for HARA utility are also independent of parameter \( \lambda \), and coincide with their counterparts under CRRA utility, which are given in (47b), (47b), (49), and (47c), respectively.

Next, we establish equation (56) that governs the investor-specific price of risk function \( \theta^u (v, y; T) \). It follows from equation (50) that, whether the interest rate is deterministic or not, \( \theta^u (v, y, \lambda; T) \) under HARA utility is characterized by

\[
\theta^u (v, y, \lambda; T) = \frac{\sigma(v, y) + \sigma(v, y) - I_d}{E[\hat{Q}_{v,T}(\lambda)|Y_v = y]} \times (E[\hat{H}_{v,T}(\lambda)|Y_v = y] + E[\hat{H}_v(\lambda)|Y_v = y] + \lambda^\gamma E[\zeta_{v,T}(\lambda)|Y_v = y]),
\]

(A.45)

where \( \zeta_{v,T}(\lambda) = \zeta_{v,T}(\lambda) + \zeta_{v,T}^\theta(\lambda) \) according to (52a). With deterministic interest rate, we have \( H_{v,s} = 0 \) due to (31) and \( \nabla r(s, Y_s) = 0 \). Thus, it follows from (51a) and (52b) that \( \hat{H}_{v,T}(\lambda) = 0 \) and \( \zeta_{v,T}^\theta(\lambda) = 0 \). Also recall that \( \theta^u (v, y, \lambda; T) \) is independent of \( \lambda \) and thus simplifies to \( \theta^u (v, y; T) \). Then, we simplify equation (A.45) to

\[
\theta^u (v, y; T) = \frac{\sigma(v, y) + \sigma(v, y) - I_d}{E[\hat{Q}_{v,T}(\lambda)|Y_v = y]} \times (E[\hat{H}_{v,T}(\lambda)|Y_v = y] + \lambda^\gamma E[\zeta_{v,T}^\theta(\lambda)|Y_v = y]),
\]

(A.46)

where \( \zeta_{v,T}^\theta(\lambda) \) is defined by (52c) as 

\[
\zeta_{v,T}^\theta(\lambda) = \bar{x} \xi_{v,T}(\lambda) H_{v,T}(\lambda) + \bar{c} \int_v^T \xi_{v,s}(\lambda) H_{v,s}(\lambda) ds.
\]

By Lemma 2, its expectation is always zero under deterministic interest rate, i.e.,

\[
E_v[\xi_{v,T}(\lambda)] = \bar{x} E_v[\xi_{v,T}(\lambda) H_{v,T}(\lambda)] + \bar{c} \int_v^T E_v[\xi_{v,s}(\lambda) H_{v,s}(\lambda) ds = 0_d.
\]

(A.47)
Thus, the last term \( \lambda^\frac{1}{2} E[\zeta^\theta_{v,T}(\lambda)|Y_v = y] \) vanishes in (A.46), and the equation further simplifies to

\[
\theta^u(v, y; T) = \frac{\sigma(v, y)^+\sigma(v, y) - I_d}{E[\mathbf{Q}_{v,T}(\lambda)|Y_v = y]} \times E[\mathbf{H}^\theta_{v,T}(\lambda)|Y_v = y].
\]

By examining the definitions of \( \mathbf{h}^\theta_{v,T}(\lambda) \) and \( \mathbf{Q}_{v,T}(\lambda) \) in (51b) and (51c), as well as the SDEs of \( \xi^S_{v,s}(\lambda) \) and \( H^\theta_{v,s}(\lambda) \) in (29) and (32), we confirm that \( \mathbf{h}^\theta_{v,T}(\lambda) \) and \( \mathbf{Q}_{v,T}(\lambda) \) reduce to \( \mathbf{h}^\theta_{v,T} \) and \( \mathbf{Q}_{v,T} \) given in (47b) and (49), respectively. Hence, the parameter \( \lambda \) does not show up in either the above equation system or its solution \( \theta^u(v, y; T) \).

**Part 2:** Next, we look into the optimal policy under HARA utility with deterministic interest rate. Under this circumstance, we have \( H^\theta_{v,s} \equiv 0_d \), and thus it follows from that (51a), (52b), and (53b) the interest hedge component \( \pi^H_t(t, X_t, Y_t) = 0_m \), i.e., there is no need to hedge uncertainty in interest rate. So, we only need to focus on the mean-variance and price of risk hedge components.

First, we solve for the multiplier \( \lambda^*_t \) from the wealth equation (54), i.e., \( (\lambda^*_t)^{-\frac{1}{2}} E_t[\mathbf{G}_{t,T}] + \pi E_t[\xi^S_{t,s}] + \pi E_t[\xi^S_{t,s}] = X_t \). Here, we drop the dependence on \( \lambda^*_t \) from \( \mathbf{G}_{t,T} \) and \( \xi^S_{t,s} \). This is because we have shown in Part 1 that the investor-specific price of risk function \( \theta^u(v, y; T) \) does not depend on \( \lambda \), and neither do \( \mathbf{G}_{t,T} \) and \( \xi^S_{t,s} \) according to (55) and (29), respectively. By (A.44), we have \( E_t[\xi^S_{t,s}] = B_{t,s} \).

Plugging it to the above equation, we solve \((\lambda^*_t)^{-\frac{1}{2}}\) as

\[
(\lambda^*_t)^{-\frac{1}{2}} = \frac{X_t}{E_t[\mathbf{G}_{t,T}]}, \tag{A.48}
\]

where \( X_t \) is defined in (58a), i.e., \( X_t = X_t - \pi B_{t,T} - \pi \int_t^T B_{t,s}ds \). Plugging (A.48) into the mean-variance component in (53a) and invoking the relationship \( \mathbf{G}_{t,T} = -\gamma \mathbf{Q}_{t,T} \), we can derive

\[
\pi^H_{uv}(t, X_t, Y_t) = -\frac{1}{X_t}(\sigma(t, Y_t)^+)^\top \theta^h(t, Y_t) (\lambda^*_t)^{-\frac{1}{2}} E_t[\mathbf{G}_{t,T}]
= -\frac{1}{X_t}(\sigma(t, Y_t)^+)^\top \theta^h(t, Y_t) \frac{X_t}{E_t[\mathbf{G}_{t,T}]} E_t[\mathbf{Q}_{t,T}]
= \frac{\hat{X}_t}{\gamma X_t} (\sigma(t, Y_t)^+)^\top \theta^h(t, Y_t). \tag{A.49}
\]

Next, in (53c), we have

\[
\pi^H_0(t, X_t, Y_t) = -\frac{1}{X_t}(\sigma(t, Y_t)^+)^\top \left((\lambda^*_t)^{-\frac{1}{2}} E_t[\mathbf{H}^\theta_{t,T}] + E_t[\xi^S_{t,s}]\right)
= -\frac{1}{X_t}(\sigma(t, Y_t)^+)^\top (\lambda^*_t)^{-\frac{1}{2}} E_t[\mathbf{H}^\theta_{t,T}]
\]

for the price of risk hedge component, where the second equality follows from (A.47). Plugging (A.48) into the right-hand side, we obtain

\[
\pi^H_0(t, X_t, Y_t) = -(\sigma(t, Y_t)^+)^\top \frac{\hat{X}_t}{X_t} \frac{E_t[\mathbf{H}^\theta_{t,T}]}{E_t[\mathbf{Q}_{t,T}]}.	ag{A.50}
\]

Finally, relationships (57) follow by comparing the optimal components in (A.49) and (A.50) with their counterparts (46a) and (46b) under the CRRA utility.

\[\square\]
Figure 1: Behavior of optimal policy in the Heston-SV model

Note: These figures plot for optimal policies under both the HARA and CRRA utilities. The upper left (resp. upper right) panel plots the optimal policy $\pi(t, X_t, V_t)$ for different wealth level $X_t/\bar{X}$ (resp. interest rate $r$). The lower left (resp. lower right) panel plots the mean-variance component $\pi^{mv}(t, X_t, V_t)$ (resp. price of risk hedge component $\pi^\theta(t, X_t, V_t)$) at different investment horizon $T - t$. These figures are generated according to the closed-form formulae (62) – (67b). The parameters are set as follows in annualized form. In the upper left panel, we set $r = 0.04$ and $T - t = 10$. In the upper right panel, we set $X_t/\bar{X} = 3$ and $T - t = 10$. In the two lower panels, we set $r = 0.04$ and $X_t/\bar{X} = 3$. Besides, we set the rest of the parameters at the following representative values $\kappa = 5.07$, $\rho = -0.767$, $\lambda = 1.1$, $\theta = 0.0457$, $\sigma = 0.48$ according to the Maximum Likelihood estimation results of Aït-Sahalia and Kimmel (2007), while choosing $\gamma = 2$ and $V_t = 0.0457$. 
Figure 2: The stock price $S_t$ (blue line with left y-axis) and the optimal policy ratio $\pi_t^H/\pi_t^L$ (red line with right y-axis) in a simulated path of the Heston-SV model.

Note: This figure plots for a simulated path of stock price $S_t$ and that of the corresponding ratio $\pi_t^H/\pi_t^L$ between the optimal policies of the high– and low–wealth investors under HARA utility. The ratios are calculated according to the closed-form formulae (65) and (66). The parameters for the Heston SV model are chosen as those representative ones employed for producing Figure 1.
Figure 3: Behavior of optimal hedging component $\pi^\theta(t, V_t)$ in the CEV-SV model

Note: The three panels plot the optimal hedge component $\pi^\theta(t, V_t)$ with respect to different choices of the level of current variance $V_t$, the leverage effect parameter $\rho$, and the risk aversion parameter $\gamma$, respectively. In each panel, three values are considered: $\nu = 1/2$ (the Heston model), $\nu = 1$ (the GARCH diffusion model), and $\nu = 3/2$ (the 3/2 model). In the left panel, we set $\gamma = 2$ and $\rho = -0.8$; in the middle and right panels, we set $V_t = 0.15$. For all the results, we employ realistic annualized parameters. We set $r = 0.04$, $\lambda = 0.5$, $\kappa = 1.5$, $\theta = 0.2$, and $T - t = 1$ in common. In particular, to control the effect of volatility for different values of $\nu$, we keep $\sigma\theta^\nu$, the volatility of the variance process $V_t$ evaluated at the mean-reverting level $\theta$, at a constant level $\phi_V = \sigma\theta^\nu$ by setting $\sigma$ according to $\sigma = \phi_V / \theta^\nu$ for each value of $\nu$. Without loss of generality, we choose this constant level as $\phi_V = 0.25 \times \sqrt{0.2} \approx 0.11$ according to a realistic parameter set of the Heston model. Then we set $\sigma$ according to $\sigma = \phi_V / \theta^\nu$ for each value of $\nu$, i.e., $\sigma = 0.25$ for Heston-SV model, $\sigma \approx 0.56$ for GARCH-SV model, and $\sigma = 1.25$ for 3/2-SV model.
Table 1: Simulation results of the incomplete-market stochastic volatility model of Heston given in Example 1.

Note: For the incomplete-market Heston SV model in Example 1, we choose the following representative annualized parameter set: \( \lambda = 0.5, \rho = -0.8, \kappa = 1.5, \sigma = 0.25, \) and \( \theta = 0.2. \) These values correspond to the numbers that Aït-Sahalia and Kimmel (2007) set for Monte Carlo simulations under Heston’s model to produce their Table 2.

| \( V_t \) | \( \hat{\pi}^\theta \) (in \( 10^{-3} \)) | \( \hat{\pi}_{true}^\theta \) (in \( 10^{-3} \)) | \( \epsilon_{rel} \) (in %) | Std (in \( 10^{-5} \)) | CI\(_{95} \) (in \( 10^{-3} \)) | Std\(_B \) (in \( 10^{-5} \)) | CI\(_{95,B} \) (in \( 10^{-3} \)) | RMSE\(_B/|\hat{\pi}^\theta_{true}| \) (in %) |
|---|---|---|---|---|---|---|---|---|
| \( T - t = 0.5 \) | | | | | | | | |
| 0.10 | 2.194 | 2.223 | 1.284 | 3.690 | [2.122, 2.266] | 3.679 | [2.129, 2.265] | 2.112 |
| 0.15 | 2.199 | 2.223 | 1.051 | 3.203 | [2.136, 2.262] | 3.197 | [2.142, 2.259] | 1.795 |
| 0.20 | 2.202 | 2.223 | 0.907 | 2.878 | [2.146, 2.259] | 2.875 | [2.151, 2.257] | 1.592 |
| 0.25 | 2.205 | 2.223 | 0.807 | 2.640 | [2.153, 2.256] | 2.637 | [2.158, 2.255] | 1.446 |
| 0.30 | 2.206 | 2.223 | 0.733 | 2.454 | [2.158, 2.254] | 2.452 | [2.163, 2.253] | 1.335 |
| \( T - t = 1.0 \) | | | | | | | | |
| 0.25 | 3.286 | 3.299 | 0.376 | 2.966 | [3.228, 3.344] | 2.979 | [3.232, 3.345] | 0.984 |
| 0.30 | 3.287 | 3.299 | 0.340 | 2.778 | [3.233, 3.342] | 2.788 | [3.237, 3.342] | 0.916 |
| \( T - t = 3.0 \) | | | | | | | | |
| 0.20 | 4.276 | 4.252 | 0.551 | 3.351 | [4.210, 4.341] | 3.325 | [4.208, 4.342] | 0.959 |
| 0.25 | 4.272 | 4.252 | 0.459 | 3.113 | [4.211, 4.333] | 3.089 | [4.210, 4.333] | 0.861 |
| 0.30 | 4.269 | 4.252 | 0.391 | 2.927 | [4.212, 4.326] | 2.903 | [4.210, 4.326] | 0.788 |
Table 2: Simulation results of the incomplete-market mean-reverting return model of Kim-Omberg given in Example 2.

Note: For the incomplete-market mean-reverting return model in Example 2, we choose the following representative annualized parameter set: \( r = 0.043, \sigma = 0.15, \lambda = 0.51, \sigma_\theta = 0.48, \) and \( \theta = 0.33 \) according to its complete-market counterpart in Wachter (2002), and choose \( \rho = -0.5 \), which is a typical value according to its economic interpretation.