Asset Allocation and Pension Liabilities in the Presence of a Downside Constraint

Byeong-Je An*
Columbia University

Andrew Ang†
Columbia University and NBER

Pierre Collin-Dufresne‡
Ecole Polytechnique Federale de Lausanne and NBER

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*Email: ba2306@columbia.edu
†Email: aa610@columbia.edu
‡Email: pierre.collin-dufresne@epfl.ch
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Abstract

We develop a separation approach to study a pension’s optimal contribution and portfolio policy when the pension has a downside constraint at the terminal date. We transform the pension’s problem into two separate problems that solve the shadow prices for the utility maximization, and for the disutility minimization. We show that the shadow prices for two problems are identical and satisfy the initial budget constraint at the optimal solution. The separation approach allows us to characterize the pension’s value function, optimal contribution, and portfolio policy in a simple way. Policy implications of satisfying the downside constraint are also discussed in the paper.
1 Introduction

It was a large decline in pension plans’ funding status that motivated the creation of mandatory contribution rules and public institutions, which have insurance obligations to workers whose defined benefit pension promises are not fulfilled due to firm bankruptcy. For example, in the U.S. Employee Retirement Income Security Act (ERISA) in 1974 created the minimum funding contribution (MFC) and Pension Benefit Guaranty Corporation (PBGC). MFC requirements specify that sponsors of underfunded plans must contribute an amount equal to any unfunded liabilities.\(^1\) After ERISA, several changes to mandatory contribution rules have been made to require better funding of defined benefit plans. The Pension Protection Act of 1987 required the deficit reduction contribution (DRC), which is 13.75% – 30% of any underfunding. Firms were required to contribute the larger of two components, MFC and DRC.

Despite of these government interventions to save underfunded pension plans, unfortunately large number of defined benefits pension plans are still underfunded. For example, in 2013 the largest 100 corporate defined benefits pension plans in the U.S. reported 1.78 trillion USD of liabilities guaranteed with only 1.48 trillion USD of asset, which represents underfunding of more than 15%.\(^2\) Some of this underfunding crisis can be attributed to misaligned incentives of pension sponsor and government. For example, a premium that PBGC collects in exchange for insurance would have been set too low, which cause a morally hazardous reaction of pension sponsor to exploit the difference between the fair and actual value of premium. One of examples of this is to increase a risk of pension’s asset by holding more equities even though a funding status is deteriorated. Another possibility is that mandatory contribution rules are less strict than they should be to bounce back to a funded level. This induces pension sponsors to make only minimum contribution that is not enough to improve funding status, since internal resources are limited and it is better to invest in a profitable project.

Thus, we believe it is important to understand how underfunded pension plans end up with funded status through the optimal asset allocation and contribution in the first place. This paper solves for the optimal asset allocation and contribution of a defined benefit pension plan that faces a Liability Driven Investment (LDI) problem under a downside constraint in an environment without a government insurance and mandatory contribution rules. By understanding the

\(^1\) MFC also includes an amount equal to the present value of pension benefit accrued during the year (called the “normal cost”). An unfunded amount may be amortized over a long period, typically 10 years. Thus, in the first year the amount of MFC is 10% of the ERISA underfunding plus the normal cost.

optimal asset allocation and contribution in the absence of government interventions, we can better evaluate the premium associated with the insurance PBGC provides and the proper level of contribution as a function of funding status. Further, we consider a benchmark case without a downside constraint and characterize the optimal policy of investment and contribution, which are informative regarding the likely response of pension sponsors in an environment with a government insurance.

LDI problems with a downside constraint are similar with standard portfolio choice problems. The fact that a pension sponsor wants to maximize the utility from the final pension’s asset is identical with standard asset allocation problems. The rational behind this utility is that firms are more likely to terminate pensions for using excess pension assets when internal finance was scarce or external finance is too costly. We model this as a utility over pension’s terminal asset value. On the other hand, there are two important differences with standard asset allocation problems. First, pension sponsors have a downside constraint: the value of pension asset cannot fall below that of the liabilities at the terminal date. This gives a pension sponsor an incentive to construct the insured portfolio. Second, there is one more control variable, contribution rate other than a portfolio weight. We assume that a pension sponsor dislikes drawing contribution from the firm’s internal resources. We directly model this dislike as a separable disutility function, which can be interpreted as a reduced form for the foregone investment opportunities. The disutility of contribution has some analogy with the adjustment cost in the investment literature. An investment can increase a firm’s capital, but also incur a cost of adjusting. The key difference is that the adjustment cost will show up in a budget constraint directly, contrary to that contribution increases pension’s asset and a pension sponsor privately incurs disutility. This motivates us to model a separable disutility of contribution, and also to utilize a separation approach, which we describe now.

We propose a separation approach to study the optimal contribution and portfolio policy of a defined benefits pension that faces a constant investment opportunity set, as well as a downside constraint. The pension’s intermediate contribution and portfolio problem is cast in two separate shadow prices problems. The first problem solves for the shadow price of maximizing the terminal utility while satisfying the static budget constraint for the fixed present value of contribution. The second problem solves for the shadow price of minimizing intermediate disutility.

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3 See Brandt (2009) and Wachter (2010) for recent summaries on this literature.
4 Petersen (1992) used plan-level data to find evidence in support of the financing motives.
5 Rauh (2006) finds that the need to make contributions leads to a reduction in corporate investment.
6 See Caballero (1999) for summaries on this literature.
utility while funding the fixed present value of contribution. Finally, we solve for the optimal level of the present value of contribution. We interpret the shadow price of the first problem as the marginal benefit of increasing the present value of contribution. Similarly, the shadow price of the second problem is the marginal cost of it. We show that the shadow prices for two problems are identical such that the marginal benefit and cost of increasing the present value of contribution are equal at the optimal solution.

Using this framework, we make the following three points. First, we propose a novel solution technique to construct an insured portfolio while maximizing the terminal utility and minimizing the intermediate disutility at the same time. We pose the LDI problem as two separable shadow price problems. This approach allows a simple representation of the LDI problem with a downside constraint. Solving two shadow price problems is relatively straightforward compared to the stochastic dynamic programming techniques. Second, our approach allows us to characterize the optimal contribution, portfolio policy, and the value of put option in a simple way. Especially, the optimal contribution and the value of put option shed light on what the minimum contribution and the premium of PBGC should be to save unfunded plans. Also, by comparing the optimal portfolio weights with those of a benchmark case (without a downside constraint), we can predict morally hazardous reactions of pension plans in the presence of government insurance. Finally, we show that a substitution between the terminal utility and the intermediate disutility affects the effective risk aversion of pension plans over funding status. The disutility of contribution together with a downside constraint introduces a kink in the value function of the pension sponsor’s problem that causes the sponsor to become risk averse whenever the funding ratio is close to the critical value, which depends on substitution between utility and disutility. We show that this kink in the value function leads to endogenous risk taking and risk management behavior at the same time.

The investment behavior of pension plans has been studied by Sharpe (1976), Sundaresan and Zapatero (1997), Boulier, Trussant and Florens (1995), and van Binsbergen and Brandt (2007). Sharpe (1976) first recognized the value of implicit put option in pension’s asset to insure shortfall at the maturity. Sundaresan and Zapatero (1997) consider the interaction of pension sponsors and their employees. Given the marginal productivity of workers, the retirement date is endogenously determined. Then, pension sponsors solve the investment problem of maximizing the utility over excess assets in liabilities. We allow intermediate contribution such that initially underfunded plans are able to construct dynamic insured portfolio. Our focus is to derive the optimal contribution and portfolio policy of LDI problem, we model the
Our paper is closely related to Boulier, Trussant and Florens (2005). In their problem, the investment manager chooses his portfolio weights and contribution rate to minimize the quadratic disutility from contribution, with a downside constraint. However, from the perspective of pension sponsors the surplus at the end of plans also matters since it is usually refunded to sponsors and can be used to fund profitable projects. We model this motive as the utility over terminal pension asset. van Binsbergen and Brandt (2007) solve for the optimal asset allocation of an investment manager of pension plans who faces LDI problem under regulatory constraints. They assume a time-varying investment opportunity set, and explore the impact of regulatory constraints on asset allocation. However, contribution is not a control variable and a downside constraint is not explicitly specified. Instead, we assume an absence of any government regulations and derive the optimal contribution and portfolio policy. By doing this, we can have policy implications on how minimum contribution rule and premium paid to PBGC should be decided.

Our methodology is based on Karatzas, Lehoczky, Shreve (1987) and El Karoui, Jeanblanc, Lacoste (2005). Karatzas et al. (1987) solve a consumption and portfolio choice problem. They find that the initial wealth can be allocated in two problem, maximizing utility over intermediate consumption and maximizing utility over terminal wealth. The optimal allocation leads to the optimal solution to the original problem. In our model, contribution is a counterpart of consumption, but it generates disutility and the pension sponsor’s objective is to minimize disutility. Thus, the problem can be cast in a problem to decide how much to contribute to satisfy a downside constraint while minimizing disutility. Karoui et al. (2005) find a put option based solution to maximize utility over terminal wealth with a downside constraint. However, their solution can be applied to only initially overfunded pensions. We allow initially underfunded pensions to contribute in order to guarantee the terminal liability.

There are at least three important aspects of the LDI problem that we do not address explicitly. First, we do not incorporate time-varying investment opportunities. The expected returns of bonds and equities are predicted by macro variables, such as short rates, yield slopes, and dividend yields. This induces non-trivial hedging demands and liability risks, which drive a wedge between myopic and dynamic investment. Second, we do not consider the taxation issues. Drawing contributions from firm’s internal resource is costly for sure, however there is

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7 As long as the market is complete, our model can be extended to incorporate a stochastic feature of liabilities, and the solution technique goes through.

8 van Binsbergen and Brandt (2007) consider a LDI problem with time-varying expected returns.
also a benefit from tax deductions. Third, our model do not include inflation. Depending on whether a pension sponsor’s preference is in real or nominal term, the allocation to real assets such as TIPS will be affected.

The paper is organized as follows. Section 2 describes the pension benefit and return dynamics. Section 4 presents a pension sponsor’s problem without a downside constraint as a benchmark case, and a separation method for the optimal investment and contribution policy. Section 3 considers a constrained case in which there is a downside constraint. Section 5 presents our results and Section 6 concludes.

2 Model

2.1 Liability

Defined benefit pensions pay employees at their retirement according to pre-defined rule. Usually, benefits depend last 5-year average of salary and number of years of employment. We model this rule in a reduced form. Let \( L_t \) be an index of pension benefit, i.e. if a employee retires right now, she receives \( L_t \). It follows:

\[
\frac{dL_t}{L_t} = g dt.
\]

The drift is intuitive. Pension liability grows with the rate \( g \). This reflects an increase in years of employment and growth of salary. We assume that any outflow from pension plan to retiring employees, i.e. decrease in benefit balances with inflow to pension fund from new employees. Thus, we can just model the growth of benefit and don’t need to capture the inflow of pension asset. Our focus is to derive the optimal policy to end up with overfunded, rather the optimal decision of employees to retire. Thus, we don’t endogenize outflow of pension’s asset. The terminal date \( T \) is exogenously given and deterministic. This can be thought as the average duration of employment. We define the downside constraint as

\[
K = L_T = L_0 e^{gT}.
\]

Pension sponsors optimally manage assets and contribute to pension asset such that the terminal value of pension asset is greater than \( K \).
2.2 Investment Sets

Pension has two available assets, risky stock and risk-free money market account. Let $r$ be the risk-free rate. We assume that $r$ is constant. Stock price follows

$$dS_t = \mu S_t dt + \sigma S_t dZ_t,$$  \hspace{1cm} (3)

where $\mu$ is expected return of stock, $\sigma$ is volatility parameter, and $Z$ is a standard Brownian motion. Hence, in our model there is only one shock and one risky asset, and the market is complete. This implies that we have the unique pricing kernel or stochastic discount factor. We have the following dynamics of pricing kernel:

$$\frac{dM_t}{M_t} = -r dt - \eta dZ_t,$$  \hspace{1cm} (4)

where $\eta = \frac{\mu - r}{\sigma}$ is the market price of risk. We normalize the pricing kernel by $M_0 = 1$ without loss of generality. Now, pension’s asset value follows

$$dW_t = [(r + \pi_t (\mu - r)) W_t + Y_t] dt + \pi_t \sigma W_t dZ_t,$$  \hspace{1cm} (5)

where $\pi$ is a fraction of asset invested in the risky stock, and $Y_t$ is the contribution to pension asset.

2.3 Pension Sponsor’s Problem

Pension sponsor’s objective is

$$\max_{\pi, Y} \mathbb{E} \left[ e^{-\beta T} u(W_T) - \int_0^T e^{-\beta t} \phi(Y_t) \, dt \right]$$  \hspace{1cm} (6)

s.t. $W_T \geq K$.

where $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ and $\phi(x) = k x^\theta$. The first objective is standard power utility with relative risk aversion $\gamma$ over final asset of pension. The utility over final asset can be justified since pension sponsor receives a surplus of pension plan. We model the cost of contribution as a separable disutility function, and the second objective is disutility from contribution. A parameter $\theta$ will capture a desire to smooth contribution. To have convex disutility, we assume that $\theta > 1$. A parameter $k$ captures the relative importance of disutility to utility over final pension asset value. For example, financially healthy pension sponsor would have low $k$ since the impact of contribution is small and they prefer to contribute to end up with higher utility. Finally, $\beta$ is the subjective discount rate of pension sponsor. We can separate this problem into two problems:
1. A pension sponsor maximizes the expected utility over final asset given the upper bound of present value of terminal asset value, $W^u_0$ and a downside constraint:

$$\max_{\pi_u} \mathbb{E}\left[ e^{\beta T} u(W^u_T) \right]$$

subject to

$$W^u_0 \geq \mathbb{E}_Q[ e^{-r T} W^u_T ]$$

$$W^u_T \geq K,$$

where $\mathbb{E}_Q[\cdot]$ is an expectation under the equivalent martingale measure $Q$.

2. A pension sponsor minimizes the expected disutility over contribution stream given the lower bound of present value of contribution, $X_0$:

$$\min_Y \mathbb{E}\left[ \int_0^T e^{-\beta t} \phi(Y_t) \, dt \right]$$

subject to

$$X_0 \leq \mathbb{E}_Q\left[ \int_0^T e^{-rt} Y_t \, dt \right].$$

3. A budget constraint is

$$W_0 + X_0 = W^u_0.$$  \hspace{1cm} (9)

Whenever $K > 0$, we call a constrained case. When $K = 0$, there is no downside constraint and it serves as a benchmark case.

### 3 Constrained Case

#### 3.1 Utility Maximization Problem

First, we solve a standard asset allocation problem. However, now the initial endowment is $W^u_0 \geq W_0$ and the difference $W^u_0 - W_0$ is the present value of contribution. That is, a pension expects a stream of contribution in the future and thus, at time zero behaves as if taking a leverage by the present value of contribution. The optimal amount of leverage will be determined later by taking into account both utility of final asset and disutility of contribution. A downside constraint imposes a condition on the initial endowment $W^u_0 \geq Ke^{-rT}$. If this condition is not satisfied, there is no solution that guarantees the liability for sure. This implies that the present value of contribution $W^u_0 - W_0$ should be at least as large as $\max(K e^{-rT} - W_0, 0)$. For example, initially underfunded plans should have the present value of contribution greater than the initial
shortfall, \(Ke^{-rT} - W_0\). The dynamic budget constraint is

\[
dW_t^u = (r + \pi^u_t(\mu - r))W_t^u dt + \pi^u_t W_t^{u}\sigma dZ_t.
\] (10)

Note that there’s no contribution process since it’s already reflected in the increased the initial endowment. We can consider a put based strategy in which some of initial endowment is managed using the optimal strategy without a downside constraint, and the rest of initial endowment is used to construct a put option portfolio. Let \(J(W_0^u)\) be the value function of the first problem, (7), and \(I_u(\cdot)\) be the inverse function of \(u'(\cdot)\). Define \(\xi_t = M_t e^{\beta t}\), and the following functions for any \(0 < y < \infty\):

\[
\mathcal{W}_u(y) = \mathbb{E}^Q \left[ e^{-rT} I_u(y\xi_T) \right] + \mathbb{E}^Q \left[ e^{-rT} (K - I_u(y\xi_T))^+ \right],
\] (11)

where \((x)^+ = \max(x, 0)\). This function calculates the present value of insured portfolio when the terminal asset value is random variable \(I_u(y\xi_T)\) and the liability level is \(K\). The terminal asset value is set such that the marginal utility is proportional to the marginal rate of substitution of the economy at the terminal date. The parameter \(y\) is a shadow price, i.e. a marginal increase in utility when a pension’s initial endowment \(W_0^u\) is marginally increased. This interpretation will be more clear later. Proposition 1 explicitly computes this function.

**Proposition 1** The function \(\mathcal{W}_u(y)\) is given by

\[
\mathcal{W}_u(y) = y^{-\frac{1}{\gamma}} e^{-\alpha_u T N(\delta_1(y, T))} + Ke^{-rT} N(-\delta_2(y, T)),
\] (12)

where \(\alpha_u = \frac{\beta}{\gamma} + \left(1 - \frac{1}{\gamma}\right) \left(r + \frac{\eta^2}{2\gamma}\right)\). \(\delta_1\) and \(\delta_2\) can be found in Appendix. Also, the first derivative of \(\mathcal{W}_u(y)\) is given by

\[
\mathcal{W}_u'(y) = -\frac{1}{\gamma} y^{-\frac{1}{\gamma}-1} e^{-\alpha_u T N(\delta_1(y, T))} < 0.
\] (13)

Since the insured portfolio consists of the underlying asset plus the put option, the expression for \(\mathcal{W}_u(y)\) looks familiar with Black-Scholes option pricing formula. The first part is the present value of terminal asset value times the probability that the pension satisfies the downside constraint at the maturity under the forward measure. Note that the the present value is discounted with a rate \(\alpha_u\) which is a weighted average of pension sponsor’s subjective discount rate and risk-adjusted expected return. Suppose that the pension sponsor is extremely risk averse, the terminal asset value is set flat regardless of state of economy. Thus, the terminal asset can be discounted with \(r\). The second part is the present value of liability times the probability that
the put option is in-the-money under the risk-neutral measure. Since we have a concave utility function, a higher shadow price implies a lower cost of constructing the insured portfolio. Thus, we can see that $W_u(y)$ is decreasing, which implies that $W_u(y)$ is invertible. Let $\mathcal{Y}_u$ denote the inverse of this function. For a fixed initial endowment, $W^u_0 \geq Ke^{-rT}$, we introduce the following random variable

$$W^u_T = I_u (\mathcal{Y}_u(W^u_0) \xi_T) + (K - I_u (\mathcal{Y}_u(W^u_0) \xi_T))^+.$$  

(14)

The following Theorem 2 states that the constructed terminal asset value is optimal for the problem (7).

**Theorem 2** For any $W^u_0 \geq Ke^{-rT}$, $W^u_T$ is optimal for the problem (7), and the optimal portfolio weight is given by

$$\pi^u_t = \frac{\eta}{\gamma \sigma} (1 - \varphi_t),$$  

(15)

where $\varphi_t = \frac{Ke^{-r(T-t)}}{W^u_t} N (-\delta_2 (y_t, T-t)) < 1$ and $y_t = \mathcal{Y}_u(W^u_0) \xi_t$. The intuition is very clear. Since the insured portfolio is constructed by combining the underlying asset and its put option, the downside constraint is always satisfied not only at the terminal date, but also along the horizon. The thing is how much a pension sponsor should hold the underlying asset to achieve the maximum, i.e. what is the optimal shadow price, $y$? Theorem 2 tells that the optimal shadow price should be $\mathcal{Y}_u(W^u_0)$ such that the cost of constructing the insured portfolio is exactly the initial endowment. The optimal portfolio weight is the weighted average of mean-variance efficient portfolio and zero investment in equity. The weight on mean-variance efficient portfolio is $1 - \varphi_t$. The parameter $\varphi_t$ measures how far away the current asset value is from the present value of liability. The closer the asset is to the present value of liability, the less fraction of asset is invested in equity.

Define the following function $G(y)$ for $0 < y < \infty$:

$$G(y) = \mathbb{E} \left[ e^{-\beta T u} (I_u(y\xi_T) + (K - I_u(y\xi_T))^+) \right].$$  

(16)

This computes the expected utility when the terminal asset value is set to $I_u(y\xi_T) + (K - I_u(y\xi_T))^+$ as a function of $y$. At the optimal solution, we choose $y = \mathcal{Y}_u(W^u_0)$ so that we can obtain the value function $J(W^u_0)$ by substituting $y$ in $G(y)$ with $\mathcal{Y}_u(W^u_0)$. Following Proposition 3 states that the first derivative of the value function, i.e. the shadow price is indeed $\mathcal{Y}_u(W^u_0)$.  

9
Proposition 3 The function $G(y)$ is given by

$$G(y) = \frac{y^{1-\frac{1}{\gamma}} e^{-\alpha u T} N (\delta_1(y, T))}{1 - \gamma} + e^{-\beta T} \frac{K^{1-\gamma}}{1 - \gamma} N (-\delta_3(y, T)),$$  

(17)

where $\delta_3$ can be found in Appendix. Also, $G(y)$ satisfies

$$G'(y) = y W_u'(y)$$  

(18)

$$J(W_0^n) = G(Y_u(W_0^n))$$  

(19)

$$J'(W_0^n) = Y_u(W_0^n).$$  

(20)

3.2 Disutility Minimization Problem

The second problem is to decide how the present value of contribution should be contributed along the horizon to minimize a disutility. The problem can be put in a different perspective that the pension sponsor has the initial endowment $X_0$ in its internal liquidity to hedge the future contribution, i.e. the pension sponsor uses this internal resource to contribute to the plan. The assumption is that the pension sponsor only considers self-financing strategy, i.e. there’s no inflow or outflow to this fund. Let $X_t$ be the time $t$ value of this fund. Then, the dynamic budget constraint is

$$dX_t = \left[ (r + \pi_t^\phi (\mu - r)) X_t - Y_t \right] dt + \pi_t^\phi \sigma X_t dZ_t,$$  

(21)

where $\pi_t^\phi$ is the portfolio weight used to manage this fund. Now, the problem becomes exactly same with an asset allocation problem with intermediate consumption and no bequest utility. However, there are two important differences. First, contribution (consumption) doesn’t increase pension sponsor’s utility, but increase disutility. Thus, the pension sponsor’s objective is to minimize this disutility. Second, the static budget constraint is that the present value of contribution should be greater than the initial endowment. The optimal solution, of course, equates two values, in particular, $X_t$ is non-negative and vanishes at the terminal date, $X_T = 0$.

The problem is stated in (8). Let $L(X_0)$ be the value function of the second problem, and $I_\phi(\cdot)$ be the inverse function of $\phi'(\cdot)$. Then, we define the following function for any $0 < y < \infty$:

$$W_\phi(y) = \mathbb{E}^Q \left[ \int_0^T e^{-rt} I_\phi (y\xi_t) dt \right].$$  

(22)

The function $W_\phi(y)$ computes the present value of contribution stream from time zero to the terminal date when an intermediate contribution is set to be $I_\phi(y\xi_t)$, i.e. the marginal disutility.
is proportional to the marginal rate of substitution of the economy for each time. As the first problem, the parameter $y$ is a shadow price, i.e. a marginal increase in disutility when the lower bound for the present value of contribution $X_0$ is marginally increased. This interpretation will be more clear later. Proposition 4 explicitly computes this function.

**Proposition 4** The function $\mathcal{W}_\phi(y)$ is given by

$$
\mathcal{W}_\phi(y) = \left(\frac{y}{k}\right)^{\frac{1}{\theta}} \frac{1 - e^{-\alpha_\phi T}}{\alpha_\phi},
$$

(23)

where $\alpha_\phi = \frac{\theta}{\theta - 1} \left( r - \frac{\eta^2}{2(\theta - 1)} \right) - \frac{\beta}{\theta - 1}$. Also, the first derivative of $\mathcal{W}_\phi(y)$ is given by

$$
\mathcal{W}'_\phi(y) = \frac{1}{y(\theta - 1)} \mathcal{W}_\phi(y) > 0.
$$

(24)

The present value of contribution stream has a form of annuity with a factor $\alpha_\phi$, which is a weighted average of pension sponsor’s subjective discount rate and risk adjusted expected return. An incentive to smooth contribution over time (high $\theta$) implies that contribution stream can be discounted with a rate $r$. Since we have a convex disutility function, a higher shadow price implies a higher present value of contribution. Thus, we can see that $\mathcal{W}_\phi(y)$ is increasing, which implies that $\mathcal{W}_\phi(y)$ is invertible. Let us denote $\mathcal{Y}_\phi$ be the inverse of the function $\mathcal{W}_\phi$. For a fixed number $X_0 > 0$, we introduce the contribution process

$$
Y_t = I_\phi \left( \mathcal{Y}_\phi \left( X_0 \right) \xi_t \right).
$$

(25)

The following Theorem 5 states that the constructed intermediate contribution policy is optimal for the problem (8).

**Theorem 5** For any $X_0 > 0$, $Y_t$ constructed above is optimal for the problem (8), and the optimal hedging policy is

$$
\pi^\phi = -\frac{\eta}{(\theta - 1)\sigma}.
$$

(26)

By setting the marginal disutility of contribution to be proportional to the marginal rate of substitution of the economy, the minimum disutility can be achieved. How proportional it should be is determined such that the present value of contribution stream is exactly the initial endowment $X_0$. The optimal hedging policy is to short equities, since intermediate contribution is increasing in marginal rate of substitution or decreasing in stock return. Higher desire to smooth contribution (higher $\theta$) implies that smaller shorting in equity.
Define the following function $C(y)$ for $0 < y < \infty$:

$$
C(y) = \mathbb{E} \left[ \int_0^T e^{-\beta t} \phi \left( I_\phi(y, \xi_t) \right) dt \right].
$$

(27)

This computes the expected disutility when intermediate contribution is set to $I_\phi(y, \xi_t)$ as a function of $y$. At the optimal solution, we choose $y = Y_\phi(X_0)$ so that we can obtain the value function $L(X_0)$ by substituting $y$ in $C(y)$ with $Y_\phi(X_0)$. Following Proposition 6 states that the first derivative of $L(X_0)$ is indeed $Y_\phi(X_0)$.

**Proposition 6** The function $C(y)$ is given by

$$
C(y) = \frac{k}{\theta} \left( \frac{y}{k} \right)^{\theta-1} \frac{1 - e^{-\alpha T}}{\alpha},
$$

and satisfies

$$
C'(y) = y W_\phi'(y)
$$

(29)

$$
L(X_0) = C(Y_\phi(X_0))
$$

(30)

$$
L'(X_0) = Y_\phi(X_0)
$$

(31)

### 3.3 Optimality of Separation

We now show that the proper present value of contribution $X_0$ leads that separately solved solutions are indeed solutions to the original problem. Pension sponsor behaves as if taking a leverage, $W_0^u = W_0 + X_0$ at time zero. With $W_0^u$, the agent solves the utility maximization problem with a downside constraint. The agent solves the disutility minimization problem to meet $X_0$ through contribution. It will be shown that how $X_0$ is decided to achieve the optimality of the original problem.

**Theorem 7** Consider an arbitrary portfolio policy and contribution pair $(\tilde{\pi}, \tilde{Y})$ satisfying a downside constraint. Then, there exists a pair $(\pi, Y)$ dominating $(\tilde{\pi}, \tilde{Y})$. In particular, the value function of the original problem $V(W_0)$ is

$$
V(W_0) = \max_{X_0} J(W_0 + X_0) - L(X_0) = \max_{W_u(y_u) - W_\phi(y_\phi) = W_0} G(y_u) - C(y_\phi).
$$

(32)

The intuition is following. For an arbitrary portfolio and contribution policy pair, we can take the present value of contribution stream, $X_0 = \mathbb{E}^Q \left[ \int_0^T e^{-rt} \tilde{Y}_t dt \right]$. Then, for the given $X_0$, $\tilde{\pi}$ becomes a feasible strategy to the problem (7), and $\tilde{Y}$ becomes a feasible strategy to the
problem (8). We can find the optimal solutions to each problem and they will (weakly) dominate \((\tilde{\pi}, \tilde{Y})\). Thus, finding the optimal solution to the original problem (6) can be translated to finding the optimal \(X_0\) maximizing the difference between two value functions of (7) and (8), \(J(W_0 + X_0) - L(X_0)\). Suppose that (32) has an interior solution. This implies that

\[ J'(W_0 + X_0) = L'(X_0). \]  

(33)

This condition states that at the optimal solution, the marginal increase in value function of utility maximization problem should be identical with the marginal increase in value function of disutility minimization problem. Thus, we can interpret LHS as the marginal benefit of increasing the present value of contribution, and RHS as the marginal cost of increasing the present value of contribution. Recall that the shadow prices of both problems are solving the static budget constraints. Hence, we have \(y = \mathcal{Y}_u(W_0 + X_0) = \mathcal{Y}_\phi(X_0)\), which is determined by

\[ \mathcal{W}_u(y) - \mathcal{W}_\phi(y) = W_0. \]  

(34)

Define the following function for \(0 < y < \infty\):

\[ \mathcal{W}(y) = \mathcal{W}_u(y) - \mathcal{W}_\phi(y). \]  

(35)

Following Proposition shows that there exists a unique \(y\) solving \(\mathcal{W}(y) = W_0\), and thus we obtain the optimal solution to the original problem.

**Proposition 8** The function \(\mathcal{W}(y)\) is decreasing in \(y\) and

\[ \lim_{y \to 0} \mathcal{W}(y) = \infty \]  

(36)

\[ \lim_{y \to \infty} \mathcal{W}(y) = -\infty. \]  

(37)

Hence, there exists a unique \(y\) satisfying \(\mathcal{W}(y) = W_0\).

Suppose that we find \(y\) solving (34). Then, the time \(t\) pension plan’s asset can be expressed as

\[ W_t = W_t^u - X_t. \]  

(38)

The above equation implies that the current pension plan’s asset plus the sponsor’s internal fund for hedging future contribution equal to the present value of the terminal pension plan’s asset value, since \(X_T = 0\), i.e. \(W_T = W_T^u\). Following Proposition describes the optimal portfolio weight and contribution rate to the original problem.
Proposition 9 The optimal portfolio weight is given by

$$\pi_t = \pi^u_t \rho_t + \pi^\phi_t (1 - \rho_t),$$

and the optimal contribution rate is given by

$$\frac{Y_t}{W_t} = (\rho_t - 1) \frac{\alpha_\phi}{1 - e^{-\alpha_\phi(T-t)}},$$

where $$\rho_t = \frac{W_t^u}{W_t} = 1 + \frac{x_t}{W_t}.$$ 

The optimal portfolio weight is the weighted average of two weights, $$\pi^u_t$$ and $$\pi^\phi_t$$. The weight is the ratio of the present value of terminal plan’s asset to current plan’s asset, or one plus the ratio of internal resource for hedging of contribution over current pension asset value. When the weight $$\rho$$ is close to one, then the state of economy is good and expected contribution is small. Also, $$\pi^u_t$$ becomes the mean-variance efficient portfolio since it is more likely that a downside constraint is not binding. Thus, the optimal portfolio weight, $$\pi_t$$ is close to the mean-variance efficient portfolio. An increase in $$\rho$$ indicates that the pension sponsor holds large internal resource to hedge large contemporaneous and future contributions, which are induced by bad state of economy (high $$\xi$$). That is, the pension sponsor expects that large contribution is coming in the future, and increases equity weight, which is hedged by future contribution. At the same time, the present value of terminal pension asset, $$W_t^u$$ approaches to the present value of liability, which results the optimal equity weight of the first problem, $$\pi^u_t$$ to decrease as in (15). If the latter effect dominates the first one, then a risk management behavior can be observed, i.e. a decrease in equity weight as the economy gets worse. On the other hand, if the first effect dominates, we can see a risk taking behavior. However, note that this risk taking incentive is not induced by a moral hazard problem, but by a commitment to contribute in the future.

4 Benchmark Case

Now, we consider a benchmark case, which serve as a prediction of pension sponsor’s reaction to a situation in which there is a government insurance so that a downside constraint doesn’t play a role. Pension sponsor’s objective becomes

$$\max_{\pi,Y} \mathbb{E} \left[ e^{-\beta T} u(W_T) - \int_0^T e^{-\beta t} \phi(Y_t) dt \right],$$

where $$\mathbb{E}$$ denotes the expectation, $$\pi$$ is the portfolio weight, $$Y$$ is the contribution rate, and $$u$$ is the utility function. The objective function represents the expected utility of the terminal wealth discounted by the cost of risk. The risk aversion parameter $$\beta$$ determines the trade-off between expected return and risk, with a higher $$\beta$$ indicating a higher level of risk aversion. The function $$\phi(Y_t)$$ represents the cost of risk at time $$t$$, which is a function of the contribution rate $$Y_t$$. The integral term $$\int_0^T e^{-\beta t} \phi(Y_t) dt$$ represents the discounted expected cost of risk over the planning horizon. This benchmark case provides a framework to analyze the behavior of pension sponsors under different economic conditions and risk scenarios.
Everything we derive for a case with a downside constraint goes through, except the first problem, since a downside constraint disappears. Now, let $W_u^{BC}(y)$ be the counterpart of $W_u(y)$ in a case with downside constraint, i.e. the present value of terminal asset, which is set to $I_u(y \xi_T)$:

$$W_u^{BC}(y) = \mathbb{E}^Q \left[ e^{-rT} I_u(y \xi_T) \right]. \quad (42)$$

Following Proposition computes this function and compare with $W_u(y)$.

**Proposition 10** The function $W_u^{BC}(y)$ is given by

$$W_u^{BC}(y) = y^{\frac{1}{\gamma}} e^{-\alpha u T}. \quad (43)$$

Also, the first derivative is given by

$$W_u^{BC}(y)' = -\frac{1}{\gamma} y^{\frac{1}{\gamma} - 1} e^{-\alpha u T} < 0. \quad (44)$$

For a given $y$, we have

$$W_u^{BC}(y) < W_u(y). \quad (45)$$

Without a downside constraint, the present value of terminal wealth, which is set such that the marginal utility is proportional to marginal rate of substitution is smaller than that with a downside constraint. The intuition behind is that to achieve the same level of marginal utility a benchmark case requires smaller initial wealth since an insured portfolio doesn’t have to be constructed. Let $\gamma_u^{BC}(W_u^0)$ be the inverse of $W_u^{BC}(y)$. For a fixed initial endowment, $W_u^0$, we introduce the random variable

$$W_T = I_u(\gamma_u^{BC}(W_u^0) \xi_T). \quad (46)$$

Theorem 11 proves that the constructed terminal asset is optimal for the utility maximizing problem.

**Theorem 11** For any $W_u^0$, $W_T$ is optimal for the utility maximization problem, and the optimal portfolio weight is given by

$$\pi_u^{BC} = \frac{\eta}{\gamma \sigma}. \quad (47)$$

As we expect, the optimal portfolio weight is standard mean-variance efficient portfolio. Define the following function $G^{BC}(y)$ for $0 < y < \infty$:

$$G^{BC}(y) = \mathbb{E} \left[ e^{-\beta T} u(I_u(y \xi_T)) \right]. \quad (48)$$

Following Proposition relates $\gamma_u^{BC}(W_u^0)$ to the shadow price of the problem.
**Proposition 12**  The function $G^{BC}(y)$ is given by

$$G^{BC}(y) = \frac{y^{1-\gamma}}{1-\gamma} e^{-\alpha u T},$$  \hspace{1cm} (49)$$

and satisfies

$$G^{BC}(y)' = yW^{BC}_u(y)'$$  \hspace{1cm} (50)$$

$$J^{BC}(W^u_0) = G^{BC}(Y^{BC}_u(W^u_0))$$  \hspace{1cm} (51)$$

$$J^{BC}(W^u_0)' = Y^{BC}_u(W^u_0).$$  \hspace{1cm} (52)$$

Now, Theorem 7, Proposition 8 and 9 can be stated for the benchmark case by substituting corresponding counterparts with $J^{BC}(W^u_0)$, $G^{BC}(y)$, $W^{BC}_u(y)$, $Y^{BC}_u(W^u_0)$, and $\pi^{u}_{BC}$.

## 5  Quantitative Analysis

We now turn to quantitative analysis of the model. For a baseline case, we use 10-year for the investment horizon of pension sponsor. According to Bureau of Labor Statistics, as of 2014 the median years of tenure with current employer for workers with age over 65 years is 10.3-year. Also, we use 0.4 for the market price of risk, 20% as the volatility of equity, 2% as the short rate, and 1% for the sponsor’s subjective discount rate. These numbers are standard assumptions in the literature. The expected excess return of equity is $\sigma \eta = 8\%$. We use $\gamma = 5$, which implies the equity weight of mean-variance efficient portfolio is $\frac{\eta}{\gamma \sigma} = 40\%$. For the disutility function, we use $k = 100$ and $\theta = 2$. The quadratic disutility assumption is common in the investment literature, in which a firm is assumed to be risk-neutral and faces quadratic costs of investment adjustment.\textsuperscript{9} Finally, we use two values of initial funding ratio, $\lambda_0 = \frac{W_0}{Ke^{\alpha T}} = 80\%$ or 120%.

We will vary preference parameters, ($\gamma, k, \theta$), price of risk, and the initial funding ratio to see the impacts on the optimal present value of contribution, portfolio and contribution policy. Table 1 summarizes all the key variables and parameters in the model.

### 5.1  Present Value of Contribution

Figure 1 plots the determination of $X_0$ by equating shadow prices of first and second problem. Panel A is when a pension is initially underfunded, $\lambda_0 = 80\%$, and Panel B is when overfunded, $\lambda_0 = 120\%$. We plot a benchmark case also. Since we assume a quadratic disutility

\textsuperscript{9} See Gould (1968); more recently Bolton, Chen, and Wang (2011); among others.
function, shadow price of second problem is linear in present value of contribution, i.e. as the present value of contribution increases, the marginal cost linearly increases. Shadow price of first problem is decreasing in present value of contribution. Also, shadow price of first problem for a constraint case is always above that for a benchmark case since a constrained case enjoys only an upside of a downside constraint. When the present value of contribution is marginally increased, the marginal benefit is an increase in expected utility, which is concave, and thus marginal benefit curve is decreasing in $X_0$. We can see that the present value of contribution is $X_0 = 3.68\%$ of initial asset and shadow price is $y = 0.18$ for a benchmark case. For an underfunded pension, present value of contribution is $X_0 = 25.10\%$ of initial asset and shadow price is $y = 1.22$. Compared to a benchmark case, initially underfunded status makes a pension contribute more to save a pension at the maturity. For an overfunded pension, present value of contribution is $X_0 = 4.15\%$ of initial asset and shadow price is $y = 0.20$. This implies that only $0.47\%$ of additional contribution is required to guarantee the liability for an initially overfunded pension.

Figure 2 plots the the cost of constructing the put-based strategy for the constrained pension’s first problem. Again, Panel A is when a pension is initially underfunded, and Panel B is when overfunded. We can see that for a initially underfunded pension, without contribution there’s no solution of put-based strategy. That is, the present value of liability at time zero is greater than the initial asset value so that a put-based strategy can not be constructed. However, with the optimal present value of contribution, $X_0 = 25.10\%$ the initial endowment of first problem is increased to $W_0^u = 125.10\%$ and there is the optimal put-based strategy whose cost of constructing is equal to the increased initial endowment. Effectively, allocation to the mean-variance efficient portfolio is $70.60\%$, and the rest of $W_0^u$, $54.50\%$ is used to replicate a put option underlied by $70.60\%$ of mean-variance efficient portfolio. On the other hand, for an overfunded pension, there’s a put-based solution even without contribution, which is $95.92\%$ in the mean-variance efficient portfolio, and $4.08\%$ for a put option. With contribution, allocation to the mean-variance efficient portfolio is increased to $101.21\%$, which is greater than the original initial asset value. This also decreases put option value to $2.94\%$, and the total initial asset is increased to $104.15\%$.

### 5.2 Portfolio Weights and Contribution Rate

Figure 3 plots portfolio weights in equity and contribution rate at time $t = 5$-year as a function of annualized equity return over last five years. We fix the initial pension asset and vary the
terminal liability, $K$. We set $K = 153\%$ for Panel A and B, and $K = 102\%$ for Panel C and D, so that the initial funding ratios are 80\% and 120\%. First, we can see that portfolio weights of a benchmark case are decreasing in past equity returns. Low equity returns over time zero to 5-year indicates that the state of economy is bad, i.e. marginal rate of substitution is high. The optimal contribution rule is to increase contribution in such state. A pension sponsor expects that future contribution will be made, and thus can take more risk by increasing equity weight. Say differently, when the state of economy is bad, future contribution can hedge positions in equity, and thus a pension sponsor can take more risks. As the state of economy gets better, equity weights of a benchmark case is approaching to the mean-variance efficient portfolio, which is $\frac{\mu}{\sigma} = 40\%$. We can see that equity weights of benchmark case are identical for initially underfunded and overfunded. This is obvious since how far away from the present value of liability doesn’t matter for a sponsor without a downside constraint.

Next, equity weights of a constrained case display an U-shaped pattern. Generally, when the state of economy is bad (so that underfunded), a pension tries to enhance a funding status by taking more risks. Higher risk taking is hedged by future contribution. On the other hand, when the state of economy is good (so that overfunded), the risk management incentive arises to avoid costly contribution. Thus, as the state of economy gets worse, a pension decreases portfolio weights. When a pension sponsor switches from risk management to risk taking depends on the initial funding status. For initially underfunded plans, risk taking incentives dominate risk management incentives. The intuition is that for same negative shocks to the economy, the impact is much greater for initially underfunded plans so that they contribute more contemporaneously and in the future, which enables higher risk taking.

By comparing the benchmark case and the constrained case, we can predict a situation in which a government insurance exists. In the benchmark case, the pension sponsor only has risk taking incentives, which are hedged by future contribution. Even if the pension ends up with underfunded, the government agency, such as PBGC will guarantee the liability. Thus, as the economy get worse the pension sponsor would increase the risk by gambling on high equity return. On the other hand, the pension sponsor without government insurance would avoid large contribution as much as it can by managing risk, i.e. decreasing equity weight. However, when the pension’s asset is severely deteriorated the pension sponsor will take more risk than the benchmark case since increased risk is hedged by larger contribution than the benchmark case in the future and if equity return is high, the pension asset can bounce back to funded level.

Panel B and D plot contribution rate, $Y_t/W_t$ as a function of annualized equity return over
time zero to 5-year. We can see again that contribution rates of benchmark case are decreasing in the state of economy and identical across initial funding status. A pension sponsor with a downside constraint behaves differently depending on the initial funding status. Initially underfunded pension sponsor contribute much more than a benchmark case for the same state of economy. The effect of negative shock to the economy is much greater to initially underfunded plans, and thus to satisfy the downside constraint much higher contribution should be made.

5.3 Effect of Initial Funding Ratio

In Figure 4, we vary the initial funding ratio by changing the terminal liability, \( K \), and find the optimal present value of contribution (Panel A). Also, based on that, we plot portfolio weights at time zero (Panel B), contribution rate at time zero (Panel C), and certainty equivalent of constrained case compared to a benchmark case (Panel D). We can see that a benchmark case has constant present value of contribution across funding ratio. Present value of contribution is \( X_0 = 3.68\% \). Also, portfolio weights is higher than the mean-variance efficient weight, \( \frac{n}{\gamma\sigma} = 40\% \) since future contribution hedges higher risky position. Contribution rate is 0.18% of the original initial asset. Note that these are just pictures at time zero so that a benchmark case is flat. However, as time passes, portfolio weight and contribution rate depend on the state of economy as we see in Figure 3.

Next, consider a constrained case. Present value of contribution is decreasing in initial funding ratio and approaching to a benchmark case. For a \( \lambda_0 = 70\% \) funded pension, \( X_0 = 42.86\% \) of the original initial asset should be made by contribution during entire horizon. In the first year, \( Y_0/W_0 = 2.09\% \) should be made. Put option value is also decreasing in initial funding ratio, and is greater than the present value of contribution for low funding ratio, and vice versa. For a \( \lambda_0 = 70\% \) funded pension, the put option value is 80%, and thus the pension sponsor should use 37.14% of original initial asset to construct the put option. For a 130% funded pension, the put option value is less than the present value of contribution, and thus the pension sponsor can use the rest of contribution to invest in the mean-variance efficient portfolio.

Portfolio weights are U-shaped, which implies that for initially underfunded pensions, taking more risk and hedging with future contribution is optimal. For overfunded pensions, risk management by decreasing equity risk is optimal. Contribution rate is decreasing in initial funding ratio. A 70% funded pension should contribute around 2% of original asset in the first year. Certainty equivalent is decreasing in initial funding ratio. For 70% funded pension, a pension sponsor with a downside constraint needs 32.55% more initial asset to have the same level of
value function with a benchmark case. This number can be interpreted as a present value of premium that a pension sponsor should pay to PBGC. The pension’s liability is guaranteed by PBGC, and in exchange for that, the pension sponsor should give up 32.55% of their asset.

5.4 Effect of Relative Importance of Disutility

Figure 5 plots the optimal present value of contribution (Panel A), put option value at time zero (Panel B), portfolio weights at time zero (Panel C), and certainty equivalent of constrained case compared to a benchmark case (Panel D) as we vary the relative importance of disutility, $k$. In Panel A, low $k$ implies that drawing contribution from pension sponsor’s internal resource does not result large disutility, and thus the pension sponsor can contribute large amount. However, as $k$ increases, contribution becomes more costly in a sense that disutility of contributing the same amount increases. Thus, the pension sponsor decreases $X_0$. The key difference between initially underfunded and overfunded pensions is whether there exists a put-based solution without contribution. As we see in Figure 2, overfunded pension has a put-based solution even without contribution. Hence, when $k$ is sufficiently large, the pension sponsor won’t contribute and just use the put-based solution without contribution. However, underfunded pension can not construct a put-based solution without contribution. Thus, we can see that even if $k$ is sufficiently large, underfunded pension take the present value of contribution, which is equal to time zero shortfall.

In Panel B, we plot the put option value at time zero. As the present value of contribution decreases, the pension should use more initial asset to construct the put option, which also results a decrease in allocation to the mean-variance efficient portfolio. This increases the put option value. For underfunded pensions, large $k$ induces the pension to take the present value of contribution as much as time zero shortfall. However, we can see that put option values are increasing. The reason is that a very small decrease in the present value of contribution can reduce a large pension’s upside benefit, i.e. utility from satisfying the liability.

In Panel C, we plot portfolio weights at time zero. The benchmark case decreases portfolio weights to the mean-variance efficient portfolio, 40% as $k$ increases, i.e. higher equity weight can not be hedged since contribution is decreasing. Overfunded pension takes lower equity weight than the benchmark since it holds a put option, which can be constructed by holding negative amount of underlying. Underfunded pension holds lowest equity weight for low $k$, but for high $k$ its equity weight is the highest. The reason is that to guarantee the liability, even for high $k$ contribution will be made so that the pension can take more risk. In Panel D, we can
see that high $k$ prevents pensions to contribute and makes a downside constraint more costly relative to the benchmark case.

### 5.5 Effect of Elasticity of Disutility

The elasticity of disutility, $\theta$ has impacts on the determination of the optimal present value of contribution. In Panel A of Figure 6, we vary $\theta$ from 1.5 to 3 and see the optimal $X_0$. We can see that contribution is increasing in $\theta$. This is counter-intuitive since high $\theta$ implies a high desire to smooth contribution. However, this argument only holds when the optimal shadow price is greater than $k$, i.e. marginal cost of contribution not scaled by $k$ is greater than one. This can be seen clearly through $W_\phi(y)$:

\[
\frac{\partial W_\phi(y)}{\partial \theta} = -W_\phi(y) \left[ \log \frac{\theta}{k} \left( \frac{\partial \phi}{\partial \theta} \right)^2 + \frac{\partial \alpha \phi \ln (1 + (1 - \alpha T) e^{-\alpha T})}{\alpha \phi} \right].
\] (53)

Since the elasticity of disutility only moves the marginal cost curve, i.e. $W_\phi(y)$, given $y$ whether an increase in $\theta$ moves the marginal cost curve upward or downward is our interest. If the term in the bracket is positive, the marginal cost curve moves upward and the optimal contribution decreases. The first part is the effect of contribution smoothing. When $y > k$, an increase in $\theta$ moves the marginal cost curve upward, and thus decreases the optimal contribution. However, with our parameter values it turns out that the optimal shadow price is less than $k$, which makes the first term in the bracket negative, i.e. downward move of marginal cost curve. The second part is the effect of $\theta$ on the annuity term, $(1 - e^{-\alpha T})/\alpha \phi$. We find that for low $\theta$ the second term in the bracket is positive and dominates the first term, and thus the marginal cost curve moves upward and the optimal contribution decreases. (Panel E) On the other hand, for high $\theta$, the opposite happens. (Panel F) The intuition is that the optimal contribution, $Y_t = I_\phi(y_\xi_t) = \left( \frac{y_\xi_t}{k} \right)^{\theta - 1}$ is convex in $\xi_t$ for $1 < \theta < 2$ and an increase in $\theta$ makes contribution more volatile, and in turn reduces the annuity. For $\theta > 2$, the optimal contribution is concave in $\xi_t$ and an increase in $\theta$ makes stable contribution, which increases the annuity.

The rest of Panels are easy to interpret. Since the optimal $X_0$ is increasing in the elasticity of disutility, put option value is decreasing, i.e. underlying asset of put option is increasing. Portfolio weights are increasing. However, underfunded pensions should take time zero shortfall even at low $\theta$, which makes decreasing equity weight for low $\theta$. As high contribution is available, the certainty equivalent is decreasing.
5.6 Effect of Risk Aversion and Price of Risk

Now, we investigate the effect of risk aversion and price of risk. High risk aversion implies that the mean-variance efficient portfolio holds less equity. (Panel C) The risk of uninsured portfolio is reduced, and thus put option value decreases (Panel B) and less contribution is required. (Panel A) For underfunded pension, no matter what risk aversion is, at least contribution of time zero shortfall 25% is required. This makes that a pension holds higher equity weight than the benchmark or overfunded pension when a risk aversion is high. Also, certainty equivalent is decreasing in risk aversion for overfunded pension, but is increasing for large risk aversion for underfunded pension due to the minimum contribution to guarantee the liability.

A change in price of risk moves both marginal benefit ($W_u(y)$) and cost ($W_o(y)$) curves. The marginal benefit curve can be moved downward since same level of terminal asset can be financed with low initial asset using high expected equity return. At the same time, high volatility of mean-variance efficient portfolio increases the put option value, and thus moves the marginal benefit curve upward. High price of risk makes hedging of contribution exposed to large risks, since it involves shorting of equity. This induces for a given contribution policy the present value of contribution to be increased, i.e. downward move of marginal cost curve. We see in Panel A of Figure 8 that the net effect is an increase in $X_0$. In Panel B, for underfunded pension an increase in contribution reduces the put option value, which implies that the marginal benefit curve is moved downward. However, for overfunded pension the marginal benefit curve is moved upward, which indicates that the put option value is increased. Equity weight of underfunded pension can be higher than the benchmark and overfunded pension since it takes higher contribution, which hedge high equity weights. Overfunded pension’s contribution is similar to the benchmark, but it holds the put option so that equity weight is lower than the benchmark. With better investment opportunity, guaranteeing the liability is more costly, since it draws higher contribution than the benchmark.

6 Conclusion

We develop a separation approach to analyze a pension sponsor’s intertemporal contribution and portfolio policy when a downside constraint at the terminal date is present. The problem can be cast in two separate shadow price problems. At the optimal solution, two shadow prices are identical. We show that while guaranteeing the liability both risk management and risk taking behaviors emerge. When pension’s asset decreases, pension sponsors decrease equity weight
to avoid costly contribution first, and then increase equity weight which is hedged by large
collection in the future. Risk taking behavior is not induced by moral hazard problem, but by
commitment to contribute in the future. We hope to extend our analysis to include time-varying
expected returns, and stochastic liability in future research.
Appendix

A Proofs

Proof of Proposition 1. By Girsanov’s Theorem, there exists a unique equivalent measure $Q$ in which all traded assets earn the risk-free rate, and under $Q$ measure the following stochastic process is a standard Brownian motion.

$$dZ_t^Q = dZ_t + \eta dt. \quad (A.1)$$

To compute $W_u(y)$, we can derive the dynamics of $\xi_t = M_te^{\beta t}$ under $Q$ measure.

$$\frac{d\xi_t}{\xi_t} = (\beta - r)dt - \eta dZ_t$$
$$= (\beta - r + \frac{\eta^2}{2})dt - \eta dZ_t^Q. \quad (A.2)$$

The random variable $I_u(y\xi_T)$ can be expressed as

$$I_u(y\xi_T) = y^{-\frac{1}{2}} \exp \left( -\frac{T}{\gamma} (\beta - r + \frac{1}{2}\eta^2) + \frac{\eta}{\gamma} (Z_T^Q - Z_0^Q) \right), \quad (A.3)$$
given that $\xi_0 = 1$. Let $A$ be the event in which $K > I_u(y\xi_T)$. The event $A$ is equivalent to

$$x < -\delta_2(y, T), \quad (A.4)$$
where $x$ is a standard normal random variable, and $\delta_2(y, T)$ is given by

$$\delta_2(y, T) = \frac{\log \frac{y^{-\frac{1}{2}}}{\sqrt{\gamma}} + \frac{T}{\gamma} (r - \beta - \frac{\eta^2}{2})}{\eta \sqrt{T}}, \quad (A.5)$$
since $Z_T^Q - Z_0^Q$ is normally distributed with zero mean and variance of $T$. Then, $W_u(y)$ can be expressed as

$$W_u(y) = E^Q \left[ e^{-rT} I_u(y\xi_T)(1 - 1(A)) \right] + Ke^{-rT} N(-\delta_2(y, T)), \quad (A.6)$$
where $1(A)$ is an indicator function of event $A$ and $N(\cdot)$ is a cumulative distribution function of standard normal random variable. The first part can be easily computed:

$$E^Q \left[ e^{-rT} I_u(y\xi_T)(1 - 1(A)) \right] = \exp \left( - \left( r + \frac{1}{\gamma} (\beta - r + \frac{1}{2}\eta^2) \right) T \right) y^{-\frac{1}{2}}$$
$$\int_{-\delta_2(y, T)}^\infty \exp \left( \frac{\eta \sqrt{T}}{\gamma} x \right) n(x) dx$$
$$= y^{-\frac{1}{2}} e^{-\alpha_u T} \int_{-\delta_2(y, T)}^\infty n \left( x - \frac{\eta \sqrt{T}}{\gamma} \right) dx$$
$$= y^{-\frac{1}{2}} e^{-\alpha_u T} N(\delta_1(y, T)), \quad (A.7)$$
where $n(\cdot)$ is a probability distribution function of standard normal random variable, $\alpha_u$ and $\delta_1(y, T)$ are given by

$$\alpha_u = \frac{\beta}{\gamma} + \left( 1 - \frac{1}{\gamma} \right) \left( r + \frac{\eta^2}{2\gamma} \right) \quad (A.8)$$
$$\delta_1(y, T) = \delta_2(y, T) + \frac{\eta \sqrt{T}}{\gamma}. \quad (A.9)$$

Now, the first derivative of $W_u(y)$ can be computed as

$$W_u'(y) = -\frac{1}{\gamma} y^{-\frac{1}{2} - 1} e^{-\alpha_u T} N(\delta_1(y, T)) + y^{-\frac{1}{2}} e^{-\alpha_u T} n(\delta_1(y, T)) \frac{\partial \delta_1(y, T)}{\partial y}$$
$$- Ke^{-rT} n(-\delta_2(y, T)) \frac{\partial \delta_2(y, T)}{\partial y}. \quad (A.10)$$
Note that \( \frac{\partial \delta_1(y,T)}{\partial y} = \frac{\partial \delta_2(y,T)}{\partial y} \) and

\[
y^{-\frac{1}{2}} e^{-\alpha_u T} n(\delta_1(y,T)) = y^{-\frac{1}{2}} e^{-\alpha_u T} n \left( \delta_2(y,T) + \frac{\eta \sqrt{T}}{\gamma} \right) \tag{A.11}
\]

\[
y^{-\frac{1}{2}} \exp \left( -\alpha_u T - \frac{\eta \sqrt{T}}{\gamma} \delta_2(y,T) - \frac{\eta^2 T}{2\gamma^2} \right) n(\delta_2(y,T)) = Ke^{-rT} n(-\delta_2(y,T)). \tag{A.12}
\]

Hence, last two terms cancel out.

**Proof of Theorem 2.** Consider any other random variable \( \hat{W}^u_T \geq K \), which is feasible by a self financing trading strategy and the initial endowment \( \hat{W}^u_0 \). This implies that

\[
\hat{W}^u_0 \geq \mathbb{E}^Q \left[ e^{-rT \hat{W}^u_T} \right]. \tag{A.13}
\]

Then, we want to show that

\[
\mathbb{E} \left[ e^{-\beta T} u (W^u_T) \right] \geq \mathbb{E} \left[ e^{-\beta T} u \left( \hat{W}^u_T \right) \right]. \tag{A.14}
\]

Since \( u \) is a concave utility, we have

\[
u(\hat{W}^u_T) - u \left( \hat{W}^u_T \right) \geq u' \left( \hat{W}^u_T \right) \left( W^u_T - \hat{W}^u_T \right). \tag{A.15}
\]

We can compute \( u' (W^u_T) \):

\[
u' \left( W^u_T \right) = u' \left( \max (I_u (\mathcal{Y}_u (W^u_0) \xi_T), K) \right) \tag{A.16}
\]

\[
= \min (\mathcal{Y}_u (W^u_0) \xi_T, u'(K)) \tag{A.17}
\]

\[
= \mathcal{Y}_u (W^u_0) \xi_T - (\mathcal{Y}_u (W^u_0) \xi_T - u'(K))^+. \tag{A.18}
\]

Substitute in (A.14), we have

\[
\hat{W}^u_0 \geq \mathcal{Y}_u (W^u_0) \mathbb{E}^Q \left[ e^{-\beta T \hat{W}^u_T} \right] = \mathcal{Y}_u (W^u_0) \mathbb{E}^Q \left[ e^{-rT (W^u_T - \hat{W}^u_T)} \right] \geq \mathcal{Y}_u (W^u_0) \left( \hat{W}^u_0 - \mathbb{E}^Q \left[ e^{-rT \hat{W}^u_T} \right] \right) \geq 0. \tag{A.19}
\]

Hence, we obtain the desired inequality. Now, the optimal portfolio weight can be obtained by matching volatility of (10) and \( W^u_T = \mathbb{E}^Q \left[ e^{-r(T-t)W^u_T} \right] \). By Proposition 1, we can easily compute the latter:

\[
\hat{W}^u_t = y^t \frac{1}{\gamma} e^{-\alpha_u (T-t)} N (\delta_1(y_t, T-t)) + K e^{-r(T-t)} N (\delta_2(y_t, T-t)), \tag{A.20}
\]

where \( y_t = \mathcal{Y}_u (W^u_0) \xi_t \). The diffusion part of the above is

\[
diff (d\hat{W}^u_t) = \frac{\eta}{\gamma} y_t \frac{1}{\gamma} e^{-\alpha_u (T-t)} N (\delta_1(y_t, T-t)) \tag{A.21}
\]

This should be equal to the diffusion part of (10), \( \pi^u_t W^u_t \sigma \). Hence, we have

\[
\pi^u_t = \frac{\eta}{\gamma \sigma} (1 - \varphi_t), \tag{A.22}
\]
where \( \varphi_t = \frac{Ke^{-r(T-t)}}{\sqrt{2\pi}W_t}N(-\delta_2(y_t, T-t)) < 1. \)

**Proof of Proposition 3.** We first compute \( G(y) \):

\[
G(y) = \mathbb{E} \left[ e^{-\beta T} (y\xi_T)^{1-\frac{1}{\gamma}} (1 - 1(\overline{A})) + e^{-\beta T} \frac{K^{1-\gamma}}{1-\gamma} (1 - 1(\overline{A})) \right] \tag{A.21}
\]

Note that the expectation is under the physical measure. The random variable \( I_u(y\xi_T) \) can be expressed as

\[
I_u(y\xi_T) = y^{-\frac{1}{\theta}} \exp \left( -\frac{T}{\theta} (\beta - r - \frac{1}{2} \eta^2) + \eta (Z_{T} - Z_0) \right). \tag{A.22}
\]

The event \( A \) is equivalent to

\[
x < -\delta_3(y, T), \tag{A.23}
\]

where \( x \) is a standard normal random variable, and \( \delta_3(y, T) \) is given by

\[
\delta_3(y, T) = \delta_2(y, T) + \eta \sqrt{T}, \tag{A.24}
\]

since \( Z_T - Z_0 \) is normally distributed with zero mean and variance of \( T \). If we follow similar steps as Proposition 1, we can obtain (17). (19) is the direct result of Theorem 1. Take the first derivative of (16) is

\[
G'(y) = \mathbb{E} \left[ e^{-\beta T} y^{1-\frac{1}{\gamma}} \xi_T \right] \tag{A.25}
\]

From (19), we have

\[
J'(W_0^u) = G'(y) \text{ } (W_0^u) \tag{A.26}
\]

**Proof of Proposition 4.** We can interchange the integral and expectation.

\[
\mathcal{W}_\phi(y) = \int_0^T e^{-rt} \mathbb{E}^Q[I_\phi(y\xi_t)] dt, \tag{A.27}
\]

where

\[
I_\phi(y\xi_t) = \left( \frac{y}{K} \right)^{\frac{1}{\gamma}} \exp \left( -\frac{t}{\theta - 1} \left( \beta - r + \frac{1}{2} \eta^2 \right) - \frac{\eta}{\theta - 1} (Z_{t}^Q - Z_0^Q) \right). \tag{A.28}
\]

Hence, the inner expectation is

\[
\mathbb{E}^Q[I_\phi(y\xi_t)] = \left( \frac{y}{K} \right)^{\frac{1}{\gamma}} \exp \left( -\frac{t}{\theta - 1} \left( \beta - r + \frac{\theta \eta^2}{2(\theta - 1)} \right) \right). \tag{A.29}
\]

Now, we can express \( \mathcal{W}_\phi(y) \) as

\[
\mathcal{W}_\phi(y) = \left( \frac{y}{K} \right)^{\frac{1}{\gamma}} \int_0^T e^{-\alpha \phi t} dt \tag{A.30}
\]

\[
= \left( \frac{y}{K} \right)^{\frac{1}{\gamma}} \frac{1 - e^{-\alpha \phi T}}{\alpha \phi},
\]
\[ \alpha = \frac{\vartheta}{\varphi - 1} \left( r - \frac{\varphi^2}{2(\varphi - 1)} \right) - \frac{\beta}{\varphi - 1}. \] The first derivative is straightforward. \[ \square \]

**Proof of Theorem 5.** Consider any other random variable \( \tilde{Y} \), whose present value is greater than \( X_0 \). This implies that
\[
E^Q \left[ \int_0^T e^{-rt} \tilde{Y}_t \, dt \right] \geq X_0. \tag{A.31}
\]
Then, we want to show that
\[
E \left[ \int_0^T e^{-\beta t} \phi (Y_t) \, dt \right] \leq E \left[ \int_0^T e^{-\beta t} \phi (\tilde{Y}_t) \, dt \right]. \tag{A.32}
\]
Since \( \phi \) is a convex disutility, we have
\[
\phi (Y_t) \leq \phi (\tilde{Y}) + \phi' (Y_t) (Y_t - \tilde{Y}) \tag{A.33}
\]
Multiplying \( e^{-\beta t} \) and taking integral and expectation under the physical measure of the second term of RHS yields
\[
\mathcal{Y}_\phi (X_0) E \left[ \int_0^T e^{-\beta t} \xi_t (Y_t - \tilde{Y}) \, dt \right] = \mathcal{Y}_\phi (X_0) \left( E^Q \left[ \int_0^T e^{-rt} Y_t \, dt \right] - E^Q \left[ \int_0^T e^{-rt} \tilde{Y} \, dt \right] \right) \tag{A.34}
\]
\[
= \mathcal{Y}_\phi (X_0) \left( X_0 - E^Q \left[ \int_0^T e^{-rt} \tilde{Y} \, dt \right] \right) \leq 0.
\]
Hence, we obtain the desired inequality. Now, the optimal hedging of contribution can be obtained by matching volatility of (21) and \( X_t = E^Q_t \left[ \int_t^T e^{-r(s-t)} Y_s \, ds \right] \). By Proposition 4, we can easily compute the latter:
\[
X_t = \left( \frac{y_t}{k} \right)^{\alpha/\varphi} \frac{1 - e^{-\alpha_0 (T-t)}/\alpha_0}{\alpha_0}, \tag{A.35}
\]
where \( y_t = \mathcal{Y}_\phi (X_0) \xi_t \). The diffusion part of the above is
\[
\text{diff} (dX_t) = -\frac{\eta}{\vartheta - 1} X_t. \tag{A.36}
\]
This should be equal to the diffusion part of (21), \( \pi^\phi X_t \sigma \). Hence, we have
\[
\pi^\phi_t = -\frac{\eta}{(\vartheta - 1)\sigma}. \tag{A.37}
\]
\[ \square \]

**Proof of Proposition 6.** We first compute \( C(y) \):
\[
C(y) = E \left[ \int_0^T e^{-\beta t} \frac{k}{\vartheta} \left( \frac{y \xi_t}{k} \right)^{\varphi/\varphi - 1} \, dt \right] \tag{A.38}
\]
\[
= \frac{k}{\vartheta} \left( \frac{y}{k} \right)^{\varphi/\varphi - 1} \int_0^T e^{-\beta t} E \left[ \xi_t^{\varphi/\varphi - 1} \right] \, dt
\]
\[
= \frac{k}{\vartheta} \left( \frac{y}{k} \right)^{\varphi/\varphi - 1} \int_0^T e^{-\alpha_0 t} \, dt
\]
\[
= \frac{k}{\vartheta} \left( \frac{y}{k} \right)^{\varphi/\varphi - 1} \frac{1 - e^{-\alpha_0 T}}{\alpha_0}. \tag{A.39}
\]
(30) is the direct result of Theorem 5. Take the first derivative of (27) is

\[ C'(y) = E\left[ \int_0^T e^{-\beta t} \phi'(I_\phi(y\xi_t)) I_{\phi}'(y\xi_t) \xi_t dt \right] \]  \hspace{1cm} (A.39)

\[ = yE^Q\left[ \int_0^T e^{-rt} \xi_t I_\phi'(y\xi_t) dt \right] \]

\[ = yW_0'(y). \]

From (30), we have

\[ L'(X_0) = C'(Y_\phi(X_0)) Y_\phi'(X_0) \]

\[ = Y_\phi(X_0). \]  \hspace{1cm} (A.40)

**Proof of Theorem 7.** We can compute the present value of arbitrary contribution policy. Let

\[ X_0 = E^Q\left[ \int_0^T e^{-rt} \tilde{Y}_t dt \right] \]  \hspace{1cm} (A.41)

\[ W_0^u = W_0 + X_0. \]  \hspace{1cm} (A.42)

Then, \((\tilde{\pi}, \tilde{Y})\) satisfies the following static budget constraint:

\[ W_0^u \geq E^Q\left[ e^{-rT} W_T \right], \]  \hspace{1cm} (A.43)

where \(W_T\) is a corresponding wealth process to \((\tilde{\pi}, \tilde{Y})\). Hence, \(\tilde{\pi}\) is a feasible trading strategy to the first problem with the initial wealth \(W_0^u\), and \(\tilde{Y}\) is a feasible contribution policy to the second problem with the present value of contribution \(X_0\). Let \(\pi^u_t\) and \(\pi^\phi\) be the optimal trading strategy to the first and the optimal hedging strategy to the second problem, respectively. Also, let \(W_t^u\) and \(X_t\) be the optimal path of asset value to the first problem, and the optimal path of internal resource for hedging contribution to the second problem, respectively. Finally, let \(Y\) denote the optimal contribution policy to the second problem. Then, we can construct following portfolio policy, contribution policy, and path of pension asset value:

\[ \pi_t = \frac{\pi^u_t W_t^u - \pi^\phi X_t}{W_t^u - X_t} \]  \hspace{1cm} (A.44)

\[ Y_t = Y_t \]  \hspace{1cm} (A.45)

\[ W_t = W_t^u - X_t. \]  \hspace{1cm} (A.46)

We need to prove that these policies are feasible for the original problem. Consider the discounted pension asset:

\[ e^{-rt} W_t = e^{-rt} W_t^u - e^{-rt} X_t \]

\[ = W_0 + X_0 + \int_0^t e^{-rs} \pi^u_s \sigma W_s^u dZ_s^Q - X_0 + \int_0^t e^{-rs} \pi^\phi \sigma X_s dZ_s^Q \]

\[ = W_0 + \int_0^t e^{-rs} \pi^u_s \sigma W_s^u dZ_s^Q + \int_0^t e^{-rs} \pi^\phi \sigma X_s dZ_s^Q \]

\[ = E^Q_t\left[ e^{-rT} W_T \right] - E^Q_t\left[ \int_t^T e^{-rY_s} ds \right], \]

since \(W_T = W_T^u - X_T = W_T^u\). Hence, \((\pi, Y)\) is a admissible portfolio and contribution policy to the original problem. Then, we have

\[ E\left[ e^{\beta T u(W_T)} \right] \geq E\left[ e^{\beta T u(W_T)} \right] \]  \hspace{1cm} (A.48)

\[ E\left[ \int_0^T e^{-\beta \phi(Y_t)} dt \right] \leq E\left[ \int_0^T e^{-\beta \phi(\tilde{Y}_t)} dt \right]. \]  \hspace{1cm} (A.49)
Hence, we have a desired inequality. Then, (32) is straightforward.

**Proof of Proposition 8.** From (12), we can easily see that \( W_u(y) \) is decreasing, \( \lim_{y \to 0} W_u(y) = \infty \), and \( \lim_{y \to \infty} W_u(y) = Ke^{-rT} \). Also, from (23) we can see that \( W_\phi(y) \) is increasing, \( \lim_{y \to 0} W_\phi(y) = 0 \), and \( \lim_{y \to \infty} W_\phi(y) = \infty \).

**Proof of Proposition 9.** Suppose that we find \( y \) solving (34), i.e. the optimal present value contribution, \( X_0 \). Then, we can set the optimal portfolio policy, contribution policy, and path of pension asset value to the original problem as (A.44), (A.45), (A.46) using solutions to the first and second problem. Then, the optimal portfolio weight is straightforward. Note that by (A.35) the optimal path of internal resource for hedging contribution is

\[
X_t = \left( \frac{y\xi_t}{k} \right) \frac{1 - e^{-\alpha_\phi(T-t)}}{\alpha_\phi} = Y_t \frac{1 - e^{-\alpha_\phi(T-t)}}{\alpha_\phi}.
\]

Hence, the optimal contribution rate is

\[
\frac{Y_t}{W_t} = \frac{X_t}{W_t} \frac{\alpha_\phi}{1 - e^{-\alpha_\phi(T-t)}}.
\]

**Proof of Proposition 10.** The first part of (12) is the present value of terminal wealth, \( I_u(y\xi_T) \) if \( I_u(y\xi_T) > K \), otherwise zero. Hence, \( W^{BC}_u(y) \) can be easily computed from that. The first derivative is straightforward. Now, we can express \( W_u(y) \) as

\[
W_u(y) = y^{-\alpha}e^{-\alpha_uT} + Ke^{-rT}N(-\delta_2(y,T)) - y^{-\alpha}e^{-\alpha_uT}N(-\delta_1(y,T))
\]

The last two are the present value of \( (K - I_u(y\xi_T))^+ \), and thus positive.

**Proof of Theorem 11.** Proof is similar to that of Theorem 2. We omit here.

**Proof of Proposition 12.** Proof is similar to that of Proposition 3. We omit here.
References


Table 1: Summary of key variables and parameters

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Parameters</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Terminal liability</td>
<td>$K$</td>
<td>Pension’s investment horizon</td>
<td>$T$</td>
<td>10-year</td>
</tr>
<tr>
<td>Pension’s asset</td>
<td>$W$</td>
<td>Price of Risk</td>
<td>$\eta$</td>
<td>0.4</td>
</tr>
<tr>
<td>Present value of terminal asset</td>
<td>$W^u$</td>
<td>Risk-free rate</td>
<td>$r$</td>
<td>2%</td>
</tr>
<tr>
<td>Pension sponsor’s internal resource for hedging contribution</td>
<td>$X$</td>
<td>Pension sponsor’s subjective discount rate</td>
<td>$\beta$</td>
<td>1%</td>
</tr>
<tr>
<td>Shadow price</td>
<td>$y$</td>
<td>Pension sponsor’s risk aversion</td>
<td>$\gamma$</td>
<td>5</td>
</tr>
<tr>
<td>Pension sponsor’s marginal rate of substitution</td>
<td>$\zeta$</td>
<td>Pension sponsor’s elasticity of disutility</td>
<td>$\theta$</td>
<td>2</td>
</tr>
<tr>
<td>Portfolio weight of equity</td>
<td>$\pi$</td>
<td>Relative importance of disutility</td>
<td>$k$</td>
<td>100</td>
</tr>
<tr>
<td>Contribution flow</td>
<td>$Y$</td>
<td>Initial funding ratio</td>
<td>$\lambda_0$</td>
<td>80% (underfunded)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Initial funding ratio</td>
<td>$\lambda_0$</td>
<td>120% (overfunded)</td>
</tr>
</tbody>
</table>

This table summarizes the symbols for the key variables used in the model and the parameter values in the baseline case.
Figure 1: Determination of Present Value of Contribution

Panel A: Initially Underfunded Pension

Panel B: Initially Overfunded Pension

This figure plots shadow prices of first and second problem as a function of present value of contribution. Panel A is for an initially underfunded pension with 80% funding ratio, and Panel B is for an initially overfunded pension with 120% funding ratio.
This figure plots costs of put-based strategy as a function of fraction of allocation to the mean-variance efficient portfolio. Panel A is for an initially underfunded pension with 80% funding ratio, and Panel B is for an initially overfunded pension with 120% funding ratio.
This figure plots portfolio weights and contribution rate at time $t = 5$-year as a function of annualized equity return over last five years. Panel A and B are for an initially underfunded pension with 80% funding ratio, and Panel C and D for an initially overfunded pension with 120% funding ratio. We fix the initial asset value at one and vary the terminal downside constraint, $K$. 
Figure 4: Effect of Initial Funding Ratio

Panel A: Present Value of Contribution

Panel B: Portfolio Weights at Time Zero

Panel C: Contribution Rate at Time Zero

Panel D: Certainty Equivalent

This figure plots present value of contribution (Panel A), portfolio weights at time zero (Panel B), contribution rate at time zero (Panel C), and certainty equivalent (Panel D) as a function of initial funding ratio. We fix the initial wealth at one and vary the terminal downside constraint, $K$. 

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Figure 5: Effect of Relative Importance of Disutility

Panel A: Present Value of Contribution

Panel B: Put Option Value at Time Zero

Panel C: Portfolio Weights at Time Zero

Panel D: Certainty Equivalent

This figure plots present value of contribution (Panel A), put option value at time zero (Panel B), portfolio weights at time zero (Panel C), and certainty equivalent (Panel D) as a function of relative importance of disutility \(k\). Underfunded pension has 80% funding ratio, and overfunded pension has 120% funding ratio.
Figure 6: Effect of Elasticity of Disutility

Panel A: Present Value of Contribution

Panel B: Put Option Value at Time Zero

Panel C: Portfolio Weights at Time Zero

Panel D: Certainty Equivalent

Panel E: Determination of $X_0$
(Concave Marginal Disutility)

Panel F: Determination of $X_0$
(Convex Marginal Disutility)
Note to Figure 6
This figure plots present value of contribution (Panel A), put option value at time zero (Panel B), portfolio weights at time zero (Panel C), and certainty equivalent (Panel D) as a function of elasticity of disutility ($k$). Also, we plot the determination of present value of contribution for different $\theta$ and underfunded pension with a downside constraint in Panel E and F. Underfunded pension has 80% funding ratio, and overfunded pension has 120% funding ratio.
Figure 7: Effect of Relative Risk Aversion

Panel A: Present Value of Contribution

Panel B: Put Option Value at Time Zero

Panel C: Portfolio Weights at Time Zero

Panel D: Certainty Equivalent

This figure plots present value of contribution (Panel A), put option value at time zero (Panel B), portfolio weights at time zero (Panel C), and certainty equivalent (Panel D) as a function of relative risk aversion ($\gamma$). Underfunded pension has 80% funding ratio, and overfunded pension has 120% funding ratio.
Figure 8: Effect of Price of Risk

This figure plots present value of contribution (Panel A), put option value at time zero (Panel B), portfolio weights at time zero (Panel C), and certainty equivalent (Panel D) as a function of price of risk (\( \eta \)). Underfunded pension has 80% funding ratio, and overfunded pension has 120% funding ratio.