Inventory and Financial Strategies with Capital Constraints and Limited Joint Liability

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We study the financial and operational decisions of two capital-constrained firms via a limited joint liability (LJL) financing scheme offered by a bank. To explicitly assess the value of the LJL financing scheme, we assume that the firms either use their own initial capital (also called the self-financing scheme) or the LJL financing scheme to support their operations. We provide a framework on how the firms react to the LJL financing scheme and how the bank sets the loan terms (e.g., the credit line) based on the the capital in the pool committed by the firms. We derive a two-stage model in which the firms separately determine their individual ordering decisions according to the prior joint liability agreement between the firms and the bank. Applying a non-cooperative game to analyze the decision-making problems of the two firms, we establish the existence of equilibrium decisions for the two firms. We characterize the mild conditions under which the LJL financing scheme is simultaneously preferred by the two firms. We show that the two firms' strategies are complementary and the firms' equilibrium order quantities are always positively influenced by the risk-sharing terms. We find that a greater bank loan leverage ratio may not simultaneously improve the two firms' performance. We examine how the firms' retail prices, unit purchase costs, and demand variability affect their equilibrium inventory decisions, and the corresponding expected terminal capital and bankruptcy-related risks. When the credit line and interest rate are endogenized by the bank, we provide insights on the relationship between optimal interest rate and bank loan leverage ratio through a risk hedging and the impact of demand correlation on the bank.

Key words: capital-constrained firms; capital pool; limited joint liability; bank loan leverage ratio; supermodular game.
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1. Introduction

In today's highly competitive market environment, working capital plays a crucial role in enabling firms to rapidly respond to customer needs in order to gain a "foothold" in the market. However, it is common that many enterprises, especially small- and medium-sized enterprises (SMEs), face the dual problem of a shortage of working capital and very limited access to financing sources, e.g., bank loans (Kouvelis and Zhao 2012). In recent years, many innovative financing schemes have emerged to enable SMEs or cash-strapped firms to obtain financing for their operations, such as purchase order financing (e.g., Tang et al. 2018), buyer direct financing (e.g., Tunca and Zhu 2017), and reverse factoring (e.g., Wu 2017). Compared with these well-known financing schemes, a budding financing scheme that is commonly used in practice is the limited joint liability (LJL) financing scheme, which was launched by China Citic Bank (CCB) in 2007.[1] Under this financing scheme, small firms can obtain higher available credit lines (than before) to significantly mitigate their financial distresses through committing their on-hand initial capital to a specified capital pool. For the bank, since the potential bankruptcy risks will be necessarily shared by the firms in the group, it has the incentive to offer LJL financing to support them.

So far the LJL financing scheme has been hailed as a success in helping firms to address their liquidity problem since its launch. For instance, a car manufacturer in Zhejiang Province in China, an SME that mainly produces electric motorcycles and electric scooters for export to the U.S., was in financial distress in 2012 because it invested most of its working capital in expanding new plants and purchasing machinery, so that it did not have enough capital to start a new production line to satisfy the orders from the U.S. Under the LJL financing scheme, the manufacturer submitted a financing report to the chamber of commerce(CC)[2] it belongs to, and then the latter recommended it and eight other firms (also CC members) that also were in financial distress to CCB simultaneously. After the recommendation was accepted by CCB, these firms committed a total amount of RMB 10.60 million to a capital pool, while the bank provided them with a loan amount equal to five times of the capital in the pool, i.e., RMB 53 million, as these firms' credit lines. In this way, the car manufacturer obtained a credit line of RMB $6 million to support its operations by committing RMB 1.2 million to the capital pool, curtailing its financial difficulty. According to the reports of CCB and other news, there are many similar cases.[3] Also, the Hangzhou Branch of

[1] In fact, the LJL financing scheme is called the "seed fund" financing mode or Zhurong lending. In this paper we formally refer to it as LJL financing, which captures the key trait of the financing scheme directly. For an introduction of LJL financing, the reader may refer to http://www.citicbank.com/enterprise/smallbusiness/zrd/. The original idea of this innovative financing mode stems from the successful practice of group lending that took place in Bangladesh in the 1990s (Stiglitz 1990).

[2] The role of chamber of commerce in this example is to just provide an intermediary service for its members and the bank, rather than influence the operations managements of the related firms.

[3] http://esearch.citicbank.com/search/index.jsp.

CCB reported that by the end of July 2013, CCB had provided a total amount of RMB 50.111 billion to SMEs via the LJL financing scheme, which was RMB 6.645 billion more than the previous year and was about 74.34% of the total loan amount provided by the branch.[4] Nevertheless, the LJL financing scheme is still in the nascent stage of development. To the best of our knowledge, there is a lack of research on examining the relative merit of the LJL financing scheme compared with the no-financing (or self-financing) case. To fill this research gap, we conduct this research to address the following research questions: Does the limited joint liability arrangement between the capital-constrained firms make the LJL financing scheme superior to other financing schemes? If so, under what conditions? Under the LJL financing scheme, how would the firms react to the joint liability contract and how does joint liability affect the firms' operations decisions? How are the bank's loan terms affected by the capital in the pool committed by the firms?

To address the above questions, we develop a stylized newsvendor-type model with two financially-constrained firms that differ in cash position, cost, revenue, and demand parameters. We assume that the demands faced by the firms are non-negative random variables. To explicitly focus on the value of the LJL financing scheme, we further assume that the firms have no access to other financing options, which means they can either use their own initial capital (which is called the self-financing scheme in the literature) or borrow through the LJL financing scheme to support their operations. Under the self-financing scheme, the problems of the two firms are the same as that of the classical newsvendor with financial constraints, which we use as the benchmark case for comparison. This benchmark case manifests in our model when the marginal value of the additional initial capital to each firm is very limited.

Under the LJL financing scheme, we formulate the operation and financial decision-making problems of the two firms as a two-stage model in which the firms separately determine their individual ordering decisions according to the prior joint liability agreement between the firms and the bank. In the first stage, the firms commit their own on-hand initial capital to the capital pool managed by the bank, and then the bank sets a bank loan leverage ratio on the capital pool as the firms' credit lines. Within its credit line, each firm decides how much to order in a non-cooperative manner. To simplify the analysis, we assume that the two firms make their ordering decisions at the same time. In the second stage, after demand realization, the capital in the pool will be used to pay for the remaining unpaid debts of the bankrupted firms, if any. In this regard, each firm's bankruptcy risk is shared to a limited extent by its partner with its initial capital in the pool. We show that such a two-player non-cooperative game is supermodular. As such, the best (joint) response function of each firm is positively related to the other firm's strategy, which means that

[4] http://hzdaily.hangzhou.com.cn/hzrb/html/2013-09/12/content_1576769.htm.

the firms' strategies are complementary. Consequently, it follows that an equilibrium of the game exists, and there are a greatest and a smallest equilibrium point. Moreover, we show that the firms' equilibrium inventory decisions are always positively associated with the endogenously (limited) risk-sharing or joint-liability term. This means that the two firms simultaneously become more aggressive in ordering as such shared risk increases.

We find that, in equilibrium, if one firm orders up to the level where it uses up its initial capital, then cooperation in financing will not occur. So we are interested in analyzing the equilibrium outcomes above such a level, and we refer to those outcomes as non-trivial equilibria. However, the selection of financing schemes for the two firms at the non-trivial equilibria is complicated, depending on the firms' financial status, the bank's interest rate, and the bank loan leverage ratio. Specifically, when the bank loan leverage ratio is considerable, the interest rate is relatively low, and the initial capital of the firms is below a certain threshold, the LJL financing scheme outperforms the self-financing scheme. However, when at least one firm is not extremely-financially constrained, cooperation under the LJL financing scheme will not occur even though the bank loan leverage ratio is relatively high and the interest rate is low. This result also holds under the condition where the bank loan leverage ratio is relatively small and the interest rate is high.

Given the role of the risk-sharing terms, the equilibrium inventory levels and the corresponding expected terminal capital of the firms are not always monotonously increasing/decreasing as the initial capital committed to the pool increases. Yet we intriguingly find that if at least one firm that uses up all its available credit line to order can bear a higher proportion of the liability, then the two firms will borrow more loans from the bank, which is not necessarily beneficial to their partners; otherwise, the results are reversed. We further show that a firm with a relatively low (moderate) level of initial capital will have larger (lower) over-investment in ordering as it has a greater liability in the pool.

Furthermore, we investigate how the bank loan leverage ratio affects the firms' strategies. Intuitively, as the bank increases the bank loan leverage ratio, the two firms' equilibrium inventory levels and thus their bankruptcy risks will increase; meanwhile, they will over-invest more in ordering. But we find that increasing the bank loan leverage ratio may not be beneficial to the two firms simultaneously because their strategies are complementary. We further investigate by numerical studies the impacts of the market parameters on the firms' equilibrium order quantities, and the corresponding expected terminal capital and bankruptcy-related risks. We find that varying a market parameter of each firm weakly affects its partner's strategy, so the expected cash level because the joint liability or (bankruptcy) risk sharing is very limited. This observation shows that the final bankruptcy risks in the case of negative demand correlation are in general lower than that of the case of positive demand correlation. This implies that the bank prefers to support firms whose demand correlation is negative. Moreover, we find that higher liability is beneficial to a firm, which however may be detrimental to the other firm's performance.

Finally, we numerically analyze the bank's problem, assuming that the bank loan market is competitive, and the problem of generalizing the allocation of the initial capital. Interestingly, we find that as the bank offers a higher leverage ratio, it will increase the interest rate to control the risks from the firms. In addition, our main findings above continue to hold in the endogenized interest rate and leverage ratio case. In the extension, our result demonstrate that committing all the initial capital to the capital pool required by the bank is not necessarily an inferior choice for the firms, especially when they are relatively financially constrained and the leverage ratio is small.

We organize the rest of the paper as follows: In Section 2 we review the related literature to identify the research gap and position our work. In Section 3 we introduce the notation, assumptions, and event sequences of the game model, and formulate the no-financing (self-financing) scheme as the benchmark case. In Section 3.2 we analyze the limited joint liability financing model and prove the existence of an equilibrium. In Section 4 we investigate the impacts of the bank loan leverage ratio and the firms' liability in the pool on the firms' equilibrium inventory decisions. In Section 5 we compare the LJL financing scheme with the benchmarking self-financing scheme. In Section 6 we discuss the results of numerical studies conducted to analyze how the market parameters affect the firms' decision-making and the merits of the LJL financing scheme. In Section 7 we consider the bank's decision-making problem and generalize the allocation of the initial capital. Finally, we conclude the paper and suggest future research topics in Section 8. We place all the proofs in Appendix C.

2. Literature Review

Our work lies in the interface of research on operations and financial decisions. Specifically, we are interested in studying the impact of financial constraints on single-firm operations strategies, and how financial constraints and financing sources affect supply chain performance. Motivated by practice and the seminal work of Modigliani and Miller (1958), many papers including Babich and Sobel (2004), Xu and Birge (2004, 2006, 2008), Chao et al. (2008), Gupta and Wang (2009), Li et al. (2013), Gong et al. (2014), and Luo and Shang (2015) have studied whether/how financial constraints affect the operations decisions of firms under given financing schemes (e.g., self-finance, bank loan, trade credit etc). Based on either the single-period or multi-period newsvendor model, they show that financial arrangements significantly affect firms' performance. Different from them, we investigate the effects of risk sharing (or joint liability) on the operations and financial decisions of multiple firms that separately determine their own order quantities and sell their products to their own uncertain markets to maximize their profits.

In this review we mainly focus on reviewing the related works that investigate the impacts of financial constraints and financing sources on supply chain (multi-firm) performance. To the best of our knowledge, Buzacott and Zhang (2004) was the first study to address the problem of how financing sources affect the operations decisions of a financially-constrained firm and a profit-maximizing bank in the setting where the bank acts as the leader and the firm acts as the follower. Adopting a similar model, Dada and Hu (2008) showed the uniqueness of the Stackelberg equilibrium and provided a non-linear loan schedule to coordinate the channel between the newsvendor firm and the bank. Zhou and Groenevelt (2008) studied the case where the supplier provides financial subsidies to a financially-constrained retailer, and showed that the retailer's capital structure has a significant impact on the overall supply chain performance. Lai et al. (2009) examined the impact of financial constraints on the choice of the supply chain mode, i.e., pre-order mode, consignment mode, or a combination of the two modes, and contracting, and showed that sharing the inventory risk in the supply chain improves system efficiency. Recently, with consideration of financial constraints and bankruptcy costs, Kouvelis and Zhao (2011, 2012, 2016, 2018) studied the optimal price-only contract, the optimal trade credit contract, contract design, and coordination of a supply chain when selling to a cash-constrained newsvendor, and the impact of credit ratings on the operations and financial decisions of a supply chain, respectively. Chen et al. (2018) investigated the efficacy of the cash-flow dynamics of third-party logistics (3PL) procurement service in a three-player supply chain that includes one manufacturer, one 3PL provider, and one buyer. They derived the conditions under which the payment timing arrangement for the 3PL provider's procurement service benefits all the parties in the supply chain. For more related works, the reader may refer to Caldentey and Chen (2010), Yang and Birge (2011, 2017), Jing et al. (2012), Cai et al. (2014), and Deng et al. (2018). Note that these studies consider a Stackelberg game in which the downstream firm, i.e., the follower (e.g., retailer/buyer), partially shares the demand risk with the upstream firm, i.e., the leader (e.g., bank/manufacturer/supplier).

Several recent works complement the above studies by considering that the demand/supply risk of the upstream firm is shared by the downstream firm. Chen (2016), and Tunca and Zhu (2017) investigated the impact of reverse factoring (also called buyer intermediated financing or retailer-direct financing) on the operations decisions of supply chain participants in the presence of demand risk. In addition, Tang et al. (2018) studied the impacts of different financing schemes on supply risk in the symmetric information, and both moral hazard and information asymmetry settings.

It is noted that in the above studies on the impacts of financial constraints and financing sources on supply chain performance, the demand/supply risk of the follower is unidirectionally shared by the leader in the supply chain. Also, they assume that the capital-constrained firms in the supply chain always individually have access to financing sources (e.g., bank loans or trade credit). Thus, there are key differences between their and our works as follows: First, the participants in our paper are not only financially-constrained but may also be outside the supply chain. Second, the firms share their demand/bankruptcy risks, which means that the demand risks are bidirectionally shared. Third, we also consider that firms having limited capital cannot obtain the needed financing alone, but they can obtain bank loans by forming a joint liability group.

To the best of our knowledge, the paper most closely related to our work is Chod (2017). By assuming that the wholesale prices are exogenously given and the bank loan/trade credit is fairly priced, Chod (2017) investigated how debt financing distorts the retailer's inventory decisions and argued whether this distortion is mitigated by using trade credit, in which the retailer is capital-constrained and orders the inventories of two products while facing uncertain demand. He showed that a debt-financed retailer should favour items (i) with a low salvage value, (ii) with a high profit margin, and (iii) that represent a large portion of the total inventory investment. In addition, the firm may be better off by using a mixed financing strategy involving some bank credit, complemented with additional supplier financing. Different from Chod (2017), the demand risks of the two products are shared between the firms in our study. In addition, we focus on investigating the impact of the LJL financing scheme on multiple capital-constrained firms' operations decisions where the firms are separately profit-maximizing and sell their (different) products to their own uncertain markets. We also show that under some mild conditions the LJL financing scheme is beneficial to both firms simultaneously.

Our paper is also related to the literature on group lending that has received considerable empirical and theoretical attention over the past two decades, especially in the broad area of micro-finance. These studies focus on the moral hazard and monitoring problems (e.g., Chowdhury 2005, Stiglitz 1990); specifically on the role of social ties in improving group members' repayment performance (e.g., Wydick 2001), and comparison between group lending and individual lending in terms of strategic default, repayment rate, and efficiency (e.g., Ghatak 1999, Bhole and Ogden 2010), peer selection (Ghatak 1999, 2000), and optimal group size in joint liability contracts (e.g., Rezaei et al. 2017), etc. Unlike these studies, we use a non-cooperative game to explicitly model the interactions of the operations and financial decisions of two capital-constrained firms that make their inventory decisions under LJL financing, from the perspective of operations management.

3. Problem Formulation

Consider two firms (denoted by $i = 1, 2$) that are financially-constrained and purchase products from their own suppliers with unlimited capacity. The demand of firm i, denoted by ξ_i, is uncertain and continuous with support on $[0, +\infty)$. Let $\boldsymbol{\xi} = (\xi_1, \xi_2)$ be a vector corresponding to the demands for the two firms' products with cumulative distribution function (CDF) $F_{\boldsymbol{\xi}}(\cdot, \cdot)$ and probability density function $f_{\boldsymbol{\xi}}(\cdot,\cdot)$. In addition, the marginal cumulative and probability distribution functions of $\boldsymbol{\xi}$ in terms of ξ_i are given by $F_{\xi_i}(\cdot)$ and $f_{\xi_i}(\cdot)$, respectively, i.e., $F_{\xi_i}(\cdot) = F_{\boldsymbol{\xi}}(\cdot, +\infty)$. Let $F_{\boldsymbol{\xi}}(\cdot,\cdot)$ and $F_{\xi_i}(\cdot)$ be differentiable and increasing, $\bar{F}_{\xi_i} = 1 - F_{\xi_i}(\xi_i)$ and $F_{\boldsymbol{\xi}}(0,0) = F_{\xi_i}(0) = 0$. We denote the mean vector and covariance matrix of the demands vector $\boldsymbol{\xi}$ as $\boldsymbol{\mu} = (\mu_1, \mu_2)$ and

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix},$$

respectively. The retail price of product i is p_i and the unit purchase cost of firm i is c_i. We denote B_i as firm i's limited initial capital (or, interchangeably, limited liability), which we assume to be not enough to make an order quantity q_i, so we impose the constraint $B_i \le c_i q_i$. As argued above, the firms either use their own limited initial capital (which we refer to as the no-financing mode) or seek external financing (e.g., a bank loan) to execute their order decisions. In the former case, firm i just has the available capital B_i to invest in its order quantity, i.e., $c_i q_i \le B_i$. However, as discussed above, these firms have limited access to financing sources individually. Thus, in practice, they may consider opting for the LJL financing scheme together to finance their inventory decisions.

Under such a scheme, the two firms at the beginning of the sales season commit their initial capital, i.e., assuming joint liability, to a capital pool managed by the bank. Then, the bank sets a leverage ratio $\beta > 1$ on the capital (limited liability) in the pool as the two firms' credit lines, i.e., amplifies the capital pool by β times as the two firms' credit lines, at an interest rate r_b. In this way, if firm i commits initial capital B_i to the pool, then it obtains the credit line βB_i to support its ordering decision. At the end of the sales season, the bank can deduct the capital in the pool to pay for the remaining debts of bankrupted firms, if any. Consequently, for $i = 1, 2$, firm i needs to decide its order quantity within its credit line, i.e.,

$$B_i \le c_i q_i \le \beta B_i. \tag{1}$$

We assume that all the unmet demand is lost and all leftover inventory is salvaged at the price $s = 0$; and, the two firms and the bank are risk-neutral and all the parameters are common knowledge among them. To avoid the trivial cases, we assume $p_i \ge (1 + r_b)c_i$. In line with real practice, it is reasonable to assume that the capital pool is managed by the bank and only utilized to pay off the loan obligations unpaid. Because the firms are strictly identified by the bank or another credit institution, we assume that the firms are credit-worthy and will meet their loan obligations to the extent possible. Table 7 summarizes all the notation used in the paper (see Appendix A).

We discuss the sequence of the events of the game model of the two firms that choose the LJL financing scheme as follows:

1. Prior to the sales season, each firm commits its own initial capital B_i to form a capital pool. Given the capital pool formed and information about the two firms' parameters, the bank offers a credit line to each firm i at the interest rate r_b;

2. Then firm i places an order quantity q_i via borrowing a short-term loan c_iq_i from the bank within its credit line βB_i and pays its supplier the amount c_iq_i. The supplier immediately delivers the product to firm i and the latter sells it to its uncertain market;

3. At the end of the sales season, firm i receives its all the sales revenue $p_i \min\{q_i, \xi_i\}$ and uses it to repay its loan obligation. If firm i's on-hand revenue is not enough to meet its loan obligation, according to the arrangement, the bank first deducts firm i's initial capital in the pool, if enough. If firm i's initial capital in the pool is not enough to repay its remaining debt, the bank can deduct firm j's remaining initial capital in the pool, if enough, to repay firm i's remaining debt.

As argued above, under the LJL financing scheme each firm shares its bankruptcy risk with the other firm (through using the initial capital in the pool). In this regard, each firm when making its operations decision does not only consider its own risk but also its partner's, which is indeed different from the literature on the interface between operations and financial decisions. However, if the two firms do not choose the LJL financing scheme, each makes its own order quantity decision ignoring the other firm's information, which is the same as the classical newsvendor problem with financial constraints. For comparison, we consider this as the benchmark case in Section 3.1. Then we focus on analyzing the LJL financing scheme in Section 3.2. For any real number x, we denote $x^+ = \max\{x, 0\}$, use E to denote expectation, and express vectors in the bold font.

3.1. Benchmark Case: No-financing Scheme

We first consider the benchmark case in which the two capital-constrained firms have no access to bank loans alone. Thus, they have to use their own initial capital to order and the remaining capital after purchasing the product will be deposited in the bank to obtain the risk-free interest r_f. For convenience, we use the superscript "NF" to represent this no-financing case (or the self-financing case). Hence, we formulate the optimization problem of firm i, for $i = 1, 2$, as follows:

$$\max_{q_i \geq 0} \pi_i^{NF}(q_i) = E_{\xi_i}[p_i \min\{q_i, \xi_i\} + (1 + r_f)(B_i - c_iq_i)], \quad \text{subject to} \quad c_iq_i \leq B_i. \tag{2}$$

Let $c_i' = (1 + r_f)c_i$. Clearly, firm i's objective function $\pi_i^{NF}(q_i)$ is strictly concave in q_i. We characterize the optimal order quantity and the corresponding expected terminal capital of firm i in the following result.

LEMMA 1. *If $B_i \geq c_i \bar{F}_{\xi_i}^{-1}(c_i'/p_i)$, firm i's optimal order quantity is $q_i^{NF*} = \bar{F}_{\xi_i}^{-1}(c_i'/p_i)$ and the corresponding expected terminal capital is $\pi_i^{NF*} = (p_i - c_i')\bar{F}_{\xi_i}^{-1}(c_i'/p_i) - \int_0^\infty \int_0^{\bar{F}_{\xi_i}^{-1}(c_i'/p_i)} F_{\boldsymbol{\xi}}(x, y)dxdy + B_i$; otherwise, firm i's optimal order quantity is $q_i^{NF*} = B_i/c_i$ and the corresponding expected terminal capital is $\pi_i^{NF*} = p_i\big(B_i/c_i - \int_0^\infty \int_0^{B_i/c_i} F_{\boldsymbol{\xi}}(x, y)dxdy\big)$.*

Intuitively, if the firms' initial capital is relatively high, then their optimal order quantities are the same as that of the unconstrained newsvendor problem. Otherwise, they have to use up all their on-hand initial capital to order. In this case, the first part of Lemma 2 shows that each firm's expected terminal capital is increasing and convex in B_i, which means that the firms significantly suffer from the limited working capital. This result also indicates that the marginal value of the additional initial capital increases if a firm has more initial capital, which is intuitive since the probability of using the additional capital to purchase its product increases when the firm has more capital. The second part of the lemma further implies that adding a unit of initial capital for ordering does not lead to a unit incremental value in the expected terminal capital.

LEMMA 2. *If $B_i < c_i \bar{F}_{\xi_i}^{-1}(c_i'/p_i)$, then* (i) *firm i's optimal expected terminal capital π_i^{NF*} is strictly increasingly convex in B_i, and* (ii) $\frac{d\pi_i^{NF*}}{dB_i}$ *is continuous in B_i and is less than 1.*

3.2. Limited Joint Liability Financing

In this section we analyze the case where the two capital-constrained firms together finance their ordering decisions with limited joint liability, i.e., under the LJL financing scheme. In the following we explicitly incorporate such joint liability, which is the key factor that distinguishes our financing model from the others in the literature, into the discussion. From the model description in the previous section, after demand realization, firm i's realized sales revenue is $p_i \min\{q_i, \xi_i\}$, which is used to meet its loan obligation. Hence, for $i = 1, 2$, firm i's terminal cash position is

$$x_i^{JF}(q_i, \xi_i) = p_i \min\{q_i, \xi_i\} - (1 + r_b) c_i q_i, i = 1, 2,$$

where superscript "JF" denotes the LJL financing case. Note that if $x_i^{JF}(q_i, \xi_i) \geq 0$, firm i's sales revenue can repay its entire principal and interest of the loan. If $x_i^{JF}(q_i, \xi_i) < 0$, according to the joint-liability arrangement, the bank first deducts firm i's initial capital in the pool to repay its remaining debt $((1 + r_b) c_i q_i - p_i \xi_i)^+$. When firm i's initial capital in the pool can completely cover its own remaining debt, the remaining initial capital in the pool is used to hedge against the (potential) bankruptcy risk of firm j. Therefore, for $i = 1, 2$, we can write firm i's expected terminal cash flow as follows:

$$\begin{aligned}\pi_i^{JF}(q_i, q_j) = &E_{\xi_i, \xi_j}\Big\{ \underbrace{[p_i \min\{q_i, \xi_i\} - (1 + r_b) c_i q_i]^+}_{\text{the first term}} \\ &+ \underbrace{\big[B_i - ((1 + r_b) c_i q_i - p_i \xi_i)^+ - ((1 + r_b) c_j q_j - p_j \xi_j - B_j)^+\big]^+}_{\text{the second term}} \Big\}, \quad j \neq i. \qquad (3)\end{aligned}$$

Note that the first term on the right hand of Equation (3) is firm i's terminal cash flow after its own loan repayment, if any, and the second term is the remaining initial capital captured by firm i

from the capital pool after shouldering the liability of firm j. Then, we state firm i's problem, for $i=1,2$ and $j=3-i$, as the following optimization model:

$$\max_{q_i\geq 0}\ \pi_i^{JF}(q_i,q_j),\ \ \text{subject to Constraint (1).} \tag{4}$$

Constraint (1) states that the capital-constrained firm i makes its ordering decision within its credit line βB_i. To simplify the analysis, we assume that the two firms make their ordering decisions simultaneously. As discussed above, we analyze the problems faced by the two firms at the start of the sales season as a non-cooperative game. From Equation (3), it seems that $\pi_i^{JF}(q_i,q_j)$ is relatively complex and, in general, non-concave (or even non-quasiconcave) in the order quantity q_i for given q_j. However, in the following, we solve the above optimization problem and show that the non-cooperative game played by the two firms is supermodular, which establishes the existence of an equilibrium of the game. To this end, we first introduce some meaningful notation. Denote S_i as the set of feasible strategies for firm i, given strategy q_j. So $S_i=\{q_i:\ B_i\leq c_iq_i\leq \beta B_i\}$. Obviously, S_i is not associated with firm j's strategies. Let S be the set of feasible joint strategies, i.e., $S=S_1\times S_2$. Thus, it is easy to derive the properties of these sets.

Lemma 3. *S_i and S are compact sublattices of R and R^2, respectively.*

For given q_j in S_j, the best response function of firm i is the set $q_i(q_j)=\arg\max_{q_i\in S_i}\pi_i^{JF}(q_i,q_j)$ of all the strategies that are optimal for firm i. For each feasible joint strategy $\mathbf{q}=(q_i,q_j)$, $\mathbf{Q}=(Q_i,Q_j)$ in S, define

$$\Pi(\mathbf{Q},\mathbf{q})=\pi_i^{JF}(Q_i,q_j)+\pi_j^{JF}(q_i,Q_j).$$

For each feasible joint strategy $\mathbf{q}$ in S, the best joint response function is the set

$$\mathbf{Q}(\mathbf{q})=\arg\max_{\mathbf{Q}\in S}\Pi(\mathbf{Q},\mathbf{q}).$$

Since $S=S_1\times S_2$, we have $\mathbf{Q}(\mathbf{q})=Q_i(q_j)\times Q_j(q_i)$.

We then explicitly describe the bankruptcy and non-bankruptcy events. On the one hand, if $\xi_i<\frac{(1+r_b)c_iq_i}{p_i}$, then $x_i^R(q_i,\xi_i)<0$, i.e., firm i does not have enough sales revenue to meet its loan obligation completely. Thus, $\frac{(1+r_b)c_iq_i}{p_i}$ is the minimum demand for firm i to repay the bank with its on-hand realized revenue. For ease of exposition, we refer to this value as firm i's *breakeven threshold*, denoted by $k_{1i}(q_i)$. Clearly, $k_{1i}(q_i)\leq q_i$ since $p_i\geq(1+r_b)c_i$. On the other hand, if $\xi_i<k_{1i}(q_i)$, then firm i's remaining debt, i.e., $(1+r_b)c_iq_i-p_i\xi_i$, will be paid by the initial capital B_i in the capital pool. When firm i's initial capital in the pool is not enough to repay its remaining debt, i.e., $B_i\leq(1+r_b)c_iq_i-p_i\xi_i$, firm i will go bankrupt. Similarly, let

$$k_{2i}(q_i)=\frac{(1+r_b)c_iq_i-B_i}{p_i}$$

be the minimum demand for firm i to completely meet its loan obligation with its on-hand realized revenue plus its initial capital, which we refer to as firm i's *effective bankruptcy threshold*. Obviously, for any $B_i \geq 0$, we have $k_{2i}(q_i) \leq k_{1i}(q_i)$. In addition, if $q_i \leq k_{2i}(q_i)$, then firm i is eventually bankrupt even though it sells all its product. As a result, firm i will not borrow, so it is expected that $q_i > k_{2i}(q_i)$. When $B_i > (1+r_b)c_iq_i - p_i\xi_i$, which means that firm i's initial capital in the pool can cover its remaining debt, firm i's remaining initial capital is used to shoulder the liability of firm j according to LJL financing arrangement. Therefore, suppose that the realized demand of firm j is zero, then the bank deducts firm i's remaining initial capital in the pool to meet the entire obligation of firm j, i.e., $(1+r_b)c_jq_j - B_j$. If $B_i - ((1+r_b)c_iq_i - p_i\xi_i) \geq (1+r_b)c_jq_j - B_j$, firm i has enough initial capital to repay the other firm's debt and survives; otherwise, firm i has to declare bankruptcy. For convenience, denote

$$k_{bi}(q_i,q_j) = \frac{(1+r_b)c_iq_i - B_i + (1+r_b)c_jq_j - B_j}{p_i} = k_{2i}(q_i) + \frac{p_j}{p_i}k_{2j}(q_j)$$

as the demand for firm i to finally survive when it shoulders the liability of firm j in the worst case (whose realized demand is 0). Clearly, if $\xi_i \geq k_{bi}(q_i,q_j)$, firm i will not be bankrupt. Moreover, we have $k_{2i}(q_i) < k_{bi}(q_i,q_j)$ and $k_{1i}(q_i) < k_{bi}(q_i,q_j)$ if and only if $B_i + B_j < (1+r_b)c_jq_j$.

Following the above reasoning, we consider the bankruptcy and non-bankruptcy events of each firm, and divide the feasible regions of the two demands into six sub-regions, which is illustrated in Figure 1. In this figure, we use the shaded areas to represent the bankruptcy event, which we denote by the superscript "b". (For more details on the six sub-regions, we refer the readers to see Appendix B.) Then, we can re-state firm i's objective function $\pi_i^{JF}(q_i,q_j)$ in Equation (3), for

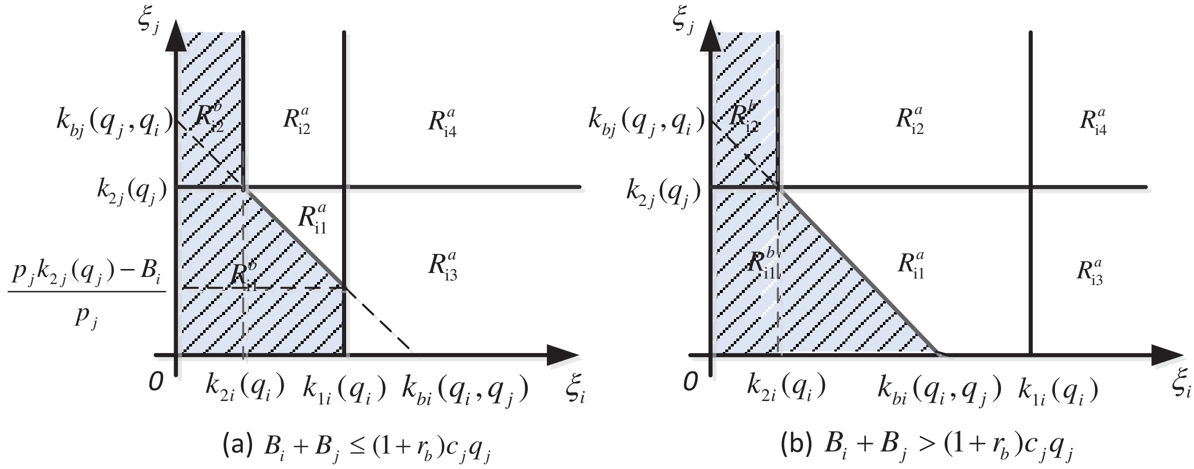

Figure 1 **Partition of the demand state space** (ξ_i,ξ_j)**.**

$i=1,2$ and $j=3-i$, as follows:

$$\begin{aligned}\pi_i^{JF}(q_i,q_j) = &\Pr(R^a_{i1})E\left(B_i-(1+r_b)c_iq_i+p_i\xi_i-(1+r_b)c_jq_j+p_j\xi_j+B_j|R^a_{i1}\right)\\&+\Pr(R^a_{i2})E\left(B_i-(1+r_b)c_iq_i+p_i\xi_i|R^a_{i2}\right)\\&+\Pr(R^a_{i3})E\left(p_i\min\{q_i,\xi_i\}-(1+r_b)c_iq_i+(B_i-(1+r_b)c_jq_j+p_j\xi_j+B_j)^+|R^a_{i3}\right)\\&+\Pr(R^a_{i4})E\left(p_i\min\{q_i,\xi_i\}-(1+r_b)c_iq_i+B_i|R^a_{i4}\right).\end{aligned} \tag{5}$$

For given q_j, by taking the first partial derivative of $\pi_i^{JF}(q_i,q_j)$ in Equation (5) with respect to q_i and simplifying, we have

$$\frac{\partial\pi_i^{JF}(q_i,q_j)}{\partial q_i}=p_i\bar{F}_{\xi_i}(q_i)-(1+r_b)c_i\bar{F}_{\xi_i}\left(k_{2i}(q_i)\right)+\phi(q_i,q_j), \tag{6}$$

where $\phi(q_i,q_j)=(1+r_b)c_i\Pr(\widehat{R}^b_{i1})$, and if $q_j\geq\frac{B_i+B_j}{(1+r_b)c_j}$, $\widehat{R}^b_{i1}=\Big\{\xi\geq 0:k_{2i}(q_i)<\xi_i\leq k_{1i}(q_i),\text{and } p_i\xi_i+p_j\xi_j<p_ik_{bi}(q_i,q_j)\Big\}$; otherwise, $\widehat{R}^b_{i1}=\Big\{\xi\geq 0:k_{2i}(q_i)<\xi_i\leq k_{bi}(q_i,q_j),\text{and } p_i\xi_i+p_j\xi_j<p_ik_{bi}(q_i,q_j)\Big\}$.

To explicitly capture the limited joint liability between the two firms, we refer to $\phi(q_i,q_{3-i})$ as the limited *risk-sharing term* of firm i $(i=1,2)$. Thus, a higher $\phi(q_i,q_j)$ means firm i assumes a greater bankruptcy risk for firm j (which is caused by firm j's strategy). In the following we use the terms limited joint liability and limited risk sharing interchangeably. With the definition of $\phi(q_i,q_j)$ above, we derive an important property of $\phi(q_i,q_j)$ in the following, which is useful to prove the existence of an equilibrium of the game.

LEMMA 4. *For $i=1,2$ and $j=3-i$ and any given (q_i,q_j) in S,* (i) $\frac{\partial\phi(q_i,q_j)}{\partial q_j}$ *is continuous in (q_i,q_j) and* (ii) $\phi(q_i,q_j)\geq 0$.

With part (i) of Lemma 4, it follows from Equation (6) that $\frac{\partial\pi_i^{JF}(q_i,q_j)}{\partial q_i}$ is continuous in (q_i,q_j) over S. This implies that the best response function $q_i(q_j)$ (which satisfies $\frac{\partial\pi_i^{JF}(q_i,q_j)}{\partial q_i}=0$, if adopted) of firm i is continuous in its partner's strategy $q_j\in S_2$. Thus, with Constraint (1), firm i's best response function is continuous in q_j and the best joint response function $\mathbf{Q(q)}$ is continuous in $\mathbf{q}\in S$. Part (ii) of Lemma 4 also shows that the risk-sharing terms are always positive (in affecting firm i's ordering decision). In what follows, we establish the existence of an equilibrium of this two-firm non-cooperative game. With Lemma 4, we characterize an important property of $\pi_i^{JF}(q_i,q_j)$ in terms of q_i and q_j in the following result.

PROPOSITION 1. *For given interest rate r_b and bank loan leverage ratio β, $\pi_i^{JF}(q_i,q_j)$ has strictly increasing difference in (q_i,q_j) for $(q_i,q_j)\in S$ $(i=1,2$ and $j=3-i)$.*

Proposition 1 indicates that the non-cooperative game played by the two firms is a supermodular game. As a result, similar to the results in Topkis (1998), we derive some properties of the set of best response functions and the set of best joint response functions in the following.

Lemma 5. *For given interest rate r_b and bank loan leverage ratio β, we have*

(a) *The set $q_i(q_j)$ of the best response functions of each firm i is a non-empty compact sublattice of R for each $q_j \in S_j$;*

(b) *The set $\mathbf{Q}(\mathbf{q})$ of the best joint response functions is a non-empty compact sublattice of R^2 for each $\mathbf{q}$ in S;*

(c) *There exists a greatest and a least best response for each firm i and each $q_j \in S_j$, i.e., $q_i(q_j)$ has a greatest element and a smallest element;*

(d) *There exists a greatest and a smallest best joint response for each $\mathbf{q}$ in S, i.e., $\mathbf{Q}(\mathbf{q})$ has a greatest element and a least element;*

(e) *The best response function $q_i(q_j)$ is increasing in q_j on S_j for firm i;*

(f) *The best joint response function $\mathbf{Q}(\mathbf{q})$ is increasing in $\mathbf{q}$ on S.*

Lemma 5 shows that in this supermodular game, the sets of each firm's best response function and best joint response function are non-empty compact, and have a greatest element and a smallest element. Furthermore, the greatest and least best (joint) response functions are increasing functions. This implies that firm i's best response function is an increasing function of firm j's strategy. In this regard, the two firms' strategies are complementary, which means that the firms in financing cooperation mutually reinforce each other (Cooper and John 1988). With the arguments above, we establish the existence of an equilibrium of the two-firm non-cooperative game. We use the superscript "J" to denote the equilibrium outcomes under the LJL financing scheme.

Theorem 1. *For given r_b and β, there exists a pure-strategy Nash equilibrium (q_1^J, q_2^J) on the inventory level, and a greatest and a smallest equilibrium point exist.*

According to the proof of Theorem 1, we see that the set of equilibrium points of the non-cooperative game is a non-empty complete lattice, and there exists a greatest and a smallest equilibrium point. In addition, with Lemma 4, we see that the (limited) risk-sharing terms always have a positive effect on the two firms' order quantities, i.e., a higher level of risk sharing induces each firm to make a more "risky" ordering decision. The following corollary implies that if one firm's equilibrium inventory level under the LJL financing scheme is the same as the optimal order quantity under the self-financing scheme, then the firm will obtain more expected terminal capital using the latter scheme. Hence, the firm will not choose the LJL financing scheme.

Corollary 1. *Suppose that $q_i^J = B_i/c_i$. For any $B_i \geq 0$ and $r_b \geq 0$, $\pi_i^{JF}(q_i^J, q_j^J) \leq \pi_i^{NF}(q_i^{NF*})$.*

We assume that if one firm makes its ordering decision by using up its initial capital, then it will choose the self-financing scheme and cooperation in financing will not occur. This make the other firm use up its initial capital to support its ordering decision. Thus, Corollary 1 implies that once there is a firm ordering the amount of B_i/c_i, each firm's problem degenerates to the case of no-financing. Below we focus on studying the equilibrium points that are greater than B_i/c_i.

4. Analysis of LJL Financing and Its Managerial Implications

We analyze the effects of the firms' limited liability in the capital pool and the bank loan leverage ratio on the equilibrium outcomes, i.e., order quantity, bankruptcy-related risk, and expected terminal capital. Notice from Corollary 1 that it is trivial to analyze the case where $B_i \geq c_i \bar{F}_{\xi_i}^{-1}(c_i'/p_i)$ since each firm in this case does not choose the joint-liability arrangement. As such, we only focus on the case where $B_i < c_i \bar{F}_{\xi_i}^{-1}(c_i'/p_i)$ in the following. For ease of exposition, we refer to the equilibrium solution in this case as the non-trivial equilibrium inventory decisions, which can be further divided into two types, namely non-trivial interior and non-trivial boundary equilibrium points. Formally, (1) a pair of equilibrium inventory decisions is non-trivial interior, denoted by (q_i°, q_j°), if the pair satisfies the first-order conditions for the two firms' expected terminal capital, in which (q_i°, q_j°) satisfies the following equation

$$p_i \bar{F}_{\xi_i}(q_i) - (1+r_b)c_i \bar{F}_{\xi_i}(k_{2i}(q_i)) + \phi(q_i, q_j) = 0,\ i = 1,2, j = 3-i. \tag{7}$$

(2) A pair of equilibrium inventory decisions is non-trivial boundary, denoted by (q_i^d, q_j^d), if there is at least one firm (without loss of generality, firm i) optimally uses up its credit line to order, i.e., firm i's optimal inventory decision is $q_i^d = \beta B_i / c_i$ and firm j's best response inventory decision is $q_j^d = \min\{q_j(\beta), \frac{\beta B_j}{c_j}\}$, where $q_j(\beta)$ satisfies the following equation

$$p_j \bar{F}_{\xi_j}(q_j) - (1+r_b)c_j \bar{F}_{\xi_j}(k_{2j}(q_j)) + \phi(q_j, \beta B_i/c_i) = 0. \tag{8}$$

By the above definitions, we derive the following result.

Proposition 2. *For $0 < B_i < B_i^0$, if $\beta \geq \max_{i=1,2} c_i q_i^\circ / B_i$, (q_i°, q_j°) is the two firms' equilibrium inventory pair; otherwise, (q_i^d, q_j^d) is the two firms' equilibrium inventory pair, where B_i^0 is the solution of the equation $c_i q_i^\circ(B_i) = B_i$.*

Proposition 2 characterizes the condition under which the non-trivial interior (or boundary) equilibrium inventory decisions are the equilibrium decisions of the two firms. Specifically, when the bank loan leverage ratio is relatively large, i.e., $\beta \geq \max_{i=1,2} c_i q_i^\circ / B_i$, the pair (q_i°, q_j°) satisfies the constraints of the two firms, so it becomes the two firms' equilibrium inventory levels. In this case, it follows from Lemma 5 that the equilibrium outcome is that firm i orders more inventory and firm j also orders more than at any other equilibrium points. When the credit line given by the bank is relatively low, at least one firm will optimally use up its credit line to support its ordering decision. Thus, this firm orders up to the non-trivial boundary equilibrium level. In this regard, the two firms' order quantities are tightly constrained by the bank loan leverage ratio.

4.1. Limited Liability

In this subsection, we proceed to analyze the effect of the two firms' liability on their non-trivial equilibrium outcomes and divide the analysis into two cases, namely $\beta \geq \max_{i=1,2} c_i q_i^\circ / B_i$ and $\beta < \max_{i=1,2} c_i q_i^\circ / B_i$ (according to Proposition 2). The next result is for the the first case, which shows the impact of the liability of each firm in the pool on the non-trivial interior equilibrium (q_1°, q_2°).

Proposition 3. *Suppose $\beta \geq \max_{i=1,2} c_i q_i^\circ / B_i$. For any given r_b and $i = 1,2$ and $j = 3 - i$, as B_i increases,* (i) *q_i°, q_j°, $k_{2i}(q_i^\circ)$, $k_{2j}(q_j^\circ)$, and $k_{bi}(q_i^\circ, q_j^\circ)$ decrease;* (ii) *$\pi_i^{JF}(q_i^\circ, q_j^\circ)$ and $\pi_j^{JF}(q_j^\circ, q_i^\circ)$ increase.*

Proposition 3 states that if the bank loan leverage ratio is relatively high such that the credit limits can be ignored, each firm can order up to the non-trivial interior equilibrium level, which decreases with its own liability in the pool. This is because under the LJL financing scheme, as firm i assumes greater liability, it will be more concerned about the loss of its capital in the pool. To reduce such loss, the firm has to lower its inventory level in order to induce its partner to decrease the order quantity, which in turn reduces firm j's debt that may need to be paid by firm i's initial capital. Interestingly, as firm i's limited liability becomes greater, firm j will increase its order quantity, i.e., one firm's financial status (equivalently, its capital in the pool) has a positive effect on the other firm's ordering decision. As a result, the two firms' bankruptcy risks decrease in the capital in the pool since the financing needed becomes less. This further results in that the two firms' total loss caused by bankruptcy also decreases and so their expected terminal capital becomes larger. In other words, the shouldering of higher liability by each firm can make the two firms simultaneously obtain more expected terminal capital.

Now we consider the second case, i.e., $\beta < \max_{i=1,2} c_i q_i^\circ / B_i$. With Proposition 2, we see that at least one firm cannot order up to the non-trivial equilibrium inventory level $(q_i^\circ, i = 1,2)$ but uses up all its credit line $(\beta B_i, i = 1,2)$ to support its ordering decision. Consequently, for $i, j = 1,2$ and $j \neq i$, if firm i is such a firm, then its partner firm j's equilibrium order quantity is $q_j^J = \min\{q_j(\beta), \frac{\beta B_j}{c_j}\}$, where $q_j(\beta)$ satisfies Equation (8). Comparing Equations (7) and (8) yields the following result.

Lemma 6. *For any given $r_c \geq 0$ and $\beta > 1$, if $q_j^\circ \geq \frac{\beta B_j}{c_j}$, then $q_i(\beta) \leq q_i^\circ$.*

With Lemma 6, we then state the impact of the capital in the pool on the non-trivial boundary equilibrium inventory levels in the following result.

Proposition 4. *Suppose $\beta < \max_{i=1,2} c_i q_i^\circ / B_i$. For a fixed i, we have*

(i) *if $q_j^J = \beta B_j / c_j$, then* (a) *q_i^J, $k_{2i}(q_i^J)$, and $k_{bi}(q_i^J, q_j^J)$ are quasiconcave in B_i;* (b) *q_j^J and $k_{2j}(q_j^J)$ are unchanged with B_i; and* (c) *$\pi_i^{JF}(q_i^J, q_j^J)$ increases with B_i and $\pi_j^{JF}(q_j^J, q_i^J)$ is quasiconvex in B_i;*

(ii) *if $q_i^J = \beta B_i / c_i$, then* (a) *q_i^J, q_j^J, $k_{2i}(q_i^J)$, $k_{2j}(q_j^J)$, and $k_{bi}(q_i^J, q_j^J)$ increase with B_i; and* (b) *as B_i increases, $\pi_i^{JF}(q_i^J, q_j^J)$ increases and $\pi_j^{JF}(q_j^J, q_i^J)$ decreases.*

From Proposition 4, we see that when the bank loan leverage ratio is relatively low such that firm i optimally uses up its credit line to order, the two firms' equilibrium inventory levels are always increasing in their liability. This is intuitive because in this case, the more capital the firms commit to the pool, the larger credit lines the firms will obtain, so the more capital they have available for ordering. Interestingly, we further see from Proposition 4 that, as firm i that uses up its credit limit has greater liability, the two firms' *effective bankruptcy risks* also become higher. Since the bank loan leverage ratio β is greater than 1 and the two firms' equilibrium order quantities increase with B_i, the firms are induced to borrow more as B_i becomes greater. This is indeed different from the result in the literature on the interface between operations and financial decisions (e.g., Dada and Hu 2008). As a consequence, the two firms' endogenous risk-sharing terms are also increasing in their liability. We also find that firm i's expected terminal capital increases with its initial capital in the pool, which is also intuitive because firm i's financial distress can be improved. Interestingly, in this case, firm i's liability in the pool has a negative effect on firm j's performance. This is because firm i sharply increases its order quantity, which brings more harm to firm j.

However, when firm j uses up all its credit line to order, the above results do not necessarily hold. Specifically, as firm i increases its own liability in the pool, the equilibrium order quantity and the corresponding effective bankruptcy risk of firm j are unchanged, whereas those of firm i initially increase and then decrease. Moreover, higher liability of firm i in the pool is not necessarily harmful to firm j, which is different from the case above. This is because from the quasi-concavity of $\pi_j^{JF}(q_j^J, q_i^J)$, increasing the liability from a relatively high level makes firm i decrease its order quantity, which can reduce the harm to firm j.

4.2. Bank loan Leverage Ratio

We provide the impact of the bank loan leverage ratio β on the two firms' equilibrium inventory decisions and the corresponding bankruptcy-related risks in the following result.

PROPOSITION 5. *For given r_b and $i = 1, 2$ and $j = 3 - i$, q_i^J, $k_{2i}(q_i^J)$, and $k_{bi}(q_i^J, q_j^J)$ are non-decreasing in β.*

Proposition 5 indicates that if the bank loan leverage ratio is high, which means that the credit lines obtained by the two firms are larger, then they have more available capital to order. Consequently, their order quantities are more than (at least not less than) before. Meanwhile, this makes the two firms bear a greater bankruptcy risk. In addition, it is intuitive that as the bank loan leverage ratio increases, the firms' financial status is improved. Thus, it is naturally expected that their expected terminal capital is also improved. But, in fact, this is not the case because the relationships between the firms' expected terminal capital and the bank loan leverage ratio do not possess the monotonicity property. Moreover, in a latter section we will illustrate that increasing the bank loan leverage ratio may not bring more benefit to the two firms simultaneously, i.e., at least one firm's expected terminal capital decreases with the bank loan leverage ratio β. The plausible reason for this result is as follows: In this case, although increasing the bank loan leverage ratio can relax one firm's constraint on its ordering decision, it also makes its partner increase its order quantity. This is because the two firms' strategies are complementary because, under the joint-liability arrangement, they provide a guarantee for each other, i.e., sharing each other's risk. As a result, it may not be beneficial to the former firm (e.g., the former's bankruptcy risk and joint liability may increase). This means that the former firm whose financial status is improved from increasing the bank loan leverage ratio needs to bear more loss from guaranteeing for firm j, which may result in its expected terminal capital becoming less.

5. Equilibrium Financing Choice

In this section we compare the LJL financing scheme with the benchmarking self-financing scheme. To this end, we first analyze the relationship between the optimal (equilibrium) order quantities obtained under the two schemes and then analyze the equilibrium financing choice. As discussed above, each firm's non-trivial equilibrium inventory level is not less than that of the no-financing case. Then, with Propositions 3 and 4, the following corollary further shows how the initial capital of each firm and the bank loan leverage ratio affect the difference between the optimal (equilibrium) order quantities under the two schemes.

Corollary 2. *For given $\beta > 1$ and $i = 1, 2$, there exists a threshold $\hat{B}_i$ such that when $0 < B_i \leq \hat{B}_i$, $q_i^J - q_i^{NF*}$ increases with B_i; and when $\hat{B}_i < B_i < c_i \bar{F}_{\xi_i}^{-1}(c_i'/p_i)$, $q_i^J - q_i^{NF*}$ decreases with B_i.*

Corollary 2 shows that when firm i is relatively financially-constrained, i.e., $0 < B_i \leq \hat{B}_i$, it will over-invest more in ordering as its liability in the pool becomes larger (compared with the no-financing case). This is a somewhat counter-intuitive result, which can be explained as follows: In this case, firm i with very low level of initial capital has to use up all its credit line to order and its order quantity is closely constrained by its liability in the pool. In addition, the bank loan leverage ratio is always greater than 1. This implies that, under the LJL financing scheme, an additional unit of capital committed to the pool can generate more incremental order quantity than that under the self-financing scheme. When firm i's cash position is at the medium level, i.e., $\hat{B}_i < B_i < c_i \bar{F}_{\xi_i}^{-1}(c_i/p_i)$, the degree of firm i's over-investing in ordering becomes less as it has higher liability in the pool. This is because, on the one hand, firm i can get rid of the limit of the credit. On the other hand, notice from Proposition 3 that in this case, the equilibrium of firm i is decreasing in its initial capital/liability. Hence, it is natural to see that as the liability of firm i increases, it becomes more "conservative" in ordering under the LJL financing scheme.

Next, we compare the expected terminal capital between the self-financing scheme and the LJL financing scheme, and characterize the equilibrium financing scheme for the two firms. We first give the following result.

LEMMA 7. *For given r_c and $\beta > 1$, we have $\lim_{B_i \to 0} \frac{d\pi_i^{JF}(\beta B_i/c_i, q_j^d)}{dB_i} \geq \lim_{B_i \to 0} \frac{d\pi_i^{NF}(B_i/c_i)}{dB_i}$ if and only if $\beta\left(\frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1 - r_b\right) + \lim_{B_i \to 0} \bar{F}_{\xi_j}(k_{2j}(q_j^d)) \geq 0$, where q_j^d is firm j's non-trivial boundary equilibrium when firm i uses up all its credit line to order $q_i^d = \beta B_i/c_i$.*

Lemma 7 provides the necessary and sufficient condition under which firm i under the LJL financing scheme obtains more incremental expected terminal capital from the additional capital in the pool (starting at the zero initial capital) than that under the self-financing scheme. Notice that the condition component in Lemma 7, i.e., $\bar{F}_{\xi_j}(k_{2j}(q_j^d))$, is somewhat complex and difficult to manipulate. Thus, to simplify the condition, we give the following corollary.

COROLLARY 3. *For given r_c and $\beta > 1$, the following results hold:*
(i) *if $\beta \geq \frac{p_i}{p_i - c_i}$ and $0 < r_b \leq \frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1$, then $\lim_{B_i \to 0} \frac{d\pi_i^{JF}(\beta B_i/c_i, q_j^d)}{dB_i} \geq \lim_{B_i \to 0} \frac{d\pi_i^{NF}(B_i/c_i)}{dB_i}$;*
(ii) *if $(\beta - 1)\frac{p_i}{c_i} - \beta r_b \leq \beta - 1$, then $\lim_{B_i \to 0} \frac{d\pi_i^{JF}(\beta B_i/c_i, q_j^d)}{dB_i} \leq \lim_{B_i \to 0} \frac{d\pi_i^{NF}(B_i/c_i)}{dB_i}$.*

Corollary 3 implies that when the bank loan leverage ratio is not low (more than $\frac{p_i}{p_i - c_i}$) and the interest rate is not high (less than $\frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1$), the additional expected terminal capital obtained by firm i from the additional initial capital under the LJL financing scheme is more than that under the self-financing scheme. When the bank loan leverage ratio is relatively low and the interest rate is relatively high (e.g., the two factors satisfy $(\beta - 1)\frac{p_i}{c_i} - \beta r_b \leq \beta - 1$), the above result does not hold. This is because in this case, a relatively low bank loan leverage ratio and a relatively high interest rate together significantly constrain the firms' ordering decisions and reduce the marginal profit of a unit product, i.e., $p_i - (1 + r_b)c_i$. Following Corollary 3, Proposition 6 provides the mild conditions under which the LJL financing scheme is the equilibrium scheme for the two firms.

PROPOSITION 6. *For $i = 1, 2$ and $j = 3 - i$, we have*

(i) *if $\beta \geq \frac{p_i}{p_i - c_i}$ and $0 < r_b \leq \frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1$, then there exist two values $\bar{B}_i^1$ and $\bar{B}_i^2$ with $\bar{B}_i^1 \leq \bar{B}_i^2$ such that when $0 \leq B_i < \bar{B}_i$, $\pi_i^{NF*} \leq \pi_i^{JF}(q_i^J, q_j^J)$; and when $\bar{B}_i^2 \leq B_i < c_i \bar{F}_{\xi_i}^{-1}(\frac{c_i'}{p_i})$, $\pi_i^{NF*} \geq \pi_i^{JF}(q_i^J, q_j^J)$;*

(ii) *if β and r_b satisfy $(\beta - 1)\frac{p_i}{c_i} - \beta r_b \leq \beta - 1$, then there exists a value $\bar{\bar{B}}_i$ such that when $0 \leq B_i < \bar{\bar{B}}_i$, $\pi_i^{NF*} \geq \pi_i^{JF}(q_i^J, q_j^J)$.*

Part (i) of Proposition 6 states that the optimal financing scheme for each firm is closely associated with three important factors, i.e., its liability in the capital pool, the bank loan leverage ratio, and the interest rate. Specifically, when the bank loan leverage ratio is relatively large (e.g., $\beta \geq \frac{p_i}{p_i - c_i}$) and the interest rate is not very high (e.g., $0 < r_b \leq \frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1$), firm i with a relatively low cash position should choose the LJL financing scheme. We can explain this result from the following two perspectives. In this case, on the one hand, the unit financing cost of firm i is relatively low and its credit line is relatively high, which makes firm i more likely to order up to its "better" position. On the other hand, a relatively low cash position means that firm i has a very limited guarantee for its partner, so firm i will pay relatively low attention to the loss of its initial capital in the pool because of the joint-liability arrangement. Furthermore, an interesting result in this case is that when firm i's liability committed to the pool is relatively high (e.g., $B_i \geq \bar{B}_i^2$), the self-financing scheme is optimal for firm i because it has a potential high loss of the initial capital in the pool used to guarantee for its partner if it chooses the LJL financing scheme, while the nominal financing needed, i.e., $c_i q_i - B_i$, is in fact relatively low.

When the bank loan leverage ratio β and the interest rate r_b satisfy the condition $(\beta-1)\frac{p_i}{c_i} - \beta r_b \leq \beta - 1$, part (ii) of Proposition 6 gives another interesting result that firm i does not choose the LJL financing scheme even though it is relatively financially-constrained (e.g., $0 \leq B_i < \bar{\bar{B}}_i$). This is because in this case, either the bank loan leverage ratio is relatively small or the interest rate is relatively high, which makes firm i either obtain a low credit line to order or incur a high financing cost. As argued above, in such a situation, it is not advisable for firm i with financial constraints to support its ordering decision via financing. With Proposition 6, we characterize the equilibrium financing scheme in the following result. Formally, we say that the equilibrium financing scheme is the LJL financing scheme for the two firms if $\pi_i^{NF*} \leq \pi_i^{JF}(q_i^J, q_j^J)$ for all $i = 1, 2$ and $j = 3 - i$; otherwise, the self-financing scheme.

THEOREM 2. *For given $\beta > 1$ and r_c, we have*

(i) *the equilibrium financing scheme for the two firms is the LJL financing scheme if $\beta > \max_{i=1,2}\{\frac{p_i}{p_i - c_i}\}$, $r_b < \min_{i=1,2}\{\frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1\}$, and $(B_1, B_2) < (\bar{B}_1, \bar{B}_2)$;*

(ii) *the equilibrium financing scheme for the two firms is the self-financing scheme if one of the following conditions holds:*

(a) *$\beta > \max_{i=1,2}\{\frac{p_i}{p_i - c_i}\}$, $r_b < \min_{i=1,2}\{\frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1\}$, and $B_1 \geq \bar{B}_1^2$ or $B_2 \geq \bar{B}_2^2$;*

(b) *for $i = 1, 2, j = 3 - i$, $(\beta - 1)\frac{p_i}{c_i} - \beta r_b \leq \beta - 1$ and $B_1 < \bar{\bar{B}}_1$ or $B_2 < \bar{\bar{B}}_2$.*

Theorem 2 indicates that when faced with a relatively large bank loan leverage ratio (e.g., $\beta > \max\limits_{i=1,2}\{\dfrac{p_i}{p_i - c_i}\}$) and a relatively low interest rate (e.g., $r_b < \min\limits_{i=1,2}\{\dfrac{\beta-1}{\beta}\dfrac{p_i}{c_i} - 1\}$), the two firms will choose the LJL financing scheme if they are sufficiently financially-constrained (e.g., $(B_1, B_2) < (\bar{B}_1^1, \bar{B}_2^1)$). Otherwise, at least one firm has relatively high initial capital, and it follows from Proposition 6 that this firm will use up all its initial capital to order, which breaks the cooperation in financing. Thus, the firms have to choose the self-financing scheme. Moreover, when the bank loan leverage ratio is relatively small and the interest rate is very high (they satisfy the condition $(\beta-1)\frac{p_i}{c_i} - \beta r_b \leq \beta - 1$), the above result does not hold. In this case, the two firms still choose the self-financing scheme even though they are extremely capital-constrained, e.g., $(B_1, B_2) < (\bar{\bar{B}}_1, \bar{\bar{B}}_2)$.

6. Numerical Studies

We conduct numerical studies to examine the impacts of the market parameters (e.g., retail price, unit purchase cost, and demand variability) on the two firms' equilibrium outcomes (e.g., order quantity, expected terminal capital, and bankruptcy-related risk) in Section 6.1. Then, in Section 6.2 we further analyze the efficiency of the LJL financing scheme. Throughout our experiments, we adopt the base parameters as follows: for $i=1,2$, $p_i=1.2$, $c_i=\{0.4,0.6\}$, $B_i=\{2,4,6\}$, $r_b=0.05$ and $\beta=4$; and the demands follow a bivariate lognormal distribution with $\mu_i=20$, $\sigma_i=\{16,20\}$, and $\rho=\{-0.5,0,0.5\}$. For ease of exposition, for $i=1,2$ and $j=3-i$, let $\pi_i^{JF*}=\pi_i^R(q_i^J,q_j^J)$ be the expected terminal capital of firm i in equilibrium, $\delta_i=\Pr(\xi_i\geq k_{2i}(q_i^J))$ be the probability that firm i can meet its loan obligation via its own sales revenue, i.e., $p_i\min\{q_i,D_i\}$, and initial capital B_i in the pool, $BP_i=\Pr(R_{i1}^b+R_{i2}^b)$ be the final bankruptcy risk after firm i provides the guarantee for its partner, and denote $\phi_i=\phi(q_i^J,q_j^J)=(1+r_b)c_i\Pr(\widehat{R}_{i1}^b)$ as the (endogenously) risk-sharing term of firm i under the LJL financing scheme.

6.1. Impacts of Market Parameters

In the previous section we showed the impacts of the bank loan leverage ratio and the liability in the pool. Thus, in this sub-section we investigate how the market parameters (e.g., retail price, unit purchase cost, and demand variability) affect the two firms' ordering decisions, and their corresponding expected terminal capital and bankruptcy-related risks.

Table 1 Effect of retail price p_1 on the two firms' equilibrium inventory decisions, expected terminal capital, bankruptcy-related risks, and risk-sharing terms for $p_2=1.2$

	p_1	q_1^J	q_2^J	π_1^{JF*}	π_2^{JF*}	δ_1	δ_2	BP_1	BP_2	ϕ_1	ϕ_2
$\rho=-0.5$	0.8	15.0606	20.8304	7.5268	13.0258	0.9916	0.9745	0.0084	0.0255	$0.2217\cdot10^{-5}$	$0.0147\cdot10^{-5}$
	1.0	18.2726	20.8304	10.1487	13.0144	0.9802	0.9745	0.0198	0.0255	$0.3335\cdot10^{-5}$	$0.1361\cdot10^{-5}$
	1.2	20.8304	20.8304	13.0037	13.0037	0.9745	0.9745	0.0255	0.0255	$0.3259\cdot10^{-5}$	$0.3259\cdot10^{-5}$
	1.4	22.9600	20.8304	16.0166	12.9967	0.9729	0.9745	0.0271	0.0255	$0.2755\cdot10^{-5}$	$0.4885\cdot10^{-5}$
	1.6	24.7948	20.8304	19.1452	12.9931	0.9734	0.9745	0.0266	0.0255	$0.2206\cdot10^{-5}$	$0.5908\cdot10^{-5}$
$\rho=0$	0.8	15.0606	20.8304	7.5275	13.0259	0.9916	0.9745	0.0091	0.0256	$0.2782\cdot10^{-3}$	$0.0317\cdot10^{-3}$
	1.0	18.2726	20.8304	10.1497	13.0150	0.9802	0.9745	0.0205	0.0258	$0.2799\cdot10^{-3}$	$0.1374\cdot10^{-3}$
	1.2	20.8304	20.8304	13.0048	13.0048	0.9745	0.9745	0.0261	0.0261	$0.2460\cdot10^{-3}$	$0.2460\cdot10^{-3}$
	1.4	22.9600	20.8304	16.0177	12.9982	0.9729	0.9745	0.0276	0.0262	$0.2100\cdot10^{-3}$	$0.3235\cdot10^{-3}$
	1.6	24.7948	20.8304	19.1462	12.9949	0.9734	0.9745	0.0270	0.0264	$0.1785\cdot10^{-3}$	$0.3681\cdot10^{-3}$
$\rho=0.5$	0.8	15.1155	20.8370	7.5328	13.0269	0.9913	0.9745	0.0124	0.0260	0.0015	0.0002
	1.0	18.3196	20.8546	10.1562	13.0188	0.9798	0.9743	0.0232	0.0272	0.0013	0.0007
	1.2	20.8699	20.8699	13.0115	13.0115	0.9742	0.9742	0.0283	0.0283	0.0011	0.0011
	1.4	22.9940	20.8796	16.0242	13.0071	0.9727	0.9742	0.0295	0.0290	0.0009	0.0013
	1.6	24.8247	20.8846	19.1525	13.0051	0.9732	0.9741	0.0287	0.0294	0.0008	0.0015

Retail Price. The effect of the retail prices is shown in Table 1. As p_1 increases, the two firms simultaneously increase their order quantities, and the incremental size of firm 1's order quantity is more than that of firm 2' order quantity. This is because the joint liability between the firms is limited, so the direct effect of p_1 on firm 2's strategy and the indirect effect of p_1 on firm 2's strategy via firm 1's strategy are relatively low; and a relatively low retail price p_1 will significantly constrain firm 1's marginal profit and reduce its ability to repay. The two results naturally reduce the probability δ_1 that firm 1 uses its sales revenue and initial capital to repay completely, and firm 2's such probability δ_2 is very slightly affected. Furthermore, the risk-sharing terms of the two firms are relatively small and very slightly affected by the retail prices, especially when the demand correlation is negative. In particular, as firm 1 increases its retail price, such term of firm 2, i.e., ϕ_2, increases, but it is not the case for firm 1. Note that a higher retail price p_1 means a greater marginal profit for firm 1. As a result, firm 1's revenue also increases and so its expected terminal capital π_1^{JF*} increases, whereas firm 2's expected terminal capital π_2^{JF*} decreases since the risk-sharing term of firm 2 becomes larger. This implies that once the two firms choose the LJL financing scheme, one firm prefers to cooperate with the firm that has a relatively low or high retail price.

Table 2 Effect of unit purchase cost c_1 on the two firms' equilibrium inventory decisions, expected terminal capital, bankruptcy-related risks, and risk-sharing terms for $c_2 = 0.6$

	c_1	q_1^J	q_2^J	π_1^{JF*}	π_2^{JF*}	δ_1	δ_2	BP_1	BP_2	ϕ_1	ϕ_2
$\rho = -0.5$	0.4	20.8304	15.6367	12.9635	9.3533	0.9745	0.9510	0.0255	0.0490	$0.1631 \cdot 10^{-4}$	$0.1090 \cdot 10^{-4}$
	0.5	17.9925	15.6367	10.9971	9.3309	0.9605	0.9510	0.0395	0.0490	$0.3265 \cdot 10^{-4}$	$0.2982 \cdot 10^{-4}$
	0.6	15.6367	15.6367	9.3132	9.3132	0.9510	0.9510	0.0490	0.0490	$0.4969 \cdot 10^{-4}$	$0.4969 \cdot 10^{-4}$
	0.7	13.5209	15.6367	7.8613	9.3089	0.9489	0.9510	0.0512	0.0490	$0.6082 \cdot 10^{-4}$	$0.5510 \cdot 10^{-4}$
	0.8	11.4777	15.6367	6.6125	9.3229	0.9561	0.9510	0.0440	0.0490	$0.5875 \cdot 10^{-4}$	$0.3837 \cdot 10^{-4}$
$\rho = 0$	0.4	20.8304	15.6367	12.9668	9.3551	0.9745	0.9510	0.0271	0.0497	0.0007	0.0005
	0.5	18.0236	15.6631	11.0008	9.3340	0.9602	0.9506	0.0417	0.0508	0.0010	0.0009
	0.6	15.6733	15.6733	9.3173	9.3173	0.9505	0.9505	0.0515	0.0515	0.0013	0.0013
	0.7	13.5618	15.6758	7.8655	9.3130	0.9481	0.9504	0.0539	0.0517	0.0015	0.0013
	0.8	11.5205	15.6678	6.6163	9.3257	0.9552	0.9506	0.0467	0.0511	0.0016	0.0011
$\rho = 0.5$	0.4	20.9223	15.6810	12.9817	9.3635	0.9738	0.9504	0.0321	0.0521	0.0025	0.0015
	0.5	18.0908	15.7158	11.0187	9.3488	0.9594	0.9498	0.0466	0.0546	0.0031	0.0028
	0.6	15.7424	15.7424	9.3367	9.3367	0.9494	0.9494	0.0565	0.0565	0.0037	0.0037
	0.7	13.6367	15.7487	7.8852	9.3332	0.9467	0.9493	0.0591	0.0570	0.0043	0.0040
	0.8	11.6066	15.7280	6.6351	9.3419	0.9535	0.9496	0.0524	0.0556	0.0050	0.0033

Unit Purchase Cost. Table 2 illustrates the effect of the unit purchase cost. When the unit purchase cost c_1 becomes larger, firm 1 will decrease its order quantity, but firm 2 initially slowly increases and then decreases the order quantity (especially for a positive demand correlation). The reason behind the latter result is threefold. First, firm 1 sharply reduces its order quantity, which decreases the amount of its bank loan, so reducing the potential debt guaranteed by firm 2. Second, firm 2's debt, if any, is also guaranteed by firm 1. Third, firm 2 may be more concerned about a relatively high product cost of firm 1. Then, the final bankruptcy risk of firm 1, i.e., BP_1, increases, but firm 2's final bankruptcy risk BP_2 initially increases and then decreases. So is the effect of cost c_1 on the risk-sharing terms of the two firms, i.e., ϕ_1 and ϕ_2. As a consequence, the two firms' expected terminal capital decreases in the unit purchase cost c_1. In this regard, one firm is a bit more willing to cooperate with a partner that has a greater advantage in the production cost.

Demand Volatility. We finally illustrate how demand uncertainty influences the two firms' operations decisions. As shown in Table 3, as σ_1 increases, the two firms will simultaneously decrease their order quantities only for negative demand correlation, which corresponds to low profit margins. Moreover, the probabilities that the two firms use their own initial capital in the pool and sales revenues to repay completely decrease because greater demand uncertainty results in a higher risk. Interestingly, it is observed that the final bankruptcy risks of the two firms will always increase with firm 1's demand variability even though their order quantities decrease, since a higher demand volatility is more concerned by firm 1, which in turn significantly affects firm 2. Consequently, the two firms' expected terminal capital decreases in firm 1's demand uncertainty, which implies that one firm prefers to cooperating with a firm whose demand variability is lower.

Combining Tables 1, 2, and 3, we further observe that the impact of the characteristics of the two firms' products on the risk-sharing terms is significantly small, especially when the demand correlation is non-positive. This mainly results from the relatively limited liability in the capital pool. Furthermore, it is interesting to see that the final bankruptcy risks in the case of negative demand correlation are in general not higher than those in the case of positive demand correlation. This means that the bank is more willing to support firms whose demand correlation is negative.

Table 3 Effect of demand variability σ_1 on the two firms' equilibrium inventory decisions, expected terminal capital, bankruptcy-related risks, and risk-sharing terms for $\sigma_2 = 16$

	σ_1	q_1^J	q_2^J	π_1^{JF*}	π_2^{JF*}	δ_1	δ_2	BP_1	BP_2	ϕ_1	ϕ_2
$\rho = -0.5$	12	21.2914	20.8304	14.3656	13.0258	0.9950	0.9745	0.0050	0.0255	$0.0034{\cdot}10^{-4}$	$0.0038{\cdot}10^{-4}$
	14	21.0652	20.8304	13.6521	13.0178	0.9867	0.9745	0.0133	0.0255	$0.0130{\cdot}10^{-4}$	$0.0140{\cdot}10^{-4}$
	16	20.8304	20.8304	13.0037	13.0037	0.9745	0.9745	0.0255	0.0255	$0.0326{\cdot}10^{-4}$	$0.0326{\cdot}10^{-4}$
	18	20.6069	20.8304	12.4180	12.9838	0.9594	0.9745	0.0406	0.0255	$0.0633{\cdot}10^{-4}$	$0.0581{\cdot}10^{-4}$
	20	20.4036	20.8304	11.8901	12.9590	0.9425	0.9745	0.0575	0.0255	$0.1045{\cdot}10^{-4}$	$0.0876{\cdot}10^{-4}$
$\rho = 0$	12	21.2914	20.8304	14.3659	13.0260	0.9950	0.9745	0.0053	0.0257	$0.0980{\cdot}10^{-3}$	$0.1089{\cdot}10^{-3}$
	14	21.0652	20.8304	13.6528	13.0182	0.9867	0.9745	0.0137	0.0259	$0.1722{\cdot}10^{-3}$	$0.1841{\cdot}10^{-3}$
	16	20.8304	20.8304	13.0048	13.0048	0.9745	0.9745	0.0261	0.0261	$0.2460{\cdot}10^{-3}$	$0.2460{\cdot}10^{-3}$
	18	20.6069	20.8304	12.4196	12.9859	0.9594	0.9745	0.0413	0.0262	$0.3122{\cdot}10^{-3}$	$0.2883{\cdot}10^{-3}$
	20	20.4036	20.8304	11.8922	12.9622	0.9425	0.9745	0.0584	0.0262	$0.3687{\cdot}10^{-3}$	$0.3132{\cdot}10^{-3}$
$\rho = 0.5$	12	21.2914	20.8304	14.3691	13.0273	0.9950	0.9745	0.0069	0.0275	0.0008	0.0009
	14	21.0963	20.8497	13.6578	13.0218	0.9866	0.9744	0.0158	0.0281	0.0010	0.0011
	16	20.8699	20.8699	13.0115	13.0115	0.9742	0.9742	0.0283	0.0283	0.0011	0.0011
	18	20.6535	20.8962	12.4276	12.9964	0.9590	0.9740	0.0437	0.0284	0.0011	0.0010
	20	20.4562	20.9269	11.9013	12.9769	0.9418	0.9738	0.0608	0.0284	0.0012	0.0009

6.2. Efficiency of the LJL Financing Scheme

We now analyze the value of using the LJL financing scheme compared with using the self-financing scheme, which is illustrated in the following table.We use $\pi_i^{JF*} - \pi_i^{NF*} (i = 1, 2)$ to represent such value of the LJL financing scheme. As shown in Table 4, when the bank loan leverage ratio is relatively low (e.g., $\beta = 2$), one of the two firms will use up its credit line to order. In this regard, as firm 1 has greater liability in the pool, the two firms' equilibrium order quantities and the corresponding bankruptcy risks increase. When the ratio is relatively high (e.g., $\beta = 6$), the above result does not hold. These results are consistent with Propositions 3 and 4. Furthermore, as B_1 increases, the value of firm 1 under the LJL financing scheme over that under the self-financing scheme, i.e., $\pi_1^{JF*} - \pi_1^{NF*}$, will decrease, and the value of firm 2, i.e., $\pi_2^{JF*} - \pi_2^{NF*}$, will decrease for a relatively low bank loan leverage ratio but increase for a relatively high bank loan leverage ratio (which is consistent with Proposition 4). This is because higher initial capital of firm 1 means its constraint on the ordering decision can be more relaxed but leads to a greater loss because firm 1 needs to use its capital in the pool to guarantee for its partner. In addition, in this case the two firms' bankruptcy risks, i.e., BP_1 and BP_2, decrease in firm 1's liability in the pool.

Table 4 Comparison between the LJL financing scheme and the benchmarking self-financing scheme for $\rho = -0.5$ and $B_2 = 4$

B_1	β	r_b	$q_1^J - q_1^{NF*}$	$q_2^J - q_2^{NF*}$	$\pi_1^{JF*} - \pi_1^{NF*}$	$\pi_2^{JF*} - \pi_2^{NF*}$	BP_1	BP_2
2	2	0.01	5.0000	10.0000	3.0387	2.3174	0.0008	0.0151
		0.05	5.0000	10.0000	2.8737	2.0028	0.0012	0.0197
		0.10	5.0000	10.0000	2.6658	1.6117	0.0017	0.0264
	4	0.01	15.0000	11.3277	5.4524	2.2641	0.0548	0.0232
		0.05	15.0000	10.8304	5.1484	1.9166	0.0633	0.0256
		0.10	15.0000	10.2369	4.7724	1.4905	0.0748	0.0285
	6	0.01	17.1305	11.3277	1.2780	0.3982	0.3570	0.3089
		0.05	16.6732	10.8304	1.2068	0.3531	0.3701	0.3173
		0.10	16.1290	10.2369	1.1200	0.2997	0.3829	0.3270
4	2	0.01	10.0000	10.0000	2.3052	2.3052	0.0151	0.0151
		0.05	10.0000	10.0000	1.9852	1.9852	0.0197	0.0197
		0.10	10.0000	10.0000	1.5852	1.5852	0.0264	0.0264
	4	0.01	11.3277	11.3277	2.3191	2.3191	0.0232	0.0232
		0.05	10.8304	10.8304	1.9869	1.9869	0.0255	0.0255
		0.10	10.2369	10.2369	1.5831	1.5831	0.0283	0.0283
	6	0.01	11.3277	11.3277	2.3191	2.3191	0.0232	0.0232
		0.05	10.8304	10.8304	1.9869	1.9869	0.0255	0.0255
		0.10	10.2369	10.2369	1.5831	1.5831	0.0283	0.0283
6	2	0.01	6.0364	10.0000	0.5522	2.3167	0.0021	0.0151
		0.05	5.5138	10.0000	0.2150	2.0020	0.0026	0.0197
		0.10	4.8896	10.0000	-0.1970	1.6108	0.0032	0.0264
	4	0.01	6.0364	11.3277	0.5420	2.3409	0.0021	0.0232
		0.05	5.5138	10.8304	0.2072	2.0116	0.0026	0.0255
		0.10	4.8896	10.2369	-0.1998	1.6116	0.0032	0.0283
	6	0.01	6.0364	11.3277	0.5420	2.3409	0.0021	0.0232
		0.05	5.5138	10.8304	0.2072	2.0116	0.0026	0.0255
		0.10	4.8896	10.2369	-0.1998	1.6116	0.0032	0.0283

We also observe that as the bank loan leverage ratio increases, the two firms will over-invest more in their ordering. Thus, the bankruptcy risks faced by them, i.e., BP_1 and BP_2, will also increase. This is consistent with the results in Proposition 5. Interestingly, we also see that a higher bank loan leverage ratio, which means greater credit lines for the two firms, does not always bring more benefit to the firms (e.g., firm 1). Also, even when the two firms have equal proportions of the liability in the pool, the values of the two firms adopting the LJL financing scheme do not increase with the bank loan leverage ratio simultaneously. In sum, we see that increasing the credit lines is not necessarily beneficial to the two firms simultaneously. Moreover, the numerical results also show that when the initial capital is at a relatively high level (e.g., $B_1 \geq 6$) and the interest rate is considerable (e.g., $r_b \geq 0.1$), the value of firm 1 using the LJL financing is negative, which means that adopting the no-finance scheme is preferable. This holds for the case when the interest rate is relatively high, even though the two firms are very financially constrained (e.g., $1 \leq r_b < 2$, $1 < \beta \leq 1.5$). These results are consistent with Proposition 6.

7. Discussions

7.1. Endogenizing the Bank-loan Leverage Ratio and the Interest Rate

So far we have analyzed the firms' operations and financial decisions for exogenously given bank loan leverage ratio and interest rate. However, in anticipation of the best responses of the two firms in Section 3.2, the bank needs to determine the bank loan leverage ratio β (or credit line) and interest rate r_b. We refer to this as the bank's problem, which we consider in this section. In line with the literature (e.g., Kouvelis and Zhao 2012, Chod 2017, Yang and Birge 2017), we assume that the bank loan market is competitive, which means that the loan terms (β and r_b) are offered at which the bank expects to make zero profit. Since under the LJL financing scheme, the two firms should bear each other's bankruptcy risk, at the end of the selling period, the bank's has three possible outcomes for its terminal revenue as follows: (a) it completely recoups its loans to the two firms, (b) it completely recoups the loan to one firm but only partially recoups the loan to the other firm, and (c) it fails to recoup the loans to the two firms. We can classify the three outcomes into the following four regions:

$$\Omega_1 = \Big\{ \boldsymbol{\xi} \geq 0 : \xi_i \leq k_{1i}(q_i), \xi_j \leq k_{1j}(q_j), p_i\xi_i + p_j\xi_j \leq p_i k_{bi}(q_i, q_j) \Big\},$$
$$\Omega_2 = \Big\{ \boldsymbol{\xi} \geq 0 : \xi_i \leq \frac{p_i k_{2i}(q_i) - B_j}{p_i}, \xi_j \geq k_{1j}(q_j) \Big\},$$
$$\Omega_3 = \Big\{ \boldsymbol{\xi} \geq 0 : \xi_j \leq \frac{p_j k_{2j}(q_j) - B_i}{p_j}, \xi_i \geq k_{1i}(q_i) \Big\},$$
$$\Omega_4 = \Big\{ \boldsymbol{\xi} \geq 0 : p_i\xi_i + p_j\xi_j \geq p_i k_{bi}(q_i, q_j), \xi_i \geq \frac{p_i k_{2i}(q_i) - B_j}{p_i}, \xi_j \geq \frac{p_j k_{2j}(q_j) - B_i}{p_j} \Big\},$$

where $(q_i, q_j) = (q_i^J, q_j^J)$. In Appendix B, we use Figure 2 to illustrate the four regions above, and state their event meanings. Based on the above discussion, then we express the expected terminal revenue of the bank as follows:

$$\begin{aligned} \Pi^B(\beta, r_b) =& \Pr(\Omega_1) E\left(p_i\xi_i + p_j\xi_j + B_i + B_j | \Omega_1\right) + \Pr(\Omega_2) E\left(p_j k_{1j}(q_j) + p_i\xi_i + B_i + B_j | \Omega_2\right) \\ &+ \Pr(\Omega_3) E\left(p_i k_{1i}(q_i) + p_j\xi_j + B_i + B_j | \Omega_3\right) + \Pr(\Omega_4) E\left(p_i k_{1i}(q_i) + p_j k_{1j}(q_j) | \Omega_4\right), \end{aligned} \tag{9}$$

where $(q_i, q_j) = (q_i^J, q_j^J)$ for $i = 1,2$ and $j = 3 - i$. The optimal bank loan leverage ratio β^* and interest rate r_b^*, with $0 \le r_b \le \min_{i=1,2}\{p_i/c_i - 1\}$ and $\beta > 1$, must satisfy the following equation

$$\Pi^B(\beta, r_b) = (1 + r_f)(c_i q_i + c_j q_j). \tag{10}$$

It follows from Proposition 2 that the bank's expected terminal revenue $\Pi^B(\beta, r_b)$ is closely affected by the bank loan leverage ratio if and only if when $\beta \le \max\limits_{i=1,2} c_i q_i^\circ / B_i$. However, it is somewhat complicated to analytically solve Equation (10), so we conduct numerical studies to further analyze the bank's problem. As shown in Table 5, an interesting observation is that the relationship

Table 5 The equilibrium outcomes for endogenized bank-loan leverage ratio and interest rate for $r_f = 0.03$ and $\sigma_i = 20$, $i = 1, 2$

B_1	(β^*, r_b^*)	$\Pi^B(\beta^*, r_b^*)$	q_1^J	q_2^J	π_1^{JF*}	π_2^{JF*}	π_1^{NF*}	π_2^{NF*}	k_{21}	k_{22}	k_{bi}
(a) $\rho = -0.5$											
2	(2,0.0303)	12.3600	6.6667	13.3333	5.3575	8.6557	3.9574	7.5256	1.7677	3.5354	5.3031
	(3,0.0310)	15.2250	10.0000	14.6360	6.2546	8.6356	3.9574	7.5256	3.4883	4.2115	7.6997
	(4,0.0310)	17.2930	13.3333	14.6490	6.7242	8.4757	3.9574	7.5256	5.2081	4.2198	9.4280
	(6,0.0310)	19.1845	16.3868	14.6561	6.8477	8.2134	3.9574	7.5256	6.8047	4.2433	11.0480
4	(2,0.0300)	16.4800	13.3333	13.3333	8.6076	8.6076	7.5256	7.5256	3.5334	3.5334	7.0668
	(3,0.0301)	18.4115	14.8960	14.8960	8.5817	8.5817	7.5256	7.5256	4.3387	4.3387	8.6774
	(4,0.0301)	18.4114	14.8960	14.8960	8.5817	8.5817	7.5256	7.5256	4.3387	4.3387	8.6774
	(6,0.0301)	18.4114	14.8960	14.8960	8.5817	8.5817	7.5256	7.5256	4.3387	4.3387	8.6774
6	(2,0.0300)	16.8187	13.8814	13.3333	10.5635	8.6540	10.4775	7.5256	2.1489	3.5333	5.6823
	(3,0.0300)	17.7915	13.8814	14.9074	10.5027	8.6883	10.4775	7.5256	2.1489	4.3440	6.4930
	(4,0.0300)	17.7915	13.8814	14.9074	10.5027	8.6883	10.4775	7.5256	2.1489	4.3440	6.4930
	(6,0.0300)	17.7915	13.8814	14.9074	10.5027	8.6883	10.4775	7.5256	2.1489	4.3440	6.4930
8	(2,0.0300)	16.7105	13.7063	13.3333	12.5573	8.6609	12.8476	7.5256	0.3921	3.5333	3.9254
	(3,0.0300)	17.6833	13.7063	14.9074	12.4964	8.6953	12.8476	7.5256	0.3921	4.3440	4.7361
	(4,0.0300)	17.6833	13.7063	14.9074	12.4964	8.6953	12.8476	7.5256	0.3921	4.3440	4.7361
	(6,0.0300)	17.6833	13.7063	14.9074	12.4964	8.6953	12.8476	7.5256	0.3921	4.3440	4.7361
(b) $\rho = 0.5$											
2	(2,0.0310)	12.3600	6.6667	13.3333	5.3614	8.6518	3.9574	7.5256	1.7700	3.5400	5.3100
	(3,0.0343)	15.1479	10.0000	14.5111	6.2650	8.6296	3.9574	7.5256	3.5050	4.1713	7.6763
	(4,0.0390)	17.2242	13.3333	14.5375	6.7124	8.4915	3.9574	7.5256	5.2600	4.2189	9.4788
	(6,0.0451)	18.8721	15.9278	14.6095	6.8028	8.3032	3.9574	7.5256	6.6567	4.3011	10.9578
4	(2,0.0326)	16.4800	13.3333	13.3333	8.6076	8.6076	7.5256	7.5256	3.5504	3.5504	7.1007
	(3,0.0352)	18.0950	14.6400	14.6400	8.5927	8.5927	7.5256	7.5256	4.2442	4.2442	8.4884
	(4,0.0352)	18.0950	14.6400	14.6400	8.5927	8.5927	7.5256	7.5256	4.2442	4.2442	8.4884
	(6,0.0352)	18.0950	14.6400	14.6400	8.5927	8.5927	7.5256	7.5256	4.2442	4.2442	8.4884
6	(2,0.0309)	16.8175	13.8795	13.3333	10.5671	8.6504	10.4775	7.5256	2.1540	3.5391	5.6931
	(3,0.0315)	17.7000	13.9057	14.7350	10.5194	8.6792	10.4775	7.5256	2.1719	4.2662	6.4381
	(4,0.0315)	17.7000	13.9057	14.7350	10.5194	8.6792	10.4775	7.5256	2.1719	4.2662	6.4381
	(6,0.0315)	17.7000	13.9057	14.7350	10.5194	8.6792	10.4775	7.5256	2.1719	4.2662	6.4381
8	(2,0.0301)	16.7105	13.7063	13.3333	12.5582	8.6599	12.8476	7.5256	0.3929	3.5341	3.9271
	(3,0.0303)	17.6679	13.7403	14.8486	12.5018	8.6926	12.8476	7.5256	0.4118	4.3160	4.7278
	(4,0.0303)	17.6679	13.7403	14.8486	12.5018	8.6926	12.8476	7.5256	0.4118	4.3160	4.7278
	(6,0.0303)	17.6679	13.7403	14.8486	12.5018	8.6926	12.8476	7.5256	0.4118	4.3160	4.7278

between the optimal bank loan leverage ratio β^* and interest rate r_b^* satisfying Equation (10) is always non-negative. That is, as the bank offers a higher leverage ratio to its firms, it correspondingly increases the interest rate slightly in order to control the risks from the increasing order quantities of the two firms. As shown in the previous section, the final bankruptcy risks in the case of negative demand correlation (e.g., $\rho = -0.5$) are smaller, relative to those in the other cases. This is true for the case of endogenized bank-loan leverage ratio and interest rate. As a result, comparing Table 5(a) and (b), the optimal interest rate when the demand correlation is negative is lower than that when the demand correlation is positive. Moreover, since the final bankruptcy risks are very low, the bank will set the optimal interest rate closer to the risk-free rate, while generating a slightly higher expected revenue in the case of negative demand correlation.

Table 5 also illustrates that as the capital committed by the two firms to the pool increases, the bank may initially increase and then decrease (rather than always increases) its interest rate, especially in the case of positive demand correlation. For the first half part, this is because in this case, at least one firm will use up all the credit line to order, so its order quantity linearly increases with its own capital in the pool. This results in a larger increase in its effective bankruptcy risk. To reduce the potential loss from this, it is optimal for the bank to slightly increase the interest rate. As a result, the bank's expected terminal revenue may also initially increase and then decrease.

In addition, we observe that the two firms' equilibrium order quantities, so their expected terminal capital, are not always simultaneously increasing in the capital in the pool. In particular, when the bank loan leverage ratio is relatively high (such that the two firms can order up to the non-trivial interior equilibrium level), as one firm's capital in the pool increases, its order quantity and the corresponding bankruptcy risk decrease, but it is not the case for the other firm. However, the firms' expected terminal capital will always increase with the initial capital, which is consistent with Proposition 3. When the bank loan leverage ratio is low, there is at least one firm (e.g., firm 1) using up all its credit line to order and the impacts of the capital in the pool on the firms' expected terminal capital are consistent with Proposition 4. Moreover, Table 5 shows that our main insights from Proposition 5 and Corollary 2 remain unchanged.

It is evident from Table 5 that our numerical results are also consistent with Theorem 2. When the bank-loan leverage ratio is relatively high (e.g., $\beta \geq 3$), which satisfies $\beta > \frac{p_i}{p_i - c_i} = 2$, the optimal interest rate charged by the bank satisfies $0 < r_b \leq \frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1$. Correspondingly, the values of the firms using LJL financing scheme are positive if the capital in the pool is low (e.g., $B_1 \leq 6$), so both firms prefer the LJL financing scheme; otherwise (e.g., $B_1 = 8$), there is at least one firm receiving a negative value of using the LJL financing scheme, so cooperation in financing cannot be achieved. In addition, when the bank sets a sufficiently low leverage ratio (e.g., $1 < \beta \leq 2$), the resulting interest rate will satisfy $(\beta - 1)\frac{p_i}{c_i} - \beta r_b \leq \beta - 1$. Accordingly, the value of at least one firm using the LJL financing scheme is negative (especially when the unit product costs are relatively high), so the self-financing scheme becomes the equilibrium financing scheme.

7.2. Generalizing the Allocation of Initial Capital

We now consider the case where the firms have the opportunity to allocate their initial capital. That is, each firm can commit a portion of its own initial capital to the joint capital pool to obtain external financing, while using the remaining initial capital to self-finance inventory. Note that although having the remaining initial capital to self-finance inventory can reduce their total financing costs, their credit lines may be more constrained (related to the case in Section 3.2). In this regard, the firms need to make a trade-off between the financing cost and potential loss from the limited credit line. Then, it is natural to ask the question: Under what condition is it that committing a part of the initial capital to the pool is always better?

To answer this question, following the practice in the Introduction, we consider a game among the three players, i.e., the two firms and the bank, as follows: Prior to the selling season, the two firms simultaneously decide to commit the amount of liability (denoted by T_i, $i=1,2$), from their own initial capital B_i, to the capital pool, i.e., $T_i \le B_i$. Next, the bank makes its decisions, i.e., setting the bank loan leverage ratio β and interest rate r_b, with the expectation that it makes zero profit. Upon knowing the bank's decisions, the two firms simultaneously place their order quantities by borrowing a loan $c_iq_i-(B_i-T_i)$, within the credit line plus the remaining initial capital, i.e., $\beta T_i+B_i-T_i$. The remaining events are the same as those in the second and third steps described in Section 3. Similar to Section 3.2, we formulate the expected terminal capital of firm i, for $i=1,2$ and $j=3-i$, as follows:

$$\pi_i^A(q_i,q_j,\mathbf{T}) = E_{\xi_i,\xi_j}\Big\{[p_i\min\{q_i,\xi_i\}-(1+r_b)(c_iq_i+T_i-B_i)]^+ \\ +\big[T_i-((1+r_b)(c_iq_i+T_i-B_i)-p_i\xi_i)^+-((1+r_b)(c_jq_j+T_j-B_j)-p_j\xi_j-T_j)^+\big]^+\Big\}, \quad (11)$$

where $\mathbf{T}=(T_i,T_j)$ and the superscript A denotes the case of endogenized allocation of the initial capital. Next we solve the problems faced by the two firms and the bank using the backward approach. First, given the firms' liability, i.e., T_i, $i=1,2$, and the bank's decisions β and r_b, each firm i chooses its order quantity q_i to maximize its expected terminal capital. That is,

$$q_i^A(\beta,r_b,\mathbf{T}) = \underset{B_i/c_i\le q_i\le((\beta-1)T_i+B_i)/c_i}{\arg\max}\ \pi_i^{JF}(q_i,q_j,\mathbf{T}),\ i=1,2,j=3-i. \quad (12)$$

Second, based on the firms' best order quantities above, we find the optimal leverage ratio $\beta^A(\mathbf{T})$ and interest rate $r_b^A(\mathbf{T})$ of the bank by solving the following equation

$$\Pi^B(\beta,r_b,\mathbf{T}) = (1+r_f)[(c_iq_i^A(\beta,r_b,\mathbf{T})+T_i-B_i)+(c_jq_j^A(\beta,r_b,\mathbf{T})+T_j-B_j)], \quad (13)$$

where $\Pi^B(\beta,r_b,\mathbf{T})$ is the same as that in Equation (9) by replacing B_i with T_i and c_iq_i with $c_iq_i^A(\beta,r_b,\mathbf{T})+T_i-B_i$. Third, based on the solutions of Equations (12) and (13), each firm i chooses its liability T_i to commit to the capital pool. That is,

$$T_i^A = \underset{T_i\le B_i}{\arg\max}\ \pi_i^{JF}(q_i^A(\beta^A(\mathbf{T}),r_b^A(\mathbf{T}),\mathbf{T}),q_i^A(\beta^A(\mathbf{T}),r_b^A(\mathbf{T}),\mathbf{T}),\mathbf{T}),\ i=1,2,j=3-i. \quad (14)$$

Similar to the analysis of Section 3.2, for given allocation of the initial capital and the bank's decisions, the two firms in Problem (12) play a non-cooperative game, which is a supermodular game. Thus, the existence of equilibrium of the inventory level $(q_1^A(\beta, r_b, \mathbf{T}), q_2^A(\beta, r_b, \mathbf{T}))$ is assured, and a greatest and a smallest equilibrium point exist. Unfortunately, with the firms' best responses for the inventory levels, solving Problems (13) and (14) is theoretically intractable, so we conduct numerical studies to analyze the problems. As shown in Table 6, as the bank increases the leverage ratio, it increases the optimal interest rate to create risk hedging, which is similar to the result in Section 7.1. Meanwhile, the firms will commit less liability to the pool since a "better" credit line is finally similarly obtained. Moreover, when the firms have relatively high initial capital (e.g., $B_i \geq 4$), committing a part of their initial capital can generate more expected capital and lower the bankruptcy risks, so it is a better option for the firms. However, when the firms are very financially constrained (and the leverage ratio offered by the bank is small), it is optimal for them to commit all their initial capital to the capital. In this case, the probability that the bank receives all the firms' loan obligations is greater.

Table 6 The equilibrium outcomes for endogenized allocation of the initial capital for $\rho = 0.5$ and $\sigma_i = 20$, $i = 1, 2$

		Endogenizing T_i				Fixing $T_i = B_i$			
B_i	(β^*, r_b^*)	T_i^A	q_i^A	π_i^{A*}	k_{2i}^A	q_i^J	π_i^{JF*}	k_{2i}^J	π_i^{NF*}
2	(3,0.0334)	2.0000	10.0000	6.2975	3.5002	10.0000	6.2975	3.5002	3.9574
	(4,0.0469)	2.0000	13.3333	6.6076	5.3128	13.3333	6.6076	5.3128	3.9574
	(6,0.0639)	1.3881	14.9012	6.5998	6.2272	15.3867	6.5536	6.5099	3.9574
3.5	(3,0.0384)	2.5902	14.4673	8.1259	4.5657	14.9041	8.0813	4.8249	6.6890
	(4,0.0404)	1.7149	14.4079	8.1539	4.5181	14.9041	8.0813	4.8249	6.6890
	(6,0.0435)	1.0188	14.3235	8.1771	4.4668	14.9041	8.0813	4.8249	6.6890
6	(3,0.0303)	1.0778	13.5926	10.7579	1.8778	13.8666	10.6100	2.1426	10.4775
	(4,0.0304)	0.7314	13.6568	10.7685	1.9026	13.8666	10.6100	2.1426	10.4775
	(6,0.0307)	0.4441	13.7010	10.7772	1.9187	13.8666	10.6100	2.1426	10.4775

8. Conclusion

In this paper we study an innovative financing scheme with two financially-constrained firms that differ in financial status, cost, revenue or demand parameters. Under such a scheme, the two firms provide limited joint liability for each other by committing their initial capital to a capital pool. We refer to the scheme as the limited joint liability (LJL) financing scheme. Using a non-cooperative game, we analyze the LJL financing scheme under which the two firms separately determine their order quantities to maximize their expected terminal capital. We show that the game is a supermodular game, and prove the existence of equilibrium and establish that the game has a least and a greatest equilibrium point. We then generate practical insights from the analytical and numerical findings to address the research questions we set out to answer.

We find that using the LJL financing scheme does not always dominate the self-finance scheme, especially when the bank loan leverage ratio is relatively small and the interest rate is high, or at least one firm is not extremely-financially constrained. Under the LJL financing scheme, the two firms' strategies are complementary. Moreover, we analyze the impacts of each firm's liability in the pool and the bank-loan leverage ratio on the two firms' equilibrium inventory levels, expected terminal capital, and bankruptcy-related risks. Interestingly, we show that when a firm that uses up all its credit line to order, it has a larger proportion of the liability in the pool, the two firms will borrow more bank loans, and the expected terminal capital of the other firm becomes less, i.e., it may be more harmful to the other firm. In addition, it is somewhat surprising that a higher bank-loan leverage ratio provided by the bank cannot always simultaneously improve the two firms' performance because the firms under the LJL financing scheme mutually reinforce one another.

We also find that the bank in general prefers negative correlation between the firms' demands because such demands correlation yields lower final bankruptcy risks from the firms. Due to the characteristic of the LJL financing scheme, varying a market parameter of each firm weakly affects its partner's strategy, so the expected cash level. Furthermore, when the bank loan market is competitive, the interest rate and bank-loan leverage ratio are mutually affected as the bank seeks to create risk hedging; meanwhile, the capital committed to the pool does not induce the bank to always reduce the interest rate. In particular, if the firms are very financially constrained, which means that they use up all their credit lines to order, increasing the interest rate can mitigate the risks from the linearly increasing order quantities of the firms. Otherwise, as they commit more initial capital to the pool, their equilibrium order quantities will remain unchanged but the corresponding final bankruptcy risks will decrease. As a consequence, the bank will decrease the interest rate. Last but not least, our findings also demonstrate that committing all the initial capital to the capital pool is not necessarily an inferior choice for the firms that are relatively financially constrained (especially for a relatively small bank-loan leverage ratio).

While our study is an initial attempt to explore the role of joint liability in capital-constrained firms' operational decisions and performance, future research may extend our work in several directions. Considering the case where the suppliers strategically set the wholesale prices is of interest, which represents the first step towards studying the (limited) joint liability financing scheme in a supply chain. Another interesting research direction is to consider multiple firms under the LJL financing scheme and study how each firm in the joint-liability group should bear the bankruptcy risks of the other firms. Recall that we assume that the firms considered in this paper are credit-worthy. As argued in the related literature on group lending (e.g., Chowdhury 2005), a moral hazard between the group-lending firms and the bank exists, so incorporating this factor into our model will be another interesting and challenging topic for future research.

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Appendix A: Notation

Table 7 Notation ($i = 1, 2$)

B_i	Firm i's initial capital
p_i	Retail price per unit product of firm i
c_i	Purchase cost per unit product of firm i
q_i	Firm i's order quantity (a decision variable)
r_b	Interest rate on the bank loan under the LJL financing scheme, where $c_i(1+r_b) \le p_i$
r_f	Risk-free interest rate
β	Bank loan leverage ratio set by the bank depending on the amount of the initial capital in the capital pool
$k_{1i}(q_i)$	Firm i's *breakeven threshold*, i.e., the minimum demand for firm i's product to meet its loan obligation with its realized sales revenue
$k_{2i}(q_i)$	Firm i's *effective bankruptcy threshold*, i.e., the minimum demand for firm i's product to meet its loan obligation with all its available capital in the pool
$k_{bi}(q_i, q_j)$	Firm i's eventual *potential bankruptcy threshold*, i.e., the minimum demand for a breakeven firm i's product to finally survive when it shoulders the liability of a worst-case firm j
$\xi_i, \boldsymbol{\xi}$	Random demand for firm i's product and the demands vector, respectively
$f_{\boldsymbol{\xi}}(\cdot,\cdot)$, $F_{\boldsymbol{\xi}}(\cdot,\cdot)$	Probability density and cumulative distribution functions of the positive vector $\boldsymbol{\xi}$, respectively
$f_{\xi_i}(\cdot)$, $F_{\xi_i}(\cdot)$	The marginal cumulative and probability distribution functions of $\boldsymbol{\xi}$ in terms of random variable ξ_i, respectively
$\bar{F}_{\xi_i}(\cdot)$	The complementary marginal cumulative distribution function of the demand ξ_i
$\phi(q_i, q_{3-i})$	Risk-sharing term of firm i under the LJL financing scheme
π_i^{NF}, π_i^{JF}	Firm i's expected terminal capital under the no-financing and LJL financing schemes, respectively

Appendix B: More Details of Sub-regions Defined in Sections 3.2 and 7.1

In Section 3.2, according to Figure 1, the six sub-regions can be further described in detail as:

$$
\begin{aligned}
R^a_{\mathrm{i}1} &= \Big\{\boldsymbol{\xi} \ge 0 : \xi_i \le k_{1i}(q_i), \xi_j \le k_{2j}(q_j), \text{and } p_i\xi_i + p_j\xi_j \ge p_i k_{bi}(q_i, q_j)\Big\},\\
R^b_{\mathrm{i}1} &= \Big\{\boldsymbol{\xi} \ge 0 : \xi_i \le k_{1i}(q_i), \xi_j \le k_{2j}(q_j), \text{and } p_i\xi_i + p_j\xi_j < p_i k_{bi}(q_i, q_j)\Big\},\\
R^a_{\mathrm{i}2} &= \Big\{\boldsymbol{\xi} \ge 0 : k_{2i}(q_i) \le \xi_i < k_{1i}(q_i), \xi_j \ge k_{2j}(q_j)\Big\},\\
R^b_{\mathrm{i}2} &= \Big\{\boldsymbol{\xi} \ge 0 : \xi_i < k_{2i}(q_i), \xi_j \ge k_{2j}(q_j)\Big\},\\
R^a_{\mathrm{i}3} &= \Big\{\boldsymbol{\xi} \ge 0 : \xi_i \ge k_{1i}(q_i), \xi_j \le k_{2j}(q_j)\Big\},\\
R^a_{\mathrm{i}4} &= \Big\{\boldsymbol{\xi} \ge 0 : \xi_i \ge k_{1i}(q_i), \xi_j \ge k_{2j}(q_j)\Big\}.
\end{aligned}
$$

Furthermore, we state the event meanings of these sub-regions as follows:

- In sub-regions $R^b_{\mathrm{i}1}$ and $R^b_{\mathrm{i}2}$, firm i does not sell all its ordered product, and either does not have enough sales revenue and initial capital to meet its loan obligation, or does not have enough remaining initial capital in the pool after repaying its loan to cover the debt of (bankrupt) firm j. Thus, firm i's terminal capital is 0;

- In sub-region R^a_{i1}, firm i's sales revenue is not enough to meet its own loan obligation, but its initial capital in the pool can cover the remaining debt, and firm i's remaining initial capital can also cover firm j's remaining debt. In this case, firm i's terminal capital is $B_i-(1+r_b)c_iq_i+p_i\xi_i-(1+r_b)c_jq_j+p_j\xi_j+B_j$;
- In sub-region R^a_{i2}, firm j has enough sales revenue and initial capital in the pool to repay its loan, and firm i can use its initial capital to cover its remaining debt even though firm i does not have enough sales revenue to repay its loan. Thus, firm i's terminal capital is $B_i-(1+r_b)c_iq_i+p_i\xi_i$;
- In sub-region R^a_{i3}, firm i has enough sales revenue to meet its loan obligation. In this case, firm i's terminal capital is $p_i\min\{q_i,\xi_i\}-(1+r_b)c_iq_i+(B_i-(1+r_b)c_jq_j+p_j\xi_j+B_j))^+$;
- In sub-region R^a_{i4}, the two firms simultaneously have enough sales revenues and initial capital to cover their debts. Thus, firm i's terminal capital is $p_i\min\{q_i,\xi_i\}-(1+r_b)c_iq_i+B_i$.

In Section 7.1, we use Figure 2 to depict the four regions of (ξ_i,ξ_j), whose meanings we state as follows:

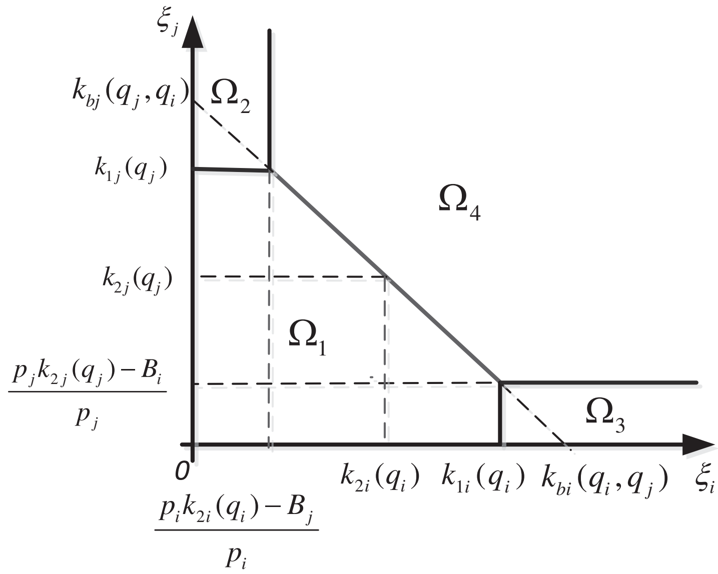

Figure 2 Partition of the demand pair (ξ_i,ξ_j)**.**

- In sub-region Ω_1, the two firms do not have enough sales revenues to completely repay their loans and their initial capital in the pool cannot pay their remaining debts. In this case, the bank receives the two firms' sales revenues and initial capital in the pool, i.e., $p_i\xi_i+p_j\xi_j+B_i+B_j$;
- In sub-region Ω_2, firm j's sales revenue can completely repay its loan, but firm i's sales revenue and initial capital in the pool are not enough to repay its loan, and firm j's initial capital in the pool can pay firm i's remaining debt. Thus, the bank receives $(1+r_b)c_jq_j+p_i\xi_i+B_i+B_j$;

- In sub-region Ω_3, firm i's sales revenue can completely repay its loan, but firm j's sales revenue and initial capital in the pool are not enough to repay its loan, and firm i's initial capital in the pool can pay firm j's remaining debt. Thus, the bank receives $(1+r_b)c_iq_i+p_j\xi_j+B_i+B_j$;
- In sub-region Ω_4, the two firms have enough sales revenues and initial capital to pay their own loans. In this case, the bank's terminal revenue is $(1+r_b)(c_iq_i+c_jq_j)$.

Appendix C: Proofs

Proof of Lemma 1: By the concavity of $\pi_i^{NF}(q_i)$, this result is easy to show, so we omit the proof here. ■

Proof of Lemma 2: By Lemma 1, when $B_i<c_i\bar{F}_{\xi_i}^{-1}(\frac{c_i'}{p_i})$, firm i's optimal expected terminal capital is $\pi_i^{NF*}=p_i\big(B_i/c_i-\int_0^\infty\int_0^{B_i/c_i}F_{\boldsymbol{\xi}}(x,y)dxdy\big)$. Thus, we have

$$\frac{d\pi_i^{NF*}}{dB_i}=\frac{p_i}{c_i}(1-\int_0^\infty F_{\boldsymbol{\xi}}(B_i/c_i,y)dxd)=\frac{p_i}{c_i}\bar{F}_{\xi_i}(B_i/c_i),$$

which is positive and not less than 1 since $B_i<c_i\bar{F}_{\xi_i}^{-1}(\frac{c_i'}{p_i})<c_i\bar{F}_{\xi_i}^{-1}(\frac{c_i}{p_i})$. In addition, it follows that $\frac{d^2\pi_i^{NF*}}{dB_i^2}=-\frac{p_i}{c_i^2}f_{\xi_i}(B_i/c_i)<0$. Obviously, $\frac{d\pi_i^{NF*}}{dB_i}$ is continuous in B_i. ■

Proof of Lemma 4: It is straightforward to derive the results in Lemma 4 by the definition of $\phi(q_i,q_j)$. ■

Proof of Proposition 1: Notice that the first and second terms on the right-hand-side of Equation (6) are not associated with the decision variable q_j. So taking the first-order partial derivatives of the two sides of Equation (6) with respect to q_j yields $\frac{\partial^2\pi_i^{JF}(q_i,q_j)}{\partial q_j\partial q_i}=\frac{\partial\phi(q_i,q_j)}{\partial q_j}$. To prove the increasing-difference property of $\pi_i^{JF}(q_i,q_j)$, it suffices to show that $\frac{\partial\phi(q_i,q_j)}{\partial q_j}\geq 0$. From the definition of $\widehat{R}_{i1}^b$ in Equation (6), $\phi(q_i,q_j)=(1+r_b)c_i\Pr(\widehat{R}_{i1}^b)$. Then, when $q_j\geq\frac{B_i+B_j}{(1+r_b)c_j}$, we have

$$\widehat{R}_{i1}^b=\Big\{\xi\geq 0:k_{2i}(q_i)<\xi_i\leq k_{1i}(q_i),\text{and } p_i\xi_i+p_j\xi_j<p_ik_{bi}(q_i,q_j)\Big\}.$$

This directly yields

$$\begin{aligned}\frac{\partial\phi(q_i,q_j)}{\partial q_j}&=(1+r_b)c_i\frac{\partial\Pr(\widehat{R}_{i1}^b)}{\partial q_j}\\&=\frac{(1+r_b)^2c_ic_j}{p_j}\int_{k_{2i}(q_i)}^{k_{1i}(q_i)}f_{\boldsymbol{\xi}}\Big(x,\frac{p_ik_{bi}(q_i,q_j)-p_ix}{p_j}\Big)dx>0,\end{aligned}$$

where $f_{\boldsymbol{\xi}}(x,y)\geq 0$ for any $(x,y)\geq 0$. Similarly, when $q_j<\frac{B_i+B_j}{(1+r_b)c_j}$, we have

$$\frac{\partial\phi(q_i,q_j)}{\partial q_j}=\frac{(1+r_b)^2c_ic_j}{p_j}\int_{k_{2i}(q_i)}^{k_{bi}(q_i,q_j)}f_{\boldsymbol{\xi}}\Big(x,\frac{p_ik_{bi}(q_i,q_j)-p_ix}{p_j}\Big)dx>0.$$

Note that from Lemma 4(i), $\frac{\partial\phi(q_i,q_j)}{\partial q_j}$ is continuous at the point $q_j=\frac{B_i+B_j}{(1+r_b)c_j}$ since, at this point, $k_{1i}(q_i)=k_{bi}(q_i,q_j)$, i.e., for any $q_j\geq 0$, $\frac{\partial\phi(q_i,q_j)}{\partial q_j}\geq 0$. In addition, noting Lemma 3, we complete the proof. ■

Proof of Lemma 5: Note that from Proposition 1 the non-cooperative game that the two firms play is a supermodular game, and from Lemma 3 the feasible joint strategy is non-empty compact. In addition, for $i=1,2$ and each $q_j \in S_j$, firm i's expected terminal capital $\pi_i^{JF}(q_i,q_j)$ is continuous in q_i, so it is lower-semicontinuous. Hence, the proofs of part (a)-(f) are similarly established by Lemma 4.2.2 in Topkis (1998). ■

Proof of Theorem 1: By the definition of the set of feasible joint strategies, it is easy to show that S is a complete lattice. In addition, with the definition of $\pi_i^{JF}(q_i,q_j)$, it is also easy to show that $\pi_i^{JF}(q_i,q_j)$ is continuous, for $i=1,2$ and $j=3-i$. Hence, $\pi_i^{JF}(q_i,q_j)$ is also upper-semicontinuous. As argued in Lemma 5, we see that each best joint response function $\mathbf{Q}(\mathbf{q})$ is increasing from S into S and $\mathbf{Q}(\mathbf{q})$ is a compact sublattice of R^2 for each $\mathbf{q}\in S$. Thus, $\mathbf{Q}(\mathbf{q})$ is a subcomplete sublattice of R^2. As a result, $\mathbf{Q}(\mathbf{q})$ is a subcomplete sublattice of S for each $\mathbf{q}\in S$. Then, the set of fixed points of the best joint response function $\mathbf{Q}(\mathbf{q})$ is a non-empty complete lattice and has a greatest element and a least element. In addition, Lemma 4.2.1 of Topkis (1998) states that the set of the equilibrium points is identical to the set of fixed points of $\mathbf{Q}(\mathbf{q})$ in S. Therefore, a pure-strategy Nash equilibrium of the inventory levels exists, and a greatest element and a smallest element exist. ■

Proof of Corollary 1: By Lemma 1, it follows that when $B_i \geq c_i\bar{F}_{\xi_i}^{-1}(c_i'/p_i)$, i.e., firm i has relatively enough initial capital, $\pi_i^{NF}(q_i^{NF*})=\pi_i^{NF}(\bar{F}_{\xi_i}^{-1}(c_i'/p_i))\geq \pi_i^{JF}(B_i/c_i,q_j^J)$. Next we analyze another case where $B_i<c_i\bar{F}_{\xi_i}^{-1}(c_i'/p_i)$. In this case, $\pi_i^{NF}(q_i^{NF*})=\pi_i^{NF}(B_i/c_i)$ under the no-financing scheme. Under the LJL financing scheme, if $q_i^J=B_i/c_i$, then firm j's equilibrium inventory level is denoted by q_j^J. Correspondingly, firm i's expected terminal capital is given by

$$\begin{aligned}\pi_i^{JF}(q_i^J,q_j^J)=&E_{\xi_i,\xi_j}\Big\{[p_i\min\{B_i/c_i,\xi_i\}-(1+r_b)B_i]^+\\&+\big[B_i-((1+r_b)B_i-p_i\xi_i)^+-((1+r_b)c_jq_j^J-p_j\xi_j-B_j)^+\big]^+\Big\}.\end{aligned}$$

First, we show that for any $B_i\geq 0$, $p_iE_{\xi_i}[\min\{B_i/c_i,\xi_i\}]-B_i\geq 0$. To see this, let $H_i(B_i)=p_iE_{\xi_i}[\min\{B_i/c_i,\xi_i\}]-B_i$. By taking the first-order derivative of $H_i(B_i)$ with respect to B_i, we have $H_i'(B_i)=p_i\bar{F}_{\xi_i}(B_i/c_i)/c_i-1>0$, where the inequality holds since $B_i<c_i\bar{F}_{\xi_i}^{-1}(c_i'/p_i)<c_i\bar{F}_{\xi_i}^{-1}(c_i/p_i)$. Then, $H_i(B_i)\geq H_i(0)=0$. As a result, we get

$$\begin{aligned}\pi_i^{JF}(q_i^J,q_j^J)&<E_{\xi_i}\Big\{[p_i\min\{B_i/c_i,\xi_i\}-r_bB_i]\cdot 1_{\{\xi_i\geq k_{1i}(B_i/c_i)\}}+(B_i+p_i\xi_i-(1+r_b)B_i)\cdot 1_{\{\xi_i<k_{1i}(B_i/c_i)\}}\Big\}\\&=E_{\xi_i}[p_i\min\{B_i/c_i,\xi_i\}-r_bB_i]\\&\leq E_{\xi_i}[p_i\min\{B_i/c_i,\xi_i\}]\\&=\pi_i^{NF}(B_i/c_i),\end{aligned}$$

where 1_A is the indicative function, i.e., $1_A=1$ if $x\in A$ and 0 otherwise. Therefore, for any $B_i\geq 0$, we have $\pi_i^{JF}(B_i/c_i,q_j^J)\leq \pi_i^{NF}(q_i^{NF*})$. ■

Proof of Proposition 3: To prove the first part, we first prove that for each firm $k(k=i,j)$ and each q_{-k} in S_{-k}, $\pi_k^{JF}(q_k,q_{-k})$ is increasing in the difference $(q_k,-B_i)$. To see this, by taking the partial derivatives of $\frac{\partial \pi_j^{JF}(q_j,q_i,B_i)}{\partial q_j}$ and $\frac{\partial \pi_i^{JF}(q_i,q_j,B_i)}{\partial q_i}$ with respect to B_j, respectively, we get that when $B_i+B_j\le(1+r_b)c_jq_j$,

$$\begin{aligned}\frac{\partial^2\pi_i^{JF}(q_i,q_j,B_i)}{\partial q_i\partial B_i}&=-\frac{(1+r_b)c_i}{p_i}\Big(\frac{p_i}{p_j}\int_{k_{2i}(q_i)}^{k_{1i}(q_i)}f_{\boldsymbol{\xi}}\Big(x,\frac{p_ik_{bi}(q_i,q_j)-p_ix}{p_j}\Big)dx\\&\qquad-f_{\xi_i}(k_{2i}(q_i))+\int_0^{k_{2j}(q_j)}f_{\boldsymbol{\xi}}\left(k_{2i}(q_i),y\right)dy\Big)\\&=-\frac{(1+r_b)c_i}{p_i}\Big(\frac{p_i}{p_j}\int_{k_{2i}(q_i)}^{k_{1i}(q_i)}f_{\boldsymbol{\xi}}\Big(x,\frac{p_ik_{bi}(q_i,q_j)-p_ix}{p_j}\Big)dx+\int_{k_{2j}(q_j)}^{+\infty}f_{\boldsymbol{\xi}}\left(k_{2i}(q_i),y\right)dy\Big)\\&<0,\end{aligned}$$

and $\frac{\partial^2\pi_j^{JF}(q_j,q_i,B_i)}{\partial q_j\partial B_i}=-\frac{(1+r_b)c_i}{p_j}\int_{k_{2i}(q_i)}^{k_{1i}(q_i)}f_{\boldsymbol{\xi}}\Big(x,\frac{p_ik_{bi}(q_i,q_j)-p_ix}{p_j}\Big)dx<0$. When $B_i+B_j>(1+r_b)c_jq_j$,

$$\begin{aligned}\frac{\partial^2\pi_i^{JF}(q_i,q_j,B_i)}{\partial q_i\partial B_i}&=-\frac{(1+r_b)c_i}{p_i}\Big(\frac{p_i}{p_j}\int_{k_{2i}(q_i)}^{k_{bi}(q_i,q_j)}f_{\boldsymbol{\xi}}\Big(x,\frac{p_ik_{bi}(q_i,q_j)-p_ix}{p_j}\Big)dx+\int_{k_{2j}(q_j)}^{+\infty}f_{\boldsymbol{\xi}}\left(k_{2i}(q_i),y\right)dy\Big)\\&<0,\end{aligned}$$

and

$$\frac{\partial^2\pi_j^{JF}(q_j,q_i,B_i)}{\partial q_j\partial B_i}=-\frac{(1+r_b)c_i}{p_j}\int_{k_{2i}(q_i)}^{k_{bi}(q_i,q_j)}f_{\boldsymbol{\xi}}\Big(x,\frac{p_ik_{bi}(q_i,q_j)-p_ix}{p_j}\Big)dx<0.$$

In addition, it is easy to show that $\frac{\partial \pi_j^{JF}(q_j,q_i,B_i)}{\partial q_j}$ and $\frac{\partial \pi_i^{JF}(q_i,q_j,B_i)}{\partial q_i}$ are continuous at the point $q_j=\frac{B_i+B_j}{(1+r_b)c_j}$. Hence, for any $k=i,j$, $\frac{\partial^2\pi_k^{JF}(q_k,q_{-k},B_i)}{\partial q_k\partial B_i}<0$. Thus, $\pi_k^{JF}(q_k,q_{-k})$ is increasing in the difference $(q_k,-B_i)$. Notice that $\pi_k^{JF}(q_k,q_{-k})$ is continuous and thus upper-semicontinuous in q_k on S_k for each B_i. Moreover, from Proposition 1, we know that the non-cooperative game that the two firms play is supermodular. Therefore, following Theorem 4.2.2. in Topkis (1998), we conclude that the greatest (smallest) equilibrium point (q_i°,q_j°) is increasing in $-B_i$, i.e., q_i° and q_j° decrease with B_i. As a result, from the definitions of $k_{2i}(q_i^\circ)$, $k_{2j}(q_j^\circ)$, and $k_{bi}(q_i^\circ,q_j^\circ)$, it easy to derive that they are decreasing in B_i. Then, the prood of part (i) is completed.

Next we show the remaining result of Proposition 3. Taking the total derivative of $\pi_i^{JF}(q_i^J,q_j^J,B_k)$ with respect to $B_k(k=i,j)$, we have

$$\begin{aligned}\frac{d\pi_i^{JF}(q_i,q_j,B_k)}{dB_k}&=\frac{\partial\pi_i^{JF}(q_i,q_j,B_k)}{\partial q_i}\frac{dq_i}{dB_k}+\frac{\partial\pi_i^{JF}(q_i,q_j,B_k)}{\partial q_j}\frac{dq_j}{dB_k}+\frac{\partial\pi_i^{JF}(q_i,q_j,B_k)}{\partial B_k}\\&=\frac{\partial\pi_i^{JF}(q_i,q_j,B_k)}{\partial q_j}\frac{dq_j}{dB_k}+\frac{\partial\pi_i^{JF}(q_i,q_j,B_k)}{\partial B_k}\\&\ge0,\end{aligned}$$

where $(q_i, q_j) = (q_i^J, q_j^J)$, and the second equality holds because q_i^* satisfies the first-order condition for $\pi_i^C(q_i, q_j)$ for any given q_j and the last inequality is due to part (i) of Proposition 3 and the following fact that, for any $(q_i, q_j) \in S$ and $B_i \geq 0$,

$$\frac{\partial \pi_i^{JF}(q_i, q_j, B_k)}{\partial q_j} = -(1+r_b)c_j \left[\Pr(R_{\text{i1}}^a) + \Pr(R_{\text{i3}}^a) E_{\xi_i,\xi_j} \left(1_{\{\xi_j > \frac{(1+r_b)c_j q_j - B_i - B_j}{p_j}\}} | R_{\text{i3}}^a \right) \right] \leq 0,$$

$$\frac{\partial \pi_i^{JF}(q_i, q_j, B_j)}{\partial B_j} = \Pr(R_{\text{i1}}^a) + \Pr(R_{\text{i3}}^a) E_{\xi_i,\xi_j} \left(1_{\{\xi_j > \frac{(1+r_b)c_j q_j - B_i - B_j}{p_j}\}} | R_{\text{i3}}^a \right) \geq 0$$

and

$$\frac{\partial \pi_i^{JF}(q_i, q_j, B_i)}{\partial B_i} = \Pr(R_{\text{i1}}^a) + \Pr(R_{\text{i2}}^a) + \Pr(R_{\text{i4}}^a) + \Pr(R_{\text{i3}}^a) E_{\xi_i,\xi_j} \left(1_{\{\xi_j > \frac{(1+r_b)c_j q_j - B_i - B_j}{p_j}\}} | R_{\text{i3}}^a \right) \geq 0.$$

Hence, the second part is proved. ■

Proof of Lemma 6: Note from part (e) of Lemma 5 that the best response function of firm i, i.e., $q_i(q_j)$, is increasing in q_j. Then, when $q_j^\circ \geq \frac{\beta B_j}{c_j}$, it directly follows that $q_i(\beta) \doteq q_i(\frac{\beta B_j}{c_j}) \leq q_i(q_j^\circ) = q_i^\circ$. ■

Proof of Proposition 4: In what follows, it suffices to prove part (i), and the second part can be similarly proved. Note that if $q_j^J = \beta B_j / c_j$, q_j^J and $k_{2j}(q_j^J)$ are unchanged as B_i varies. In addition, as argued in Section 4, we get that firm i's best response order quantity is $q_i^J = q_i^d = \min\{q_i(\beta), \frac{\beta B_i}{c_i}\}$, where $q_i(\beta)$ satisfies Equation (8). Then, we can further derive that $q_i(\beta)$ is decreasing in B_i. To see this, substituting $q_j = \beta B_j / c_j$ into $\pi_i^{JF}(q_i, q_j)$ and taking the partial derivatives of $\pi_i^{JF}(q_i, \beta B_j / c_j)$ with respect to q_i and b_i yields

$$\frac{\partial^2 \pi_i^{JF}(q_i, \beta B_j / c_j)}{\partial q_i \partial B_i} = -\frac{(1+r_b)c_i}{p_i} \Big(\frac{p_i}{p_j} \int_{k_{2i}(q_i)}^{k_{1i}(q_i)} f_{\boldsymbol{\xi}} \Big(x, \frac{p_i k_{bi}(q_i, \beta B_j / c_j) - p_i x}{p_j} \Big) dx + \int_{k_{2j}(\beta B_j / c_j)}^{+\infty} f_{\boldsymbol{\xi}} \left(k_{2i}(q_i), y \right) dy \Big) < 0.$$

If $q_i(\beta)$ is the optimal inventory level of firm i ignoring the constraint $c_i q_i \leq \beta B_i$, then it follows that $q_i(\beta)$ decreases with B_i. On the one hand, note that as $B_i = 0$, $\frac{\beta B_i}{c_i} = 0$ and $q_i(\beta)|_{B_i=0} = y_i^U > 0$, where y_i^U is the solution of the equation

$$p_i \bar{F}_{\xi_i}(q_i) - (1+r_b)c_i \bar{F}_{\xi_i}\left(k_{1i}(q_i)\right) = 0.$$

On the other hand, $q_i(\beta)$ is continuous in B_i. With the above results, there exists a threshold $\bar{B}_i$ that satisfies $c_i q_i(\beta) = \beta B_i$ such that if $B_i < \bar{B}_i$, $q_i^J = \frac{\beta B_i}{c_i}$ increases with B_i; otherwise, $q_i^J = q_i(\beta)$ decreases with B_i. That is, q_i^J is initially increasing and then decreasing (quasi-concave) in B_i. So are $k_{2i}(q_i^J)$ and $k_{bi}(q_i^J, q_j^J)$.

Next we prove the monotonicity of $\pi_i^{JF}(q_i^J, q_j^J)$ and $\pi_j^{JF}(q_j^J, q_i^J)$ with respect to B_i. To see this, we divide the analysis into two cases, i.e., $q_i^J = \frac{\beta B_i}{c_i}$ and $q_i^J = q_i(\beta)$. If $q_i^J = \frac{\beta B_i}{c_i}$, we have

$$\begin{aligned}
\frac{d\pi_i^{JF}(q_i^J, q_j^J, B_i)}{dB_i} &= \frac{\partial \pi_i^{JF}(q_i, q_j)}{\partial q_i} \frac{dq_i}{dB_i}\Big|_{(q_i^J, q_j^J)=(\frac{\beta B_i}{c_i}, \frac{\beta B_j}{c_j})} + \frac{\partial \pi_i^{JF}(q_i, q_j)}{\partial B_i}\Big|_{(q_i^J, q_j^J)=(\frac{\beta B_i}{c_i}, \frac{\beta B_j}{c_j})} \\
&\geq \frac{\partial \pi_i^{JF}(q_i, q_j)}{\partial B_i}\Big|_{(q_i^J, q_j^J)=(\frac{\beta B_i}{c_i}, \frac{\beta B_j}{c_j})} \\
&= \Pr(R_{i1}^a) + \Pr(R_{i2}^a) + \Pr(R_{i4}^a) + \Pr(R_{i3}^a) E_{\xi_i, \xi_j}\left(1_{\{\xi_j > \frac{((1+r_b)\beta - 1)B_j - B_i}{p_j}\}} | R_{i3}^a\right) \\
&\geq 0,
\end{aligned}$$

and

$$\begin{aligned}
\pi_j^R(q_j^J, q_i^J) = & E_{\xi_i, \xi_j}\Big\{[p_j \min\{\frac{\beta B_j}{c_j}, \xi_i\} - (1+r_b)\beta B_j]^+ \\
& + \big[B_j - ((1+r_b)\beta B_j - p_j \xi_j)^+ - ((1+r_b)\beta B_i - B_i - p_i \xi_i)^+\big]^+ \Big\},
\end{aligned}$$

which is decreasing in B_i since the function $\max\{x, 0\}$ is increasing in x. If $q_i^J = q_i(\beta)$, it follows that

$$\begin{aligned}
\frac{d\pi_i^{JF}(q_i^J, q_j^J, B_i)}{dB_i} &= \frac{\partial \pi_i^{JF}(q_i, q_j)}{\partial q_i} \frac{dq_i}{dB_i}\Big|_{(q_i^J, q_j^J)=(q_i(\beta), \frac{\beta B_j}{c_j})} + \frac{\partial \pi_i^{JF}(q_i, q_j)}{\partial B_i}\Big|_{(q_i^J, q_j^J)=(q_i(\beta), \frac{\beta B_j}{c_j})} \\
&= \frac{\partial \pi_i^{JF}(q_i, q_j)}{\partial B_i}\Big|_{(q_i^J, q_j^J)=(q_i(\beta), \frac{\beta B_j}{c_j})} \\
&= \Pr(R_{i1}^a) + \Pr(R_{i2}^a) + \Pr(R_{i4}^a) + \Pr(R_{i3}^a) E_{\xi_i, \xi_j}\left(1_{\{\xi_j > \frac{((1+r_b)\beta - 1)B_j - B_i}{p_j}\}} | R_{i3}^a\right) \\
&\geq 0,
\end{aligned}$$

and

$$\begin{aligned}
\pi_j^R(q_j^J, q_i^J) = & E_{\xi_i, \xi_j}\Big\{[p_j \min\{\frac{\beta B_j}{c_j}, \xi_i\} - (1+r_b)\beta B_j]^+ \\
& + \big[B_j - ((1+r_b)\beta B_j - p_j \xi_j)^+ - ((1+r_b)c_i q_i(\beta) - B_i - p_i \xi_i)^+\big]^+ \Big\},
\end{aligned}$$

which is increasing in B_i since $q_i(\beta)$ is decreasing B_i as proved above. In addition, $\pi_i^{JF}(q_i^d, q_j^d)$ and $\pi_j^{JF}(q_j^d, q_i^d)$ are continuous in B_i. Hence, $\pi_i^{JF}(q_i^J, q_j^J)$ is always increasing in B_i, and $\pi_j^R(q_j^J, q_i^J)$ is initially decreasing and then increasing (quasi-convex) in B_i. ■

Proof of Proposition 5: Clearly, q_i^J, $k_{2i}(q_i^J)$, and $k_{bi}(q_i^J, q_j^J)$ are continuous in $\beta > 1$. Similar to the analysis in Section 4.1, we divide the analysis into two cases, i.e., $\beta > \max_{i=1,2} c_i q_i^\circ / B_i$ and $\beta \leq \max_{i=1,2} c_i q_i^\circ / B_i$. In the first case, it follows from Proposition 2 that the equilibrium inventory levels of firms i and j are $q_i^J = q_i^\circ$ and $q_j^J = q_j^\circ$, respectively, which are not associated with β. Then, $k_{2i}(q_i^J)$, $k_{bi}(q_i^J, q_j^J)$, and $\pi_i^{JF}(q_i^J, q_j^J)$ are unchanged as β varies. In the second case, i.e., $\beta \leq \max_{i=1,2} c_i q_i^\circ / B_i$, there is at least one firm using up its own credit limit to order; without loss of generality, suppose that firm i is such firm, then $q_i^J = \frac{\beta B_i}{c_i}$, which is strictly increasing in β. Then, it is easy to show that firm j's best response inventory level is $q_j^J = \min\{q_j(\beta), \frac{\beta B_j}{c_j}\}$. In what follows, we first show that $q_j(\beta)$ is increasing in β. To see this, note that $q_j(\beta)$ is the solution of Equation (8) and let $H(q_j,\beta) = p_j\bar{F}_{\xi_j}(q_j) - (1+r_b)c_j\bar{F}_{\xi_j}(k_{2j}(q_j)) + \phi(q_j, \beta B_i/c_i)$. Then, taking the partial derivative of $H(q_j,\beta)$ with respect to β, we have

$$\begin{aligned}\frac{\partial H(q_j,\beta)}{\partial \beta} &= \frac{\partial \phi(q_j,\beta B_i/c_i)}{\partial \beta}\\ &= \frac{(1+r_b)^2 c_i B_i}{p_i}\int_{k_{2j}(q_j)}^{k_{1j}(q_j)} f_{\boldsymbol{\xi}}\Big(\frac{p_j k_{bj}(q_j,\beta B_i/c_i) - p_j y}{p_i}, y\Big)dy > 0.\end{aligned}$$

If $q_j(\beta)$ becomes the optimal order quantity of firm i, then $q_j(\beta)$ increases with β. As a result, q_i^J increases as β increases. Thus, it follows that $k_{2i}(q_i^J)$ and $k_{bi}(q_i^J, q_j^J)$ are increasing in β. If $q_j^J = \frac{\beta B_j}{c_j}$, it is obvious that firm i's equilibrium order quantity is strictly increasing in β. In other words, q_j^J is non-decreasing in β. ■

Proof of Corollary 2: We first prove Corollary 2 when $q_i^J = \min\{q_i^\circ, \beta B_i/c_i\}$. Let $H_i(B_i) = q_i^\circ - \beta B_i/c_i$. Obviously, on the one hand, $H_i(0) = q_i^\circ|_{B_i=0} \geq 0$. On the other hand, since q_i° decreases with B_i from Proposition 3 and $\beta B_i/c_i$ is strictly increasing in B_i, there exists a threshold $\hat{B}_i$ such that if $B_i \leq \hat{B}_i$, $H_i(B_i) \geq 0$ and thus $q_i^J = \beta B_i/c_i$; otherwise, $q_i^J = q_i^\circ$. In the former case, $q_i^J - q_i^{NF*} = (\beta-1)B_i/c_i$ is strictly increasing in B_i for any $\beta > 1$. In the latter case, because again q_i° decreases with B_i, $q_i^J - q_i^{NF*} = q_i^\circ - B_i/c_i$ is decreasing in B_i. Following the result in Proposition 4, $q_i(\beta)$ is decreasing in B_i. As such, we can similarly prove the other case, i.e., $q_i^J = \min\{q_i(\beta), \beta B_i/c_i\}$, where B_i satisfies the equation $q_i(\beta) = \beta B_i/c_i$. In addition, firm i's equilibrium inventory level q_i^J is continuous in B_i. Thus, the proof is completed. ■

Proof of Lemma 7: As shown in Section 4, when $q_i^J = \beta B_i/c_i$, firm j's best response inventory decision is $q_j^J = q_j^d = \min\{q_j(\beta), \frac{\beta B_j}{c_j}\}$, where $q_j(\beta)$ is increasing in B_i (see the proof of Proposition 4). Correspondingly, firm i's expected terminal capital is given by

$$\begin{aligned}\pi_i^{JF}(\beta B_i/c_i, q_j^d) =& E_{\xi_i,\xi_j}\Big\{[p_i\min\{\beta B_i/c_i,\xi_i\} - (1+r_b)\beta B_i]^+\\ &+ \big[B_i - ((1+r_b)\beta B_i - p_i\xi_i)^+ - ((1+r_b)c_j q_j^d - p_j\xi_j - B_j)^+\big]^+\Big\}\\ &= p_i\int_{\frac{(1+r_b)\beta B_i}{p_i}}^{\beta B_i/c_i} \bar{F}_{\xi_i}(x)dx + \int_0^{+\infty}\int_0^{\frac{(1+r_b)\beta B_i}{p_i}} \Big(B_i - ((1+r_b)\beta B_i - p_i x)\\ &- ((1+r_b)c_j q_j^d - p_j y - B_j)^+\Big)^+ f_{\boldsymbol{\xi}}(x,y)dxdy\\ &+ \int_0^{+\infty}\Big(B_i - ((1+r_b)c_j q_j^d - p_j y - B_j)^+\Big)^+\Big(1 - F_{\boldsymbol{\xi}}(\frac{(1+r_b)\beta B_i}{p_i}, y)\Big)dy.\end{aligned}$$

For any given $y \geq 0$, let

$$\phi_1(B_i, y) = \int_0^{\frac{(1+r_b)\beta B_i}{p_i}} \Big(B_i - ((1+r_b)\beta B_i - p_i x) - ((1+r_b)c_j q_j^d - p_j y - B_j)^+\Big)^+ f_{\boldsymbol{\xi}}(x, y)dx$$

and

$$\phi_2(B_i, y) = \Big(B_i - ((1+r_b)c_j q_j^d - p_j y - B_j)^+\Big)^+ \left(1 - F_{\boldsymbol{\xi}}(\frac{(1+r_b)\beta B_i}{p_i}, y)\right).$$

By taking the partial derivatives of $\phi_1(B_i, y)$ and $\phi_2(B_i, y)$ with respect to B_i, respectively, we have

$$\begin{aligned}\frac{d\phi_1(B_i, y)}{dB_i} &= \int_0^{\frac{(1+r_b)\beta B_i}{p_i}} \frac{\partial \varphi(B_i, x, y)}{\partial B_i} f_{\boldsymbol{\xi}}(x, y)dx \\ &\quad + \frac{(1+r_b)\beta}{p_i}(B_i - ((1+r_b)c_j q_j^d - p_j y - B_j)^+)^+ f_{\boldsymbol{\xi}}(\frac{(1+r_b)\beta B_i}{p_i}, y),\end{aligned}$$

where

$$\varphi(B_i, x, y) = \Big(B_i - ((1+r_b)\beta B_i - p_i x) - ((1+r_b)c_j q_j^d - p_j y - B_j)^+\Big)^+,$$

and for any given $(x, y) \geq 0$

$$\frac{\varphi(B_i, x, y)}{\partial B_i} = \left(1 - (1+r_b)\beta - (1+r_b)c_j \frac{dq_j^d}{dB_i} \cdot 1_{\{y < k_{2j}(q_j^d)\}}\right) \cdot 1_{\{p_i x + p_j y > p_i k_{bi}(\beta \frac{B_i}{c_i}, q_j^d)\}},$$

and

$$\begin{aligned}\frac{d\phi_2(B_i, y)}{dB_i} &= -\frac{(1+r_b)\beta}{p_i}(B_i - ((1+r_b)c_j q_j^d - p_j y - B_j)^+)^+ f_{\boldsymbol{\xi}}(\frac{(1+r_b)\beta B_i}{p_i}, y) \\ &\quad + \left(1 - (1+r_b)c_j \frac{dq_j^d}{dB_i} \cdot 1_{\{y < k_{2j}(q_j^d)\}}\right) 1_{\{y > \frac{(1+r_b)c_j q_j^d - B_j - B_i}{p_i}\}} \left(1 - F_{\boldsymbol{\xi}}(\frac{(1+r_b)\beta B_i}{p_i}, y)\right).\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\lim_{B_i \to 0} \frac{d\phi_1(B_i, y)}{dB_i} + \frac{d\phi_2(B_i, y)}{dB_i} \\ &= \lim_{B_i \to 0} \left(1 - (1+r_b)c_j \frac{dq_j^d}{dB_i} \cdot 1_{\{y < k_{2j}(q_j^d)\}}\right) 1_{\{y > \frac{(1+r_b)c_j q_j^d - B_j - B_i}{p_i}\}} \left(1 - F_{\boldsymbol{\xi}}(\frac{(1+r_b)\beta B_i}{p_i}, y)\right) \\ &= \left(1 - (1+r_b)c_j \lim_{B_i \to 0} \frac{dq_j^d}{dB_i} \cdot 1_{\{y < k_{2j}(q_j^d)\}}\right) 1_{\{y > k_{2j}(q_j^d)\}} \\ &= \lim_{B_i \to 0} 1_{\{y > k_{2j}(q_j^d)\}},\end{aligned}$$

where the first equality holds because when $B_i \to 0$, $\frac{(1+r_b)\beta B_i}{p_i} \to 0$, and for any $y \geq 0$, we have

$$(B_i - ((1+r_b)c_j q_j^d - p_j y - B_j)^+)^+ \to 0;$$

the second equality is true since $\lim_{B_i \to 0} \bar{F}_{\xi_i}(\frac{(1+r_b)\beta B_i}{p_i}) = 1$; the third one is from the fact that the event $\{y : y < k_{2j}(q_j^d)\}$ and the event $\{y : y > k_{2j}(q_j^d)\}$ do not exist simultaneously, so

$$\lim_{B_i \to 0} \frac{dq_j^d}{dB_i} \cdot 1_{\{y < k_{2j}(q_j^d)\}} \cdot \lim_{B_i \to 0} 1_{\{y > k_{2j}(q_j^d)\}} = \lim_{B_i \to 0} \frac{dq_j^d}{dB_i} \cdot 1_{\{y < k_{2j}(q_j^d)\}} \cdot 1_{\{y > k_{2j}(q_j^d)\}} = 0.$$

Notice that for any given $y \geq 0$, $\phi_1(B_i, y)$ and $\phi_2(B_i, y)$ are continuous in $B_i \geq 0$. Then, we have

$$\begin{aligned}
\lim_{B_i \to 0} \frac{d\pi_i^{JF}(\beta B_i/c_i, q_j^d)}{dB_i} &= \lim_{B_i \to 0} \frac{p_i\beta}{c_i} \bar{F}_{\xi_i}(\frac{\beta B_i}{c_i}) - (1+r_b)\beta \bar{F}_{\xi_i}(\frac{(1+r_b)\beta B_i}{p_i}) \\
&\quad + \int_0^{+\infty} \Big[\frac{d\phi_1(B_i, y)}{dB_i} + \frac{d\phi_2(B_i, y)}{dB_i}\Big] dy \\
&= \lim_{B_i \to 0} \frac{p_i\beta}{c_i} \bar{F}_{\xi_i}(\frac{\beta B_i}{c_i}) - (1+r_b)\beta \bar{F}_{\xi_i}(\frac{(1+r_b)\beta B_i}{p_i}) + E_y[1_{\{y > k_{2j}(q_j^d)\}}] \\
&= \beta(p_i/c_i - (1+r_b)) + \lim_{B_i \to 0} \bar{F}_{\xi_j}(k_{2j}(q_j^d)),
\end{aligned}$$

where the last equality holds because as $B_i \to 0$, $\frac{(1+r_b)\beta B_i}{p_i} \to 0$. On the other hand, it is easy to show that

$$\lim_{B_i \to 0} \frac{d\pi_i^{NF}(B_i/c_i)}{dB_i} = \lim_{B_i \to 0} \frac{p_i}{c_i} \bar{F}_{\xi_i}(B_i/c_i) = \frac{p_i}{c_i}.$$

Hence,

$$\lim_{B_i \to 0} \frac{d\pi_i^{JF}(\beta B_i/c_i, q_j^d)}{dB_i} - \lim_{B_i \to 0} \frac{d\pi_i^{NF}(B_i/c_i)}{dB_i} = \beta\Big(\frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1 - r_b\Big) + \lim_{B_i \to 0} \bar{F}_{\xi_j}(k_{2j}(q_j^d)).$$

Then, we derive Lemma 7. ■

Proof of Corollary 3: When $\beta \geq \frac{p_i}{p_i - c_i}$ and $0 < r_b \leq \frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1$, we have

$$\beta\big(\frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1 - r_b\big) + \lim_{B_i \to 0} \bar{F}_{\xi_j}(k_{2j}(q_j^d)) \geq \beta\big(\frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1 - r_b\big) \geq 0,$$

where the first inequality holds since for any $B_i \geq 0$, we have $\bar{F}_{\xi_j}(k_{2j}(q_j^d)) \geq 0$. Following Lemma 7, we have

$$\lim_{B_i \to 0} \frac{d\pi_i^{JF}(\beta B_i/c_i, q_j^d)}{dB_i} \geq \lim_{B_i \to 0} \frac{d\pi_i^{NF}(B_i/c_i)}{dB_i}.$$

When $(\beta - 1)\frac{p_i}{c_i} - \beta r_b \leq \beta - 1$,

$$\beta\big(\frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1 - r_b\big) + \lim_{B_i \to 0} \bar{F}_{\xi_j}(k_{2j}(q_j^d)) \leq \beta\big(\frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1 - r_b\big) + 1 \leq 0,$$

where the first inequality is true since for any $B_i \geq 0$, we have $\bar{F}_{\xi_j}(k_{2j}(q_j^d)) \leq 1$. Again, from Lemma 7, we derive that

$$\lim_{B_i \to 0} \frac{d\pi_i^{JF}(\beta B_i/c_i, q_j^d)}{dB_i} \leq \lim_{B_i \to 0} \frac{d\pi_i^{NF}(B_i/c_i)}{dB_i}.$$

Then, we complete the proof. ■

Proof of Proposition 6: To prove the first part, we consider the analysis in two steps. First, according to Propositions 3 and 4, for given r_c and $\beta > 1$, and firm j's strategy, as the initial capital B_i varies from 0 to a relatively high value (e.g., $c_i\bar{F}_{\xi_i}^{-1}(c_i'/p_i)$), firm i's non-trivial equilibrium order quantity changes from the non-trivial boundary equilibrium $\beta B_i/c_i$ to the non-trivial interior equilibrium q_i°. Let $\widehat{B}_i$ is the solution of the equation $\beta B_i/c_i = q_i^\circ$. As a result, when $0 < B_i \leq \widehat{B}_i$, firm i's non-trivial equilibrium order quantity is $q_i^J = \beta B_i/c_i$ and the corresponding expected terminal capital is $\pi_i^{JF}(q_i^J, q_j^J) = \pi_i^{JF}(\beta B_i/c_i, q_j^d)$; when $\widehat{B}_i < B_i < B_i^0$, firm i's non-trivial equilibrium order quantity is $q_i^J = q_i^\circ$ and the corresponding expected terminal capital is $\pi_i^{JF}(q_i^J, q_j^J) = \pi_i^{JF}(q_j^\circ, q_j^\circ)$, where B_i^0 satisfies the equation $c_i q_i^\circ(B_i^0) = B_i^0$. Notice that as $B_i \to B_i^0$, firm i's non-trivial interior equilibrium inventory level q_i° will decreasingly tend to the point B_i/c_i. Following Corollary 1, when $q_i^J = B_i/c_i$, we have $\pi_i^{JF}(q_i^J, q_j^J) \le \pi_i^{NF}(q_i^{NF*})$. In other words, for $B_i \in (B_i^0 - \varepsilon, B_i^0)$, we have $\pi_i^{JF}(q_i^J, q_j^J) \le \pi_i^{NF}(B_i/c_i)$, where ε is arbitrarily small. In addition, if $\beta \ge \frac{p_i}{p_i - c_i}$ and $0 < r_b \le \frac{\beta-1}{\beta}\frac{p_i}{c_i} - 1$, then it follows from Corollary 3 that when $B_i \to 0$, we have

$$\frac{d}{dB_i}(\pi_i^{JF}(\beta B_i/c_i, q_j^d) - \pi_i^{NF}(B_i/c_i)) \to \lim_{B_i \to 0} \frac{d\pi_i^{JF}(\beta B_i/c_i, q_j^d)}{dB_i} - \lim_{B_i \to 0} \frac{d\pi_i^{NF}(B_i/c_i)}{dB_i} \ge 0.$$

That is, for any sufficiently small ε and $B_i \in (0, \varepsilon)$, $\pi_i^{JF}(\beta B_i/c_i, q_j^d) - \pi_i^{NF}(B_i/c_i)$ is a continuously increasing function of B_i. Thus, for any $B_i \in (0, \varepsilon)$, we obtain that $\pi_i^{JF}(\beta B_i/c_i, q_j^d) \ge \pi_i^{NF}(B_i/c_i)$.

Second, on the one hand, according to Lemma 2, we see that $\pi_i^{NF}(B_i/c_i) = \pi_i^{NF*}$ is increasingly concave in B_i over the interval $(0, c_i \bar{F}_{\xi_i}^{-1}(c_i'/p_i))$ with the increasing rate less than 1. On the other hand, it is easy to show that (q_i^J, q_j^J) is continuous at the point $B_i = \widehat{B}_i$. Hence, for $B_i \ge 0$, $\pi_i^{JF}(q_i^J, q_j^J)$ is continuous in B_i. Combining with the results in the second parts of Propositions 3 and 4, we get that $\pi_i^{JF}(q_i^J, q_j^J)$ is increasing in B_i over the interval $(0, B_i^0)$. With the results above and the fact that $\pi_i^{JF}(\beta B_i/c_i, q_j^d) - \pi_i^{NF}(B_i/c_i)$ is continuous in B_i, there exist two thresholds $\bar{B}_i^1$ and $\bar{B}_i^2$ such that when $0 \le B_i < \bar{B}_i$, $\pi_i^{NF*} \le \pi_i^{JF}(q_i^J, q_j^J)$, and when $\bar{B}_i^2 \le B_i < c_i \bar{F}_{\xi_i}^{-1}(c_i'/p_i)$, $\pi_i^{NF*} \ge \pi_i^{JF}(q_i^J, q_j^J)$. Here $\bar{B}_i^1 \le \bar{B}_i^2$, where $\bar{B}_i^1$ and $\bar{B}_i^2$ are determined as follows: If $\pi_i^{JF}(\beta \widehat{B}_i/c_i, q_j^\circ(\widehat{B}_i)) > \pi_i^{NF}(\widehat{B}_i/c_i)$, then $\bar{B}_i^1 = \bar{B}_i^2$ satisfies the equation $\pi_i^{JF}(q_i^\circ, q_j^\circ) = \pi_i^{NF}(B_i/c_i)$; if $\pi_i^{JF}(\beta \widehat{B}_i/c_i, q_j^\circ(\widehat{B}_i)) \le \pi_i^{NF}(\widehat{B}_i/c_i)$, then $\bar{B}_i^1$ satisfies the equation $\pi_i^{JF}(\beta B_i/c_i, q_j^d) = \pi_i^{NF}(B_i/c_i)$, and

$$\bar{B}_i^2 = \sup\{B_i : \pi_i^{JF}(q_i^\circ, q_j^\circ) - \pi_i^{NF}(B_i/c_i) = 0\}.$$

Then, we show the second part. If β and r_b satisfy $(\beta - 1)\frac{p_i}{c_i} - \beta r_b \le \beta - 1$, it follows from Corollary 3 that as $B_i \to 0$

$$\frac{d}{dB_i}(\pi_i^{JF}(\beta B_i/c_i, q_j^d) - \pi_i^{NF}(B_i/c_i)) \to \lim_{B_i \to 0} \frac{d\pi_i^{JF}(\beta B_i/c_i, q_j^d)}{dB_i} - \lim_{B_i \to 0} \frac{d\pi_i^{NF}(B_i/c_i)}{dB_i} \le 0.$$

That is, for $B_i \in (0, \varepsilon)$, $\pi_i^{JF}(\beta B_i/c_i, q_j^d) - \pi_i^{NF}(B_i/c_i)$ is continuously decreasing in B_i. Thus, for any $B_i \in (0, \varepsilon)$, we have $\pi_i^{JF}(\beta B_i/c_i, q_j^d) \le \pi_i^{NF}(B_i/c_i)$. According to the proof of the first part above, for $B_i \in (B_i^0 - \varepsilon, B_i^0)$, $\pi_i^{JF}(q_i^J, q_j^J) \le \pi_i^{NF}(B_i/c_i)$. If

$$\Omega \doteq \{B_i : \pi_i^{JF}(\beta B_i/c_i, q_j^d) - \pi_i^{NF}(B_i/c_i) = 0, \text{ or } \pi_i^{JF}(q_i^\circ, q_j^\circ) - \pi_i^{NF}(B_i/c_i) = 0\} \neq \emptyset,$$

then we set $\bar{\bar{B}}_i = \inf \Omega$; otherwise, we set $\bar{\bar{B}}_i = B_i^0$. Therefore, when $0 \le B_i < \bar{\bar{B}}_i$, we obtain that $\pi_i^{JF}(q_i^J, q_j^J) \le \pi_i^{NF}(B_i/c_i)$. ■

Proof of Theorem 2: By the definition of the equilibrium financing mode and the results in Proposition 6, it is straightforward to show this theorem. ■