Integrating Inventory Replenishment and Cash Payment Decisions in Supply Chains

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The recent financial crisis demonstrates that an upstream member in a supply chain usually has a weaker cash liquidity. Cash shortage affects material supply to a downstream member, which, in turn, affects the performance of the entire chain. This paper provides a supply chain model that integrates material and cash flows and investigates the impact of payment policies on the system performance. Specifically, we consider a two-stage system in which a retailer replenishes inventory from a supplier in a finite horizon. The retailer has stronger cash liquidity in the sense that it can transfer cash from or to an investment account in each period. To quantify the value of payment flexibility, we consider two payment schemes. For the flexible payment (FP) scheme, the retailer may delay the payment or subsidize cash to the supplier; for the strict payment (SP) scheme, the retailer pays exactly what it orders. We prove that the optimal joint policy for the FP model has a surprisingly simple structure – both stages implement an echelon base-stock policy for inventory replenishment; the retailer monitors the system's working capital and implements a two-threshold policy for cash transfers and a pay-up-to policy for payment. Solving the SP model is more involved. We first provide a lower bound on the optimal cost by connecting the SP model to an assembly system. We then propose a simple and intuitive heuristic. Our numerical study characterizes circumstances under which the value of flexible payment is most significant and identifies factors that affect the optimal cash to inventory ratio. In addition, the volatility of payment from the retailer to the supplier is larger (smaller) than that from the supplier to the external vendor under the FP (SP) scheme. Thus, the material and financial bullwhip effects may not amplify in the same direction.

Key words: multi-echelon, inventory replenishment, payment policies, supply chain finance

1 Introduction

The fundamental purpose of supply chain management is to efficiently coordinate material, information, and financial flows so as to reduce risks of demand-supply mismatches. These risks can be mitigated through implementing optimal or near-optimal inventory policies (e.g., Clark and Scarf 1960, Federgruen and Zipkin 1984, Shang and Song 2003), coordinating supply chain members (e.g., Lee and Whang 1999, Cachon and Lariviere 2005, Shang et al. 2009), sharing demand and projected order information (e.g., Lee et al. 2000, Aviv 2001, Chen and Lee 2009), and exercising operational hedging strategies (e.g., Huchzermeier and Cohen 1996, Harrison and Van Mieghem 1999). While the supply chain literature on the aforementioned topics is quite extensive, these studies essentially focus on achieving a more effective integration of material and information flows. Interestingly, the literature on studying the integration of material and financial flows is relatively sparse, even though these two flows are closely related and affect each other. For example, in the recent financial crisis, many upstream suppliers who provide trade credit contracts to their downstream retailers suffered from cash shortage because of the difficulty of getting funds from banks. Lacking cash will affect the suppliers' normal operations, which, in turn, affects material supply to their downstream retailers. Such a supply disruption risk can be mitigated, however, if a more effective cash payment policy is in place. The purpose of this paper is to provide a framework that integrates financial flows into an existing supply chain model and to explore the impact of cash payment policies on the system performance. To this end, we shall demonstrate that an effective cash payment policy can mitigate the supply disruption risk and improve supply chain efficiency.

We consider a two-stage inventory system in which an upstream supplier provides materials to a downstream retailer, who faces stochastic customer demand. The demands are independent from period to period but not necessarily identical. There are positive lead times for both stages. Linear inventory holding and backorder costs are incurred in each period. In addition, there is a linear purchase cost incurred for each inventory ordered at each stage. Consistent with a cited phenomenon that the downstream retailer usually has stronger cash liquidity (e.g., Lester 2002, Boute et al. 2011), we assume that the retailer can either *dispose* cash to invest, for example, in equity markets or *retrieve* cash by selling equities to assist operations, if necessary. We call this cash disposal or retrieval *investment* decision (either invest in equities or inventory). This modeling approach is commonly seen in the cash management literature since large companies usually hold two distinct accounts for cash usage: an operating account (cash balance account) and an investment account (portfolio of liquid assets); see, for example, Baumol (1952), Tobin (1956), and Miller and Orr (1966). On the other hand, the supplier mainly focuses on using cash for operations, i.e., paying inventory, without major investment activities. In each period, both stages decide how much inventory to order; the retailer also makes payment and investment decisions. The objective is to obtain the optimal joint inventory order, cash payment, and investment policies such that the system-wide discounted expected cost in a finite horizon is minimized.

To quantify the value of payment flexibility, we consider two payment schemes for the retailer. For the flexible payment (FP) scheme, the retailer may delay the payment or subsidize cash to the supplier; for the strict payment (SP) scheme, the retailer pays exactly what it orders. In both schemes, the supplier pays exactly what it orders. Clearly, the benefit gained from the FP scheme over the SP scheme is the value of flexible payment.

We first formulate a dynamic program which includes (two) inventory and (two) cash states for the considered two-stage system under the FP scheme. This problem is complicated as one cannot directly obtain a structured joint optimal policy from the dynamic program. Nevertheless, by redefining the state variables into *echelon* terms, we can transform the original two-stage system into a four-stage system, under which the optimal joint policy can be characterized. The optimal joint policy is surprisingly simple. The inventory policy has the same structure as the traditional multi-echelon system (c.f., Clark and Scarf 1960): each stage reviews the echelon inventory position at the beginning of a period and orders up to a target echelon base-stock level. For the cash payment and investment policies, the retailer reviews the *echelon working capital position* (echelon inventory position plus inventory-equivalent cash level) at the beginning of each period and disposes (retrieves) cash down (up) to a threshold level; then the retailer pays the supplier up to a target level (or the supplier receives cash payment up to a target level). Technically, we simplify the computation by decoupling the original dynamic program with four states into four separate dynamic programs, each with one state variable. Thus, the optimal policy parameters can be easily obtained.

Solving the model with the SP scheme is more involved. Simply speaking, the problem is similar to a serial capacitated system (Parker and Kapuscinski 2004) in the sense that the on-hand cash level can be viewed as a capacity constraint for inventory ordering. However, the major difference between the traditional capacitated system and ours is that the cash level constraint is *endogenously* determined by the inventory, payment, and investment decisions. Thus, we are not able to characterize the optimal policy. Nevertheless, we provide a lower bound to the optimal cost by mapping the considered SP model into an assembly system with two component flows – one is retailer's cash flow and the other is the system's material flow. In addition, we propose a simple and effective heuristic policy for the SP model. We use the heuristic policy to reveal qualitative insights.

We summarize several major analytical and qualitative findings. First, by adopting a flexible payment policy, the inventory decision is independent of the cash payment decision; however, the cash payment decision depends on the echelon working capital position. An implication is that the accounting department should consider the inventory level when making cash payment decisions. Second, the optimal echelon working capital position depends on the length of the total inventory lead time plus financial payment lead time. In other words, when the retailer plans for its optimal cash level in the current period, it has to consider the demand occurring in the aforementioned lead time length in the future. Third, the retailer should subsidize the supplier when the supplier profit margin is low and the demand is increasing. In this case, the flexible payment policy has a significant value and can mitigate the risk of supply shortage. Nevertheless, the value of flexible payment diminishes faster than the reduction of capital liquidity. On the other hand, the retailer should delay the payment to the supplier when the supplier's profit margin is high and demand tends to be stationary. In this case, delaying payment is equivalent to allocating more cash to the retailer who has a better investment capability, and thus the value of flexible payment is significant. This finding also suggests that the cash payment policy can be a useful mechanism to share the financial default risk between the supply chain parties. Fourth, when flexible payment is in place, we find that both stages' optimal cash and inventory levels are non-decreasing with demand volatility, but the change of inventory level is larger than that of cash level, making the cash to inventory ratio decrease. In addition, the optimal cash to inventory ratio of the supplier tends to be more stable than that of the retailer. Lastly, the variability of cash payment from the retailer to the supplier is larger (smaller) than that from the supplier to the outside vendor under the FP (SP) scheme. On the other hand, the variability of inventory shipment grows when moving upstream. Thus, the material and financial bullwhip effects may not amplify in the same direction in an integrated supply chain.

2 Literature Review

Our work is related to four streams of research in the literature: cash management, multi-echelon inventory models, capacitated inventory models, and inventory model with financial flows.

For the cash management literature, most papers treated cash as inventory and used inventory control tools to find the optimal cash balance for firms. Baumol (1952) studied the optimal cash level for a firm that uses cash either for paying transactions or for investment. Baumol's model is similar to the retailer in our setting. Tobin (1956) considered a more refined model, focusing on the impact of interest rate on the demand for cash at a given volume of transactions. In both works, the cash flows were modeled as a deterministic stream. In contrast, Miller and Orr (1966) highlighted the stochastic nature of transactions by assuming that the net cash flows are generated by a stationary random walk. This line of research was further extended by Frenkel and Jovanovic (1980), who analyzed the optimal cash holdings when net disbursements are governed by a standard Wiener process. For dynamic, periodic-review cash balance problems, Girgis (1968) modeled the selection of a cash level in anticipation of future net expenses as a single product multi-period inventory system. Porteus (1972) showed that the marginal cost and total cost formulation of these problems are equivalent under appropriate cost constructions. Heyman (1973) presented a model to minimize the average cash balance subject to a constraint on the probability of stock-out. The biggest difference between these studies and ours is that these authors focus on cash dynamics without considering detailed operational decisions, while we specifically model the cash and inventory dynamics as two inter-related flows.

Our research is also related to the multi-echelon literature. In particular, our model incorporates cash flows into the seminal supply chain model developed by Clark and Scarf (1960), who proved that an echelon base-stock policy is optimal. Furthermore, they showed that the problem can be decoupled into a series of separate one-dimensional dynamic programs by introducing the notion of echelon inventories. Federgruen and Zipkin (1984) extended their results to an infinite horizon model and showed that a stationary order-up-to policy is optimal. Chen and Zheng (1994) simplified the optimality proof and provided a lower bound on the long run costs. Angelus (2011) considered a variant of the Clark-Scarf model by allowing each stage to dispose excess inventory to a secondary market. He introduced a class of heuristic policies, called disposal saturation policies, which can be obtained using the Clark-Scarf decomposition. In our model, we allow both cash disposal and injection.

The capacitated inventory problem is related to our model since the cash constraint on inventory replenishment can be viewed as the supply capacity. For single-stage systems, Federgruen and Zipkin (1986) showed that the modified base-stock policy is optimal. Angelus and Porteus (2002) derived the optimal joint capacity adjustment and production plan with and without carryover of unsold inventory units. Their capacity adjustment decision is similar to our cash investment decision, but our cash capacity is also affected by payment decisions and random sales. For serial systems, Roundy and Muckstadt (2000) assumed base-stock policy and proposed an efficient approximation. Parker and Kapuscinski (2004) demonstrated that a modified echelon base-stock policy is optimal in a two-stage system when there is a smaller capacity at the downstream facility. Glasserman and Tayur (1995) studied the stability issue of the system and provided a heuristic for the optimal base-stock policy. Huh et al. (2010) studied the stability properties of base-stock policies with a sample path approach. The main difference between the serial capacitated models and ours is that the cash constraint is endogenously determined by inventory and cash decisions.

Finally, there have been several recent studies to incorporate financial decisions or budget constraints into inventory models. Most of these papers are based on single-stage systems. Buzacott and Zhang (2004) incorporated asset-based financing into production decisions. They demonstrated the importance of joint consideration of production and financing decisions to capital constrained firms. Babich and Sobel (2004) coordinated the financial and operational decisions to maximize the present value of proceeds from an IPO. They characterized an optimal capacity-expansion policy by formulating the IPO event as a stopping time in an infinite-horizon Markov decision process. Li et al. (2005) studied a dynamic model in which inventory and financial decisions are made simultaneously in the presence of uncertain demand. The objective is to maximize the expected present value of dividends. The authors proved that the myopic policy is optimal. Ding et al. (2007) studied the integrated operational and financial hedging decisions faced by a global firm which sells to both home and foreign markets. They showed that the firm's financial hedging strategy is closely tied to its operational strategy. Chao et al. (2008) modeled a self-financing retailer's inventory replenishment decisions with a cash budget constraint. They characterized the optimal inventory control policy. Gupta and Wang (2009) presented a discrete-time inventory model with trade credit and showed that the problem can be converted into a single-stage system model with refined holding cost rates. Babich (2010) studied a manufacturer's joint inventory and financial subsidy decisions when facing a supplier whose financial state is governed by a firm-value model. He showed that an order-up-to policy and subsidize-up-to policy are optimal for the manufacturer. Our model is different from Babich's as we explicitly model the interrelated dynamics of inventory and cash flows between the retailer and supplier, while the supplier in Babich's is exogenous to the manufacturer. Yang and Birge (2011) modeled a Stackelberg game between a retailer and a supplier with the use of a trade credit contract. They demonstrated that an effective trade credit policy can enhance supply chain efficiency. Protopappa and Seifert (2010) conducted a simulation study on a two-stage supply chain to reveal qualitative insights on the allocation of working capital between the supply chain partners.

The rest of this paper is organized as follows. §3 describes the FP model and formulates the corresponding dynamic program. §4 proves the optimal joint policy by transforming the original two-stage system into a four-stage model. §5 focuses on the SP model. We provide lower bounds to the optimal cost and suggest a heuristic. §6 examines the effectiveness of the heuristic for the SP model, and discusses the qualitative insights through a numerical study. §7 concludes. Appendix A extends our model to incorporate a penalty cost charged on the debt between the supplier and the retailer. Appendix B provides proofs. Throughout this paper, we define $x^+ = \max(x, 0)$, $x^- = -\min(x, 0)$, $a \vee b = \max(a, b)$, and $a \wedge b = \min(a, b)$.

3 The Model

We consider a periodic-review, two-stage serial inventory system where a retailer (stage 1) orders from a supplier (stage 2), which orders from an outside ample source. The supplier pays exactly what it orders. The retailer receives payment from customers and decides the payment amount to the supplier. We consider two payment schemes for the retailer, flexible payment (FP) and strict payment (SP). For the FP scheme, the retailer can decide any positive payment amount to the supplier. If the payment is greater than what is orders, the additional amount can be seen as a *financial subsidy* to the supplier. On the other hand, if the payment is less than what is orders, the retailer is purchasing on open account, i.e., *delayed payment* in trade finance. The SP scheme is a special case of the FP scheme as the retailer pays exactly what it orders. The retailer faces a stochastic demand D_t in period t. The demands are independent between periods, but the demand distributions may differ from period to period. We assume that unsatisfied demand is fully backlogged. Figure 1 shows the material and cash flows in solid and dashed arrows, respectively. Without loss of generality, we assume that the material lead time is one period for both stages.

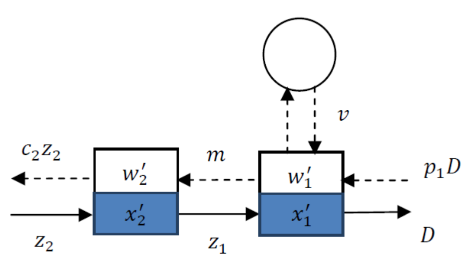


Figure 1: The two-stage FP model with material and cash flows.

To reflect that the retailer has a stronger cash liquidity, we assume that the retailer's role is similar to an investment center in that it can either invest excess cash in equity markets or retrieve cash by selling equities, if necessary. As commonly seen in the cash management literature, large firms usually manage cash in two separate accounts: an investment account where firms hold a portfolio of invested assets (e.g., stock equities) and an operating account from which inventory payment is drawn and to which sales revenue is deposited. Transferring funds between these two accounts is permissible and assumed to be instantaneous yet with costs incurred proportional to the amount transferred. The circle in Figure 1 represents the investment account; the top white rectangle represents the operating account, i.e., retailer's cash balance. On the other hand, the supplier's role is similar to a cost center in a sense that it only uses cash for inventory procurement without investment activities.

The sequence of events is as follows: At the beginning of the period, (1) both stages receive shipments; (2) stage 2 and outside source receive payment from stage 1 and stage 2, respectively; (3) stage 1 makes an investment decision; (4) both stages make an order decision; (5) stage 1 makes a payment decision. During the period, demand is realized and stage 1 receives revenue. At the end of the period, all inventory and cash related costs are calculated. The planning horizon is T periods, and we count the time backwards, i.e., $T, T-1, ..., 1$. The objective is to minimize the system's total discounted cost over the entire horizon.

We assume that the actual payment transaction occurs upon the receipt of shipments. That is, the supplier will not receive the payment determined by the retailer in period t until period $t-1$ when the retailer receives the shipment (placed in period t). This payment practice is similar to a Letter of Credit (LC), in which the retailer holds the exact cash amount in a bank for the inventory ordered. The bank will guarantee the payment to the supplier after the retailer receives the shipment. In other words, we can view that there is a one-period lead time for the cash payment for both stages. For stage $i = 1, 2$, let p_i be the unit selling price, c_2 the unit procurement cost of stage 2, and c_1 the inter-stage transportation cost. We assume $c_2 < p_2$ and $p_2 + c_1 < p_1$ to ensure the profitability for both stages. We now define state and decision variables. Here and in the sequel, we use prime to indicate local variables and parameters. For stage $i = 1, 2$ and period t, let

$$
\begin{aligned}
x'_{1,t} &= \text{net inventory level at stage 1 after Event (1);} \\
x'_{2,t} &= \text{on hand inventory level at stage 2 after Event (1);} \\
w'_{i,t} &= \text{cash level in stage } i \text{ after Event (2);} \\
v_t &= \text{amount of cash transferred into stage 1's operating account made in Event (3);} \\
z_{i,t} &= \text{order quantity for stage } i \text{ made in Event (4);} \\
m_t &= \text{payment amount from stage 1 to stage 2 made in Event (5).}
\end{aligned}
$$

Note that v_t^+ is the cash amount that flows into the retailer's operating account and v_t^- is the cash amount that flows out to the investment account.

The dynamics of states between two periods are shown below:

$$x'_{1,t-1} = x'_{1,t} + z_{1,t} - D_t, \tag{1}$$
$$x'_{2,t-1} = x'_{2,t} + z_{2,t} - z_{1,t}, \tag{2}$$
$$w'_{2,t-1} = w'_{2,t} + m_t - c_2 z_{2,t}, \tag{3}$$
$$w'_{1,t-1} = w'_{1,t} + v_t - m_t + p_1 D_t. \tag{4}$$

For stage 1's cash dynamic in (4), we assume that the customer will prepay the item even if there is a stock out. This assumption is reasonable as all demand will be filled in the future under the backorder model. It is also commonly seen in practice and in the dynamic pricing literature, e.g., Federgruen and Heching (1999). The cash dynamics in (3) and (4) do not include holding and backorder cost, or inter-stage transportation cost because they usually are not incurred in the daily cash transactions. More specifically, the inventory holding and transportation costs are usually calculated biannually or annually; backorder cost is a concept representing loss of goodwill, and cash holding cost is viewed as an opportunity loss of capital. Furthermore, the cash transaction costs are often minimal and therefore ignored in the daily cash calculations. Nevertheless, these costs are considered in the total cost function as they will be realized at the end of the planning horizon.

Define $\mathbf{x}' = (x'_1, x'_2)$, $\mathbf{w}' = (w'_2, w'_1)$, and $\mathbf{z} = (z_1, z_2)$. The feasible set in each period is

$$\hat{S}(\mathbf{x}', \mathbf{w}') = \left\{ \mathbf{z}, m, v \,|\, 0 \leq z_1 \leq x'_2, 0 \leq z_2 \leq w'_2/c_2, 0 \leq m \leq w'_1 + v,\ v \leq K' \right\}.$$

The first constraint states that stage 1's order quantity cannot exceed stage 2's on-hand inventory; the second constraint states that stage 2's order quantity is constrained by its cash available; the third constraint specifies stage 1's maximum payment to stage 2, which also implies that the cash put into the investment account in each period cannot exceed its on-hand cash level, i.e., $v \geq -w_1$. Finally, the last constraint specifies an upper limit (denoted by K') on the amount of cash that can flow into stage 1's operating account in each period. This upper bound can be viewed as a proxy of the liquidity level of the retailer's investment account.

We introduce the cost parameters. Following the inventory literature, we charge a linear local holding cost h'_i for each unit of inventory held at stage i in each period, and a backorder cost b for each unit of backorder incurred at the retailer in each period. In addition, there is a linear holding cost η'_i per dollar per period for the cash held in the stage i's operating account. Empirical research suggests that there is an opportunity cost of holding cash for operations. For example, η'_1 may be viewed as the potential loss on the return if the cash were transferred to the investment account. Here, we assume that $h'_1 > h'_2 > \eta'_2 c_2$, i.e., holding a unit of inventory at downstream is more costly than that at upstream, and holding an unit of inventory is more costly than holding the same value amount of cash. The later is generally true since inventory holding cost consists of both financial opportunity cost and physical shelf cost. Let β'_i and β'_o denote the unit cash transfer cost to and from the retailer's operating account, respectively. In practice, the cost charged on transferring funds can be regarded as brokerage fees. Finally we denote γ' as the unit payment transaction cost between stages.

The single-period expected cost function is

$$\begin{aligned}\hat{G}_t(\mathbf{x}', \mathbf{w}', \mathbf{z}, m, v) = \mathsf{E}_{D_t}\left[h_1^{'}(x_1^{'} - D_t)^+ + b(x_1^{'} - D_t)^-\right] + h_2^{'}x_2^{'} + c_1 z_1 + c_2 z_2 \\ + \eta_2^{'}w_2^{'} + \eta_1^{'}\mathsf{E}_{D_t}\left[w_1^{'} + v + p_1 D_t\right] + \gamma' m + \beta_i^{'}v^+ + \beta_o^{'}v^-. \qquad (5)\end{aligned}$$

The first line in the cost function is the inventory-related cost, which includes inventory holding, backlogging and procurement costs. By convention, we charge $h_2^{'}$ to the pipeline inventory so $h_2^{'}x_2^{'}$ is the cost for the inventories held at stage 2 plus those in the pipeline. The second line is the cash-related cost, which includes cash holding and transaction costs. As shown, we charge $\eta_2^{'}$ for $w_2^{'}$ because stage 2 still holds the inventory payment (determined in period t) until it receives the shipment in next period. A similar idea applies to stage 1.

We make a remark here. Under the FP scheme, the retailer's payment decision can be any non-negative value. On the other hand, for the SP scheme, the payment m is equal to $p_2 z_1$. Also, under the FP scheme, there might be additional interest costs for the retailer (supplier) in each period due to delayed payment (subsidy). Nevertheless, from the entire supply chain perspective, the retailer's interest loss is equal to the supplier's interest gain. Thus, we do not specify these interest costs as they will be canceled out in a period.

Let α denote the single-period discount rate. Recall that the objective is to minimize the expected discounted total cost in T periods. Denote $\hat{J}_t(\mathbf{x}', \mathbf{w}', \mathbf{z}, m, v)$ as the minimum expected discounted cost over period t to 0, at the beginning of period t, with given states $(\mathbf{x}', \mathbf{w}', \mathbf{z}, m, v)$. And denote $\hat{V}_t(\mathbf{x}', \mathbf{w}')$ as the optimal total discounted cost over all feasible decisions. The dynamic program is

$$\begin{aligned}\hat{J}_t(\mathbf{x}', \mathbf{w}', \mathbf{z}, m, v) &= \hat{G}_t(\mathbf{x}', \mathbf{w}', \mathbf{z}, m, v) + \alpha\mathsf{E}_{D_t}\Big[\hat{V}_{t-1}(x_1^{'} + z_1 - D_t,\ x_2^{'} + z_2 - z_1, \\ &\qquad w_2^{'} + m - c_2 z_2,\ w_1^{'} + v - m + p_1 D_t)\Big], &(6)\\ \hat{V}_t(\mathbf{x}', \mathbf{w}') &= \min_{\mathbf{z}, m, v \in \hat{S}(\mathbf{x}', \mathbf{w}')} \hat{J}_t(\mathbf{x}', \mathbf{w}', \mathbf{z}, m, v), &(7)\\ \hat{V}_0(\mathbf{x}', \mathbf{w}') &= 0. &(8)\end{aligned}$$

Here we assume a zero terminating cost for simplicity. Appendix A extends this model to incorporate a penalty cost charged on the expected end debt between the retailer and the supplier. One can expect that if the penalty cost is significantly high, the expected end debt tends to be zero.

The local formulation in (6)-(8) is difficult to solve. One can prove that $\hat{J}_t(\cdot)$ is jointly convex in all state and decision variables. Thus, the global minimum solution is state-dependent. However, computing the solution is quite hard due to the curse of dimensionality. In the next section, we transform the original problem into a new system, from which we can obtain a simple and implementable optimal joint policy.

4 Flexible Payment Scheme

4.1 Echelon Formulation

We transform the original two-stage system into a four-stage serial model by introducing new system variables. Define the following *echelon* variables:

$$x_1 = x_1', \;\; x_2 = x_1' + x_2',$$
$$w_2 = x_1' + x_2' + w_2'/c_2, \;\; w_1 = x_1' + x_2' + (w_1' + w_2')/c_2.$$

Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{w} = (w_2, w_1)$. We refer to $\mathbf{x}$ as the *echelon net inventory level*, and $\mathbf{w}$ as the *echelon net working capital level* measured in inventory unit, which is obtained by converting cash to inventory at the value of c_2. This state transformation explicitly treats cash as inventory. More specifically, the financial flow in the system can be seen as an extension of the material flow after "flipping" the operating accounts of the supplier and the retailer. These two operating accounts serve as new upstream stages. We define the corresponding echelon decision variables:

$$y_1 = x_1' + z_1, \;\; y_2 = x_1' + x_2' + z_2,$$
$$r_2 = x_1' + x_2' + (w_2' + m)/c_2, \;\; r_1 = x_1' + x_2' + (w_1' + w_2' + v)/c_2.$$

Let $\mathbf{y} = (y_1, y_2)$ and $\mathbf{r} = (r_2, r_1)$. Figure 2 shows the transformed FP system.

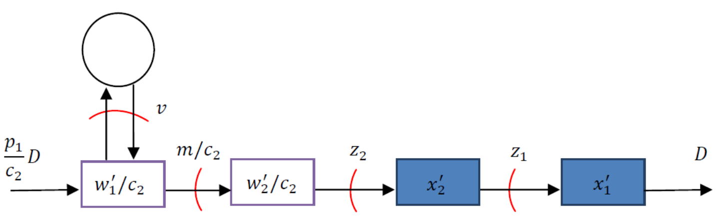


Figure 2: The four-stage transformed FP system.

With this transformation, supplier's operating account becomes stage 3 in the new system, directly supplying its own inventory. Symmetrically, retailer's operating account turns into stage 4, the most upstream stage in the transformed system. Beyond inventory echelons x_1 and x_2, we define echelon 3 (with state variable w_2) as stage 2's echelon working capital, which includes inventory at both stages and supplier's cash expressed in inventory units. We define echelon 4 (with state variable w_1) as stage 1's echelon working capital, which is equal to w_2 plus stage 1's cash expressed in inventory units. Clearly, w_1 is equivalent to the total system working capital in inventory units.

Similar to a multi-echelon inventory model, we denote echelon holding cost rate $\eta_1 = \eta_1' c_2$, $\eta_2 = (\eta_2' - \eta_1')c_2$, $h_2 = h_2' - \eta_2' c_2$, and $h_1 = h_1' - h_2'$. Since $h_1' > h_2' > \eta_2' c_2$ by assumption, we have $\eta_1 > 0$, and $h_i > 0$, $i = 1, 2$. Furthermore, let $\beta_i = \beta_i' c_2$, $\beta_o = \beta_o' c_2$, $\gamma = \gamma' c_2$, $\theta = p_1/c_2 - 1 > 0$, and $K = K'/c_2$.

With these echelon terms, the state dynamics in (1)-(4) become

$$x_{1,t-1} = y_{1,t} - D_t, \quad x_{2,t-1} = y_{2,t} - D_t, \quad w_{2,t-1} = r_{2,t} - D_t, \quad w_{1,t-1} = r_{1,t} + \theta D_t,$$

and the constraint set becomes

$$S(\mathbf{x}, \mathbf{w}) = \{\mathbf{y}, \mathbf{r} \,|\, x_1 \le y_1 \le x_2 \le y_2 \le w_2 \le r_2 \le r_1 \le w_1 + K\}.$$

We further specify the holding and backorder cost associated with each echelon as:

$$\begin{aligned}
&H_{1,t}(x_1) = \mathsf{E}_{D_t}\left[(h_1 + h_2 + \eta_2 + \eta_1 + b)(D_t - x_1)^+ + h_1(x_1 - D_t)\right],\\
&H_{2,t}(x_2) = \mathsf{E}_{D_t} h_2(x_2 - D_t), \quad H_{3,t}(w_2) = \mathsf{E}_{D_t}\eta_2(w_2 - D_t), \quad H_{4,t}(r_1) = \mathsf{E}_{D_t}\eta_1(r_1 + \theta D_t).
\end{aligned}$$

Then, we can rewrite the dynamic program in (6)-(8) as follows:

$$J_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, \mathbf{r}) = G_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, \mathbf{r}) + \alpha \mathsf{E}_{D_t}\left[V_{t-1}(y_1 - D_t, y_2 - D_t, r_2 - D_t, r_1 + \theta D_t)\right], \tag{9}$$

$$V_t(\mathbf{x}, \mathbf{w}) = \min_{\mathbf{y}, \mathbf{r} \in S(\mathbf{x}, \mathbf{w})} J_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, \mathbf{r}), \tag{10}$$

$$V_0(\mathbf{x}, \mathbf{w}) = 0, \tag{11}$$

where the single-period cost function can be shown as

$$\begin{aligned}
G_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, \mathbf{r}) = {} & H_{1,t}(x_1) + H_{2,t}(x_2) + H_{3,t}(w_2) + H_{4,t}(r_1)\\
& + c_1(y_1 - x_1) + c_2(y_2 - x_2) + \gamma(r_2 - w_2) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^-.
\end{aligned}$$

We refer to the dynamic program in (9)-(11) as the echelon formulation of the model.

4.2 The Optimal Policy

We first state the optimal joint policy for the FP model, which includes three decisions made through five control parameters $(y_1^*, y_2^*, r_2^*, l^*, u^*)$ in each period. For the inventory ordering decision, each stage implements an echelon base-stock policy. That is, stage i reviews its x_i at the beginning of each period. If $x_i < y_i^*$, stage i orders up to y_i^* or as close as possible to y_i^* if its upstream does not have sufficient stock; otherwise, it does not order. For the investment decision, stage 1 reviews w_1. If $w_1 > u^*$, stage 1 disposes cash down to the maximum of u^* and w_2; if $w_1 < l^*$, stage 1 retrieves cash up to l^* or as close as possible to l^* (due to the upper bound K); otherwise, it does not transfer cash between accounts. Finally, for the payment decision, stage 2 reviews w_2. If $w_2 < r_2^*$, stage 2 receives stage 1's payment up to r_2^* or as close as possible to r_2^* if stage 1 does not have sufficient cash; it does not receive payment otherwise.

We next explain how the optimal policy is derived and how to calculate these policy parameters. The following proposition decouples inventory decisions from the rest of the system.

Proposition 1. *For all t and states $(\mathbf{x}, \mathbf{w})$, $V_t(\mathbf{x}, \mathbf{w}) = f_{1,t}(x_1) + f_{2,t}(x_2) + F_t(\mathbf{w})$, where $f_{i,t}(x_i)$ is convex in x_i and $F_t(\mathbf{w})$ is joint convex in $\mathbf{w}$.*

We define $f_{i,t}(\cdot)$ as the expected optimal cost for echelon i, and $F_t(\cdot)$ as the expected optimal cost for echelon 3 and 4 combined, at the beginning of period t. These functions can be expressed as

$$f_{i,t}(x_i) = H_{i,t}(x_i) + \Gamma_{i,t}(x_i) + \min_{x_i \le y_i} \{c_i(y_i - x_i) + \alpha \mathsf{E}_{D_t} f_{i,t-1}(y_i - D_t)\}, \tag{12}$$

$$F_t(\mathbf{w}) = H_{3,t}(w_2) + \Gamma_{3,t}(w_2) + \min_{w_2 \le r_2 \le r_1 \le w_1 + K} \left\{ \begin{array}{l} \gamma(r_2 - w_2) + H_{4,t}(r_1) \\ +\beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^- \\ +\alpha \mathsf{E}_{D_t} F_{t-1}(r_2 - D_t, r_1 + \theta D_t) \end{array} \right\}, \tag{13}$$

where the $\Gamma_{i,t}(\cdot)$ functions are the so-called induced penalty cost functions defined in Clark and Scarf (1960). More specifically, $\Gamma_{1,t}(\cdot) \equiv 0$, and

$$\Gamma_{2,t}(x_2) = \left\{ \begin{array}{ll} c_1(x_2 - y_{1,t}^*) + \alpha \mathsf{E}_{D_t} \left[f_{1,t-1}(x_2 - D_t) - f_{1,t-1}(y_{1,t}^* - D_t) \right], & x_2 \le y_{1,t}^*, \\ 0, & \text{otherwise}, \end{array} \right\} \tag{14}$$

$$\Gamma_{3,t}(w_2) = \left\{ \begin{array}{ll} c_2(w_2 - y_{2,t}^*) + \alpha \mathsf{E}_{D_t} \left[f_{2,t-1}(w_2 - D_t) - f_{2,t-1}(y_{2,t}^* - D_t) \right], & w_2 \le y_{2,t}^*, \\ 0, & \text{otherwise}. \end{array} \right\} \tag{15}$$

The optimal control parameters $y_{1,t}^*$ and $y_{2,t}^*$ can be obtained by solving the minimization problem in (12). That is, let $g_{i,t}(y_i) = c_i y_i + \alpha \mathsf{E}_{D_t} f_{i,t-1}(y_i - D_t)$, then

$$y_{i,t}^* = \arg\min_y \left\{ g_{i,t}(y) \right\}.$$

Here, $\Gamma_{2,t}(\cdot)$ represents the penalty cost charged to echelon 2 if the supplier cannot ship up to retailer's target base-stock level $y_{1,t}^*$; $\Gamma_{3,t}(\cdot)$ represents the penalty cost charged to echelon 3 if the supplier fails to hold sufficient cash to pay for its inventory procurement up to the target echelon base-stock level $y_{2,t}^*$.

The next proposition further decouples the $F_t(\mathbf{w})$ function in (13). The analysis appears to be new and different from Clark and Scarf's decomposition scheme.

Proposition 2. *For all t and states $\mathbf{w}$, $F_t(\mathbf{w}) = f_{3,t}(w_2) + f_{4,t}(w_1)$, where $f_{3,t}(\cdot)$ and $f_{4,t}(\cdot)$ are convex functions.*

We express the expected optimal cost for echelon 3 and 4 as follows

$$f_{3,t}(w_2) = H_{3,t}(w_2) + \Gamma_{3,t}(w_2) + \Lambda_{3,t}(w_2) + \min_{w_2 \le r_2} \{\gamma(r_2 - w_2) + \alpha \mathsf{E}_{D_t} f_{3,t-1}(r_2 - D_t)\}, \tag{16}$$

$$f_{4,t}(w_1) = \Lambda_{4,t}(w_1) + \left\{ \begin{array}{ll} L_t(w_1), & \text{if } w_1 \le l_t^* \\ H_{4,t}(w_1) + \Gamma_{4,t}(w_1) + \alpha \mathsf{E}_{D_t} f_{4,t-1}(w_1 + \theta D_t), & \text{if } l_t^* < w_1 \le u_t^* \\ U_t(w_1), & \text{if } u_t^* < w_1 \end{array} \right\}, \tag{17}$$

where $\Gamma_{4,t}(\cdot)$ has the same structure as the standard induced-penalty cost function, i.e.,

$$\Gamma_{4,t}(r_1) = \left\{ \begin{array}{ll} \gamma(r_1 - r_{2,t}^*) + \alpha \mathsf{E}_{D_t}\left[f_{3,t-1}(r_1 - D_t) - f_{3,t-1}(r_{2,t}^* - D_t)\right], & r_1 \leq r_{2,t}^*, \\ 0, & \text{otherwise.} \end{array} \right\} \tag{18}$$

However, it has a different economic meaning: it is the penalty cost changed to echelon 4 if the retailer does not have sufficient post-investment cash to pay up to the target echelon working capital level $r_{2,t}^*$. Similarly, let $g_{3,t}(r_2) = \gamma r_2 + \alpha \mathsf{E}_{D_t} f_{3,t-1}(r_2 - D_t)$, then,

$$r_{2,t}^* = \arg\min_{r_2} \left\{ g_{3,t}(r_2) \right\}.$$

There are *new* penalty cost functions $\Lambda_{3,t}(\cdot)$ and $\Lambda_{4,t}(\cdot)$ appearing in (16) and (17). To illustrate their meanings, we define

$$g_{4,t}(w_1) = H_{4,t}(w_1) + \Gamma_{4,t}(w_1) + \alpha \mathsf{E}_{D_t} f_{4,t-1}(w_1 + \theta D_t), \tag{19}$$

$$L_t(w_1) = -\beta_i(w_1 - l_t^*) + g_{4,t}(l_t^*), \tag{20}$$

$$U_t(w_1) = \beta_o(w_1 - u_t^*) + g_{4,t}(u_t^*). \tag{21}$$

One can view $g_{4,t}(w_1)$ as the optimal cost for echelon 4 when the system working level w_1 is in $[l_t^*, u_t^*]$. Under the optimal policy, when $w_1 < l_t^*$, the retailer should retrieve cash from the investment account until w_1 reaching l_t^*. Thus, $L_t(w_1)$ can be viewed as the optimal cost when $w_1 < l_t^*$. Similarly, $U_t(w_1)$ can be viewed as the optimal cost when $w_1 > u_t^*$. In such case, the retailer should dispose cash down to u_t^*. With these explanations, the two new penalty cost functions can be defined as follows:

$$\Lambda_{3,t}(w_2) = \left\{ \begin{array}{ll} 0, & \text{if } w_2 \leq u_t^*, \\ g_{4,t}(w_2) - U_t(w_2), & \text{otherwise,} \end{array} \right\} \tag{22}$$

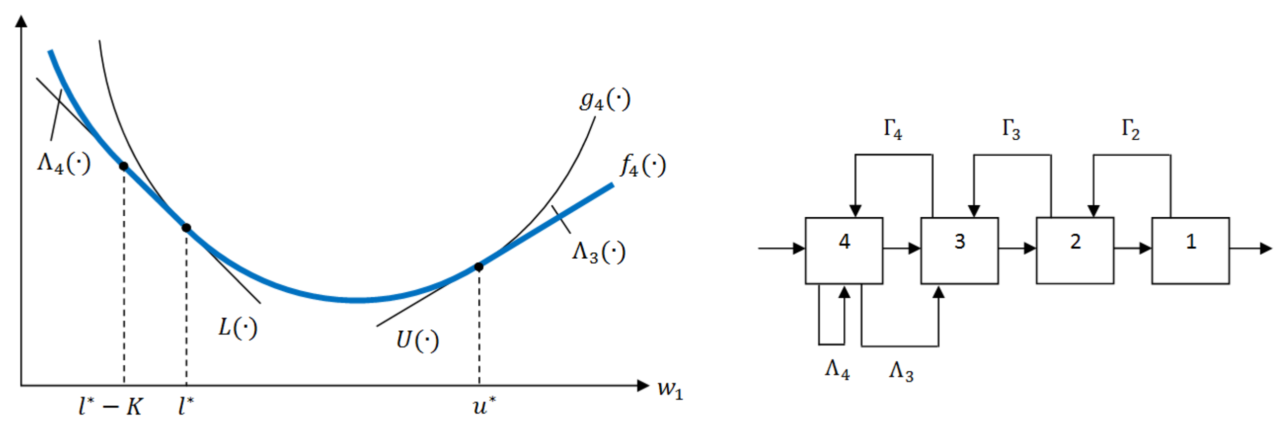
$$\Lambda_{4,t}(w_1) = \left\{ \begin{array}{ll} g_{4,t}(w_1 + K) + \beta_i K - L_t(w_1), & w_1 \leq l_t^* - K, \\ 0, & \text{otherwise.} \end{array} \right\} \tag{23}$$

Let us first consider $\Lambda_{3,t}(w_2)$ in (22). This is a penalty cost charged to echelon 3 for carrying too much working capital. Intuitively, if echelon working capital w_2 is less than or equal to u_t^*, echelon 4 can always maintain a system total working capital between l_t^* and u_t^*. However, if $w_2 > u_t^*$, the best the retailer can do is to dispose all cash on hand, making $w_1 = w_2$. In such case, the extra cost $g_{4,t}(w_2) - U_t(w_2)$ incurred in echelon 4 should be charged back to echelon 3 due to its excess working capital. For this reason, we call $\Lambda_{3,t}(w_2)$ the *excess capital penalty.* (Recall that $\Gamma_{3,t}(w_2)$ is the penalty cost charged to echelon 3 due to insufficient cash holding.) The cash transfer control thresholds can be obtained from the following equations:

$$l_t^* = \sup\left\{ r_1 : \frac{\partial}{\partial r_1} g_{4,t}(r_1) \leq -\beta_i \right\}, \quad u_t^* = \sup\left\{ r_1 : \frac{\partial}{\partial r_1} g_{4,t}(r_1) \leq \beta_o \right\}.$$

With a similar logic, $\Lambda_{4,t}(w_1)$ in (23) can be explained: this is a self-induced penalty cost charged to echelon 4 if the system total working capital w_1 is less than $l_t^* - K$ due to too much investment in the previous period. In such a case, the retailer is penalized with the extra cost $g_{4,t}(w_1+K)+\beta_i K - L_t(w_1)$ for over-disposing cash.

Figure 3(a) depicts functions $L(\cdot)$, $U(\cdot)$, $g_4(\cdot)$, $f_4(\cdot)$, as well as induced penalty functions $\Lambda_3(\cdot)$ and $\Lambda_4(\cdot)$ created while decoupling echelon 3 and 4 (with time subscripts suppressed). The optimal control threshold l^* (u^*) is derived as the tangent point of curve $g_{4,t}(\cdot)$ and a line with slope $-\beta_i$ (β_o). Function $f_4(\cdot)$ is shown as the bold convex curve connected by four different functions, which are, from the right to the left, the linear function $U(\cdot)$, the convex function $g_4(\cdot)$, the linear function $L(\cdot)$, and the convex function g_4 shifted from point $(l^*, g_4(l^*))$ to point $(l^* - K, L(l^* - K))$; the induced penalty function $\Lambda_4(w_1)$ is the difference between $f_4(w_1)$ and $L(w_1)$ to the left of $l^* - K$; the induced penalty function $\Lambda_3(w_2)$ is the difference between $g_4(w_2)$ and $U(w_2)$ to the right of u^*. Figure 3(b) illustrates the relationship between four echelons and five penalty cost functions in our problem. The direction of the arrow indicates to which echelon the penalty cost should be charged.



(a) Induced penalty functions at echelon 3 and 4.

(b) Relationship between echelons and the induced penalty functions.

Figure 3: Induced penalty functions of the FP model.

We summarize our main result in Theorem 1.

Theorem 1. *For all t and $(\mathbf{x}, \mathbf{w})$, $V_t(\mathbf{x}, \mathbf{w}) = f_{1,t}(x_1) + f_{2,t}(x_2) + f_{3,t}(w_2) + f_{4,t}(w_1)$.*

Theorem 1 indicates that we have transformed a four-state dynamic program into four, single-dimensional dynamic programs.

5 Strict Payment Scheme

We now consider the SP model, which is a special case of the FP model by replacing the payment decision m with the ordered value $p_2 z_1$, and adding a budget constraint at stage 1. For simplicity, we will keep the same notation to describe the SP model without confusion.

The constraint set for the SP model is

$$\hat{S}(\mathbf{x}',\mathbf{w}') = \left\{ \mathbf{z}, v \mid 0 \leq z_1 \leq \min\left(\frac{w_1^{'} + v}{p_2}, x_2^{'} \right), 0 \leq z_2 \leq w_2^{'}/c_2,\, v \leq K' \right\}. \tag{24}$$

As shown in the first inequality, $p_2 z_1$ cannot exceed the available cash $w_1^{'} + v$. The dynamic program for the SP model formulated by the local state and decision variables is

$$\hat{J}_t(\mathbf{x}',\mathbf{w}',\mathbf{z},v) = \hat{G}_t(\mathbf{x}',\mathbf{w}',\mathbf{z},v) + \alpha \mathsf{E}_{D_t}\Big[\hat{V}_{t-1}(x_1^{'} + z_1 - D_t, x_2^{'} + z_2 - z_1,$$
$$w_2^{'} + p_2 z_1 - c_2 z_2, w_1^{'} + v - p_2 z_1 + p_1 D_t)\Big], \tag{25}$$
$$\hat{V}_t(\mathbf{x}',\mathbf{w}') = \min_{\mathbf{z},v \in \hat{S}(\mathbf{x}',\mathbf{w}')} \hat{J}_t(\mathbf{x}',\mathbf{w}',\mathbf{z},v), \tag{26}$$
$$\hat{V}_0(\mathbf{x}',\mathbf{w}') = 0, \tag{27}$$

where the single-period expected cost is

$$\hat{G}_t(\mathbf{x}',\mathbf{w}',\mathbf{z},v) = \mathsf{E}_{D_t}\Big[h_1^{'}(x_1^{'} - D_t)^+ + b(x_1^{'} - D_t)^-\Big] + h_2^{'}x_2^{'} + (c_1 + \gamma' p_2)z_1$$
$$+ c_2 z_2 + \eta_2^{'} w_2^{'} + \eta_1^{'} \mathsf{E}_{D_t}\Big(w_1^{'} + v + p_1 D_t\Big) + \beta_i^{'} v^+ + \beta_o^{'} v^-.$$

The dynamic model described in (25)-(27) can be interpreted as a serial inventory problem with capacities (in the form of cash constraints) at both stages. However, these constraints are random and *endogenous*, hence very different from those in the traditional capacitated inventory model (Parker and Kapuscinski 2004). More specifically, from the state transitions in (25), stage 2's future cash level $w_{2,t-1}^{'}$ is determined by both stages' current order quantity $z_{1,t}$ and $z_{2,t}$; stage 1's future cash level $w_{1,t-1}$ is determined by the current period's order decision $z_{1,t}$, investment decision v_t as well as the demand realization D_t.

5.1 Echelon Formulation

Because of the strict payment, we need to create a different echelon transformation scheme. Define

$$x_1 = x_1^{'},\ \ y_1 = x_1^{'} + z_1;$$
$$x_2 = x_1^{'} + x_2^{'},\ \ y_2 = x_1^{'} + x_2^{'} + z_2;$$
$$w_1 = x_1^{'} + w_1^{'}/p_2,\ \ r_1 = x_1^{'} + (w_1^{'} + v)/p_2;$$
$$w_2 = x_1^{'} + x_2^{'} + w_2^{'}/c_2.$$

Here, x_i, y_i, and w_2 have the same definition and meaning as those in the FP model; w_1 is stage 1's net working capital level (in inventory units), rather than both stages' combined. With these state transformations, we redefine the echelon holding cost parameters for the SP model: $\eta_2 = \eta_2^{'}c_2$, $h_2 = h_2^{'} - \eta_2^{'}c_2$, $\eta_1 = \eta_1^{'}p_2$, and $h_1 = h_1^{'} - h_2^{'} - \eta_1^{'}p_2$. Also redefine $\beta_i = p_2\beta_i^{'}$, $\beta_o = p_2\beta_o^{'}$, $\gamma = p_2\gamma'$,

$\theta = p_1/p_2 - 1 > 0$, $K = K'/p_2$, and finally $\rho = p_2/c_2$.

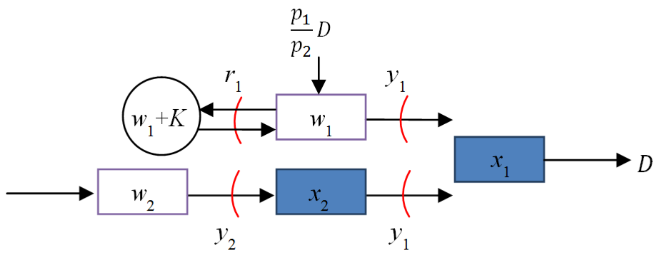


Figure 4: The transformed SP System.

Figure 4 shows the transformed SP system, which is similar to an *assembly* system. With the new echelon terms, the feasible set becomes

$$S(\mathbf{x}, \mathbf{w}) = \{\mathbf{y}, r_1 \mid x_1 \le y_1 \le r_1 \le w_1 + K,\ x_1 \le y_1 \le x_2 \le y_2 \le w_2\}.$$

The echelon formulation of the SP model becomes

$$\begin{aligned} J_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, r_1) &= G_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, r_1) \\ &\quad + \mathsf{E}_{D_t}\big[V_{t-1}(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t, r_1 + \theta D_t)\big], && (28) \\ V_t(\mathbf{x}, \mathbf{w}) &= \min_{\mathbf{y}, r_1 \in S(\mathbf{x}, \mathbf{w})} J_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, r_1), && (29) \\ V_0(\mathbf{x}, \mathbf{w}) &= 0, && (30) \end{aligned}$$

where the single-period cost function can be shown as

$$\begin{aligned} G_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, r_1) &= H_{1,t}(x_1) + H_{2,t}(x_2) + H_{3,t}(w_2) + H_{4,t}(r_1) \\ &\quad + (c_1 + \gamma)(y_1 - x_1) + c_2(y_2 - x_2) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^-. \qquad (31) \end{aligned}$$

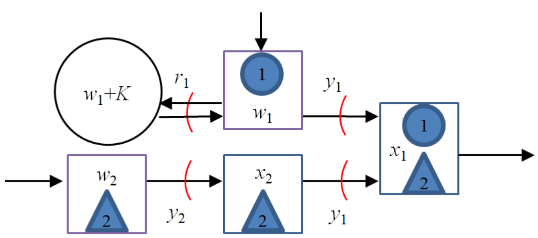
After the new transformation, some of the complexities caused by the endogenous constraints are reduced. More specifically, the dynamics of the new echelon variable w_1 no longer depend on z_1. However, the dynamics of echelon w_2 still depend on the decision $y_1 - x_1$ associated with echelon 1, as shown in (28). This unique feature of strict payment undermines the decomposition structure in the FP model and differentiates the SP model from the traditional assembly system (Rosling 1989). Below, we provide an approach to derive lower bounds to the optimal cost of the SP model.

5.2 Lower Bounds

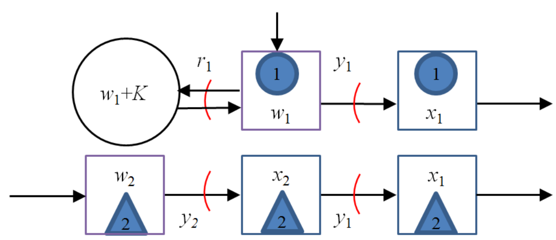
This subsection establishes two lower bounds to the optimal cost for the SP model. Recall that the SP model is similar to an assembly system. The main idea of constructing these lower bounds is to decompose this assembly system. More specifically, the expression of $S(\mathbf{x}, \mathbf{w})$ indicates that the retailer's decision y_1 is subject to two constraints: one is $y_1 \le r_1 \le w_1 + K$, which represents the cash constraint on the order quantity; the other is $y_1 \le x_2 \le y_2 \le w_2$, which can be viewed as a material order constraint in a two-stage system with an endogenous, random capacity w_2 at the upstream stage 2. Figure 5(a) shows these two sets of constraints.

Now, imagine that the final product sold at stage 1 consists of two components: a physical component (depicted by triangles) supplied from stage 2's stock, and a "cash" component (depicted by circles) supplied from stage 1's operating account. The constraint $0 \le z_1 \le \min\{(w_1^{'} + v)/p_2, x_2^{'}\}$ in (24) (or, equivalently, $x_1 \le y_1 \le \min\{r_1, x_2\}$) implies a similar structure to an assembly system: the same amount of inventory and cash equivalent are matched through replenishment at stage 1.

To derive a lower bound to the optimal cost, we relax the above matching constraint by assuming that the components can be ordered and sold separately. As a result, the original system is decoupled into two independent subsystems as shown in Figure 5(b) – subsystem 1 represents the cash flows; subsystem 2 represents the material flow. The sum of the minimum costs of subsystems is a lower bound on the minimum cost of the original system.



(a) The original SP system



(b) Subsystem 1 (top) and Subsystem 2 (bottom)

Figure 5: Decomposition of the SP system.

We specify the total cost function for each of the subsystems. Let h_1^1 and h_1^2 be the inventory holding cost for Subsystem 1 and 2, respectively, where $h_1^1 + h_1^2 = h_1$. Let b_1 and b_2 be the backorder cost for Subsystem 1 and 2, respectively, where $b_1 + b_2 = b$.

$$H_{1,t}^1(x_1) = \mathsf{E}_{D_t}\left[(h_1^1 + \eta_1 + b_1)(D_t - x_1)^+ + h_1^1(x_1 - D_t)\right], \tag{32}$$

$$H_{1,t}^2(x_1) = \mathsf{E}_{D_t}\left[(h_1^2 + h_2 + \eta_2 + b_2)(D_t - x_1)^+ + h_1^2(x_1 - D_t)\right]. \tag{33}$$

Now define

$$G_t^1(x_1, w_1, y_1, r_1) = H_{1,t}^1(x_1) + H_{4,t}(r_1) + \gamma(y_1 - x_1) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^-, \tag{34}$$

$$G_t^2(\mathbf{x}, w_2, \mathbf{y}) = H_{1,t}^2(x_1) + H_{2,t}(x_2) + H_{3,t}(w_2) + c_1(y_1 - x_1) + c_2(y_2 - x_2). \tag{35}$$

Note that $H_{1,t}(x_1) = H_{1,t}^1(x_1) + H_{1,t}^2(x_1)$, hence we have

$$G_t^1(x_1, w_1, y_1, r_1) + G_t^2(\mathbf{x}, w_2, \mathbf{y}) = G_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, r_1).$$

With this cost allocation, the dynamic program for Subsystem 1 can be expressed as

$$V_t^1(x_1, w_1) = \min_{x_1 \le y_1 \le r_1 \le w_1 + K} \left\{ G_t^1(x_1, w_1, y_1, r_1) + \alpha \mathsf{E}_{D_t} \left[V_{t-1}^1(y_1 - D_t, r_1 + \theta D_t) \right] \right\}, \tag{36}$$

$$V_0^1(x_1, w_1) = 0. \tag{37}$$

And the dynamic program for Subsystem 2 is

$$\begin{aligned} V_t^2(x_1, x_2, w_2) = & \min_{x_1 \le y_1 \le x_2 \le y_2 \le w_2} \left\{ G_t^2(\mathbf{x}, w_2, \mathbf{y}) \right. \\ & \left. + \alpha \mathsf{E}_{D_t} \left[V_{t-1}^2(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t) \right] \right\}, \end{aligned} \tag{38}$$

$$V_0^2(x_1, x_2, w_2) = 0. \tag{39}$$

Proposition 3. *$V_t(\mathbf{x}, \mathbf{w}) \ge V_t^1(x_1, w_1) + V_t^2(x_1, x_2, w_2)$ for all $(\mathbf{x}, \mathbf{w})$ and t.*

Proposition 3 shows that for any combination of (h_1^1, h_1^2) and (b_1, b_2), the sum of the two subsystems forms a cost lower bound to the original system. Maximizing expected cost over all parameter combinations yields the best lower bound.

The remaining question is how to find the optimal cost of these subsystems. A careful examination of Subsystem 1 described in (36) and (37) reveals that it is the echelon transformation for a single-stage system with inventory and cash investment decisions. Specifically, the system has to decide how much to invest and how much to order in each period (and pay exactly what it orders to the outside vendor). This is actually a special case of the FP model in which there is only one stage (the retailer, stage 1). Thus, we have characterized the optimal joint policy, i.e., using the base-stock policy to control the inventory replenishment and the two-threshold policy to control the cash investment.

Solving Subsystem 2 is much harder. The dynamic problem described in (38) and (39) is the echelon expression of a two-stage inventory model with random, endogenous capacity at the upstream stage. There exists no known optimal policy for this model. Thus, we provide two approaches to further develop a lower bound to the optimal cost for Subsystem 2.

Constraint Relaxation (CR) Bound

As the name suggests, we form the lower bound by relaxing the constraint $y_2 \le w_2$ at stage 2. Once w_2 is removed from the constraint set, it only appears in the expected cost function of each period. The following lemma characterizes the expected value of w_2 through the flow conservation.

Lemma 1. *Given the initial states $w_{2,T}$ and $x_{1,T}$, for any policy we have*

$$\mathop{\mathsf{E}}_{D_T, \ldots, D_{t+1}} w_{2,t} = \rho \cdot \mathop{\mathsf{E}}_{D_T, \ldots, D_{t+1}} x_{1,t} + B_t,$$

where $B_t = (\rho - 1) \sum_{s=t+1}^{T} \mu_s + w_{2,T} - \rho x_{1,T}$.

Recall that in the periodic cost function (35), the function $H_{3,t}(w_2)$ is a linear function of w_2. Therefore, by using Lemma 1, we can replace $H_{3,t}(w_2)$ with $H_{3,t}(\rho x_1 + B_t)$ without affecting the optimal decision in each period. With this construction, w_2 can be replaced by x_1 and Subsystem 2 becomes a classic two-stage serial system in which Clark and Scarf's algorithm can be applied to find the optimal echelon base-stock levels for both stages. The CR bound generally works well when the constraint $y_2 \leq w_2$ is not binding, i.e., when stage 2 holds sufficient cash. This occurs if the supplier's markup $(p_2/c_2 - 1)$ is high and demand tends to be stationary. However, under increasing demand, it is optimal for the supplier to order more in anticipation of future demand uprise. In such case, supplier's cash constraint could become binding, especially if its markup is low. Thus, we need another lower bound to complement the performance of the CR bound.

Sample Path (SA) Bound

The difficulty of solving Subsystem 2 comes from keeping track of the state $w_{2,t}$. As stated earlier, the current period's $w_{2,t}$ depends on the previous period's order quantity and the demand realization. Nevertheless, if we consider a specific demand sample path, $w_{2,t}$ can be fully characterized by the flow conservation.

Lemma 2. *Let $d_t(\omega)$ represent the demand realization in period t given a demand sample path ω. With initial stages $w_{2,T}$ and $x_{1,T}$, we have $w_{2,t} = \rho x_{1,t} + B_t(\omega)$, where*

$$B_t(\omega) = (\rho - 1) \sum_{s=t+1}^{T} d_s(\omega) + w_{2,T} - \rho x_{1,T}.$$

The proof of Lemma 2 is similar to that of Lemma 1, and thus omitted. Given the initial states and a demand sample path, $B_t(\omega)$ is a constant. If we replace $w_{2,t}$ (according to Lemma 2) in both the constraint set and the periodic cost function, Subsystem 2 can be reduced to a two-stage serial system with deterministic demand subject to the following constraint (at time t):

$$S_t^d(x_1, x_2 \mid \omega) = \{y_1, y_2 \mid x_1 \leq y_1 \leq x_2 \leq y_2 \leq \rho x_1 + B_t(\omega)\}.$$

The constraints state that stage 1's order decision y_1 is affected by stage 2's echelon inventory level x_2; stage 2's order decision y_2 is affected by a linear function of stage 1's inventory level x_1. The optimal y_1^* and y_2^* can be obtained by solving a two-dimensional convex program in each period. To facilitate the computation, we prove that this problem can be decoupled into two one-dimensional convex programs. Let $V_t^d(x_1, x_2 \mid \omega)$ represent the optimal cost for Subsystem 2 for any demand sample path ω after $w_{2,t}$ is substituted with $\rho x_{1,t} + B_t(\omega)$. The following proposition shows the decoupling result.

Proposition 4. $V_t^d(x_1, x_2 \mid \omega) = v_t^1(x_1 \mid \omega) + v_t^2(x_2 \mid \omega)$, *where $v_t^i(x_i \mid \omega)$ is a convex function.*

We refer the reader to the proof for the detailed formulation of v_t^1 and v_t^2 functions. A lower bound to the optimal cost of the Subsystem 2 under the SA approach can be found by averaging total costs over all demand sample paths.

In summary, we are able to generate two lower bounds – the sum of the optimal cost obtained from Subsystem 1 and the optimal cost obtained from either the CR approach or the SA approach.

5.3 Heuristic

We suggest a simple heuristic for the SP model. Let $y^u_{1,t}$, $y^u_{2,t}$, and $r^u_{1,t}$ be the heuristic solution (at time t) to the echelon SP formulation in (28)-(30). The heuristic policy is executed as follows. First, we implement an echelon base-stock policy at both stages for the inventory replenishment. The base-stock levels are set to be the optimal echelon base-stock levels obtained from the FP model, y^*_1 and y^*_2. More specifically, each stage orders up to y^*_i or as close as possible to y^*_i subject to upstream stage's stock availability and the stage's cash availability. For the cash investment decision, stage 1 disposes excess cash to the investment account after the inventory payment or retrieves cash if it does not have sufficient cash to pay for the inventory order. Mathematically, the resulting heuristic solution (suppressing the time subscript) is

$$y^u_1 = x_1 \vee (y^*_1 \wedge x_2 \wedge w_1 + K), \qquad y^u_2 = x_2 \vee (y^*_2 \wedge w_2), \qquad r^u_1 = y^u_1.$$

The rationale implied by the SP heuristic is that we aim to set the inventory decision as a primary one and subordinate the investment decision to the inventory decision. In practice, such heuristic policy should be effective as the cash transfer cost rates β_o and β_i are often small. As we shall see later, it remains effective even when the transfer cost rates are reasonably large.

6 Numerical Study

We proceed to show the numerical results of both FP and SP models studied above. In §6.1 the heuristic performance is evaluated in a numerical study. The value of cash payment flexibility is discussed in §6.2. Other qualitative properties and managerial insights are presented in §6.3.

6.1 Effectiveness of the Heuristic

Let C_U be the cost of the heuristic which serves as an upper bound cost of the SP model. We compare C_U with a lower bound cost C_L, where

$$C_L = \max\{C_{CR}, C_{SA}\}.$$

Here, C_{CR} and C_{SA} represent the cost of the constraint relaxation bound and the sample path bound, respectively. Notice that the optimal cost of the FP model is also a lower bound. However, in our numerical study, the optimal FP cost is always smaller than either C_{CR} or C_{SA}. To evaluate the effectiveness of the heuristic, we define the percentage error as

$$\%\text{ error} = \frac{C_U - C_L}{C_L} \times 100\%.$$

We conduct a numerical study by starting with a basic case which has the time horizon of 10 periods, total simulation of 1000 scenarios (in computing the SA bound), and fixed parameters $\alpha = 0.95$, $c_2 = 1$, $c_1 = 0.25$, $\gamma' = 0.05$, $\eta_1' = 0.05$, and $h_1' = 1$. We vary the other parameters with each taking two values: $p_2 = (1.2, 2)$, $p_1 = (2.5, 4)$, $b = (5, 10)$, $\eta_2' = (0.05, 0.2)$, $h_2' = (0.25, 0.75)$, $\beta_o' = (0.05, 0.15)$, and $\beta_i' = (0.05, 0.2)$. In addition, two demand forms are considered. In the i.i.d. demand case, D_t is Poisson distributed with mean $\mu_t = 10$ for all t; in the increasing demand case, D_t is Poisson distributed with the first period mean $\mu_T = 10$ (we count time backward) and μ_t increasing at a rate of 1.2. In both demand cases, we fix the liquidity level $K_t' = \mu_t$ (non-stationary in the increasing demand cases). The combination of these parameters covers a wide range of different system characteristics. For example, when $(p_1, p_2) = (2.5, 2)$ ($(4, 1.2)$, respectively) the supplier markup ($= p_2/c_2 - 1$) is low (high, respectively). For each demand form, we generate 128 instances. The total number of instances in our test bed is 256. For all cases we assume the initial on-hand inventory and cash level $(x_{1,T}', x_{2,T}', w_{2,T}', w_{1,T}') = (16, 10, 10, 10)$, roughly equal to the steady-state inventory/cash level under i.i.d. demand with the aforementioned parameters.

The average (maximum, minimum) performance error in our test bed is 1.71% (4.00%, 0.31%) for the i.i.d. demand and 2.72% (6.81%, 0.15%) for the increasing demand case. The proposed heuristic performs well in general. Nevertheless, we suggest conditions under which our heuristic performs less effectively: (1) p_1 is significantly larger than p_2 and β_o' is large; (2) p_1 is close to p_2 and β_i' is large. In our heuristic policy, stage 1 disposes excess cash to the investment account after inventory payment or retrieves to the operating account just enough cash for the inventory payment. In other words, stage 1 does not hold any cash after payment. When p_1 is significantly larger than p_2, stage 1 tends to have excess cash to dispose. Thus, the heuristic performs less effectively when β_o' is high. This explains condition (1). Similarly, when p_1 is close to p_2, stage 1 is likely to have cash shortage, thus has to retrieve cash from the investment account. So a higher β_i' will make the heuristic less effective. This explains condition (2).

In practice, the 15%-20% transaction fees are very unlikely to happen. If we use a more practical transaction percentage $\beta_o' = \beta_i' = 0.05$, the average percentage error will reduce to 0.86% for the i.i.d. demand case and 1.81% for the increasing demand case.

6.2 Value of Flexible Payment

To assess the value of flexible payment, we compare the optimal cost of the FP model, C_{FP}, with the heuristic cost, C_U. Define the value of payment flexibility as

$$\% \text{ value} = \frac{C_U - C_{FP}}{C_U} \times 100\%.$$

We compute the percentage value for the same 256 cases in §6.1. Table 1 (left) summarizes the value of payment flexibility under i.i.d. demand (128 cases). The results are aggregated into 4 quadrants, each displaying the average value of 32 cases with the same p_2 and η_2' inputs. Since we fix $c_2 = 1$, p_2 represents the supplier markup. As shown in Table 1 (left), payment flexibility does not add much value if the supplier markup is low (e.g., $p_2/c_2 = 1.2$). This is because under the i.i.d. demand, the optimal m_t is likely to be $z_{1,t}$. When p_2 gets close to 1, the strict payment ($m_t = p_2 z_{1,t}$) policy can be near optimal. On the other hand, if the supplier markup is high (e.g., $p_2/c_2 = 2$), flexible payment will then play a significant role – it will be better off for the retailer to delay the payment so less cash will be accumulated at the supplier's. This value of flexible payment would be more significant when supplier cash holding cost η_2' is high.

Under increasing demand, the strict payment scheme will make the system perform poorly when the supplier markup is low. More specifically, as the retailer order size increases with the demand, ideally the supplier should in turn increase its inventory stocking to prepare for the future bigger orders. However, under the strict payment, the supplier might not have sufficient cash to do so due to its low markup ($p_2 = 1.2$). This vicious circle will make the supply chain very inefficient. Table 1 (right) demonstrates this inefficiency. As shown, when the supplier markup is low and demand is increasing, the value of flexible payment can be very significant. This value is even higher when backorder cost is larger: restricting upstream capacity is more costly when unsatisfied demand is penalized more.

	Unit price p_2	
Cash holding cost η_2'	1.2	2
0.05	2.18%	7.65%
0.2	5.84%	21.94%

	Unit price p_2	
Backorder cost b	1.2	2
5	63.61%	11.40%
10	78.23%	11.92%

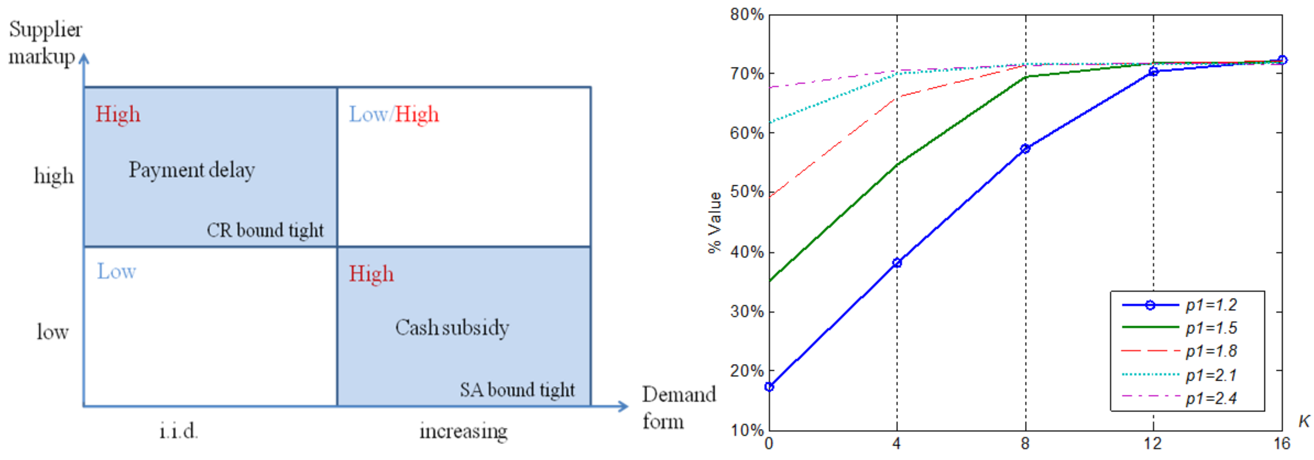
Table 1: Value of payment flexibility - i.i.d. demand (left) and increasing demand (right)

Figure 6(a) summarizes the conditions under which the flexible payment scheme has a significant value[1]. When demand is stationary and the supplier markup is high, payment flexibility adds substantial value by delaying the retailer's payment, making more cash disposed to the downstream investment account. On the other hand, when the demand is increasing and the supplier markup is low, the role of flexible payment is to allocate more cash to the supplier for stocking inventory. We notice that our conclusion is consistent with a practice of supply chain finance at Caterpillar: In 2009, the equipment maker took the unusual step of visiting with key suppliers to ensure they had the quick resources to boost output. According to Aeppel (2010), Caterpillar was in extreme cases helping suppliers get financing, just to restock dealer inventories and meet on going demand.

Figure 6(b) illustrates the impact of retailer's liquidity level on the value of flexible payment when demand is increasing. Here we fix $p_2 = 1.05$, $b = 5$, $\eta_2' = 0.2$, $h_2' = 0.25$, $\beta_o' = \beta_i' = 0.05$ and keep other parameter values the same as in §6.1. The liquidity of retailer's operating and investment account is represented by p_1 and K, respectively, with each taking 5 values. (We keep both p_1 and K stationary in each of the 25 cases.) As shown in the figure, the value of flexible payment is limited when both

[1]Figure 6 also shows that CR (SA) bound is tight in the upper left (lower right) quadrant

p_1 and K are small, i.e., when downstream liquidity is weak. In addition, p_1 and K complement each other's role as a liquidity source. For example, $(K, p_1) = (8, 1.2)$ and $(4, 1.5)$ yield a similar value of flexible payment. Notice that the *marginal* value of payment flexibility decreases in p_1 or K. For instance, when $p_1 = 1.2$, the % value increases from 56% to 68% when K increases from 8 to 12, and increases from 68% to 70% when K increases from 12 to 16. This observation suggests that the value of flexible payment diminishes faster than the reduction of the capital liquidity.



(a) Demand form, supplier markup, and FP value.

(b) Downstream liquidity and FP value.

Figure 6: Value of flexible payment.

6.3 Other Qualitative Insights

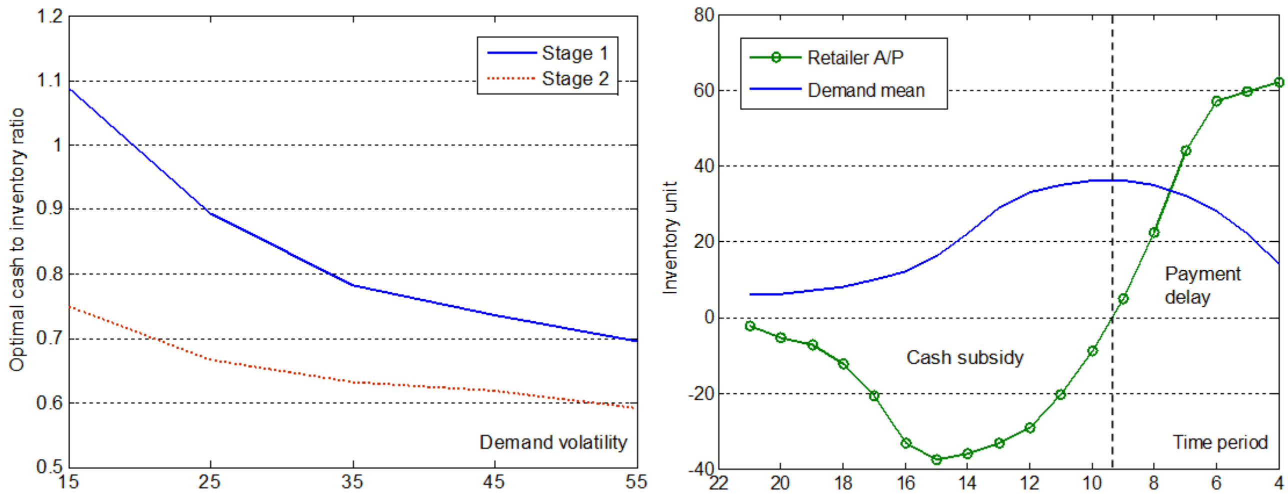
Optimal Cash to Inventory Ratio

The FP model allows us to study the impact of demand volatility on the optimal cash to inventory ratio at both stages. We compute the optimal control policy of the FP model with an i.i.d. negative binomial demand. Let $\overline{x}_i$ and $\overline{w}_i$ $(i = 1, 2)$ be the long-run optimal inventory and cash level at stage i, respectively. At steady state, we have

$$\overline{x}_1 = y_1^* - \mu, \;\; \overline{x}_2 = y_2^* - y_1^*,$$
$$\overline{w}_2 = \min\{r_2^*, u^*\} - y_2^*, \;\; \overline{w}_1 = \frac{p_1}{p_2}\mu,$$

where μ is the demand mean. The optimal cash to inventory ratio of stage i is defined as $\overline{w}_i/\overline{x}_i$. In conducting the comparative statics, we fix $\mu = 10$ and change the demand variation from 15 to 55. In addition, we fix $p_2 = 1.2$, $p_1 = 2.5$, $b = 55$, $\eta_2' = 0.15$, $h_2' = 0.2$, $\beta_o' = \beta_i' = 0.05$, and keep all the other parameters the same as in §6.1.

Figure 7(a) shows that the optimal cash to inventory ratio for both stages is decreasing with demand volatility. This is because when demand becomes more volatile, the stage 1 inventory $\overline{x}_1$ increases most, followed by $\overline{x}_2$, and by $\overline{w}_2$, whereas $\overline{w}_1$ remains fairly stable. Overall, the change of inventory level is significantly larger than that of cash level. The insight of the observation is that when the demand becomes more variable, the inventory stocking should increase, especially at the most downstream stage. However, the cash level can be maintained at a fairly stable level.



(a) Impact of demand volatility on optimal cash to inventory ratio. (b) Retailer A/P under product life cycle demand.

Figure 7: Qualitative properties of the FP model.

Product Life Cycle Demand

Now we focus on the dynamics of cash flows between stages when FP is in place and when demand follows a product life cycle form. More specifically, we consider a time horizon of 22 periods with independent Poisson demand in each period. The demand mean starts from 6, peaks at 36, drops to 14 and remains there for the last 5 periods. We compare the flexible payment m and the strict payment $p_2 z_1$ if SP were in place, and keep track of the retailer's accumulated account payable (A/P), i.e., $\sum_{k=t}^{T}(p_2 z_{1,k} - m_k)$ (account receivable if negative) in each period. The system parameters remain the same value as in computing the optimal cash to inventory ratio.

From Figure 7(b) we can see that the retailer's (accumulated) A/P becomes negative when demand increases, and becomes positive when demand decreases. This result implies that, when facing a product life cycle demand, the retailer should subsidize the supplier during the product introduction and growth stages to help the supplier maintain sufficient inventory supply. On the other hand, the retailer should delay payment during the mature and decline stages so cash could be better used via investment. This sheds light on large manufacturing firms and their smaller suppliers in the post-crisis time. When demand starts to pick up, the downstream firm should make financial subsidy, rather than push delayed payment contracts, such as requesting trade credits from its supplier.

Financial Bullwhip Effect

Bullwhip effect is a phenomenon that the order variability amplifies when moving along the supply chain from downstream toward upstream stages (Lee et al. 1997). It describes a phenomenon of order information distortion. In the empirical studies, bullwhip phenomenon is also observed in material shipments (e.g., Blanchard 1983, Cachon et al. 2007). Interestingly, in the financial literature, a similar phenomenon called "financial contagion" has also been observed in practice (e.g., Allen and Gale 2000). The financial contagion describes that the risk of financial payment defaults amplifies when moving toward upstream in a supply chain. One reason that causes the financial contagion is the material bullwhip effect: the financial payment amount is usually consistent with the order/shipment size. When order/shipment amplifies, an upstream stage requires more capital for its inventory payments. As stated, the upstream member in general has weaker cash liquidity, which results in a higher risk of financial defaults. Nevertheless, the financial contagion may be mitigated by a flexible payment policy. To see this, we conduct a simulation study. In the SP model, we find that the variability of payment indeed amplifies: the coefficient of variation (c.v.) of the payment from the retailer to the supplier is smaller than that of the payment from the supplier to the outside ample source. However, in the FP model, we observe the opposite – the c.v. of the payment from the retailer to the supplier is larger. Our study suggests that the financial bullwhip effect may not be consistent with the material bullwhip effect and the payment policy can be an effective tool to shift the financial risk between the supply chain partners.

7 Concluding Remarks

This paper integrates financial flows into a two-stage supply chain model, in which the downstream retailer has a higher cash liquidity in the sense that it can dispose cash to invest in the equity market or retrieve cash by selling equities. In each period, both the retailer and the supplier make inventory replenishment decisions. In addition, the retailer also makes cash payment and investment decisions. The objective is to minimize the total supply chain cost in a finite horizon. To assess the impact of payment flexibility, we compare two schemes. For the flexible payment scheme, the retailer can either delay payment to or subsidize the supplier; for the strict payment policy, the retailer has to pay what it orders in the same period. We prove the joint optimal inventory and cash policy for the flexible payment scheme and suggest a simple and effective heuristic for the strict payment scheme. We characterize the conditions under which the value of payment flexibility is most significant. We also discuss qualitative insights regarding the optimal inventory to cash ratio, the product life cycle demand, as well as the financial bullwhip effect.

For the flexible payment model, we can modify the terminating condition to incorporate a penalty cost charged on any positive debt (between the retailer and the supplier) at the end of the horizon. As shown in Appendix A, changing the terminating condition will not affect the structure of the optimal policy as well as the qualitative insights. Finally, the optimality results of the flexible payment scheme apply to the following generalizations: general number of stages, general lead times, and Markov modulated demand. We refer the reader to the authors for these extensions.

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Appendix A: Model Extension

We consider an extension where a penalty cost is charged on the end-of-horizon debt between the retailer and the supplier. This can be done by modifying the terminating condition in equation (11). First, let us assume at the end of horizon, i.e., $t = 0$, (1) necessary inventory replenishment has to be made to satisfy the backlogged demand; (2) the retailer can return the excess inventory to the supplier at the price p_2. Let R_t denote the retailer's account receivable (A/R) at time t (if $R_t < 0$, then $-R_t$ is the retailer's A/P). The following proposition derives the expected value of R_0 through flow conservation.

Proposition 5. *Under assumption (1) and (2), and given the initial states $x_{1,T}$ and $w_{2,T}$ at the beginning of the horizon, we have*

$$\mathop{\mathsf{E}}_{D_T,\dots,D_1} R_0 = \mathop{\mathsf{E}}_{D_T,\dots,D_1} c_2 w_{2,0} - C,$$

where $C = c_2 w_{2,T} - p_2 x_{1,T} + (p_2 - c_2)\sum_{t=1}^{T} \mu_t$ is a constant.

Denoting M as the unit penalty cost on the non-zero A/R or A/P at the retailer's, we change the terminating condition to the following:

$$V_0(\mathbf{x}, \mathbf{w}) = M \left| c_2 w_2 - C \right|,$$

which is a convex function of w_2. Therefore, all analytical results derived in §4.2 remain unaffected. This approach saves us from keeping track of another state variable R_t, but at the same time ignores the historical demand information captured in R_t. Since the terminating cost is non-linear in w_2, the resulting system forms an upper bound to the system where R_t is in place. Thus, the value of payment flexibility will be under-estimated when comparing with the SP model.

Appendix B: Proofs

Lemma 3 (Karush 1959) shows the additive separation of a function value.

Lemma 3. *If a function $f(y)$ is convex on $(-\infty, \infty)$ and attains its minimum at y^*, then*

$$\min_{a \le y \le b} f(y) = f_L(a) + f_U(b),$$

where $f_L(a) = \min_{a \le y} f(y)$ *is convex non-decreasing in* a*, and* $f_U(b) = f(b) - f(b \vee y^*)$ *is convex non-increasing in* b*.*

Proof of Proposition 1.

Proof. Prove by induction. The claim trivially holds for $t = 0$. Assume $V_{t-1}(\mathbf{x}, \mathbf{w}) = f_{1,t-1}(x_1) + f_{2,t-1}(x_2) + F_{t-1}(\mathbf{w})$, then

$$V_t(\mathbf{x}, \mathbf{w}) = \min_{\mathbf{y},\mathbf{r} \in S(\mathbf{x},\mathbf{w})} \left\{ \begin{array}{l} \sum_{i=1}^{2} [H_{i,t}(x_i) + c_i(y_i - x_i) + \alpha \mathsf{E}_{D_t} f_{i,t-1}(y_i - D_t)] \\ +H_{3,t}(w_2) + H_{4,t}(r_1) + \gamma(r_2 - w_2) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^- \\ +\alpha \mathsf{E}_{D_t} F_{t-1}(r_2 - D_t, r_1 + \theta D_t) \end{array} \right\}.$$

For $i = 1, 2$, let $g_{i,t}(y_i) = c_i y_i + \alpha \mathsf{E}_{D_t} f_{i,t-1}(y_i - D_t)$. Since $f_{i,t-1}(\cdot)$ is convex (from the induction assumption), by Lemma 3 we can decompose the cost functions of echelon 1 and 2 as follows:

$$\begin{aligned} \min_{x_1 \le y_1 \le x_2} g_{1,t}(y_1) &= \min_{x_1 \le y_1} \left\{ c_1 y_1 + \alpha \mathsf{E}_{D_t} f_{1,t-1}(y_1 - D_t) \right\} + \Gamma_{2,t}(x_2), \\ \min_{x_2 \le y_2 \le w_2} g_{2,t}(y_2) &= \min_{x_2 \le y_2} \left\{ c_2 y_2 + \alpha \mathsf{E}_{D_t} f_{2,t-1}(y_2 - D_t) \right\} + \Gamma_{3,t}(w_2), \end{aligned}$$

where the induced penalty functions $\Gamma_{2,t}(x_2)$ and $\Gamma_{3,t}(w_2)$ are expressed in (14) and (15), respectively. Now, define $f_{i,t}(x_i)$ and $F_t(\mathbf{w})$ as in (12) and (13). From Lemma 3 and Proposition B-4 in Heyman and Sobel (1984), it can be seen that $f_{i,t}(x_i)$ is convex and $F_t(\mathbf{w})$ is joint convex. In addition, $V_t(\mathbf{x}, \mathbf{w}) = f_{1,t}(x_1) + f_{2,t}(x_2) + F_t(\mathbf{w})$, completing the proof. □

Proof of Proposition 2.

Proof. Prove by induction. The claim trivially holds for $t = 0$. Assume $F_{t-1}(\mathbf{w}) = f_{3,t-1}(w_2) + f_{4,t-1}(w_1)$, then

$$F_t(\mathbf{w}) = \min_{w_2 \le r_2 \le r_1 \le w_1 + K} \left\{ \begin{array}{l} H_{3,t}(w_2) + \Gamma_{3,t}(w_2) + \gamma(r_2 - w_2) + \alpha \mathsf{E}_{D_t} f_{3,t-1}(r_2 - D_t) \\ +H_{4,t}(r_1) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^- + \alpha \mathsf{E}_{D_t} f_{4,t-1}(r_1 + \theta D_t) \end{array} \right\}.$$

Let $g_{3,t}(r_2) = \gamma r_2 + \alpha \mathsf{E}_{D_t} f_{3,t-1}(r_2 - D_t)$. Since $f_{3,t-1}(\cdot)$ is convex (from the induction assumption), by Lemma 3 we can decompose the cost function of echelon 3:

$$\min_{w_2 \le r_2 \le r_1} g_{3,t}(r_2) = \min_{w_2 \le r_2} \left\{ \gamma r_2 + \alpha \mathsf{E}_{D_t} f_{3,t-1}(r_2 - D_t) \right\} + \Gamma_{4,t}(r_1).$$

where the induced penalty function $\Gamma_{4,t}(r_1)$ is expressed in (18). Define $g_{4,t}(r_1) = H_{4,t}(r_1) + \Gamma_{4,t}(r_1) + \alpha \mathsf{E}_{D_t} f_{4,t-1}(r_1 + \theta D_t)$. Then

$$\begin{aligned} F_t(\mathbf{w}) = {} & H_{3,t}(w_2) + \Gamma_{3,t}(w_2) + \min_{w_2 \le r_2} \left\{ \gamma(r_2 - w_2) + \alpha \mathsf{E}_{D_t} f_{3,t-1}(r_2 - D_t) \right\} \\ & + \min_{w_2 \le r_1 \le w_1 + K} \left\{ g_{4,t}(r_1) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^- \right\}. \end{aligned} \tag{40}$$

Let $\tilde{r}_{1,t} = \arg\min_{r_1} \{g_{4,t}(r_1)\}$, and $r^*_{1,t} = \arg\min_{r_1} \{g_{4,t}(r_1) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^-\}$. The convexity of $g_{4,t}(r_1)$ implies the existence of the one-sided derivative $\partial g_{4,t}(r_1)/\partial r_1$. Define $l^*_t = \sup\{r_1 : \partial g_{4,t}(r_1)/\partial r_1 \leq -\beta_i\}$ and $u^*_t = \sup\{r_1 : \partial g_{4,t}(r_1)/\partial r_1 \leq \beta_o\}$. The monotonicity of $\partial g_{4,t}(r_1)/\partial r_1$ implies $l^*_t \leq \tilde{r}_{1,t} \leq u^*_t$. Using Proposition B-7 in Heyman and Sobel (1984), we have

$$r^*_{1,t} = \left\{ \begin{array}{ll} l^*_t, & \text{if } w_1 \leq l^*_t, \\ w_1, & \text{if } l^*_t < w_1 \leq u^*_t, \\ u^*_t, & \text{if } u^*_t < w_1. \end{array} \right\} \tag{41}$$

Define $L_t(w_1) = -\beta_i(w_1 - l^*_t) + g_{4,t}(l^*_t)$, $U_t(w_1) = \beta_o(w_1 - u^*_t) + g_{4,t}(u^*_t)$, and let

$$W_t(w_1) = \left\{ \begin{array}{ll} L_t(w_1), & \text{if } w_1 \leq l^*_t, \\ g_{4,t}(w_1), & \text{if } l^*_t < w_1 \leq u^*_t, \\ U_t(w_1), & \text{if } u_t < w_1. \end{array} \right\} \tag{42}$$

From (41) it can be easily shown that $W_t(w_1) = \min_{r_1}\{g_{4,t}(r_1) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^-\}$. Now, we impose the constraint $w_2 \leq r_1 \leq w_1 + K$. First, let

$$r^{**}_{1,t} = \arg \min_{w_2 \leq r_1 \leq w_1 + K} \left\{g_{4,t}(r_1) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^-\right\}, \tag{43}$$

and define the induced penalty functions $\Lambda_{3,t}(w_2)$ and $\Lambda_{4,t}(w_1)$ according to (22) and (23), respectively. Then, we define $f_{4,t}(w_1)$ as in (17). The convexity of $f_{4,t}(w_1)$ can be easily proved by showing that $\partial f_{4,t}(w_1)/\partial w_1$ is non-decreasing in w_1. Next, we prove the decomposition of echelon 4:

$$\min_{w_2 \leq r_1 \leq w_1 + K} \left\{g_{4,t}(r_1) + \beta_i(r_1 - w_1)^+ + \beta_o(r_1 - w_1)^-\right\} = f_{4,t}(w_1) + \Lambda_{3,t}(w_2). \tag{44}$$

The echelon system dynamics and constraint guarantee that $w_2 \leq w_1$ holds for all periods. To prove (44), we consider all possible relationships between w_2, w_1, l^*_t and u^*_t, as extensively described in the 4 cases below.

Case 1. When $w_1 \leq l^*_t - K$, we have $r^{**}_{1,t} = w_1 + K \leq l^*_t$, $f_{4,t}(w_1) = g_{4,t}(w_1 + K) + \beta_i K$ and $\Lambda_{3,t}(w_2) = 0$. Thus, $f_{4,t}(w_1) + \Lambda_{3,t}(w_2) = g_{4,t}(w_1 + K) + \beta_i K = g_{4,t}(r^{**}_{1,t}) + \beta_i(r^{**}_{1,t} - w_1)^+ + \beta_o(r^{**}_{1,t} - w_1)^-$, i.e., (44) holds.

Case 2. When $l^*_t - K < w_1 \leq l^*_t$, we have $r^{**}_{1,t} = r^*_{1,t} = l^*_t$, $f_{4,t}(w_1) = L_t(w_1)$ and $\Lambda_{3,t}(w_2) = 0$. Clearly (44) holds.

Case 3. When $l^*_t < w_1$, and $w_2 \leq u^*_t$, we have $r^{**}_{1,t} = r^*_{1,t} = \tilde{r}_{1,t}$, $f_{4,t}(w_1) = g_{4,t}(w_1)$, and $\Lambda_{3,t}(w_2) = 0$. Clearly, (44) holds.

Case 4. When $u^*_t < w_2 \leq w_1$, we have $r^{**}_{1,t} = w_2$, $f_{4,t}(w_1) = U_t(w_1)$ and $\Lambda_{3,t}(w_2) = g_{4,t}(w_2) - U_t(w_2)$. Thus, $f_{4,t}(w_1) + \Lambda_{3,t}(w_2) = g_{4,t}(w_2) + U_t(w_1) - U_t(w_2) = g_{4,t}(w_2) + \beta_i(w_1 - w_2) = g_{4,t}(r^{**}_{1,t}) + \beta_i(r^{**}_{1,t} - w_1)^+ + \beta_o(r^{**}_{1,t} - w_1)^-$, i.e., (44) holds.

Therefore, we verified that (44) holds in all cases. Substituting (44) into (40), and defining $f_{3,t}(w_2)$

as in (16), we complete the induction $F_t(\mathbf{w}) = f_{3,t}(w_2) + f_{4,t}(w_1)$. Using Lemma 3, all induced penalty functions are convex, thus, $f_{i,t}(\cdot)$ is convex $(i = 3, 4)$, accomplishing the proof. $\square$

Proof of Proposition 3.

Proof. Prove by induction. $V_0(\mathbf{x}, \mathbf{w}) = 0 = V_0^1(x_1, w_1) + V_0^2(x_1, x_2, w_2)$. Suppose $V_{t-1}(\mathbf{x}, \mathbf{w}) \geq V_{t-1}^1(x_1, w_1) + V_{t-1}^2(x_1, x_2, w_2)$ for all $(\mathbf{x}, \mathbf{w})$, then

$$\begin{aligned}
V_t(\mathbf{x}, \mathbf{w}) = & \min_{\mathbf{y}, r_1 \in S(\mathbf{x}, \mathbf{w})} \{G_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, r_1) && (45)\\
& + \alpha \mathsf{E}_{D_t} [V_{t-1}(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t, r_1 + \theta D_t)]\} \\
& \geq \min_{\mathbf{y}, r_1 \in S(\mathbf{x}, \mathbf{w})} \{G_t(\mathbf{x}, \mathbf{w}, \mathbf{y}, r_1) && (46)\\
& + \alpha \mathsf{E}_{D_t} \left[V_{t-1}^1(y_1 - D_t, r_1 + \theta D_t)\right] + \alpha \mathsf{E}_{D_t} \left[V_{t-1}^2(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t)\right]\} \\
& \geq \min_{\mathbf{y}, r_1 \in S(\mathbf{x}, \mathbf{w})} \left\{G_t^1(x_1, w_1, y_1, r_1) + \alpha \mathsf{E}_{D_t} \left[V_{t-1}^1(y_1 - D_t, r_1 + \theta D_t)\right]\right\} && (47)\\
& + \min_{\mathbf{y}, r_1 \in S(\mathbf{x}, \mathbf{w})} \left\{G_t^2(\mathbf{x}, w_2, \mathbf{y}) + \alpha \mathsf{E}_{D_t} \left[V_{t-1}^2(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t)\right]\right\} \\
& \geq \min_{x_1 \leq y_1 \leq r_1 \leq w_1 + K} \left\{G_t^1(x_1, w_1, y_1, r_1) + \alpha \mathsf{E}_{D_t} \left[V_{t-1}^1(y_1 - D_t, r_1 + \theta D_t)\right]\right\} && (48)\\
& + \min_{x_1 \leq y_1 \leq x_2 \leq y_2 \leq w_2} \left\{G_t^2(\mathbf{x}, w_2, \mathbf{y}) + \alpha \mathsf{E}_{D_t} \left[V_{t-1}^2(y_1 - D_t, y_2 - D_t, w_2 + \rho(y_1 - x_1) - D_t)\right]\right\} \\
& = V_t^1(x_1, w_1) + V_t^2(x_1, x_2, w_2). && (49)
\end{aligned}$$

The inequality in (46) and (48) are due to induction and constraint relaxation, respectively. The above relationship holds for all $(\mathbf{x}, \mathbf{w})$ in period t, completing the induction. $\square$

Proof of Lemma 1.

Proof. We write out the flow conservation of w_2 and x_1 from $s = T$ to $s = t + 1$.

$$\mathop{\mathsf{E}}_{D_T, \ldots, D_{t+1}} w_{2,t} - w_{2,T} = \sum_{s=t+1}^{T} \rho z_{1,s} - \sum_{s=t+1}^{T} \mu_s, \tag{50}$$

$$\mathop{\mathsf{E}}_{D_T, \ldots, D_{t+1}} x_{1,t} - x_{1,T} = \sum_{s=t+1}^{T} z_{1,s} - \sum_{s=t+1}^{T} \mu_s. \tag{51}$$

The result is shown by subtracting $\rho \times$(51) from (50). $\square$

Proof of Proposition 4.

Proof. We first specify the cost functions when demand is deterministic. Define

$$H_{1,t}^{2d}(x_1) = (h_1^2 + h_2 + \eta_2 + b_2)(d_t(\omega) - x_1)^+ + h_1^2(x_1 - d_t(\omega)),$$
$$H_{2,t}^d(x_2) = h_2(x_2 - d_t(\omega)), \quad H_{3,t}^d(a) = \eta_2(a - d_t(\omega)).$$

We then prove by induction. The claim trivially holds for $t=0$. Assume $V_{t-1}^d(x_1, x_2 \mid \omega) = v_{t-1}^1(x_1 \mid \omega) + v_{t-1}^2(x_2 \mid \omega)$, and let $g_{i,t}^d = c_i y_i + \alpha v_{t-1}^i(y_i - d(\omega) \mid \omega)$. From the convexity of $v_{t-1}^i(\cdot)$ and Lemma 3, we can decompose the cost functions of echelon 1 and 2 as follows:

$$\begin{aligned}
\min_{x_1 \le y_1 \le x_2} g_{1,t}^d(y_1) &= \min_{x_1 \le y_1} \left\{ c_1 y_1 + \alpha v_{t-1}^1(y_1 - d(\omega) \mid \omega) \right\} + \Gamma_{2,t}^d(x_2), \\
\min_{x_2 \le y_2 \le a} g_{2,t}^d(y_2) &= \min_{x_2 \le y_2} \left\{ c_2 y_2 + \alpha v_{t-1}^2(y_2 - d(\omega) \mid \omega) \right\} + \Gamma_{1,t}^d(a),
\end{aligned}$$

where $a = \rho x_1 + B_t(\omega)$. Let $y_{1,t}^*$ minimize $g_{i,t}^d(y_i)$, the induced penalty functions are

$$\begin{aligned}
\Gamma_{2,t}^d(x_2) &= \left\{ \begin{array}{ll} c_1(x_2 - y_{1,t}^*) + \alpha \left[v_{t-1}^1(x_2 - d_t(\omega) \mid \omega) - v_{t-1}^1(y_{1,t}^* - d_t(\omega) \mid \omega) \right], & x_2 \le y_{1,t}^*, \\ 0, & \text{otherwise}, \end{array} \right\} \\
\Gamma_{1,t}^d(x_1) &= \left\{ \begin{array}{ll} c_2(a - y_{2,t}^*) + \alpha \left[v_{t-1}^2(a - d_t(\omega) \mid \omega) - v_{t-1}^2(y_{2,t}^* - d_t(\omega) \mid \omega) \right], & a \le y_{2,t}^*, \\ 0, & \text{otherwise}. \end{array} \right\}
\end{aligned}$$

From Lemma 3, the functions above are convex. Therefore, the following functions are convex:

$$\begin{aligned}
v_t^1(x_1 \mid \omega) &= H_{1,t}^{2d}(x_1) + H_{3,t}^d(a) + \Gamma_{1,t}^d(a) + \min_{x_1 \le y_1} \left\{ c_1(y_1 - x_1) + \alpha v_{t-1}^1(y_1 - d_t(\omega) \mid \omega) \right\}, \\
v_t^2(x_2 \mid \omega) &= H_{2,t}^d(x_2) + \Gamma_{2,t}^d(x_2) + \min_{x_2 \le y_2} \left\{ c_2(y_2 - x_2) + \alpha v_{t-1}^2(y_2 - d_t(\omega) \mid \omega) \right\}.
\end{aligned}$$

Furthermore, $V_t^d(x_1, x_2 \mid \omega) = v_t^1(x_1 \mid \omega) + v_t^2(x_2 \mid \omega)$, completing the proof. □

Proof of Proposition 5.

Proof. We write out the flow conservation of w_2 and x_1, from $t = T$ to $t = 0$.

$$\sum_{t=1}^T m_t / c_2 - \sum_{t=1}^T \mu_t = \mathsf{E} w_{2,0} - w_{2,T}, \tag{52}$$

$$\sum_{t=1}^T z_{1,t} - \sum_{t=1}^T \mu_t = \mathsf{E} x_{1,0} - x_{1,T}, \tag{53}$$

where all expectations are taken over $D_T, ..., D_1$. Let (52)$\times c_2$ -(53)$\times p_2$, we have

$$\sum_{t=T}^1 m_t - \sum_{t=T}^1 p_2 z_{1,t} + p_2 \mathsf{E} x_{1,0} = c_2 \mathsf{E} w_{2,0} - C, \tag{54}$$

where $C = c_2 w_{2,T} - p_2 x_{1,T} + (p_2 - c_2) \sum_{t=1}^T \mu_t$. Note here that $\sum_{t=T}^1 m_t - \sum_{t=T}^1 p_2 z_{1,t}$ is the retailer's A/R at the beginning of time 0. Now, if $x_{1,0} > 0$, by assumption (2) the retailer can return this excess inventory, thus, $p_2 x_1$ is added to the retailer's A/R; if $x_{1,0} < 0$, by assumption (1) an additional replenishment will be made and $-p_2 x_1$ is subtracted from the retailer's A/R. Therefore, the left hand side of (54) equals to the retailer's expected A/R at the end of time 0. □