Optimal Dual Sourcing Strategy with Capacity Constraint and Fixed Bilateral Adjustment Costs

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Abstract

This paper studies a finite horizon, single product, periodic review inventory system with two supply nodes. Supply nodes are differentiated by their fixed and variable costs, delivery lead times, as well as constraints on the order sizes. The buyer faces interesting tradeoffs: Procuring from one of the supplier, referred to "onshore supplier", involves a high variable cost but a shorter delivery delay, while procuring from the other supplier, "offshore supplier", involves a low variable cost but a longer delivery delay. Each order from both suppliers incurs a fixed cost as well as a constraint on the maximum order size. Excess inventories could be salvaged at the end of each period with a fixed cost or be carried over to the next period. We show that the cost-to-go function satisfies a certain convexity, called (CK_1, K_2)-convexity introduced in this paper. We also show that the (CK_1, K_2)-convexity is preserved in certain types of optimizations and derive structural results for solutions of these optimizations involving (CK_1, K_2)-convex functions. Based on these structural results, we partially characterize the optimal sourcing and salvaging decisions.

Keywords: Dual sourcing, capacity constraints, convexity, fixed cost

1 Introduction

Offshore outsourcing/production has been increasingly popular in the past one and half decades. The benefits of offshore sourcing include lower costs, better availability of skilled people, and lower foreign corporate tax rate. For example, about two-thirds of Apple's \$97.6 billion cash pile is offshore in year 2011. However, the cost savings of the offshore reduces the inventory flexibility in a sense that the delivery takes a longer time and the procurement has capacity limits due to the financial budget or the regulation of trading in the foreign countries. In contrast, the onshore procurement in the local market is fast but incurring higher cost. How to trade-off between the onshore production and the offshore outsourcing motivates this paper.

This paper studies the inventory replenishment policy in a capacitated inventory system with two supply nodes, a slow one (offshore supplier) and a fast node (onshore supplier). An order placed from the fast supplier at the beginning of a period is delivered at the end of the period, whereas an order from the slow node is delivered at the end of the next period. The procurement from each node incurs both a variable cost per unit and a fixed cost per order. There is limits on the order size for each order from either slow node or fast node due to suppliers' production capacity or the budget constraint. Leftover inventory at the end of each period can either be carried over to the next period incurring inventory holding cost, or be salvaged immediately with a fixed cost.

We begin with the analysis for the corresponding no-capacity inventory system by introducing a concept called *strong* (K_1, K_2)*-convexity*. The strong (K_1, K_2)-convexity includes several commonly used convexity as special cases, such as, the convex function is strong $(0, 0)$-convex function, and the strong K-convex function is a strong $(K, 0)$-convex function. Besides, we show that the strong (K_1, K_2)-convexity is preserved under expectation and linear combination with non-negative weight. Using these two properties, we show that the cost-to-go functions for the multiple period inventory system without capacity limits are strong (K_1, K_2)-convex. We characterize the optimal sourcing strategy from each supplier and the salvaging decision based on properties of the strong (K_1, K_2)-convexity. In particular, the ordering strategy from the fast supplier at each period can be characterized by four parameters $(b, \bar{b}, \underline{s}, s)$, where $(b \leq \bar{b} \leq \underline{s} \leq s)$. The ordering decisions are partially determined by $(b, \bar{b})$, namely, no order when the inventory level is larger than $\bar{b}$, ordering up-to a constant level when it is below b, and either purchasing or staying put when its value is between $\bar{b}$ and b. The $(\underline{s}, s)$ characterize the threshold to salvage or not, i.e., always selling down-to a constant level when the starting inventory level is larger then s, staying put when it is below $\underline{s}$, and either selling or staying put when it is between s and $\underline{s}$. The overall inventory policy for fast node is a combination of the above two ordering and salvaging policy. As for the slow node, the ordering policy has a similar structure to the ordering policy for the fast note.

We then generalize the above convexity concept to the (CK_1, K_2)-convexity. We show that the (CK_1, K_2)-convexity is preserved under linear transformations and is also preserved under certain optimizations whereby the decision variable is constrained from above. We show that the cost-to-go function for a multi-period capacitated inventory system with two supplier nodes are (CK_1, K_2)-convex. (For conciseness and ease of presentation, the main body of the paper focuses on the case when each replenishment order from the fast supplier has a maximum size limit. The case when orders from both the fast supplier and the slow supplier have maximum size limit is included in Section 6.) Therefore, we could partially characterize the optimal sourcing strategy and the ordering quantities from each supply

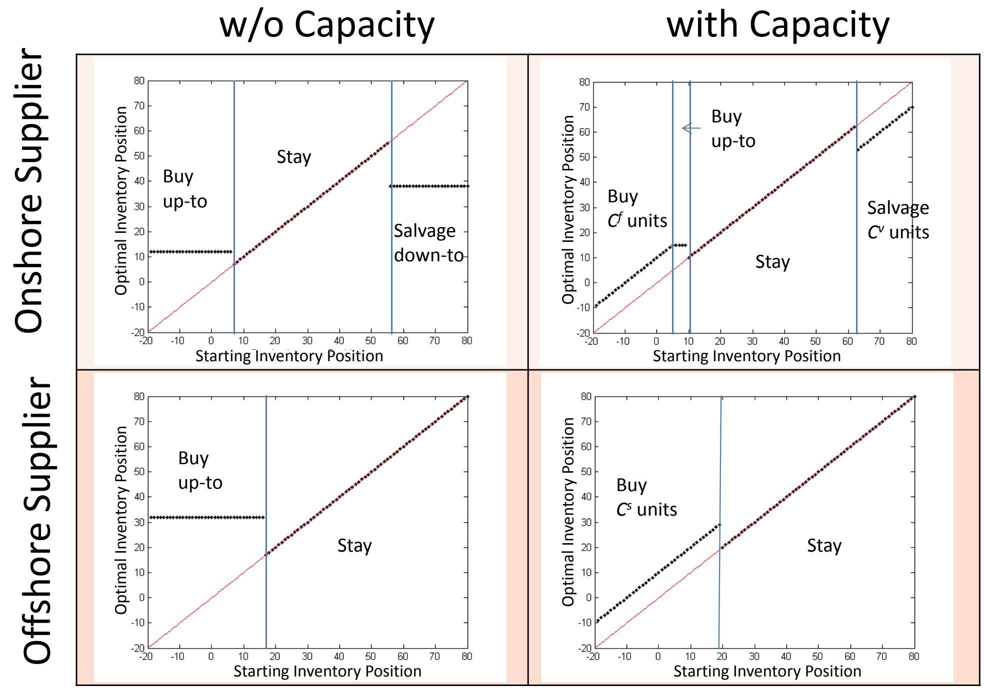


Figure 1: Structure of the Optimal Policies. (Fixed cost equals to 1$ per order and maximum size limit equals to 10 units per order.)

node on the basis of properties of the (CK_1, K_2)-convexity. The optimal policies are similar, but take a more complicated structure, to the above inventory system without a capacity constraint on the order size. For the fast node, the parameter space of the starting inventory level is divided into five regions, characterized by $(b, \bar{b}, \underline{s}, s)$. The optimal policy for the fast node is fully characterized in the three regions, ordering with the maximum size when the starting inventory level is below b, staying put when it is between $\bar{b}$ and $\underline{s}$, and salvaging down-to a constant level (or with the maximum quantity) when it is above s. In the other two regions, $(b, \bar{b})$ and $(\underline{s}, s)$, the optimal policy is either staying put or ordering and salvaging, respectively.

Finally, we show how our model could be generalized to other important scenarios, such as including the terminal value at the end of horizon as well as constraints on the maximum order size for both fast and slow nodes and a constraint on the size of salvaging quantity.

This paper makes the following contributions.

(i) We introduce a new convexity concept, i.e., strong (K_1, K_2)-convexity, and we show that the strong (K_1, K_2)-convexity is preserved under linear transformations, therefore, its properties are also preserved in many applications in the context of dynamic programming. We show that the cost-to-go function for a multiple-period inventory replenishment problem with two supplier nodes, different delivery lead time, fixed-cost associated with ordering or salvaging, is a strong (K_1, K_2)-convex function. We derive the properties of solutions to certain type of optimizations with a strong (K_1, K_2)-convex function. Applying these properties of strong (K_1, K_2)-convexity, we could therefore characterize the optimal inventory replenishment and inventory salvaging policy for such a system.

(ii) We generalize the above strong (K_1, K_2)-convexity to (CK_1, K_2)-convexity, which is also preserved under linear transformation. In addition, the (CK_1, K_2)-convexity is preserved under certain optimization when the decision variable is constrained from above. We characterize the structure of optimal solutions for these optimizations that involves a (CK_1, K_2)-convex function.

(iii) We show that the cost-to-go function for a capacitated, multi-period, dual sourcing inventory system satisfies the (CK_1, K_2)-convexity. Applying the properties of the (CK_1, K_2)-convexity, we could partially characterize the optimal sourcing and inventory salvaging strategy for this inventory system.

(iv) Finally, we generalize the above (CK_1, K_2)-convexity to the (C_1K_1, C_2K_2)-convexity. We show that the cost-to-go function for the above system with an additional constraint that each salvaging order has a maximum size limit is (C_1K_1, C_2K_2)-convex.

The remainder of this paper is organized as follows. Section 2 provides review on the related literatures. The model is introduced in Section 3. Section 4 develops properties of a strong (K_1, K_2)-convexity and applies it to study the inventory system without capacity limit, and Section 5 studies the capacitated inventory system using properties of (CK_1, K_2)-convexity. Section 6 concludes the paper with several discussions. All proofs except that of theorems are provided in the Appendix.

2 Literature Review

Our paper is closely related to the literature on the inventory management. In most inventory systems, finished products are outsourced from outside suppliers and sold to end customers through retailers. The problem of interest is how to replenish inventory in the logistic network, see Zipkin (2000) for a comprehensive review. Song and Zipkin (1993, 1996) have shown that the base stock policy is optimal when ordering cost is linear in order quantity, Scarf (1960), Veinott and Wagner (1965), and Sethi and Cheng (1997) show that the (s, S) policy is optimal when a fixed-ordering cost is incurred with each order, whereby, an order is placed to rise inventory level up-to S when the inventory level at the beginning is less than s and stay put otherwise. Chen and Lambrecht (1996), Gallego and Scheller-Wolf (2000), and Chen (2004) extend the above settings to incorporate the capacity constraints for each ordering quantity. However, these papers are confined themselves to unilateral inventory adjustment, namely, inventory could only be procured from suppliers and unused inventory is carried over to the next period. We consider bilateral inventory adjustments that the buyer could raise inventory from multiple suppliers and/or reduce inventory level by salvaging partial of inventory to the market.

Our paper is also closely related to the growing literature dealing with inventory control with multiple suppliers. Most of this literature considers inventory systems where all suppliers have negligible fixed costs but are differentiated instead in their variable costs and their delivery delay. The tradeoff firms face is whether to source from the slow but cheap supplier or from the fast but expensive supplier; see for example Moinzadeh and Schmidt (1991), Song and Zipkin (2009). Sethi et al. (2003) extends the above results to the scenarios when ordering from both fast supplier and slow one incurs a fixed cost. Zhang et al. (2012) study the dual sourcing with heterogeneous suppliers, whereas the buyer tradeoff between a supplier with higher variable cost against a supplier with a lower variable cost but with a fixed cost and capacity limit for each order. However, they assume that the suppliers' deliveries have identical and negligible delays. We consider an inventory system where the inventory could be replenished from fast and/or slow suppliers that each order incurs a fixed cost and has a maximum order limit. In additional, the excess inventory could be salvaged incurring a fixed cost per salvaging order or be carried over to the next period.

Table 1: Summary of Commonly Used Convexity

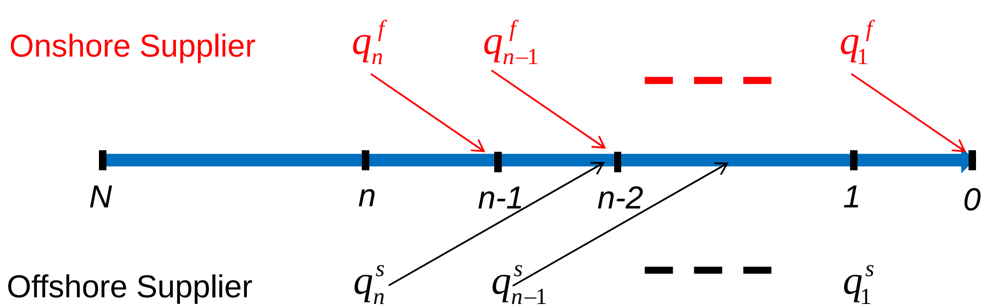
Concept	Definition	Related papers
convex	$f(x+a) \geq f(x) + \frac{a}{b}[f(y) - f(y-b)]$, for any $y \leq x,\ a \geq 0,\ b > 0$	Song and Zipkin (1993, 1996, 2009) Moinzadeh and Schmidt (1991)
K-convex	$f(x+a) + K \geq f(x) + \frac{a}{b}[f(x) - f(x-b)]$, for any $a \geq 0,\ b > 0$	Scarf (1960), Sethi and Cheng (1997) Sethi et al. (2003)
CK-convex	$f(x+a) + K \geq f(x) + \frac{a}{b}[f(x) - f(x-b)]$, for any $a \in [0, C],\ b > 0$	Gallego and Scheller-Wolf (2000)
strong K-convex	$f(x+a) + K \geq f(x) + \frac{a}{b}[f(y) - f(y-b)]$, for any $y \leq x,\ a \geq 0,\ b > 0$	Gallego and Scheller-Wolf (2000)
strong CK-convex	$f(x+a) + K \geq f(x) + \frac{a}{b}[f(y) - f(y-b)]$, for any $y \leq x,\ a \in [0, C],\ b > 0$	Gallego and Scheller-Wolf (2000) Chen and Lambrecht (1996), Chen (2004)
(K_1, K_2)-convex	$f(x+a) + K_1 \geq f(x) + \frac{a}{b}[f(x) - f(x-b) - K_2]$, for any $a \geq 0,\ b > 0$	Semple (2007) Ye and Duenyas (2007)
strong (K_1, K_2)-convex	$f(x+a) + K_1 \geq f(x) + \frac{a}{b}[f(y) - f(y-b) - K_2]$, for any $y \leq x,\ a \geq 0,\ b > 0$	This paper
strong (CK_1, K_2)-convex	$f(x+a) + K_1 \geq f(x) + \frac{a}{b}[f(y) - f(y-b) - K_2]$, for any $y \leq x,\ a \in [0, C],\ b > 0$	This paper
strong (C_1K_1, C_2K_2)-convex	$f(x+a) + K_1 \geq f(x) + \frac{a}{b}[f(y) - f(y-b) - K_2]$, for any $y \leq x,\ a \in [0, C_1],\ b \in (0, C_2]$	This paper

Another field that is also closely related to our paper is the capacity expansion. Most papers in the literature of capacity expansion assume that capacity is durable, see Van Mieghem (2003, 2007). Ye and Duenyas (2007) consider a finite-horizon period-review, single-product model with two-sided fixed-capacity adjustment cost. They introduce the (K_1, K_2)-concavity to show the optimal strategy for capacity expansion. In particular, the capacity strategy at each period can be characterized in a similar way to the ordering policy for the fast node in a inventory system without capacity limits. A same result is obtained by Semple (2007) with the concept of the weak (K_1, K_2)-concavity. However, these papers focus exclusively on single supplier/production which does not have the capacity limit for quantity purchased or salvaged in each order. Table 1 summarizes concepts of various convexities used in existing literatures and the relationship between our paper with the above literatures.

3 Model Formulation

This paper studies a capacitated inventory system with two supply nodes, an offshore supplier and an onshore supplier. The procurement from both suppliers incurs both a variable cost per unit and a fixed cost per order. Order from the fast supplier has a capacity limit due to suppliers' production capacity or the budget constraints. Leftover inventory at the end of each period can either be carried over to the next period incurring inventory holding cost or be salvaged immediately with a fixed cost. An order placed at the fast supplier at the beginning of a period is delivered at the end of the period, whereas an order from the slow supplier is delivered at the end of the next period. The framework of our model and the flow of order delivery are illustrated in Figure 2. For the convenience of analysis, the periods are numbered in a backward way.

Figure 2: Model



For each period $n = 1, 2, \ldots, N$, let

K_n^s = a fixed cost associates with each order from the slow supplier, with a unit cost c_n^s

K_n^f = a fixed cost with each order from the fast supplier, with a unit cost c_n^f

K_n^v = a fixed cost with each order salvaged, with a unit revenue c_n^v

q_n^s = order quantity that is ordered from the slow supplier

q_n^f = order quantity that is ordered from the fast supplier (if it is positive), or is salvaged in the local market (if it is negative)

α = the discount rate, $\alpha \in [0, 1]$

$L_n(y)$ = expected one-period cost with y units of on-hand inventory at the beginning

$f_n(x)$ = the cost-to-go function for a n period problem with an initial on-hand inventory x

There is capacity limits on each orders from fast supplier, namely, an expediting quantity from the fast supplier can not exceed its capacity C, i.e, $q_n^f \leq C$. Without loss of generality, we assume

$$c_n^f \geq c_n^v \quad \text{and} \quad c_n^s \geq \alpha c_{n-1}^v, \quad \text{for any } n.$$

The first one is intuitive and it merely represents that the unit procurement cost is larger than the unit salvage value. The second one requires that the unit purchasing cost should be no less than the unit salvage value discounted one period later. These two conditions exclude the arbitrage opportunity, namely, a negative infinite cost or positive infinity profit can be achieved by purchasing infinite orders from the slow supplier when $c_n^s < \alpha c_{n-1}^v$ and salvaging all these amounts at the end of next period. In our analysis, we assume that the one-period inventory cost $L_n(\cdot)$ is a convex function, for e.g., most inventory literatures assume that the inventory cost includes the inventory holding cost and the backlog cost as $L_n(y) = h_n \mathbb{E}[y - D_n]^+ + p_n \mathbb{E}[y - D_n]^-$. As a start point, we assume that $f_0(x) = 0$, and we will relax this assumption later to include salvaging value or the penalty cost at the end of horizon when its inventory is positive or negative, respectively.

Using the approach of the standard dynamic programming, one can easily show that the cost-to-go functions satisfy the following dynamic equations: Given the inventory position (on-hand inventory + pipeline inventory from the slow node ordered one period earlier − backlog) at the beginning of period n is x, the cost-to-go functions satisfy

$$f_n^b(x) = \min_{q_n^s \geq 0,\ q_n^f \in [0, C]} \{K_n^s \delta(q_n^s) + c_n^s q_n^s + K_n^f \delta(q_n^f) + c_n^f q_n^f + L_n(x + q_n^f) + \alpha \mathsf{E} f_{n-1}(x + q_n^f + q_n^s - D_n)\}, \quad (1)$$

$$f_n^v(x) = \min_{q_n^s \geq 0,\ q_n^f \leq 0} \{K_n^s \delta(q_n^s) + c_n^s q_n^s + K_n^v \delta(-q_n^f) + c_n^v q_n^f + L_n(x + q_n^f) + \alpha \mathsf{E} f_{n-1}(x + q_n^f + q_n^s - D_n)\}, \quad (2)$$

and

$$f_n(x) = \min\{f_n^b(x), f_n^v(x)\}. \tag{3}$$

If we change the decision variables by $y = x + q_n^f$ and $z = x + q_n^f + q_n^s$, we get an equivalent formulation as follows:

$$\begin{aligned}
f_n^b(x) &= \min_{z\geq y,\ y\in[x,x+C]}\{K_n^s\delta(z-y) + c_n^s(z-y) + K_n^f\delta(y-x) + c_n^f(y-x) + L_n(y) + \alpha\mathsf{E}f_{n-1}(z-D_n)\} \\
&= \min_{y\in[x,x+C]}\{K_n^f\delta(y-x) + c_n^f(y-x) + g_n(y)\}, && (4)\\
f_n^v(x) &= \min_{z\geq y,\ y\leq x}\{K_n^s\delta(z-y) + c_n^s(z-y) + K_n^v\delta(x-y) + c_n^v(y-x) + L_n(y) + \alpha\mathsf{E}f_{n-1}(z-D_n)\} \\
&= \min_{y\leq x}\{K_n^v\delta(x-y) + c_n^v(y-x) + g_n(y)\}, && (5)\\
f_n(x) &= \min\{f_n^b(x), f_n^v(x)\}, && (6)
\end{aligned}$$

where

$$g_n(y) = L_n(y) + f_n^s(y) \quad \text{and} \quad f_n^s(y) = \min_{z\geq y}\{K_n^s\delta(z-y) + c_n^s(z-y) + \alpha\mathsf{E}f_{n-1}(z-D_n)\}. \tag{7}$$

In the remaining sections of our paper, we will study the properties of the cost-to-go functions given by Equations (4)-(7) and after that, we characterize the inventory policy for such systems. To that end, we first study the system without capacity limit associated with each order and then we characterize the optimal policies for systems with capacity limit associated with each order from the fast node. In a later section, Section 6, we show how our analysis could be extended to cases when orders from the slow node has a maximum order size and when each salvage also has a maximum size.

4 Systems Without Capacity Limits

This section investigates the case when there is no capacity limit for orders from the fast supplier. Alternatively, one can think of the case when $C = \infty$. Before we give the analysis for optimal inventory management policy, we first introduce the concept of a strong (K_1, K_2)-convex function.

Definition 1 *A real value function $f(x)$ is called strong (K_1, K_2)-convex, if it satisfies*

$$f(x+a) + K_1 \geq f(x) + \frac{a}{b}[f(y) - f(y-b) - K_2], \quad \textit{for any } y \leq x,\ a \geq 0,\ b > 0. \tag{8}$$

It is easy to verify that the strong (K_1, K_2)-convexity include the convexity and the strong K-convexity as special cases with $K_1 = K_2 = 0$ and $K_2 = 0$, respectively, see Table 1. An equivalent definition for the strong (K_1, K_2)-convexity is

$$f(x+a) + K_1 \geq f(x) + \frac{a}{b}\sup_{y\leq x}[f(y) - f(y-b) - K_2], \quad \text{for any } a \geq 0,\ b > 0.$$

To characterize the optimal inventory policy, we first study the properties of a strong (K_1, K_2)-convex function. These properties will then be used to show that cost-to-go functions in the dynamic programming are strong (K_1, K_2) convex. Using the properties of strong (K_1, K_2)-convexity, we are then able to characterize the optimal inventory policy for the un-capacitated inventory systems. The following definition from Porteus (1971) is used later to characterize the properties of cost-to-go functions.

Definition 2 *A function $F(x)$ is non-K-decreasing if $F(x) \le F(y) + K$ for all $x \le y$. A function $F(x)$ is non-K-increasing if $F(x) + K \ge F(y)$ for all $x \le y$.*

4.1 Properties of (K_1, K_2)-Convexity

In this section, we will study the following optimization problem

$$\begin{cases} f_1(x) = \min_{y \ge x}\{K_1\delta(y-x) + c(y-x) + g(y)\}, \\ f_2(x) = \min_{y \le x}\{K_2\delta(x-y) + v(y-x) + g(y)\}, \\ f(x) = \min\{f_1(x), f_2(x)\}. \end{cases} \tag{9}$$

In particular, we will study the properties of the above value function and characterize the solution to the above minimization problem when g is a strong (K_1, K_2)-convex function. The properties of strong (K_1, K_2)-convexity are summarized in the following lemma.

Lemma 1 *(a) If $f(x)$ is strong (K_1, K_2)-convex, then $f(x-a)$ is also strong (K_1, K_2)-convex for any a. Moreover, for any random variable $\widetilde{X}$, $\mathsf{E}f(x-\widetilde{X})$ is also strong (K_1, K_2)-convex provided $\mathsf{E}f(x-\widetilde{X}) < \infty$.*

(b) If $f(x)$ is strong (K_1, K_2)-convex, then $f(x)$ is strong (K_1', K_2')-convex for any $K_1' \ge K_1, K_2' \ge K_2$.

(c) If $f(x)$ and $g(x)$ are strong (K_1, K_2)-convex, strong (G_1, G_2)-convex, respectively, then for any $\alpha, \beta \ge 0$, $\alpha f(x) + \beta g(x)$ is strong $(\alpha K_1 + \beta G_1, \alpha K_2 + \beta G_2)$-convex.

The beauty of normal convexity or K-convexity is its preservation of convexity or K-convexity under dynamic minimization. In the following proposition, we show that strong (K_1, K_2)-convexity is also preserved under minimization.

Proposition 1 *If a function $g(x)$ is strong (K_1, K_2)-convex, then*

$$\begin{aligned} f_1(x) &= \min_{y \ge x}\{K_1\delta(y-x) + g(y)\}, \\ f_2(x) &= \min_{y \le x}\{K_2\delta(x-y) + g(y)\}, \\ f(x) &= \min\{f_1(x), f_2(x)\}, \end{aligned}$$

are also strong (K_1, K_2)-convex.

Furthermore, we can get a stronger result that shows the preservation of optimization of (K_1, K_2)-convexity with linear operations. Namely,

Corollary 1 *If a function $g(x)$ is strong (K_1, K_2)-convex, then*

$$f_1(x) = \min_{y \geq x}\{K_1\delta(y-x) + c(y-x) + g(y)\},$$
$$f_2(x) = \min_{y \leq x}\{K_2\delta(x-y) + v(y-x) + g(y)\},$$

are also strong (K_1, K_2)-convex, for any c, v.

Different to Proposition 1, the function $f(x) = \min\{f_1(x), f_2(x)\}$ may not be a strong (K_1, K_2)-convex function depending on the value of c and v. In the problem of our interest, $c \geq v$ is satisfied, and we can show $f(x) = \min\{f_1(x), f_2(x)\}$ is indeed a strong (K_1, K_2)-convex function. The proof needs some properties of the solutions to the above two minimization problems in Corollary 1, so it is presented later in Proposition 3. To solve the above minimization problems, we first show that the value functions satisfy a certain monotonicity in the following lemma.

Lemma 2 *$f_1(x) + cx$ is non-K_1-decreasing, $f_2(x) + vx$ is non-K_2-increasing.*

Now, we are ready to characterize the solutions to the minimization $f(x) = \min\{f_1(x), f_2(x)\}$ where f_1, f_2 are given in Corollary 1.

Lemma 3 *If $c \geq v$, then all buy regions are to the left of all sell region. Saying it in other words,* (I) *If it is optimal to sell at x, then it is never optimal to purchase at y for any $y \geq x$;* (II) *If it is optimal to purchase at x, then it is never optimal to sell at y for any $y \leq x$;*

Let

$$B = \sup\left\{\operatorname{argmin}_y\{cy + g(y)\}\right\}, \quad b = \inf\{x: \ \widetilde{A}_1(x) \geq 0\}, \quad \bar{b} = \sup\{x: \ \widetilde{A}_1(x) < 0\}, \tag{10}$$
$$S = \inf\left\{\operatorname{argmin}_y\{vy + g(y)\}\right\}, \quad s = \sup\{x: \ \widetilde{A}_2(x) \geq 0\}, \quad \underline{s} = \inf\{x: \ \widetilde{A}_2(x) < 0\}, \tag{11}$$

where

$$\widetilde{g}_1(x) = K_1 + \inf_{y \geq x}\{c(y-x) + g(y)\}, \qquad \widetilde{A}_1(x) = \widetilde{g}_1(x) - g(x),$$
$$\widetilde{g}_2(x) = K_2 + \inf_{y \leq x}\{v(y-x) + g(y)\}, \qquad \widetilde{A}_2(x) = \widetilde{g}_2(x) - g(x),$$

Obviously, by the definitions, we have

$$f_1(x) = \min\{g(x), \ \widetilde{g}_1(x)\}, \quad \widetilde{A}_1(x) < 0 \ \text{ for any } x < b, \quad \widetilde{A}_1(x) \geq 0 \ \text{ for any } x > \bar{b},$$
$$f_2(x) = \min\{g(x), \ \widetilde{g}_2(x)\}, \quad \widetilde{A}_2(x) < 0 \ \text{ for any } x > s, \quad \widetilde{A}_2(x) \geq 0 \ \text{ for any } x < \underline{s}.$$

The following lemma characterizes the relationship between these points, which are critical to characterize the structure of optimal policy.

Lemma 4 *The critical points have the following orders*

(i) $b \le \bar{b} \le \underline{s} \le s$.

(ii) $b \le B \le S \le s$.

(iii) *If* $K_1 \ge K_2$, $\bar{b} \le B$; *otherwise if* $K_1 \le K_2$, $\underline{s} \ge S$.

Now, we are ready to characterize the solution to (9) and its value function in the following proposition.

Proposition 2 *The structure of the optimal solution to (9) and its corresponding optimal value function has the following structure*

(i) *if* $K_1 \ge K_2$

$$y^*(x) = \begin{cases} B, & x < b \\ \{B, x\}, & x \in [b, \bar{b}) \\ x, & x \in [\bar{b}, \underline{s}) \\ \{S(x), x\}, & x \in [\underline{s}, s) \\ S, & x \ge s \end{cases}, \quad f(x) = \begin{cases} K_1 + g(B) + c(B-x), & x < b \\ \min\{K_1 + g(B) + c(B-x),\ g(x)\}, & x \in [b,\ \bar{b}) \\ g(x), & x \in [\bar{b},\ \underline{s}) \\ \min\{\widetilde{g}_2(x),\ g(x)\}, & x \in [\underline{s},\ s) \\ K_2 + g(S) + v(S-x), & x \ge s \end{cases} \quad (12)$$

(ii) *if* $K_2 \ge K_1$

$$y^*(x) = \begin{cases} B, & x < b \\ \{B(x), x\}, & x \in [b, \bar{b}) \\ x, & x \in [\bar{b}, \underline{s}) \\ \{S, x\}, & x \in [\underline{s}, s) \\ S, & x \ge s \end{cases}, \quad f(x) = \begin{cases} K_1 + g(B) + c(B-x), & x < b \\ \min\{\widetilde{g}_1(x),\ g(x)\}, & x \in [b,\ \bar{b}) \\ g(x), & x \in [\bar{b},\ \underline{s}) \\ \min\{K_2 + g(S) + v(S-x),\ g(x)\}, & x \in [\underline{s},\ s) \\ K_2 + g(S) + v(S-x), & x \ge s \end{cases} \quad (13)$$

where $B(x) = \operatorname{argmin}_{y \ge x}\{g(y) + cy\}$ *and* $S(x) = \operatorname{argmin}_{y \le x}\{g(y) + vy\}$.

The next proposition shows that the function $f(x)$ also preserves the (K_1, K_2)-convexity.

Proposition 3 (Preservation) *If* $g(x)$ *is strong* (K_1, K_2)*-convex,* $f(x)$ *is also strong* (K_1, K_2)*-convex.*

We have thus shown that the strong (K_1, K_2)-convexity is indeed preserved in a minimization that has a form of (9). With the help of these results, we are now ready to present the main results for the un-capacitated inventory systems.

4.2 Optimal Policies

In this section, we employ properties of the strong (K_1, K_2)-convexity obtained in the previous section to study inventory systems that do not have capacity constraints on each order. Regarding to the fixed cost, we assume

$$\text{(A1)} \qquad \alpha K^f_{n-1} \le K^s_n \le K^f_n \quad \text{and} \quad K^v_n \ge \alpha K^v_{n-1}.$$

The former condition says that the fixed cost associated with the slow order is smaller than the fixed cost with the fast order but is larger than the one–period discounted fixed cost with the fast order. Similarly, the second condition means that the fixed cost associated with the salvaging is larger than the one–period discounted fixed salvaging cost. These assumptions guarantee the strong (K_1, K_2)-convexity of cost-to-go functions are preserved in the dynamic programming, and they are not uncommon in the literature, for e.g., Sethi et al. (2003).

Theorem 1 *Assume that* (A1) *holds.* $f_n(x)$ *is a strong* (K^f_n, K^v_n)*-convex function for any* $n \ge 0$.

Proof. Obviously, the result holds for $n = 0$. Next, we establish the proof by induction. Assuming that $f_{n-1}(x)$ is (K^f_{n-1}, K^v_{n-1}) convex, we next prove that $f_n(x)$ is (K^f_n, K^v_n) convex. By Lemma 1, we know that $\alpha \mathsf{E} f_{n-1}(y - D)$ is a $(\alpha K^f_{n-1}, \alpha K^v_{n-1})$-convex function. Since $\alpha K^f_{n-1} \le K^s_n$, according to Lemma 1, $\alpha \mathsf{E} f_{n-1}(y - D)$ is also a $(K^s_n, \alpha K^v_{n-1})$ convex function. Thus, by Corollary 1,

$$f^s_n(y) = \max_{z \ge y} \{K^s_n \delta(z - y) + c^s_n(z - y) + \alpha \mathsf{E} f_{n-1}(z - D_n)\},$$

is $(K^s_n, \alpha K^v_{n-1})$ convex. Since $L_n(y)$ is convex function, $g_n(y) = L_n(y) + f^s_n(y)$ is also $(K^s_n, \alpha K^v_{n-1})$ convex by Lemma 1. By Lemma 1, $g_n(y)$ is also (K^f_n, K^v_n) convex since $\alpha K^v_{n-1} \le K^v_n$ and $K^s_n \le K^f_n$. Therefore, by Corollary 1 and Proposition 3, $f^b_n(x), f^v_n(x), f_n(x)$ given by equations (4), (5), (6), respectively, are also (K^f_n, K^v_n) convex. □

The optimal inventory replenishment policy could be determined in a similar way to Proposition 2. In particular, for $n = 1, 2, \ldots, N$, let

$$B^s_n = \sup \left\{ \operatorname{argmin}_z \{c^s_n z + \alpha \mathsf{E} f_{n-1}(z - D_n)\} \right\}, \quad b^s_n = \inf\{x: \ \widetilde{A}^s_n(x) \ge 0\}, \quad \bar{b}^s_n = \sup\{x: \ \widetilde{A}^s_n(x) < 0\}, \tag{14}$$

$$B^f_n = \sup \left\{ \operatorname{argmin}_y \{c^f_n y + g_n(y)\} \right\}, \quad b^f_n = \inf\{x: \ \widetilde{A}^f_n(x) \ge 0\}, \quad \bar{b}^f_n = \sup\{x: \ \widetilde{A}^f_n(x) < 0\}, \tag{15}$$

$$S^v_n = \inf \left\{ \operatorname{argmin}_y \{c^v_n y + g_n(y)\} \right\}, \quad s^v_n = \sup\{x: \ \widetilde{A}^v_n(x) \ge 0\}, \quad \underline{s}^v_n = \inf\{x: \ \widetilde{A}^v_n(x) < 0\}. \tag{16}$$

Here $g_n(y)$ is given by (7) and

$$\widetilde{g}_n^s(y) = K_n^s + \inf_{z\geq y}\{c_n^s(z-y) + \alpha \mathsf{E} f_{n-1}(z-D_n)\}, \qquad \widetilde{A}_n^s(y) = \widetilde{g}_n^s(y) - \alpha \mathsf{E} f_{n-1}(y-D_n),$$
$$\widetilde{g}_n^f(x) = K_n^f + \inf_{y\geq x}\{c_n^f(y-x) + g_n(y)\}, \qquad \widetilde{A}_n^f(x) = \widetilde{g}_n^f(x) - g_n(x),$$
$$\widetilde{g}_n^v(x) = K_n^v + \inf_{y\leq x}\{c_n^v(y-x) + g_n(y)\}, \qquad \widetilde{A}_n^v(x) = \widetilde{g}_n^v(x) - g_n(x),$$

Obviously, by above definitions, one has

$$f_n^s(y) = \min\{\alpha \mathsf{E} f_{n-1}(y-D),\ \widetilde{g}_n^s(y)\}, \quad \widetilde{A}_n^s(y) < 0, \forall y < b_n^s, \quad \widetilde{A}_n^s(y) \geq 0, \forall y > \bar{b}_n^s,$$
$$f_n^b(x) = \min\{g_n(x),\ \widetilde{g}_n^f(x)\}, \quad \widetilde{A}_n^f(x) < 0, \forall x < b_n^f, \quad \widetilde{A}_n^f(x) \geq 0, \forall x > \bar{b}_n^f,$$
$$f_n^v(x) = \min\{g_n(x),\ \bar{g}_{n2}(x)\}, \quad \bar{A}_{n2}(x) < 0, \forall x > s_n^v, \quad \bar{A}_{n2}(x) \geq 0, \forall x < \underline{s}_n^v.$$

Similar to Lemma 4, it can be proved that: For any $n = 1, 2, \ldots, N$, we have (i) $b_n^f \leq \bar{b}_n^f \leq \underline{s}_n^v \leq s_n^v$ and $b_n^s \leq \min\{\bar{b}_n^s, B_n^s\}$; (ii) $b_n^f \leq B_n^f \leq S_n^v \leq s_n^v$; (iii) $\bar{b}_n^f \leq B_n^f$ if $K_n^f \geq \alpha K_{n-1}^v$, $\underline{s}_n^v \geq S_n^v$ if $K_n^f \geq \alpha K_{n-1}^v$, and $\bar{b}_n^s \leq B_n^s$ if $K_n^s \geq \alpha K_{n-1}^v$.

Theorem 2 *The structure of the optimal inventory replenishment/salvaging policy has the following structure: For any $n = 1, 2, \ldots, N$,*

1. (The Onshore Supplier and Salvaging Policy) *Given the starting inventory level x, the optimal ordering decision with the fast node and the optimal salvaging decision are characterized as follows:*

   (i) *if $K_n^f \geq K_n^v$, the optimal policy is determined by*

$$y_n^*(x) = \begin{cases} B_n^f, & x < b_n^f \\ \{B_n^f, x\}, & x \in [b_n^f, \bar{b}_n^f) \\ x, & x \in [\bar{b}_n^f, \underline{s}_n^v) \\ \{S_n^v(x), x\}, & x \in [\underline{s}_n^v, s_n^v) \\ S_n^v, & x \geq s_n^v \end{cases}, \tag{17}$$

   (ii) *if $K_n^v \geq K_n^f$, the optimal policy could be characterized by*

$$y_n^*(x) = \begin{cases} B_n^f, & x < b_n^f \\ \{B_n^f(x), x\}, & x \in [b_n^f, \bar{b}_n^f) \\ x, & x \in [\bar{b}_n^f, \underline{s}_n^v) \\ \{S_n^v, x\}, & x \in [\underline{s}_n^v, s_n^v) \\ S_n^v, & x \geq s_n^v \end{cases}, \tag{18}$$

2. (The Offshore Supplier) *Given the inventory level after ordering is placed from fast node or after salvaging, the optimal ordering decision with the slow node is as follows:*

$$z_n^*(y) = \begin{cases} B_n^s, & y < b_n^s \\ \{B_n^s, y\}, & y \in [b_n^s, \bar{b}_n^s) \\ y, & y \geq \bar{b}_n^s \end{cases} \quad \text{if } K_n^s \geq \alpha K_{n-1}^v, \tag{19}$$

$$\text{and} \qquad z_n^*(y) = \begin{cases} B_n^s, & y < b_n^s \\ \{B_n^s(y), y\}, & y \in [b_n^s, \bar{b}_n^s) \\ y, & y \geq \bar{b}_n^s \end{cases} \quad \text{if } K_n^s < \alpha K_{n-1}^v.$$

where $B_n^s(y) = \operatorname{argmin}_{z \geq y}\{c_n^s(z-y) + \alpha \mathsf{E} f_{n-1}(z - D_n)\}$, $B_n^f(x) = \operatorname{argmin}_{y \geq x}\{c_n^f y + g_n(y)\}$ *and* $S_n^v(x) = \operatorname{argmin}_{y \leq x}\{c_n^v y + g_n(y)\}$.

The optimal policy with the slow node has a similar structure with the well-known (s, S) policy. An order is placed to raise the inventory up to a constant level B_n^s when the starting inventory is below level b_n^s, and no order is placed when its starting inventory is above a certain level $\bar{b}_n^s$. While when the starting inventory is in the middle between b_n^s and $\bar{b}_n^s$, the optimal ordering policy is only partially characterized, namely, either staying put or ordering with quantity equals to an local (or global) point. The ordering policy for the fast node and the optimal salvaging policy could be characterized similarly by parameters $(b_n^f, \bar{b}_n^f, B_n^f)$ and $(\underline{s}_n^v, s_n^v, S_n^v)$, respectively. With the optimal policies characterized in the above proposition, the cost-to-go functions could be determined in a similar way to Proposition 2.

5 Systems with Capacity Limits

To analyze the optimal inventory policy for the above capacitated dual-sourcing inventory systems, we first give a definition, called strong (CK_1, K_2)-convexity, that incorporates capacity limit constraint into the strong (K_1, K_2)-convexity defined in the Definition 1.

Definition 3 *A real value function $f(x)$ is called strong (CK_1, K_2)-convex, if it satisfy*

$$f(x+a) + K_1 \geq f(x) + \frac{a}{b}[f(y) - f(y-b) - K_2], \quad \forall y \leq x, a \in [0, C], b > 0. \tag{20}$$

Our next Lemma shows that most properties of strong (K_1, K_2) convexity still hold for the strong (CK_1, K_2)-convexity. Namely, strong (CK_1, K_2) convexity is preserved under linear operations, expectation, and optimization, see Lemma 1, Corollary 1, Propositions 1 and 3.

Lemma 5 *(a) If $f(x)$ is strong (CK_1, K_2)-convex, then $f(x-a)$ is also strong (CK_1, K_2) for any a. Moreover, for any random variable $\widetilde{X}$, $\mathsf{E} f(x - \widetilde{X})$ is also strong (CK_1, K_2)-convex provided $\mathsf{E} f(x - \widetilde{X}) < \infty$.*

(b) *If $f(x)$ is strong (CK_1, K_2)-convex, then $f(x)$ is strong $(C'K_1', K_2')$-convex for any $K_1' \geq K_1, K_2' \geq K_2$, and $C' \leq C$.*

(c) *If $f(x)$ and $g(x)$ are strong (CK_1, K_2)-convex, strong (CG_1, G_2)-convex, respectively, then for any $\alpha, \beta \geq 0$, $\alpha f(x) + \beta g(x)$ is strong $\big(C(\alpha K_1 + \beta G_1), \alpha K_2 + \beta G_2\big)$-convex.*

Obviously, a strong (K_1, K_2)-convex is also strong (CK_1, K_2)-convex for any $C \geq 0$, and a convex function is also strong (CK_1, K_2)-convex for any $K_1 \geq 0, K_2 \geq 0, C \geq 0$.

We first study a variant of minimization problem (9):

$$\begin{cases} f_1(x) = \min_{x \leq y \leq x+C}\{K_1\delta(y-x) + c(y-x) + g(y)\}, \\ f_2(x) = \min_{y \leq x}\{K_2\delta(x-y) + v(y-x) + g(y)\}, \\ f(x) = \min\{f_1(x), f_2(x)\}. \end{cases} \tag{21}$$

Similar to Lemma 3, the next lemma shows that the sell region is always to the right of purchase region.

Lemma 6 *If $c \geq v$, then all buy regions are to the left of all sell regions.*

An important issue of the convexity optimization is preservation of the convexity after one-step optimization? The next lemma shows that the strong (CK_1, K_2) convexity is indeed preserved after one-step optimization.

Lemma 7 (Preservation) *If a function $g(x)$ is strong (CK_1, K_2)-convex, then*

$$f_1(x) = \min_{y \in [x, x+F]}\{K_1\delta(y-x) + c(y-x) + g(y)\},$$
$$f_2(x) = \min_{y \leq x}\{K_2\delta(x-y) + v(y-x) + g(y)\},$$

are also strong (CK_1, K_2)-convex for any $F \geq C$. Furthermore, $f(x) = \min\{f_1(x), f_2(x)\}$ is strong (CK_1, K_2)-convex if $c \geq v$.

Similar to Theorem 1, our next theorem shows that the cost-to-go functions in a capacitated inventory system with two supply nodes are strong (CK_1, K_2)-convex.

Theorem 3 *Assume that* (A1) *holds. $f_n(x)$ is a strong (CK_n^f, K_n^v)-convex function for any $n \geq 0$.*

Proof. The result holds trivially for $n = 0$. Next, we prove the result by induction. Assuming that $f_{n-1}(x)$ is strong (CK_{n-1}^f, K_{n-1}^v) convex, we show that $f_n(x)$ is also strong (CK_n^f, K_n^v) convex. By Lemma 5, we know that $\alpha\mathsf{E}f_{n-1}(y-D)$ is a strong $(C\alpha K_{n-1}^f, \alpha K_{n-1}^v)$-convex function. Since $\alpha K_{n-1}^f \leq K_n^s$, according to Lemma 5, $\alpha\mathsf{E}f_{n-1}(y-D)$ is also a strong $(CK_n^s, \alpha K_{n-1}^v)$-convex function. Thus, by Lemma 7 (with $F = \infty$),

$$f_n^s(y) = \max_{z \geq y}\{K_n^s\delta(z-y) + c_n^s(z-y) + \alpha\mathsf{E}f_{n-1}(z-D_n)\},$$

is strong $(CK^s_n, \alpha K^v_{n-1})$ convex. Since $L_n(y)$ is a convex function, $g_n(y) = L_n(y) + f^s_n(y)$ is also strong $(CK^s_n, \alpha K^v_{n-1})$ convex by Lemma 5. Applying Lemma 5 once again, $g_n(y)$ is strong (CK^f_n, K^v_n) convex since $\alpha K^v_{n-1} \le K^v_n$ and $K^s_n \le K^f_n$. Thus, by Lemma 7, $f^b_n(x), f^v_n(x), f_n(x)$ are also strong (CK^f_n, K^v_n) convex. □

In the following, we turn to characterize the optimal policies. Let

$$\hat{B}^s_n = \sup\left\{\text{argmin}_z\{c^s_n z + \alpha \mathsf{E} f_{n-1}(z - D_n)\}\right\}, \quad \hat{b}^s_n = \inf\{x: \hat{A}^s_n(x) \ge 0\}, \quad \hat{\bar{b}}^s_n = \sup\{x: \hat{A}^s_n(x) < 0\}, \tag{22}$$

$$\hat{B}^f_n = \sup\left\{\text{argmin}_y\{c^f_n y + g_n(y)\}\right\}, \quad \hat{b}^f_n = \inf\{x: \hat{A}^f_n(x) \ge 0\}, \quad \hat{\bar{b}}^f_n = \sup\{x: \hat{A}^f_n(x) < 0\}, \tag{23}$$

$$\hat{S}^v_n = \inf\left\{\text{argmin}_y\{c^v_n y + g_n(y)\}\right\}, \quad \hat{s}^v_n = \sup\{x: \hat{A}^v_n(x) \ge 0\}, \quad \hat{\underline{s}}^v_n = \inf\{x: \hat{A}^v_n(x) < 0\}. \tag{24}$$

Where

$$\begin{aligned}
\hat{g}^s_n(y) &= K^s_n + \inf_{z \ge y}\{c^s_n(z - y) + \alpha \mathsf{E} f_{n-1}(z - D_n)\}, \qquad \hat{A}^s_n(y) = \hat{g}^s_n(y) - \alpha \mathsf{E} f_{n-1}(y - D_n), \\
\hat{g}^f_n(x) &= K^f_n + \inf_{y \in [x, x+C]}\{c^f_n(y - x) + g_n(y)\}, \qquad \hat{A}^f_n(x) = \hat{g}^f_n(x) - g_n(x), \\
\hat{g}^v_n(x) &= K^v_n + \inf_{y \le x}\{c^v_n(y - x) + g_n(y)\}, \qquad \hat{A}^v_n(x) = \hat{g}^v_n(x) - g_n(x).
\end{aligned}$$

Obviously, one has

$$\begin{aligned}
f^s_n(y) &= \min\{\alpha \mathsf{E} f_{n-1}(y - D_n),\ \hat{g}^s_n(y)\}, \quad \hat{A}^s_n(y) < 0, \forall y < \hat{b}^s_n, \quad \hat{A}^s_n(y) \ge 0, \forall y > \hat{\bar{b}}^s_n, \\
f^b_n(x) &= \min\{g_n(x),\ \hat{g}^f_n(x)\}, \quad \hat{A}^f_n(x) < 0, \forall x < \hat{b}^f_n, \quad \hat{A}^f_n(x) \ge 0, \forall x > \hat{\bar{b}}^f_n, \\
f^v_n(x) &= \min\{g_n(x),\ \hat{g}^v_n(x)\}, \quad \hat{A}^v_n(x) < 0, \forall x > \hat{s}^v_n, \quad \hat{A}^v_n(x) \ge 0, \forall x < \hat{\underline{s}}^v_n.
\end{aligned}$$

Similar to Lemma 4, it can be shown that: For any $n = 1, 2, \ldots, N$, we have (i) $\hat{b}^f_n \le \hat{\bar{b}}^f_n \le \hat{\underline{s}}^v_n \le \hat{s}^v_n$, and $\hat{b}^s_n \le \min\{\hat{\bar{b}}^s_n, \hat{B}^s_n\}$; (ii) $\hat{b}^f_n \le \hat{B}^f_n \le \hat{S}^v_n \le \hat{s}^v_n$, (iii) $\hat{\bar{b}}^f_n \le \hat{B}^f_n$ if $K^f_n \ge K^v_n$, $\hat{S}^v_n \le \hat{\underline{s}}^v_n$ if $K^v_n \ge K^f_n$, and $\hat{\bar{b}}^s_n \le \hat{B}^s_n$ if $K^s_n \ge \alpha K^v_{n-1}$.

The optimal policies in the capacitated inventory system have a same structure to those for uncapacitated inventory system characterized in Theorem 2. The ordering policy with the slow supplier and the salvaging decisions are exactly same to those in Theorem 2. The only difference is for the ordering quantity with the fast supplier, i.e., the order quantity from the fast node has to be less than its maximum capacity limit C, rather than raise its inventory up-to its global maximum in the uncapacitated inventory system. Our next theorem provides a condition under which it is optimal to order with a maximum quantity from the fast node when the starting inventory is below $\hat{b}^f_n - C$ and raise up the inventory above $\hat{b}^f_n$ with an order from the fast supplier.

Theorem 4 *Assume that $K_n^v = 0$. It is optimal to order with maximum size C from the fast supplier when the beginning inventory is below $\hat{b}_n^f - C$, and raise the inventory above $\hat{b}_n^f$ when the beginning inventory is between $\hat{b}_n^f - C$ and $\hat{b}_n^f$.*

Proof. Since it is optimal to order something from the fast supplier when the beginning inventory is lower than $\hat{b}_n^f$, it suffices to show that the function $H_n(y) \equiv c_n^f y + g_n(y)$ is decreasing when $y < \hat{b}_n^f$. For any $y_1 < y_2 < \hat{b}_n^f$, by (23), we have $\hat{A}_n^f(y_2) < 0$, i.e., $g_n(y_2) > \hat{g}_n^f(y_2) = K_n^f + c_n^f z_0 + g_n(y_2 + z_0)$ or $H_n(y_2) > K_n^f + H_n(y_2 + z_0)$ for some $z_0 \in (0, C]$. By Theorem 3 and Lemma 5, $H_n(y)$ is strong (CK_n^f, K_n^v)-convex, therefore, with $a = z_0, b = y_2 - y_1$ in (20), one has

$$H_n(y_2) > K_n^f + H_n(y_2 + z_0) \geq H_n(y_2) + \frac{z_0}{y_2 - y_1}[H_n(y_2) - H_n(y_1) - K_n^v].$$

Or equivalently, $H_n(y_2) < H_n(y_1)$ since $K_n^v = 0$. We have thus shown that the function $H_n(y) \equiv c_n^f y + g_n(y)$ is strictly decreasing when $y < \hat{b}_n^f$. □

Remark 1 *For any $K_n^v > 0$, it can be shown that the function $c_n^f y + g_n(y)$ is decreasing on $y \leq 0$, therefore, it is optimal to order as much as possible whenever ordering from the fast supplier when the starting inventory is less than $-C$.*

6 Discussions

We conclude our paper in this section with several discussions. In particular, we show that our analysis could be generalized to the following scenarios. First, we show that our model could extend to include a salvage value for the left-over inventory or a penalty cost for unsatisfied demands at the end of horizon. We also show that our analysis is easily extended to incorporate a capacity limit for order with the slow supplier and/or maximum size to be salvaged in each order.

Salvage Value or Penalty Cost at the End of Horizon

We first relax the assumption of no-value at the end of horizon. That is, we consider the case when the leftover inventory has a salvage value, or when the unsatisfied demand incurs penalty cost, or both, at the end of period. For example, the leftover inventory can be resold to a second market or the undelivered customer demand requires to be cleared at the end of horizon. All our results still hold provided the end horizontal value function is a (K_0^f, K_0^v)-convex function. For example, the leftover inventory can be salvaged at unit price c_0^v with a fixed cost K_0^v, and the uncleared demand requires to be cleared from expediting supplier at unit price c_0^f with fixed cost K_0^f. In this case, the end-period value function is given by

$$f_0(x) = \begin{cases} \min\left\{K_0^v \delta(x) - c_0^v x, 0\right\}, & \text{if } x \geq 0, \\ K_0^f \delta(-x) + c_0^f(-x), & \text{otherwise.} \end{cases}$$

It is not difficult to show that $f_0(x)$ given above is a strong (K_0^f, K_0^v)-convex function, therefore, is also a strong (FK_0^f, K_0^v)-convex function for any $F \geq 0$. Thus, all our analysis in the previous sections, such as Theorems 1 and 3, hold without any modification.

Capacity limits for Orders with Offshore Supplier and Salvaging Order

In this section, we show how our analysis in the above sections could be extended to cases when each order with the slow suppliers and/or quantities to be salvaged each time also have capacity limit, referred to as C^s and/or C^v, respectively. We refer the capacity limit for orders with the fast supplier to as C^f. The dynamic formulations (4)–(7) require slight modifications to incorporate these capacity limits as follows:

$$\begin{aligned}
f_n^b(x) &= \min_{z\in[y,y+C^s],\ y\in[x,x+C^f]} \{K_n^s\delta(z-y) + c_n^s(z-y) + K_n^f\delta(y-x) + c_n^f(y-x) + L_n(y) + \alpha\mathsf{E}f_{n-1}(z-D_n)\} \\
&= \min_{y\in[x,x+C^f]} \{K_n^f\delta(y-x) + c_n^f(y-x) + g_n(y)\}, \qquad (25) \\
f_n^v(x) &= \min_{z\in[y,y+C^s],\ y\leq[x-C^v,x]} \{K_n^s\delta(z-y) + c_n^s(z-y) + K_n^v\delta(x-y) + c_n^v(y-x) + L_n(y) + \alpha\mathsf{E}f_{n-1}(z-D_n)\} \\
&= \max_{y\leq[x-C^v,x]} \{K_n^v\delta(x-y) + c_n^v(y-x) + g_n(y)\}, \qquad (26) \\
f_n(x) &= \max\{f_n^b(x), f_n^v(x)\}, \qquad (27) \\
g_n(y) &= L_n(y) + \max_{z\in[y,y+C^s]} \{K_n^s\delta(z-y) + c_n^s(z-y) + \alpha\mathsf{E}f_{n-1}(z-D_n)\}. \qquad (28)
\end{aligned}$$

We also modify the strong (CK_1, K_2)-convexity in Definition 3 as follows,

Definition 4 *A real value function $f(x)$ is called strong (C_1K_1, C_2K_2)-convex, if it satisfy*

$$f(x+a) + K_1 \geq f(x) + \frac{a}{b}[f(y) - f(y-b) - K_2], \quad \forall y \leq x, a \in [0, C_1], b \in (0, C_2]. \qquad (29)$$

Most properties for a strong (CK_1, K_2)-convexity, such properties in Lemmas 5–7, could also be established for the strong (C_1K_1, C_2K_2)-convexity. For example, Lemma 5(b) could be stated as follows: If $f(x)$ is (C_1K_1, C_2K_2)-convex, then $f(x)$ is also (D_1K_1', D_2K_2')-convex for any $K_i' \geq K_i$ and $D_i \leq C_i, i = 1, 2$. With help of these properties, we show that the cost-to-go functions are strong (C_1K_1, C_2K_2)-convex functions in the following Theorem.

Theorem 5 *Assume that* (A1) *holds. $f_n(x)$ is a strong $(CK_n^f, C^vK_n^v)$-convex function for any $n \geq 0$, where $C = \min\{C^f, C^s\}$.*

As a concluding remark, our model is also applicable to cases when capacity limits are non-stationary, for e.g., the capacity limit for the fast supplier, the slow supplier, and the salvaging order is time dependent, denoted as C_n^f, C_n^s, C_n^v, respectively. It is not difficult to verify that all our results, such as Theorem 5, still hold provided that $\min\{C_n^f, C_n^s\} \leq \min\{C_{n-1}^f, C_{n-1}^s\}$ and $C_n^v \leq C_{n-1}^v$. Recall that our period is numbered in a backward way, the above two conditions represent settings where a firm is given a larger maximum size limit per order by its suppliers and also a larger capability in salvaging its left-over inventory in the market as the firm stays in business for a longer time. In practice, these conditions might be a result of the improved relationship between the firm and its suppliers, enhanced liquidity conditions, and so on.

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Appendix: Proofs

Proof of Lemma 1. The proof of part (a) and part (b) follows immediately from (20) in the Definition 1. Next, we prove part (c). Assuming that $f(\cdot)$ is strong (K_1, K_2)-convex and $g(\cdot)$ is strong (G_1, G_2)-convex, by (20), for any $y \le x,\ a \ge 0,\ b > 0$, one has

$$f(x+a) + K_1 \ge f(x) + \frac{a}{b}[f(y) - f(y-b) - K_2], \quad \text{and}$$
$$g(x+a) + G_1 \ge g(x) + \frac{a}{b}[g(y) - g(y-b) - G_2].$$

Thus, for any $\alpha, \beta \ge 0$, one has

$$\alpha[f(x+a) + K_1] + \beta[g(x+a) + G_1] \ge \alpha\left(f(x) + \frac{a}{b}[f(y) - f(y-b) - K_2]\right) + \beta\left(g(x) + \frac{a}{b}[g(y) - g(y-b) - G_2]\right).$$

Let $F(x) = \alpha f(x) + \beta g(x)$, the above inequality is equivalent to

$$F(x+a) + \alpha K_1 + \beta G_1 \ge F(x) + \frac{a}{b}[F(y) - F(y-b) - (\alpha K_2 + \beta G_2)].$$

This completes the proof of the part (c). □

Proof of Proposition 1.

(I) First, we show $f_1(x)$ is strong (K_1, K_2)-convex. For $\forall y \le x, a \ge 0, b > 0$, let

$$\Delta_1 = K_1 + f_1(x+a) - f_1(x) - \frac{a}{b}[f_1(y) - f_1(y-b) - K_2].$$

It suffices to show $\Delta_1 \ge 0$. To this end, we consider the following four cases.

(a) <u>$f_1(x+a) = g(x+a), f_1(y-b) = g(y-b)$</u>. In this case, we have

$$\begin{aligned}\Delta_1 &= K_1 + g(x+a) - f_1(x) - \frac{a}{b}[f_1(y) - g(y-b) - K_2] \\ &\ge K_1 + g(x+a) - g(x) - \frac{a}{b}[g(y) - g(y-b) - K_2] \ge 0,\end{aligned}$$

where, the first inequality is by the definition of $f_1(x)$, and the second inequality follows from the definition of strong (K_1, K_2)-convexity of $g(\cdot)$.

(b) <u>$f_1(x+a) = g(x+a), f_1(y-b) = g(y-b+u) + K_1$</u>. In this case, we get

$$\Delta_1 = K_1 + g(x+a) - f_1(x) - \frac{a}{b}[f_1(y) - g(y-b+u) - K_1 - K_2].$$

Considering the following two subcases.

(b.1) $\underline{f_1(y) \le g(y-b+u)+K_1+K_2}$. Obviously,

$$\Delta_1 \ge K_1 + g(x+a) - f_1(x) \ge 0.$$

(b.2) $\underline{f_1(y) > g(y-b+u)+K_1+K_2}$. First, we show $u \le b$. Assume $u > b$, then

$$g(y-b+u)+K_1+K_2 < f_1(y) \le g(y-b+u)+K_1, \quad \text{i.e.,} \quad K_2 < 0.$$

This contradicts the fact that $K_2 \ge 0$, therefore, $u \le b$. Thus,

$$\begin{aligned}\Delta_1 &\ge K_1 + g(x+a) - f_1(x) - \frac{a}{b-u}[f_1(y) - g(y-b+u) - K_1 - K_2] \\ &\ge K_1 + g(x+a) - g(x) - \frac{a}{b-u}[g(y) - g(y-b+u) - K_2] \ge 0\end{aligned}$$

(c) $\underline{f_1(x+a) = K_1 + g(x+a+u), f_1(y-b) = g(y-b)}$, we have

$$\begin{aligned}\Delta_1 &= K_1 + K_1 + g(x+a+u) - f_1(x) - \frac{a}{b}[f_1(y) - g(y-b) - K_2] \\ &\ge K_1 + g(x+a+u) - g(x+u) - \frac{a}{b}[g(y) - g(y-b) - K_2] \ge 0.\end{aligned}$$

(d) $\underline{f_1(x+a) = K_1 + g(x+a+u), f_1(y-b) = K_1 + g(y-b+w)}$, we have

$$\Delta_1 = K_1 + K_1 + g(x+a+u) - f_1(x) - \frac{a}{b}[f_1(y) - g(y-b+w) - K_1 - K_2].$$

We consider the following two subcases.

(d.1) $\underline{f_1(y) \le f(y-b+w)+K_1+K_2}$, we get

$$\Delta_1 \ge K_1 + K_1 + g(x+a+u) - f_1(x) \ge K_1 \ge 0.$$

(d.2) $\underline{f_1(y) > g(y-b+w)+K_1+K_2}$, first, we show $w \le b$. Otherwise, assume $w > b$, we have

$$g(y-b+w)+K_1+K_2 < f_1(y) \le K_1 + g(y-b+w), \quad \text{i.e.,} \quad K_2 < 0,$$

this contradicts the fact that $K_2 \ge 0$.

$$\begin{aligned}\Delta_1 &\ge K_1 + K_1 + g(x+a+u) - f_1(x) - \frac{a}{b-w}[f_1(y) - g(y-b+w) - K_1 - K_2] \\ &\ge K_1 + g(x+a+u) - g(x+u) - \frac{a}{b-w}[g(y) - g(y-b+w) - K_2] \ge 0.\end{aligned}$$

Therefore, we have shown that $f_1(x)$ is strong (K_1, K_2)-convex.

(II) Next, we prove $f_2(x)$ is strong (K_1, K_2)-convex. Similarly, for $\forall y \le x, a \ge 0, b > 0$, let

$$\Delta_2 = K_1 + f_2(x+a) - f_2(x) - \frac{a}{b}[f_2(y) - f_2(y-b) - K_2].$$

we consider the following four cases to show $\Delta_1 \ge 0$.

(a) $\underline{f_2(x+a) = g(x+a), f_2(y-b) = g(y-b)}$, same to the above Case (I)(a), we can show $\Delta_2 \geq 0$.

(b) $\underline{f_2(x+a) = g(x+a), f_2(y-b) = g(y-b-u) + K_2}$, we get

$$\begin{aligned}\Delta_2 &= K_1 + g(x+a) - f_2(x) - \frac{a}{b}[f_2(y) - g(y-b-u) - K_2 - K_2] \\ &\geq K_1 + g(x+a) - g(x) - \frac{a}{b}[g(y-u) - g(y-b-u) - K_2] \geq 0,\end{aligned}$$

where the first inequality is because $f_2(x) \leq g(x)$ and $f_2(y) \leq g(y-u) + K_2$, and the last inequality follows from the strong (K_1, K_2)-convexity.

(c) $\underline{f_2(x+a) = g(x+a-u) + K_2, f_2(y-b) = g(y-b)}$, we get

$$\Delta_2 = K_1 + K_2 + g(x+a-u) - f_2(x) - \frac{a}{b}[f_2(y) - g(y-b) - K_2].$$

we consider two subcases,

(c.1) $\underline{K_1 + K_2 + g(x+a-u) \geq f_2(x)}$, we obtain

$$\Delta_2 \geq -\frac{a}{b}[f_2(y) - g(y-b) - K_2] \geq 0.$$

where the last inequality follows from the definition of $f_2(y)$.

(c.2) $\underline{K_1 + K_2 + g(x+a-u) < f_2(x)}$, in this case, we can show $a \geq u$, since otherwise, $f_2(x) \leq K_2 + f(x+a-u)$, i.e., $K_1 < 0$. This contradicts the fact $K_1 \geq 0$. Thus, $a \geq u$. Thus

$$\begin{aligned}\Delta_2 &\geq K_1 + K_2 + g(x+a-u) - f_2(x) - \frac{a-u}{b}[f_2(y) - g(y-b) - K_2] \\ &\geq K_1 + g(x+a-u) - g(x) - \frac{a-u}{b}[g(y) - g(y-b) - K_2] \geq 0.\end{aligned}$$

where, the first inequality follows from the fact that $f_2(y) - g(y-b) - K_2 \leq 0$.

(d) $\underline{f_2(x+a) = g(x+a-u) + K_2, f_2(y-b) = g(y-b-w) + K_2}$, we have

$$\Delta_2 = K_1 + K_2 + g(x+a-u) - f_2(x) - \frac{a}{b}[f_2(y) - g(y-b-w) - K_2 - K_2].$$

Noted that $f_2(y) - g(y-b-w) - K_2 - K_2 \leq 0$, we consider two subcases.

(d.1) $\underline{K_1 + K_2 + g(x+a-u) \geq f_2(x)}$, obviously, $\Delta_2 \geq 0$.

(d.2) $\underline{K_1 + K_2 + g(x+a-u) < f_2(x)}$, it is easy to show $a \geq u$, thus

$$\Delta_2 \geq K_1 + g(x+a-u) - g(x) - \frac{a-u}{b}[g(y-w) - g(y-b-w) - K_2] \geq 0$$

(III) Finally, we will show $f(x)$ is also strong (K_1, K_2)-convex. For $\forall y \leq x, a \geq 0, b > 0$, let

$$\Delta = K_1 + f(x+a) - f(x) - \frac{a}{b}[f(y) - f(y-b) - K_2].$$

we will consider the following four cases to show $\Delta \geq 0$.

(a) $\underline{f(x+a) = f_1(x+a), f(y-b) = f_1(y-b)}$, we have

$$\begin{aligned}\Delta &= K_1 + f_1(x+a) - f(x) - \frac{a}{b}[f(y) - f_1(y-b) - K_2] \\ &\geq K_1 + f_1(x+a) - f_1(x) - \frac{a}{b}[f_1(y) - f_1(y-b) - K_2] \geq 0.\end{aligned}$$

(b) $\underline{f(x+a) = f_1(x+a), f(y-b) = f_2(y-b)}$, $\Delta = K_1 + f_1(x+a) - f(x) - \frac{a}{b}[f(y) - f_2(y-b) - K_2]$. We consider two cases

(b.1) $\underline{f_2(y-b) = g(y-b)}$. In this case, since $g(y-b) \geq f_1(y-b)$, thus, $f(y-b) = f_1(y-b)$ by case (a), $\Delta \geq 0$.

(b.2) $\underline{f_2(y-b) = K_2 + g(y-b-u)}$.

$$\begin{aligned}\Delta &= K_1 + f_1(x+a) - f(x) - \frac{a}{b}[f(y) - g(y-b-u) - K_2 - K_2] \\ &\geq K_1 + f_1(x+a) - f(x) - \frac{a}{b}[g(y-u) - g(y-b-u) - K_2],\end{aligned}$$

where the inequality follows from $f(y) \leq f_2(y) \leq g(y-u) + K_2$. When $f_1(x+a) = g(x+a)$, by strong (K_1, K_2)-convexity of $g(\cdot)$ and $f(x) \leq g(x)$, $\Delta \geq 0$. Otherwise, if $f_1(x+a) = K_1 + g(x+a+w)$, we have

$$\Delta \geq K_1 + K_1 + g(x+a+w) - g(x) - \frac{a}{b}[g(y-u) - g(y-b-u) - K_2] \geq 0.$$

(c) $\underline{f(x+a) = f_2(x+a), f(y-b) = f_1(y-b)}$. $\Delta = K_1 + f_2(x+a) - f(x) - \frac{a}{b}[f(y) - f_1(y-b) - K_2]$. We consider two cases

(c.1) $\underline{f_1(y-b) = g(y-b)}$. In this case, since $g(y-b) \geq f_2(y-b)$, thus, $f(y-b) = f_2(y-b)$ by the following case (d), $\Delta \geq 0$.

(c.2) $\underline{f_1(y-b) = K_1 + g(y-b+w)}$.

$$\Delta = K_1 + f_2(x+a) - f(x) - \frac{a}{b}[f(y) - g(y-b+w) - K_1 - K_2].$$

Noted that $f(y) \leq g(y-b+w) + K_1 + K_2$. When $f_2(x+a) = g(x+a)$, $f(x) \leq f_1(x) \leq K_1 + g(x+a)$, thus, $\Delta \geq 0$. Otherwise, if $f_2(x+a) = K_2 + g(x+a-u)$, we have $f(x) \leq K_1 + K_2 + g(x+a-u)$, thus, $\Delta \geq 0$.

(d) $\underline{f(x+a) = f_2(x+a), f(y-b) = f_2(y-b)}$, similar to case (a), we get

$$\begin{aligned}\Delta &= K_1 + f_2(x+a) - f(x) - \frac{a}{b}[f(y) - f_2(y-b) - K_2] \\ &\geq K_1 + f_2(x+a) - f_2(x) - \frac{a}{b}[f_2(y) - f_2(y-b) - K_2] \geq 0.\end{aligned}$$

□

Proof of Corollary 1. By Lemma1, $cy + g(y)$ and $vy + g(y)$ is strong (K_1, K_2)-convex, thus, by Proposition1, $f_1(x) + cx, f_2(x) + vx$ are strong (K_1, K_2)-convex. Again, by Lemma1, $f_1(x), f_2(x)$ are strong (K_1, K_2)-convex.

Proof of Lemma 2. We need to prove $f_1(x) + cx \le K_1 + f_1(y) + cy$ for any $x \le y$, this is trivial for $y = x$, therefore, we only need to show $x < y$. Assume

$$z(y) = \text{argmin}_{z \ge y}\{K_1\delta(z - y) + cz + g(z)\}.$$

we have

$$f_1(x) + cx \quad \le \quad K_1 + cz(y) + g(z(y)) \;\le\; K_1 + K_1\delta(z(y) - y) + cz(y) + g(z(y)) \;=\; K_1 + f_1(y) + cy.$$

Similarly, we show $f_2(x) + vx + K_2 \ge f_2(y) + vy, \quad$ for any $x \le y$. □

Proof of Lemma 3. We prove case (I) by contradiction. Suppose there exist $x < y$ such that it is optimal to sell to s at x and optimal to purchase to b at y, where $s < x < y < b$. Then we have

$$g(x) > K_2 + g(s) + v(s - x), \quad \text{(since it is optimal to sell to } s \text{ at } x\text{)},$$
$$g(y) > K_1 + c(b - y) + g(b), \quad \text{(it is optimal to purchase to } b \text{ at } y\text{)},$$

by (K_1, K_2) convexity of $g(\cdot)$, we have

$$K_1 + g(b) - g(y) \ge \frac{b - y}{x - s}[g(x) - g(s) - K_2] > -v(b - y) \ge -c(b - y)$$

This is a contradiction, since $g(y) > K_1 + c(b - y) + g(b)$. Thus, we have proved the case (I). Case (II) can be proved in the same way. □

Proof of Lemma 4. (i). First, we show $\bar{b} \le \underline{s}$ by contradiction. Assume $\bar{b} > \underline{s}$, by definition of $\bar{b}$ and $\underline{s}$ in (10) and (11), there exists x, y such that $\underline{s} < x < y < \bar{b}$, $\widetilde{A}_2(x) < 0$ and $\widetilde{A}_1(y) < 0$. Saying it in other ways, it is optimal to sell at x, and to purchase at y. This violates the Lemma 3.

Next, we turn to show $b \le \bar{b}$ by contradiction. Assume $b > \bar{b}$, by definition of b, for any $z \in (\bar{b}, b)$, $\widetilde{A}_1(z) < 0$, this contradicts the definition of $\bar{b}$. Similarly, it is easy to prove $\underline{s} \le s$.

In fact, we can prove a stronger results. Let

$$\hat{b} = \inf\{x : f_1(x) \ge f_2(x)\}, \quad \hat{s} = \sup\{x : f_1(x) \le f_2(x)\}. \tag{A-1}$$

We will prove a stronger results by showing $b \leq \hat{b} \leq \bar{b}$, $\underline{s} \leq \hat{s} \leq s$. For any $x < b$, we have $f_1(x) = \widetilde{g}_1(x) < g(x)$. Since $b < \underline{s}$, $f_2(x) = g(x)$, as a result, $f_1(x) < f_2(x)$. Thus, by definition of $\hat{b}$ in (A-1), we obtain $b \leq \hat{b}$. For any $x > \bar{b}$, we have $f_1(x) = g(x) \geq f_2(x)$, thus, by definition of $\hat{b}$, we obtain $\hat{b} \leq \bar{b}$. Similarly, we can prove $\underline{s} \leq \hat{s} \leq s$.

(ii). Since $c \geq v$, from (10)-(11), $B \leq S$ is trivial. In the following, we show $b \leq B$ by contradiction. Assume $b > B$, by (10), we have $\widetilde{A}_1(B) < 0$, i.e.,

$$g(B) \;>\; \widetilde{g}_1(B) = K_1 + \inf_{y \geq B}\{g(y) + cy\} - cB \;=\; K_1 + g(B),$$

where the equality is from the fact that B is the global minimizer of $g(y) + cy$, see (10). This contradicts the fact that $K_1 \geq 0$. Thus, $b \leq B$. In a similar way, we can show $S \leq s$.

(iii). We prove the case when $K_1 \geq K_2$ by contradiction, the other case when $K_1 \leq K_2$ can be proved in the same way. Assume $\bar{b} > B$, then there exists $x \in (B, \bar{b})$ such that

$$g(x) > \widetilde{g}_1(x) = K_1 + g(B_x) + c(B_x - x),$$

where $B_x > x$ is the optimal purchase-up-to level at x. By the definition of B in (10), we have

$$g(B_x) + cB_x > g(B) + cB.$$

Since $g(x)$ is a (K_1, K_2)-convex function, we obtain

$$K_1 + g(B_x) \geq g(x) + \frac{B_x - x}{x - B}[g(x) - g(B) - K_2],$$

or equivalently,

$$\begin{aligned} g(x) \;&\leq\; \frac{x - B}{B_x - B}[K_1 + g(B_x)] + \frac{B_x - x}{B_x - B}[g(B) + K_2] \\ &=\; K_1 + g(B_x) + \frac{B_x - x}{B_x - B}[g(B) + K_2 - g(B_x) - K_1] \\ &\leq\; K_1 + g(B_x) + \frac{B_x - x}{B_x - B}[g(B) - g(B_x) + c(B_x - B)] \\ &<\; K_1 + g(B_x) + c(B_x - x), \end{aligned}$$

where the second inequality is because $K_2 \leq K_1$. This is a contradiction, thus $\bar{b} \leq B$. □

Proof of Proposition 2. By Lemmas 3 and 4, the value function has the following form

$$f(x) = \min\{f_1(x), f_2(x)\} = \begin{cases} K_1 + g(B) + c(B - x), & x < b \\ \min\{\widetilde{g}_1(x),\ g(x)\}, & x \in [b,\ \bar{b}) \\ g(x), & x \in [\bar{b},\ \underline{s}) \\ \min\{\widetilde{g}_2(x),\ g(x)\}, & x \in [\underline{s},\ s) \\ K_2 + g(S) + v(S - x), & x \geq s \end{cases} \tag{A-2}$$

The rest of proof follows immediately by the definitions. $\square$

Proof of Proposition 3. Similar to Proposition 1, for $\forall y \leq x, u \geq 0, t > 0$, let

$$\Delta = K_1 + f(x+u) - f(x) - \frac{u}{t}[f(y) - f(y-t) - K_2].$$

it suffices to show $\Delta \geq 0$. By Lemma 3, we only need to consider three cases:

$$\begin{array}{ll} \text{(a)} & f(y-t) = f_1(y-t), f(x+u) = f_1(x+u); \\ \text{(b)} & f(y-t) = f_1(y-t), f(x+u) = f_2(x+u); \\ \text{(c)} & f(y-t) = f_2(y-t), f(x+u) = f_2(x+u); \end{array}$$

According to Corollary 1, $f_1(x), f_2(x)$ are both (K_1, K_2)-convex. Therefore, case (a) and case (c) could be easily proved in the same way as (III.a) and (III.d) of Proposition 1, respectively. In the following, we prove the case (b). If $f_1(y-t) = g(y-t)$, then $f_1(y-t) = f_2(y-t)$ and this is the case (c), which is true. Similarly, if $f_2(x+u) = g(x+u)$, the result is true. In the following we show $\Delta \geq 0$ when $f_1(y-t) = K_1 + g(B(y-t)) + c(B(y-t) - y + t)), f_2(x+u) = K_2 + g(S(x+u)) + v(S(x+u) - x - u)$. where $B(\cdot), S(\cdot)$ are given in Proposition 2. Obviously, we have

$$f(y) \leq \left\{ \begin{array}{ll} K_1 + c(B(y-t) - y) + g(B(y-t)), & B(y-t) \geq y \\ K_2 + v(B(y-t) - y) + g(B(y-t)), & B(y-t) < y \end{array} \right. ,$$

$$f(x) \leq \left\{ \begin{array}{ll} K_1 + c(S(x+u) - x) + g(S(x+u)), & S(x+u) \geq x \\ K_2 + v(S(x+u) - x) + g(S(x+u)), & S(x+u) < x \end{array} \right. .$$

We only present the proof for $K_1 \geq K_2$ here to show $\Delta \geq 0$ by considering four cases separately. Note that the order-up-to level is B if it is optimal to order according to By Proposition 2.

- Case 1. $y \leq B, x \leq S(x+u)$. In this case, we have $f(y) \leq K_1 + c(B-y) + g(B) = f_1(y-t) - ct$ and $f(x) \leq K_1 + c(S(x+u) - x) + g(S(x+u))$. Thus,

$$\begin{aligned} \Delta &\geq K_1 + K_2 + g(S(x+u)) + v(S(x+u) - x - u) - K_1 - c(S(x+u) - x) - g(S(x+u)) \\ &\quad - \frac{u}{t}[f_1(y-t) - ct - f_1(y-t) - K_2] \\ &\geq K_2 - cu + \frac{u}{t}[ct + K_2] \geq 0, \end{aligned}$$

  where the second inequality follows from the fact $v(S(x+u) - x - u) - c(S(x+u) - x) = (v - c)(S(x+u) - x - u) - cu \geq -cu$ by the facts that $c \geq v$ and $S(x+u) \leq x + u$.

- Case 2. $y \leq B, x > S(x+u)$. In this case, we have

$$\begin{aligned} \Delta &\geq K_1 + K_2 + g(S(x+u)) + v(S(x+u) - x - u) - K_2 - v(S(x+u) - x) - g(S(x+u)) \\ &\quad - \frac{u}{t}[f_1(y-t) - ct - f_1(y-t) - K_2] \\ &= K_1 - vu + \frac{u}{t}[ct + K_2] \geq 0, \end{aligned}$$

- <u>Case 3. $y > B, x > S(x+u)$</u>.

$$\begin{aligned}\Delta &\geq K_1 + K_2 + g(S(x+u)) + v(S(x+u)-x-u) - K_2 - v(S(x+u)-x) - g(S(x+u)) \\ &\quad -\frac{u}{t}[K_2 + v(B-y) + g(B) - K_1 - g(B) - c(B-y+t) - K_2] \\ &\geq K_1 - vu + \frac{u}{t}(vt + K_1) \geq 0,\end{aligned}$$

where the second inequality is because $v(B-y) - c(B-y+t) = (v-c)(B-y+t) - vt \leq -vt$.

- <u>Case 4. $y > B, x \leq S(x+u)$</u>. In this case, we get

$$\begin{aligned}\Delta &\geq K_1 + K_2 + g(S(x+u)) + v(S(x+u)-x-u) - K_1 - c(S(x+u)-x) - g(S(x+u)) \\ &\quad -\frac{u}{t}[K_2 + v(B-y) + g(B) - K_1 - g(B) - c(B-y+t) - K_2] \\ &\geq K_2 - cu + \frac{u}{t}(K_1 + vt)\end{aligned}$$

We consider the following subcases.

4.1. <u>$t+u \leq S(x+u) - B$</u>. In this case, since $y - t \leq B$ and $S(x+u) \leq x+u$, we have $f(x) \leq K_2 + g(S(x+u)-u) + v(S(x+u)-u-x)$ and $f(y) \leq K_1 + g(B+t) + c(B+t-y)$, thus, we get

$$\Delta \geq K_1 + g(S(x+u)) - g(S(x+u)-u) - \frac{u}{t}[g(B+t) - g(B) - K_2] \geq 0,$$

where the last inequality follows from the (K_1, K_2)-convexity of $g(\cdot)$.

4.2. <u>$t+u > S(x+u) - B$</u>.

$$\begin{aligned}\Delta &= K_1 + K_2 + g(S(x+u)) + v(S(x+u)-x-u) - f(x) \\ &\quad -\frac{u}{t}[f(y) - g(B) - c(B-y+t) - K_1 - K_2]\end{aligned}$$

$\Delta \geq 0$ is equivalent to

$$\frac{f(y) + cy - g(B) - cB - K_1 - K_2}{t} - \frac{g(S) + vS - f(x) - vx + K_1 + K_2}{u} \leq c - v$$

4.2.1. <u>If $f(y) + cy - g(B) - cB - K_1 - K_2 \leq 0$</u>, then

$$\begin{aligned}&\frac{f(y) + cy - g(B) - cB - K_1 - K_2}{t} - \frac{g(S) + vS - f(x) - vx + K_1 + K_2}{u} \\ \leq\ & -\frac{g(S) + vS - f(x) - vx + K_1 + K_2}{u} \\ \leq\ & -\frac{g(S) + vS - [K_1 + g(S) + c(S-x)] - vx + K_1 + K_2}{u} \\ =\ & \frac{(c-v)(S-x-u) - K_2}{u} + (c-v) \\ \leq\ & c - v\end{aligned}$$

4.2.2. If $g(S)+vS-f(x)-vx+K_1+K_2 \geq 0$, then

$$\begin{aligned}
& \frac{f(y)+cy-g(B)-cB-K_1-K_2}{t} - \frac{g(S)+vS-f(x)-vx+K_1+K_2}{u} \\
\leq\ & \frac{f(y)+cy-g(B)-cB-K_1-K_2}{t} \\
\leq\ & \frac{[K_2+g(B)+v(B-y)]+cy-g(B)-cB-K_1-K_2}{t} \\
=\ & \frac{(c-v)(y-t-B)-K_1}{u} + (c-v) \\
\leq\ & c-v
\end{aligned}$$

4.2.3. If $f(y)+cy-g(B)-cB-K_1-K_2 > 0$ and $g(S)+vS-f(x)-vx+K_1+K_2 < 0$. Let

$$t_0 = \inf\{v \geq y-B: \ g(y-v) > K_1+g(B)+c(B-y+v)\},$$
$$u_0 = \inf\{v \geq S-x: \ g(x+v) > K_2+g(S)+v(S-x-v)\}.$$

Obviously, $t \geq t_0, u \geq u_0$. By the continuity, $g(y-t_0) = K_1+g(B)+c(B-y+t_0)$ and $g(x+u_0) = K_2+g(S)+v(S-x-u_0)$. Thus,

$$\begin{aligned}
& \frac{f(y)+cy-g(B)-cB-K_1-K_2}{t} - \frac{g(S)+vS-f(x)-vx+K_1+K_2}{u} \\
\leq\ & \frac{f(y)+cy-g(B)-cB-K_1-K_2}{t_0} - \frac{g(S)+vS-f(x)-vx+K_1+K_2}{u_0} \\
=\ & \frac{f(y)-g(y-t_0)-K_2}{t_0} - \frac{g(x+u_0)-f(x)+K_1}{u_0} + c - v \\
\leq\ & c-v
\end{aligned}$$

where the last inequality follows from the (K_1, K_2)-convexity of $g(\cdot)$ and definition of $f(\cdot)$, which imply

$$\begin{aligned}
& K_1+g(x+u_0)-f(x) - \frac{u_0}{t_0}[f(y)-g(y-t_0)-K_2] \\
\geq\ & K_1+g(x+u_0)-g(x) - \frac{u_0}{t_0}[g(y)-g(y-t_0)-K_2] \geq 0
\end{aligned}$$

□

Proof of Lemma 5. The proof follows line by line to the proof of Lemma 1. □

Proof of Lemma 6. The proof is exactly the same to the proof of Lemma 3 with an additional requirement of $b-y \leq C$. □

Proof of Lemma 7. The proof is similar to Propositions 1 and 3, with slight modifications.

(I) First, we show $f_1(x)$ is (CK_1, K_2)-convex if $F \geq C$. Without loss of generality, we show the results for $c = 0$. For $\forall y \leq x, a \in [0, C], b > 0$, let

$$\Delta_1 = K_1 + f_1(x+a) - f_1(x) - \frac{a}{b}[f_1(y) - f_1(y-b) - K_2].$$

It suffices to show $\Delta_1 \geq 0$. To this end, we consider the following four cases.

(a) $\underline{f_1(x+a) = g(x+a), f_1(y-b) = g(y-b)}$. In this case, we have

$$\begin{aligned} \Delta_1 &= K_1 + g(x+a) - f_1(x) - \frac{a}{b}[f_1(y) - g(y-b) - K_2] \\ &\geq K_1 + g(x+a) - g(x) - \frac{a}{b}[g(y) - g(y-b) - K_2] \geq 0, \end{aligned}$$

where, the first inequality is by the definition of $f_1(x)$, and the second inequality follows from the definition of (CK_1, K_2)-convexity of $g(\cdot)$.

(b) $\underline{f_1(x+a) = g(x+a), f_1(y-b) = g(y-b+u) + K_1,\ u \in [0, F]}$. In this case, we get

$$\Delta_1 = K_1 + g(x+a) - f_1(x) - \frac{a}{b}[f_1(y) - g(y-b+u) - K_1 - K_2].$$

We consider the following two subcases.

(b.1) $\underline{f_1(y) \leq g(y-b+u) + K_1 + K_2}$. Since $a \leq C \leq F$, we have

$$\Delta_1 \geq K_1 + g(x+a) - f_1(x) \geq 0.$$

(b.2) $\underline{f_1(y) > g(y-b+u) + K_1 + K_2}$. First, we show $u \leq b$. Assume $u > b$, then

$$g(y-b+u) + K_1 + K_2 < f_1(y) \leq g(y-b+u) + K_1, \quad \text{i.e.,} \quad K_2 < 0.$$

This contradicts the fact that $K_2 \geq 0$, therefore, $u \leq b$. Thus,

$$\begin{aligned} \Delta_1 &\geq K_1 + g(x+a) - f_1(x) - \frac{a}{b-u}[f_1(y) - g(y-b+u) - K_1 - K_2] \\ &\geq K_1 + g(x+a) - g(x) - \frac{a}{b-u}[g(y) - g(y-b+u) - K_2] \geq 0 \end{aligned}$$

(c) $\underline{f_1(x+a) = K_1 + g(x+a+u), f_1(y-b) = g(y-b)}$, where $u \leq F$, we have

$$\begin{aligned} \Delta_1 &= K_1 + K_1 + g(x+a+u) - f_1(x) - \frac{a}{b}[f_1(y) - g(y-b) - K_2] \\ &\geq K_1 + g(x+a+u) - g(x+u) - \frac{a}{b}[g(y) - g(y-b) - K_2] \geq 0. \end{aligned}$$

(d) $\underline{f_1(x+a) = K_1 + g(x+a+u), f_1(y-b) = K_1 + g(y-b+w)}$, $u, w \in [0, F]$, we have

$$\Delta_1 = K_1 + K_1 + g(x+a+u) - f_1(x) - \frac{a}{b}[f_1(y) - g(y-b+w) - K_1 - K_2].$$

We consider the following two subcases.

(d.1) $\underline{f_1(y) \le f(y-b+w)+K_1+K_2.}$ (d.1.1) if $a+u \le F$, we get

$$\Delta_1 \ge K_1 + K_1 + g(x+a+u) - f_1(x) \ge K_1 \ge 0.$$

(d.1.2) Otherwise, if $a+u > F$, we get

$$\begin{aligned}\Delta_1 &\ge K_1 + g(x+a+u) - g(x+F_1) - \frac{a+u-F_1}{b}[f_1(y) - g(y-b+w) - K_1 - K_2] \\ &\ge K_1 + g(x+a+u) - g(x+F_1) - \frac{a+u-F_1}{b}[g(y+w) - g(y-b+w) - K_2] \ \ge \ 0\end{aligned}$$

(d.2) $\underline{f_1(y) > g(y-b+w)+K_1+K_2}$, first, we show $w \le b$. Otherwise, assume $w > b$, we have

$$g(y-b+w) + K_1 + K_2 < f_1(y) \le K_1 + g(y-b+w), \quad \text{i.e.,} \quad K_2 < 0,$$

this contradicts the fact that $K_2 \ge 0$.

$$\begin{aligned}\Delta_1 &\ge K_1 + K_1 + g(x+a+u) - f_1(x) - \frac{a}{b-w}[f_1(y) - g(y-b+w) - K_1 - K_2] \\ &\ge K_1 + g(x+a+u) - g(x+u) - \frac{a}{b-w}[g(y) - g(y-b+w) - K_2] \ \ge \ 0.\end{aligned}$$

Therefore, we have shown that $f_1(x)$ is (CK_1, K_2)-convex.

(II) Next, we prove $f_2(x)$ is (CK_1, K_2)-convex. Without loss of generality, we show the results for $c = 0$. For $\forall y \le x, a \in [0, C], b > 0$, let

$$\Delta_2 = K_1 + f_2(x+a) - f_2(x) - \frac{a}{b}[f_2(y) - f_2(y-b) - K_2].$$

we consider the following four cases to show $\Delta_1 \ge 0$.

(a) $\underline{f_2(x+a) = g(x+a), f_2(y-b) = g(y-b)}$, same to the above Case (I)(a), we can show $\Delta_2 \ge 0$.

(b) $\underline{f_2(x+a) = g(x+a), f_2(y-b) = g(y-b-u) + K_2}$, $u \ge 0$, we get

$$\begin{aligned}\Delta_2 &= K_1 + g(x+a) - f_2(x) - \frac{a}{b}[f_2(y) - g(y-b-u) - K_2 - K_2] \\ &\ge K_1 + g(x+a) - g(x) - \frac{a}{b}[g(y-u) - g(y-b-u) - K_2] \ \ge \ 0,\end{aligned}$$

where the first inequality is because $f_2(x) \le g(x)$ and $f_2(y) \le g(y-u) + K_2$, and the last inequality follows from the strong (K_1, K_2)-convexity.

(c) $\underline{f_2(x+a) = g(x+a-u) + K_2, f_2(y-b) = g(y-b)}$, $u \ge 0$, we get

$$\Delta_2 = K_1 + K_2 + g(x+a-u) - f_2(x) - \frac{a}{b}[f_2(y) - g(y-b) - K_2].$$

we consider two subcases,

(c.1) $\underline{K_1 + K_2 + g(x+a-u) \geq f_2(x)}$, we obtain

$$\Delta_2 \geq -\frac{a}{b}[f_2(y) - g(y-b) - K_2] \geq 0.$$

where the last inequality follows from the definition of $f_2(y)$.

(c.2) $\underline{K_1 + K_2 + g(x+a-u) < f_2(x)}$, in this case, we can show $a \geq u$, since otherwise, $f_2(x) \leq K_2 + f(x+a-u)$, i.e., $K_1 < 0$. This contradicts the fact $K_1 \geq 0$. Thus, $a \geq u$. Thus

$$\begin{aligned}\Delta_2 &\geq K_1 + K_2 + g(x+a-u) - f_2(x) - \frac{a-u}{b}[f_2(y) - g(y-b) - K_2] \\ &\geq K_1 + g(x+a-u) - g(x) - \frac{a-u}{b}[g(y) - g(y-b) - K_2] \geq 0.\end{aligned}$$

where, the first inequality follows from the fact that $f_2(y) - g(y-b) - K_2 \leq 0$.

(d) $\underline{f_2(x+a) = g(x+a-u) + K_2, f_2(y-b) = g(y-b-w) + K_2}$, $u, w \geq 0$, we have

$$\Delta_2 = K_1 + K_2 + g(x+a-u) - f_2(x) - \frac{a}{b}[f_2(y) - g(y-b-w) - K_2 - K_2].$$

We consider two subcases. Obviously, $f_2(y) - g(y-b-w) - K_2 - K_2 \leq 0$.

(d.1.1) $\underline{\text{if } K_1 + K_2 + g(x+a-u) \geq f_2(x)}$, thus, $\Delta_2 \geq 0$.

(d.1.2) $\underline{\text{if } K_1 + K_2 + g(x+a-u) < f_2(x)}$, it is easy to show $a \geq u$.

$$\Delta_2 \geq K_1 + g(x+a-u) - g(x) - \frac{a-u}{b}[g(y-w) - g(y-b-w) - K_2] \geq 0.$$

Thus, $f_2(x)$ is (CK_1, K_2)-convex.

(III) The proof for $f(x)$ is exact the same to the proof of Proposition 3.

□

Proof of Theorem 5. To prove the theorem, we restate the preservation Lemma 7 in the following lemma.

Lemma A-1 *If a function $g(x)$ is strong (C_1K_1, C_2K_2)-convex, then*

$$\begin{aligned}f_1(x) &= \min_{y \in [x, x+F_1]} \{K_1\delta(y-x) + c(y-x) + g(y)\}, \\ f_2(x) &= \min_{y \in [x-F_2, x]} \{K_2\delta(x-y) + v(y-x) + g(y)\},\end{aligned}$$

are also strong (D_1K_1, D_2K_2)-convex for any c, v with $D_i = \min\{C_i, F_i\}, i = 1, 2$. Furthermore, $f(x) = \min\{f_1(x), f_2(x)\}$ is strong (D_1K_1, D_2K_2)-convex if $c \geq v$.

The proof of the above lemma could be obtained in the same way to those of Lemma 7. The reminder proof of the theorem follows line by line in the proof of Theorem 3, and is thus omitted. □