Estimating Buyer Willingness-to-Pay and Seller Reserve Prices from Negotiation Data and the Implications for Pricing

Robert Phillips  
Graduate School of Business, Columbia University, New York, NY 10027 and Nomis Solutions, rp2051@columbia.edu

A. Serdar Şimşek
School of Operations Research and Information Engineering, Cornell University, Ithaca, NY 14853, as2899@cornell.edu

Garrett van Ryzin
Graduate School of Business, Columbia University, New York, NY 10027, gjv1@columbia.edu

We consider a company selling heterogeneous products where the final price is set by negotiation between sales agents and customers. Negotiation is common in many industries including retail automotive sales, consumer lending, insurance and many business-to-business markets. We assume that each sales agent has a minimum reserve price that he/she will accept and each customer has a maximum willingness-to-pay; however, these values are not directly observable. If the customer’s willingness-to-pay is greater than the sales agent’s reserve price, then a sale takes place at the Nash bargaining equilibrium price, otherwise no trade takes place. Given a record of the outcomes of a series of such negotiations, we would like to estimate the underlying joint distribution of willingness-to-pay and reserve prices. To do so, we propose and analyze a structural estimation method based on the expectation-maximization (EM) algorithm that allows for the willingness-to-pay and reserve price distributions to depend on an arbitrary set of covariates (e.g. characteristics of the customer, product or sales agent) and be potentially correlated. Using a real-world data set, we show, via out-of-sample tests, that our estimation method generates accurate estimates of customer-price response and the prices of lost deals. We also show how our estimated distributions can be used to optimize controls on negotiated prices that can significantly increase revenues relative to both unconstrained negotiations and centrally-optimized fixed prices. These results provide a justification for the policy commonly observed in practice of providing sales staff limited discretion to change prices relative to a centrally-established price list.

Key words: negotiation; structural estimation; Nash bargaining solution; limited sales force price discretion; consumer lending
1. Introduction

Price negotiations are often associated with markets such as fine art and real estate in which the items for sale are unique. However, negotiations are also found in many other markets in which the items sold are identical or similar. Examples include retail automotive sales, insurance, consumer loans, mortgages, and many business-to-business markets. In these markets, the price paid for identical or similar items may be quite different depending upon the outcome of the negotiation. Moreover, the seller and the customer may not be able to agree on a price, resulting in a lost deal. Whether or not a deal occurs and, if it does, the price that results will depend upon the willingness-to-pay of the customer, the reserve price of the seller and the negotiating power of each party.

Understanding the variation of willingness-to-pay among customers is important to most firms in order to segment customers and intelligently set prices for different products and customer segments. Estimating the willingness-to-pay distribution for a particular product within a particular customer segment based on historical sales data is always challenging and a variety of approaches have been developed. (See Breidert et al. 2006, for a good review.) Estimating price response in cases where negotiation is involved poses two specific challenges. First, price information is typically unobserved for lost deals, since sales staff may be unwilling or simply unable to accurately estimate the last price under consideration before a customer walked away. Second, when a trade does take place, the resulting price is correlated with the seller’s reserve price as well as the customer’s willingness-to-pay.

To address these challenges, we develop a structural estimation method to determine the joint distribution of customer willingness-to-pay and seller reserve prices in a market with negotiation. Specifically, we postulate a model of a heterogeneous sellers selling to heterogeneous buyers via bilateral price negotiations. Each seller has a private reserve price, $R$, for each deal, and will not sell at a price lower than $R$. Similarly, each buyer has a private willingness-to-pay, $W$, and will not buy at a price higher than $W$. Hence, no sale will occur if $R > W$. If $W \geq R$, we assume that the parties will reach agreement and a sale will occur at the symmetric Nash bargaining equilibrium price $p = (R + W)/2$. Given a data set on both completed and lost deals (and the prices at which deals occur), we address the problem of how to estimate the parameters of the joint distribution of $R$ and $W$. We allow for the fact that the distribution of $(R, W)$ may depend on a set of covariates describing the product, buyer and seller and that the values $R$ and $W$ may be correlated with each other.

A novel feature of our model is the assumption of a reserve price distribution among sellers. This reserve price distribution can reflect sales force heterogeneity; that is, a more skilled sales agent should be able to extract a higher price from the same customer, which could manifest itself
as a higher reserve price. Additionally, sales staff might be more or less willing to accept specific deals at lower prices based on idiosyncratic incentives, e.g. how close they are to meeting a sales quota. Such differences would also be reflected as differences in reserve prices. If sales staff were homogeneous in terms of their abilities and incentives to make deals, this would be reflected in a common reserve price among similar deals – or a reserve price that is largely predictable as a function of the characteristics of the deal. In our real-world auto lending data, this is not the case: reserve prices appear to vary significantly among sales representatives for similar deals.

We consider three approaches to estimating the parameters of the joint distribution of $R$ and $W$ given data on historical deals: direct maximization of the likelihood function and two variations on the expectation-maximization (EM) algorithm, which is commonly used to estimate model parameters given censored data. We show that the expectation-maximization algorithm is more robust and generally provides faster convergence, particularly on large data sets. We therefore use the EM algorithm to estimate the joint distribution of $R$ and $W$ as a function of customer and deal characteristics for a large data set of auto lending deals.

Lastly, we illustrate how our estimated joint distribution can be used to improve negotiated pricing and help compensate for heterogeneity in the sales force. We assume that “headquarters” wishes to maximize expected profit across all deals and sets the parameters within which local sales agents must work and compare three policies: centralized deal-specific optimal pricing, negotiated pricing with centrally set deal-specific minimum prices, and optimized seller reserve prices. Applying these three policies to a real-world auto lending data set we find that optimized seller reserve prices outperforms centralized deal-specific optimal pricing which outperforms negotiated pricing with minimum prices. We discuss the implications of these findings in Section 6.

2. Literature Review

Starting with Weinberg (1975), a number of researchers have considered the question of how much pricing discretion (if any) should be granted to a sales force. A number of these papers have assumed that local sales representatives have local market information that they can use to set better prices than headquarters (Lal 1986). Mishra and Prasad (2004) show that, even when a customer possesses private information, if the sales agent first observes the private information, then signs the contract, a company can set prices centrally and capture the full benefits available from discretionary pricing by using an appropriate contract structure – though compensation may be a complicated function of quantity and price. They then analyze competitive markets and show that, under asymmetric information, an equilibrium always exists where all firms use centralized pricing, regardless of the intensity of competition (Mishra and Prasad 2005).

Joseph (2001) develops a model that explicitly incorporates the sales agent’s superior information about customer willingness-to-pay and the possibility of sub-optimal trade-offs between price and
Phillips, Şimşek, and van Ryzin: Estimating Buyer WTP and Seller RP from Negotiation Data and the Implications

effort (using discounting rather than expending effort on selling). In his model, centralization of pricing authority is sometimes preferable to full decentralization, depending on the effort cost of following a high-quality prospecting strategy and the structure of customer segments. Bhardwaj (2001) reaches a similarly mixed conclusion in the case of competing firms when principals and agents have the same information. Roth et al. (2006) consider the case of bargaining versus fixed (or posted) price for customized services and show that, if bargaining costs are low, bargaining can be a preferred approach for the seller. Kuo et al. (2011) present a model that incorporates the interactions among dynamic pricing, negotiation, and inventory and show that negotiation is an effective tool to achieve price discrimination, particularly when the inventory level is high and/or the remaining selling season is short.

However, all of these models have led to results concluding that either full pricing centralization or full pricing decentralization is optimal (or showing the equivalence of the two for specified contract structures). Yet we observe in many markets with decentralized sales forces, that local sales staff has discretion to negotiate prices for individual deals within pre-set limits. A recent Ernst & Young Consulting survey showed that about 75% of business-to-business companies surveyed in the United States allowed their sales staff limited discretion to adjust prices (Williams 2014). One conclusion of our work is that, in general, neither fully centralized nor fully decentralized pricing maximizes profits. Fully centralized pricing is not optimal because it does not allow sales staff to adjust prices toward the willingness-to-pay of individual customers. Fully decentralized pricing is not optimal because it becomes too dependent on the idiosyncratic reserve prices of sales staff. Sales staff with very low reservation prices will systematically under-price deals. Rather than relying on full price centralization or decentralization, our work suggests firms can maximize profits by setting list prices and giving sales representatives limited discretion to adjust the prices for individual deals. In this way, our work provides a heretofore missing justification for the common practice of list pricing with discretion.

In the remainder of the paper, we describe our structural model and how the EM method can be used to estimate the parameters of the model given deal and lost-deal data. We then use synthetic data to show that our estimates converge to the underlying true parameters when willingness-to-pay and reserve prices are drawn from normal distributions. Next, we apply the EM algorithm to a large data set of negotiated deals and lost deals obtained from a major North American auto lender and evaluate the predictions produced by our method relative to standard regression-based predictors. Our method also provides better estimates of the prices associated with lost deals1.

We then show how the distributions of reserve price and willingness-to-pay can be used to determine the optimal amounts of pricing discretion to provide to the sales force. Finally, we

1 When a deal is lost, the last price (APR of the loan) offered to the customer is still recorded in this data set.
discuss the robustness of the model to the specification of the underlying distributions and to the assumption about the relative bargaining power of sales staff and customers.

3. Model Description

Consider a setting in which a group of heterogeneous sellers sells a (potentially heterogeneous) product to a heterogeneous group of buyers. Prices are negotiated bi-laterally between a single buyer and single seller in each transaction. We assume that in each negotiation, the seller has a reserve price $R$ and the buyer has a willingness-to-pay $W$. The values of $R$ and $W$ are both unobservable, but they both may be correlated with observable attributes of the seller, the buyer and the product being sold. If the reserve price is greater than the willingness-to-pay, then no trade takes place. If the willingness-to-pay is greater than the reserve price, then the outcome of the negotiation is assumed to be the generalized Nash bargaining solution. (See Binmore et al. 1986, Muthoo 1999, for reviews of Nash bargaining.) Nash bargaining is a widely-used model of negotiations (c.f. Desai and Purohit 2004, Roth et al. 2006, Kuo et al. 2011).

In the Nash solution, both parties divide the total surplus from trade (the difference between the sum of their utilities before and after the bargaining) according to their relative bargaining powers. Let $s_1$ and $s_2$ be the utilities to players 1 and 2 derived from the bargaining outcome and let $s^0_1$ and $s^0_2$ be, respectively, the corresponding disagreement utilities (utilities to each party when no trade takes place). Then the generalized Nash bargaining solution outcome maximizes $(s_1 - s^0_1)\alpha(s_2 - s^0_2)(1-\alpha)$, where $\alpha$ is a measure of the relative bargaining power of the buyer and seller (see Binmore et al. 1986).

We assume that both the seller’s and buyer’s utilities will be zero if no trade takes place, i.e., $s^0_1 = s^0_2 = 0$. If trade takes place at a negotiated price $p$, then the seller’s utility will be $p - R$ and the buyer’s utility will be $W - p$. Therefore, when $W \geq R$ the generalized Nash bargaining solution solves:

$$\max_p (p - R)\alpha(W - p)(1-\alpha)$$

The solution is

$$p^eq = \alpha R + (1 - \alpha)W$$

which gives the equilibrium price of the bargaining outcome$^2$. In our model, we assume that the seller and buyer have equal bargaining powers, i.e., $\alpha = 0.5$. Hence, if a trade takes place, it does so at the price determined by the symmetric Nash bargaining equilibrium $p^eq = (R + W)/2$. We show in Section 7.2 that $\alpha$ is in fact not identifiable from the data and moreover the estimation results are robust to different choices of $\alpha$. Lastly, in the case where no trade takes place, then we only know that $W < R$.

$^2$Note that we are assuming an efficient bargaining outcome in a full information setting. Modeling an imperfect bargaining situation in an asymmetric (or private) information setting would require additional assumptions (see, for example, Chatterjee and Samuelson 1983).
4. Structural Estimation Problem

Assume we have a data set $D$ of the outcomes of $N$ negotiations indexed by $i$, where the values $(R_i, W_i)$ are IID draws from a (unknown) bi-variate normal joint distribution $F(R, W)$. For each negotiation $i$ in the data set, we observe only i) whether trade takes place or not, ii) if trade takes place, the outcome price $p_i$ of the negotiation, and iii) a set of covariates describing the seller, buyer and product involved in the negotiation. Given these assumptions and inputs, we want to estimate the joint distribution $F$.

Define

- $R_i =$ reserve price of seller in negotiation $i$ (unobservable random variable).
- $W_i =$ willingness-to-pay of buyer in negotiation $i$ (unobservable random variable).
- $p_i =$ final price of negotiation $i$ when trade takes place (data).
- $S_i =$ vector of seller characteristics for negotiation $i$ (data).
- $B_i =$ vector of buyer characteristics for negotiation $i$ (data).
- $P_i =$ vector of product characteristics for negotiation $i$ (data).
- $N =$ total number of negotiations (data).

We assume that $R_i$ and $W_i$ are joint normally distributed random variables, defined as

\[
R_i = X_i^T \beta_r + \epsilon_i \\
W_i = Y_i^T \beta_w + \delta_i
\]

where $X_i = (S_i, P_i)$, $Y_i = (B_i, P_i)$ are the vector of seller and buyer covariates, respectively, $\beta_r$ and $\beta_w$ are the vectors of coefficients on these covariates, and $(\epsilon_i, \delta_i)$, $i = 1, \ldots, N$ are IID error terms which are bivariate normal with mean 0 and covariance matrix

\[
\Sigma = \begin{pmatrix}
\sigma^2_r & \sigma_{rw} \\
\sigma_{rw} & \sigma^2_w
\end{pmatrix}
\]

with corresponding correlation coefficient $\rho = \sigma_{rw}/(\sigma_r \sigma_w)$. Note because we assume the error terms have zero mean, $E[R_i] = X_i^T \beta_r$ and $E[W_i] = Y_i^T \beta_w$; hence, the covariates influence only the means of the willingness-to-pay and reserve price distribution. Without loss of generality, assume that $X_i$ and $Y_i$ have the same dimension $M$ and let $X_{ij}$ and $Y_{ij}$ denote the $j$-th component of $X_i$ and $Y_i$, respectively. For notational convenience, also define $\mathbf{R} = (R_1, \ldots, R_N)$, $\mathbf{W} = (W_1, \ldots, W_N)$, $\mathbf{X} = (X_1, \ldots, X_N)$ and $\mathbf{Y} = (Y_1, \ldots, Y_N)$. We denote the marginal cdf of $\epsilon$ and $\delta$ by $F_{\epsilon}$ and $F_{\delta}$, respectively. We denote the marginal cdf of $R$ by $F_r(r, \theta_r)$ and the associated marginal pdf $f_r(r, \theta_r)$. Similarly, the marginal cdf of $W$ is denoted $F_w(w, \theta_w)$ with associated pdf $f_w(w, \theta_w)$. 
Finally, we let $\theta = (\beta_r, \sigma_r, \beta_w, \sigma_w, \rho)$ denote the vector of parameters to be estimated from the data set $D$ and define the sets

$$T = \{i: \text{trade takes place}\}$$

$$\bar{T} = \{i: \text{no trade takes place}\}$$

We seek to estimate $\theta$ from the data $D$.

### 4.1. Non-Quasi-Concavity of the Log-Likelihood Function

Consider the simplest case where there are no covariates and $R$ and $W$ are independent. In this case, both the reserve price and willingness-to-pay follow independent normal distributions with $\theta_r = (\mu_r, \sigma_r)$ and $\theta_w = (\mu_w, \sigma_w)$, then $\theta = (\mu_r, \sigma_r, \mu_w, \sigma_w)$. In this case, the log-likelihood function is

$$L(D, \theta) = \sum_{i \in T} \ln \{P(R + W = 2p_i, R \leq p_i \leq W)\} + \sum_{i \in \bar{T}} \ln \{P(W < R)\}$$

$$= -\frac{|T|}{2} \ln (2\pi) - \frac{|T|}{2} \ln (\sigma_r^2 + \sigma_w^2) - \frac{1}{2(\sigma_r^2 + \sigma_w^2)} \sum_{i \in T} (2p_i - \mu_r - \mu_w)^2$$

$$+ \sum_{i \in \bar{T}} \ln \Phi \left( \frac{(p_i - \mu_r)\sigma_w^2 - (p_i - \mu_w)\sigma_r^2}{\sigma_r\sigma_w\sqrt{\sigma_r^2 + \sigma_w^2}} \right) + |T| \ln \Phi \left( \frac{\mu_r - \mu_w}{\sqrt{\sigma_r^2 + \sigma_w^2}} \right)$$  \hspace{1cm} (1)

**Proposition 1.** The function $L(D, \theta)$ in equation (1) is not quasi-concave in general.

**Proof:** We prove the proposition by providing a counterexample. Consider a case where there are nine negotiations and trade happened in the first five of them at prices $(60, 70, 75, 80, 88)$, hence $D = (60, 70, 75, 80, 88, NT, NT, NT, NT, NT)$, where $NT$ indicates that no trade took place. Take the following two points: $\theta^1 = (60, 10, 60, 36)$ and $\theta^2 = (90, 35, 90, 10)$. Then, $L(D, \theta^1) = -29.36$ and $L(D, \theta^2) = -29.40$. If we take $\alpha = 0.5$, we get an intermediate point $\theta^\alpha = \alpha \theta^1 + (1 - \alpha) \theta^2 = (75, 22.5, 75, 23)$, with $L(D, \theta^\alpha) = -29.74$, which is smaller than the previous two. Therefore, quasi-concavity is violated. Q.E.D.

### 4.2. An EM Estimation Method

The expectation-maximization (EM) method is a standard approach to parameter estimation with censored data. Introduced by Dempster et al. (1977), the EM method is useful when the incomplete data log-likelihood function is difficult to maximize directly but the complete data log-likelihood function has a simple form, as we show is the case in our model. The book by McLachlan and Krishnan (1996) is a good resource for details of the method and its extensions. The EM method is used widely in the empirical operations management literature for estimation problems in choice-based revenue management (Vulcano et al. 2010), censored sales (Stefanescu 2012), substitutable
products with stock-outs (Anupindi et al. 1998), and selling with incomplete product availability (Conlon and Mortimer 2007). Monte Carlo implementations of the EM algorithm have been developed for more complex likelihood functions (see, for example, Levine and Casella 2001, Chan and Ledolter 1995).

Consider an ideal data set which records every pair \((R_i, W_i)\) for each negotiation \(i = 1, \ldots, N\). Given these exact observations, as we show below, the log-likelihood function to estimate \(F\) is straightforward. We can consider our data set \(D\) to be a partial observation of these complete data. The idea of the EM method is to work with the conditional expected value of the simpler complete-data likelihood rather than the more complex incomplete-data likelihood. Specifically, the EM algorithm starts with initial estimates of the parameters \(\theta\) (an initial "guess") and then computes the expected value of the complete data log-likelihood function conditional on \(\theta\) and \(D\). This is the "E" (expectation) step. The resulting expected log-likelihood is of the same simple form as the complete-data log-likelihood function and can be easily maximized to find updated parameter estimates \(\theta\). This is the "M" (maximization) step. The process is then repeated until the estimates \(\theta\) converge. The generalized EM algorithm improves the complete data problem’s conditional expectation of the log-likelihood rather than maximizing it at every iteration.

In summary, the steps in the algorithm are:

- **Initialization:** Set \(\theta = \theta^{(0)}\)

On the \((k+1)\)th iteration:

- **E-step:** Calculate \(Q(\theta, \theta^{(k)}) = E_{\theta^{(k)}}[\mathcal{L}_c|D, \theta^{(k)}]\)

- **M-step:** Choose \(\theta^{(k+1)}\) such that

\[
Q(\theta^{(k+1)}, \theta^{(k)}) \geq Q(\theta, \theta^{(k)}) \quad \forall \theta \in \Omega
\]

Repeat these E and M steps until convergence (e.g. until \(\theta^{(k+1)} \approx \theta^{(k)}\)).

4.2.1. The EM Method Applied to the Structural Bargaining Model Suppose that all of the values of \(R_i\) and \(W_i\) in our negotiating model are known for \(i = 1, 2, \ldots, N\). Then, the maximum likelihood estimates of the model parameters are the solution to the normal equations for the bivariate normal distribution:

\[
(X^T X) \hat{\beta}_r = X^T R
\] (2)

\[
(Y^T Y) \hat{\beta}_w = Y^T W
\] (3)

\[
\hat{\sigma}_r^2 = \frac{1}{N} \sum_{i=1}^{N} (R_i - X_i^T \hat{\beta}_r)^2
\] (4)

\[
\hat{\sigma}_w^2 = \frac{1}{N} \sum_{i=1}^{N} (W_i - Y_i^T \hat{\beta}_w)^2
\] (5)
\[
\hat{\sigma}_{rw} = \frac{1}{N} \sum_{i=1}^{N} (R_i - X_i^T \hat{\beta}_r)(W_i - Y_i^T \hat{\beta}_w)
\]  

(6)

Consider the E-step on the \((k+1)\)th iteration of the EM algorithm, where \(\theta^{(k)} = (\hat{\beta}_r, \sigma_r, \hat{\beta}_w, \hat{\beta}_w, \sigma_w, \rho)\) denotes the incumbent estimate of the parameter values \(\theta\) after the \(k\)th EM iteration. The conditional expectation of the complete-data log-likelihood is then

\[
E_{\theta^{(k)}}[L_c|D, \theta^{(k)}] = \sum_{i \in T} \frac{-1}{2} \ln(2\pi) - \frac{1}{2} \ln((\sigma_r^{(k)})^2) - \frac{1}{2(\sigma_r^{(k)})^2} E_{\theta^{(k)}}[(R - X_i^T \hat{\beta}_r)^2|R + W = 2p_i, R \leq p_i]
+ \sum_{i \in T} \frac{-1}{2} \ln(2\pi) - \frac{1}{2} \ln((\sigma_r^{(k)})^2) - \frac{1}{2(\sigma_r^{(k)})^2} E_{\theta^{(k)}}[(R - X_i^T \hat{\beta}_r)^2|W < R]
+ \sum_{i \in T} \frac{-1}{2} \ln(2\pi) - \frac{1}{2} \ln((\sigma_w^{(k)})^2) - \frac{1}{2(\sigma_w^{(k)})^2} E_{\theta^{(k)}}[(W - Y_i^T \hat{\beta}_w)^2|R + W = 2p_i, W \geq p_i]
+ \sum_{i \in T} \frac{-1}{2} \ln(2\pi) - \frac{1}{2} \ln((\sigma_w^{(k)})^2) - \frac{1}{2(\sigma_w^{(k)})^2} E_{\theta^{(k)}}[(W - Y_i^T \hat{\beta}_w)^2|W < R]
\]  

(7)

The M-step on the \((k+1)\)th iteration is implemented by replacing \(R_i, W_i, (R_i - X_i^T \hat{\beta}_r)^2\) and \((W_i - Y_i^T \hat{\beta}_w)^2\) by their current conditional expectations in (2)-(6). Hence, the \((k+1)\)th estimates of \(\beta_r\) and \(\beta_w\), \(\beta_r^{(k+1)}\) and \(\beta_w^{(k+1)}\), are specified by solving the following modified normal equations:

For all \(j = 1, 2, \ldots, M\)

\[
\sum_{i=1}^{N} \sum_{\ell=1}^{M} X_{ij} X_{i\ell} \beta_r^{(k+1)} = \sum_{i \in T} X_{ij} E_{\theta^{(k)}}[R|R + W = 2p_i, R \leq p_i] + \sum_{i \in T} X_{ij} E_{\theta^{(k)}}[R|W < R]
\]

\[
\sum_{i=1}^{N} \sum_{\ell=1}^{M} Y_{ij} Y_{i\ell} \beta_w^{(k+1)} = \sum_{i \in T} Y_{ij} E_{\theta^{(k)}}[W|R + W = 2p_i, W \geq p_i] + \sum_{i \in T} Y_{ij} E_{\theta^{(k)}}[W|W < R]
\]

And, the \((k+1)\)th estimates of \(\sigma_r\), \(\sigma_w\), and \(\sigma_{rw}\), \(\sigma_r^{(k+1)}\), \(\sigma_w^{(k+1)}\) and \(\sigma_{rw}^{(k+1)}\) are

\[
(\sigma_r^{(k+1)})^2 = \frac{1}{N} \left\{ \sum_{i \in T} E_{\theta^{(k)}}[(R - X_i^T \hat{\beta}_r)^2|R + W = 2p_i, R \leq p_i] \right\}
+ \sum_{i \in T} E_{\theta^{(k)}}[(R - X_i^T \hat{\beta}_r)^2|W < R]
\]

\[
(\sigma_w^{(k+1)})^2 = \frac{1}{N} \left\{ \sum_{i \in T} E_{\theta^{(k)}}[(W - Y_i^T \hat{\beta}_w)^2|R + W = 2p_i, W \geq p_i] \right\}
+ \sum_{i \in T} E_{\theta^{(k)}}[(W - Y_i^T \hat{\beta}_w)^2|W < R]
\]

\[
\sigma_{rw}^{(k+1)} = \frac{1}{N} \left\{ \sum_{i \in T} E_{\theta^{(k)}}[(R - X_i^T \hat{\beta}_r)(W - Y_i^T \hat{\beta}_w)|R + W = 2p_i, R \leq p_i] \right\}
+ \sum_{i \in T} E_{\theta^{(k)}}[(R - X_i^T \hat{\beta}_r)(W - Y_i^T \hat{\beta}_w)|W < R]
\]
4.2.2. Convergence and Theoretical Properties

All limit points of the EM algorithm for our model are provably stationary points of the incomplete log-likelihood function because the expected log-likelihood function (7) is continuous in \( \beta_r^{(k)}, \sigma_r^{(k)}, \beta_w^{(k)}, \sigma_w^{(k)}, \) and \( \rho^{(k)} \) (see Wu 1983). While the EM algorithm may not converge for a particular instance, if it does converge, it converges to a stationary point. Moreover, one can easily detect during the course of the algorithm whether or not it is converging. In practice, the EM algorithm has proved to be a robust and efficient way to compute maximum likelihood estimates for incomplete data problems. Also, since the EM estimates are maximum likelihood estimates, the statistical properties of maximum likelihood estimators are also valid for them, i.e., they are consistent, asymptotically unbiased and asymptotically efficient.

4.3. Numerical Tests on Synthetic Data

We conducted a series of numeric tests on synthetic data using direct maximization of the maximum likelihood function in (1) as well as the EM method with and without using Monte Carlo simulation to estimate the conditional expected values required in the E-Step. Our purpose in performing these experiments was two-fold: first, to determine whether or not the EM method converged to reasonable estimates of the underlying parameters and, second to compare the performance and robustness of the EM method to direct maximization.

We constructed 24 examples by combining three different problem dimensions:

- Number of observations: 100; 1,000; 10,000 and 100,000,
- Percentage of negotiations that resulted in a deal: \( \sim 40\% \) and \( \sim 80\% \),
- Number of covariates: 2, 5, and 10.

The mean willingness-to-pay and mean reserve price for each deal were linear in attributes. The errors were assumed to be independent with zero means and known standard deviations.

We estimated the parameters for each of the 24 instances using the EM method with two different approaches to calculating the conditional expected values required in the E-Step. In the first approach (EM), we calculated the conditional expected value numerically. In the second approach (Monte Carlo EM), we used Monte Carlo simulation to estimate conditional expectation. We also estimated the parameters by directly maximizing the incomplete log-likelihood function in MATLAB using its \textit{fmincon} function. In all cases, we initiated the methods using 10 as the initial intercept value and 0 as the initial estimates of the other coefficients. We ran the EM algorithm until either the difference of two consecutive log-likelihood function was less than \( 10^{-2} \) or the number of iterations exceeded 100. We used the default stopping criteria of the \textit{fmincon} function. In all cases, we terminated operations after 20,000 seconds (about 5 and 1/2 hours). The \textit{fmincon} function and the Monte Carlo EM method did not converge in some cases in which cases, we restarted the algorithm using a different randomly-chosen initial point. Table 1 reports the total
amount of computation time for each method applied to each instance (including multiple starting points). The times are for a laptop with Intel Core i7-3537U CPU with 2.00 GHz and 2.50 GHz processor, 8 GB RAM and 64-bit Windows operating system.

<table>
<thead>
<tr>
<th>Number of Observations</th>
<th>Instance Number</th>
<th>Runtime (sec)</th>
<th>EM</th>
<th>Monte Carlo EM</th>
<th>Direct Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1</td>
<td>3.15</td>
<td>3.48</td>
<td>1.48</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.33</td>
<td>4.50</td>
<td>0.37</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.18</td>
<td>0.78</td>
<td>0.46</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3.08</td>
<td>1.51</td>
<td>0.14</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>5.34</td>
<td>4.39</td>
<td>51.95</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7.31</td>
<td>3.64</td>
<td>0.70</td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>3.73</td>
<td>3.05</td>
<td>9.18</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| 1,000                  | 1               | 46.38         | 65.37 | 0.75           |             |
|                        | 2               | 28.09         | 16.65 | 9.45           |             |
|                        | 3               | 10.12         | 7.25  | 0.17           |             |
|                        | 4               | 24.92         | 7.70  | 0.16           |             |
|                        | 5               | 157.79        | 32.38 | 1.47           |             |
|                        | 6               | 86.01         | 48.85 | 73.46          |             |
| Average                | 58.89           | 29.70         | 14.24 |

| 10,000                 | 1               | 430.89        | 570.43 | 123.23         |             |
|                        | 2               | 1,102.12      | 155.71 | 1,034.20       |             |
|                        | 3               | 110.94        | 83.90  | 118.62         |             |
|                        | 4               | 341.30        | 92.05  | 19,659.12      |             |
|                        | 5               | 1,548.98      | 318.14 | 377.65         |             |
|                        | 6               | 1,159.15      | 540.31 | >20,000        |             |
| Average                | 782.24          | 293.42        | >6,885.47 |

| 100,000                | 1               | 4,596.66      | 5,190.19 | >20,000        |             |
|                        | 2               | 11,012.59     | 2,447.16 | 18.03          |             |
|                        | 3               | 2,125.79      | 768.79  | 2,076.20       |             |
|                        | 4               | 4,194.65      | 845.05  | 5.02           |             |
|                        | 5               | 15,480.73     | 4,158.28 | >20,000        |             |
|                        | 6               | 11,535.06     | 5,358.41 | >20,000        |             |
| Average                | 8,150.08        | 3,127.98      | >10,349.88 |

Table 1: Computation Time Comparison of the Three Estimation Methods: EM, Monte Carlo EM, and Direct Maximization of the Log-Likelihood Function

For all problem instances, the Monte Carlo EM algorithm converged to a stationary point in under 100 iterations. The EM algorithm converged in under 100 iterations for 11 instances, but it was very close to convergence when we stopped the procedure at 100 iterations for the other seven instances. For three instances with 100,000 data points and one instance with 10,000 observations, the direct maximization procedure could not find a solution after 20,000 seconds. When the methods converged (or stopped in the EM method), they always converged to parameter estimates with very similar likelihoods. The average deviation from the highest log-likelihood was 0.1% and the maximum deviation of any method on any instance was 0.79%. Hence, we did not observe a significant advantage of any procedure in terms of the quality of the parameter estimates. However,
both implementations of the EM algorithm were more reliable in converging to a solution and, on average, converged more quickly for the larger instances.

Tables 2, 3, and 4 show the true and estimated parameters of the willingness-to-pay and reserve price values for 100, 1,000, and 10,000 observations, respectively. They report the EM algorithm results for 100 and 1,000 observations (since the algorithm converged in a reasonable time) and the Monte Carlo EM algorithm results for the 10,000 observation cases (since the EM algorithm required significantly longer computation times). The average mean squared errors between the true and estimated parameters were 10.4991, 2.8293, and 0.4085 for 100, 1,000, and 10,000 observations, respectively. For these cases, the EM methods appear to be converging to the true parameter values.

<table>
<thead>
<tr>
<th>Trade pc: 30%</th>
<th>Trade pc: 50%</th>
<th>Trade pc: 70%</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>true prms</td>
<td>true prms</td>
<td>true prms</td>
</tr>
<tr>
<td>est. prms</td>
<td>est. prms</td>
<td>est. prms</td>
</tr>
<tr>
<td>W</td>
<td>W</td>
<td>W</td>
</tr>
<tr>
<td>true prms</td>
<td>true prms</td>
<td>true prms</td>
</tr>
<tr>
<td>est. prms</td>
<td>est. prms</td>
<td>est. prms</td>
</tr>
</tbody>
</table>

Table 2 True and Estimated Parameters of Willingness-to-Pay ($W$) and Reserve Price ($R$) for 100 Observations

5. Application to Auto Lending Data

We applied our structural EM method to a data set supplied by a major auto lender operating in the United States. This data set is also used in Phillips et al. (2014). The lender sells to its end customers through dealers who are located throughout the country. Although there is a centralized annual percentage rate (APR) set for each loan, in most cases the dealer has discretion to negotiate the final interest rate with customers. We excluded about 55.5% of the deals in the original data base from the analysis because they were offered at promotional rates (so-called subvened rates) over which the dealers had little or no authority to negotiate. Our structural EM method provides

---

3 We are using the word “negotiation” in a broad sense, since it is possible that some customers do not realize that the rate is negotiable and may accept (or reject) the lender’s first offer without any “back and forth discussion”.

---
Table 3 True and Estimated Parameters of Willingness-to-Pay ($W$) and Reserve Price ($R$) for 1,000 Observations

<table>
<thead>
<tr>
<th>Trade pc</th>
<th>True prms</th>
<th>Trade pc</th>
<th>True prms</th>
</tr>
</thead>
<tbody>
<tr>
<td>35%</td>
<td>$\beta_0, \beta_1, \beta_2, \sigma_r$</td>
<td>76%</td>
<td>$\beta_0, \beta_1, \beta_2, \sigma_r$</td>
</tr>
<tr>
<td>true prms</td>
<td>5 0.1 0.5 3</td>
<td>true prms</td>
<td>5 0.1 0.5 3</td>
</tr>
<tr>
<td>est. prms</td>
<td>5 0.1 0.5 3</td>
<td>est. prms</td>
<td>5 0.1 0.5 3</td>
</tr>
<tr>
<td>W</td>
<td>$\beta_0, \beta_1, \beta_2, \sigma_r$</td>
<td>W</td>
<td>$\beta_0, \beta_1, \beta_2, \sigma_r$</td>
</tr>
<tr>
<td>true prms</td>
<td>5 0.1 0.5 3</td>
<td>true prms</td>
<td>5 0.1 0.5 3</td>
</tr>
<tr>
<td>est. prms</td>
<td>5 0.1 0.5 3</td>
<td>est. prms</td>
<td>5 0.1 0.5 3</td>
</tr>
</tbody>
</table>

Table 4 True and Estimated Parameters of Willingness-to-Pay ($W$) and Reserve Price ($R$) for 10,000 Observations

<table>
<thead>
<tr>
<th>Trade pc</th>
<th>True prms</th>
<th>Trade pc</th>
<th>True prms</th>
</tr>
</thead>
<tbody>
<tr>
<td>35%</td>
<td>$\beta_0, \beta_1, \beta_2, \sigma_r$</td>
<td>76%</td>
<td>$\beta_0, \beta_1, \beta_2, \sigma_r$</td>
</tr>
<tr>
<td>true prms</td>
<td>5 0.1 0.5 3</td>
<td>true prms</td>
<td>5 0.1 0.5 3</td>
</tr>
<tr>
<td>est. prms</td>
<td>5 0.1 0.5 3</td>
<td>est. prms</td>
<td>5 0.1 0.5 3</td>
</tr>
<tr>
<td>W</td>
<td>$\beta_0, \beta_1, \beta_2, \sigma_r$</td>
<td>W</td>
<td>$\beta_0, \beta_1, \beta_2, \sigma_r$</td>
</tr>
<tr>
<td>true prms</td>
<td>5 0.1 0.5 3</td>
<td>true prms</td>
<td>5 0.1 0.5 3</td>
</tr>
<tr>
<td>est. prms</td>
<td>5 0.1 0.5 3</td>
<td>est. prms</td>
<td>5 0.1 0.5 3</td>
</tr>
</tbody>
</table>

us estimates of the distributions of willingness-to-pay and reserve price as well as estimates of the last prices (APRs) for lost deals – that is, failed transactions. We evaluate the results of using the EM approach in two ways. First, we compare the take-up rates and APRs of both successful deals and lost deals with a hold-out sample. Our benchmark is the set of APRs predicted using linear regression on deal characteristics. We also compare the take up rates predicted by the EM method with the take-up rates predicted by standard probit regression. Second, we compute the price elasticities implied by our estimates and compare these to the endogeneity-corrected estimates
in Phillips et al. (2014) for the same auto lender data. In both cases, the EM method compares favorably with the alternatives.

5.1. Data Set and Model

Our data set includes information on the outcome of all approved applications for auto loans received by a major North American indirect auto lender data during a multi-year period starting in January 2009\(^4\). As noted previously, we considered only the non-subvened deals – that is those in which rates were negotiable. There are 950,985 approvals in this data set of which 719,835 (75.69\%) were taken up. All of these loans were offered through the finance and insurance (F&I) departments of local dealerships and, in each case, the F&I manager had the authority to negotiate the APR of the loan with the customer. Each record includes the APR at which the loan was offered, the final disposition of the loan (accepted by the customer or not accepted) and information about the customer (FICO score), information about the product (make and model of car, size of loan, new or used car, term of loan, etc.), and the identity and location of the dealer.

The variables in the data and their statistics for the sample that we used are shown in Tables 5 and 6. FICO score is an industry-standard measure of individual default risk that ranges from 300 to 850 with higher values representing lower risk. The term and amount of the loan and whether or not the car being purchased is new or used are characteristics of the loan being requested. The customer rate is the final rate charged to the customer in the case of a successful negotiation and the last rate offered in the case of a lost deal. LIBOR is the interbank lending rate and is closely correlated with the lender’s cost of funds and with the primer rate. The dealer offered cash incentives for some deals. The amount of this incentive – if offered – is also recorded in the data. If the incentive was not offered, this value is 0. The lender applied a proprietary methodology to classify potential customers into five risk tiers with risk tier 1 being the least risky and risk tier 5 the most risky. Risk tier is correlated with FICO score but is not fully determined by it because the lender used additional information to classify customers into risk tiers. To create an individual risk measure that is not correlated with risk tier, we created a normalized FICO score for each customer, \( \hat{\text{FICO}} \), defined by \( \hat{\text{FICO}} = \text{Actual FICO Score} - \text{Mean FICO Score for risk tier} \). \( \hat{\text{FICO}} \) is uncorrelated with risk tier.

Applying the EM method to the entire data set would require an excessive amount of computational time. More specifically, estimating the parameters using the full data set would require 57 hours of CPU time on a laptop with Intel(R) Core(TM) i7-3537U CPU @ 2.00 GHz and 2.50 GHz processor, 8GB RAM and 64-bit Windows operating system. (We ran the EM algorithm up to

\(^4\) The lender has requested anonymity and certain details about the data have been suppressed to conceal its identity.

\(^5\) This variable is defined in the same way in Phillips et al. (2014) as well.
1,000 iterations to ensure convergence and computation time for each iteration was 206 seconds.) Therefore, we randomly selected a sample of 200,000 observed negotiation outcomes from the original data (21.03% of the “non-subvened” deals) for computations (this required 11 hours of CPU time on the same platform – 1,000 iterations with 40 sec/iteration). We randomly split the data set into estimation and holdout samples of equal size. As a first step, we estimated a single model with separate coefficients for each of the variables listed in Tables 5 and 6 allowing \( R \) and \( W \) to be correlated using the algorithm described in Section 4.2. To estimate the standard errors, we used the bootstrapping approach described in Chapter 4.6 of McLachlan and Krishnan (1996). Table 7 shows the results of the estimation.

<table>
<thead>
<tr>
<th>Variable</th>
<th>( \beta_r )</th>
<th>SE</th>
<th>t-stat</th>
<th>( \beta_w )</th>
<th>SE</th>
<th>t-stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>1.87</td>
<td>0.30</td>
<td>6.27</td>
<td>18.78</td>
<td>0.33</td>
<td>57.35</td>
</tr>
<tr>
<td>Tier 1</td>
<td>1.73</td>
<td>0.04</td>
<td>42.02</td>
<td>1.48</td>
<td>0.05</td>
<td>32.45</td>
</tr>
<tr>
<td>Tier 2</td>
<td>3.58</td>
<td>0.05</td>
<td>75.60</td>
<td>3.29</td>
<td>0.05</td>
<td>69.04</td>
</tr>
<tr>
<td>Tier 3</td>
<td>5.32</td>
<td>0.12</td>
<td>45.92</td>
<td>4.03</td>
<td>0.09</td>
<td>43.39</td>
</tr>
<tr>
<td>Tier 4</td>
<td>7.46</td>
<td>0.14</td>
<td>53.20</td>
<td>5.65</td>
<td>0.17</td>
<td>33.48</td>
</tr>
<tr>
<td>Tier 5</td>
<td>5.11</td>
<td>0.08</td>
<td>63.08</td>
<td>-1.17</td>
<td>0.08</td>
<td>-14.47</td>
</tr>
<tr>
<td>LIBOR</td>
<td>1.19</td>
<td>0.02</td>
<td>54.61</td>
<td>1.06</td>
<td>0.02</td>
<td>55.34</td>
</tr>
<tr>
<td>( FICO )</td>
<td>-1.3</td>
<td>0.03</td>
<td>-49.59</td>
<td>-1.25</td>
<td>0.03</td>
<td>-45.16</td>
</tr>
<tr>
<td>( \log \text{Amount} )</td>
<td>-0.17</td>
<td>0.03</td>
<td>-5.53</td>
<td>-1.39</td>
<td>0.03</td>
<td>-44.24</td>
</tr>
<tr>
<td>( \log \text{Std. Dev.} )</td>
<td>2.82</td>
<td>0.03</td>
<td>88.77</td>
<td>2.57</td>
<td>0.03</td>
<td>77.03</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variable</th>
<th>( \beta_r )</th>
<th>SE</th>
<th>t-stat</th>
<th>( \beta_w )</th>
<th>SE</th>
<th>t-stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term Class 2</td>
<td>0.23</td>
<td>0.06</td>
<td>3.96</td>
<td>0.72</td>
<td>0.06</td>
<td>11.33</td>
</tr>
<tr>
<td>Term Class 3</td>
<td>0.47</td>
<td>0.05</td>
<td>9.99</td>
<td>0.37</td>
<td>0.05</td>
<td>7.18</td>
</tr>
<tr>
<td>Term Class 4</td>
<td>0.67</td>
<td>0.05</td>
<td>12.77</td>
<td>0.62</td>
<td>0.06</td>
<td>10.47</td>
</tr>
<tr>
<td>( FICO ) /100</td>
<td>-1.3</td>
<td>0.03</td>
<td>-49.59</td>
<td>-1.25</td>
<td>0.03</td>
<td>-45.16</td>
</tr>
<tr>
<td>( \log \text{Amount} )</td>
<td>-0.17</td>
<td>0.03</td>
<td>-5.53</td>
<td>-1.39</td>
<td>0.03</td>
<td>-44.24</td>
</tr>
<tr>
<td>( \log \text{Cash Incen.} )</td>
<td>-0.03</td>
<td>0.04</td>
<td>-22.48</td>
<td>0.98</td>
<td>0.04</td>
<td>22.53</td>
</tr>
<tr>
<td>( \log \text{Std. Dev.} )</td>
<td>2.82</td>
<td>0.03</td>
<td>88.77</td>
<td>2.57</td>
<td>0.03</td>
<td>77.03</td>
</tr>
</tbody>
</table>

Covariance \( \sigma_{RW} \) = 0.01 (SE: 0.07; t-stat: 0.14)

Correlation Coeff. for \( R \) and \( W \) = 0.0014 (SE: 0.01; t-stat: 0.14)

Log-Likelihood (for holdout) = -28,392.9

Table 7 Coefficient Estimates for the Structural EM Method. All values are significant at the .01 level.
The values in Table 7 generally accord with intuition. The coefficient $\beta_w$ increases with risk tier and decreases with $\hat{FICO}$, indicating that customer willingness-to-pay increases with risk. This is consistent with previous findings that more risky customers are less sensitive to the price of loans than less risky borrowers (Phillips and Raffard 2011). Average willingness-to-pay decreases with $Log\ Amount$, consistent with the intuition that customers are often more likely to shop for alternatives if the amount of the loan – and hence the monthly payment – is higher. $Term$ has no consistent effect on average willingness-to-pay. On the other hand, the reserve price consistently increases in term. This may be due to the fact that customers preferring longer-term loans tend to be higher risk than those preferring shorter-term loans. Average seller reserve price increases with risk tier. This is intuitive because risk represents a cost to the seller and increased cost needs to be covered by an increased price. Seller reserve price decreases in $Log\ Amount$ – this is consistent with the common expectation that sellers should be more willing to discount larger deals. It may also reflect a revenue-bias on the part of the sales force.

The results in Table 7 indicate that reserve price and willingness-to-pay are not highly correlated with each other – the correlation coefficient between $R$ and $W$ is .0014 and the covariance is .01. This suggests an alternative model in which we estimate a different model for each risk tier assuming that the covariance between $R$ and $W$ is zero. The results for this second approach are shown in Table 8, which also shows the empirical means and standard deviations of $R$ and $W$ for each risk tier for the hold-out sample (numbers in parentheses show the bootstrapped standard errors – all the variables except $Term\ Class\ 3/4$ and $Used\ of\ Tier\ 5$ are statistically significant at the .01 significance level). As expected, the mean values of both $R$ and $W$ increase by risk tier. Henceforth, we use this model as our standard model for comparison.

The results in Table 8 show some of the same patterns as Table 7. Notably, both reserve price and willingness-to-pay increase with risk. This is evident from the fact that both the intercepts and the mean values increase with tier and the coefficients for $\hat{FICO}$ are less than zero for both $R$ and $W$. There is no longer any consistent pattern evident in the coefficients for term class for either $R$ or $W$ suggesting that the pattern in the previous model was indeed due to the correlation between risk and term. The coefficients for $Log\ Amount$ are negative for all categories except Tier 1 of $R$. The coefficients for $Used$ show a striking pattern in that they are all positive for $R$ and all negative for $W$. We conjecture that the positive coefficients for $R$ reflect the fact that used-car APR’s are rarely subvened in the marketplace and hence lenders feel they can extract a higher APR than for new cars. On the other hand, the negative coefficient for $W$ may reflect the fact that used car buyers are, in general, more price-sensitive than new car buyers. The pattern for $Cash\ Incentive$ is also interesting. Offering a cash incentive increases the willingness of borrowers to accept a higher APR for the loan, which is intuitive. It is more difficult to explain the consistently
negative coefficients for $R$. The coefficients for $LIBOR$ are all positive and are generally close to 1, reflecting that it is a cost that needs to be reflected in the APR.

### 5.2. Quality of Predictions

One of the advantages of our structural estimates is that they predict both take-up and APR. Hence, the estimates can be used not only to estimate take-up and take-up sensitivity to price, but also to impute APRs for lost deals (the last APR offered to a loan that a customer rejected) as well as predict the APRs for future deals. Below, we compare the quality of these predictions to standard regression-based approaches applied to the same data. In particular, we compare APRs predicted by our structural EM method to the actual APRs of the deals and to the APRs predicted by linear regression. We also compare the take-up predictions of our method to the take-up predictions of a two-stage endogeneity-corrected probit model, as well as to the actual realized value recorded in the data set.

#### 5.2.1. APR Predictions

Table 9 compares the APR’s predicted for deals in the test data generated by our method to the actual APRs and to the APRs predicted using linear regression applied using APR as the target variable and all of the variables listed in Tables 5 and 6 (except APR) as independent variables. This is the same approach used to generate the first-stage estimates of APR’s described in more detail in Phillips et al. (2014). The structural EM method more
accurately forecasts APRs for all risk tiers for both won and lost deals than linear regression. The amount of improvement generally increases with risk tier. This probably reflects the fact that the range of APRs offered for different deals increases in risk tier, providing more room for improvement in RMSE.

5.2.2. Take-Up Predictions

Table 10 shows the take-up rates estimated by our method compared to the observed take-up and the take-up forecast by two different probit models. In the first probit model (Probit A), we assume that prices are not available for lost deals and we use the prices imputed by linear regression. In the second probit regression (Probit B), we utilized the prices for all recorded negotiations whether they resulted in transactions or lost deals. A lender who had access to rates for “lost deals” would typically utilize that information in modeling take-up and would utilize an approach similar to the second probit model. A lender who did not have rates for “lost deals” would typically use a method similar to the first probit model. Hence, these serve as benchmarks for two different possible informational states of a lender.

Table 10 shows the take-up prediction performance of the various methods on the hold-out sample. Note the two probit approaches did not differ significantly in terms of concordance for risk tiers 1 - 3, nor for the test data set as a whole. Probit B improved the concordances for tiers 4 and 5 compared to Probit A and had a slightly higher log-likelihood for the sample as a whole. Our structural EM estimates had higher concordance in total (.928) than either of the alternatives (.921 and .922 for Probit A and B, respectively). The EM algorithm actually performed worse (in terms of both concordance and log-likelihood) in risk tiers 4 and 5 compared to Probit B, but this was outweighed by its superior performance in the much larger higher risk tiers 1 - 3.

5.3. Comparison of Elasticities

We can use the coefficients of our structural model to compute customer and seller price-elasticities. Let $F_W$ and $F_R$ denote the marginal cumulative distribution functions and $f_w$ and $f_R$ the marginal density functions for $W$ and $R$ respectively and let $\bar{F}_W = 1 - F_W$. Then, the customer price elasticity at price $p$ is $\epsilon_W(p) = -p f_W(p) / \bar{F}_W(p)$. This corresponds to the standard definition of price elasticity;
it is the percentage change per unit percentage change in price. By analogy, we can also calculate a seller price elasticity as $\epsilon_{R}(p) = \frac{pf_{R}(p)}{F_{R}(p)}$. We computed the elasticity values for both $R$ and $W$ using the empirical mean and standard deviations of $R$ and $W$ reported in Table 8 and the mean customer rate reported in Table 5. Table 11 shows both the customer and seller price elasticities for each risk tier. For comparison, Table 11 also shows the customer price elasticities estimated via the method in Phillips et al. (2014) using an endogeneity-corrected probit model for the same auto lender data.

The price-elasticities from Phillips et al. (2014) are not directly comparable because they were calculated using the entire data set while our elasticities were calculated from the data set excluding subvened deals. However, our estimates follow the expected pattern of decreasing in magnitude for larger risk tiers – again reflecting the fact that higher-risk customers are less price-sensitive than lower-risk customers. Seller price-elasticity increases in tier as well. This would indicate that sellers are, in general, less-willing to negotiate with higher risk customers, all else being equal.

6. Implications for Pricing Policy

We next consider how estimated willingness-to-pay and reserve price distributions can be used to improve profits. We show –first theoretically and then empirically– that, it can be profitable for headquarters to impose a minimum price by deal category on its sales force. Properly set, these limits can generate significant profit improvements relative to unconstrained negotiation and even relative to data-driven optimized fixed prices.
6.1. Theoretical Analysis

When the sales force is given full pricing discretion, the expected revenue from a negotiation is

\[
E(Revenue) = E\left(\frac{R+W}{2} | W \geq R \right) P\{W \geq R\}
\]

We assume that “headquarters” can impose a minimum price for each transaction on the sales force. That is, for a given negotiation, the sales agent cannot agree to a price less than \( p_{\text{min}} \). We note that \( p_{\text{min}} \) can be a function of observable covariates of a deal. The reserve price of each individual sales agent is independent of the minimum price imposed by headquarters. In this case, headquarters wishes to derive the \( p_{\text{min}}^* \) that will maximize expected revenue. We note that there are four cases given \( W \geq R \):

- Case 1: \( R \leq W \leq p_{\text{min}} \). In this case no trade will occur.
- Case 2: \( R \leq p^{eq} \leq p_{\text{min}} \leq W \). Trade will occur at \( p_{\text{min}} \)
- Case 3: \( R \leq p_{\text{min}} \leq p^{eq} \leq W \). Trade will occur at \( p^{eq} \)
- Case 4: \( p_{\text{min}} \leq R \leq W \). Trade will occur at \( p^{eq} \)

Figure 1 illustrates these cases. For cases 3 and 4, the imposition of a minimum price does not change revenue while revenue may decrease for case 1 and increase for case 2.

![Figure 1](image-url)  
**Figure 1** Revenue Change Zones Based on the Value or \( R \) and \( W \)

The expected revenue from a transaction under a minimum price policy (MPP) can be written as

\(6\) In this section we assume w.l.o.g. that incremental costs are zero so that maximizing revenue is the same as maximizing profit.
\[ E(Revenue_{MPP}) = 0 \cdot P\{R \leq W \leq p_{\text{min}}\} \]
\[ + p_{\text{min}} \cdot P\{R \leq p^{eq} \leq p_{\text{min}} \leq W\} \]
\[ + E\left(\frac{R + W}{2} | R \leq p_{\text{min}} \leq p^{eq} \leq W\right) \cdot P\{R \leq p_{\text{min}} \leq p^{eq} \leq W\} \]
\[ + E\left(\frac{R + W}{2} | p_{\text{min}} \leq R \leq W\right) \cdot P\{p_{\text{min}} \leq R \leq W\} \]

With a little bit of algebra, we can write this expression as
\[
E(Revenue_{MPP}) = p_{\text{min}} \int_{-\infty}^{\text{p}_{\text{min}}} F_r(x) f_w(2p_{\text{min}} - x) \, dx
\]
\[ + \frac{1}{2} \left[ \int_{\text{p}_{\text{min}}}^{\infty} x f_w(x) F_r(x) \, dx - \int_{\text{p}_{\text{min}}}^{\infty} x f_w(x) F_r(2p_{\text{min}} - x) \, dx \right]
\]
\[ - \frac{1}{2} \left[ \int_{\text{p}_{\text{min}}}^{\infty} x f_r(x) F_w(x) \, dx + \int_{-\infty}^{\text{p}_{\text{min}}} x f_r(x) F_w(2p_{\text{min}} - x) \, dx \right] + \frac{\mu_r}{2} \quad (8) \]

**Lemma 1.** If \( F_r \) and \( F_w \) have support on \((0, \infty)\) and if \( f_w \) is non-increasing in an interval \((a, 2a)\), then:
\[
\frac{\partial E(Revenue_{MPP})}{\partial p_{\text{min}}} \bigg|_{p_{\text{min}}=a} \leq 0
\]

**Proof:** The derivative of (8) is
\[
\frac{\partial E(Revenue_{MPP})}{\partial p_{\text{min}}} = \int_{-\infty}^{\text{p}_{\text{min}}} F_r(x) f_w(2p_{\text{min}} - x) \, dx - p_{\text{min}}f_w(p_{\text{min}})F_r(p_{\text{min}})
\]
\[ = \int_{\text{p}_{\text{min}}}^{\infty} F_r(2p_{\text{min}} - x) f_w(x) \, dx - p_{\text{min}}f_w(p_{\text{min}})F_r(p_{\text{min}}) \]

Because \( F_r \) and \( F_w \) have support only on \((0, \infty)\),
\[
\frac{\partial E(Revenue_{MPP})}{\partial p_{\text{min}}} = \int_{0}^{\text{p}_{\text{min}}} F_r(x) f_w(2p_{\text{min}} - x) \, dx - p_{\text{min}}f_w(p_{\text{min}})F_r(p_{\text{min}})
\]

By the Mean Value Theorem, there must be some \( \hat{p} \in (0, p_{\text{min}}) \) such that:
\[
\int_{0}^{\text{p}_{\text{min}}} F_r(x) f_w(2p_{\text{min}} - x) \, dx = p_{\text{min}}F_r(\hat{p}) f_w(2p_{\text{min}} - \hat{p})
\]

So,
\[
\frac{\partial E(Revenue_{MPP})}{\partial p_{\text{min}}} = p_{\text{min}} \left[ F_r(\hat{p}) f_w(2p_{\text{min}} - \hat{p}) - f_w(p_{\text{min}})F_r(p_{\text{min}}) \right]
\]

Since \( F_r(\hat{p}) \leq F_r(p_{\text{min}}) \) and \( f_w(2p_{\text{min}} - \hat{p}) \leq f_w(p_{\text{min}}) \), the partial derivative is less than or equal to 0. Q.E.D.
Corollary 1. If $W$ is distributed exponentially, then $p_{\text{min}}^* = 0$. If $R \sim \text{Uniform}(a_r, b_r)$ and $W \sim \text{Uniform}(a_w, b_w)$ with some $a_r, a_w \geq 0$, then:

- If $a_w \leq a_r$, then $p_{\text{min}}^* = 0$
- If $a_w > a_r$, then $p_{\text{min}}^* = a_w$

**Proof:** The first part follows immediately from Proposition 1. The second part follows since in case $a_w > a_r$, if $p_{\text{min}} > a_w$, then \( \frac{\partial \mathbb{E}(\text{Revenue}_{\text{MPP}})}{\partial p_{\text{min}}} \leq 0 \) by Proposition 1 and if $p_{\text{min}} < a_w$, then \( \frac{\partial \mathbb{E}(\text{Revenue}_{\text{MPP}})}{\partial p_{\text{min}}} \geq 0 \) and non-decreasing (zero if $p_{\text{min}} < \frac{a_w}{2}$ and increasing if $\frac{a_w}{2} \leq p_{\text{min}} \leq a_w$) in $p_{\text{min}}$. Finally, in case $a_w \leq a_r$, if $p_{\text{min}} > a_w$, then $p_{\text{min}}^* = a_w$

**Corollary 2.** If $W$ follows a normal distribution, then $p_{\text{min}}^* \leq \mu_W$.

When $R$ and $W$ are both normally distributed, there is no closed-form solution for $p_{\text{min}}^*$ but it can be calculated numerically. For the auto lending data, Figure 2 shows the graph of expected customer rate from a negotiation given by (8) as a function of $p_{\text{min}}$ when $R$ and $W$ are normally distributed with the estimated parameters given previously in Table 8. Note that revenue in each tier could be increased by imposing a minimum price. The increase is most significant for Tier 1 customers due to the fact that the mean reserve price is lowest for this tier. Note that, as required by Corollary 2, $p_{\text{min}}^* \leq \mu_W$ in each tier.

6.2. Policy Comparisons

Using our estimates of the willingness-to-pay and reserve price distribution, we can estimate the counter-factual change in expected profit that would have occurred from alternative policies. For a given deal $j$, we calculate the expected profit at price $p$ by:

\[
\Pi_j(p) = POP_j \ast (\text{TotalPayment}_j(p) - \text{CapitalCost}_j) - (1 - POP_j) \ast LGD \ast \text{CapitalCost}_j
\]

where $POP$ denotes the Probability of Payment which is the probability that the borrower will repay the loan in full as estimated by the lender for each prospective deal (this variable was calculated by the auto lender and recorded in the data), and $LGD$ denotes the loss-given-default ratio – that is, the expected fraction of the capital cost of the loan that will need to be written off if the borrower defaults. We set $LGD = .5$, which is the value used by the lender. The $\text{TotalPayment}_j$ is calculated by multiplying the monthly payments (calculated using the standard equation $p = Pr(1 + r)^n/[(1 + r)^n - 1]$, where $p$, $P$, $r$, and $n$ are monthly payment, initial principal, monthly APR, and term in months, respectively) by the term of the loan and $\text{CapitalCost}_j$ is obtained by calculating the total payment at the LIBOR, which is an indicator of the lender’s capital cost.

\[\text{In this example, “revenue” corresponds to the customer rate.}\]
For all three of the policies that we consider, we assume that headquarters wants to maximize total expected profit – that is, the sum of $\Pi_j(p)$ over all deals – and will set centralized prices and associated parameters that achieve that goal. Field sales will negotiate individual deals within the parameters specified by headquarters. We consider the following three policies:

1. **Centralized Fixed Pricing (CFP)**. In this policy, we assume the headquarters does not allow negotiation and sets a deal-specific rate based on observable customer and product characteristics. The price for each deal is given by:

   $$p_{j}^{CFP} = \arg\max_{p_j} \Pi_j(p_j) \cdot \mathbb{P}\{W_j \geq p_j\}. \hspace{1cm} (10)$$

2. **Negotiated Rate with Floor (NRF)**. In this policy, headquarters imposes a lower bound (or floor) on the final rate. Agents negotiate individual deals but cannot commit to a final price lower than the minimum negotiated rate. The headquarters calculates the optimal lower bound for each deal, $p_{j}^{min}$, to maximize expected profit from each deal:

   $$p_{j}^{min} = \arg\max_{p} \left\{ 0 \cdot \mathbb{P}\{R \leq W \leq p\} + \Pi_j(p) \cdot \mathbb{P}\{R \leq p^{eq} \leq p \leq W\} \right. \right.$$
   $$\left. + \mathbb{E}(\Pi_j(p^{eq})|R \leq p^{eq} \leq W)) \cdot \mathbb{P}\{R \leq p \leq p^{eq} \leq W\} \right\} \cdot \mathbb{P}\{p \leq R \leq W\},$$

   where $p_{j}^{eq} = (R + W)/2$. We denote the price for deal $j$ under the NRF policy as $p_{j}^{NRF}$. As before, if $R > W$, there will be no deal, otherwise, $p_{j}^{NRF} = \max[p_{j}^{eq}, p_{j}^{min}]$. 

---

**Figure 2** Expected Customer Rate as a Function of the Minimum Price Threshold ($p_{min}$) by Risk Tier
3. **Imposed Reserve Price (IRP) Policy.** All of our models so far have assumed that the sales agent reserve prices are exogenous. Under the IRP policy, we consider the case where headquarters sets the reserve price for each deal. To the extent that the reserve price incorporates elements of sales agent negotiating skill, this may not be feasible. However, it is certainly feasible that headquarters could influence the reserve price by, for example, setting deal-specific commission structures (e.g., headquarters could provide a bonus to sales agents based on the spread between the negotiated rate and the imposed reserve price). In this case, headquarters sets an optimal reserve price $\ell_j^*$ as:

$$
\ell_j^* = \arg \max_{\ell_j} E[\Pi_j((\ell_j + W_j)/2)|W_j \geq \ell_j] \cdot \tilde{P}\{W_j \geq \ell_j\}
$$

(12)

The expected price for each deal under the IRP policy is then $p_j^{IRP} = E[(\ell_j^* + W_j)/2|W_j \geq \ell_j^*]$.

For the CFP and NRF policies, we compute the expected counter-factual profit for each deal by $E[\Pi_j(p_i^j)] = \Pi_j(p_i^j) \cdot \tilde{P}\{W_j \geq p_i^j\}$, where $p_i^j$ denotes the price for deal $j$ under policy $i$ where $i \in \{CFP, NRF\}$, $\Pi_j(p)$ is calculated as in equation (9), and $\tilde{P}$ is the conditional take-up probability, i.e., $\tilde{P}\{W_j \geq p_i^j\} = P\{W_j \geq p_i^j | \text{customer’s decision at rate } = \text{Customer Rate}\}$. For the IRP policy, the expected counter-factual profit is computed by $E[\Pi_j((\ell_j^* + W_j)/2)|W_j \geq \ell_j^*] \cdot \tilde{P}\{W_j \geq \ell_j^*\}$.

The results of applying the three policies to the prospective deals in the test data set are shown in Table 12. “Mean Final Price” is the average APR offered across all prospective deals, the “Mean Take-up Probability” is the expected fraction of the prospective deals that would be accepted under each policy, the average profit per negotiation is the average profit realized over all negotiations (both successful and unsuccessful). All three policies showed significant increases in profit relative to what was actually achieved. We did not have full information about the actual costs per deal so some of this increase may be artificial. However, this limitation does not affect the relative results from the three alternative policies considered.

<table>
<thead>
<tr>
<th>Policy</th>
<th>Mean Final Price (APR)</th>
<th>Mean Take-Up Probability</th>
<th>Average Profit per Negotiation</th>
<th>Increase From Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centralized Fixed Pricing (CFP)</td>
<td>9.89%</td>
<td>.76</td>
<td>$3,288</td>
<td>18.10%</td>
</tr>
<tr>
<td>Negotiated Rate with Floor (NRF)</td>
<td>9.64%</td>
<td>.70</td>
<td>$3,216</td>
<td>15.52%</td>
</tr>
<tr>
<td>Imposed Reserve Price (IRP)</td>
<td>8.82%</td>
<td>.87</td>
<td>$3,973</td>
<td>42.71%</td>
</tr>
<tr>
<td>Actual</td>
<td>8.49%</td>
<td>.76</td>
<td>$2,784</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 12  Final prices, take-up probabilities and average profits for the three policies compared with the actual.

The Imposed Reserve Price policy achieved significantly higher revenue than the other two policies (and the status quo) by a large margin. It was able to both achieve higher average margins for funded deals as well as increase the expected number of total deals by lowering rates for deals
that were not accepted by customers. For deals that were actually booked, the mean customer rate was 7.6%. For the same deals, the mean rate offered under the IRP policy was 9.03% with a corresponding take-up rate of .96. For these deals, the mean profit under IRP was $4,446 compared with the actual mean profit of $3,688. IRP also lowered prices on average for deals that were lost and hence increased the acceptance rate from those deals. IRP offered an average rate of 8.17% for these deals (the actual mean rate of them was 11.22%) and converted 57.9% of them leading to additional expected profit of $2,516 for these deals, which contributed no profit in actuality.

The Centralized Fixed Pricing policy performed slightly better than the Negotiated Rate with Floor policy. This would suggest that, in this case, the benefits of centralized pricing control outweigh the benefits of allowing sales staff to negotiate prices with a simple lower bound. Still, it is worth noting that imposing an optimized lower bound on negotiations provides a significant improvement in profits (15.52%) over the status quo. This shows that the simple change of imposing a price limit on negotiations can provide significant pricing improvement, provided the negotiating limits are intelligently computed.

7. Robustness
The EM method applied to the structural negotiation model described in this paper results in estimated parameters for the distributions of \( W \) and \( R \) that meet intuition and lead to APR and take-up predictions as good or better than competing models. However, the estimation approach is based on several important structural assumptions. For one, it requires \textit{ex ante} specification of the form of the distributions of reserve price and willingness-to-pay. If these distributions are misspecified, this could lead to incorrect results. Second, it specifies that, when a transaction takes place, it takes place at the symmetric Nash bargaining equilibrium price – halfway between the willingness-to-pay and the reserve price. While the symmetric Nash bargaining equilibrium price has appealing theoretical qualities, it is conceivable that individual deals take place at prices that are closer to either the reserve price or the customer willingness-to-pay depending upon the respective bargaining skills of the customer and seller and possibly other factors. In this section we investigate the robustness of our approach to these assumptions.

7.1. Misspecification of the Joint Distribution
Our analysis of the auto lender data assumed that \( R \) and \( W \) were jointly normally distributed. To assess the robustness of our results with respect to this assumption, we generated synthetic data from different distributions with known parameters, and applied our EM algorithm to estimate the willingness-to-pay and reserve price distribution parameters, assuming (incorrectly) that the data was normally distributed. Specifically, we drew 1,000 willingness-to-pay and reserve price
observations from normal, exponential, and logistic distributions, and, for each distribution combination, estimated the willingness-to-pay and reserve price distribution parameters assuming both the willingness-to-pay and reserve price are distributed normally. We evaluated the nine instances shown in Table 13. We fixed the mean and standard deviation of the willingness-to-pay to 100, and mean and standard deviation of the reserve price to 50, and generated data from the following random variables:

\[
W_1 \sim N(100, 100), \quad W_2 \sim \text{Exp}(\lambda = 1/100), \quad \text{and} \quad W_3 \sim \text{Logistic}(\mu = 100, s = 100\sqrt{3}/\pi)
\]

where \(\mu\) is the location and \(s\) is the scale parameter; similarly, \(R_1 \sim N(50, 50), \quad R_2 \sim \text{Exp}(\lambda = 1/50), \quad \text{and} \quad R_3 \sim \text{Logistic}(\mu = 50, s = 50\sqrt{3}/\pi)\). We used the same criteria for the stopping conditions of the EM algorithm as before.

Table 13 shows the results. In Table 13, \(\mu_p\) and \(\sigma_p\) correspond to the mean and standard deviation of the prices realized among the 1,000 observations (for the deals where the trade occurs), the column labeled “Tr. %” shows the corresponding trade percentage among these 1,000 observations. \(\hat{\mu}_p = E(p|W \geq R)\) and \(\hat{\sigma}_p = \sqrt{\text{Var}(p|W \geq R)}\) and the last column corresponds to the predicted trade probabilities when \(R\) and \(W\) are both normally distributed with mean and standard deviations that are taken as the estimated parameters in columns 6-9. In all cases, the parameters estimated by the EM method generated price distributions and trade percentages that closely matched those in the holdout sample. Estimation of the means and standard deviations of the underlying distributions were less robust to the forms of the distributions. When the actual distributions were normal, the errors in estimating the means of the reserve price and willingness-to-pay distributions were 5.04% and 2.94% respectively. In the worst cases, the corresponding errors were 50.92% (Exponential/Logistic) and 20.64% (Normal/Exponential) respectively. This degradation in performance is likely due to the very different form of the exponential distribution relative to the normal.
7.2. Asymmetric Bargaining

In our model, we assumed that the status quo utilities (i.e. the utility obtained if one decides not to bargain with the other player) of the seller and the customer are equal, so that for cases where the trade takes place, the Nash bargaining solution is symmetric, i.e.,

\[ R \leq W \implies p = 0.5R + 0.5W \]

The generalized Nash bargaining solution (Binmore et al. 1986) specifies that the price for a particular deal will be given by

\[ p = \alpha R + (1 - \alpha)W \]

for some \( \alpha \in (0,1) \). If \( \alpha < 0.5 \), it can be interpreted as the sellers’ having systematically more bargaining power than the customers and, if \( \alpha > 0.5 \), the customers would have more bargaining power. Ideally, we would like to estimate the value of \( \alpha \) from the data. Unfortunately, the following proposition shows that \( \alpha \) is not identifiable from data and therefore must be assumed.

**Proposition 2.** Assume that the true values of the model primitives are \( R_1, R_2, \ldots, R_N \) and \( W_1, W_2, \ldots, W_N \) and \( \alpha \) and that a trade occurs for observations \( 1, 2, \ldots, n \) at rate \( p_i = (1 - \alpha)R_i + \alpha W_i \) with \( W_i \geq R_i \), and no trade occurs for the remaining observations implying that \( W_i < R_i \) for \( i = n + 1, \ldots, N \). Then, the model primitives are not jointly identifiable.

**Proof:** Fix the \( W \)'s and assume that we consider some \( \hat{\alpha} \neq \alpha \) for some \( 0 \leq \hat{\alpha} \leq 1 \). Define

\[ \hat{R}_i = \frac{(\alpha - \hat{\alpha})W_i + (1 - \alpha)R_i}{1 - \hat{\alpha}} \]

We will show that the underlying variables \( \hat{R}_1, \hat{R}_2, \ldots, \hat{R}_N \) and \( W_1, W_2, \ldots, W_N \) and \( \hat{\alpha} \) will produce exactly the same set of observations (prices and no-trade outcomes) as the “real” values. For \( i = 1, 2, \ldots, n \):

\[
(1 - \hat{\alpha})\hat{R}_i + \hat{\alpha}W_i = (1 - \hat{\alpha})\left(\frac{(\alpha - \hat{\alpha})W_i + (1 - \alpha)R_i}{1 - \hat{\alpha}}\right) + \hat{\alpha}W_i = \alpha W_i - \hat{\alpha}W_i + R_i - \alpha R_i + \hat{\alpha}W_i = \alpha W_i + (1 - \alpha)R_i = p_i
\]

And for \( i = n + 1, \ldots, N \), let \( \gamma = \frac{\alpha - \hat{\alpha}}{1 - \hat{\alpha}} \). Then, \( \gamma \leq 1 \) and:

\[
\hat{R}_i = \gamma W_i + (1 - \gamma)R_i = R_i + \gamma(W_i - R_i) \geq W_i \text{ since } W_i - R_i < 0 \text{ and } \gamma \leq 1
\]

**Q.E.D.**
This raises the question of whether or not an incorrectly specified value of $\alpha$ would lead to significant misestimation of the parameters of the underlying distribution. To address this question, we generated synthetic data for three values of $\alpha = 0.2, 0.5, \text{ and } 0.8$ and normally distributed values for $R$ and $W$. We then estimated the model parameters $\mu_r, \mu_w, \sigma_r, \text{ and } \sigma_w$ using the EM algorithm and assumed values of $\hat{\alpha}$ from 0.1 to 0.9 at increments of 0.1. The results are shown in Table 14 for true values $(\mu_r, \sigma_r, \mu_w, \sigma_w) = (58, 6, 55, 10)$ and $(\mu_r, \sigma_r, \mu_w, \sigma_w) = (50, 5, 60, 10)$. The former set of parameter values corresponds to 40% trade probability and the latter set corresponds to 80% trade probability.

<table>
<thead>
<tr>
<th>$\alpha = 0.2$</th>
<th>True Parameters</th>
<th>$\mu_r$</th>
<th>$\sigma_r$</th>
<th>$\mu_w$</th>
<th>$\sigma_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}$</td>
<td>EM Estimations</td>
<td>$\hat{\mu}_r$</td>
<td>$\hat{\sigma}_r$</td>
<td>$\hat{\mu}_w$</td>
<td>$\hat{\sigma}_w$</td>
</tr>
<tr>
<td>0.1</td>
<td>56.99</td>
<td>4.89</td>
<td>54.22</td>
<td>9.27</td>
<td>-6672.10</td>
</tr>
<tr>
<td>0.2</td>
<td>56.99</td>
<td>4.81</td>
<td>54.17</td>
<td>10.07</td>
<td>-6718.01</td>
</tr>
<tr>
<td>0.3</td>
<td>55.28</td>
<td>4.25</td>
<td>53.21</td>
<td>12.07</td>
<td>-6774.54</td>
</tr>
<tr>
<td>0.4</td>
<td>54.46</td>
<td>4.23</td>
<td>52.74</td>
<td>13.71</td>
<td>-6897.76</td>
</tr>
<tr>
<td>0.5</td>
<td>54.19</td>
<td>4.04</td>
<td>51.49</td>
<td>17.06</td>
<td>-7071.26</td>
</tr>
<tr>
<td>0.6</td>
<td>56.27</td>
<td>4.28</td>
<td>50.97</td>
<td>20.37</td>
<td>-7304.71</td>
</tr>
<tr>
<td>0.7</td>
<td>56.21</td>
<td>4.02</td>
<td>49.54</td>
<td>26.34</td>
<td>-7501.00</td>
</tr>
<tr>
<td>0.8</td>
<td>56.80</td>
<td>4.47</td>
<td>47.01</td>
<td>36.46</td>
<td>-7931.36</td>
</tr>
<tr>
<td>0.9</td>
<td>58.95</td>
<td>5.50</td>
<td>48.02</td>
<td>42.58</td>
<td>-8294.12</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha = 0.5$</th>
<th>True Parameters</th>
<th>$\mu_r$</th>
<th>$\sigma_r$</th>
<th>$\mu_w$</th>
<th>$\sigma_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}$</td>
<td>EM Estimations</td>
<td>$\hat{\mu}_r$</td>
<td>$\hat{\sigma}_r$</td>
<td>$\hat{\mu}_w$</td>
<td>$\hat{\sigma}_w$</td>
</tr>
<tr>
<td>0.1</td>
<td>57.63</td>
<td>6.46</td>
<td>55.22</td>
<td>6.44</td>
<td>-6565.73</td>
</tr>
<tr>
<td>0.2</td>
<td>55.55</td>
<td>4.33</td>
<td>52.81</td>
<td>9.57</td>
<td>-6561.17</td>
</tr>
<tr>
<td>0.3</td>
<td>55.29</td>
<td>4.08</td>
<td>52.31</td>
<td>11.14</td>
<td>-6654.65</td>
</tr>
<tr>
<td>0.4</td>
<td>55.79</td>
<td>4.31</td>
<td>52.53</td>
<td>12.13</td>
<td>-6794.94</td>
</tr>
<tr>
<td>0.5</td>
<td>55.48</td>
<td>4.21</td>
<td>51.42</td>
<td>15.31</td>
<td>-7066.49</td>
</tr>
<tr>
<td>0.6</td>
<td>55.61</td>
<td>4.20</td>
<td>50.62</td>
<td>19.26</td>
<td>-7230.15</td>
</tr>
<tr>
<td>0.7</td>
<td>55.44</td>
<td>4.98</td>
<td>47.98</td>
<td>28.48</td>
<td>-7568.02</td>
</tr>
<tr>
<td>0.8</td>
<td>57.57</td>
<td>4.88</td>
<td>49.80</td>
<td>30.40</td>
<td>-7837.32</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha = 0.8$</th>
<th>True Parameters</th>
<th>$\mu_r$</th>
<th>$\sigma_r$</th>
<th>$\mu_w$</th>
<th>$\sigma_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}$</td>
<td>EM Estimations</td>
<td>$\hat{\mu}_r$</td>
<td>$\hat{\sigma}_r$</td>
<td>$\hat{\mu}_w$</td>
<td>$\hat{\sigma}_w$</td>
</tr>
<tr>
<td>0.1</td>
<td>55.92</td>
<td>7.06</td>
<td>53.69</td>
<td>5.81</td>
<td>-6550.62</td>
</tr>
<tr>
<td>0.2</td>
<td>54.49</td>
<td>5.23</td>
<td>52.24</td>
<td>7.15</td>
<td>-6459.69</td>
</tr>
<tr>
<td>0.3</td>
<td>54.09</td>
<td>4.80</td>
<td>51.70</td>
<td>8.24</td>
<td>-6515.71</td>
</tr>
<tr>
<td>0.4</td>
<td>54.20</td>
<td>4.63</td>
<td>51.62</td>
<td>9.05</td>
<td>-6573.37</td>
</tr>
<tr>
<td>0.5</td>
<td>53.69</td>
<td>4.28</td>
<td>50.57</td>
<td>11.35</td>
<td>-6721.09</td>
</tr>
<tr>
<td>0.6</td>
<td>54.42</td>
<td>4.52</td>
<td>50.84</td>
<td>12.22</td>
<td>-6850.56</td>
</tr>
<tr>
<td>0.7</td>
<td>53.76</td>
<td>4.21</td>
<td>49.35</td>
<td>17.02</td>
<td>-7110.46</td>
</tr>
<tr>
<td>0.8</td>
<td>53.91</td>
<td>4.27</td>
<td>47.82</td>
<td>23.47</td>
<td>-7445.59</td>
</tr>
<tr>
<td>0.9</td>
<td>54.87</td>
<td>4.57</td>
<td>47.00</td>
<td>31.46</td>
<td>-7806.27</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha = 0.1$</th>
<th>True Parameters</th>
<th>$\mu_r$</th>
<th>$\sigma_r$</th>
<th>$\mu_w$</th>
<th>$\sigma_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha}$</td>
<td>EM Estimations</td>
<td>$\hat{\mu}_r$</td>
<td>$\hat{\sigma}_r$</td>
<td>$\hat{\mu}_w$</td>
<td>$\hat{\sigma}_w$</td>
</tr>
<tr>
<td>0.1</td>
<td>58.95</td>
<td>5.50</td>
<td>48.02</td>
<td>42.58</td>
<td>-8294.12</td>
</tr>
<tr>
<td>0.2</td>
<td>56.80</td>
<td>4.47</td>
<td>47.01</td>
<td>36.46</td>
<td>-7931.36</td>
</tr>
<tr>
<td>0.3</td>
<td>55.61</td>
<td>4.20</td>
<td>47.98</td>
<td>28.48</td>
<td>-7568.02</td>
</tr>
<tr>
<td>0.4</td>
<td>55.44</td>
<td>3.98</td>
<td>47.39</td>
<td>23.47</td>
<td>-7445.59</td>
</tr>
<tr>
<td>0.5</td>
<td>54.87</td>
<td>4.57</td>
<td>47.00</td>
<td>31.46</td>
<td>-7806.27</td>
</tr>
</tbody>
</table>

*LL: Log-Likelihood

Table 14 Asymmetric Bargaining Synthetic Data Test
In these cases, assuming $\alpha = 0.5$ does a reasonable job of estimating the underlying parameters even when the true value of $\alpha$ is 0.2 or 0.8. In fact, for some cases in which the actual value of $\alpha$ is not 0.5, using a value of $\alpha = 0.5$ in the estimation actually results in estimates of parameters that are closer to their true values than using the true value of $\alpha$. Additionally, the log-likelihood is not always the highest for the estimations using the true values of $\alpha$, reflecting the fact that $\alpha$ is not identifiable from the data. It also notable that, for each set of estimated parameters, the predicted values of $\hat{\mu}_p = E(p|W \geq R)$, $\hat{\sigma}_p = \sqrt{\text{Var}(p|W \geq R)}$, and trade probabilities are very close to their true values.

8. Discussion and Conclusions

In this paper, we proposed a structural EM method to estimate the joint distribution of customer willingness-to-pay and seller reserve prices from a series of bi-lateral price negotiations. The resulting distributions can be used to predict both the take-up probability and the price of negotiations. We showed that our method successfully uncovers the parameters of willingness-to-pay and reserve price distributions from synthetic data given a sufficient number of observations. When applied to a real-world auto-lending data set, our method generally predicted take-up and APRs on a test data set better than standard regression-based approaches. Furthermore, our estimates provide interesting insights into the reserve price distribution among sales representatives and how it varied as a function of deal characteristics.

While our structural EM method has its strengths, the EM algorithm is notoriously computationally intensive and can be slow to converge. Computational considerations limited us to using only 200,000 observations out of 950,985 available in the estimation process. In this case, the results using the smaller sample were still an improvement over direct maximization of the likelihood, but the computational intensity of the approach could be a limitation in other settings. In addition, our approach requires a priori specification of the bargaining parameter $\alpha$ and the underlying form of the distributions of $R$ and $W$. While robustness tests suggest that the results are reasonably insensitive to the form of the joint distribution and the value of $\alpha$, severe misspecification (e.g. normal for exponential) can lead to misleading results. Both the development of improved algorithms and techniques for choosing appropriate distributions and values of the bargaining parameter are useful directions for future research.

Lastly, we used our estimation method to perform a counter-factual analysis of pricing policies on a real-world data set of auto loans. The analysis showed if local sales staff are given pricing discretion, significant profit improvements can be achieved by providing an optimized lower bound on the negotiated price. The lower-bound prevents sales staff with very low reservation prices from dropping the final price too low. These results also provide an explanation for the commonly-observed
practice of providing sales staff limited discretion to change prices relative to a centrally-established price list. Yet even with optimized lower bounds, deal-specific optimized fixed prices generated slightly higher profits than negotiated prices. However, the highest level of profits observed in our counterfactual analysis occurred when headquarters was allowed to set optimized reserve prices for each deal and individual sales staff then negotiated deals given those optimized reserve prices. This last result suggests that a combination of data-driven pricing controls and sales agent negotiation can produce higher profits than either approach alone. How to achieve such results in practice through appropriate training, controls, and sales force incentives is an open topic for further study.

**References**


