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We study firms that sell multiple differentiated substitutable products and customers whose purchase behavior follows a Nested Logit model, of which the Multinomial Logit model is a special case. Customers make purchasing decision sequentially under the Nested Logit model: they first select a nest of products and subsequently purchase a product within the selected class. We consider the general Nested Logit model with product-dependent price sensitivities and general nest coefficients. The problem is to price the products to maximize expected profits. We show that the *adjusted markup*, defined as price minus cost minus the reciprocal of the product's price sensitivity, is constant for all products within a nest at optimality. This reduces the problem's dimension to a single variable per nest. We also show that each nest has an *adjusted nest-level markup* that is nest invariant, which further reduces the problem to a single variable optimization of a continuous function over a bounded interval. We provide conditions for this function to be uni-modal. We also use this result to simplify the oligopolistic price competition and characterize the Nash equilibrium (NE) and provide conditions under which the Tatonnement process converges to the unique NE. Furthermore, we investigate its application to multi-product dynamic pricing and revenue management, as well as extensions to more general attraction functions including linear and constant elasticity of substitution.

Key words: multi-product pricing; Attraction model; Nested Logit model; Multinomial Logit model; product-differentiated price sensitivity; substitutable products

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1. Introduction

Firms offering a menu of differentiated substitutable products face the problem of pricing them to maximize profits. This becomes more complicated with rapid technology development as new products are constantly introduced into the market and typically have a short life cycle. In this paper we are concerned with the problem of maximizing expected profits when customers follow a nested choice model where they first select a nest of products and then a product within the nest. The selection of nests and products depend on brand, product features, quality and price. The Nested Logit (NL) model and its special case the Multinomial Logit (MNL) model are among the most popular models to study purchase behavior of customers who face multiple substitutable products. The main contribution of this paper is to find very efficient solutions for a very general class of NL models and to explore the implications for oligopolistic competition and dynamic pricing.

The MNL model has received significant attention by researchers from economics, marketing, transportation science and operations management, and it has motivated tremendous theoretical research and empirical validations in a large range of applications since it was first proposed by McFadden (1974), who was later awarded the 2000 Nobel Prize in Economics. The MNL model has been derived from an underlying random utility model, which is based on a probabilistic model of individual customer utility. Probabilistic choice can model customers with inherently unpredictable behavior that shows probabilistic tendency to prefer one alternative to another. When there is a random component in a customer' utility or a firm has only probabilistic information on the utility function of any given customer, the MNL model describes customers' purchase behavior very well.

The MNL model has been widely used as a model of customer choice, but it severely restricts the correlation patterns among choice alternatives and may behave badly under certain conditions (Williams and Ortuzar 1982), in particular when alternatives are correlated. This restrictive property is known as the *independence of irrelevant alternatives* (IIA) property (see Luce 1959). If the choice set contains alternatives that can be grouped such that alternatives within a group are more similar than alternatives outside the group, the MNL model is not realistic because adding new alternative reduces the probability of choosing similar alternatives more than dissimilar alternatives. This is often explained with the famous "red-bus/blue-bus" paradox (see Debreu 1952).

The NL model has been developed to relax the assumption of independence between all the alternatives, modeling the "similarity" between "nested" alternatives through correlation on utility components, thus allowing differential substitution patterns within and between nests. The NL model has become very useful on contexts where certain options are more similar than others, although the model lacks computational and theoretical simplicity. Williams (1977) first formulated the NL model and introduced structural conditions associated with its inclusive value parameters, which are necessary for the compatibility of the NL model with utility maximizing theory. He formally derived the NL model as a descriptive behavioral model completely coherent with basic micro-economic concepts. McFadden (1980) generated the NL model as a particular case of the generalized extreme value (GEV) discrete-choice model family and showed that it is numerically equivalent to Williams (1977). The NL model can also be derived starting from Gumbel marginal functions. Later on, Daganzo and Kusnic (1993) pointed out that although the conditional probability may be derived from a logit form, it is not necessary that the conditional error distribution be Gumbel. To keep consistent with micro-economic concepts, like random utility maximization, certain restrictions on model parameters that control the correlation among unobserved attributes have to be satisfied. One of the restrictions is that nest coefficients are required to lie within the unit interval, i.e., $0 \le \gamma_i \le 1 \ \forall i$.

Multi-product price optimization under the NL model and the MNL model has been the subject of active research since the models were first developed. Hanson and Martin (1996) show that the profit function for a firm selling multiple differentiated substitutable products under the MNL model is not jointly concave in the price vector. While the objective function is not concave in prices, it turns out to be concave with respect to the market share vector, which is in one-to-one correspondence with the price vector. To the best of our knowledge, this result is first established by Song and Xue (2007) and Dong et al. (2009) in the MNL model and by Li and Huh (2011) in the NL model. In all of their models, the price-sensitivity parameters are assumed identical for all the products within a nest and the nest coefficients are restricted to be in the unit interval. Empirical studies have shown that the product-specified price sensitivity may vary widely and recognized the importance of allowing different price sensitivities in the MNL model (see Berry et al. 1995 and Erdem et al. 2002). Borsch-Supan (1990) points out that the restriction for nest coefficients in the unit interval leads too often rejection of the NL model. Unfortunately, the concavity with respect to the market share vector is lost when price-sensitivity parameters are product-differentiated or nest coefficients are greater than one as shown through an example in Appendix A.

Under the MNL model with identical price-sensitivity parameters, it has been observed that the *markup*, that is price minus cost, is constant across all the products of the firm at the optimal solution (see Anderson and de Palma 1992, Aydin and Ryan 2000, Hopp and Xu 2005 and Gallego and Stefanescu 2011). The profit function is uni-modal and there exist a unique optimal solution, which can be found by solving the first order conditions (see Aydin and Porteus 2008, Akcay et al. 2010 and Gallego and Stefanescu 2011). In this paper, we consider the general NL model with product-differentiated price-sensitivity parameters and general nest coefficients. We show that the *adjusted markup*, which is defined as price minus cost minus the reciprocal of the price sensitivity, is constant across all the products in each nest at optimal (locally or globally) prices. When optimizing multiple nests of products, the *adjusted nest-level markup*, which is an adjusted average markup for all the products in the same nest, is also constant for each nest. By using this result, the multi-product and the multi-nest optimizations can be reduced to a single-dimensional problem of maximizing a continuous function over a bounded interval. We also provide mild conditions under which the single-dimensional problem is uni-modal, further simplifying the problem.

In a game-theoretic decentralized framework, the existence and uniqueness of a pure Nash equilibrium in a price competition model depend fundamentally on the demand functions as well as the cost structure. Milgrom and Roberts (1990) identify a rich class of demand functions, including the MNL model, and point out that the price competition game is supermodular, which guarantees the existence of a pure Nash equilibrium. Bernstein and Federgruen (2004) and Federgruen and Yang (2009) extend this result for a generalization of the MNL model referred to as the attraction model. Gallego et al. (2006) provide sufficient conditions for the existence and uniqueness of a Nash equilibrium under the cost structure that is increasing convex in the sale volume. Liu (2006), Cachon and Kok (2007) and Kok and Xu (2011) consider the NL model with identical price sensitivities for the products of the same firm and have characterized the Nash equilibrium. Moreover, Li and Huh (2011) study the same model with nest coefficients $0 \leq \gamma_i \leq 1$ and have derived the unique equilibrium in a closed-form expression involving the Lambert W function (see Corless et al. 1996). In all these models, the product-specified price-sensitivity parameters for the products of the same firm are assumed identical. This paper considers competition under the general NL model and shows that the multi-product price competition is equivalent to a log-supermodular game in a single-dimensional strategy space.

In addition to monopoly and oligopoly pricing under the NL model, we also consider an application to dynamic pricing and an extension to more general nested attraction models. Multi-product dynamic pricing problems have been popular (see Gallego and van Ryzin 1997) and some heuristics have been developed to solve this complicated dynamic program. We show how dynamic pricing can be done under the NL model by using the fact that the *adjusted markup* is constant for all the products in each nest. This allow us to simplify the multi-product dynamic pricing problem under the NL to the classic dynamic pricing problem for a single product with concave revenue rate. We also consider the extension to more general attraction functions other than the exponential one that leads to the NL model. In particular, we show that the optimization also reduces to a single variable when the attractions are linear or have constant elasticity of substitution.

The remainder of this paper is organized as follows. In Section 2, we consider the general Nested Logit model and show that the *adjusted markup* is constant across all the products of a nest. Moreover, the *adjusted nest-level markup* is also constant for each nest in a multi-nest optimization problem. In Section 3, we investigate the oligopolistic price competition problem, where each firm controls a nest of substitutable products. A Nash equilibrium exists for the general NL model and sufficient conditions for the uniqueness of the equilibrium are also provided. Section 4 is the application in multi-product dynamic pricing under the framework of revenue management. In Section 5, we consider an extension to other Nested Attraction models and conclude with a summary of our main results and useful management insights for application in business.

2. Nested Logit Model

The Multinomial Logit (MNL) model has been widely used to study customer choice behavior in marketing, economics, transportation science and operations management. However, it exhibits the *Independence of Irrelevant Alternatives* (IIA) property, which implies that the ratio of probabilities of choosing any two alternatives is independent of the availability and attributes of a third alternative. In reality, adding new alternative or reducing the price of an alternative hurt the similar alternatives more than dissimilar alternatives and the empirical studies have shown that the MNL model doesn't work well when a firm has multiple substitutable products, in particularly when the products are correlated. The Nested Logit (NL) model is a popular generalization of the standard MNL model and its structure with a two-stage process alleviates the IIA property. Under the NL model, the customers make product selection decisions sequentially: at the upper level, they first select a branch, called a "nest" that includes multiple similar products; at the lower level, their subsequent selection is within that chosen nest (see McFadden 1976, Carrasco and Ortuzar 2002 and Greene 2007). The IIA property no longer holds when the two alternatives don't belong to the same nest.

Suppose that the substitutable products constitute n nests and nest i has m_i products. Customers' product selection behavior follows the NL model: they first select a nest and then choose a product within their chosen nest. Let $Q_i(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ be the probability that a customer selects nest i at the upper level; and let $q_{k|i}(\mathbf{p}_i)$ denote the probability that product k of nest i is selected at the lower level, given that the customer selects nest i at the upper level, where $\mathbf{p}_i = (p_{i1}, p_{i2}, \ldots, p_{im_i})$ is the price vector for all the products in nest i. Following Williams (1977), McFadden (1980) and Greene (2007), $Q_i(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ and $q_{k|i}(\mathbf{p}_i)$ are defined as follows:

$$Q_i(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \frac{e^{\gamma_i I_i}}{1 + \sum_{l=1}^{n} e^{\gamma_l I_l}}, \tag{1}$$

$$q_{k|i}(\mathbf{p}_i) = \frac{e^{\alpha_{ik} - \beta_{ik} p_{ik}}}{\sum_{s=1}^{m_i} e^{\alpha_{is} - \beta_{is} p_{is}}}, \tag{2}$$

where α_{is} can be interpreted as the "quality" of product s in nest i, $\beta_{is} \geq 0$ is the product-specified price sensitivity for that product, $I_l = \log \sum_{s=1}^{m_l} e^{\alpha_{ls} - \beta_{ls} p_{ls}}$ represents the attractiveness of nest l, which is the expected value of the maximum of the utilities of all the products in nest l (see Anderson et al. 1992), and nest coefficient γ_i can be viewed as the degree of inter-nest heterogeneity. When $0<\gamma_i<1$, products are more similar within nest i than cross nests; when $\gamma_i=1$, products in nest i have the same degree of similarity as products in other nests, and the NL model reduces to the standard MNL model; when $\gamma_i>1$, products are more similar to the ones in other nests. The probability that a customer will select product k of nest i, which can also be considered the market share of that product, is

$$\pi_{ik}(\mathbf{p}_1,\dots,\mathbf{p}_n)=Q_i(\mathbf{p}_1,\dots,\mathbf{p}_n)\cdot q_{k|i}(\mathbf{p}_i). \tag{3}$$

Note that $\sum_{k=1}^{m_i} q_{k|i}(\mathbf{p}_i)=1$ and $\sum_{k=1}^{m_i}\pi_{ik}(\mathbf{p}_1,\dots,\mathbf{p}_n)=Q_i(\mathbf{p}_1,\dots,\mathbf{p}_n)$. Liu (2006) and Li and Huh (2011) consider a special case of the NL model, where the price-sensitivity parameters β_{is} are identical for all the products of nest i.

It is easily verified that, the price sensitivities of the market shares with respect to prices are given by

$$\begin{aligned}
\frac{\partial\pi_{ij}(\mathbf{p}_1,\dots,\mathbf{p}_n)}{\partial p_{ij}} &= -\beta_{ij}\pi_{ij}(\mathbf{p}_1,\dots,\mathbf{p}_n)\big((1-q_{j|i}(\mathbf{p}_i))+\gamma_i(1-Q_i(\mathbf{p}_1,\dots,\mathbf{p}_n))q_{j|i}(\mathbf{p}_i)\big)\le 0,\\
\frac{\partial\pi_{ij}(\mathbf{p}_1,\dots,\mathbf{p}_n)}{\partial p_{ik}} &= \beta_{ik}\pi_{ij}(\mathbf{p}_1,\dots,\mathbf{p}_n)q_{k|i}(\mathbf{p}_i)\big(1-\gamma_i(1-Q_i(\mathbf{p}_1,\dots,\mathbf{p}_n))\big),\ \forall\ k\neq i,\\
\frac{\partial\pi_{ij}(\mathbf{p}_1,\dots,\mathbf{p}_n)}{\partial p_{ls}} &= \beta_{ls}\gamma_l\pi_{ij}(\mathbf{p}_1,\dots,\mathbf{p}_n)\pi_{ls}(\mathbf{p}_1,\dots,\mathbf{p}_n)\ge 0,\ \forall\ l\neq i.
\end{aligned}$$

Each product's market share is decreasing in its own price and increasing in the prices of the products in other nests. The sign of the derivative of a product's market share with respect to the prices of other products in the same nest depends on the sign of $1-\gamma_i(1-Q_i(\mathbf{p}_1,\dots,\mathbf{p}_n))$: it is positive if $\gamma_i\le 1$; it is negative if γ_i is sufficiently large.

The NL model is a generalization of the standard MNL model, which corresponds to the case when $\gamma_i=1$ for all $i=1,2,\dots,n$, and

$$\pi_{ik}(\mathbf{p}_1,\dots,\mathbf{p}_n)=\frac{e^{\alpha_{ik}-\beta_{ik}p_{ik}}}{1+\sum_{l=1}^{n}\sum_{s=1}^{m_l}e^{\alpha_{ls}-\beta_{ls}p_{ls}}}.$$

Without loss of generality, assume that the market size is normalized to 1. For the NL model, the monopolist's problem is to determine the price vectors $(\mathbf{p}_1,\dots,\mathbf{p}_n)$ to maximize the total expected profit

$$R(\mathbf{p}_1,\dots,\mathbf{p}_n)\stackrel{\text{def}}{=}\sum_{i=1}^{n}\sum_{k=1}^{m_i}(p_{ik}-c_{ik})\pi_{ik}(\mathbf{p}_1,\dots,\mathbf{p}_n). \tag{4}$$

We will later also consider the oligopolist problem where each firm controls one or more nests.

2.1. Optimization over a Single Nest

Here we consider the problem of optimizing the profits of nest i assuming the prices of all the other nests are fixed. This problem may arise if a firm controls a single nest and the other nests are controlled by other firms, or if a nest is controlled by a manager within a firm that takes the other nest prices as fixed. We will later leverage the results obtained for nest i to deal with the monopolist's problem of maximizing $R(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ and to deal with the oligopolist problem where each firm controls one or more nests. Given the price vectors of other nests $\mathbf{p}_{\text{-}i} = (\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n)$, the problem for nest i is to maximize her expected profit:

$$R_i(\mathbf{p}_i, \mathbf{p}_{\text{-}i}) \stackrel{\text{def}}{=} \sum_{k=1}^{m_i} (p_{ik} - c_{ik}) \pi_{ik}(\mathbf{p}_i, \mathbf{p}_{\text{-}i}). \tag{5}$$

The profit function $R_i(\mathbf{p}_i, \mathbf{p}_{\text{-}i})$ is not quasi-concave in $\mathbf{p}_i$ (see Hanson and Martin 1996 for a counterexample), so other researchers, including Song and Xue (2007), Dong et al. (2009), have taken different approaches to establish the structure of the MNL profit function. They express profit as a function of market shares and show that it is jointly concave with respect to market shares. These authors assume identical price sensitivities with each nest, but the profit function is not jointly concave when the price sensitivities are allowed to be product dependent, as we do, within each nest. The Appendix A provides the analysis and an example where the objective function fails to be jointly concave.

We will next take a different approach to consider the price optimization problem under the general NL model. The first order condition (FOC) of the profit function (5) is

$$\begin{aligned}\frac{\partial R_i(\mathbf{p}_i, \mathbf{p}_{\text{-}i})}{\partial p_{ij}} = \pi_{ij}(\mathbf{p}_i, \mathbf{p}_{\text{-}i}) \cdot \Big[& 1 - \beta_{ij}(p_{ij} - c_{ij}) + \beta_{ij}\big(1 - \gamma_i(1 - Q_i(\mathbf{p}_i, \mathbf{p}_{\text{-}i}))\big) \\ & \cdot \sum_{s=1}^{m_i} (p_{is} - c_{is}) q_{s|i}(\mathbf{p}_i) \Big] = 0. \end{aligned} \tag{6}$$

Roots of the FOC (6) can be obtained by either setting the inner of the square bracket term to zero, resulting in

$$1 - \beta_{ij}(p_{ij} - c_{ij}) + \beta_{ij}\big(1 - \gamma_i(1 - Q_i(\mathbf{p}_i, \mathbf{p}_{\text{-}i}))\big) \cdot \sum_{s=1}^{m_i} (p_{is} - c_{is}) q_{s|i}(\mathbf{p}_i) = 0, \tag{7}$$

or by setting $\pi_{ij}(\mathbf{p}_i, \mathbf{p}_{\text{-}i}) = 0$ which requires $p_{ij} = \infty$. There are 2^{m_i} potential solutions to the FOC depending on the set F_i of products with finite prices. We first consider price optimization given the set F_i of products with finite prices and will later show that is optimal to select $F_i = \{1, \ldots, m_i\}$. We will also show that the problem of finding optimal finite prices $p_{ij}, j = 1, \ldots, m_i$ for all the products in the nest can be reduced to the problem of maximizing a single-dimensional continuous function over a bounded interval and will present sufficient conditions for this function to be uni-modal. This will tremendously simplify the problem of maximizing the products of a single nest and will be leveraged later to price the products of multiple nests.

Equation (7) can be rewritten as follows:

$$p_{ij} - c_{ij} - \frac{1}{\beta_{ij}} = \big(1 - \gamma_i(1 - Q_i(\mathbf{p}_i, \mathbf{p}_{\text{-}i}))\big) \sum_{s \in F_i} (p_{is} - c_{is}) q_{s|i}(\mathbf{p}_i), \ \forall j \in F_i. \tag{8}$$

Because the right hand side (RHS) of equation (8) is independent of product index j, then $p_{ij} - c_{ij} - \frac{1}{\beta_{ij}}$ is constant for each $j \in F_i$. We will call $\theta_i = p_{ij} - c_{ij} - \frac{1}{\beta_{ij}}$ the *adjusted markup* for nest i, which is constant for all the products in each nest at optimality (local or global). A similar result for the standard MNL model has been observed by Aydin and Ryan (2000), Hopp and Xu (2005) and Gallego and Stefanescu (2011). Li and Huh (2011) also point out that the *markup* is constant in the NL model with identical price-sensitivity parameters.

We will abuse the notations a bit below without much ambiguity. The original pricing problem (5) can be simplified to the following optimization problem with single decision variable:

$$R_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i}) \overset{\text{def}}{=} Q_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i})(\theta_i + w_i^{F_i}(\theta_i)) \tag{9}$$

where

$$\begin{aligned}
&Q_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i}) = \tfrac{e^{\gamma_i I_i}}{1 + a_{\text{-}i} + e^{\gamma_i I_i}}, && q_{k|i}^{F_i}(\theta_i) = \tfrac{e^{\tilde{\alpha}_{ik} - \beta_{ik}\theta_i}}{\sum_{s \in F_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}}, \\
&w_i^{F_i}(\theta_i) = \textstyle\sum_{k \in F_i} \frac{1}{\beta_{ik}} \cdot q_{k|i}^{F_i}(\theta_i), && \pi_{ik}^{F_i}(\theta_i, \mathbf{p}_{\text{-}i}) = Q_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i}) \cdot q_{k|i}^{F_i}(\theta_i), \\
&I_i = \log \textstyle\sum_{s \in F_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}, && I_l = \log \textstyle\sum_{s=1}^{m_l} e^{\alpha_{ls} - \beta_{ls} p_{ls}}, \ l \neq i, \\
&\tilde{\alpha}_{is} = \alpha_{is} - \beta_{is} c_{is} - 1, \ \forall s, && a_{\text{-i}} = \textstyle\sum_{l \neq i} \log I_l.
\end{aligned}$$

The value $Q_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i})$ is the probability for nest i to be selected given that all the products in set F_i are priced with constant *adjusted markup* θ_i, or equivalently with prices $p_{ij} = c_{ij} + \frac{1}{\beta_{ij}} + \theta_i$ for all $j \in F_i$. $\theta_i + w_i^{F_i}(\theta_i) = \sum_{k \in F_i} (p_{ik} - c_{ik}) q_{k|i}^{F_i}(\theta_i)$ can be considered the *average markup* for all the products in set F_i. It is easy to verify the following:

$$\begin{aligned}
\frac{\partial Q_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i})}{\partial \theta_i} &= -\gamma_i Q_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i})(1 - Q_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i})) v_i^{F_i}(\theta_i) \leq 0, \\
\frac{\partial q_{k|i}^{F_i}(\theta_i)}{\partial \theta_i} &= (v_i^{F_i}(\theta_i) - \beta_{ik}) q_{k|i}^{F_i}(\theta_i),
\end{aligned}$$

where $v_i^{F_i}(\theta_i) = \sum_{k \in F_i} \beta_{ik} q_{k|i}^{F_i}(\theta_i)$. The total market share of nest i is decreasing in the *adjusted markup* θ_i. The monotonicity of a specific product's chosen probability within a nest with respect to the *adjusted markup* of that nest, given that the nest is selected at the upper level, depends on the comparison of the price sensitivities: if the product k is the least (most) price sensitive, i.e., $\beta_{ik} \leq (\geq) \beta_{ij} \forall\, j \in F_i$, then $\beta_{ik} \leq (\geq) v_i^{F_i}(\theta_i) \forall\, \theta_i$ from Lemma 1 and $q_{k|i}^{F_i}(\theta_i)$ is increasing (decreasing) in her *adjusted markup* θ_i; otherwise, the monotonicity is not clear.

There are some monotonic properties for functions $w_i^{F_i}(\theta_i)$ and $v_i^{F_i}(\theta_i)$.

Lemma 1 *(a)* $w_i^{F_i}(\theta_i)$ *is increasing in* θ_i *and* $\frac{1}{\max_{s\in F_i}\beta_{is}} \leq w_i^{F_i}(\theta_i) \leq \frac{1}{\min_{s\in F_i}\beta_{is}}$.
(b) $v_i^{F_i}(\theta_i)$ *is decreasing in* θ_i *and* $\min_{s\in F_i}\beta_{is} \leq v_i^{F_i}(\theta_i) \leq \max_{s\in F_i}\beta_{is}$. *Furthermore,* $w_i^{F_i}(\theta_i)v_i^{F_i}(\theta_i) \geq 1\ \forall\ \theta_i, F_i$, *and all the inequalities become equalities when* β_{is} *is identical for all* $s \in F_i$.

Notice that if set F_i contains more than one product with different price sensitivities then $w_i^{F_i}(\theta_i)v_i^{F_i}(\theta_i) < \frac{\max_{s\in F_i}\beta_{is}}{\min_{s\in F_i}\beta_{is}}$, because $w_i^{F_i}(\theta_i)$ is increasing and bounded by $\frac{1}{\min_{s\in F_i}\beta_{is}}$, and $v_i^{F_i}(\theta_i)$ is decreasing and bounded by $\max_{s\in F_i}\beta_{is}$.

Will the optimal profit increase if another product, say product z, is added to the product set F_i? We will show that the answer to this question is yes at optimally chosen adjusted markups, but no at arbitrarily chosen adjusted markups. Let $F_i^+ := F_i \cup \{z\}$. We will show that $\max_{\theta_i} R_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i}) \leq \max_{\theta_i} R_i^{F_i^+}(\theta_i, \mathbf{p}_{\text{-}i})$. Example 1 below shows that $R_i^{F_i^+}(\theta_i, \mathbf{p}_{\text{-}i})$ is not always greater than $R_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i})$ for all θ_i.

Example 1 *Suppose that there are three available products with parameters* $\tilde{\boldsymbol{\alpha}}_i = (0.7256, 6.3544, 8.0862)$ *and* $\boldsymbol{\beta}_i = (0.6422, 1.0721, 1.7322)$ *and the nest coefficient is* $\gamma_i = 0.8945$. *Assume* $a_{\text{-}i} = \sum_{l\neq i} I_l = 0$ *for simplicity. Let* $F_i = \{1,2\}$ *and* $F_i^+ = \{1,2,3\}$. *Figure 1 demonstrates the comparison between offering all the three products and offering products* 1 *and* 2. *When* $\theta < 3.53$, $R_i^{F_i^+}(\theta_i, \mathbf{p}_{\text{-}i}) < R_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i})$; *when* $\theta > 3.53$, $R_i^{F_i^+}(\theta_i, \mathbf{p}_{\text{-}i}) > R_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i})$.

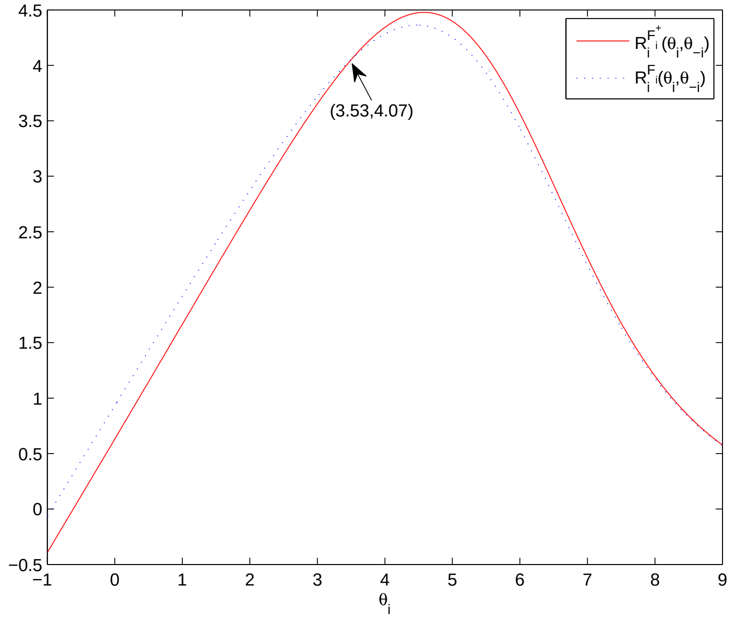
We will show that $\max_{\theta_i} R_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i})$ is increasing in F_i by showing that $\max_{0\leq\rho_i\leq 1} r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i})$ is increasing in F_i where

$$\begin{aligned} r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i}) &:= \max\nolimits_{\mathbf{p}_i<\infty} \sum\nolimits_{k\in F_i}(p_{i,k} - c_{i,k})\pi_{i,k}^{F_i}(\mathbf{p}_i, \mathbf{p}_{\text{-}i}) \\ s.t.,\ Q_i^{F_i}&(\mathbf{p}_i, \mathbf{p}_{\text{-}i}) = \rho_i \end{aligned} \tag{10}$$

is the maximum profit that we can obtain from set F_i when we set the market share of nest i to ρ_i. Clearly, $\max_{\mathbf{p}_i} R_i^{F_i}(\mathbf{p}_i, \mathbf{p}_{\text{-}i}) = \max_{0\leq\rho_i\leq 1} r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i})$. Similarly, we can show that the *adjusted markups* for all the products in set F_i are constant at optimality (local or global), denoted by θ_i. Note that θ_i and ρ_i are one-to-one mapping because $Q_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i}) = \frac{\left(\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}}{1+a_{\text{-}i}+\left(\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}}$ is strictly decreasing in θ_i. Consequently,

$$r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i}) = \rho_i(\theta_i + w_i^{F_i}(\theta_i)), \tag{11}$$

Figure 1 **Comparison between Offering all and Offering partial**



where θ_i is the unique solution to $Q_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i}) = \rho_i$.

We are now ready to compare $r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i})$ and $r_i^{F_i^+}(\rho_i, \mathbf{p}_{\text{-}i})$.

Proposition 1 *The function $r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i})$ is strictly monotone increasing in F_i for all $\rho \in (0,1)$, i.e., $r_i^{F_i^+}(\rho_i, \mathbf{p}_{\text{-}i}) > r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i})$.*

As a consequence of Proposition 1 we can do at least as well by adding more products to F_i until F_i contains all the products in the nest! This means that at optimality the prices of all products are finite and among the 2^{m_i} solutions to the first order conditions, the solution associated with $F_i = \{1, \ldots, m_i\}$, where all products in the nest have a common, finite adjusted markup, is globally optimal. Without further notice we will omit the notation F_i from now on unless otherwise stated. We now state our main condition for nest-price optimization under the general NL model:

Condition 1 $\gamma_i \geq 1$ *or* $\frac{\max_s \beta_{is}}{\min_s \beta_{is}} \leq \frac{1}{1-\gamma_i}$.

Notice that both the standard MNL model ($\gamma_i = 1$) and the NL model with identical price-sensitivity parameters and $\gamma_i < 1$ satisfy Condition 1. When $\gamma_i > 1$, it corresponds to the scenario where products are more similar cross nests; when $0 < \gamma_i < 1$, it refers to the case where products within the same nest are more similar, so the price coefficients of the products in the same nest should not vary too much and the condition $\frac{\max_s \beta_{is}}{\min_s \beta_{is}} \leq \frac{1}{1-\gamma_i}$ is reasonable. Condition 1 will be used in Theorem 1 to establish important structural results.

We remark that $(1-\gamma_i)w_i(\theta_i)v_i(\theta_i) < 1$ for all θ_i under Condition 1. If there are more than one products with different price sensitivities, $w_i(\theta_i)v_i(\theta_i) < \frac{\max_s \beta_{is}}{\min_s \beta_{is}}$, then $\frac{\max_s \beta_{is}}{\min_s \beta_{is}} \leq \frac{1}{1-\gamma_i}$ implies that $(1-\gamma_i)w_i(\theta_i)v_i(\theta_i) < 1$ for all θ_i; otherwise $w_i(\theta_i)v_i(\theta_i) = 1$ for all θ_i.

Theorem 1 *(a) The adjusted markups for all the products in nest i are constant and the optimal price vector, denoted by $\mathbf{p}_i^*$, can be expressed as follows*

$$p_{ij}^* = c_{ij} + \frac{1}{\beta_{ij}} + \theta_i^* \ \forall j = 1, 2, \ldots, m_i, \tag{12}$$

where θ_i^* is a root of

$$\theta_i + (1 - \frac{1}{\gamma_i})w_i(\theta_i) = \frac{1}{(1+a_{\text{-}i})\gamma_i} \cdot \sum_{k=1}^{m_i} \frac{e^{\tilde{\alpha}_{ik} - \beta_{ik}\theta_i}}{\beta_{ik}} \left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i} \right)^{\gamma_i - 1}. \tag{13}$$

If Condition 1 is satisfied, θ_i^* is the unique root to equation (13).

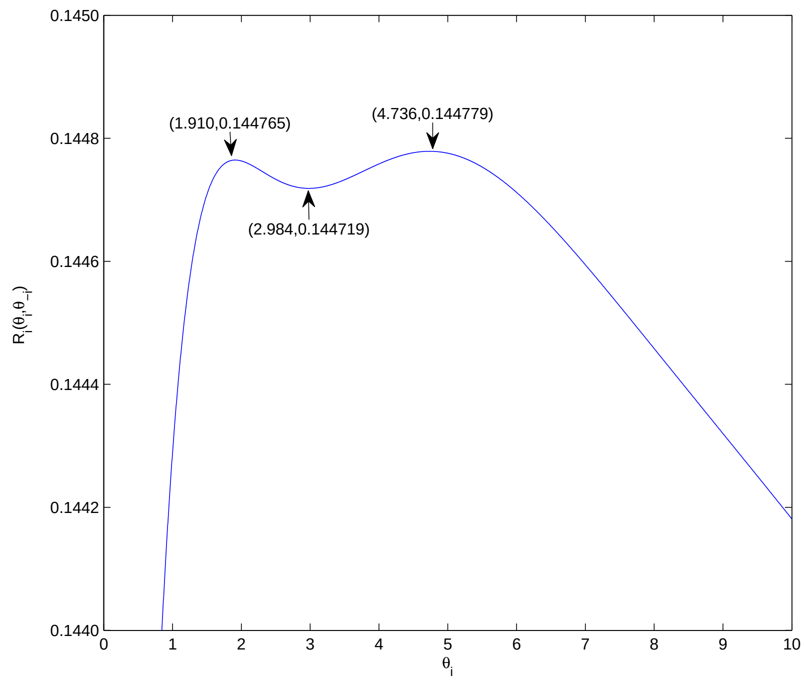
(b) Under Condition 1, $R_i(\theta_i, \mathbf{p}_{\text{-}i})$ is strictly uni-modal with respect to θ_i and $r_i(\rho_i, \mathbf{p}_{\text{-}i})$ is strictly concave in ρ_i.

Note that θ_i^* doesn't have to be positive in general, but it must be strictly positive when the nest coefficient $\gamma_i \leq 1$ because the total profit can be expressed as $\theta_i + (1 - \frac{1}{\gamma_i})w_i(\theta_i)$, where θ_i is a solution to equation (13). Consequently, when $\gamma_i > 1$, it may be optimal to include "loss-leaders" as part of the optimal pricing strategy. More specifically, it may be optimal to include products with negative *adjusted markups* or even negative margins for the purpose of attracting attention to the nest.

If Condition 1 is satisfied, the profit function $R_i(\theta_i, \mathbf{p}_{\text{-}i})$ is uni-modal in θ_i and $r_i(\rho_i, \mathbf{p}_{\text{-}i})$ is concave in ρ_i, so the FOC is sufficient to determine the optimal prices and the optimal solution is unique, which can be easily found by several well known algorithms for uni-modal or concave functions, like binary search and golden section search; if Condition 1 is not satisfied, $R_i(\theta_i, \mathbf{p}_{\text{-}i})$ may not be uni-modal in θ_i as illustrated in the following example.

Example 2 *Assume that fnest i contains five products with parameters for the NL model: $\tilde{\boldsymbol{\alpha}}_i =$ $(1.9769, 0.5022, 0.6309, 0.6013, 0.0841)$ and $\boldsymbol{\beta}_i = (0.6720, 1.1249, 1.0247, 0.7968, 0.0150)$. The nest coefficient $\gamma_i = 0.9150$. The total attractiveness of non-purchase and other nests is $1 + a_{\text{-}i} = 500$. Note that $\frac{\max_s \beta_{is}}{\min_s \beta_{is}} = \frac{1.1249}{0.0150} = 74.99 > \frac{1}{1-\gamma_i} = 11.76$ so Condition 1 is not satisfied. Figure 2 clearly shows that $R_i(\theta_i, \theta_{\text{-}i})$ is not uni-modal in θ_i and there are three solutions to equation (13) in the*

Figure 2 **Non-unimodality of** $R_i(\theta_i, \theta_{\text{-}i}))$



interval $(1,10)$*:* $(1.910, 0.144765)$*,* $(2.984, 0.144719)$ *and* $(4.736, 0.144779)$*. Observe that the maximum relative profit difference is very small:* $(0.144779 - 0.144719)/0.144719 = 0.04\%$*. This suggests that* $R_i(\theta_i, \mathbf{p}_{\text{-}i})$ *is very flat at the peak and any solution to the FOC can be considered a good approximation to the optimal adjusted markup.*

We next show that a global optimal *adjusted markup* can be found in a bounded interval even if Condition 1 fails.

Proposition 2 *Denote* $\theta_{i,\min} = \frac{1-\gamma_i}{\gamma_i \min_s \beta_{is}}$ *and* $\theta_{i,\max} = \frac{1-\gamma_i + \left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_{i,\min}}\right)^{\gamma_i} \Big/ (1+a_{\text{-}i})}{\gamma_i \min_s \beta_{is}}$*, and let* $\theta'_{i,\min} = \frac{1-\gamma_i}{\gamma_i \max_s \beta_{is}}$ *and* $\theta'_{i,\max} = \frac{1-\gamma_i + \left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta'_{i,\min}}\right)^{\gamma_i} \Big/ (1+a_{\text{-}i})}{\gamma_i \min_s \beta_{is}}$*. The optimal adjusted markup* θ_i^* *is in the interval* $[\theta_{i,\min},\ \theta_{i,\max}]$ *if* $\gamma_i \geq 1$*, and it is in the interval* $[\theta'_{i,\min},\ \theta'_{i,\max}]$ *if* $0 < \gamma_i < 1$*.*

The price optimization problem in an m_i-dimensional space is reduced to the problem of maximizing a continuous single-dimensional function over a bounded interval. There are several well developed algorithms that can be employed to solve it efficiently.

The Corollary follows immediately for the special cases: the standard MNL model and the NL model with identical price-sensitivity parameters.

Corollary 1 *If $\gamma_i = 1$ or β_{is} is identical for all $s = 1, 2, \ldots, m_i$, denoted by β_i, the optimal prices are unique and can be expressed in (12), where θ_i^* is, respectively, the unique solution to one of the following equations:*

$$\theta_i = \frac{1}{(1 + a_{\text{-}i})} \cdot \sum_{k=1}^{m_i} \frac{e^{\tilde{\alpha}_{ik} - \beta_{ik}\theta_i}}{\beta_{ik}},$$
$$\theta_i + (1 - \frac{1}{\gamma_i})/\beta_i = \frac{(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}})^{\gamma_i}}{(1 + a_{\text{-}i})\gamma_i\beta_i} \cdot e^{-\beta_i\theta_i}.$$

2.2. Optimization of Multiple Nests

Nest, we consider the centralized system, where all the nests are controlled by a central planner with objective to maximize the total profit $R(\mathbf{p}_1, \ldots, \mathbf{p}_n)$, as expressed in equation (4).

The FOC of function $R(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is

$$\frac{\partial R(\mathbf{p}_1, \ldots, \mathbf{p}_n)}{\partial p_{ij}} = \pi_{ij}(\mathbf{p}_1, \ldots, \mathbf{p}_n) \cdot \Bigg[1 - \beta_{ij}(p_{ij} - c_{ij}) + \beta_{ij}(1 - \gamma_i) \sum_{s=1}^{m_i} (p_{is} - c_{is}) q_{s|i}(\mathbf{p}_i)$$
$$+ \beta_{ij}\gamma_i \sum_{l=1}^{n} \sum_{s=1}^{m_i} (p_{is} - c_{is}) \pi_{is}(\mathbf{p}_1, \ldots, \mathbf{p}_n)\Bigg] = 0.$$

Roots of the FOC can be found by either setting $\pi_{ij}(\mathbf{p}_1, \ldots, \mathbf{p}_n) = 0$, which requires $p_{ij} = \infty$ or letting the inner term of the square bracket equal 0, which is equivalent to

$$p_{ij} - c_{ij} - \frac{1}{\beta_{ij}} = (1 - \gamma_i) \sum_{s=1}^{m_i} (p_{is} - c_{is}) q_{s|i}(\mathbf{p}_i) + \gamma_i \sum_{l=1}^{n} \sum_{s=1}^{m_l} (p_{ls} - c_{ls}) \pi_{ls}(\mathbf{p}_1, \ldots, \mathbf{p}_n). \tag{14}$$

Similar to Theorem 1, it is optimal to sell all the products at finite prices such that the *adjusted markups* are constant for all the products in the same nest as suggested in Theorem 2. Then, problem (4) is equivalent to determining the *adjusted markups* $\boldsymbol{\theta} := (\theta_1, \ldots, \theta_n)$ to maximize the total expected profit,

$$\max_{\boldsymbol{\theta}} R(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \sum_{i=1}^{n} Q_i(\theta_1, \ldots, \theta_n)(\theta_i + w_i(\theta_i)). \tag{15}$$

where

$$Q_i(\theta_1, \ldots, \theta_n) = \tfrac{e^{\gamma_i I_i}}{1 + \sum_{l=1}^{n} e^{\gamma_l I_l}}, \; q_{k|i}(\theta_i) = \tfrac{e^{\tilde{\alpha}_{ik} - \beta_{ik}\theta_i}}{\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}},$$
$$w_i(\theta_i) = \textstyle\sum_{s=1}^{m_i} \tfrac{1}{\beta_{ik}} \cdot q_{k|i}(\theta_i), \quad \pi_{ik}(\theta_1, \ldots, \theta_n) = Q_i(\theta_1, \ldots, \theta_n) \cdot q_{k|i}(\theta_i),$$
$$I_l = \log \textstyle\sum_{s=1}^{m_l} e^{\tilde{\alpha}_{ls} - \beta_{ls} p_{ls}}, \qquad \tilde{\alpha}_{is} = \alpha_{is} - \beta_{is} c_{is} - 1, \; \forall s.$$

When the price-sensitivity parameters are identical for all the products in each nest, i.e., $\beta_{is} = \beta_i$ for all $s = 1, 2, \ldots, m_i$, then $w_i(\theta_i) = \frac{1}{\beta_i}$ and the optimization problem (15) can be rewritten as follows:

$$\max_{\boldsymbol{\theta}} R(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \hat{Q}_i(\theta_1, \ldots, \theta_n)(\theta_i + \frac{1}{\beta_i})$$

where $\hat{Q}_i(\theta_1,\ldots,\theta_n) = \frac{e^{\hat{\alpha}_i-\beta_i\gamma_i\theta_i}}{1+\sum_{l=1}^n e^{\hat{\alpha}_l-\beta_l\gamma_l\theta_l}}$ and $\hat{\alpha}_l = \gamma_l \log(\sum_{s=1}^{m_l} e^{\tilde{\alpha}_{ls}})$. This reduces to the standard MNL model, where the nests play the role of products. This problem is easy to solve as shown in Corollary 1.

For the general NL model with product-differentiated price-sensitivity parameters, the FOC of the total profit $R(\boldsymbol{\theta})$ in (15) is

$$\frac{\partial R(\boldsymbol{\theta})}{\partial \theta_i} = \gamma_i Q_i(\theta_1,\ldots,\theta_n) v_i(\theta_i)\left[\sum_{j=1}^n Q_j(\theta_1,\ldots,\theta_n)(\theta_j + w_j(\theta_j)) - \left(\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i)\right)\right] = 0.$$

Again, because $v_i(\theta_i) \geq \min_s \beta_{is} > 0$, the solutions to the above FOC can be found by either letting $Q_i(\theta_1,\ldots,\theta_n) = 0$, which requires $\theta_i = \infty$ or setting the inner term of the square bracket equal to zero, which is equivalent to

$$\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i) = \sum_{j=1}^n Q_j(\theta_1,\ldots,\theta_n)(\theta_j + w_j(\theta_j)) \tag{16}$$

Theorem 2 says that no nest would be priced out by charging infinite prices and equation (16) is satisfied for each nest at the optimal *adjusted markups*. The RHS of equation (16) is independent of nest index i, so at the optimal solutions $\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i)$ is constant for all i, denoted by ϕ. Note that $\theta_i + w_i(\theta_i) = \sum_{k=1}^{m_i}(\theta_i + \frac{1}{\beta_{ik}})q_{k|i}(\theta_i)$ is the average markup for all the products in nest i, so we call $\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i)$ the *adjusted nest-level markup* for nest i. Thus, problem (15) can be reduced to an optimization problem with respect to *adjusted nest-level markup* in a single-dimensional space,

$$\begin{aligned} &\max_\phi \ R(\phi) \stackrel{\text{def}}{=} \textstyle\sum_{i=1}^n Q_i(\theta_1,\ldots,\theta_n)(\theta_i + w_i(\theta_i)), \\ &\text{where} \ \ \theta_i + (1-\tfrac{1}{\gamma_i})w_i(\theta_i) = \phi, \ \forall i = 1,2,\ldots,n. \end{aligned} \tag{17}$$

Profit $R(\phi)$ is an implicit function expressed in terms of θ_i, but there is a one-to-one mapping between θ_i and ϕ under Condition 1 for each i because $\frac{\partial}{\partial\theta_i}\left(\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i)\right) = \frac{1}{\gamma_i}(1-(1-\gamma_i)w_i(\theta_i)v_i(\theta_i)) > 0$.

The price optimization can also be transformed to an optimization problem with respect to the total market share. Let $R(\rho)$ be the maximum achievable total expected profit given that the total market share $\sum_{i=1}^n Q_i(\mathbf{p}_1,\ldots,\mathbf{p}_n) = \rho$.

$$\begin{aligned} R(\rho) &:= \max_{\mathbf{p}} \textstyle\sum_{i=1}^n \sum_{k=1}^{m_i} (p_{ik} - c_{ik})\pi_{ik}(\mathbf{p}_1,\ldots,\mathbf{p}_n) \\ &\quad s.t., \textstyle\sum_{i=1}^n Q_i(\mathbf{p}_1,\ldots,\mathbf{p}_n) = \rho. \end{aligned} \tag{18}$$

Theorem 2 *(a) It is optimal to offer all the products in each nest at prices such that equation (14) is satisfied, which implies that* $p_{ij} - c_{ij} - \frac{1}{\beta_{ij}}$*, called adjusted markup, is constant for all* j *of each nest* i*.*

(b) Under Condition 1 for each $i = 1, 2, \ldots, n$, it is optimal to charge the adjusted markup for each nest such that equation (16) is satisfied, which implies that $\theta_i + (1 - \frac{1}{\gamma_i})w_i(\theta_i)$, called adjusted nest-level markup, is constant for all i.

(c) Under Condition 1 for each $i = 1, 2, \ldots, n$, function $R(\phi)$ is strictly uni-modal in ϕ and the optimal solution to problem (17) is unique, denoted by ϕ^*. The the optimal solution to the original problem (4) is also unique and the optimal prices can be written in terms of

$$p^*_{ij} = c_{ij} + \frac{1}{\beta_{ij}} + \theta^*_i \tag{19}$$

where θ^*_i is the unique solution to

$$\theta_i + (1 - \frac{1}{\gamma_i})w_i(\theta_i) = \phi^*. \tag{20}$$

(d) Under Condition 1 for each $i = 1, 2, \ldots, n$, the maximum achievable profit $R(\rho)$ is strictly concave in the aggregate market share ρ.

An interesting observation is that the optimal ϕ^* is equal to the optimal profit and ϕ^* is the maximum fixed point of $R(\phi)$. If Condition 1 is satisfied for each i, the uni-modal or concave optimization can be used to find the optimal ρ^* and ϕ^* as well as the optimal *adjusted markups* $\boldsymbol{\theta}^*$ and optimal prices $(\mathbf{p}^*_1, \ldots, \mathbf{p}^*_n)$. Moreover, the optimal ϕ^* is in a bounded interval so it is easy to find even if Condition 1 fails for some or all nests.

Proposition 3 *Denote* $\hat{\theta}_{i,\min} = \frac{1}{\gamma_i \max_s \beta_{is}} - \frac{1}{\min_s \beta_{is}}$ *and* $\phi_{\max} = \sum_{i=1}^n \frac{\left(\sum_{s=1}^{m_i} e^{\alpha_{is} - \beta_{is}\hat{\theta}_{i,\min}}\right)^{\gamma_i}}{\gamma_i \min_s \beta_{is}}$. *The optimal ϕ^* is in the interval* $[0,\ \phi_{max}]$.

3. Oligopolistic Competition

We will next consider oligopolistic price competition where each firm controls one or more nests. This is consistent with an NL model where customers first select a brand and then a product within a brand. We will later consider the case where a firm controls several nests, e.g., a firm owns more than one brand. The oligopolistic price (Bertrand) competition with single and multiple products under the standard MNL model has been widely examined and the existence and uniqueness of Nash equilibrium have been established (see Gallego et al. 2006, Allon et al. 2011). Liu (2006) and Li and Huh (2011) have studied price competition under the NL model with identical price sensitivities for all the products of each firm. However, their approach cannot easily extend to the general NL model with product-differentiated price sensitivities. To the best of our knowledge, our paper is the first to study oligopolistic competition with multiple products under the general NL model with product dependent price-sensitivities and arbitrary nest coefficients.

In the price competition game, the expected profit for firm i is

$$\textbf{Game I:} \qquad R_i(\mathbf{p}_i, \mathbf{p}_{\text{-}i}) = \sum_{k=1}^{m_i} (p_{ik} - c_{ik}) \cdot \pi_{ik}(\mathbf{p}_i, \mathbf{p}_{\text{-}i})$$

where $\mathbf{p}_i = (p_{i1}, p_{i2}, \ldots, p_{im_i})$ and $\mathbf{p}_{\text{-}i} = (\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n)$.

From Theorem 1, each firm's problem can be reduced to a problem with single decision variable as follows

$$\textbf{Game II:} \qquad R_i(\theta_i, \theta_{\text{-}i}) = Q_i(\theta_i, \theta_{\text{-}i})(\theta_i + w_i(\theta_i)).$$

where $Q_i(\theta_1, \ldots, \theta_n) = \frac{e^{\gamma_i I_i}}{1 + \sum_{l=1}^{n} e^{\gamma_l I_l}}$, $I_l = \log \sum_{s=1}^{m_l} e^{\tilde{\alpha}_{ls} - \beta_{ls} p_{ls}}$ for $l = 1, \ldots, n$.

We remark that $R_i(\theta_i, \theta_{\text{-}i})$ is log-separable. Because the profit function $R_i(\theta_i, \theta_{\text{-}i})$ is uni-modal with respect to θ_i under Condition 1, then it is also quasi-concave in θ_i because quasi-concavity and uni-modality are equivalent in a single-dimensional space. The quasi-concavity can guarantee the existence of the Nash equilibrium (see, e.g., Nash 1951 and Anderson et al. 1992), but there are some stronger results without requiring Condition 1 because of the special structure of the NL model.

Theorem 3 *(a)* **Game I** *is equivalent to* **Game II***, i.e., they have the same equilibria.*

(b) **Game II** *is strictly* log*-supermodular; the equilibrium set is a nonempty complete lattice and, therefore, has the componentwise largest and smallest elements, denoted by* $\overline{\boldsymbol{\theta}}^*$ *and* $\underline{\boldsymbol{\theta}}^*$ *respectively. Furthermore, the largest equilibrium* $\overline{\boldsymbol{\theta}}^*$ *is preferred by all the firms.*

The multi-product price competition game has been reduced to an equivalent game with single decision variable for each firm. The existence of Nash equilibrium has been guaranteed and the largest one is a Pareto improvement among the equilibrium set.

3.1. Uniqueness of Equilibrium

To examine the uniqueness of the Nash equilibrium, we will concentrate on **Game II**, which is equivalent to **Game I** from Theorem 3. First, we consider a special case: the symmetric game. Suppose that there are n firms and that all the parameters $(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, \gamma_i)$ in the NL model and the cost vector $\mathbf{c}_i$ are the same for each firm i. Some further properties of the equilibrium set can be derived.

Condition 2 $\gamma_i \geq \frac{n}{n-1}$ *or* $\frac{\max_s \beta_{is}}{\min_s \beta_{is}} \leq \frac{1}{1 - \frac{n-1}{n} \cdot \gamma_i}$.

We remark that Condition 2 is a bit stronger than Condition 1 and they are closer for larger n (they coincide when n goes infinite).

Theorem 4 *(a) Only symmetric equilibria exist for the symmetric game discussed above.*
(b) The equilibrium is unique under Condition 2.

Under Condition 2, the equilibrium is the unique solution to

$$\theta_i + \left(1 - \frac{1}{\gamma_i(1 - Q_i(\theta_i, n))}\right) w_i(\theta_i) = 0. \tag{21}$$

where $Q_i(\theta_i, n) = \frac{\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}\right)^{\gamma_i}}{1+n\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}\right)^{\gamma_i}}$. It has been shown how to obtain equation (21) in the proof of Theorem 4.

We will next state some sufficient conditions for the uniqueness of the Nash equilibrium in the general case:

Condition 3 *(a) Denote* Ψ *as the region such that*

$$-\frac{\partial Q_i(\theta_i, \theta_{\text{-}i})}{\partial \theta_i} > \sum_{j \neq i} \frac{\partial Q_i(\theta_i, \theta_{\text{-}i})}{\partial \theta_j}, \ \boldsymbol{\theta} \in \Psi, \ i = 1, 2, \ldots, n.$$

(b) Denote Ω_i *as the region such that* $\theta_i + w_i(\theta_i)$ *is* log-*concave in* $\theta_i \in \Omega_i$, $i = 1, 2, \ldots, n$.

Notice that the NL model with product independent price-sensitivity parameters within a nest and homogeneous nest coefficients, satisfies Condition 3 for any $\boldsymbol{\theta}$. Condition 3(a) is a standard diagonal dominant condition (see e.g., Vives 2001) and it says that a uniform increase of the *adjusted markups* by all the n firms would result in a decrease of any firm's market share. In the NL model, Condition 3(a) is equivalent to

$$\gamma_i v_i(\theta_i) > \sum_{j=1}^{n} \gamma_j Q_j(\theta_j, \theta_{\text{-}j}) v_j(\theta_j). \tag{22}$$

From Lemma 1, inequality (22) can be implied by the following condition that is stronger but easier to be verified:

$$\min_i \gamma_i \min_{l,s} \beta_{l,s} > \max_i \gamma_i \max_{l,s} \beta_{l,s} \sum_{j=1}^{n} Q_j(\theta_j, \theta_{\text{-}j}),$$

which is equivalent to

$$\sum_{l=1}^{n} \left(\sum_{s=1}^{m_l} e^{\tilde{\alpha}_{ls} - \beta_{ls}\theta_l} \right)^{\gamma_l} < \frac{\min_i \gamma_i \min_{l,s} \beta_{ls}}{\max_i \gamma_i \max_{l,s} \beta_{ls} - \min_i \gamma_i \min_{l,s} \beta_{ls}}. \tag{23}$$

From inequalition (23), Condition 3(b) can be satisfied when the *adjusted markups* θ_i are sufficiently large for all the firms.

Apparently, $\theta_i + w_i(\theta_i) > 0$ because each firm sells all her products at a positive average margin. Then, Condition 3(b) can be implied by a stronger condition that $\theta_i + w_i(\theta_i)$ or $w_i(\theta_i)$ is concave in θ_i for each $i = 1, 2, \ldots, n$ because if $w_i(\theta_i)$ is concave, then

$$\frac{\partial^2 \log(\theta_i + w_i(\theta_i))}{\partial \theta_i^2} = \frac{w_i''(\theta_i)(\theta_i + w_i(\theta_i)) - (w_i'(\theta_i))^2}{(\theta_i + w_i(\theta_i))^2} \leq 0,$$

where $w_i'(\theta_i) = \partial w_i(\theta_i)/\partial \theta_i$ and $w_i''(\theta_i) = \partial^2 w_i(\theta_i)/\partial \theta_i^2$.

When θ_i is large enough, Condition 3(b) can also be satisfied without requiring the concavity of $\theta_i + w_i(\theta_i)$ or $w_i(\theta_i)$.

Lemma 2 *There exist a threshold $\tilde{\theta}_i$ for each firm i such that $\theta_i + w_i(\theta_i)$ is* log-*concave in θ_i for $\theta_i \geq \tilde{\theta}_i$.*

Tatonnement process can reach equilibrium under some mild conditions. In the basic *tatonnement* process, firms take turns in adjusting their price decisions and each firm reacts optimally to all other firms' prices without anticipating others' response, which can be interpreted as a way of expressing bounded rationality of agents. In each iteration, firms respond myopically to the choices of other firms in the previous iteration and the dynamic process can be expressed below.

Tatonnement Process: Select a feasible vector $\boldsymbol{\theta}^{(0)}$; in the k^{th} iteration determine the optimal response for each firm i as follows:

$$\theta_i^{(k)} = \arg\max_{\theta_i \in \Omega_i \cap \Psi} R_i(\theta_i, \theta_{-i}^{(k-1)}). \tag{24}$$

Theorem 5 *Suppose $\boldsymbol{\theta}^*$ is an equilibrium under Condition 3,*

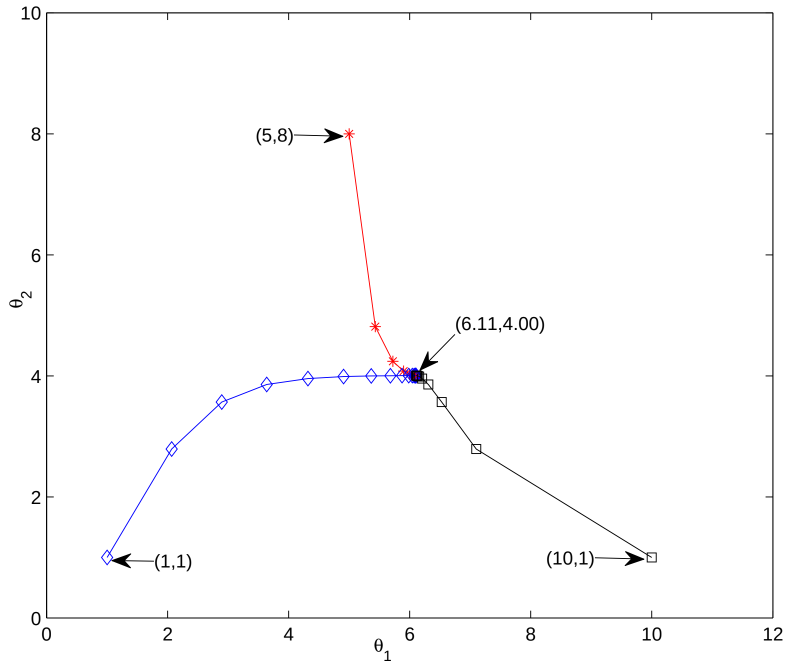
(a) $\boldsymbol{\theta}^*$ is the unique pure Nash equilibrium of **Game II** *in region $(\bigcap_{i=1}^n \Omega_i) \bigcap \Psi$.*

(b) The unique pure Nash equilibrium $\boldsymbol{\theta}^*$ can be computed by the tatonnement scheme, starting from an arbitrary price vector $\boldsymbol{\theta}^{(0)}$ in the region $(\bigcap_{i=1}^n \Omega_i) \bigcap \Psi$, i.e., $\boldsymbol{\theta}^{(k)}$ converges to $\boldsymbol{\theta}^*$.

Example 3 *Consider an example with two firms and each firm sells two products. The demand follows the NL model. For firm* 1*, $\tilde{\boldsymbol{\alpha}}_1 = (1.0, 2.0)$, $\boldsymbol{\beta}_1 = (0.6, 0.8)$ and $\gamma_1 = 0.75$; for firm* 2*, $\boldsymbol{\alpha}_2 = (0.8, 1.1)$, $\boldsymbol{\beta}_2 = (0.7, 1.2)$ and $\gamma_2 = 0.5$.*

The Nash equilibrium of **Game II** *is $\boldsymbol{\theta}^* = (6.11, 4.00)$. Figure 3 shows the tatonnement process converges to the Nash equilibrium $\boldsymbol{\theta}^*$ from three different initial points $(1,1)$, $(10,1)$ and $(5,8)$ respectively and all the pathes converge to $\boldsymbol{\theta}^*$ very fast.*

Figure 3 Convergence of Tatonnement Process



3.2. Competition with Multiple Nests

This subsection considers the oligopolistic price competition in a more complicated environment, where each firm controls multiple nests of products. Let N_i be the set of nests controlled by firm i, then her price optimization problem is

$$\max_{\mathbf{p}_{N_i}} \; R_{N_i}(\mathbf{p}_{N_i}, \mathbf{p}_{\text{-}N_i}) = \sum_{l \in N_i} \sum_{k=1}^{m_l} (p_{lk} - c_{lk}) \pi_{lk}(\mathbf{p}_l, \mathbf{p}_{\text{-}l}),$$

where $\mathbf{p}_{N_i} = (\mathbf{p}_l)_{l \in N_i}$ is firm i's decision variables, that are the price vectors for the nests controlled by firm i, and $\mathbf{p}_{\text{-}N_i}$ includes all the price vectors for the nests charged by other firms. From Theorem 2, it is equivalent for each firm to consider the following simplified optimization problem in a single-dimensional strategy space:

$$\max_{\phi_i} R_{N_i}(\phi_i, \phi_{\text{-}i}) \overset{\text{def}}{=} \sum_{l \in N_i} Q_l(\theta_l, \theta_{\text{-}l})(\theta_l + w_l(\theta_l)),$$
$$\text{where} \;\; \theta_l + (1 - \tfrac{1}{\gamma_l}) w_l(\theta_l) = \phi_i, \;\; \forall l \in N_i.$$

By a similar argument to Theorem 3, the above two games are equivalent. Moreover, from Theorem 2, $R_{N_i}(\phi_i, \phi_{\text{-}i})$ is uni-modal and quasi-concave in ϕ_i under Condition 1 for each $l \in N_i$, so there exits a Nash equilibrium for the simplified game as well as the original game.

4. Dynamic Pricing in Revenue Management

In this section, we consider the application of the NL model to the traditional revenue management problem, where a firm sells multiple substitutable products over a finite horizon and the inventory cannot be replenished during the selling season (see Gallego and van Ryzin 1997). For notational simplicity, the cost for each product is assumed to be zero and the salvage value of remaining capacity is also assumed to be zero. The problem with a constant cost or a constant salvage value per unit would result in a similar formulation. The time horizon is discretized to T periods and each time interval is tiny enough that the probability that more than one customers arrive is negligible. Customers' purchase behavior is influenced by the price vector in that period and their product selection follows the NL model.

Assume that the firm is a monopolist and we will omit the firm index in this section. Suppose the firm sells m products that constitute a nest. The model and results are similar if the products form multiple nests. Assume that the customer arrival process is a nonhomogeneous Poisson process with rate λ_t in period t. Without loss of generality, we assume that the total attractiveness from non-purchase and all the competitors is normalized to 1. Then, the probability that a customer chooses product k, given the price vector $\mathbf{p} := (p_1, p_2, \ldots, p_m)$, is

$$\pi_k(\mathbf{p}) = Q(\mathbf{p}) \cdot q_k(\mathbf{p}) = \frac{\left(\sum_{s=1}^m e^{\alpha_s - \beta_s p_s}\right)^\gamma}{1 + \left(\sum_{s=1}^m e^{\alpha_s - \beta_s p_s}\right)^\gamma} \cdot \frac{e^{\alpha_k - \beta_k p_k}}{\sum_{s=1}^m e^{\alpha_s - \beta_s p_s}} \tag{25}$$

4.1. Single Resource

First, we consider the problem where all the products consume a common resource (see, e.g., Maglaras and Meissner 2006). Let x denote the number of remaining units of capacity at the beginning of period t, and t be the time-to-go. Let $J(x,t)$ be the expected revenue-to-go function starting at state (x,t). The Bellman equation is the following,

$$J(x,t) = \lambda_t \left\{ \max_{\mathbf{p}} \sum_{k=1}^m \big(p_k - \Delta J(x,t-1)\big) \cdot \pi_k(\mathbf{p}) \right\} + J(x,t-1), \tag{26}$$

where $\Delta J(x,t-1) = J(x,t-1) - J(x-1,t-1)$ is the marginal value of the resource at state $(x,t-1)$. The boundary conditions are $J(0,t) = 0$ and $J(x,0) = 0$.

Let $R(\mathbf{p}) = \sum_{k=1}^m p_k \pi_k(\mathbf{p})$ and $\rho = \sum_{j=1}^m \pi_k(\mathbf{p})$. Then, $\lambda_t \cdot R(\mathbf{p})$ is the total revenue rate and $\lambda_t \cdot \rho$ is the aggregate rate of capacity consumption in period t. Abusing notations a bit, we define

$$\begin{aligned} r(\rho) &:= \max_{\mathbf{p}} \textstyle\sum_{k=1}^m p_k \pi_k(\mathbf{p}) \\ s.t.,& \textstyle\sum_{k=1}^m \pi_k(\mathbf{p}) = \rho. \end{aligned} \tag{27}$$

Similarly to Theorem 1, the optimal prices to problem (27), denoted by $(p_1^*, p_2^*, \ldots, p_m^*)$, are unique for each $\rho \in (0,1)$ and can be expressed as $p_k^* = c_k + \frac{1}{\beta_k} + \theta^*(\rho)$, where $\theta^*(\rho)$ is the unique solution to

$$\sum_{s=1}^{m} e^{\tilde{\alpha}_s - \beta_s \theta} = \left(\frac{\rho}{1-\rho}\right)^{\frac{1}{\gamma}},$$

where $\tilde{\alpha}_s = \alpha_s - \beta_s c_s - 1$ for $s = 1, 2, \ldots, m$.

The value $\lambda_t \cdot r(\rho)$ is the maximum achievable revenue rate at period t subject to the constraint that all products jointly consume the common resource at rate $\lambda_t \cdot \rho$. Then, the Bellman equation (26) can be rewritten as follows

$$J(x,t) = \lambda_t \max_{0 \le \rho \le 1} \{r(\rho) - \rho \Delta J(x, t-1)\} + J(x, t-1). \tag{28}$$

The multi-product dynamic pricing problem has been reduced to a dynamic pricing problem (28) in a single-dimensional space. If there exists an inverse demand function that maps the market shares into a corresponding vector of prices, the revenue function $R(\mathbf{p})$ can be expressed in terms of demand rate. Maglaras and Meissner (2006) point out that if the revenue function is continuous, bounded and strictly jointly concave in demand rates, the maximum achievable revenue $r(\rho)$ is concave with respect to the aggregate rate of capacity consumption ρ. However, we have shown that the revenue rate is not jointly concave in market shares (which refer to the demand rates here) under the NL model in Section 2. But, the maximal achievable revenue is concave in the aggregate rate under the NL model under Condition 1 from Theorem 1, so the multi-product dynamic pricing program has been reduced to the classic dynamic program for single product with a concave demand rate (see Gallego and van Ryzin 1994 and Maglaras and Meissner 2006). Furthermore, as shown in Section 2, the optimal prices exist and are unique at each state.

4.2. Multiple Resources

In the previous subsection, all the products consume the same common resource. In this subsection, the products are stocked at the finished product level and cannot be replenished during the selling season. Dong et al. (2009) and Akcay et al. (2010) have studied the problem under the standard MNL model and Li and Huh (2011) have considered the NL model with identical price sensitivity for the products of the same firm. We will next investigate the problem under the general NL model with product-differentiated price sensitivities.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ denote the vector of the inventory levels. The Bellman equation is the following

$$J(\mathbf{x}, t) = \lambda_t \left\{ \max_{\mathbf{p}} \sum_{k=1}^{m} \big(p_k - \Delta_k J(\mathbf{x}, t-1)\big) \cdot \pi_k(\mathbf{p}) \right\} + J(\mathbf{x}, t-1), \tag{29}$$

where $\Delta_k J(\mathbf{x}, t-1) = J(\mathbf{x}, t-1) - J(\mathbf{x} - \mathbf{e}_k, t-1)$ if $x_k \geq 1$; $\Delta_k J(\mathbf{x}, t-1) = \infty$ otherwise, and $\mathbf{e}_k$ is an all-zero vector except the k^{th} entry of 1. When a product is sold out it can be eliminated from the consideration set, which is equivalent to pricing it at infinity. At state (x, t), the optimization problem is

$$\max_{\mathbf{p}} R(\mathbf{p}) \overset{\text{def}}{=} \sum_{k=1}^{m} \big(p_k - \Delta_k J(\mathbf{x}, t-1)\big) \cdot \pi_k(\mathbf{p}).$$

Notice that in the above optimization, the stock-out product is automatically priced at infinity and the consideration set only includes products with positive inventory. From Theorem 1, the optimal price vector can be expressed as follows

$$p_k^* = \Delta_k J(\mathbf{x}, t-1) + \frac{1}{\beta_k} + \theta^*,$$

where θ^* is one of the roots to the following

$$\theta + (1 - \frac{1}{\gamma}) w(\theta) = \frac{1}{\gamma} \cdot \sum_{s=1}^{m} \frac{e^{\bar{\alpha}_s - \beta_s \theta}}{\beta_s} \left(\sum_{s=1}^{m} e^{\tilde{\alpha}_s - \beta_s \theta} \right)^{\gamma - 1}, \tag{30}$$

that maximizes $R(\theta)$, where $\bar{\alpha}_s = \alpha_s - \beta_s \Delta_s J(\mathbf{x}, t-1) - 1$. If Condition 1 is satisfied, θ^* is the unique root to equation (30); if it is not satisfied, the optimal θ^* can be found in a bounded interval from Proposition 2.

5. Extension and Discussion

Discrete choice model is one of the most popular models to study customer choice behavior when multiple substitutable products are available. While the acceptance and application of the popular MNL model are adversely affected by the IIA property, the NL model with a two-stage process has been generalized and it alleviates the IIA property.

5.1. Extension: Non-purchase in a Nest

In the NL model, at the lower stage customers are assumed to select one product within the nest they chose at the upper stage. As an extension, non-purchase may also be an option at the lower stage. Let a_{i0} refer to the attractiveness of non-purchase option in nest i and the choice probabilities $Q_i^N(\mathbf{p}_i, \mathbf{p}_{\text{-}i})$ and $q_{k|i}^N(\mathbf{p}_i)$ can be redefined as follows:

$$Q_i^N(\mathbf{p}_i, \mathbf{p}_{\text{-}i}) = \frac{e^{\gamma_i I_i^N}}{1 + \sum_{l=1}^{n} e^{\gamma_l I_l^N}},$$

$$q_{k|i}^N(\mathbf{p}_i) = \frac{e^{\alpha_{ik} - \beta_{ik} p_{ik}}}{a_{i0} + \sum_{s=1}^{m_i} e^{\alpha_{is} - \beta_{is} p_{is}}},$$

where $I_l^N = \log\big(a_{l0} + \sum_{s=1}^{m_l} e^{\alpha_{ls} - \beta_{ls} p_{ls}}\big)$.

The FOC of the profit function (5) under the NL model with non-purchase option in a nest is

$$\frac{\partial R_i(\mathbf{p}_i,\mathbf{p}_{\text{-}i})}{\partial p_{ij}} = \pi_{ij}^N(\mathbf{p}_i,\mathbf{p}_{\text{-}i})\left[1-\beta_{ij}(p_{ij}-c_{ij})+\beta_{ij}\big(1-\gamma_i(1-Q_i^N(\mathbf{p}_i,\mathbf{p}_{\text{-}i}))\big)\sum_{s=1}^{m_i}(p_{is}-c_{is})q_{s|i}^N(\mathbf{p}_i)\right]=0,$$

where $\pi_{ij}^N(\mathbf{p}_i,\mathbf{p}_{\text{-}i}) = Q_i^N(\mathbf{p}_i,\mathbf{p}_{\text{-}i})q_{k|i}^N(\mathbf{p}_i)$.

Similar to Theorem 1, we can prove that the *adjusted markup* is also constant for all the products in the same nest at optimality and the multi-product price optimization can be reduced to an optimization problem with single decision variable, the *adjusted markup*.

5.2. Extension: Nested Attraction Model

Market share attraction models have received increasing attention in the marketing literature and it specifies that a market share of a firm is equal to its attraction divided by the total attraction of all the firms in the market, including the non-purchase attraction, where a firm's attraction is a function of the values of its marketing instruments, e.g., brand value, advertising, product features and variety, etc. As an extension, we will consider the generalized Nested Attraction models, of which the MNL model and the NL model are special cases. Again, its two-stage structure can alleviate the IIA property, imposed by the MNL model and other Attraction models.

In this subsection, we extend to the general Nested Attraction model:

$$\begin{aligned}
Q_i(\mathbf{p}_i,\mathbf{p}_{\text{-}i}) &= \frac{e^{\gamma_i I_i}}{1+\sum_{l=1}^n e^{\gamma_l I_l}},\\
q_{k|i}(\mathbf{p}_i) &= \frac{a_{ik}(p_{ik})}{\sum_{s=1}^{m_i} a_{is}(p_{is})},\\
\pi_{ik}(\mathbf{p}_i,\mathbf{p}_{\text{-}i}) &= Q_i(\mathbf{p}_i,\mathbf{p}_{\text{-}i})\cdot q_{k|i}(\mathbf{p}_i),
\end{aligned}$$

where $a_{is}(p_{is})$ is the attractiveness of product s of nest i at price p_{is} and it is continuously twice-differentiable in p_{is}, and $I_l = \log\sum_{s=1}^{m_l} a_{ls}(p_{is})$ is the total attractiveness of nest i. Note that $a_{is}(p_{is}) = e^{\alpha_{is}-\beta_{is}p_{is}}$ for the NL model discussed above; for the linear model $a_{is}(p_{is}) = \alpha_{is}-\beta_{is}p_{is}$, $\alpha_{is},\beta_{is}>0$; for the modified constant elasticity of substitution (CES) model $a_{is}(p_{is}) = \alpha_{is}p_{is}^{-\beta_{is}}$, $\alpha_{is}>0,\beta_{is}>1$. (Here, we call it the modified CES model because the standard CES model is not a probabilistic choice model.)

Consider the FOC for the profit $R_i(\mathbf{p}_i,\mathbf{p}_{\text{-}i})$ in function (5) under the Nested Attraction model:

$$\frac{\partial R_i(\mathbf{p}_i,\mathbf{p}_{\text{-}i})}{\partial p_{ij}} = \frac{\beta_{ij}\pi_{ij}(\mathbf{p}_i,\mathbf{p}_{\text{-}i})a'_{ij}(p_{ij})}{a_{ij}(p_{ij})}\cdot\left[(p_{ij}-c_{ij})+\frac{a_{ij}(p_{ij})}{a'_{ij}(p_{ij})}-\big(1-\gamma_i(1-Q_i(\mathbf{p}_i,\mathbf{p}_{\text{-}i}))\big)\sum_{s=1}^{m_i}(p_{is}-c_{is})q_{s|i}(\mathbf{p}_i)\right]=0.$$

The above FOC is satisfied when either $\frac{\beta_{ij}\pi_{ij}(\mathbf{p}_i,\mathbf{p}_{\text{-}i})a'_{ij}(p_{ij})}{a_{ij}(p_{ij})}=0$, which requires $a'_{ij}(p_{ij})=0$, or the inner term of the square bracket is equal to zero

$$(p_{ij}-c_{ij})+\frac{a_{ij}(p_{ij})}{a'_{ij}(p_{ij})}-\big(1-\gamma_i(1-Q_i(\mathbf{p}_i,\mathbf{p}_{\text{-}i}))\big)\cdot\sum_{s=1}^{m_i}(p_{is}-c_{is})q_{s|i}(\mathbf{p}_i)=0. \tag{31}$$

The following Theorem 6 says that it is optimal to offer all the products at prices such that equation (31) holds, which implies that $(p_{ij}-c_{ij})+\frac{a_{ij}(p_{ij})}{a'_{ij}(p_{ij})}$ is constant for all j, denoted by η_i.

Condition 4 *(a)* $a'_{ij}(p_{ij})\le 0,\ 2(a'_{ij}(p_{ij}))^2 > a_{ij}(p_{ij})a''_{ij}(p_{ij})\ \forall j,\ p_{ij}$.
(b) That $a'_{ij}(p_{ij})=0$ *implies that* $(p_{ij}-c_{ij})a_{ij}(p_{ij})=0$.

That $a'_{ij}(p_{ij})\le 0$ says that each product's attractiveness is decreasing in its price; that $2(a'_{ij}(p_{ij}))^2 > a_{ij}(p_{ij})a''_{ij}(p_{ij})$ can be implied by a stronger condition that $a_{ij}(p_{ij})$ is log-concave in p_{ij}, which is equivalent to $(a'_{ij}(p_{ij}))^2 > a_{ij}(p_{ij})a''_{ij}(p_{ij})$ for all p_{ij}. Condition 4(b) requires that $a_{ij}(p_{ij})$ converges to zero at a faster rate than linear functions when $a'_{ij}(p_{ij})$ converges to zero. In other words, when $a'_{ij}(p_{ij})=0$, product j of nest i doesn't contribute any profit so it can be eliminated from the profit function.

It is straightforward to verify that the MNL model, the linear attraction model and the modified CES model all satisfy Condition 4. Furthermore, there is a one-to-one mapping between p_{ij} and η_i for all j because $\frac{\partial}{\partial p_{ij}}\left((p_{ij}-c_{ij})+\frac{a_{ij}(p_{ij})}{a'_{ij}(p_{ij})}\right)=\frac{2(a'_{ij}(p_{ij}))^2-a_{ij}(p_{ij})a''_{ij}(p_{ij})}{a'_{ij}(p_{ij})}<0$ under Condition 4.

Then, problem (5) under the general Nested Attraction model can be reduced to the optimization problem in a single-dimensional space as follows

$$\begin{aligned}&\max_{\eta_i}\ R_i(\eta_i,\eta_{-i})\stackrel{\text{def}}{=}Q_i(\mathbf{p}_i,\mathbf{p}_{-i})\cdot\textstyle\sum_{k=1}^{m_i}(p_{ik}-c_{ik})q_{k|i}(\mathbf{p}_i),\\&\text{where } (p_{ij}-c_{ij})+\frac{a_{ij}(p_{ij})}{a'_{ij}(p_{ij})}=\eta_i.\end{aligned}\tag{32}$$

Theorem 6 *Under Condition 4, it is optimal to offer all the products at prices such that equation (31) is satisfied, which implies that* $(p_{ij}-c_{ij})+\frac{a_{ij}(p_{ij})}{a'_{ij}(p_{ij})}$ *is constant for all* $j=1,2,\dots,m_i$.

The Corollary follows immediately for the special cases: the Nested linear attraction model and the Nested modified CES model.

Corollary 2 *The following quantities are constant at optimal prices for the Nested linear attraction model and the Nested modified CES model, respectively:*

$$2p_{ij}-c_{ij}-\frac{\alpha_{ij}}{\beta_{ij}},\quad (1-\frac{1}{\beta_{ij}})p_{ij}-c_{ij}.$$

The multi-product pricing problem can be simplified to an optimization problem in a single-dimensional space. It is not hard to show that $R_i(\eta_i,\eta_{-i})$ is log-supermodular for the general Nested Attraction model under Condition 4, so Theorem 3 also holds here.

5.3. Discussion and Conclusion

Discrete choice modeling has become a popular vehicle to study purchase behavior of customers who face multiple substitutable products. The standard Multinomial Logit (MNL) discrete choice model has been well studied and widely used in marketing, economics, transportation science and operations management, but it suffers the IIA property, which limits its application and acceptance, especially in the scenarios with correlated products. The Nested Logit (NL) model with a two-stage process has been generalized and it can alleviate the IIA property. Empirical studies have shown that the NL model works well in the environment with differentiated substitutable products.

This paper considers price optimization and competition with multiple substitutable products under the general NL model. We investigate the general NL model with product-dependent price-sensitivity parameters and general nest coefficients. Optimization analysis shows that the *adjusted markup* is constant for all products within a nest. In addition, the *adjusted nest-level markup* is constant for each nest when optimizing over multiple nests. By using this result, the multi-product and multi-nest optimization problems can be simplified to a single-dimensional maximization of a continuous function over a a bounded interval. Mild conditions are provided for this function to be uni-modal. We also use this result to characterize the Nash equilibrium and the equilibrium can be quickly found by the *Tatonnement* process.

Furthermore, we consider its application in multi-product dynamic pricing under the framework of revenue management, and establish structural results of the optimal pricing policy. Revenue management and dynamic pricing have been well investigated in the last couple of decades and it has been widely used in practice for management of airlines, hotels, rental cars, cruises, etc. Significant revenue benefits have been documented from this scientific management. With the help of our theoretical analysis on the general NL model, customer purchase behavior will be deeply investigated and a new direction in marketing management will arise. Our research work will shine bright light on the application of the NL model and deliver important management insights in practice.

We have also studied the general Nested Attraction model, of which the NL model and the MNL model are special cases, and have shown how it can be transformed to an optimization problem in a single-dimensional space. The two-stage model can alleviate the IIA property and derive high acceptance and wide use in practice. In the future, the research and practice on customers' selection behavior with three or even higher stages may attract more attention because it may be closer to customers' rationality. One of other future research directions may consider the heterogeneity of customers and investigate the discrete choice model in the context with multiple heterogenous market segments.

Appendix A: Non-concavity of Market Share Transformation

In the NL model, denote $a_{\text{-}i} = \sum_{l \neq i} e^{\gamma_l I_l}$ for notational convenience. From equation (1),

$$\frac{Q_i}{1-Q_i} = \frac{\left(\sum_{s=1}^{m_i} e^{\alpha_{is}-\beta_{is}p_{is}}\right)^{\gamma_i}}{1+a_{\text{-}i}}.$$

Combining with equation (2) results in

$$e^{\alpha_{ik}-\beta_{ik}p_{ik}} = \frac{\pi_{ik}}{Q_i} \cdot \left(\frac{Q_i(1+a_{\text{-}i})}{1-Q_i}\right)^{\frac{1}{\gamma_i}}$$

Then, p_{ik} can be expressed in terms of $\boldsymbol{\pi}_i := (\pi_{i1}, \pi_{i2}, \ldots, \pi_{im_i})$ as follows

$$p_{ik}(\boldsymbol{\pi}_i) = \frac{1}{\beta_{ik}}\big(\log Q_i - \log \pi_{ik}\big) + \frac{1}{\beta_{ik}\gamma_i}\big(\log(1-Q_i) - \log Q_i\big) + \frac{\alpha_{ik}}{\beta_{ik}} - \frac{\log(1+a_{\text{-}i})}{\beta_{ik}\gamma_i}. \tag{33}$$

The profit of firm i can be rewritten as a function of market shares:

$$R_i(\boldsymbol{\pi}_i, \mathbf{p}_{\text{-}i}) = \sum_{k=1}^{m_i}\left(\frac{1}{\beta_{ik}}\big(\log Q_i - \log \pi_{ik}\big) + \frac{1}{\beta_{ik}\gamma_i}\big(\log(1-Q_i) - \log Q_i\big) - \tilde{c}_{ik}\right) \cdot \pi_{ik}, \tag{34}$$

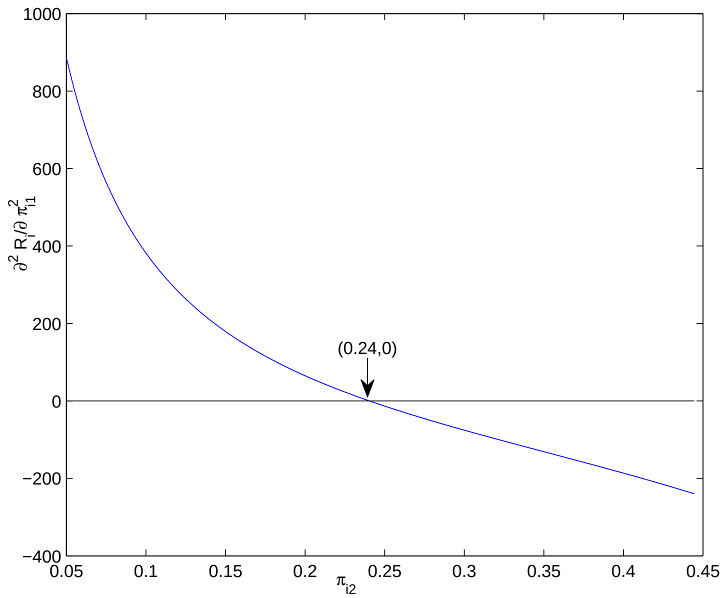
where $Q_i = \sum_{s=1}^{m_i} \pi_{is}$ and $\tilde{c}_{ik} = c_{ik} - \frac{\alpha_{ik}}{\beta_{ik}} + \frac{\log(1+a_{\text{-}i})}{\beta_{ik}\gamma_i}$.

The price optimization under the NL model has been transformed to the optimization problem in market shares as discussed above. Li and Huh (2011) have examined the NL model with nest coefficient $\gamma_i \leq 1$ and identical price coefficients within each firm (maybe different across firms) and proven that the total profit is jointly concave with respect to market shares and used this result to analytically compare the optimal monopolistic solution to oligopolistic equilibrium solutions. However, their approach cannot easily extend to the NL model with $\gamma_i > 1$ or product-differentiated price coefficients. We will present an example to show that the profit function is not jointly concave in market shares for the general NL model and then develop a new approach which exploits the structure of optimal prices.

Taking the first and second order derivatives of the profit function (34) with respect to π_{ij} results in

$$\begin{aligned}
\frac{\partial R_i(\boldsymbol{\pi}_i, \mathbf{p}_{\text{-}i})}{\partial \pi_{ij}} &= \frac{1}{\beta_{ij}}\big(\log(Q_i) - \log(\pi_{ij}) - 1\big) + \frac{1}{\beta_{ij}\gamma_i}\big(\log(1-Q_i) - \log(Q_i)\big) - \tilde{c}_{ij} \\
&\quad + \sum_{k=1}^{m_i}\left(\frac{1}{\beta_{ik}Q_i} - \frac{1}{\beta_{ik}\gamma_i}\left(\frac{1}{1-Q_i} + \frac{1}{Q_i}\right)\right) \cdot \pi_{ik}, \\
\frac{\partial^2 R_i(\boldsymbol{\pi}_i, \mathbf{p}_{\text{-}i})}{\partial \pi_{ij}^2} &= \frac{1}{\beta_{ij}}\left(\frac{2}{Q_i} - \frac{1}{\pi_{ij}}\right) - \frac{2}{\beta_{ij}\gamma_i}\left(\frac{1}{1-Q_i} + \frac{1}{Q_i}\right) - \sum_{k=1}^{m_i}\left(\frac{1}{\beta_{ik}Q_i^2} + \frac{1}{\beta_{ik}\gamma_i}\left(\frac{1}{(1-Q_i)^2} - \frac{1}{Q_i^2}\right)\right) \cdot \pi_{ik}
\end{aligned}$$

Figure 4 **Non-concavity of** $R_i(\pi_i, \mathbf{p}_{\text{-}i}))$



Observe that the second order derivative $\frac{\partial^2 R_i(\boldsymbol{\pi}_i, \mathbf{p}_{\text{-}i}))}{\partial \pi_{ij}^2}$ is independent of $a_{\text{-}i}$, the attractiveness of other firms and even α_{is} $\forall$ s, the qualities of all the products in the same nest. A necessary condition for the concavity of $R_i(\pi_i, \mathbf{p}_{\text{-}i})$ in $\boldsymbol{\pi}_i$ is: in the feasible region,

$$\frac{\partial^2 R_i(\boldsymbol{\pi}_i, \mathbf{p}_{\text{-}i}))}{\partial \pi_{ij}^2} \leq 0, \ \forall \ j, \ k, \ \pi_{ik}. \tag{35}$$

Example 4 *Assume that firm i sells two products with product-differentiated price coefficients* $\boldsymbol{\beta}_i = (0.9, 0.1)$ *and nest coefficient* $\gamma_i = 0.1$*. Figure 4 demonstrates the relationship between the second order derivative* $\frac{\partial^2 R_i(\boldsymbol{\pi}_i, \mathbf{p}_{\text{-}i}))}{\partial \pi_{i1}^2}$ *with respect to the market share* π_{i2} *of product 2, fixing the market shore of product 1 at* $\pi_{i1} = 0.0054$*. It shows that* $\frac{\partial^2 R_i(\boldsymbol{\pi}_i, \mathbf{p}_{\text{-}i}))}{\partial \pi_{i1}^2} > 0$ *for* $0.05 \leq \pi_{i2} < 0.24$*, then* $R_i(\boldsymbol{\pi}_i, \mathbf{p}_{\text{-}i}))$ *is not always concave in the feasible region.*

Appendix B: Proofs

Proof of Lemma 1. (a) Consider the first order derivative of $w_i^{F_i}(\theta_i)$. Then

$$\frac{\partial w_i^{F_i}(\theta_i)}{\partial \theta_i} = -1 + \frac{\sum_{k \in F_i} e^{\tilde{\alpha}_{ik} - \beta_{ik}\theta_i} / \beta_{ik} \sum_{s \in F_i} \beta_{is} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}}{\left(\sum_{s \in F_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}\right)^2}.$$

That $\frac{\partial w_i^{F_i}(\theta_i)}{\partial \theta_i} \geq 0$ can be shown by Cauchy-Schwarz inequality that is $\sum_{i=1}^m x_i \sum_{i=1}^m y_i \geq \left(\sum_{i=1}^m \sqrt{x_i y_i}\right)^2$ for any $x_i, y_i \geq 0$. Because

$$\sum_{k\in F_i} \frac{e^{\tilde{\alpha}_{ik}-\beta_{ik}\theta_i}}{\max_{s\in F_i}\beta_{is}} \leq \sum_{k\in F_i} \frac{e^{\tilde{\alpha}_{ik}-\beta_{ik}\theta_i}}{\beta_{ik}} \leq \sum_{k\in F_i} \frac{e^{\tilde{\alpha}_{ik}-\beta_{ik}\theta_i}}{\min_{s\in F_i}\beta_{is}}$$

then, $\frac{1}{\max_{s\in F_i}\beta_{is}} \leq w_i^{F_i}(\theta_i) \leq \frac{1}{\min_{s\in F_i}\beta_{is}}$ and the inequalities become equalities when β_{is} is constant for all $s \in F_i$.

(b) Consider the first order derivative of $v_i^{F_i}(\theta_i)$. Then

$$\frac{\partial v_i^{F_i}(\theta_i)}{\partial \theta_i} = \frac{-\sum_{k\in F_i}\beta_{ik}^2 e^{\tilde{\alpha}_{ik}-\beta_{ik}\theta_i}\cdot\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i} + \left(\sum_{s\in F_i}\beta_{is}e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^2}{\left(\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^2}.$$

It can be shown that $\frac{\partial v_i(\theta_i)}{\partial \theta_i} \leq 0$ by a similar argument to part (a).

$$w_i^{F_i}(\theta_i)v_i^{F_i}(\theta_i) = \frac{\sum_{k\in F_i} e^{\tilde{\alpha}_{ik}-\beta_{ik}\theta_i}/\beta_{ik}\sum_{s\in F_i}\beta_{is}e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}}{\left(\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^2} = \frac{\partial w_i^{F_i}(\theta_i)}{\partial \theta_i} + 1 \geq 1$$

The inequality holds because of part (a). □

Proof of Proposition 1. From

$$Q_i^{F_i}(\theta_i, \mathbf{p}_{\text{-}i}) = \frac{\left(\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}}{1 + a_{\text{-}i} + \left(\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}} = \rho_i,$$

then, $\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i} = \left(\frac{\rho_i(1+a_{\text{-}i})}{1-\rho_i}\right)^{\frac{1}{\gamma_i}}$ and $r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i})$ can be written as follows

$$r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i}) = \rho_i\left(\theta_i + \frac{\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}/\beta_{is}}{\left(\frac{\rho_i(1+a_{\text{-}i})}{1-\rho_i}\right)^{\frac{1}{\gamma_i}}}\right),$$

where θ_i is the unique solution to $\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i} = \left(\frac{\rho_i(1+a_{\text{-}i})}{1-\rho_i}\right)^{\frac{1}{\gamma_i}}$, denoted by $\theta_i^{F_i}$. Denote $H^{F_i}(\theta_i)$ as follows:

$$H^{F_i}(\theta_i) = \rho_i\left(\theta_i + \frac{\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}/\beta_{is}}{\left(\frac{\rho_i(1+a_{\text{-}i})}{1-\rho_i}\right)^{\frac{1}{\gamma_i}}}\right), \tag{36}$$

Note that function (36) is convex in θ_i and

$$\left.\frac{\partial H^{F_i}(\theta_i)}{\partial \theta_i}\right|_{\theta_i=\theta_i^{F_i}} = \rho_i\left.\left(1 - \frac{\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}}{\left(\frac{\rho_i(1+a_{\text{-}i})}{1-\rho_i}\right)^{\frac{1}{\gamma_i}}}\right)\right|_{\theta_i=\theta_i^{F_i}} = 0.$$

The last inequality holds because $\sum_{s\in F_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}|_{\theta_i=\theta_i^{F_i}} = \left(\frac{\rho_i(1+a_{\text{-}i})}{1-\rho_i}\right)^{\frac{1}{\gamma_i}}$. So, $r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i})$ is equal to the minimum of $H^{F_i}(\theta_i)$ in θ_i, i.e., $r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i}) = \min_{\theta_i} H^{F_i}(\theta_i)$.

Suppose that another product z is added to set F_i and denote $F_i^+ := F_i \cup \{z\}$. Similarly, we have $r_i^{F_i^+}(\rho_i, \mathbf{p}_{\text{-}i}) = \min_{\theta_i} H^{F_i^+}(\theta_i) = H^{F_i^+}(\theta_i^{F_i^+})$.

Note that $H^{F_i^+}(\theta_i) > H^{F_i}(\theta_i)$ for all θ_i and $\theta_i^{F_i} < \theta_i^{F_i^+}$. Therefore,

$$r_i^{F_i^+}(\rho_i, \mathbf{p}_{\text{-}i}) = H^{F_i^+}(\theta_i^{F_i^+}) > H^{F_i}(\theta_i^{F_i^+}) > H^{F_i}(\theta_i^{F_i}) = r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i}).$$

The first inequality holds because $H^{F_i^+}(\theta_i) > H^{F_i}(\theta_i)$ for all θ_i; the second inequality holds because $\theta_i^{F_i^+}$ is the minimizer of $H^{F_i^+}(\theta_i)$.

Therefore, $r_i^{F_i}(\rho_i, \mathbf{p}_{\text{-}i})$ is strictly increasing in F_i for any $0 < \rho_i < 1$. $\square$

Proof of Theorem 1. (a) Because Proposition 1 says that it is optimal to offer all the products at finite prices, from equation (7),

$$p_{ij} - c_{ij} - \frac{1}{\beta_{ij}} = (1 - \gamma_i(1 - Q_i(\mathbf{p}_i, \mathbf{p}_{\text{-}i}))) \sum_{s=1}^{m_i} (p_{is} - c_{is}) q_{s|i}(\mathbf{p}_i). \tag{37}$$

Since the right hand side (RHS) of equation (37) is independent of j, then $p_{ij} - c_{ij} - \frac{1}{\beta_{ij}}$ is constant for each $j = 1, 2, \ldots, m_i$ from equations (37). Let $\theta_i = p_{ij} - c_{ij} - \frac{1}{\beta_{ij}}$. Equation (37) can be rewritten as follows:

$$\left(1 - \frac{(1 + a_{\text{-}i})\gamma_i}{1 + a_{\text{-}i} + (\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i})^{\gamma_i}}\right) w_i(\theta) - \frac{(1 + a_{\text{-}i})\gamma_i\theta_i}{1 + a_{\text{-}i} + (\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i})^{\gamma_i}} = 0,$$

where $\tilde{\alpha}_{is} = \alpha_{is} - \beta_{is}c_{is} - 1$. Since $(1 + a_{\text{-}i}) + (\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i})^{\gamma_i} < \infty$, the above equation can be rewritten as equation (13). After some algebra, equation (37) can also be rewritten as follows

$$Q_i(\theta_i, \mathbf{p}_{\text{-}i})(\theta_i + w_i(\theta_i)) = \theta_i + (1 - \frac{1}{\gamma_i}) w_i(\theta_i).$$

So, the total expected profit can be expressed as follows

$$R_i(\theta_i, \mathbf{p}_{\text{-}i}) = \theta_i + (1 - \frac{1}{\gamma_i}) w_i(\theta_i), \tag{38}$$

where θ_i is a solution to equation (13). Furthermore, the optimal adjusted markup is the solution that maximizes the expression (38). The proof to the uniqueness of the solution to equation (13) can be found in part (b) below.

(b) Consider the first order derivative of $R_i(\theta_i, \mathbf{p}_{\text{-}i})$ as follows

$$\begin{aligned} \frac{\partial R_i(\theta_i, \mathbf{p}_{\text{-}i})}{\partial \theta_i} &= \sum_{s=1}^{m_i} \beta_{is}\pi_{is}(\theta_i, \mathbf{p}_{\text{-}i}) \Big[-\gamma_i(1 - Q_i(\theta_i, \mathbf{p}_{\text{-}i}))\theta_i + (1 - \gamma_i(1 - Q_i(\theta_i, \mathbf{p}_{\text{-}i})) w_i(\theta_i)) \Big] \\ &= \gamma_i Q_i(\theta_i, \mathbf{p}_{\text{-}i}) \big(1 - Q_i(\theta_i, \mathbf{p}_{\text{-}i})\big) v_i(\theta_i) \Bigg[- \left(\theta_i + (1 - \frac{1}{\gamma_i}) w_i(\theta)\right) \\ &\quad + \frac{1}{\gamma_i(1 + a_{\text{-}i})} \cdot \left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i} / \beta_{is}\right) \left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}\right)^{\gamma_i - 1} \Bigg]. \end{aligned} \tag{39}$$

Let $X(\theta_i) = -\Big(\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i)\Big) + \frac{1}{\gamma_i(1+a_{\text{-}i})} \cdot \Big(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}/\beta_{is}\Big)\Big(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\Big)^{\gamma_i-1}$. Consider its first order derivative

$$\frac{\partial X(\theta_i)}{\partial\theta_i} = \frac{1}{\gamma_i(1-Q_i(\theta_i,\mathbf{p}_{\text{-}i}))} \cdot \Big((1-\gamma_i)w_i(\theta_i)v_i(\theta_i) - 1\Big).$$

We claim that $(1-\gamma_i)w_i(\theta_i)v_i(\theta_i) < 1$ for all θ_i under Condition 1. It is clearly true if $\gamma_i \geq 1$. If there are multiple products with different price sensitivities then $w_i(\theta_i)v_i(\theta_i) < \frac{\max_s \beta_{is}}{\min_s \beta_{is}}$ together with $\frac{\max_s \beta_{is}}{\min_s \beta_{is}} \leq \frac{1}{(1-\gamma_i)}$ imply that $(1-\gamma_i)w_i(\theta_i)v_i(\theta_i) < 1$; otherwise $w_i(\theta_i)v_i(\theta_i) = 1$ for all θ_i, then, clearly, $(1-\gamma_i)w_i(\theta_i)v_i(\theta_i) < 1$.

Therefore, $\frac{\partial X(\theta_i)}{\partial\theta_i} < 0$ for all θ_i under Condition 1, and $X(\theta_i)$ is decreasing from positive to negative as θ_i goes from $-\infty$ to ∞. Hence, $R_i(\theta_i,\mathbf{p}_{\text{-}i})$ is strictly uni-modal with respect to θ_i and there exists a unique solution to $\frac{\partial R_i(\theta_i,\mathbf{p}_{\text{-}i})}{\partial\theta_i} = 0$.

Let $\rho(\theta_i) = Q_i^{F_i}(\theta_i,\mathbf{p}_{\text{-}i})$. For the concavity of $r_i(\rho_i,\mathbf{p}_{\text{-}i})$, from equation (11),

$$\begin{aligned}\frac{\partial r_i(\rho_i,\mathbf{p}_{\text{-}i})}{\partial\theta_i} &= \gamma_i(1-Q_i(\theta_i,\mathbf{p}_{\text{-}i}))Q_i(\theta_i,\mathbf{p}_{\text{-}i})v_i(\theta_i)\Bigg[-\Big(\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i)\Big) \\ &\quad + \frac{1}{\gamma_i(1+a_{\text{-}i})} \cdot \left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}/\beta_{is}\right)\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i-1}\Bigg], \\ \frac{\partial\rho_i(\theta_i)}{\partial\theta_i} &= -\gamma_i(1-Q_i(\theta_i,\mathbf{p}_{\text{-}i}))Q_i(\theta_i,\mathbf{p}_{\text{-}i})v_i(\theta_i).\end{aligned}$$

Then,

$$\begin{aligned}\frac{\partial r_i(\rho_i,\mathbf{p}_{\text{-}i})}{\partial\rho_i} &= \frac{\partial r_i(\rho_i,\mathbf{p}_{\text{-}i})/\partial\theta_i}{\partial\rho_i(\theta_i)/\partial\theta_i} \\ &= \left(\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i)\right) - \frac{1}{\gamma_i(1+a_{\text{-}i})} \cdot \left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}/\beta_{is}\right)\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial^2 r_i(\rho_i,\mathbf{p}_{\text{-}i})}{\partial\rho_i^2} &= \frac{\partial}{\partial\rho_i}\left(\frac{\partial r_i(\rho_i,\mathbf{p}_{\text{-}i})}{\partial\rho_i}\right) = \frac{\partial\left(\frac{\partial r_i(\rho_i,\mathbf{p}_{\text{-}i})}{\partial\rho_i}\right)/\partial\theta_i}{\partial\rho_i(\theta_i)/\partial\theta_i} \\ &= \frac{1}{\left(\gamma_i(1-Q_i(\theta_i,\mathbf{p}_{\text{-}i}))\right)^2 Q_i(\theta_i,\mathbf{p}_{\text{-}i})v_i(\theta_i)} \cdot \Big((1-\gamma_i)\cdot w_i(\theta_i)v_i(\theta_i) - 1\Big) < 0.\end{aligned}$$

The inequality holds because $(1-\gamma_i)w_i(\theta_i)v_i(\theta_i) < 1$ for all θ_i under Condition 1. Thus, $r_i(\rho_i,\mathbf{p}_{\text{-}i})$ is strictly concave in ρ_i under Condition 1. □

Proof of Proposition 2. The profit can be rewritten in function (38): $R_i(\theta_i) = \theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i)$, where θ_i is a solution to equation (13). If $\gamma_i \geq 1$, that $\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i) \geq 0$ implies that $\theta_i \geq -(1-\frac{1}{\gamma_i})w_i(\theta_i) \geq$

$\frac{1-\gamma_i}{\gamma_i \min_s \beta_{is}} \stackrel{\text{def}}{=} \theta_{i,\min}$; if $0 < \gamma_i < 1$, that $\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i) \geq 0$ implies that $\theta_i \geq -(1-\frac{1}{\gamma_i})w_i(\theta_i) \geq \frac{1-\gamma_i}{\gamma_i \max_s \beta_{is}} \stackrel{\text{def}}{=} \theta'_{i,\min}$.

From the FOC (13),

$$\theta_i = \frac{1}{\gamma_i}\left((1-\gamma_i) + \frac{1}{(1+a_{\text{-}i})}\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}\right) w_i(\theta_i) \leq \frac{1-\gamma_i + \left(\sum_{s=1}^{m_s} e^{\tilde{\alpha}_{is}\beta_{is}\theta_i}\right)^{\gamma_i}/(1+a_{\text{-}i})}{\gamma_i \min_s \beta_{is}}.$$

Because the RHS of the inequality is decreasing in θ_i, we have the upper bounds $\theta_{i,\max}$ and $\theta'_{i,\max}$.

Actually, the upper bound can be a bit tighter if Condition 1 is satisfied. Let $X_2(\theta_i) = \frac{1}{\gamma_i}\left((1-\gamma_i) + \frac{1}{(1+a_{\text{-}i})}\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}\right) w_i(\theta_i)$. Then, under Condition 1,

$$\frac{\partial X_2(\theta_i)}{\partial \theta_i} = \frac{Q_i(\theta_i, \theta_{\text{-}i})}{\gamma_i(1-Q_i(\theta_i,\theta_{\text{-}i}))} \cdot \Big((1-\gamma_i)w_i(\theta_i)v_i(\theta_i) - 1\Big) < 0.$$

So, $X_2(\theta_i)$ is decreasing in θ_i under Condition 1 and it is straightforward to find a tighter upper bound. □

Proof of Theorem 2. (a) Let F_i be the set of products in nest i to offer at finite prices. The *adjusted markup* of each product $k \in F_i$ is constant, denoted by θ_i. Suppose the prices of all products in other nests are given and denote total attractiveness of nest j as $a_j = \sum_{s=1}^{m_j} e^{\alpha_{js}-\beta_{js}p_{js}}$ $\forall j \neq i$. The total market share can be expressed as follows

$$\rho = \sum_{k=1}^{n} Q_k(\theta_i, \mathbf{p}_{\text{-}i}) = \frac{\sum_{j\neq i} a_j + \left(\sum_{s\in F_i} e^{\alpha_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}}{1 + \sum_{j\neq i} a_j + \left(\sum_{s\in F_i} e^{\alpha_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}}.$$

Then, the total attractiveness of nest i is

$$\left(\sum_{s\in F_i} e^{\alpha_{is}-\beta_{is}\theta_i}\right)^{\gamma_i} = \frac{1}{1-\rho} - (1+a_{\text{-}i}) \tag{40}$$

where $a_{\text{-}i} = \sum_{j\neq i} a_j$. Given that the total market share is ρ and the offered product set in nest i is F_i, the *adjusted markup* is the unique solution to equation (40), denoted by $\theta_i^{F_i}$, and the total profit can be expresses as follows

$$\begin{aligned} R^{F_i}(\rho) &= Q_i(\theta_i, \mathbf{p}_{\text{-}i})(\theta_i + w_i(\theta_i)) + \sum_{j\neq i} Q_j(\theta_i, \mathbf{p}_{\text{-}i}) \sum_{s=1}^{m_j} (p_{js} - c_{js}) q_{s|j}(\mathbf{p}_j) \\ &= \big(1-(1+a_{\text{-}i})(1-\rho)\big)\cdot\left(\theta_i + \frac{\sum_{s\in F_i} e^{\alpha_{is}-\beta_{is}\theta_i}/\beta_{is}}{\big(1/(1-\rho)-(1+a_{\text{-}i})\big)^{1/\gamma_i}}\right) + \sum_{j\neq i} a_j(1-\rho)\cdot\sum_{s=1}^{m_j}(p_{js}-c_{js})q_{s|j}(\mathbf{p}_j), \end{aligned}$$

where $\theta_i = \theta_i^{F_i}$. Let $H_2^{F_i}(\theta_i)$ be

$$H_2^{F_i}(\theta_i) = \big(1-(1+a_{\text{-}i})(1-\rho)\big)\cdot\left(\theta_i + \frac{\sum_{s\in F_i} e^{\alpha_{is}-\beta_{is}\theta_i}/\beta_{is}}{\big(1/(1-\rho)-(1+a_{\text{-}i})\big)^{1/\gamma_i}}\right) + \sum_{j\neq i} a_j(1-\rho)\cdot\sum_{s=1}^{m_j}(p_{js}-c_{js})q_{s|j}(\mathbf{p}_j).$$

Note that $H_2^{F_i}(\theta_i)$ is convex in θ_i and

$$\left.\frac{\partial H_2^{F_i}(\theta_i)}{\partial \theta_i}\right|_{\theta_i=\theta_i^{F_i}} = 0.$$

Then, $R^{F_i}(\rho) = \min_{\theta_i} H_2^{F_i}(\theta_i) = H_2^{F_i}(\theta_i^{F_i})$. Similarly to the proof of Proposition 1, we can show that $R^{F_i}(\rho) < R^{F_i^+}(\rho)$ for any $0 < \rho < 1$. Therefore, it is optimal to offer all the products at prices such that the *adjusted markup* is constant in each nest.

(b) Let E be the set of nests whose *adjusted markup* satisfies equation (16). The total market share can be expressed as follows

$$\rho = \sum_{i\in E} Q_i(\theta_i, \theta_{\text{-}i}) = \frac{\sum_{i\in E}\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}}{1+\sum_{i\in E}\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}}, \tag{41}$$

where $\theta_{\text{-}i}$ is the vector of the adjusted markups for all other nests in set E excluding nest i. Then,

$$\sum_{i\in E}\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i} = \frac{\rho}{1-\rho}, \tag{42}$$

$$\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i) = \phi, \ \forall i = 1, 2, \ldots, n. \tag{43}$$

The solution to (42) and (43) is unique, which will be shown later, denoted by ϕ^E and $\boldsymbol{\theta}^E$. The total profit can be expressed

$$\begin{aligned} R^E(\rho) &= \sum_{i\in E} Q_i(\theta_i, \theta_{\text{-}i})\left(\phi^E + \frac{w_i(\theta_i)}{\gamma_i}\right) \\ &= \rho\phi + \sum_{i\in E}\frac{\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}\cdot w_i(\theta_i)}{\gamma_i/(1-\rho)}, \end{aligned}$$

where $\phi = \phi^E$ and $\boldsymbol{\theta} = \boldsymbol{\theta}^E$. Denote $H_3^E(\phi)$ as follows

$$H_3^E(\phi) = \rho\phi + \sum_{i\in E}\frac{\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}\cdot w_i(\theta_i)}{\gamma_i/(1-\rho)}.$$

where θ_i satisfies (43). We will next show that $H_3^E(\phi)$ is convex in ϕ under Condition 1 for each i.

$$\frac{\partial H_3^E(\phi)}{\partial\phi} = \rho + \sum_{i\in E}\frac{\partial G_3^E(\phi)/\partial\theta_i}{\partial\phi/\partial\theta_i} = \rho - (1-\rho)\sum_{i\in E}\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i},$$

where $G_3^E(\phi) = \sum_{i\in E}\frac{\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}\cdot w_i(\theta_i)}{\gamma_i/(1-\rho)}$. The second equality holds because

$$\begin{aligned} \frac{\partial G_3^E(\phi)}{\partial\theta_i} &= -\frac{1-\rho}{\gamma_i}\cdot\big(1-(1-\gamma_i)w_i(\theta_i)v_i(\theta_i)\big)\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}, \\ \frac{\partial\phi}{\partial\theta_i} &= \frac{1}{\gamma_i}\big(1-(1-\gamma_i)w_i(\theta_i)v_i(\theta_i)\big). \end{aligned}$$

And $\frac{\partial H_3^E(\phi)}{\partial\phi}|_{\phi=\phi^E} = 0$ because of equation (42). The second order derivative of $H_3^E(\phi)$ is

$$\frac{\partial^2 H_3^E(\phi)}{\partial\phi^2} = \sum_{i\in E}\frac{\frac{\partial}{\partial\theta_i}(\partial H_3^E(\phi)/\partial\phi)}{\frac{\partial\phi}{\partial\theta_i}} = \sum_{i\in E}\frac{\gamma_i^2(1-\rho)v_i(\theta_i)\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}}{1-(1-\gamma_i)w_i(\theta_i)v_i(\theta_i)} > 0$$

Thus, $H_3^E(\phi)$ is convex in ϕ, and the solution to (42) and (43) is unique. Let E^+ be the new nest set if the *adjusted markup* of another nest satisfies equation (16). We can prove that $R^{E^+}(\rho) > R^E(\rho)$ for all $\rho \in (0,1)$ by the same argument as in the proof of Theorem 1. Therefore, it is optimal to keep the *adjusted markup* of all the nests to satisfy equation (16).

(c) Consider the FOC of $R(\phi)$,

$$\frac{\partial R(\phi)}{\partial \phi} = \sum_{i=1}^{n} \frac{\partial R(\boldsymbol{\theta})/\partial \theta_i}{\partial \phi / \partial \theta_i} = (R(\phi) - \phi) \sum_{i=1}^{n} \frac{\gamma_i^2 Q_i(\theta_1, \ldots, \theta_n) v_i(\theta_i)}{1 - (1-\gamma_i) w_i(\theta_i) v_i(\theta_i)} = 0, \tag{44}$$

where θ_i is the solution to equation (43). Because $\sum_{i=1}^{n} \frac{\gamma_i^2 Q_i(\theta_1,\ldots,\theta_n) v_i(\theta_i)}{1-(1-\gamma_i) w_i(\theta_i) v_i(\theta_i)} > 0$ under Condition 1 for each i, then, $R(\phi)$ is increasing (decreasing) in ϕ if and only if $R(\phi) \geq (\leq) \phi$.

(i) Case I: there is only one solution to equation (44), denoted by ϕ^*. Apparently $R(\phi)$ is increasing in ϕ for $\phi \leq \phi^*$ and is decreasing in ϕ for $\phi > \phi^*$.

(ii) Case II: there are multiple solutions to equation (44). Suppose that there are two consecutive solutions $\phi_1 = R(\phi_1) < \phi_2 = R(\phi_2)$ and there is no solution to equation (44) between ϕ_1 and ϕ_2. It must hold that $R(\phi) < R(\phi_2)$ for any $\phi_1 < \phi < \phi_2$; otherwise, there must be another solution to equation (44) between ϕ_1 and ϕ_2, which contradicts that ϕ_1 and ϕ_2 are two consecutive solutions. We claim that $R(\phi)$ is increasing in ϕ for $\phi \in [\phi_1, \phi_2]$. Assume there are two points $\phi_1 < \phi_1' < \phi_2' < \phi_2$ such that $R(\phi_1') > R(\phi_2')$. Then, there must be a solution to equation (44) between ϕ_1' and ϕ_2, which also contradicts that ϕ_1 and ϕ_2 are two consecutive solutions. Thus, $R(\phi)$ is increasing between any two solutions to equation (44) and $R(\phi)$ may be decreasing after the largest solution. Therefore, $R(\phi)$ is unimodual with respect to ϕ under Condition 1 for each i.

(d) Let $\rho(\phi) = \sum_{i=1}^{n} Q_i(\theta_1, \ldots, \theta_n)$, where θ_i is the solution to $\theta_i + (1 - \frac{1}{\gamma_i}) w_i(\theta_i) = \phi$, $\forall i = 1, 2, \ldots, n$. Then,

$$\begin{aligned}
\frac{\partial \rho(\phi)}{\partial \phi} &= \sum_{i=1}^{n} \frac{\partial \rho(\phi)/\partial \theta_i}{\partial \phi / \partial \theta_i} = -Q_0(\theta_1, \ldots, \theta_n) \sum_{i=1}^{n} \frac{\gamma_i^2 Q_i(\theta_1, \ldots, \theta_n) v_i(\theta_i)}{1 - (1-\gamma_i) w_i(\theta_i) v_i(\theta_i)}, \\
\frac{\partial R(\phi)}{\partial \rho} &= \frac{\partial R(\phi)/\partial \phi}{\partial \rho(\phi)/\partial \phi} = -\frac{R(\boldsymbol{\theta}) - \phi}{1-\rho}.
\end{aligned}$$

We can easily show that $R(\rho)$ is unimodual in ρ by a similar argument to part (c). Moreover, we consider the second order derivative under Condition 1 for all i,

$$\begin{aligned}
\frac{\partial^2 R(\rho)}{\partial \rho^2} &= -\frac{\partial}{\partial \rho} \cdot \frac{R(\boldsymbol{\theta}) - \phi}{1-\rho} = -\frac{R(\boldsymbol{\theta}) - \phi}{(1-\rho)^2} + \frac{1}{1-\rho} \cdot \frac{\frac{\partial R(\boldsymbol{\theta})}{\partial \phi} - 1}{\frac{\partial \rho}{\partial \phi}} \\
&= -\frac{1}{(1-\rho)^2 \sum_{i=1}^{n} \frac{\gamma_i^2 Q_i(\theta_1,\ldots,\theta_n) v_i(\theta_i)}{1-(1-\gamma_i) w_i(\theta_i) v_i(\theta_i)}} < 0.
\end{aligned}$$

The last equality hold because $(1-\gamma_i) w_i(\theta_i) v_i(\theta_i) < 1$ for all θ_i and each i under Condition 1. Therefore, $R(\rho)$ is concave in ρ under Condition 1 for each i. $\square$

Proof of Proposition 3 From equation (16), the optimal ϕ^* is one of the fixed points of $R(\phi)$ and apparently $\phi^* > 0$. Because $\theta_i^* + (1 - \frac{1}{\gamma_i}) w_i(\theta_i^*) = \phi^*$, then $\theta_i^* \geq \frac{1}{\gamma_i \max_s \beta_{is}} - \frac{1}{\min_s \beta_{is}} \stackrel{\text{def}}{=} \hat{\theta}_{i,\min}$.

Equation (16) can be rewritten as follows

$$\phi = \sum_{i=1}^{n} Q_i(\theta_i, \theta_{\text{-}i}) \left(\phi + \frac{w_i(\theta_i)}{\gamma_i}\right),$$

which can be simplified to

$$\phi = \sum_{i=1}^{n} \left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i} \cdot \frac{w_i(\theta_i)}{\gamma_i} \le \sum_{i=1}^{n} \frac{(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i})^{\gamma_i}}{\gamma_i \min_s \beta_{is}} \le \sum_{i=1}^{n} \frac{\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\hat{\theta}_{i,\min}}\right)^{\gamma_i}}{\gamma_i \min_s \beta_{is}} \stackrel{\text{def}}{=} \phi_{max}.$$

The first inequality holds because $w_i(\theta_i) \le \frac{1}{\min_s \beta_{is}}$ for each i from Lemma 1 and the second inequality holds because $(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i})^{\gamma_i}$ is decreasing in θ_i for each i.

If Condition 1 is satisfied for some i, a tighter upper bound can be easily found because $(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i})^{\gamma_i} \cdot \frac{w_i(\theta_i)}{\gamma_i}$ is decreasing in θ_i under Condition 1. □

Proof of Theorem 3. (a) Suppose that $(\mathbf{p}_i^*, \mathbf{p}_{\text{-}i}^*)$ is an equilibrium of **Game I**. From Theorem 1, the *adjusted markup* for all the products of each firm is constant, i.e., $p_{ik} - c_{ik} - \frac{1}{\beta_{ik}}$ is constant for all k, denoted by θ_i^*. We will argue that $(\theta_i^*, \theta_{\text{-}i}^*)$ must be the equilibrium of **Game II**. If firm i is better-off to deviate to $\hat{\theta}_i$, then firm i will also be better-off to deviate to $\hat{\mathbf{p}}_i$ in **Game I**, where $\hat{\mathbf{p}}_i = (\hat{p}_{i1}, \ldots, \hat{p}_{im_i})$ and $\hat{p}_{ik} = \hat{\theta}_i + c_{ik} + \frac{1}{\beta_{ik}}$. It contradicts that $(\mathbf{p}_i^*, \mathbf{p}_{\text{-}i}^*)$ is equilibrium of **Game I**.

Suppose that $(\theta_i^*, \theta_{\text{-}i}^*)$ is an equilibrium of **Game II**. We will argue that $(\mathbf{p}_i^*, \mathbf{p}_{\text{-}i}^*)$ is an equilibrium of **Game I**, where $p_{ik} = \theta_i^* + c_{ik} + \frac{1}{\beta_{ik}}$ for all k. If firm i is better-off to deviate to $\hat{\mathbf{p}}_i := (\hat{p}_{i1}, \hat{p}_{i2}, \ldots, \hat{p}_{im_i})$ in **Game I**, $\hat{p}_{ik} - c_{ik} - \frac{1}{\beta_{ik}}$ must be constant from Theorem 1, denoted by $\hat{\theta}_i$. Then, firm i must be better-off to deviate to $\hat{\theta}_i$ in **Game II**, which contradicts that $(\theta_i^*, \theta_{\text{-}i}^*)$ is an equilibrium of **Game II**.

(b) Consider the derivatives of $\log R_i(\theta_i, \theta_{\text{-}i})$:

$$\begin{aligned}
\frac{\partial \log R_i(\theta_i, \theta_{\text{-}i})}{\partial \theta_i} &= -\gamma_i(1 - Q_i(\theta_i, \theta_{\text{-}i}))v_i(\theta_i) + \frac{w_i(\theta_i)v_i(\theta_i)}{\theta_i + w_i(\theta_i)}, \\
\frac{\partial \log R_i(\theta_i, \theta_{\text{-}i})}{\partial \theta_j} &= \gamma_j Q_j(\theta_j, \theta_{\text{-}j})v_j(\theta_j) \ge 0, \ \forall j \ne i, \\
\frac{\partial^2 \log R_i(\theta_i, \theta_{\text{-}i})}{\partial \theta_i \partial \theta_j} &= \gamma_i \gamma_j Q_i(\theta_i, \theta_{\text{-}i}) Q_j(\theta_j, \theta_{\text{-}j}) v_i(\theta_i) v_j(\theta_j) \ge 0, \ \forall j \ne i.
\end{aligned}$$

Then, **Game II** is a log-supermodular game. Note that the strategy space for each firm is the real line. From Topkis (1998) and Vives (2001), the equilibrium set is a nonempty complete lattice and, therefore, has the componentwise largest and smallest elements, denoted by $\overline{\boldsymbol{\theta}}^*$ and $\underline{\boldsymbol{\theta}}^*$ respectively.

For any equilibrium $\boldsymbol{\theta}^*$, $\overline{\boldsymbol{\theta}}^* \ge \boldsymbol{\theta}^* \ge \underline{\boldsymbol{\theta}}^*$ and

$$\log R_i(\theta_i^*, \theta_{\text{-}i}^*) \le \log R_i(\theta_i^*, \overline{\theta}_{\text{-}i}^*) \le \log R_i(\overline{\theta}_i^*, \overline{\theta}_{\text{-}i}^*).$$

The first inequality holds because $\frac{\partial \log R_i(\theta_i,\theta_{\text{-}i})}{\partial \theta_j} \geq 0$; the second inequality holds because $(\overline{\theta}_i^*, \overline{\theta}_{\text{-}i}^*)$ is a Nash equilibrium of the log-supermodular game. Because logarithm is an monotonic increasing transformation,

$$R_i(\theta_i^*, \theta_{\text{-}i}^*) \leq R_i(\theta_i^*, \overline{\theta}_{\text{-}i}^*) \leq R_i(\overline{\theta}_i^*, \overline{\theta}_{\text{-}i}^*).$$

Therefore, the largest equilibrium $\overline{\boldsymbol{\theta}}^*$ is preferred by all the firms. □

Proof of Theorem 4 (a) Suppose that there exists an asymmetric equilibrium, denoted by $(\theta_1^*, \theta_2^*, \theta_3^*, \ldots, \theta_n^*)$. Suppose that θ_1^* is the largest and θ_2^* is the smallest without loss of generality, then $\theta_1^* > \theta_2^*$. Because the game is symmetric, $(\theta_2^*, \theta_1^*, \theta_3^*, \ldots, \theta_n^*)$ is also an equilibrium. In other words, the best strategies for firm 1 are θ_1^* and θ_2^* respectively corresponding to other firms' strategies $(\theta_2^*, \theta_3^*, \ldots, \theta_n^*)$ and $(\theta_1^*, \theta_3^*, \ldots, \theta_n^*)$. Since the game is strictly supermodular and $(\theta_2^*, \theta_3^*, \ldots, \theta_n^*) < (\theta_1^*, \theta_3^*, \ldots, \theta_n^*)$, then $\theta_1^* \leq \theta_2^*$, which is a contradiction.

(b) Recall the first order derivative of $R_i(\theta_i, \theta_{\text{-}i})$ equation (39)

$$\frac{\partial R_i(\theta_i,\theta_{\text{-}i})}{\partial \theta_i} = \gamma_i Q_i(\theta_i,\theta_{\text{-}i})\big(1-Q_i(\theta_i,\theta_{\text{-}i})\big)v_i(\theta_i)\Bigg[-\left(\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta)\right) + \frac{1}{\gamma_i(1+a_{\text{-}i})}\cdot\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}/\beta_{is}\right)\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i-1}\Bigg].$$

For the symmetric equilibria in an n-firm game, $a_{\text{-}i} = (n-1)\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}$. Let $Y(\theta_i) = -\left(\theta_i + (1-\frac{1}{\gamma_i})w_i(\theta_i)\right) + \frac{\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}}{\gamma_i\left(1+(n-1)\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}\right)}\cdot w_i(\theta_i)$ and it can be simplified to

$$Y(\theta_i) = -\theta_i - \left(1 - \frac{1}{\gamma_i(1-Q_i(\theta_i,n))}\right)w_i(\theta_i) = 0.$$

Consider its first order derivative

$$\frac{\partial Y(\theta_i)}{\partial \theta_i} = \frac{1}{\gamma_i(1-Q_i(\theta_i,n))}\cdot\left(-1 + w_i(\theta_i)v_i(\theta_i)\left(1-\gamma_i+\gamma_i\frac{(n-1)(Q_i(\theta_i,n))^2}{1-Q_i(\theta_i,n)}\right)\right),$$

where $Q_i(\theta_i,n)$ is the market share for firm i when all firms charge the same *adjusted markup* θ_i, i.e., $Q_i(\theta_i,n) = \frac{\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}}{1+n\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is}-\beta_{is}\theta_i}\right)^{\gamma_i}}$. Clearly, $Q_i(\theta_i,n) < \frac{1}{n}$. Then,

$$\frac{\partial Y(\theta_i)}{\partial \theta_i} < \frac{1}{\gamma_i(1-Q_i(\theta_i,n))}\cdot\left(-1 + w_i(\theta_i)v_i(\theta_i)\left(1-\frac{n-1}{n}\gamma_i\right)\right).$$

Therefore,

(i) If $\gamma_i \geq \frac{n}{n-1}$, $\frac{\partial Y(\theta_i)}{\partial \theta_i} < 0$ for all θ_i.

(ii) If $0 < \gamma_i < \frac{n}{n-1}$ and $\frac{\max_s \beta_{is}}{\min_s \beta_{is}} \leq \frac{1}{1-\frac{n-1}{n}\cdot\gamma_i}$, we claim that $w_i(\theta_i)v_i(\theta_i)\left(1-\frac{n-1}{n}\gamma_i\right) < 1$. If there are more than one products with different price coefficients, $w_i(\theta_i)v_i(\theta_i) < \frac{\max_s \beta_{is}}{\min_s \beta_{is}}$; otherwise $w_i(\theta_i)v_i(\theta_i) = 1$ for all θ_i. In both cases, $w_i(\theta_i)v_i(\theta_i)\left(1-\frac{n-1}{n}\gamma_i\right) < 1$.

Thus, $\frac{\partial Y(\theta_i)}{\partial \theta_i} < 0$ for all θ_i under Condition 2. Then, $Y(\theta_i)$ is strictly decreasing from positive to negative as θ_i goes from $-\infty$ to ∞. Hence, there exists a unique solution to $Y(\theta_i) = 0$ and it is also the unique equilibrium to the symmetric game. $\square$

Proof of Lemma 2. Consider the derivatives of $\log(\theta_i + w_i(\theta_i))$,

$$\begin{aligned}
\frac{\partial \log(\theta_i + w_i(\theta_i))}{\partial \theta_i} &= \frac{w_i(\theta_i)v_i(\theta_i)}{\theta_i + w_i(\theta_i)}, \\
\frac{\partial^2 \log(\theta_i + w_i(\theta_i))}{\partial \theta_i^2} &= \frac{w_i(\theta_i)}{\theta_i + w_i(\theta_i)} \cdot \frac{\partial v_i(\theta_i)}{\partial \theta_i} + v_i(\theta_i) \cdot \frac{\partial \frac{w_i(\theta_i)}{\theta_i + w_i(\theta_i)}}{\partial \theta_i} \\
&= \frac{w_i(\theta_i)}{\theta_i + w_i(\theta_i)} \cdot \frac{\partial v_i(\theta_i)}{\partial \theta_i} + v_i(\theta_i) \cdot \frac{\theta_i \cdot \big(-1 + w_i(\theta_i)v_i(\theta_i)\big) - w_i(\theta_i)}{(\theta_i + w_i(\theta_i))^2}
\end{aligned}$$

The log-concavity of $\theta_i + w_i(\theta_i)$ can be guaranteed by $\theta_i \cdot \big(-1 + w_i(\theta_i)v_i(\theta_i)\big) - w_i(\theta_i) \le 0$ because $v_i(\theta_i)$ is decreasing from Lemma 1. We will next show that $\theta_i \cdot \big(-1 + w_i(\theta_i)v_i(\theta_i)\big) \to 0$ as $\theta_i \to \infty$. Denote $\underline{\beta}_i = \min_s \beta_{is}$ and let Ξ_i be the set $\Xi_i = \{s : \beta_{is} = \underline{\beta}_i\}$. Then,

$$\begin{aligned}
-1 + w_i(\theta_i)v_i(\theta_i) &= \frac{-\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}\right)^2 + \left(\sum_{s=1}^{m_i} \frac{1}{\beta_{is}} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}\right) \cdot \left(\sum_{s=1}^{m_i} \beta_{is} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}\right)}{\left(\sum_{s=1}^{m_i} e^{\tilde{\alpha}_{is} - \beta_{is}\theta_i}\right)^2} \\
&= \frac{1}{\left(\sum_{s\in\Xi_i} e^{\tilde{\alpha}_{is}} + \sum_{s\notin\Xi_i} e^{\tilde{\alpha}_{is} - (\beta_{is} - \underline{\beta}_i)\theta_i}\right)^2} \cdot \Bigg(-\Bigg(\sum_{s\in\Xi_i} e^{\tilde{\alpha}_{is}} + \sum_{s\notin\Xi_i} e^{\tilde{\alpha}_{is} - (\beta_{is} - \underline{\beta}_i)\theta_i} \Bigg)^2 \\
&\quad + \Bigg(\sum_{s\in\Xi_i} \frac{1}{\underline{\beta}_i} e^{\tilde{\alpha}_{is}} + \sum_{s\notin\Xi_i} \frac{1}{\beta_{is}} e^{\tilde{\alpha}_{is} - (\beta_{is} - \underline{\beta}_i)\theta_i} \Bigg) \cdot \Bigg(\sum_{s\in\Xi_i} \underline{\beta}_i e^{\tilde{\alpha}_{is}} + \sum_{s\notin\Xi_i} \beta_{is} e^{\tilde{\alpha}_{is} - (\beta_{is} - \underline{\beta}_i)\theta_i} \Bigg) \Bigg) \\
&\approx \frac{\sum_{s\in\Xi_i} e^{\tilde{\alpha}_{is}} \cdot \left(\sum_{s\notin\Xi_i} \left(\frac{\beta_{is}}{\underline{\beta}_i} + \frac{\underline{\beta}_i}{\beta_{is}} - 2\right) e^{\tilde{\alpha}_{is} - (\beta_{is} - \underline{\beta}_i)\theta_i} \right)}{\left(\sum_{s\in\Xi_i} e^{\tilde{\alpha}_{is}}\right)^2 + 2\left(\sum_{s\in\Xi_i} e^{\tilde{\alpha}_{is}}\right) \cdot \left(\sum_{s\notin\Xi_i} e^{\tilde{\alpha}_{is} - (\beta_{is} - \underline{\beta}_i)\theta_i}\right)}
\end{aligned}$$

In the above approximation, the higher order terms are ignored. Because $\frac{\beta_{is}}{\underline{\beta}_i} + \frac{\underline{\beta}_i}{\beta_{is}} - 2 > 0$ and $\beta_{is} - \underline{\beta}_i > 0$, then

$$\theta_i \cdot \left(\left(\frac{\beta_{is}}{\underline{\beta}_i} + \frac{\underline{\beta}_i}{\beta_{is}} - 2\right) e^{\tilde{\alpha}_{is} - (\beta_{is} - \underline{\beta}_i)\theta_i} \right) = \frac{\theta_i \cdot \left(\frac{\beta_{is}}{\underline{\beta}_i} + \frac{\underline{\beta}_i}{\beta_{is}} - 2\right)}{e^{-\tilde{\alpha}_{is} + (\beta_{is} - \underline{\beta}_i)\theta_i}} \to 0, \text{ as } \theta_i \to \infty.$$

The above convergence holds because the exponential function is increasing faster than the linear function. Since $\left(\sum_{s\in\Xi_i} e^{\tilde{\alpha}_{is}}\right)^2 + 2\left(\sum_{s\in\Xi_i} e^{\tilde{\alpha}_{is}}\right) \cdot \left(\sum_{s\notin\Xi_i} e^{\tilde{\alpha}_{is} - (\beta_{is} - \underline{\beta}_i)\theta_i}\right) \to \left(\sum_{s\in\Xi_i} e^{\tilde{\alpha}_{is}}\right)^2$ as $\theta_i \to \infty$, therefore, $\theta_i \cdot \big(-1 + w_i(\theta_i)v_i(\theta_i)\big) \to 0$. There exists $\tilde{\theta}_i$ such that

$$\theta_i \cdot \big(-1 + w_i(\theta_i)v_i(\theta_i)\big) \le \frac{1}{\max_s \beta_{is}} \le w_i(\theta_i), \text{ for } \theta_i \ge \tilde{\theta}_i.$$

Thus, $\theta_i + w_i(\theta_i)$ is log-concave for $\theta_i \ge \tilde{\theta}_i$. $\square$

Proof of Theorem 5. Consider the first order and second order derivatives of $\log R_i(\theta_i, \theta_{-i})$ with respect to θ_i:

$$\frac{\partial \log R_i(\theta_i,\theta_{-i})}{\partial \theta_i} = \frac{\partial \log(Q_i(\theta_i,\theta_{-i}))}{\partial \theta_i} + \frac{\partial \log(\theta_i + w_i(\theta_i))}{\partial \theta_i} = -\gamma_i(1-Q_i(\theta_i,\theta_{-i}))v_i(\theta_i) + \frac{\partial \log(\theta_i + w_i(\theta_i))}{\partial \theta_i},$$
$$\frac{\partial^2 \log R_i(\theta_i,\theta_{-i})}{\partial \theta_i^2} = -\gamma_i^2 Q_i(\theta_i,\theta_{-i})(1-Q_i(\theta_i,\theta_{-i}))\big(v_i(\theta_i)\big)^2 + \frac{\partial^2 \log(\theta_i + w_i(\theta_i))}{\partial \theta_i^2}.$$

The cross-derivative of $\log R_i(\theta_i, \theta_{-i})$ is

$$\frac{\partial^2 \log R_i(\theta_i,\theta_{-i})}{\partial \theta_i \partial \theta_j} = \gamma_i \gamma_j Q_i(\theta_i,\theta_{-i}) Q_j(\theta_j,\theta_{-j}) v_i(\theta_i) v_j(\theta_j) \geq 0, \ \forall j \neq i.$$

Then,

$$\sum_{j\neq i} \frac{\partial^2 \log R_i(\theta_i,\theta_{-i})}{\partial \theta_i \partial \theta_j} = \gamma_i Q_i(\theta_i,\theta_{-i}) v_i(\theta_i) \sum_{j\neq i} \gamma_j Q_j(\theta_j,\theta_{-j}) v_j(\theta_j).$$

Under Condition 3, $R_i(\theta_i, \theta_{-i})$ is log-dominant diagonal,

$$-\frac{\partial^2 \log R_i(\theta_i,\theta_{-i})}{\partial \theta_i^2} \geq \sum_{j\neq i} \frac{\partial^2 \log R_i(\theta_i,\theta_{-i})}{\partial \theta_i \partial \theta_j} \tag{45}$$

The inequality holds because

$$\gamma_i^2 Q_i(\theta_i,\theta_{-i})(1-Q_i(\theta_i,\theta_{-i}))\big(v_i(\theta_i)\big)^2 \geq \gamma_i Q_i(\theta_i,\theta_{-i}) v_i(\theta_i) \sum_{j\neq i} \gamma_j Q_j(\theta_j,\theta_{-j}) v_j(\theta_j)$$

under Condition 3(a), and

$$\frac{\partial^2 \log(\theta_i + w_i(\theta_i))}{\partial \theta_i^2} \leq 0$$

under Condition 3(b). The inequality (45) establishes the uniqueness of the Nash equilibrium to **Game II** (see e.g., Vives 2001).

If the equilibrium of a log-supermodular game with continuous payoff is unique, it is globally stable and a tatonnement process with dynamic response (24) converges to it from any initial point in the feasible region.

□

Proof of Theorem 6. Suppose that F_i is the set of product to offer at prices such that equation (31) is satisfied. Without loss of generality, assume that $a_{-i} = 0$. Consider the price optimization problem with the market share constraint as follows:

$$r_i^{F_i}(\rho_i, \mathbf{p}_{-i}) := \max_{\mathbf{p}_i < \infty} \textstyle\sum_{k\in F_i} (p_{ik} - c_{ik}) \pi_{ik}(\mathbf{p}_i, \mathbf{p}_{-i})$$
$$s.t., \ Q_i(\mathbf{p}_i, \mathbf{p}_{-i}) = \rho_i.$$

We can easily show that at optimal prices $(p_{ij}-c_{ij})+\frac{a_{ij}(p_{ij})}{a'_{ij}(p_{ij})}$ is constant for all $j\in F_i$, denoted by η_i. Then,

$$r_i^{F_i}(\rho_i,\mathbf{p}_{\text{-}i})=\rho_i\sum_{k\in F_i}(p_{ik}-c_{ik})q_{k|i}(\mathbf{p}_i)$$

where p_{ij} satisfies that $(p_{ij}-c_{ij})+\frac{a_{ij}(p_{ij})}{a'_{ij}(p_{ij})}=\eta_i$ for all $j\in F_i$ and η_i is the unique solution, denoted by $\eta_i^{F_i}$, to

$$\frac{\left(\sum_{s\in F_i}a_{is}(p_{is})\right)^{\gamma_i}}{1+\left(\sum_{s\in F_i}a_{is}(p_{is})\right)^{\gamma_i}}=\rho_i,$$

which is equivalent to $\sum_{s\in F_i}a_{is}(p_{is})=\left(\frac{\rho_i}{1-\rho_i}\right)^{1/\gamma_i}$. Then, $r_i^{F_i}(\rho_i,\mathbf{p}_{\text{-}i})$ can be rewritten as follows

$$r_i^{F_i}(\rho_i,\mathbf{p}_{\text{-}i})=\rho_i\eta_i-\rho_i^{1-\frac{1}{\gamma_i}}(1-\rho_i)^{\frac{1}{\gamma_i}}\sum_{s\in F_i}\frac{(a_{is}(p_{is}))^2}{a'_{is}(p_{is})},$$

where $\eta_i=\eta_i^{F_i}$ and $(p_{ij}-c_{ij})+\frac{a_{ij}(p_{ij})}{a'_{ij}(p_{ij})}=\eta_i^{F_i}$. Denote $H_4^{F_i}(\eta_i)$ as follows:

$$H_4^{F_i}(\eta_i)=\rho_i\eta_i-\rho_i^{1-\frac{1}{\gamma_i}}(1-\rho_i)^{\frac{1}{\gamma_i}}\sum_{s\in F_i}\frac{(a_{is}(p_{is}))^2}{a'_{is}(p_{is})}.$$

We will next show that $H_4^{F_i}(\eta_i)$ is convex in η_i under Condition 4. Consider the first order derivative:

$$\begin{aligned}\frac{\partial H_4^{F_i}(\eta_i)}{\partial\eta_i}&=\sum_{s\in F_i}\frac{\partial H_4^{F_i}(\eta_i)/\partial p_{is}}{\partial\eta_i/\partial p_{is}}=\rho_i-\rho_i^{1-\frac{1}{\gamma_i}}(1-\rho_i)^{\frac{1}{\gamma_i}}\sum_{s\in F_i}\frac{\frac{2a_{is}(p_{is})(a'_{is}(p_{is}))^2-(a_{is}(p_{is}))^2a''_{is}(p_{is})}{(a'_{is}(p_{is}))^2}}{1+\frac{(a'_{is}(p_{is}))^2-a_{is}(p_{is})a''_{is}(p_{is})}{(a'_{is}(p_{is}))^2}}\\&=\rho_i-\rho_i^{1-\frac{1}{\gamma_i}}(1-\rho_i)^{\frac{1}{\gamma_i}}\sum_{s\in F_i}a_{is}(p_{is})\end{aligned}$$

And $\left.\frac{\partial H_4^{F_i}(\eta_i)}{\partial\eta_i}\right|_{\eta_i=\eta_i^{F_i}}=0$ because $\sum_{s\in F_i}a_{is}(p_{is})=\left(\frac{\rho_i}{1-\rho_i}\right)^{1/\gamma_i}$. The second order derivative of $H_4^{F_i}(\eta_i)$ is

$$\frac{\partial^2H_4^{F_i}(\eta_i)}{\partial\eta_i^2}=\sum_{s\in F_i}\frac{\frac{\partial}{\partial p_{is}}(\partial H_4^{F_i}(\eta_i)/\partial\eta_i)}{\frac{\partial\eta_i}{\partial p_{is}}}=-\rho_i^{1-\frac{1}{\gamma_i}}(1-\rho_i)^{\frac{1}{\gamma_i}}\sum_{s\in F_i}\frac{(a'_{is}(p_{is}))^3}{2(a'_{is}(p_{is}))^2-a_{is}(p_{is})a''_{is}(p_{is})}\geq 0.$$

The last inequality holds because $a'_{is}(p_{is})\leq 0$ and $2(a'_{is}(p_{is}))^2-a_{is}(p_{is})a''_{is}(p_{is})>0$ from Condition 4. So, $H_4^{F_i}(\eta_i)$ is convex in η_i and $r_i^{F_i}(\rho_i,\mathbf{p}_{\text{-}i})=\min_{\theta_i}H_4^{F_i}(\eta_i)=H_4^{F_i}(\eta_i^{F_i})$.

Suppose another product z is added to the set F_i and denote the new set as $F_i^+=F_i\cup\{z\}$. By the same argument as in the proof of Proposition 1, we can show that $r_i^{F_i^+}(\rho_i,\mathbf{p}_{\text{-}i})>r_i^{F_i}(\rho_i,\mathbf{p}_{\text{-}i})$ for any $0<\rho_i<1$. Therefore, it is optimal to offer all the product at finite prices such that equation (31) is satisfied and the multi-product pricing problem has been reduced to an optimization problem in a single-dimensional space.

□

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