The Allocational Effects of Reporting
the Precision of Accounting Estimates

Ronald A. Dye and Sri S. Sridhar

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Abstract

This paper studies the consequences of reporting the precisions of accounting estimates, when the choice of precisions is endogenous. We show that: broadly, reporting the precisions of estimates reduces the equilibrium precisions that firms choose; if firms are sufficiently similar to each other, reporting precisions reduces the equilibrium investments of all firms; and any given firm’s equilibrium investment level increases as the precision of its own estimate increases and decreases as the precisions of other firms’ estimates rise.

These results also establish that firms’ expected equilibrium prices often will be highest when information about the precisions of estimates is suppressed, both because the ensuing higher precision levels induce higher equilibrium investments and because higher precisions also reduce the informational risk premia demanded by investors.
1 Introduction

Many accounting disclosures are estimates - certainly, each of: earnings, sales, cost of goods sold, bad debt expense, pension liabilities, and depreciation expense are estimates, as are other items that are explicitly labeled as estimates, such as earnings forecasts, estimated contingent liabilities, and the like. In order to interpret estimates, investors either must receive information from firms about the precision (i.e., inverse of variance) of the estimates or else they must make inferences about the precision of the estimates on their own. The first part of this paper examines how a firm’s choice of the precision of its estimates is affected by whether the firm discloses the precision of its estimates or, instead, leaves it to investors to make inferences about the precision of the estimates, in the presence of other firms who make similar choices. The second part of the paper examines how changes in the precision of estimates can affect firms’ investment decisions. When combined with the first part of the paper, this part establishes some of the resource allocation implications generated by the reporting or non-reporting of the precision of estimates.

Our first main result is the following. Contrast two disclosure regimes in both of which, say, earnings forecasts are being disclosed. In the “one dimensional”
disclosure regime, only point forecasts of earnings are feasible. In the “two dimensional” disclosure regime, point forecasts of earnings are accompanied by credible disclosures of the precision of those point forecasts. In both regimes, the choice of precisions of the forecasts is endogenous, but only in the two-dimensional regime is the choice of precision disclosed to investors. In the one-dimensional regime, investors must infer the precision of the forecasts. We show, broadly, that the equilibrium precisions of the estimates chosen in the two-dimensional disclosure regime are never higher than (and often are strictly lower than) in the one-dimensional regime: that is, disclosure of the precision of estimates reduces the equilibrium precision of the estimates.

Our second main result is this: suppose the market prices of firms depend upon the reported values of their estimates of their future cash flows/liquidating dividends. And suppose the firms make investment decisions in anticipation of the impact those investment decisions will have on the distributions of the subsequently reported estimates. We ask: how are those investment choices affected by changes in the precisions of firms’ estimates? We show that increases in the precision of a given firm’s estimate always strictly increase that firm’s equilibrium level of investment; further, we also show that increases in the precision
of other firms’ estimates always weakly decrease the given firm’s level of investment. These results are quite robust: within the class of multivariate normally distributed cash flows and investors with constant absolute risk averse preferences, these results hold for all structures of covariances of firms’ cash flows, and for most investment functions that exhibit positive and decreasing marginal returns to investment. The result that a given firm’s equilibrium investment level is always decreasing in other firms’ precisions of estimates indicates what we believe is a previously undocumented externality among firms’ precision choices.

Our third main result builds on the first two results. We ask: what is the impact of converting from a regime in which precisions of estimates are public to one in which precisions of estimates are private on firms’ equilibrium investment levels? In general, the impact is ambiguous. This follows, because the first result above indicates that making precisions private increases the equilibrium precision levels of all firms, and the second result shows that increases in "own" firm precision has a positive impact on a given firm's investment level, but increases in "other" firm precisions decrease the given firm’s investment level. But, we show that, in going from a regime in which precisions are public to a regime in which precisions are private, if all firms increase the precision of their
estimates by the same amount, then equilibrium levels of investment of all firms increase.

Our fourth main result adds to this last result by demonstrating that if firms are sufficiently homogenous (in terms of their risk aversion and their feasible sets of precision choices) and the direct costs of increasing precision over the feasible range are negligible, then in going from a regime in which precisions are public to a regime in which precisions are private, all firms’ equilibrium investment levels increase.

We now give some background for these results. In some settings the choice of precision of estimates is a matter of indifference to firms. For instance, when both a firm and its investors are risk-neutral, and the choice of precision is public information, then any level of precision is an equilibrium. This follows because, regardless of the choice of precision, the firm’s average selling price will equal the firm’s unconditional expected value. Thus, a policy of disclosing highly precise estimates is an equilibrium, as is a policy of disclosing very imprecise estimates, and so is a policy of disclosing estimates of intermediate precision. All such policies result in the firm’s expected market value equalling the expected discounted value of its cash flows, and so the firm (its managers and its owners)
will be indifferent among such policies.

When firms are risk-averse, but investors remain risk-neutral, the equilibrium choice of precisions narrows. Dye [1990] considers a model where firms precommit to disclosures with known levels of precision, and firms have access to equally costly alternative estimates of varying precision. In this model, Dye shows that firms’ equilibrium precisions in general depend upon (a) how risk averse the firms are relative to the (aggregate) risk-aversion of investors and (b) the covariances between the distributions of the firms’ cash flows. In the typical case, where investors in aggregate are nearly risk-neutral and each firm is (either slightly or significantly) risk-averse, Dye [1990] shows that firms in equilibrium will choose to disclose estimates of minimal precision. This is because, as the precision of an estimate increases, investors respond by putting more weight on the firm’s estimates, and this increases the variance in each firm’s market value. This is an undesirable result for a risk-averse firm.

But in practice it appears that, in many cases, firms make no disclosures about the precision of their estimates. They do not disclose confidence intervals or t-statistics accompanying their disclosures of earnings, sales, or other reports. In these circumstances, investors have no choice but to make conjectures about
the level of precision of the estimates. In this situation, which is central to the present study, one might conjecture that there would be a “race to the bottom” toward minimal precisions, when firms are not compelled to adhere to previously enunciated precision levels. However, we find, contrary to this intuition, that firms tend to increase the precision of their estimates when the precision choices are not publicly observable.

The explanation for the result is the following. When investors are forced to infer the precision of a firm’s estimates, the amount by which any given estimate will change investors’ prior beliefs about the firm’s value depends upon investors’ conjectures about the precision of the estimate. In deciding what precision to adopt, then, firms take the way investors are perceived to process/react to their (subsequent) disclosures as given, and then maximize with respect to their actual choice of precision. Because how investors process the information the firms reveal is beyond the firm’s control, but the actual precision/variance in their estimates is under their control, risk-averse firms can reduce the variance of the price reaction to their disclosures by maximizing the precision of their disclosures.

The impact of precisions on firms’ investment choices is also intuitive. In-
creasing the precision of a firm’s estimate invariably increases its equilibrium investment choice, since more accurate disclosures result in market prices more accurately reflecting the firm’s investment decisions. Since the natural tendency of firms is to underinvest when their investment choices are not public, increases in the precision of disclosures serve to partially correct these underinvestment incentives. The result that increases in the precision of one firm’s estimate serves to reduce the equilibrium investment in other firms is also intuitive, and is based on the following result from regression theory: when investors attempt to estimate the value of a firm cash flows, the weight they will assign to the firm’s own disclosures will decrease as the precision of other firms’ estimates increases, regardless of the covariance structure describing the relationship among the firms’ cash flows. This is a strong, and potentially testable, implication of the theory here.

We also provide evidence of how important this last effect is quantitatively. When the precision of a given firm’s estimate is low, we show that increases in other firms’ precision levels can reduce the given firm’s investment by more than 80%. But, when the precision of the given firm’s estimate is high, we show that the impact of other firms’ precisions on the given firm’s investment is reduced
substantially (for the parametrizations we study, the impact drops to less than 15%).

There are several strands of literature related to this paper. This paper builds on Dye [1990], who contrasts the disclosure policies firms would make under a voluntary disclosure regime to disclosures that might be mandatorily imposed; he assumes throughout that firms’ disclosure policies are public; and he takes firms’ investments to be exogenously given. In contrast, we compare allocations that result from disclosure policies being public to allocations that result from disclosure policies being private, and much of the present analysis posits that firms’ investments are endogenous.

Since the present paper emphasizes the precision of accounting estimates, and precision is a measure of the quality of accounting reports, our paper is related to the theoretical literature on information quality. While we take quality/precision to be one-dimensional, Antle and Demski [1989] note that the quality of accounting information can often be a multi-dimensional concept, a point they examine in addressing alternative revenue recognition policies. Kanodia [1980] and Kanodia and Mukherji [1996] show how another aspect of the quality of accounting information - how disaggregated the information is - can
lead to greater efficiency and higher levels of investment. These papers, like ours, posit that the value of accounting disclosures derives from firms’ anticipation of how their investments will affect the distributions of subsequent accounting disclosures. A primary point of contrast between these papers and ours is that the present paper demonstrates that non-disclosure (of the precision of estimates) may sometimes elevate enhanced investment levels.

Our analysis is also related to the literature on multi-tasking (see, e.g., Holmstrom and Milgrom [1991], Feltham and Xie [1994], and Datar, Kulp, and Lambert [2001]), although the multi-tasking literature initially seems quite independent of the present work, since it is framed in an agency context whereas the present paper is developed in a market context. But, interesting ties between these works can be found once one views (1) the firm in the present paper as an agent of prospective investors and (2) the firm’s two tasks to consist of selecting an investment level and selecting a precision level. One of the main results of the multi-tasking literature is that principals must be aware of the tendency of agents to direct their efforts toward tasks that are easy to measure, and away from tasks that are difficult to measure, if principals emphasize the measurable tasks in the agents’ compensation schemes. This result would
seem to suggest that in the present context, when precisions of estimates are not observable, it will be more difficult for principals (i.e., future investors) to induce the firm to produce precise estimates than when the precisions of estimates are observable, and that, when precisions are observable, the firm will reduce its equilibrium investment level. Perhaps surprisingly, neither of these intuitions from the multi-tasking literature is supported by the predictions of our model. Rather, our results, when cast in agency-theoretic terms, show that the disclosure of a performance measure sometimes can adversely affect performance. This result is also reminiscent of Christensen and Feltham[2000], who find conditions under which agency problems are sometimes improved by the absence of disclosure.

Credibly reporting the precision of estimates is analogous to centralizing the choice of precision, since a principal directly can control only what he can observe. So, our results obtained in a non-contracting setting merit contrasting with Demski, Patell, and Wolfson [1984], who observe, in a contracting setting, that decentralizing the choice of information systems may be preferred to centralizing that choice.

The paper proceeds as follow. The basic setup is described in the Section
2. In that section, as well as Section 3, investment decisions are taken as exogenous. Section 3 contrasts the equilibrium precisions chosen when precisions are reported with the precisions chosen when precisions are private. Section 4 expands upon the preceding sections by examining what happens when firms’ investment decisions and their precision choices are linked. Section 5 contains a brief conclusion followed by an appendix which contains the proofs of the main results.

2 Model Setup with Exogenous Investment

The base model is similar to that of Dye [1990]. There are \( n \) firms indexed by \( i = 1, 2, \ldots, n \). The risk-averse initial owner/manager of firm \( i \) is intent upon maximizing his expected utility from selling his firm to a collection of \( m \) risk-averse investors. The motive for sale is exogenous, perhaps involving life cycle considerations. If firm \( i \) sells for price \( P_i \), then \( i \)'s owner’s utility is given by the constant absolute risk averse (CARA) preferences \( U_i(P_i) = -e^{-\beta_i P_i} \), \( i = 1, \ldots, n \). Likewise, investor \( j \)'s utility for final consumption \( w \) is given by \( V_j(w) = -e^{-\gamma_j w}, j = 1, \ldots, m \), so if investor \( j \)'s fractional ownership in each of the \( n \) firms is \( (\theta_{j1}, \ldots, \theta_{jn}) \), the price of firm \( i \) is \( P_i \), and the liquidating dividend (equivalently, the present value of the future cash flows) of firm \( i \) turns out to
be $z_i$, then investor $j$’s utility is given by $-e^{-\gamma_j \sum_{i=1}^n \theta_j (z_i - P_i)}$.

We next describe the joint distribution of the firms’ liquidating dividends $\tilde{z}_i$, $i = 1, \ldots, n$. The vector $\tilde{\mathbf{z}} = \begin{bmatrix} \tilde{z}_1 \\ \vdots \\ \tilde{z}_n \end{bmatrix}$ denotes the random vector of all $n$ firms’ liquidating dividends. The expected value of $\tilde{z}_i$ is normally distributed with an unknown mean represented by the random variable $\tilde{\mu}_i$ and known variance. More precisely, the joint distribution of $\tilde{\mathbf{z}}$ given $\tilde{\mathbf{\mu}}$ is assumed to be multivariate normal with mean $\tilde{\mathbf{\mu}}$ and covariance matrix $\Phi$. In the following, we sometimes write this as $\tilde{\mathbf{z}} \sim N(\tilde{\mathbf{\mu}}, \Phi)$.

Investors’ and initial owners’ common prior beliefs about the vector $\tilde{\mathbf{\mu}} = \begin{bmatrix} \tilde{\mu}_1 \\ \vdots \\ \tilde{\mu}_n \end{bmatrix}$ of all $n$ firms’ random means are that $\tilde{\mathbf{\mu}} \sim N(\mathbf{m}, \Omega)$, where $\mathbf{m} = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}$ denotes the prior joint mean. Throughout the following, we assume that the covariance matrix $\Omega$ is strictly positive-definite.\(^1\)

To summarize, the distribution of cash flows is $\tilde{\mathbf{z}} = \tilde{\mathbf{\mu}} + \tilde{\mathbf{\phi}}$, where $\tilde{\mathbf{\phi}} = \begin{bmatrix} \tilde{\phi}_1 \\ \vdots \\ \tilde{\phi}_n \end{bmatrix}$ is distributed $N(\mathbf{0}, \Phi)$ and $\tilde{\mathbf{\mu}} = \mathbf{m} + \tilde{\mathbf{\omega}}$, where $\tilde{\mathbf{\omega}} = \begin{bmatrix} \tilde{\omega}_1 \\ \vdots \\ \tilde{\omega}_n \end{bmatrix}$ is distributed $N(\mathbf{0}, \Omega)$.

Prior to selling his firm, the owner of firm $i$ obtains and discloses an unbiased estimate $\tilde{x}_i$ of $\tilde{\mu}_i$ with precision (inverse of variance) $r_i$. The precision $r_i$ is a choice variable for the owner of firm $i$, and belongs to some set $[\underline{r}_i, \overline{r}_i]$.\(^{1}\) Positive semi-definiteness is automatic, since $\Omega$ is a covariance matrix.
The vector $\mathbf{r} = \begin{bmatrix} r_1 \\
 \bullet \\
 \bullet \\
 r_n \end{bmatrix}$ denotes the set of precisions of estimates chosen by all firms, and the vector $\mathbf{\tilde{x}} = \begin{bmatrix} \tilde{x}_1 \\
 \bullet \\
 \bullet \\
 \tilde{x}_n \end{bmatrix}$ denotes the set of disclosed estimates.

Given $\mathbf{r}$ and $\boldsymbol{\mu}$, $\mathbf{\tilde{x}}$ is $N(\boldsymbol{\mu}, \mathbf{R}(\mathbf{r})^{-1})$, where $\mathbf{R}(\mathbf{r}) = \begin{bmatrix} r_1 & 0 & 0 & 0 \\
 0 & \bullet & 0 & 0 \\
 0 & 0 & \bullet & 0 \\
 0 & 0 & 0 & r_n \end{bmatrix}$ denotes the precision matrix corresponding to the $n$ firms' disclosures. Equivalently, $\mathbf{\tilde{x}} = \tilde{\mathbf{\mu}} + \tilde{\mathbf{\varepsilon}}$, where $\tilde{\mathbf{\varepsilon}} \sim N(0, \mathbf{R}(\mathbf{r})^{-1})$. To allow for the possibility that it is costly for firms to produce estimates, with more precise estimates being more costly to produce, we introduce $c_i(r_i)$ as the cost to firm $i$ of choosing precision $r_i$.

As a special case, we sometimes consider $c_i(r_i)$ being identically zero over the interval $r_i \in [\underline{r}_i, \overline{r}_i]$. In all cases, we assume that $c_i(r_i)$ is nonnegative, weakly increasing, and differentiable on its domain, with $\lim_{r_i \to 0} c'_i(r_i)$ being finite. When costs of precision are incorporated into the model, the final consumption of the owner of firm $i$ is modified to $-e^{-\beta_i(P_i - c_i(r_i))}$.

To display the precision choices of all firms other than firm $i$, we write $(\mathbf{r} \backslash r_i) = (r_1, r_2, ..., r_{i-1}, r_{i+1}, ..., r_n)$. The concatenation $(\mathbf{r} \backslash r_i, \mathbf{\hat{r}}_i)$ denotes the vector of precisions $(r_1, r_2, ..., \hat{r}_i, r_{i+1}, ..., r_n)$. 

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2.1 Equilibrium Prices

Let $P_i(x|\mathbf{r})$ be the equilibrium price of firm $i$, given the vector of disclosures $\mathbf{x}$ made when investors believe the firms’ precisions of estimates are collectively given by $\mathbf{r}$, and let $\mathbf{P}(x|\mathbf{r}) = \begin{bmatrix} P_1(x|\mathbf{r}) \\ \vdots \\ P_n(x|\mathbf{r}) \end{bmatrix}$ be the vector of all firms’ prices. Dye [1990] shows, with $E[\tilde{z}|x, \mathbf{r}]$ denoting the conditional expected value of $\tilde{z}$ given $\mathbf{x}$ and $\mathbf{r}$, $\gamma = 1/\sum_{j=1}^{m} \frac{1}{\gamma_j}$ representing investors’ aggregate risk-aversion, and $\mathbf{1}$ denoting an $n \times 1$ vector of 1’s:

$$\mathbf{P}(x|\mathbf{r}) = E[\tilde{z}|x, \mathbf{r}] - \gamma \times (\Phi + (\mathbf{R}(\mathbf{r}) + \Omega^{-1})^{-1}) \mathbf{1},$$

(1) indicates that the equilibrium prices of the firms are equal to the expected values of their cash flows conditional on whatever disclosures are made net of two kinds of risk premia, one risk premium related to the covariances $\Phi$ of the underlying distribution of cash flows conditional on a known distribution of the cash flows, and a second risk premium related to uncertainty about the distribution of the cash flows, as summarized by the posterior covariance matrix $(\mathbf{R}(\mathbf{r}) + \Omega^{-1})^{-1}$ for $\tilde{\mathbf{u}}$. (2) describes the standard result that, when disclosures
are unbiased estimates of the firms’ mean cash flows, then the posterior assessment of the joint expected cash flows is a weighted average of the prior means of these cash flows and the realized estimates, with the weights proportional to the relative precision of the prior means and realized disclosures.

2.2 The Univariate Case

The economics of the subsequent results is sometimes made clearer by considering the case of a single firm in isolation. When we consider the univariate case, we replace the precision matrix $R(r)$ associated with the vector $\tilde{x}$ of disclosed information with the (univariate) precision $r$ associated with estimate of $\tilde{x}$. (When we are considering the univariate case, we do not bother to index the firms by $i$.) Also we replace the precision matrix $\Omega^{-1}$ associated with the priors on $\tilde{\mu}$ with the precision $\tau$ associated with $\tilde{\mu}$, and finally, we replace the covariance matrix $\Phi$ for the vector of cash flows $\tilde{z}$ (conditional on $\tilde{\mu}$) with the variance $\phi$ of the cash flows $\tilde{z}$ (conditional on $\tilde{\mu}$). With these substitutions, the univariate counterparts to (1) and (2) are given by:

$$P(x|r) = E[\tilde{z}|x, r] - \gamma \times (\phi + \frac{1}{r + \tau})$$  \hspace{1cm} (3)

and

$$E[\tilde{z}|x,r] = \frac{rx + \tau m}{r + \tau}. \hspace{1cm} (4)$$
2.3 The Definition(s) of Equilibrium

As discussed in the Introduction, we consider two regimes. In the first, one-dimensional, disclosure regime, each firm $i$ discloses only its estimate $\hat{x}_i$ of $\mu_i$. In this regime, it is up to each investor and every owner other than the owner of firm $i$ to make inferences about the precision of the estimate $\hat{x}_i$ firm $i$ chooses.

In the second, two-dimensional, disclosure regime, in addition to disclosing its estimate $\hat{x}_i$, the owner of firm $i$ discloses the actual precision $r_i$ of the estimate $\hat{x}_i$.

When no precisions of estimates are disclosed, the sequence of events is as follows: investors conjecture that the firms collectively adopt precisions $r^*$. The owner of firm $i$ takes these investors’ conjectures as given, and also takes the conjectures $r^* \setminus r_i$ about other firms’ precisions as given. Then, firm $i$ chooses its own estimate’s precision $r_i$ to maximize $E[-e^{-\beta_i(P_i(\hat{x}_i|r^*)-c_i(r_i))} | r^* \setminus r_i, r_i]$. In the following formal definition of equilibrium, we let $r_i^{pri}(r^* \setminus r_i^*)$ be the ”private precision policy” that maximizes this expectation. Also, we let $1_i$ denote a column vector consisting of all zeroes other than a one in the $i^{th}$ coordinate.

**Definition 1** When precisions are privately chosen, a Nash equilibrium consists of a collection of precisions $r^* = (r_1^*, ..., r_n^*)$ such that, for all firms $i = 1, 2, ..., n$, ...
1. \( r_i^{\text{pri}}(r^*|r_i^*) \in \arg \max_{r_i} E[-e^{-\beta_i(P_i(\tilde{x}|r^*)-c_i(r_i))}|r^*|r_i^*, r_i]; \)

2. \( r_i^* = r_i^{\text{pri}}(r^*|r_i^*); \)

3. for each \( x, P_i(x|r^*) = 1_i'P(x|r^*), \) where \( P(x|r^*) \) is as specified in (1) and \( E[\tilde{z}|x, r^*] \) is as specified in (2), with \( r = r^*. \)

The first part of the definition indicates that each owner chooses the precision of its estimate so as to maximize his expected utility, taking as given his conjectures about both the precisions of the estimates chosen by other firms and also his conjectures about what investors believe are the precisions of estimates of all firms. The unobservability of firms’ actual choices is embedded in the specification that the price \( P_i(\tilde{x}|r^*) \) depends only upon investors’ conjectures about the precisions chosen, not the actual precisions chosen. The fact that firm \( i \) itself knows the precision of its estimate is implicit in the specification that the expectation \( E[-e^{-\beta_i(P_i(\tilde{x}|r^*)-c_i(r_i))}|r^*|r_i^*, r_i] \) depends upon firm \( i \)'s actual choice of precision \( r_i \). The second part of the definition states that an equilibrium set of estimate policies is a fixed point of the \( n \) reaction functions \( r_i^{\text{pri}}(r^*|r_i^*). \) That is, the conjectures of all investors and all owners other than the owner of firm \( i \) about firm \( i \)'s precision coincides with the precision that firm \( i \) actually chooses. The third requirement in the definition is simply that
prices are set correctly, given the estimates disclosed and the precisions of those estimates.

When precisions are public, the sequence of events is the same as above, except that investors no longer have to make conjectures about the precisions of firms’ actual estimates. In this case, the definition of an equilibrium is similar to the preceding one, except that the choice of precisions is now available for everyone to see. In particular, as a firm’s chosen precision changes, the pricing function changes accordingly. For this definition, we let \( r_{i}^{\text{pub}}(r^{*} \backslash r_{i}^{*}) \) denote the ”public precision policy” that maximizes the owner of firm \( i \)'s expected utility, taking as given the precisions firm \( i \) observes other firms to have chosen.

**Definition 2** When precisions are publicly chosen, a Nash equilibrium consists of a collection of precisions \( r^{**} = (r_{1}^{**}, ..., r_{n}^{**}) \) such that, for all firms \( i = 1, 2, ..., n, \)

1. \( r_{i}^{\text{pub}}(r^{**} \backslash r_{i}^{*}) \in \arg \max_{r_{i}} E[-e^{-\beta_{i}(P_{i}(\bar{z} \mid r^{**} \backslash r_{i}^{*}) - c_{i}(r_{i}))} \mid r^{**} \backslash r_{i}^{*}, r_{i}] \);
2. \( r_{i}^{**} = r_{i}^{\text{pub}}(r^{**} \backslash r_{i}^{*}) \);
3. for each \( i \) and \( x \), \( P_{i}(x \mid r^{**}) = 1 \cdot P(x \mid r^{**}) \), where \( P(x \mid r^{**}) \) is as specified in (1) and \( E[\bar{z} \mid x, r^{**}] \) is as specified in (2) with \( r = r^{**} \).

Notice that the principal difference between the two definitions of equilibrium
is that the pricing function \( P_i(x|\pi^{r_i}, r_i) \) changes as firm \( i \) contemplates changing the precision of its estimate in the case where precisions are public, whereas the pricing function \( P_i(\tilde{x}|r^*) \) remains unaltered by changes in firm \( i \)'s chosen precision when precisions are private information.

Before proceeding, it is useful to describe the essential components of these equilibria more explicitly. Whether precisions are public or private, the pricing equations (1) and (2) imply that there is a vector of constants \( a = a(r) \) and an \( n \times n \) matrix \( B = B(r) \) corresponding to the equilibrium precisions \( r \) such that

\[
P(x|r) = a + Bx. \tag{5}
\]

Examining the pricing equations, we see immediately that the constants \( a \) and \( B \) are given by:

\[
a = a(r) \equiv (R(r) + \Omega^{-1})^{-1}\Omega^{-1}m - \gamma \times (\Phi + (R(r) + \Omega^{-1})^{-1})1 \tag{6}
\]

and

\[
B = B(r) \equiv (R(r) + \Omega^{-1})^{-1}R(r). \tag{7}
\]

Because the initial owners have CARA preferences, and all variables are normally distributed, it is possible to provide a certainty equivalent representation of the initial owners’ expected utilities. That is, for the owner of firm \( i \), his ex-
pected utility is given by \( CE_i = mean - .5\beta_i\text{variance} \), where the mean and variance refer to the mean and variance of firm \( i \)'s equilibrium price. Given the pricing equation (5), it is clear that the mean for firm \( i \) is \( \beta_i \times (a + Bm) \).\(^2\) The variance in the price of firm \( i \) is \( \text{var}(\beta_i P(\tilde{x}|r)) = \text{var}(\beta_i B\tilde{x}) = \beta_i B(\Omega + R^{-1}(r))B \), since we recall that the ex ante distribution of \( \tilde{x} = \tilde{\mu} + \tilde{\xi} \), where \( \tilde{\mu} \sim N(m, \Omega) \) and \( \tilde{\xi} \sim N(0, R(r)^{-1}) \).

To summarize, owner \( i \)'s certainty equivalent is given by:

\[
\beta_i \times (a + Bm) - c_i(r_i) - .5\beta_i \beta_i B(\Omega + R^{-1}(r))B_i. \quad (8)
\]

Now, in the case where precisions are private, the constants \( a \) and \( B \) depend only on investors conjectures about the precisions \( r^* \), i.e., \( a = a(r^*) \) and \( B = B(r^*) \).

The owner of firm \( i \) takes these constants as given, as well as the precisions \( r^* \setminus r^*_i \) of other firms, when choosing his own precision \( r_i \). So, the owner of firm \( i \) selects \( r_i \) to maximize

\[
\beta_i \times (a + Bm) - c_i(r_i) - .5\beta_i \beta_i B(\Omega + R^{-1}(r_i)^{-1}(r_i^* \setminus r_i^*))B_i.
\]

To present the simplest expression for an initial owner’s certainty equivalent \( ^2 \)Note that this is true whether or not investors correctly perceive the precisions firms choose. That is, even if investors think the firms collectively have chosen precision levels \( r \), when in fact the firms have chosen precision levels \( \hat{r} \), it follows that \( E[P_i(\hat{\tilde{x}}|r)] = \beta_i \times (a + Bm) \). This follows since, whether investors correctly infer precision levels or not, the estimates are unbiased - and so the means are evaluated correctly.
a and B depend upon the publicly known precision levels. Specifically, if the collective choices of the precisions are \((r_i^*, r_i)\), then \(a = a(r_i^*, r_i)\) and \(B = B(r_i^*, r_i)\). These values are related to the pricing equation (1) through

\[
a(r_i^*, r_i) + B(r_i^*, r_i)m = m - \gamma \times (\Phi + (R(r_i^*, r_i) + \Omega^{-1})^{-1})1.
\]

Notice that this relationship holds whether or not the precision levels \((r_i^*, r_i)\) are equilibrium levels, since these precision levels are directly observable by investors.\(^3\) So, the owner of firm \(i\)'s certainty equivalent is given by

\[
1_i'(m - \gamma \times (\Phi + (R(r_i^*, r_i) + \Omega^{-1})^{-1})1) - c_i(r_i^*)
\]

\[=-0.5 \beta_i 1_i'B(r_i^*, r_i)(\Omega + R^{-1}(r_i^*, r_i))B(r_i^*, r_i)1_i.\]

3 Reporting precisions when investments are exogenous

To begin our analysis, we take the means \(m\) to be exogenous. That is, we take firms’ investment decisions to be exogenously given, and in particular, to be independent of firms’ chosen precisions.

Our first main result is the following.

**Theorem 1** If \(r^* = (r_1^*, ..., r_n^*)\) and \(r^{**} = (r_1^{**}, ..., r_n^{**})\) denotes equilibrium precision policies in the cases, respectively, where precisions of estimates are

\(^3\)A counterpart to this claim when precisions are private is not true.
private or public, then

(a) if \( c_i(r_i) \equiv 0 \), then \( r_i^* = \bar{r}_i \) for every \( i = 1, 2, \ldots, n \) and \( r_i^{**} \leq r_i^* \);

(b) if \( \gamma \) is sufficiently small, i.e., if in the aggregate, the market engages in approximately risk-neutral pricing, then \( r_i^{**} \leq r_i^* \) for all \( i \).

What Theorem 1(a) asserts is that, when the incremental cost of increasing the precision of a firm’s estimate is negligible, then without further qualification, one can conclude that higher precisions are selected in equilibrium when precisions are private than when precisions are public. What Theorem 1(b) asserts is that, even when the incremental costs of choosing higher precisions are non-negligible, then the same conclusion obtains, as long as equilibrium pricing is approximately the same as risk-neutral pricing. The latter condition will hold if, for example, there are sufficiently many investors in the market and each investor \( i \)'s risk aversion \( \gamma_i \) satisfies \( \gamma_i \geq \gamma \) for some fixed \( \gamma > 0 \).

The intuition for these results was described briefly in the Introduction. We expand on that intuition here by considering the univariate case when the incremental costs of increasing precision are negligible. When precisions of disclosures are public, the owner’s choice of precision affects both the firm’s expected selling price and the variance in the selling price. The effect of precision on the
expected selling price appears through the size of the risk premium investors pay due to uncertainty about the distribution of the firm’s cash flows. This risk premium is the term $\gamma \times \frac{1}{r + \tau}$ in the pricing equation,

$$P(x|\tau) = E[\tilde{z}|x, \tau] - \gamma \times (\phi + \frac{1}{r + \tau}).$$

(10)

This risk premium clearly decreases in the precision of the estimate. The effect of the choice of the precision on the variance of the firm’s selling price arises from the effect of precision on the variance of the posterior mean $E[\tilde{z}|x, \tau] = \frac{rx + m}{r + \tau}$.

Explicitly,

$$\text{var}(P(\tilde{z}|x)) = \text{var}(\frac{r\tilde{z} + \tau m}{r + \tau}) = \left(\frac{r}{r + \tau}\right)^2 \text{Var}(\tilde{z}) = \left(\frac{r}{r + \tau}\right)^2 (\text{var}(\tilde{\mu}) + \frac{1}{r}) = \left(\frac{r}{r + \tau}\right)^2 \left(\frac{1}{r} + \frac{1}{r}\right) = \frac{r}{\tau(r + \tau)},$$

which is increasing in the precision $r$ of the estimate. That is, in the case where precisions are public, increasing the precision of the estimate increases the variance in the firm’s selling price. This is intuitive: as the information content of an estimate increases, i.e., as the precision of the estimate increases, investors revise their priors more in response to the estimate, and so the price reaction to the estimate increases. When an owner is choosing among precisions, he trades off these opposing effects. The optimal trade-off is determined so as
to maximize his certainty equivalent value from selling the firm:

\[ CE(r) = \text{mean} - 0.5\beta \text{var} = m - \gamma \times \left( \phi + \frac{1}{r + \tau} \right) - 0.5\beta \frac{r}{\tau(r + \tau)}. \]

A simple computation shows that \( CE(r) \) is increasing (decreasing) in \( r \) if \( \gamma > 0.5\beta \) (\( \gamma < 0.5\beta \)). That is, the owner is better off (resp., worse off) by increasing the precision of his estimate, depending on how risk averse he is relative to the aggregate risk aversion of investors in the market. In the typical case, we would expect \( \gamma \) to be near zero, so the theory predicts that the precision of the estimate would be as low as possible.

Now we consider what happens when the precision of the estimate is privately chosen by the owner. Holding investors’ conjectures about the firm’s chosen precision as fixed and given by \( r^* \), first note that, regardless of the actual precision \( r \) the owner chooses, the expected selling price of the firm is given by \( E[P(\tilde{x}|r^*)|r] = m - \gamma \times \left( \phi + \frac{1}{r + \tau} \right) \). This follows since the expected value of the estimate \( \tilde{x} \) is unbiased, and hence \( E[E[\tilde{x}|\tilde{x}, r^*]|r] = m \), whether or not \( r = r^* \). The variance in the firm’s selling price is given by

\[ \text{var}(P(\tilde{x}|r^*)|r) = \text{var} \left( \frac{r^* \tilde{x} + \tau m}{r^* + \tau} | r \right) = \left( \frac{r^*}{r^* + \tau} \right)^2 \text{var}(\tilde{x}) = \left( \frac{r^*}{r^* + \tau} \right)^2 \frac{1}{\tau} \left( r + \frac{1}{r} \right). \]

This variance is decreasing in the actual choice of precision \( r \). This is a funda-

\footnote{This result was reported in Dye [1990], and is included here to serve as a reference point to what happens when precisions are private.}
mental difference from the case where the choice of precision is public information. In that case, as was noted above, the variance of the firm’s selling price increased in the precision of the estimate. The reason for the difference between these two cases is that: when the precision choice is private, investors cannot adjust their interpretation of, or equivalently, their reaction to, an estimate as the firm’s actual choice of precision changes. Consequently, the owner succeeds in reducing the variance of the firm’s selling price by choosing the highest precision possible. The formal proof of this result for the general case with multiple firms making concurrent disclosures is presented in the appendix.

We now recast the above result using the nomenclature of the one-dimensional and two-dimensional disclosure regimes presented in the Introduction.

**Theorem 2** Under the hypotheses of Theorem 1, the precisions firms select in the Nash equilibrium of the two-dimensional disclosure regime are uniformly never higher than that in the Nash equilibrium of the one-dimensional regime.

4 The Effects of Precision on Endogenous Investment

In the preceding section, priors regarding the expected value of the firms’ cash flows were taken to be exogenous and summarized via the prior mean $m$. In this
section, we examine the case where these priors are endogenous and determined
by the levels of firms’ investments. The purpose of this extension is to under-
stand the role played by the precisions of firms’ disclosures in firms’ investment
decisions.

To investigate these investment effects of precision choices, we now suppose
that
\[
\begin{bmatrix}
  m_1 \\
  \vdots \\
  m_n
\end{bmatrix}
= \begin{bmatrix}
  F_1(I_1) \\
  \vdots \\
  F_n(I_n)
\end{bmatrix},
\]
where \( I_i \) is firm \( i \)'s investment in some stochastic production function which, for each \( I_i \), has expected value \( F_i(I_i) \).

We assume \( F_i(\bullet) \) exhibits positive and decreasing returns to scale \( (F_i'(\bullet) > 0, F_i''(\bullet) < 0) \). To avoid corner solutions, we posit \( F_i'(0) = \infty \). In terms of the
notation of the previous section, we posit that
\[
\begin{bmatrix}
  \hat{\phi}_1 \\
  \vdots \\
  \hat{\phi}_n
\end{bmatrix}
= \mathbf{N}(0, \Phi),
\]
where \( \hat{\phi}_i \) is \( \hat{\mu}_i + \hat{\phi}_i \), where, as before,
\[
\begin{bmatrix}
  \hat{\mu}_1 \\
  \vdots \\
  \hat{\mu}_n
\end{bmatrix}
\]
and \( m \) is \( \hat{\mu} = m + \tilde{\omega} \), where
\[
\tilde{\omega} = \begin{bmatrix}
  \tilde{\omega}_1 \\
  \vdots \\
  \tilde{\omega}_n
\end{bmatrix}
\]
is \( \mathbf{N}(0, \Omega) \), also as in the previous section. And the estimates
\[
\begin{bmatrix}
  \hat{x}_1 \\
  \vdots \\
  \hat{x}_n
\end{bmatrix}
\]
remain distributed as before as well, i.e., given \( r \) and \( \mu \), \( \hat{x} \) is \( \mathbf{N}(\mu, R(r)^{-1}) \).

We assume that the investment choices \( I_i, i = 1, 2, \ldots, n \) are not public. We
next extend the definition of an equilibrium to the present case where investment
choices are endogenous.
Definition 3 When precisions are privately chosen, a Nash equilibrium with endogenous investments consists of a collection of precisions \( r^* = (r_1^*, ..., r_n^*) \) and investment levels \( I^* = (I_1^*, ..., I_n^*) \) such that, for all firms \( i = 1, 2, ..., n, \)

1. \( I_i(I_i^\parallel, r^* \backslash r_i^*) \), \( r_i^{pri}(I_i^* \backslash I_i^*, r^* \backslash r_i^*) \) are such that \( \max_{r_i, I_i} E[-e^{-\beta_i(P_i(\hat{r}_i^* r_i^* - I_i c_i(r_i)))} | I_i^* I_i, I_i, r^* \backslash r_i^*, r_i^*] \);
2. \( I_i^* = I_i(I_i^* \backslash I_i^*, r^* \backslash r_i^*), r_i^* = r_i^{pri}(I_i^* \backslash I_i^*, r^* \backslash r_i^*) \);
3. for each \( i \) and \( x, P_i(x|r^*, I^*) = 1_{P(x|r^*, I^*)}, \) where \( P(x|r^*, I^*) \) being as specified in (1) with \( r = r^* \) and

\[
E[\hat{z}| x, r^*, I^*] = (R(r^*) + \Omega^{-1})^{-1}(R(r^*) x + \Omega^{-1} m^*), \text{ where } m^* = \begin{bmatrix}
F_1(I_1^*) \\
\vdots \\
F_n(I_n^*)
\end{bmatrix} = F(I^*). \tag{11}
\]

This is the natural counterpart to Definition 1 above, modified to incorporate the endogenous investment decisions. The corresponding definition in the case where precisions are publicly chosen is similar, and not repeated here.

4.1 How Equilibrium Investments Depends Upon Firms’ Precision Choices

It is the purpose of this subsection to understand how changes in the precision of estimates changes equilibrium investment levels.

Suppose investors believe that firms’ precision choices are given by \( r^* \). Investors will conjecture that these precision choices will result in equilibrium
investment decisions \( I^* \), leading to means \( m^* = F(I^*) \). Consequently, just as in (5), (6), and (7) above, there will be

\[
a = a(r^*) \equiv (R(r) + \Omega^{-1})^{-1} \Omega^{-1} m^* - \gamma \times (\Phi + (R(r^*) + \Omega^{-1})^{-1}) 1
\]

and

\[
B = B(r^*) = (R(r^*) + \Omega^{-1})^{-1} R(r^*)
\]

such that

\[
P(x|r^*) = a + Bx.
\]

The owner of firm \( i \)'s certainty equivalent

\[
E[P_i(\tilde{x}|r^*)] - \beta_i Var(P_i(\tilde{x}|r^*))/2 - I_i - c_i(r_i)
\]

will be affected by his actual choice \( I_i \) only to the extent that \( I_i \) affects \( E[P_i(\tilde{x}|r^*)] \) — \( I_i \), since \( Var(P_i(\tilde{x}|r^*)) \) does not vary with the investment. Thus, the first-order condition describing the optimal investment for the owner of firm \( i \) is given by:

\[
\frac{\partial E[P_i(\tilde{x}|r^*)]}{\partial I_i} = 1.
\]

Because

- \( E[P_i(\tilde{x}|r^*)] = 1_i'(a + Bm) \),

- \( \frac{\partial}{\partial m_i} 1_i'(a + Bm) = 1_i'B1_i = b_{ii} \), where \( b_{ii} \) is the \((i, i)\) element of \( B \), and
• \( \frac{\partial m_i}{\partial I_i} = F'_i(I_i) \),

this first-order condition can be rewritten as:

\[
\frac{\partial \mathbb{E}[P_i(x|r)]}{\partial I_i} = \frac{\partial \mathbb{E}[P_i(x|r)]}{\partial m_i} \frac{\partial m_i}{\partial I_i} = b_{ii}F'_i(I_i) = 1.
\]

So, the optimal choice of \( I_i \) is given by

\[
I^*_i = I^*_i(r^*) = F^{-1}_i(1/b_{ii}).
\]  

This first-order condition is crucial to establishing the dependence of the optimal investment decision on the precisions of firms’ estimates. Clearly, (14) indicates that the equilibrium investment level \( I^*_i \) is increasing in \( b_{ii} \). So, if we can establish the impact of firms’ precision choices on \( b_{ii} \), we will have determined the effect of precision choices on the size of firms’ equilibrium investment levels. The following pair of theorems addresses these issues.

**Theorem 3** (a) \( \frac{\partial b_{ii}}{\partial r_i} > 0 \);

(b) \( \frac{\partial b_{ii}}{\partial r_k} \leq 0 \) for \( i \neq k \).

**Theorem 4** (a) A firm’s equilibrium investment level always strictly increases as the precision of its own estimates increases, and

(b) a firm’s equilibrium investment level always weakly declines as the precision of other firms’ estimates increases.
In these theorems, we see that a firm’s equilibrium investment level increases in the precision of its own estimates, and that its equilibrium investment levels decrease as the precision of other firms’ estimates increases. Theorem 4(a) is intuitive: in general, we expect that, when investors do not have perfect information regarding firms’ investment choices, then the amount of investment firms make will be less than first-best, and the extent to which the investment is below first best depends upon how much information the realized estimates reveal about the firm’s return from investment. The anticipation of more informative, that is, more precise, estimates will induce a firm’s owner to make higher levels of investment.

Perhaps more surprisingly, Theorem 4(b) identifies a (to the best of our knowledge) heretofore unknown externality concerning firms’ disclosure choices: increasing the precision of one firm’s estimates has a negative impact on the investment levels chosen by other firms. This result is very general: it holds without qualification regarding the mean $\mathbf{m}$ or precision matrix $\Omega$ for the firm’s priors, and also without restrictions on the covariance matrix $\Phi$ on the distribution of firms’ cash flows.

To get at the intuition underlying this result, consider the special case where
there are two firms \((n = 1, 2)\), and suppose we concentrate on the effects of firms’ precision choices on firm 1. The disclosure-contingent price of firm 1, \(P_1(x_1, x_2|R)\), will vary with the estimates \(x_1\) and \(x_2\) produced by both firms, and will equal the expected value of \(\hat{\mu}_1\), given those disclosures, i.e.,

\[
P_1(x_1, x_2|R) = E[\hat{\mu}_1| x_1, x_2, R] = E[\hat{\mu}_1| x_1, x_2, R].
\]

Since all variables are normally distributed, the conditional expectations are linear, that is, there exists constants \(c\) and \(b_{11}\) and \(b_{12}\) such that

\[
E[\hat{\mu}_1| x_1, x_2, R] = c + b_{11}x_1 + b_{12}x_2.
\]

As noted in the discussion of (14) above, in choosing its investment \(I_1\), firm 1 cares only about the magnitude of the coefficient \(b_{11}\), since \(I_1\) only affects the realization of \(\hat{x}_1\). The claim in Theorem 3(b), which converts to a statement about equilibrium investment choices in Theorem 4(b), is that, regardless of the covariance matrix \(\Omega\),

\[
\frac{\partial b_{11}}{\partial r_2} = \frac{\partial}{\partial r_2} \frac{\partial}{\partial x_1} E[\hat{\mu}_1| x_1, x_2, R] \leq 0.
\]

That is, the weight \(b_{11}\) attached to the estimate \(x_1\) of firm 1 declines as the precision \(r_2\) of firm 2’s estimate \(\hat{x}_2\) increases.

To gain more intuition for this result, consider the special case where the
correlation between \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \) is 1.\(^5\) Effectively, \( \tilde{\mu}_1 \equiv \tilde{\mu}_2 \) in this special case, and so each of the disclosures \( \tilde{x}_1 \) and \( \tilde{x}_2 \) provides an estimate of \( \tilde{\mu}_1 \). Then, it is clear that, as the precision of the estimate \( \tilde{x}_2 \) goes up, the weight \( b_{11} \) assigned to \( \tilde{x}_1 \) goes down, when trying to estimate \( \tilde{\mu}_1 \). It turns out that this argument does not depend upon \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \) having a perfect, or even a positive, covariance, however.

Regardless of the correlation between \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \), in trying to estimate \( \tilde{\mu}_1 \) based on the observation of \( \tilde{x}_1 \) and \( \tilde{x}_2 \), investors will attach less weight to \( \tilde{x}_1 \) the greater the precision of the estimate \( \tilde{x}_2 \). The proof shows that this intuitive idea holds generally, regardless of the number \( n \) of firms, and regardless of the correlation structure linking the various conditional means \( \tilde{\mu}_i \). As the Introduction noted, this is a strong testable implication of the theory put forth here.

The impact of the precision of one firm’s estimate on another firm’s equilibrium investment level can be surprisingly strong. To illustrate this, we continue to study the two firm case. Letting \( \omega_{ij} \) be the covariance between \( \tilde{\mu}_i \) and \( \tilde{\mu}_j \), some simple algebra shows that:

\[
b_{11} = \frac{r_1 (\omega_{11} + r_2 (\omega_{11} \omega_{22} - \omega_{12}^2))}{1 + r_2 \omega_{22} + r_1 (\omega_{11} + r_2 (\omega_{11} \omega_{22} - \omega_{12}^2))},
\]

\(^5\)We wish to thank Anne Beyer for suggesting this intuition.
so

$$\frac{\partial b_{11}}{\partial r_1} = \frac{(1 + r_2 \omega_{22})(\omega_{11} + r_2(\omega_{11} \omega_{22} - \omega_{12}^2))}{(1 + r_2 \omega_{22} + r_1(\omega_{11} + r_2(\omega_{11} \omega_{22} - \omega_{12}^2)))^2}$$

and

$$\frac{\partial b_{11}}{\partial r_2} = -\frac{r_1 \omega_{12}^2}{(1 + r_2 \omega_{22} + r_1(\omega_{11} + r_2(\omega_{11} \omega_{22} - \omega_{12}^2)))^2}.$$ 

Both of these derivatives are consistent with the preceding general theoretical results, since $\omega_{11} \omega_{22} - \omega_{12}^2$ is the (positive) determinant of the positive-definite covariance matrix $\Omega$. To exhibit numerical results, we suppose $F_i(I_i) = 2\sqrt{T_i}$ and $\omega_{11} = .4$, $\omega_{22} = .9$, $\omega_{12} = .3$, and $r_1 = .01$. As $r_2$ varies from .01 to 1, the equilibrium investment level for firm 1 drops by over 80%. See Figure 1 below.

![Figure 1: Equilibrium investment by firm 1 when $r_1=0.01$](image)

In contrast, if we increase $r_1$ to $r_1 = 1$ and leave the remaining parameters
unchanged, there is less than a 15% change in the equilibrium investment level as \( r_2 \) varies from .01 to 1. This is illustrated by Figure 2 below.

These graphs confirm that the equilibrium investment in firm 1 declines in the precision of firm 2’s estimate and increases in the precision of firm 1’s estimate, and that the size of the impact of firm 2’s precision choice on firm 1’s investment is decreasing as firm 1’s precision increases. These features are general, not artifacts of the parametrizations chosen.

Next, we consider what happens to equilibrium investment levels when more than one firm’s estimate’s precision changes at the same time. If, by happenstance, one firm’s precision were to increase, while all other firms’ precisions were to decline, then Theorems 3 and 4 can be applied immediately to conclude
that the investment in the firm whose precision increases would rise. We would not think of this case as typical, though. Instead, we would expect that precisions different firms choose would tend to move in the same direction. This situation might arise, for example, because of a mandated increase in the precisions of all firms’ disclosures. In this situation, Theorems 3 and 4 might lead one to expect that the effects of increases in all firms’ precisions would have an ambiguous effect on firms’ equilibrium investment levels. What the following theorem shows is that, if the precision choices of all firms increase by the same amount, then the effect on investment choice is unambiguous.

**Theorem 5**  If all firms’ precisions increase by the same amount, say from $r = (r_1, \ldots, r_n)$ to $r = (r_1 + \Delta, \ldots, r_n + \Delta)$, for some $\Delta > 0$, then every firm’s investment level strictly increases.

This theorem suggests that, at least as judged by the metric of investment, or output, levels, uniform increases in precision have salutary effects. It is difficult to conclude, though, what the effects are when all firms’ precisions’ increase though not uniformly.
4.2 The Effects of Reporting Precisions on Endogenous Investment

Finally, we consider the effects of the observability of precisions on firms’ endogenous investment choices.

First, we consider the case where the precisions are private. Taking investors’ conjectures about $r^*$ as given, the values $a$ and $B$ are exactly as described in (12) and (13) above, and so the counterpart to (8) above, when the owner’s actual choice of precision is $r_i$, is

$$1_i' \times \{ a + BF(I') - I' \} - 0.5\beta_i 1_i' B(\Omega + R^{-1}(r^*|r_i^*, r_i))B1_i - c_i(r_i).$$

Since the equilibrium choice of $I^*$ depends only on investors’ conjectures $r^*$, the owner’s actual choice of $r_i$ will be chosen to minimize $0.5\beta_i 1_i' B(\Omega + R^{-1}(r^*|r_i^*, r_i))B1_i + c_i(r_i)$. This is exactly the same problem the owner of firm $i$ faced when investment was exogenous. This proves that:

**Theorem 6** When precisions of estimates are private information, firms’ equilibrium precision levels are independent of whether investment levels are exogenous or endogenous.

This theorem follows since, when precisions are private, a firm’s equilibrium investment choice depends only on investors’ conjectures about the precision
choice and not the actual precision choice. The owner uses the actual precision choice only to minimize the sum of the posterior variance in the firm’s price and the direct cost of precision, which is a problem he confronts whether investment is endogenous or exogenous.

Next, we consider the case where precisions are public. The only adjustment that has to be made to the certainty equivalent (9) above is to recognize that the mean \( m = F(I^\ast) \) and that account must be made for the cost \( I^\ast \) of the investment. That is, the certainty equivalent reads:

\[
1_i ^{\prime} \left( F(I^\ast) - I^\ast - \gamma \times (\Phi + (R(r_i^\ast|r_i^\ast, r_i) + \Omega^{-1})^{-1})1 \right) - c_i (r_i^\ast) \tag{15}
\]

\[
-0.5\beta_i 1_i ^{\prime} B(r_i^\ast|r_i^\ast, r_i)(\Omega + R^{-1}(r_i^\ast|r_i^\ast, r_i))B(r_i^\ast|r_i^\ast, r_i)1_i .
\]

What is critical to recognize is that the \( I^\ast \) will be chosen just as in (14) above (with \( b_{ii} \) being the \((i, i)\) element of \( B(r_i^\ast|r_i^\ast, r_i) \)). So, unlike the case where precisions are privately chosen, the investment choices will adjust as the chosen precisions change. So, by the envelope theorem, \( \frac{\partial 1_i ^{\prime}(F(I^\ast) - I^\ast)}{\partial r_i} = 0 \). Therefore, the first-order condition for firm \( i \)'s optimal choice of precision will be independent of its effect on \( F(I^\ast) - I^\ast \), and so that choice will be exactly the same as in section 3 above when investment was exogenous. This proves:

**Theorem 7** When precisions of estimates are public information, firms’ equilib-
rium precision levels are independent of whether investment levels are exogenous or endogenous.

Consequently, the conclusions of Theorems 1 and 2 remain valid in the case where investments are endogenous, and so, under the hypotheses of those theorems, the equilibrium precisions are higher when precisions are not disclosed than when they are disclosed. Furthermore, in the special case where the direct costs $c_i(r_i)$ are all identically zero, $[\bar{r}_i, \bar{r}_i] = [r_i, r_i + \Delta]$, and the risk aversions of all initial owners $\beta_i$ are the same, it follows that the change in equilibrium precision choices (when going from precisions being public to precisions being private) will be the same for all firms, and so Theorem 5 applies. That is,

**Theorem 8** If $c_i(r_i) \equiv 0, [\bar{r}_i, \bar{r}_i] = [r_i, r_i + \Delta]$, and $\beta_i = \beta$ for all $i$, then the regime in which all firms’ precisions are public has strictly lower equilibrium investment for all firms than the regime in which all firms’ precisions are private.

Theorem 8 gives rise to one additional implication of the model: firms’ expected equilibrium prices will be highest when information about the precisions of estimates is suppressed, because suppressed precisions result in higher equilibrium precision levels, and higher equilibrium precision levels induce both higher equilibrium investment levels and lower informational risk premia.
5 Conclusions

The paper studies the problem of endogenizing the precision of firms' estimates. Since virtually all accounting disclosures are estimates, this paper covers most accounting disclosures firms make. The paper has several central findings. Among them, the following stand out. First, disclosing the precision of estimates reduces the chosen precision of estimates. Second, a given firm's equilibrium investment level increases as the precision of its estimate increases, and the firm's equilibrium investment level decreases as the precision of other firms' estimates increase. Third, if the precisions of all firms' estimates increase by the same amount, then all firms' equilibrium investment levels increase. Fourth, if firms are sufficiently similar to each other (in their preferences and in the "width" of their feasible sets of precisions) and if the costs of varying precisions are negligible, then making precisions public reduces the equilibrium investment levels, and hence outputs, of all firms. These four results are robust to variations in the covariances among both firms' realized cash flows and investors' priors regarding the joint distribution of these cash flows.

One possible extension of the analysis here would be to analyze how firms control the precision of their estimates. The perspective taken in this paper is
the simplifying one that the firms get to choose the precision of their estimates exactly. In practice, they may be able to obtain only rough indications of the accuracy of their estimates. That is, the precision of the estimates may be not be under their complete control. There may, instead, be a non-degenerate distribution of possible precisions, and through the care by which they construct their estimates, firms may be able to reduce the variance in this distribution. Studying the effects of modifying, or disclosing, information about the distribution of precisions would be interesting.

6 Appendix

Proof of Theorem 1 We begin by studying the regime in which precisions are privately chosen.

Suppose investors think that all firms choose precisions \( r \), and that firm \( i \) thinks that all firms other than firm \( i \) choose precisions \( r \setminus r_i \), and that firm \( i \) in fact chooses precision \( \hat{r}_i \). Then, (8) applies to characterize investor \( i \)'s certainty equivalent, with \( a = a(r) \), \( B = B(r) \), and \( R^{-1}(r) \) replaced by \( R^{-1}(r \setminus r_i, \hat{r}_i) \).

That is, the certainty equivalent of the owner of firm \( i \) is given by:

\[
1_i' \times (a + Bm) - c_i(r_i) - \frac{1}{2} \hat{r}_i \Sigma \beta_i 1_i' B_i \Omega + R^{-1}(r \setminus r_i, \hat{r}_i) B_i 1_i. \tag{16}
\]
Taking the derivative of this last expression with respect to $\hat{r}_i$, recognizing that $a$, $B$, $m$, and $\Omega$ are constants as far as this derivative are concerned, and letting the $(i,j)^{th}$ element of $B$ be denoted by $b_{ij}$, we get the first-order condition:

\[
\frac{5\beta_i b_{ii}^2}{\hat{r}_i^2} - c_i'(\hat{r}_i) = 0 \text{ when } \underline{r}_i < \hat{r}_i < \bar{r}_i \tag{17}
\]

This is the interior first-order condition for $\hat{r}_i$ when precisions are private.

To determine the optimal choice of precision for firm $i$ when precisions are public, the certainty equivalent (9) must be maximized. By adapting (A2.6) in
Dye [1990], we conclude that the first-order condition corresponding to choosing 

\( r_i = r_i^{**} \) to maximize this certainty equivalent is given by:

\[
-a^{**ii} (\beta_i/2 - \gamma) a^{*ii} - \gamma \sum_{j \neq i} a^{*ij} = c_i'(r_i^{**}).
\]  

(19)  

(Here, \( a^{*ij} \) is the \((i, j)\)th element of the matrix \((R^{**} + \Omega^{-1})^{-1}\).) Upon comparing LHS(18) and LHS(19), we see that \( LHS(18) > LHS(19) \), when \( \gamma \) is sufficiently near zero, since each of \( a^{ii} \) and \( a^{iii} \) is positive, as each is a diagonal element of a positive-definite matrix. Thus, the value of \( \hat{r}_i \) that satisfies (18) must exceed the value \( r_i = r_i^{**} \) that satisfies (19). This proves Theorem 1(b).

\[\blacksquare\]

**Proof of Theorem 3**

Note that \( b_{ii} = 1 \cdot B_{1i} \), and that

\[
B = (R(r) + \Omega^{-1})^{-1} R(r) = (R(r)^{-1}(R(r) + \Omega^{-1}))^{-1} = (I + R(r)^{-1}\Omega^{-1})^{-1} = Q(r_k)^{-1},
\]

where \( I \) is the \( nxn \) identity matrix.

Writing this explicitly and letting \( \omega^{ij} \) be the \((i, j)\) element of \( \Omega^{-1} \), we observe
that $\mathbf{B}$ is the inverse of the matrix

$$
\mathbf{Q}(r_k) = \begin{bmatrix}
1 + \frac{\omega^{11}}{r_1} & \frac{\omega^{12}}{r_2} & \frac{\omega^{13}}{r_3} & \cdots & \cdots & \cdots & \frac{\omega^{1n}}{r_n} \\
\frac{\omega^{21}}{r_1} & 1 + \frac{\omega^{22}}{r_2} & \frac{\omega^{23}}{r_3} & \cdots & \cdots & \cdots & \frac{\omega^{2n}}{r_n} \\
\frac{\omega^{31}}{r_1} & \frac{\omega^{32}}{r_2} & 1 + \frac{\omega^{33}}{r_3} & \cdots & \cdots & \cdots & \frac{\omega^{3n}}{r_n} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\frac{\omega^{n1}}{r_1} & \frac{\omega^{n2}}{r_2} & \frac{\omega^{n3}}{r_3} & \cdots & 1 + \frac{\omega^{nk}}{r_k} & \cdots & \frac{\omega^{nn}}{r_n}
\end{bmatrix}.
$$

(20)

In this expression, we have emphasized the dependence of $\mathbf{Q}$ on $r_k$, because in performing comparative statics, we shall take $r_j$, for $j \neq k$, as given.  

Even though, in the following, we will only make use of the matrix $\mathbf{Q}$ as just defined, the next lemma applies to any $n \times n$ invertible matrix $\mathbf{Q}(t)$ whose $(i,j)^{th}$ element is the differentiable function (of $t$) $q_{ij}(t)$.

**Lemma 1** For any $n \times n$ invertible matrix $\mathbf{Q}(t)$ whose $(i,j)^{th}$ element is the differentiable function (of $t$) $q_{ij}(t)$,

$$
\mathbf{Q}^{-1}(t) = -\mathbf{Q}^{-1}(t)\mathbf{Q}'(t)\mathbf{Q}^{-1}(t).
$$

(See Gradshteyn, I.S. and I.M. Ryzhik [1980], page 1107.)

Now, returning to the specification of $\mathbf{Q}(r_k)$ in (20), we have the following technical result.

**Lemma 2** With $\mathbf{Q}(r_k)$ as specified in (20), and with the $(i,j)$ element of $\mathbf{Q}^{-1}(r_k)$ invertible because it is positive definite (see also the discussion on page 32 below.)
denoted by \( q^{ij}(r_k) \),

\[
\frac{\partial b_{ii}}{\partial r_k} = \begin{cases} 
-\frac{q^{ji}(r_k) q^{ki}(r_k)}{r_k}, & \text{if } k \neq i \\
\frac{q^{ji}(r_k)}{r_k} (1 - q^{ki}(r_k)), & \text{if } k = i 
\end{cases}.
\] (21)

**Proof:** With \( Q(r_k) \) as defined in (20), we see that

\[
Q'(r_k) = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\frac{-\omega^{k1}}{r_k^2} & \frac{-\omega^{k2}}{r_k^2} & \frac{-\omega^{k3}}{r_k^2} & \cdots & \frac{-\omega^{kn}}{r_k^2} \\
0 & 0 & 0 & \cdots & 0 
\end{bmatrix}.
\]

Since \( \frac{\partial b_{ii}}{\partial r_k} = 1_i Q^{-1}(r_k) 1_j \), Lemma 1 implies:

\[
\frac{\partial b_{ii}}{\partial r_k} = -1_i Q^{-1}(r_k) Q'(r_k) Q^{-1}(r_k) 1_i
\]

\[
= -(q^{11}(r_k), q^{12}(r_k), \ldots, q^{ij}(r_k)) \star \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\frac{-\omega^{k1}}{r_k^2} & \frac{-\omega^{k2}}{r_k^2} & \frac{-\omega^{k3}}{r_k^2} & \cdots & \frac{-\omega^{kn}}{r_k^2} \\
0 & 0 & 0 & \cdots & 0 
\end{bmatrix} \star \begin{bmatrix}
q^{11}(r_k) \\
q^{12}(r_k) \\
\cdots \\
q^{nj}(r_k) 
\end{bmatrix}
\]
\[ q^{ik}(r_k) = \frac{\omega^{k1}}{r_k}, q^{ik}(r_k) = \frac{\omega^{k2}}{r_k}, \ldots, q^{ik}(r_k) = \frac{\omega^{kn}}{r_k} \]

\[ = \left( q^{ik}(r_k) \frac{\omega^{k1}}{r_k}, q^{ik}(r_k) \frac{\omega^{k2}}{r_k}, \ldots, q^{ik}(r_k) \frac{\omega^{kn}}{r_k} \right) \left[ \begin{array}{c} q^{ij}(r_k) \\
q^{ji}(r_k) \\
\vdots \\
\vdots \\
q^{ni}(r_k) \end{array} \right] \]

\[ = q^{ik}(r_k) \sum_j \frac{\omega^{kj}}{r_k} q^{ji}(r_k) = q^{ik}(r_k) \left\{ \left( \sum_{j \neq k} \frac{\omega^{kj}}{r_k} q^{ji}(r_k) \right) + \frac{\omega^{kk}}{r_k} q^{ki}(r_k) \right\} \]

\[ = q^{ik}(r_k) \left\{ \left( \sum_j q_{kj}(r_k) q^{ji}(r_k) \right) + (q_{kk} - 1) q^{ki}(r_k) \right\} \]

\[ = q^{ik}(r_k) \left\{ \left( \sum_j q_{kj}(r_k) q^{ji}(r_k) \right) - q_{ki}(r_k) \right\} \]

\[ = \begin{cases} -q^{ik}(r_k) q^{ki}(r_k), & \text{if } k \neq i \\
q^{ik}(r_k) (1 - q^{ki}(r_k)), & \text{if } k = i \end{cases} \]

(The last equality follows because \( \sum_j q_{kj}(r_k) q^{ji}(r_k) \) is either 0 or 1, depending on whether \( k \neq i \) or \( k = i \), since this sum is product of the \( k \text{th} \) row of a matrix multiplied by the \( i \text{th} \) column of its inverse.)

**Proof of Theorem 3a**

By (21), it suffices to show that \( q^{kk} = q^{kk}(r_k) \in (0, 1) \).

We first claim that \( q^{kk} < 1 \). Let \( Q_{ij} \) be the submatrix of \( Q \) with the \( i \text{th} \) row and \( j \text{th} \) column deleted. A well-known fact in linear algebra (see, e.g., Strang [1988] (p.232)) is that

\[ q^{ik} = |Q_{ki}|/|Q|. \] (22)
In particular,

\[ q^{kk} = \frac{|Q_{kk}|}{|Q|} \]  

(23)

Apparently, \( q^{kk} < 1 \) iff \( |Q_{kk}| < |Q| \). We claim this last inequality is true.

To that end, note that, by cofactor expansion of \( |Q| \) along the \( k \)th, \( |Q| = \sum_j q_{jk}(-1)^{k+j}|Q_{jk}| \). So,

\[ |Q| - |Q_{kk}| = (q_{kk} - 1)|Q_{kk}| + \sum_{j \neq k} q_{jk}(-1)^{k+j}|Q_{jk}|. \]

Recall that, by definition of \( Q \), \( q_{kk} = 1 + \omega_{kk}/\tau_k \). Consequently,

\[ |Q| - |Q_{kk}| = (\omega_{kk}/\tau_k)|Q_{kk}| + \sum_{j \neq k} q_{jk}(-1)^{k+j}|Q_{jk}|. \]  

(24)

Notice that RHS(24) is the cofactor expansion along the \( k \)th column of the matrix \( I_k + R^{-1}\Omega^{-1} \) (here, \( I_k \) denotes the \( nxn \) matrix that is the same as the \( nxn \) identity matrix \( I \) except that the \( k \)th diagonal term is replaced by zero):

this is clear, since \( I_k + R^{-1}\Omega^{-1} \) is just the same as \( Q = I + R(r)^{-1}\Omega^{-1} \), except for its \((k, k)\) entry, which is \( \omega_{kk}/\tau_k \) rather than \( 1 + \omega_{kk}/\tau_k \). We claim that \( I_k + R^{-1}\Omega^{-1} \) is positive-definite, as the sum of the positive semi-definite matrix \( I_k \) and the positive-definite matrix \( (R^{-1}\Omega^{-1}) \).\(^7\) We conclude that RHS(24) is positive, as the determinant of a positive-definite matrix. Thus, \( |Q| - |Q_{kk}| > 0 \).

\(^7\)This claim relies on the following result, which we suspect is well known in the theory of matrices, though we have been unable to locate a statement of it.

**Lemma:** If \( A \) is a positive definite matrix, and \( R \) is a positive definite diagonal matrix, then the product \( RA \) is positive definite.

**Proof:** Recall that the principal minors of an \( nxn \) matrix consist of all square submatrices
We next claim that $q^{kk} > 0$. $|Q|$ is positive, as the determinant of a positive definite matrix. If the $k$th row and column of an $n \times n$ positive definite matrix are deleted, the result is an $(n-1) \times (n-1)$ positive definite matrix. So, $|Q_{kk}|$ is positive. Therefore, $q^{kk}$ is positive by (23). This proves Theorem 3a.

Proof of Theorem 3b

According to (21), it suffices to show that $q^{ik} = q^{ik}(r_k)$ and $q^{ki} = q^{ki}(r_k)$ both have the same sign. In turn, by (22), it suffices to show $|Q_{ik}|$ and $|Q_{ki}|$ have the same sign.

To that end, recall that if we take any square matrix (in particular, if we take matrix $Q$) and multiply any fixed row of it by a positive constant $r$, then the determinant of the resulting matrix is $r$ times the determinant of the original matrix. Of course, this implies that the determinant of the resulting matrix has

with the last $k$ rows and columns deleted, and also recall that a matrix is positive definite if the determinants of all of its principal minors are positive (see, e.g., Henderson and Quandt [1977]).

Let $r_1, r_2, \ldots, r_n$ be the diagonal elements of $R$. Since $R$ is positive definite, each $r_i$ is positive.

The matrix $RA$ is obtained from the matrix $A$ by multiplying each element of the first row of $A$ by $r_1$, each element of the second row of $A$ by $r_2$, etc. Let $\hat{A}_k$ (resp., $A_k$) denote the $k \times k$ submatrix of $RA$ (resp., of $A$) consisting of the first $k$ rows and columns of $RA$ (resp., $A$). Clearly, $\hat{A}_k$ is obtained by multiplying each element of the first row of $A_k$ by $r_1$, each element of the second row of $A_k$ by $r_2$, etc.

A well-known property of determinants is that if one takes a given square matrix and multiplies all elements of a row of that matrix by the same term, say $r$, then the determinant of the modified matrix is $r$ times the determinant of the original matrix. From this fact and the method by which $\hat{A}_k$ was derived from $A_k$, it follows that $|\hat{A}_k| = r_1 \ldots r_k |A_k|$. Since $A_k$ is positive definite (as a principal minor of the positive definite matrix $A$), $|A_k|$ is positive, and since each of the $r_i$ is positive, then $|\hat{A}_k|$ is positive. Since this is true for each principal minor of $\hat{A}_k$, it follows that $RA$ is positive definite, as claimed.
the same sign as the determinant of the original matrix. Moreover, the same
holds true for any minor of the original matrix: that is, if we take \( Q \) and multiply
any row of it by some positive constant \( r \), then all minors of the resulting matrix
will have the same sign as the corresponding minors of the original matrix.

We can repeat this process over and over: in particular, we can multiply
the first row of \( Q \) by the positive constant \( r_1 \), the second row of \( Q \) by the
positive constant \( r_2, \ldots, \) in general the \( i \)th row of \( Q \) by the positive constant \( r_i \),
\( i = 1, \ldots, n \), and all of the minors of the resulting matrix, call it \( \tilde{Q} \), will have the
same signs as the corresponding minors of the original matrix \( Q \).

Thus, all minors of \( Q \) will have the same sign as the corresponding minors

\[
\tilde{Q} = \begin{bmatrix}
    r_1 + \omega^{11} & \omega^{12} & \omega^{13} & \cdots & \omega^{1n} \\
    \omega^{21} & r_2 + \omega^{22} & \omega^{23} & \cdots & \omega^{2n} \\
    \omega^{31} & \omega^{32} & r_3 + \omega^{33} & \cdots & \omega^{3n} \\
    \omega^{k1} & \omega^{k2} & \omega^{k3} & \cdots & \omega^{kk} \\
    \omega^{n1} & \omega^{n2} & \omega^{n3} & \cdots & r_n + \omega^{nn}
\end{bmatrix}
\]

Notice that \( \tilde{Q} \) is a symmetric matrix.\(^8\) Hence, if \( \tilde{Q}_{ij} \) denotes the submatrix
of \( \tilde{Q} \) with the \( i \)th row and \( j \)th column deleted, it follows that \( \tilde{Q}_{ij} = \tilde{Q}_{ji} \), and
hence that \( |\tilde{Q}_{ij}| = |\tilde{Q}_{ji}|. \)

\(^8\)This is the reason for the preceding construction. Note that the original matrix \( A \) is not
symmetric.

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But, we have just argued that the minors of $\hat{Q}$ and $Q$ have the same sign for all $i$ and $j$. Thus,

$$\text{sgn}(Q_{ij}) = \text{sgn}(\hat{Q}_{ij}) = \text{sgn}(\hat{Q}_{kl}) = \text{sgn}(Q_{kl}).$$

\_\_\_\_

**Proof of Theorem 5** As in the proof of Theorem 3, we focus on how the change in precisions affects the term $b_{ii} = 1_i \mathbf{B}1_i$. In this case, we write the matrix $\mathbf{B} = (I + \mathbf{R}(r)^{-1}\Omega^{-1})^{-1}$ as the inverse of the matrix

$$\hat{\mathbf{Q}}(\Delta) \equiv \begin{bmatrix}
1 + \frac{\omega^{11}}{\omega_{11} + \Delta} & \frac{\omega^{12}}{\omega_{11} + \Delta} & \ldots & \ldots & \ldots & \frac{\omega^{1n}}{\omega_{11} + \Delta} \\
\frac{\omega^{21}}{\omega_{22} + \Delta} & 1 + \frac{\omega^{22}}{\omega_{22} + \Delta} & \ldots & \ldots & \ldots & \frac{\omega^{2n}}{\omega_{22} + \Delta} \\
\frac{\omega^{31}}{\omega_{33} + \Delta} & 1 + \frac{\omega^{32}}{\omega_{33} + \Delta} & \frac{\omega^{33}}{\omega_{33} + \Delta} & \ldots & \ldots & \frac{\omega^{3n}}{\omega_{33} + \Delta} \\
\frac{\omega^{k1}}{\omega_{kk} + \Delta} & \frac{\omega^{k2}}{\omega_{kk} + \Delta} & \frac{\omega^{k3}}{\omega_{kk} + \Delta} & 1 + \frac{\omega^{kk}}{\omega_{kk} + \Delta} & \ldots & \frac{\omega^{kn}}{\omega_{kk} + \Delta} \\
\frac{\omega^{n1}}{\omega_{nn} + \Delta} & \frac{\omega^{n2}}{\omega_{nn} + \Delta} & \ldots & \ldots & \ldots & 1 + \frac{\omega^{nn}}{\omega_{nn} + \Delta}
\end{bmatrix},$$

to reflect the assumption that each firm increases its precision by the same amount $\Delta$. It suffices to show that $\frac{\partial b_{ii}}{\partial \Delta}|_{\Delta=0} = 1_i \hat{\mathbf{Q}}^{-1}(\Delta)|_{\Delta=0} \mathbf{1}_i$ is positive. By Lemma 1,

$$1_i \hat{\mathbf{Q}}^{-1}(\Delta)|_{\Delta=0} \mathbf{1}_i = -1_i \hat{\mathbf{Q}}^{-1}(0)\hat{\mathbf{Q}}(\Delta)|_{\Delta=0} \hat{\mathbf{Q}}^{-1}(0) \mathbf{1}_i.$$
Thus, we see that $q^1_i(r_k), q^2_i(r_k), \ldots, q^n_i(r_k)$ is the quadratic form associated with the positive definite matrix $R^{-2} \Omega^{-1}$, which is positive, since not all elements of $(q^1_i(r_k), q^2_i(r_k), \ldots, q^n_i(r_k))$ can be zero (since $\hat{Q}(0)$ is invertible).

7 References


Strang, G. Linear Algebra and Its Applications. Harcourt Brace Jovanovich