The Timing of Analysts’ Earnings Forecasts and Investors’ Beliefs\(^1\)

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Abstract

The literature assumes that the order and timing of analysts’ earnings forecasts are determined exogenously. Ignoring strategic timing decisions of analysts may lead to inconsistent estimates of investors’ beliefs. The paper analyzes the equilibrium timing strategies for analysts, and derives the consistent estimates of investors’ beliefs. I follow the literature in assuming that analysts care foremost about the accuracy of their forecasts, but in some cases may have an incentive to bias their forecasts. I further assume that investors reward early forecasting analysts. The paper introduces a timing game with two analysts and derives its unique subgame perfect equilibrium in pure strategies. The equilibrium has two patterns: either the times of the analysts’ forecasts cluster, or there is separation in time of the forecasts. More precise private signals of the analysts induces earlier forecasts, and increases the likelihood that the analysts’ forecasts will cluster in time. The second part of the paper presents an algorithm for calculating the consistent investors’ beliefs following an analyst’s forecasts (for N analysts). It shows that the mean of investors’ beliefs is a weighted average of all the forecasts, where the weights are determined according to the recency of each forecast, the precision of the private signal of each analyst, and the precision of the investors beliefs about the analyst’s bias.
1 Introduction

Sell-side analysts are one of the most important sources of information for investors in the stock market. Among other services, they provide early forecasts of quarterly earnings, forecasts which investors use for stock valuation. Their forecasts are based on information they generate privately as well as on publicly available information, which includes prior forecasts of other analysts. This suggests that every analyst is at the same time a supplier of information to other analysts and a consumer of such information, that comes from other analysts. The degree to which he plays each part is determined by his position in the sequence of forecasts announcements - that is to say the timing of his forecast. The question that generated this paper is whether the timing and order of analysts’ forecasts is determined exogenously, as implicitly assumed in much of the literature, or whether analysts choose the timing of their forecasts strategically. The answer to this question may yield additional insights into the behavior of sell-side analysts that received so much attention recently. Specifically, the information contained in the timing and order of analysts’ forecasts may help decipher their informational content. Ignoring this information frequently leads to inconsistent estimates. The answer to this question is an empirical one. In this paper I propose a theory for the timing of analysts’ earnings forecasts and analyze the equilibrium timing and reporting strategies for the analysts. The second question that the paper addresses is how rational investors should form their beliefs following an analyst’s forecast.

I follow the literature in assuming that analysts care primarily about the accuracy of their forecasts (e.g. Mikhail, Walther, Willis [1999], Hong and Kubik [2003]), but in some cases may also have an incentive to bias their forecasts (e.g. Dugar and Nathan [1995], Hong and Kubik [2003], Bernhardt & Campello [2002], and Lim [2001]). In a conventional model based only on these two assumptions, all analysts would optimally forecast immediately before the earnings announcement by the firm, since this is when their forecasts are the most accurate. This is clearly not a reasonable outcome. Investors would be willing to reward a deviating analyst who provides an earlier signal. Hence, there must be an offsetting effect. To capture this effect, I propose an additional component to the analyst’s payoff function. I assume that the compensation of the analyst declines in the precision of the investors’ beliefs about the earnings of the firm at the time of the forecast. Thus, the payoff of an early-reporting analyst is higher than the payoff of an analyst who publish the same forecast at a later time. This assumption
finds an empirical support in Cooper, Day and Lewis (2001).

Following the incorporation of this third assumption about the analyst’s payoff function, the analyst faces the trade-off between an earlier, but less precise forecast, and a later but more precise one. I further assume that a continuous stream of information from exogenous sources (all sources other than analysts) arrives over time and affects the public’s beliefs about the firm’s earnings. Ivkovic and Jagadeesh (2004) find that the informational content of analysts’ earnings forecasts revisions, generally increases over event time. As to investors, I assume that they (as well as other analysts) do not necessarily know the actual bias of an analyst, which further complicates their inference.

I start by analyzing a single analyst case, which serves as an unconstrained optimum benchmark. The optimal forecast timing for a single analyst is determined by the precision of his private signal and by his cost of a forecast error. The intuition is straightforward: higher precision induces earlier forecast, while higher cost of an error postpones the forecast for the purpose of gaining more information over time. Next, I introduce a timing game with two analysts. The game has a unique subgame-perfect equilibrium in pure strategies, which has two patterns. The simplest case is when the two analysts are sufficiently different from each other, that each publishes his forecast at the respective unconstrained optimal time. In this case, there are no strategic interactions between the two. Alternatively, the two analysts issue forecasts one immediately after the other, creating an endogenous clustering in time of forecasts. The likelihood of the latter (clustering) equilibrium pattern declines in the distance between the unconstrained optimal timing of the two analysts, and increases in the precision of the private signals of the analysts and the precision of the investors beliefs about the bias of the analysts. At times of extensive arrival of new information (around an event), the model generates an endogenous clustering of timings of analysts’ forecasts, that are very common in the data. The model predicts the order, the timing, and the reported value of both forecasts.

A second contribution of this paper is to derive the consistent investors’ beliefs about the value of a stock, following an analyst’s forecast. This part is not limited for only two analysts, and is applicable for any number of analysts. Analysts’ forecasts of earnings are increasingly used in accounting and finance research as proxies for the unobservable ”market” expectation.

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1 In their introduction, Cooper, Day and Lewis (2001) state: "Since brokerage firms’ profits depend directly on commission revenues, analysts compensation is based, in part, on the trading volume generated by their research. This gives superior analysts an incentive to release information before other analysts in order to capture trading volume for their firms."

2 The model works the same for analysts target prices. I conjecture that a similar argument can be made about stock recommendations, however, since those are on a discrete grid, a different methodology must be employed.
of earnings. Since forecasts are not simultaneous, one has to decide how to distill the forecasts into expectations. A frequently used proxy for the investors' beliefs is the widely disseminated consensus (for example as calculated by I/B/E/S), which gives equal weight to the last forecasts of every analyst covering the stock. Equal weights imply that effectively, earlier forecasts get higher weights than the later ones, since the inferred information from the first forecast is incorporated in the following forecasts. O'Brien (1988) finds that the most current forecast available is more accurate than either the mean or the median of all available forecasts. This suggests that the timing of a forecast contains information about its precision. Moreover, conditional on only relative recent forecasts being included, means or medians increase accuracy by aggregating across idiosyncratic individual error. Brown (1991) compares the predictive accuracy of the mean and three timely composites: the most recent forecast, the average of the three most recent forecasts, and the 30-day average. The mean is shown to be less accurate than all three, and the 30-day average is shown to be the most accurate timely composite. These findings suggest the existence of a trade-off between recency and aggregation. In this paper I introduce an algorithm that derives the consistent investors' beliefs. In order to derive the investors' beliefs following an analyst's forecast, one should extract the estimate of the analyst's private signal from his forecast, and use the prior beliefs and all the extracted signals' estimates of analysts that issued their forecasts already. The expected earnings is an average of all the signals, weighted by their relative precisions and the prior mean. The reduced form of the posterior mean is presented as a weighted average of all the forecasts and the prior mean. The weight given to an analyst's forecast increases in the precision of his private signal, the precision of investors' beliefs about his bias, and his position in the sequence of forecasts. Later forecasts receive larger weight, since they incorporate the information from previous forecasts. In the particular case where the bias of each analyst is common knowledge, the forecast of the last analyst is a sufficient statistic for the mean of investors' beliefs.

The suggested algorithm for calculating investors beliefs (mean and variance), is applicable both under the assumption of endogenously or exogenously determined order of forecasts. The difference manifest itself when the parameters of the analyst's payoff function are not known to the investors. In this case, if the order is endogenous, investors can make inference from the order itself about the analysts' private signals and biases.

While the literature on the incentives of forecasters, and in particular analysts, is quite extensive, very little has been said about the order and timing of analysts' forecasts. To the best of my knowledge, the only theoretical paper that addresses the endogenous timing of
forecasters is Gul and Lundholm (1995). They present a model in which two agents choose an action and the time at which to take the action. Each agent gets the realization of one random variable, where the value of the project is the sum of the two random variables. The agents must predict the future value of a project, and given all else equal, they prefer to predict sooner rather than later. The authors show that agents’ forecasts always clusters together in time. The main ingredient of their model is the trade-off between a more informed decision and the urgency to make a decision. They assume that the urgency to forecast is independent of the precision of the investors beliefs, and accounts only for the time per-se. Moreover, they do not capture the possibility that forecasters may be biased.

Several empirical papers are of particular relevance: Cooper, Day and Lewis (2001) provide an assessment of analyst quality that differs from the standard approach, which uses survey evidence to rate analysts (e.g. Institutional Investors). They find that lead analysts, identified by their measure of forecast timeliness, have a greater impact on stock prices than follower analysts. Further, they find that performance rankings based on forecast timeliness are more informative than rankings based on abnormal trading volume and forecast accuracy. Lin, McNichols, and O’Brien (2003) provide evidence that affiliation influences analysts’ timeliness in downgrading their recommendations. Bernhardt and Campello (2002) study the relation between the forecast and its timing, but their focus is mostly on the forecast revisions towards the end of the forecasting period.

There is an extensive literature claiming that analysts may have incentives that lead to biases in their forecasts and their stock recommendations. Dugar and Nathan (1995) show that financial analysts of brokerage firms that provide investment banking services to a company are optimistic, relative to other (non-investment bankers) analysts, in their earnings forecasts and stock recommendations. Lin and McNichols (1998) find that lead and co-underwriter analysts’ growth forecasts and recommendations are significantly more favorable than those made by unaffiliated analysts, although their earnings forecasts are not generally greater. Michaely and Womack [1999] document that analysts may be too optimistic about firms from which they are trying to solicit underwriting business. Hong and Kubik (2003) show that career concerns may induce overoptimistic forecasts. Bernhardt and Campello (2002) attribute the bias in the analysts’ forecasts to expectation management by the managers, who try to

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3 A very strong implicit testimony, is the recent settlement between ten largest investment banks and the SEC agreeing to pay $1.4 billion in fines and reparations for potentially misleading investors in their analysts' reports.
avoid negative earnings surprises.\textsuperscript{4} Lim (2001) claims that an analyst’s forecast bias is fully rational because it induces the firm’s management to produce better information to optimistic analysts.\textsuperscript{5} There is an extensive literature (both theoretical and empirical) that examines the relationship between reputational concerns and herding behavior (e.g., Scharfstein and Stein (1990), Trueman (1994), Welch (2000)). In these models, the reputation arises from learning over time about agent’s exogenous characteristics (e.g. ability) through his observed behavior. Considerations for reputation or ”career concerns” can lead agents to underweight (or even ignore) private information and to herd. I use a reduced form for the analyst’s objectives, where forecasting errors induce reputation cost. I do not model the origin of the reputation costs; rather I assume that it is given exogenously.

The paper is organized as follows: Section 2 presents the setup of the model. In section 3 I derives the optimal behavior of a single analyst. Section 4 presents the timing game between two analysts. In Section 5, I derive consistent estimates of the investors’ beliefs. Section 6 concludes.

\section{Model Setup}

Most of the literature implicitly assumes that the timing of analysts’ earnings forecasts is random or exogenously determined. The main objective of analysts is assumed to be the accuracy of their forecasts, i.e. to minimize the expected squared error of their forecasts.\textsuperscript{6} However, as discussed in the introduction, analysts may have other incentives that may bias their forecasts. The incentives of analysts to bias their forecasts at a specific time (quarter) are not transparent and not perfectly known to investors. The model assumes that the actual bias in analysts’ forecast may be unknown to the investors and to the other analysts (this is similar to the assumption of Fischer and Verrecchia (2000) about managers’ reporting bias). I also solve the particular case where the bias of the analysts is known to investors (which is equivalent to unbiased analysts)). For the simplicity of the disposition, I assume that the analyst’s expected utility is linear in his bias. The model is robust to a large class of functional dependence between the analyst’s bias and his expected payoff, and is not restricted to linear


\textsuperscript{5}Irvine (2003), asserts that an analyst’s coverage of a firm induces higher commissions to his brokerage firm; nevertheless, analysts can not induce extra commissions by simply biasing their published forecast.

\textsuperscript{6}Basu and Markov (2003) argue that analysts behavior is rational if we assume that they minimize their absolute forecast error rather than a quadratic cost function.
functional dependence (later the paper will elaborate on this).\footnote{If we assume that \( \alpha_i \) - the coefficient of the bias in the analyst’s expected payoff, is not a constant and is a function of \( f(t) \), then the optimal forecast time of the analyst is: \( f(t^*_i) = \sqrt{\frac{\beta_i}{1 - \frac{\beta_i}{\alpha_i} \frac{\partial f(t)}{\partial t}}} - f_s \). The model is robust to all \( f(t) \) for which \( f(t^*_i) \) is well defined.}

The above two components of the analyst’s utility function are prevalent in the literature. In a conventional model based only on this two components, all analysts would optimally forecast immediately before the earnings announcement of the firm, since this is when their forecasts are the most accurate. This is clearly not a reasonable outcome, because investors would be willing to reward a deviating analyst who provide early forecast. So there must be an offsetting effect that is ignored. The additional assumption that I propose to capture this offsetting effect is the following: the payoff of an analyst depends on how valuable his forecast is to investors - reflected by the precision of investors’ beliefs about the firm’s earnings immediately prior to the analyst’s forecast. The less precise the investors’ beliefs about the firm’s earnings are, the higher is the payoff of an analyst for a given forecast. Following is the motivation for this new assumption.

The analyst is paid by the brokerage house he works for. Big part of the earnings of a brokerage house is from trading commissions from its investors clients. The brokerage house and the analyst want to maintain existing clients, to have new clients and to increase the volume of trade executed through the brokerage house. The benefit that investors receive from analysts’ earnings forecasts is early access to information. The most preferred clients of an analyst get his forecast first; only later do the less preferred clients get this forecast, and eventually it is publicly published. Access to the information before it becomes public is valuable to investors and they are willing to pay for that. The investors use this information in order to construct their beliefs, upon which they make their financial decisions. These decisions eventually generate trade in the stock. In the extreme case where the investors are perfectly informed about the future earnings of the firm, an analyst’s forecast is worthless to them. Moreover, in this case, an analyst’s forecast will not generate any trade in the stock. The less informed investors are - that is the lower the precision of their beliefs about the firm’s earnings is, the more valuable the analyst forecast is to investors and the higher the trade it may generate.

As time advances, more public information about the forthcoming earnings arrives. This information arrives from analysts’ forecasts as well as from any other sources of information that is relevant to the firm (Macro economics, competitors, upstream and downstream firms, conference calls etc.). I refer to all information other than analysts’ forecasts as exogenous.
information. The precision of the investors beliefs about the earnings of the firm at time \( t \) is
denoted by \( f(t) \). I assume that the arrival of the exogenous information is continuous, meaning
that the precision of the investors beliefs is assumed to be continuously increasing in time (in
all times except at a time of forecast publication where there will be a discrete increase in the
precision of the investors beliefs).  

All else equal (including the precision of the forecast), the sooner the analyst provides his
forecast, the more valuable his information to his clients is and the more trade it may generate,
hence his payoff is higher. But there is also a cost for early forecasting. The sooner the analyst
publishes his forecast, the less accurate is the public information he uses to generate his forecast;
hence, his expected forecast error is higher. This is the basic trade-off that analysts face in
determining the timing of their forecasts. Cooper Day and Lewis (2001), point out (empirically)
the willingness of lead analysts to trade accuracy for timeliness due to their desire to maximize
compensation.

I believe that the three incentives of analysts mentioned above are the central ones to their
behavior. As in every model, there might be other incentives of analysts which are not accounted
for in my model. I use a reduced form of the analyst’s objective function, which captures the
above characteristics and trade-off. The expected utility of analyst \( i \) who makes a forecast at
time \( t \) is assumed to be:

\[
EU_i^t = \alpha_i \left( \pi_{i,t}^F - E[\pi|\psi, I_t] \right) - \beta_i E \left[ (\pi_{i,t}^F - \pi)^2 |\psi, I_t \right] - f(t)
\]

where \( \alpha_i \sim N(\mu_{\alpha_i}, \sigma_{\alpha_i}^2) \) is the bias parameter, \( \beta_i \) is a positive constant, \( \pi_{i,t}^F \) is the forecast of
analyst \( i \) (published at time \( t \)), \( \psi_i \) is the private signal of the analyst, and \( I_t \) is all the public
information available at time \( t \) (immediately prior to the analyst’s forecast) which includes the
preceding analysts’ forecasts. The realization of \( \alpha_i \) is known only to the analyst himself, where
\( \beta_i \) is common knowledge. 

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8The model is robust to any process of information arrival, but the continuity assumption simplifies the
analysis by making the analysts’ utility function differentiable, and facilitate a simple analytical optimization
solution.

9A more general utility function is: \( EU_i^t = \alpha_i \left( \pi_{i,t}^F - E[\pi|\psi, I_t] \right) - \beta_i E \left[ (\pi_{i,t}^F - \pi)^2 |\psi, I_t \right] - \gamma_i g(f(t)) \) where
\( g(\cdot) \) is a continuously increasing function. I show in Appendix 1 that the results of the model holds for a very
big set of functions \( g(f(t)) \). w.l.o.g. \( \gamma_i \) is normalized to equal 1.

If the utility function is assumed to be multiplicative rather than additive, for example \( EU_i^t =
E \left[ \alpha_i \pi_{i,t}^F - \beta f(t) (\pi_{i,t}^F - \pi^R)^2 |\psi, I_t \right] \),
we get that the optimal forecasting timing is mostly a corner solution. Nevertheless, in general, the comparative
statics works (weakly) in the same directions as in the above additive utility function.

7
The utility function of the analyst has three components: The term \( \pi_{i,t}^F - E[\pi_i | \psi_i, I_t] \) captures the analyst’s incentives to bias his forecast;

The term \( E \left[ (\pi_{i,t}^F - \pi)^2 | \psi_i, I_t \right] \) is the mean squared error (MSE) of the analyst’s forecast and captures the analyst’s desire to be precise; and \( f(t) \) captures the incentive of the analyst to provide his forecast at a stage where the precision of the investors information is low.

An analyst has to publish his forecast at some point during the "forecasting period" \( t \in [0, T] \), e.g. between the earnings report of the previous quarter and the forthcoming earnings report. After the forecasting period, the firm reports its realized earnings, denoted by \( \pi \). At the beginning of the forecasting season \( (t = 0) \), investors are assumed to have normally distributed prior beliefs about the earnings of the firm - \( \pi_0 \sim N(\mu_{\pi_0}, \sigma_{\pi_0}^2) \). The precision of the prior beliefs is denoted by \( f(0) \equiv 1/\sigma_{\pi_0}^2 \). As time progresses, there is a continuous stream of information that increases the precision of public’s beliefs - \( f(t) \) (while the beliefs remain normally distributed). For all \( t_1 > t_2 \) we have \( f(t_1) > f(t_2) \), \( f(0) > 0 \) and \( f(T) < \infty \).

Analyst \( i \) gets a private signal about the earnings of the firm - \( \bar{\psi}_i = \pi + \bar{\epsilon}_i \), where \( \bar{\epsilon}_i \sim N(0, \sigma_{\bar{\epsilon}_i}^2) \) is independent of \( \pi \) (in the case of more than one analyst, for all \( i \neq j \) - \( \bar{\epsilon}_i \) is independent of \( \bar{\epsilon}_j \)). I denote the precision of analyst \( i \)'s private signal by \( f_{S_i} \equiv 1/\sigma_{\bar{\epsilon}_i}^2 \). The time at which an analyst observes his private signal does not influence the equilibrium results, as long as it happens before the equilibrium timing of his forecast. But just for simplicity, lets assume that the analysts get their private signals at \( t = 0 \).

Immediately following the analyst’s forecast, there is a discrete jump in the precision of the investors’ beliefs, and thereafter, until the next analyst’s forecast, the precision of the investors’ beliefs continuously increases according to the exogenous information process.

Using the above setup, I first solve the optimization problem of a single analyst case - his optimal forecasting timing and the optimal forecast at that point of time. Next, I study a game between two analysts, who must decide at what point of time to publish their forecasts. Since the analyst’s expected utility depends on the precision of the public’s information at the time of forecast (and not on the time per se), a strategic interaction arises between the analysts, which should be considered. I derive a pure strategies subgame perfect equilibrium of this game and prove its existence and uniqueness.

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10 Assuming that managers manipulate the reported earnings by a constant (see for example Stein (89) and Gutman, Kadan, Kandel (2004)), would not influence the results.

11 It could be an approximation of a discrete process of normally distributed signals.
3 The Single Analyst Case

In this part I model the case of a single analyst who has to choose the time of his forecast. This simple case is not a strategic game, but rather a simple optimization problem. I first derive the optimal forecast for every given forecasting time, then, given the optimal forecasts I find the optimal timing of the forecast (to be precise, I derive the precision of the investors’ beliefs at which it is optimal for the analyst to publish his forecast).

3.1 The Optimal Forecast

The analyst’s first order condition with respect to the forecast at a given time \( t \) is
\[
\alpha_i - 2\beta_i E \left( \pi_{i,t}^F - \pi | \psi_i, I_t \right) = 0.
\]

The second order condition for maximum is satisfied. Hence, for every given timing of forecast \(-t\), the analyst’s optimal forecast is:
\[
\pi_{i,t}^F = \frac{\alpha_i}{2\beta_i} + E \left( \pi | \psi_i, I_t \right)
\]
(2)
\[
= \frac{\alpha_i}{2\beta_i} + \frac{\sigma^2}{\sigma^2_{\pi_t} + \sigma^2 \mu_{\pi_t}} + \frac{\sigma^2}{\sigma^2_{\pi_t} + \sigma^2} \psi_i
\]
where \( \mu_{\pi_t} = E \left( \pi | I_t \right) \) is the public’s expectation at time \( t \) immediately prior to the forecast (which may be different from \( \mu_{\pi_0} \)).

The optimal forecast of the analyst is to bias his forecast by a constant \(-\frac{\alpha_i}{2\beta_i}\). Although the analyst’s optimal forecast is linear in his private signal, it does not fully reveal his private signal since \( \alpha_i \) is not known to investors. Only in the case where the bias parameter \( \alpha_i \) is common knowledge, the analyst’s forecast fully reveals his private signal.

Substituting the optimal forecast of the analyst into his utility function yields:\[12\]
\[
EU_i^A = \frac{\alpha_i^2}{4\beta_i} - \frac{\beta_i}{f(t)} - f(S_t) - f(t).
\]
An increase in \( f(t) \) has two opposite effects on the analyst’s expected utility. On the one hand, it increases the analyst’s information and reduces the expected cost of a forecast error. On the other hand, the analyst incurs the direct cost of a later forecast. In the next section I find the analyst’s optimal behavior in solving the above trade-off.

\[12\] \( EU_i^A = \alpha_i \frac{\alpha_i}{2\beta_i} - \beta_i E \left( E [\pi | \psi_i, I_t] + \frac{\alpha_i}{2\beta_i} - \pi \right)^2 | \psi_i, I_t \right) - f(t) = \alpha_i \frac{\alpha_i}{2\beta_i} - \frac{\alpha_i^2}{4\beta_i} - \beta_i Var(\pi | \psi, I_t) - f(t) \), in the appendix I show that \( Var(\pi | \psi, I_t) = \frac{1}{f(t) + f(S_t)} \)
3.2 The Optimal Timing of the Forecast

The time per-se is not an important factor, rather, what matters is the precision of the investors beliefs at each point in time.\textsuperscript{13} In light of the above trade-off, in order to find the precision of the investors’s beliefs at the optimal forecasting timing, I take the derivative of the analyst’s expected utility with respect to the precision of the investors beliefs.

Optimization yields
\[
\frac{\beta_i}{(f(t) + f_{S_i})^2} - 1 = 0,
\]
which suggests that the analyst should forecast at
\[
f(t) = \sqrt{\beta_i} - f_{S_i}.
\]

But the precision of the investors’ beliefs at which the analyst can forecast is constrained by the precision of the investors’ beliefs at the beginning and at the end of the forecasting season.

I denote the analyst’s optimal forecasting time by \( t_i^* \). Given the constraint of the forecasting season, we get:

**Proposition 1**

\[
t_i^* = \begin{cases} 
0 & \text{if } f_{S_i} \geq \sqrt{\beta_i} - f(0) \\
 f^{-1}(\sqrt{\beta_i} - f_{S_i}) & \text{if } \sqrt{\beta_i} - f(T) < f_{S_i} < \sqrt{\beta_i} - f(0) \\
 T & \text{if } f_{S_i} \leq \sqrt{\beta_i} - f(T)
\end{cases}
\]

where \( f^{-1}(\sqrt{\beta_i} - f_{S_i}) \) is the time at which the precision of investors’ beliefs is \( f(t) = \sqrt{\beta_i} - f_{S_i} \).

Hereafter, I refer to the time \( t_i^* \) as the “unconstrained optimum timing”.

If the analyst’s private signal is sufficiently precise, an increase in the precision of investors’ beliefs is not sufficient to compensate for the cost of late forecast. This means that his expected utility monotonically decreases in time, and hence he forecasts as early as he can. This is illustrated by the High precision case in the figure below. On the other hand, for sufficiently low precision of the analyst’s private signal he will wait to gain as much public information as he can, and will forecast at \( T \) - the Low precision case in the figure below. In the “intermediate case”, the trade-off is such that the analyst waits until time \( t_i^* \) where \( f(t_i^*) = \sqrt{\beta_i} - f_{S_i} \). After this time, the cost of late forecasting exceeds his payoff from the increase in the precision of the

\textsuperscript{13}I could account for the time value of money as well.
information he uses. This implies that for \( f_{S_i} < \sqrt{\beta_i} - f(0) \) the expected utility of the analyst monotonically increases for \( f(t) < f(t^*) \), and after the time \( t^* \) it monotonically decreases. This is illustrated by the Interior solution case in the figure below.

\[ EU_i \]

Expected utility - different cases

**Corollary 1 (ComparativeStatics)** Given the interior solution for \( t_i^* \), i.e. \( \sqrt{\beta_i} - f(T) < f_{S_i} < \sqrt{\beta_i} - f(0) \), the optimal forecasting time of the analyst decreases in the precision of his private signal; and increases in the cost of an error - \( \beta_i \).

The above corollary is quite intuitive. Less precise private signal of the analyst induces him to postpone his forecast and gain from the increased precision of the public’s information. On the other hand, the lower is his reputation cost for a given forecast error (captured by \( \beta_i \)), the higher is his propensity to risk a large error in order to provide his forecast at an early stage. Since the coefficient of the precision of the investors’ beliefs in the analyst’s utility function is normalized to equal 1, higher rewards for early forecast is equivalent to reducing both \( \alpha_i \) and \( \beta_i \), and induces an earlier forecast.

The timing of the analyst’s forecast is independent of the precision of the investors’ beliefs about his bias. The realizations of both the private signal and the bias parameter do not affect the optimal forecasting timing. The realized signal does not influence the precision of the analyst’s beliefs nor the precision of the investors beliefs following his forecast (it influences only the conditional expectations but not the conditional variance). Hence, the realized signal is does not influence the trade-off that the analyst faces while choosing the time of his forecast. As to the value of \( \alpha_i \), changes in \( \alpha_i \) linearly change the bias in the analyst’s forecast. All else equal, a change in \( \alpha_i \) affects only the first expression in (3) which is independent of \( f(t) \), and is

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out of the analyst’s timing trade-off decision. From the investors’ perspective, the closer is the realized value of $\alpha_i$ to its mean, the more precise their beliefs following the analyst’s forecast are. The more confidence the investors regarding the analyst’s bias, the more they learn from his forecast.

In the next section I introduce a timing game between two analysts. Due to the strategic interaction between the analysts, both their prior beliefs about the bias of the other analyst and the precision of the private signals will influence the equilibrium strategies of the analysts.

4 Timing Game With Two Analysts

Most stocks are covered by more than one sell-side analyst. This implies that it is important to understand how does competition alter the behavior of sell-side analysts. Recall that following an analyst’s forecast, there is a discrete increase (“jump”) in the precision of investors’ beliefs. The support of the precision of investors beliefs at which an analyst can publish his forecast is no longer $[f(0), f(T)]$, but rather it depends on the precision of the private signal of the second analyst and the time at which the second analyst publishes his forecast. One can think of two different time lines: the calendar time line and the precision of the investors’ beliefs time lines. While the calendar time line is continuous, the precision of the investors beliefs time line has a ”jump” at the time of analysts’ forecasts. Figure 1 presents the two time lines for analyst $i$. The the horizontal axes (the calendar time line) obtains continuous values, but on the vertical axes (the precision of investors beliefs at which analyst $i$ can publish his forecast) there is a discrete jump at $f(t^*)$ following the forecast of analyst $j$ at time $t^*$. The size of the discrete jump following the forecast of analyst $j$ is denoted by $\Delta f_{S_j}$. 

Since the precision of investors’ beliefs derives the optimal timing, the analysts must consider each other’s timing. This immediately introduces strategic interaction between them. Let's
assume, for example, that the unconstrained optimal time for analyst \( i \) is \( t^*_i \). If he waits till \( t^*_i \), he bares the risk that analyst \( j \) will step in front of him and forecast at \( t^*_i - \epsilon \). If analyst \( j \) does forecast at this time, analyst \( i \) will face investors’ beliefs with precision of \( (f(t^*_i - \epsilon)) + \Delta f_{S_j} \), where \( \Delta f_{S_j} \) denotes the increase in the precision of the investors’ beliefs due to the forecast of analyst \( j \). This will decrease the expected utility of analyst \( i \) relative to forecasting right before analyst \( j \). Analyst \( j \) has to take into account that analyst \( i \) may hence forecast earlier than \( t^*_i \), and should consider forecasting even earlier. This example illustrates the kind of strategic interaction that the analysts have to take into account. In this section I develop and prove the existence and uniqueness of a subgame perfect equilibrium in pure strategies of a game between two analysts.

In the game with two analysts, the second forecaster incorporates the information from the first forecast, while forming his beliefs. The higher is the precision of the investors’ beliefs regarding the bias of an analyst, the more they can infer from his forecast, and the higher is the precision of their beliefs immediately after his forecast. Hence, the magnitude of the discrete jump in the precision of the investors’ beliefs following an analyst’s forecast increases in both the precision of the analyst’s private signal and in the precision of the investors beliefs about his bias. While forming his strategy, each analyst has to take into account the increase in the precision of investors’ beliefs due to the other analyst’s forecast, and due to his own forecast. In contrast to the single analyst case, the precision of the investors’ beliefs regarding the bias of the analysts do influence the equilibrium timing of the forecasts.

To simplify the intuition, I first solve the model for the case where the bias of each analyst is common knowledge and hence the analyst’s forecast fully reveals his private signal.\(^{14}\) After this basic model is established, I introduce the case of asymmetric information regarding the analyst’s bias.

The basic setup and assumptions I use are similar to the single analyst case.

### 4.1 The Known Bias Case

Lets assume that there are two analysts \( i = 1, 2 \). The expected utility of analyst \( i \) who makes a forecast at a time \( t \) (at which the precision of the investors’ beliefs is \( f(t) \)) is assumed to be

\(^{14}\)As will be shown ahead, the equilibrium timing strategy is monotonic in the precision of the private signal, and hence the forecast fully reveals the private signal.
(as before):

\[ \text{EU}_t^i = \alpha_i \left( \pi_{i,t}^F - E[\pi|\psi_i, I_t] \right) - \beta_i E \left[ (\pi_{i,t}^F - \pi)^2 |\psi_i, I_t \right] - f(t), \]

where the parameters of the utility function (including \( \alpha_i \)) and the precision of the private signals of each of the analysts are common knowledge.

The precision of the public’s beliefs is continuously increasing in time, except at the times of the analysts’ forecasts, where it follows a discrete jump. At the beginning of the game \((t = 0)\), each analyst observes a private signal of the reported earnings - \( \tilde{\psi}_i = \pi + \tilde{\varepsilon}_i \). Each analyst has to forecast at some point during the forecasting season \( t \in [0, T] \). If the parameters of each analyst are drawn from a continuous distribution, the probability that both analysts will want to forecast at the same time \( t' \in (0, T) \) is of measure zero. Moreover, the only pure strategies subgame perfect equilibrium in which both analysts forecast simultaneously at \( t' \in (0, T) \) is the one I derive below. Hence we only have to deal with simultaneous forecasting at \( t = 0 \) and at \( t = T \). I assume that if both analysts forecast simultaneously, then the utility of each of them is exactly the same as if he was the only analyst to forecast at that point of time.

A strategy for an analyst is a function that maps from the prior parameters into a precision of investors’ beliefs at which to forecast and the forecast at that time. The prior parameters includes: the utility function of each analyst, the precision of the analysts’ private signals, and the precision of the investors’ beliefs about the bias parameters of the analysts. I denote the precision of the investors’ beliefs at the equilibrium forecasting time of analyst \( i \) by \( f(t_{i,c}^*) \) (where the subscript \( c \) indicates the constraint on the precision of the investors’ beliefs at which an analyst can forecast - due to the discrete jump in the precision of the investors’ beliefs following the other analyst’s forecast).

If the increase in the precision of investors’ beliefs following the analysts’ forecasts is sufficiently small, and the unconstrained optimal forecasting times of the analysts are sufficiently apart from each other, then it is feasible that each analyst forecasts at his unconstrained optimal forecasting time - \( t_i^* \). In this case, none of the analysts has an incentive to deviate from this strategy, hence it is an equilibrium. Moreover, I later show that in this case it is the unique Sub-game Perfect Equilibrium in pure strategies. But if it is not the case, then each analyst has to take into account the increase in the precision of investors’ beliefs due to his own and the other analyst’s forecast. The following Lemma describes the increase in the precision of investors’ beliefs due to an analyst’s forecast.

**Lemma 1** When \( \alpha_i \) is common knowledge, the increase in the precision of the investors’ beliefs
following an analyst’s forecast is constant and equals to the precision of his private signal.

For proof see Appendix 2.A.

If the precision of investors’ beliefs immediately before the analyst’s forecast is \( f(t) \), and the precision of the private signal of analyst \( i \) is \( f_{S_i} \equiv \frac{1}{\sigma_{S_i}^2} \), then the precision of the investors’ beliefs immediately after the forecast of analyst \( i \) is \( f(t) + f_{S_i} \).

I next derive the pure strategies equilibrium.

**Claim 1** For each analyst \( i = 1, 2 \):

(A) If \( f_{S_i} > \sqrt{\beta_i} - f(0) \) then he forecasts as soon as he can (at \( t = 0 \)).
(B) If \( f_{S_i} < \sqrt{\beta_i} - f(T) \) then the analyst forecasts at the latest possible time, that is at \( t = T \).

**Proof.** Since this is the unconstrained optimal strategy of an analyst, and it is feasible, by revealed preference the proof from the single analyst case holds in this case as well. ■

Let’s consider the case where the unconstrained optimum of both analysts is interior, that is for \( i = 1, 2 \) \( f(0) < f(t_i^*) < f(T) \). Given that the increase in the precision of investors’ beliefs due to the forecast of analyst 2 is \( f_{S_2} \), there is a "hole" of size \( f_{S_2} \), in the support of the precision of investors’ beliefs at which analyst 1 can publish his forecast. But the location of this "hole" depends on the timing strategy of analyst 2, which of course, takes into account the strategy of analyst 1. To resolve this strategic interaction I will define and use the notion of "indifference interval". Intuitively speaking, the indifference interval of analyst 1 is the interval of precision of investors’ beliefs, of size \( f_{S_2} \), for which analyst 1 is indifferent between forecasting at either the lower or the upper end of this interval. The expected utility of the analyst monotonically increases in the precision of the investors beliefs until it gets maximized, and from there on the expected utility monotonically decreases. This implies both the uniqueness of the indifference interval and that the indifference interval straddles the unconstrained optimum of the analyst (where its expected utility is maximized). Bellow is a formal definition of the lower end of the indifference interval \( f_{1L}^1 \), which also defines the indifference interval for more complex cases where the above "intuitively speaking definition" does not hold.

**Definition 2** Let define \( f_{1L}^1 \) as follows:

If there exist a precision of investors’ beliefs \( f' \) such that analyst 1 is indifferent between forecasting at a precision of investors’ beliefs that equals either \( f' \) or \( f' + f_{S_2} \) then \( f_{1L}^1 \equiv f' \).\(^{15}\) Figure

\(^{15}\)If such \( f' \) does exists then it is unique and \( f(0) \leq f_{1L}^1 < f(t_1^*) \).
2.1 illustrates $f^1_L$ for this case (interior indifference interval).

If an indifference interval as the above does not exist, it is one of the following two cases:

Case A - $EU^1 (f(t) = f(0)) > U^1 (f(t) = f(0) + f_{S_2})$. In this case I define $f^1_L = f(0)$. (See Figure 2.2)

Case B - $EU^1 (f(t) = f(T)) > EU^1 (f(t) = f(T) - f_{S_2})$. In this case I define $f^1_L$ to be the precision of investors’ beliefs where $EU^1 (f^1_L) = EU^1 (f(T))$ and $f^1_L < f(T)$. (See Figure 2.3)

Intuitively, $f^1_L$ is the lower value of the interval of size $f_{S_2}$, that will make the analyst indifferent between forecasting at the one or the other ends of that interval.

Given the above definition, I can now present the main proposition of this section.

---

16Up to this point, for the simplicity of disposition, I haven’t defined whether $f(T)$ is the precision of investors’ beliefs at the end of the ”forecasting season” given that the other analyst has or has not published his forecast. Here I can no longer be vague about it, and I define $f(T)$ as the precision of investors’ beliefs given that the other analyst has published his forecast.
Proposition 2 There exists a unique Subgame Perfect Equilibrium in pure strategies where the equilibrium strategies of the analysts are as follows:

For each analyst $i = 1, 2$ if $f^*_i = f(0)$ he forecasts at $t = 0$. If at least for one of the analysts $f^*_i > f(0)$, then analyst 1 is the first to forecast if and only if $f^*_L < f^*_2$. Analyst 1 forecasts at a time $t^*_{1,c}$ where the precision of the investors’ beliefs is:

$$f(t^*_{1,c}) = \text{Min}\left\{f^*_L, f(t^*_i)\right\}.$$  

If the first analyst forecasts at $f^*_2$ then the second analyst forecasts ”immediately after” that, where the precision of the investors’ beliefs is $f(t) = f^*_2 + f_S$. If the first analyst forecasts at $f(t^*_1)$ then the second analyst will forecast at $f(t^*_2)$ if it is feasible (that is if: $f(t^*_1) + f_S \leq f(t^*_2)$), or else ”immediately after” the first forecast.

The optimal forecast of each analyst $i$ is:

$$\pi^F_{i,t} = \frac{\alpha_i}{2\beta_i} + E(\pi|\psi_i, I_t).$$  

The off equilibrium beliefs are as follows. Lets assume w.l.o.g. that $f^*_L < f^*_2$. For all $f(t) > f^*_L$ analyst 1 believe that analyst 2 is going to forecast immediately.$^{18}$

Before proving the proposition I describe the equilibrium intuitively and graphically.

The equilibrium may have one of the following two patterns. Non Clustering Equilibrium Pattern (or Separation in time) - each analyst publish his forecast at his unconstrained optimum. If this is not feasible, then the equilibrium has a different pattern.$^{19}$ Clustering in Time Pattern - the first forecasters is the analyst who’s lower end of the indifference interval ($F^L_i$) is smaller. Assume this is analyst 1. Analyst 1 publishes his forecast at $F^L_2$.Following his forecast the precision of the investors beliefs increases instantaneously and becomes higher than the unconstrained optimum of analyst 2. Since the precision of the investors beliefs is past the

$^{17}$The strategy that says that the second analyst will forecast ”immediately after” the first analyst is somewhat vague. One way to have well defined strategies and outcomes is using the framework of Simon and Stinchcombe (1989). All three assumptions that they impose on the strategies (F1-F3) hold in my model. Using this framework, there can be two consecutive forecast at the same instant of time. The limit of the discrete time equilibria as the time interval goes to zero converges to the continuous time equilibrium. Another way to have well defined strategies is using the framework of Perry and Reny (1994), where restriction (S4) upon strategies, imposes some $\varepsilon$ ”lag” between agents actions and guarantees that the game is well defined. Adopting Perry and Reny’s framework requires some adjustments for $f^*_L$ (in a magnitude smaller than $\varepsilon$)

$^{18}$The equilibrium can be supported by a larger and more general set of off equilibrium beliefs, including mixed strategies off the equilibrium path.

$^{19}$The Non-Clustering Pattern my be feasible even in the case where the unconstrained optimum of an analyst is included in the indifference interval of the other analyst.
Since $F^1_L < F^2_L$, analyst 1 is the first to forecast. He publishes his forecast at $F^2_L$. Following his forecast, the precision of the investors' beliefs “jumps” to $F^1_L + f_{S2}$, which is higher than the unconstrained optimum of analyst 2. Since the expected utility of analyst 2 at this region is decreasing in the precision of the investors’ beliefs, he publishes his forecast immediately after the forecast of analyst 1.

Analyst 1 forecast first at his unconstrained optimum - $f(t_1^*)$. Following his forecast the precision of the investors’ beliefs “jumps” to $f(t_1^*) + f_{S1}$, which is still lower than the unconstrained optimum of analyst 2. Analyst 2 waits until the precision of the investors’ beliefs equals his unconstrained optimum and then publishes his forecast.
Proof of Proposition 2. Analyst 1 will never forecast earlier than $f^1_L$ since he can guarantee himself higher expected utility. If the precision of investors’ beliefs is higher than $f(t^*_1)$ (due to discrete jump after the forecast of analyst 2), then analyst 1 will forecast immediately. Hence the only interval of public precision left to investigate (from analyst 1’s perspective) is $f^1_L < f(t) < f(t^*_1)$. In the case where $f^2_L > f(t^*_1)$ analyst 2 will not forecast before $f^2_L$, hence, knowing that, analyst 1 will wait and forecast at his unconstrained optimum, where the precision is $f(t^*_1)$. If $f^1_L < f^2_L < f(t^*_1)$ then analyst 1 will wait at least until $f^2_L$.

Before proceeding the proof, I introduce the following Lemma.

Lemma 2 If $f^1_L < f^2_L < f(t^*_1)$ then there is no pure strategies subgame perfect equilibrium in which the first forecast of the analysts will be at $f(t) > f^2_L$.

Proof of the Lemma. Assume that such pure strategy subgame perfect equilibrium exists. Then, the second forecaster can deviate and forecast at a sufficiently small amount of time earlier than the first forecaster, and by doing that he strictly increases his expected utility - in contradiction to this being an equilibrium. QED Lemma.

The Lemma indicates that for $f^1_L < f^2_L < f(t^*_1)$, if analyst 2 hasn’t forecasted before $f^2_L$, analyst 1 will forecast at $f^2_L$.

So far, I have shown that if $f^1_L < f^2_L$ then analyst 1 will forecast first at a public precision of $\min\{f^2_L, f(t^*_1), f(T)\}$. It is straight forward that the second analyst to forecast, will either wait until $f(t^*_2)$ (if it is feasible, that is if: $f(t^*_2) > f^2_L + f_{S_1}$), or else he will forecast immediately after the first analyst. QED.

4.1.1 Discussion of the Known Bias Case

An interesting question that generates empirical predictions is what determines the order and timing of the analysts’ forecast. In the model, the order and timing of the forecasts is determined by the precision of the private signals of the analysts, the reputation parameters $\beta_i$, and the process of exogenous information arrival (in the Unknown bias case presented in the following section, the precision of the investors’ beliefs about the bias parameter $\alpha_i$ will also influence the order and timing of the forecasts).

An increase in the cost of error of analyst $i$ - $\beta_i$, motivates him to publish his forecast later - when there is more public information he can use. More formally, it pushes his unconstrained optimum to later in time and to higher precision of the investors’ beliefs. The size of the indifference interval of both analysts is independent of $\beta_i$. Hence, an increase in $\beta_i$ shifts the
indifference interval of analyst $i$ to the right, without influencing the indifference interval of the second analyst. This will induce later forecast by analyst $i$. Note that if analyst $i$ was the first to forecast, it may also change the order of the forecasts.

As long as the exogenous information arrival is continuous, changes in the process of exogenous information arrival will not influence the precision of the investors' beliefs at which each of the analysts will publish his forecast. That is, with respect to the "precision of the investors’ beliefs time line" there is no change in any factor. The only thing the will change is the calendar time at which each analyst will publish his forecast.

An increase in the precision of the private signal of analyst $i$ has two conflicting effects. On the one hand, it decreases the unconstrained optimum of analyst $i$, but on the other hand it increases the change in the precision of investors’ beliefs following his forecast, which increases the indifference interval of analyst $j$. This in turn, reduces the lower end of the indifference interval of analyst $j - F_j^L$. The corollary bellow indicates that the influence of the first effect on the order of forecasts always dominates.

**Corollary 2 (Comparative Statics)** Suppose that in equilibrium analyst $i$ forecasts at time $t' \in (0, T)$. An increase in the precision of the private signal of analyst $i$, will early the time of his forecast, and weakly early the forecasting time of analyst $j$. If analyst $i$ was the first forecaster, he will still forecast first, but if he was the second forecaster, then, he may now become the first to forecast. Moreover, the precision of the investors’ beliefs immediately prior the first forecast will be lower relative to before the change.\(^{20}\)

For the proof of the Corollary see Appendix 3.

An interesting feature of the equilibrium is that not necessarily the analyst with the higher precision of private signal will be the first to forecast. Even in the single analyst case, the unconstrained optimum was determined by both the precision of the private signal and the error cost parameter $\beta$. It is possible that analyst $i$ has a higher precision of private signal, but his error cost parameter - $\beta_i$, is sufficiently higher than that of analyst $j$, so that the unconstrained optimum of analyst $i$ will be at a higher precision of the investors’ beliefs. But it is also possible that even though the unconstrained optimum of analyst $i$ is at a lower precision of investors’ beliefs than the unconstrained optimum of analyst $j$, still analyst $j$ will be the first to forecast. The reason for that is that the higher precision of the private signal of analyst $i$

\(^{20}\)The precision of investors’ beliefs at the time of the second forecaster may be higher, the same or lower. Both cases are presented in the proof.
induces a higher indifference interval of analyst $j$. As shown in Proposition (2) the first analyst to forecast is the one whose lower end of the indifference interval is smaller.

A very interesting extension of the model is the multi-analysts timing game, where there are more than two analysts. Having more than two analysts complicates the model. The indifference interval of each analyst should be defined in respect to each of the other analysts, since the increase in the precision of the investors’ beliefs following forecasts of different analyst may be different. In this case, it is not clear whether the uniqueness of the equilibrium holds in the general case, and I suspect that there is no closed form solution for deriving the equilibrium. Nevertheless, I conjecture that the intuitions of the two-analysts game apply also to the multi-analysts game. The Non Clustering Equilibrium Pattern of the two-analysts game holds in the multi-analysts game in the following case. If there is "no overlap" - in the sense that non of the analysts’ unconstrained optimum is included in other analyst’s indifference interval, then it is feasible that each analyst publish his forecast at his unconstrained optimum. This is obviously a Subgame Perfect Equilibrium. If there exists an "overlap" between at least two analysts, then in equilibrium we may obtain clustering in time of the analysts involved in this "overlap". As long as the indifference interval of each analyst includes the unconstrained optimum of at most one other analyst, the model can be implemented for any number of analysts. That is to say that we will obtain a combination of clusters of at most two forecasts and separation in time between different clusters and between single forecasts that are not clustered. This may occur if the unconstrained optimum of the analysts is not ”too close” to each other, and the increase in the precision of the investors’ beliefs due to the analysts’ forecasts is not ”too big”. Roughly speaking, the more analysts there are, the more likely that there will be "overlap", hence the more likely that we will obtain clustering in time.

Another prediction from increasing the number of analysts is that the analysts will forecast weakly earlier in time. This was obtained also while adding a second analyst to the single analyst case.

4.2 The Unknown Bias Case

In this part I allow for the analyst’s bias to be his private information. I believe that in ”real life”, even for an affiliated analyst, the investors do not know his exact incentives at a given point of time, to bias his forecast. In this case, the forecasts of the analysts do not fully reveal their private signals. Contrary to the known bias case, here, the increase in the precision of

\footnote{As noted before, this "overlap" is not sufficient to prevent the Non-Clustering Pattern Equilibrium.}
investors’ beliefs due to the analysts’ forecasts is not constant, and depends on the precision of the investors’ beliefs immediately prior to the forecast. In Appendix 2.B, I show that the variance of the investors’ beliefs after the forecast of analyst \(i\) is given by:

\[
\text{Var} \left( \pi_t | \pi_{i,t}^E, I_t \right) = \text{Var} \left( \pi_t | \pi_{i,t}^E \right) = \sigma_{\pi_t}^2 \left( 1 - \rho_{\pi_{i,t}^E, \pi_t}^2 \right) \tag{4}
\]

where \(\sigma_{\pi_t}^2 = \text{Var} \left( \pi_t | I_t \right) \equiv \frac{1}{f(t)}\) is the variance of investors beliefs about the firm’s earnings at time \(t\) - before the analyst’s forecast, \(\rho_{\pi_{i,t}^E, \pi_t}^2\) is the correlation coefficient between the analyst’s forecast and the firm’s earnings, and \(\sigma_{\alpha}^2\) is the variance of investors beliefs about the analyst’s bias parameter - \(\alpha_i\).

Denoting the increase in the precision of investors’ beliefs following a forecast of analyst \(i\) at time \(t\) by \(\Delta f_{S_i}(t)\), we have:

\[
\Delta f_{S_i}(t) \equiv \frac{1}{\text{Var} \left( \pi_t | \pi_{i,t}^E \right)} - \frac{1}{\text{Var} \left( \pi_t | I_t \right)},
\]

In contrast to the known bias case where the "indifference interval" (and hence \(f_i^L\)) of each analyst was constant over time, here it is not the case. Finding an equilibrium in such case may seem complicated, and one can predict multiple equilibria. Fortunately it is not the case.

The following intuitive comparative statics can be easily obtained from equation (4): lower \(\sigma_{\alpha}^2\) and higher precision of the analyst’s private signal \((f_{S_i} \equiv \frac{1}{\sigma_{\alpha}^2})\) increase the amount of information conveyed by the analyst’s forecast, and hence increase the precision of the investors’ beliefs after his forecast.

The following Lemma presents a result which is not straightforward. The quantitative part of the Lemma will be used in deriving the equilibrium and proving its uniqueness.

**Lemma 3** \(\Delta f_{S_i}(t)\) monotonically decreases in the precision of investors’ beliefs immediately prior to the analyst’s forecast. The pace of the decrease in \(\Delta f_{S_i}(t)\) is lower than the increase in \(f(t)\), that is:

\[
0 > \frac{\partial (\Delta f_{S_i}(t))}{\partial (f(t))} > -1
\]

**Proof:** See Appendix 2.C.

As time progresses, the precision of investors’ beliefs increases and \(\Delta f_{S_i}(t)\) decreases. It means that the indifference interval (which straddles the unconstrained optimum and in a "size"
equals to $\Delta f_{S_i} (t)$, decreases. Since $\Delta f_{S_i} (t)$ depends on $t$, also the lower end of the indifference interval (which was denoted before as $f^i_L$) depends on $t$, hence, I add to its notation the subscript $t$, and it is now denoted by: $f^i_L (t)$. If the decrease in the indifference interval was sufficiently fast, then we could expect a situation in which as $f(t)$ increases $f^i_L (t)$ increases even faster. This in turn could induce that $f(t)$ will equal $f^i_L (t)$ more than once during the forecasting season, which would distort at least the uniqueness of an equilibrium similar to the known bias case. Lemma 3 implies that after the first time where $f(t) = f^i_L (t)$, we get that $f(t)$ will always be higher than $f^i_L (t)$. This proves the uniqueness of the time where $f(t) = f^i_L (t)$.

The main proposition of the model, will use the following notation: $\tau_0$ Denotes the first time where both $f(t) \geq f^i_L (t)$ and $f(t) \geq f^j_L (t)$.

**Proposition 3** There exists a unique Subgame Perfect Equilibrium in pure strategies where:
The first analyst to forecast is analyst $i$ where $f^i_L (\tau_0) < f^j_L (\tau_0)$.
In equilibrium, the precision of investors’ beliefs immediately prior to the forecast of the first analyst (assume w.l.o.g. analyst $i$) is

$$f (t^*_i) = \min \{f^j_L (\tau_0), f (t^*_i)\}.$$\textsuperscript{22}

The second analyst (analyst $j$) will forecast at his unconstrained optimum if it is feasible, and if it is not feasible, he will forecast immediately after the first forecast.\textsuperscript{23}

The forecasts of the analysts are:

$$f_k (t) = \pi^F_{k,t} = \frac{\alpha_k}{2\beta_k} + E(\pi|\psi_k, I_t)$$

**Proof.** As in the known bias case, it is straightforward that as long as for at least one of the analysts, $f^i_L (t) > f(t)$, non of the analysts publishes his forecast. As in the known bias case, if it is feasible that each analyst will publish his forecast at his unconstrained optimum - $f(t^*_i)$, doing so is an equilibrium. If it is not feasible, the first candidate for a pure strategies subgame perfect equilibrium is the following: at $\tau_0$, the analyst with the lower $f^i_L (\tau_0)$ will forecast first. I assume w.l.o.g. that it is analyst 1. If analyst 1 did not forecast at $f^i_L (\tau_0)$ then analyst 2 will forecast immediately after (this is off the equilibrium path). Since non of the analysts will want to deviate, it is a subgame perfect equilibrium.

\textsuperscript{22}The first forecaster does not necessarily publish his forecast at $t = \tau_0$. This is the case only in the clustering pattern of the equilibrium. In the separation in time pattern, the first forecaster will forecast earlier than $\tau_0$.

\textsuperscript{23}The notes about ”immediately after” and about the off equilibrium beliefs at the Known Bias Case Proposition apply here as well.
I next prove the uniqueness of this pure strategies equilibrium. Let assume that there is another pure strategies Subgame Perfect Equilibrium, in which the first analyst to forecast, publishes his forecast at a time later than $\tau_0$. In this case, the expected utility of the other analyst (the second forecaster) is lower than if he himself was forecasting at $\tau_0$ (or sufficiently small amount of time before that), hence he will want to deviate - in contradiction to the assumption of this being an equilibrium. Noting that each analyst will forecast immediately as he can after his unconstrained optimum, completes the proof for the existence and uniqueness. ■

4.2.1 Discussion of the Unknown Bias Case

Introducing a noise in the investors’ (and other analysts’) beliefs about the bias of the analysts, adds another layer of complexity, but does not qualitatively affect the results. From the investors’ (and the other analyst’s) perspective, the signal about the earnings of the firm that they get from the analyst’s forecast is noisier. But introducing the uncertainty about the analysts’ biases is not exactly equivalent to decreasing the precisions of the analysts’ private signal.

All the points raised in the discussion of the known bias cases hold in the unknown bias case as well. The only point that should be proved is Corollary 2. I order to show that Corollary 2 of the known bias case holds in this case as well, it is sufficient to show that: $0 < \frac{\partial \Delta f_{S_i}(t)}{\partial f_{S_i}} < 1$. This is proved in Appendix 2.D. As $f_{S_i}$ increases we get that $\Delta f_{S_i}(t)$ increases as well, but by less than the increase in $f_{S_i}$. It means that the indifference interval of analyst $j$ increases in $f_{S_i}$, but by less than the increase in $f_{S_i}$. Other than this qualitative change in respect to the known bias case, the rest of the proof remains the same. As to the quantitative effect, in the unknown bias case, the forecasts of the first analyst will be earlier as $f_{S_i}$ increases, but by a smaller extant than in the known bias case.

As in the known bias case, the analyst with the higher precision of private signal will not necessarily be the first to forecast. But due to the additional layer of asymmetric information, if the precision of the investors’ beliefs about the bias of analyst 1 is higher than about the bias of analyst 2, then even if the unconstrained optimum of analyst 1 is higher, and the analysts have an equal precision of private signals, it is possible that analyst 1 will be the seconds to forecast.

The equilibrium forecasting timing is invariant to the realizations of the private signals and the realized bias parameters of the analysts. Given estimates of the analysts’ parameters and the precision of the investors’ beliefs, the model generates precise predictions of the analysts
forecasting timing. These timings are observable. Since the parameters of the analysts’ utility functions, the precision of the private signals and the distribution of the bias parameters are all unobserved, they should be estimated (the estimation methods are not discussed in this paper). If one assumes that analysts behave according to the proposed model, then the observed forecasting timings of the analysts, can facilitates the estimation of the above parameters, which in return will induce better understanding of the informational content of analysts’ forecasts. The model does not impose any restrictions on the exogenous stream of information. In the data, we often see that analysts’ forecasts are clustered around an ”informational event” related to the firm. If during a given period there is an extensively incoming flow of information, we get that the growing frequency of forecasts around this period arises endogenously from the model.24

5 Analysts’ Forecasts and Investors’ Beliefs

In light of the above model of analysts’ behavior, the natural question to addresses is how do rational investors derive consistent beliefs about the value of a stock, following an analyst’s forecast. In a similar setup where there are N analysts, I propose an algorithm for deriving the consistent investors beliefs that accounts for the precisions of the analysts’ private signals and the precision of the investors beliefs about the analysts’ biases as well as for the timing of an analysts’ earnings forecasts. The proposed algorithm is innovative and may have significant contribution for financial research as well as for investment decisions, regardless of whether the timing of analysts’ earnings forecasts is determined exogenously or endogenously.

Analysts’ forecasts of earnings are increasingly used in accounting and finance research as proxies for the otherwise unobservable ”market” expectation of earnings. Since analysts’ forecasts are sequential, one has to decide how to distill the forecasts into expectations. A frequently used proxy for the investors beliefs is the widely disseminated consensus (for example as calculated by I/B/E/S), which gives equal weight to the last forecasts of every analyst covering the stock, regardless of the recency of the forecast. Equal weights implies that effectively, earlier forecasts get higher weights than the later ones, since the inferred information from the first forecast is incorporated in the later forecasts. In order for the prevailing consensus to be the Bayesian conditional mean, it must be that the precision of the signal about the firm earnings

24 In case of a discrete increase in the public information, it can be shown that the analysts are likely to cluster and forecast simultaneously.
that the investors get from an early forecast should be higher than the one they get from a later forecast. Moreover, the precisions must satisfy a very specific quantitative requirements. This of course is very unlikely. Obviously, the consensus is only a simple and applicable heuristic to proxy the mean of the investors beliefs. The algorithm that I present in this part, derives the consistent investors beliefs following every analyst’s forecast. I should note that in order to use the suggested algorithm, investors should form beliefs about three parameters of each analyst which are not straight forward.

As mentioned in the introduction, O’Brien (1988) and Brown (1991) show empirical findings that suggest that there is a trade-off between recency and aggregation. Friesen and Weller (2002) develop a rational model (in addition to a behavioral model) where analysts provide unbiased forecasts. In this extreme case, each analyst publishes a forecast that equals his expectation for the firm earnings. Since the forecasts fully reveal the private signals of the analysts, each analyst calculates his expected earnings based on all the private signals as well as the other public information. In their model, the forecast of the last analyst should reflect the mean of the investors’ beliefs following this forecast. Note that when the analysts are biased, but their bias is common knowledge, the forecast of the last analyst minus his bias is a sufficient statistic for the mean of the investors beliefs.

I first present the intuition behind the proposed investors’ expectation. Given the forecast of each analyst, investors can form their beliefs about the analyst’s private signal. Given those beliefs, investors calculate a weighted average of the inferred signals, where the weights are relative to the precisions of the inferred signals. Higher precision of investors’ beliefs about the analyst’s bias, as well as higher precision of the analyst’s private signal, increase the precision of the inferred signal that investors get from the analyst’s forecast. Investors do not observe the analyst’s private signals, rather they observe his forecast. An equivalent way to form the investors beliefs about the firm’s earnings, using the analysts’ forecasts is as follows. As in the forecasting timing model, the optimal forecast of each analyst is to forecast his expectation of the firm earnings plus a constant (which is unknown to investors). The joint distribution of the forecast of the first forecasting analyst and the prior beliefs is bivariate normal. The investors’ beliefs upon observing the first analyst’s forecast are according to the standard conditional distribution, which is also normally distributed. When the second analyst is about to forecast,
he uses all the public information which is presented by the normal distribution of earnings -
conditional on the first analyst’s forecast. Again the forecast of the second analyst equals
his expectation plus a constant. The investors now face the same situation as after the first
forecast, only that the beliefs immediately prior to the second analyst’s forecast are the beliefs
conditional on the first analyst’s forecast. This procedure is iteratively repeated for each analyst,
and produces a weighted average of the analysts’ forecasts. Note that if all the analysts are
identical in their parameters, then the weights in the expectation given to the analysts’ forecasts
is monotonically increasing in their order (the last forecast gets the highest weight). In the
more general case, the higher is the precision of the analyst’s private signal and the higher is
the precision of the investors’ beliefs about his bias, the higher the weight that his forecast gets
while forming the mean of investors’ beliefs.

The suggested algorithm for calculating investors beliefs (mean and variance), is applicable
both under the assumption of endogenously or exogenously determined order of forecasts. The
difference manifest itself when the parameters of the analyst’s payoff function are not known
to the investors. In this case, if the order is endogenous, investors can make inference from
the order itself about the analysts’ private signals and biases. This part provides guidelines for
empirical research that uses proxies for investors’ beliefs on various applications.

5.1 Model Setup

The fundamentals of this part are similar to the timing model, only that there are $n$ analysts
denoted by $i = 1, ..., n$. The information of analyst $i \in \{1, ..., n\}$ immediately prior to his
forecast, is composed of his private signal and the history at that time, which is composed of all
the previously published forecasts and the prior beliefs. In this part I abstract from incoming
public information other than the analysts’ forecasts. The history that analyst $i$ observes
immediately prior to his forecast is denoted by $H_{i-1}$, and the forecast of analyst $i$ is denoted by
$\pi_i^F$. Each analyst $i = 1, ..., n$ has two pieces of private information. The first piece is the private
signal regarding the true earnings - $\tilde{\psi}_i = \pi + \tilde{\epsilon}_i$, where $\tilde{\epsilon}_i \sim N(0, \sigma_{\epsilon}^2)$ and $\text{Cov}(\tilde{\epsilon}_i, \tilde{\epsilon}_j) = 0$ for
$i \neq j$. The second piece is an idiosyncratic bias parameter $\alpha_i$, drawn from a normal distribution
$- \tilde{\alpha}_i \sim N(\mu_{\alpha_i}, \sigma_{\alpha_i}^2)$, where $\text{Cov}(\tilde{\alpha}_i, \tilde{\alpha}_j) = 0$ for $i \neq j$, $\text{Cov}(\tilde{\alpha}_i, \tilde{\pi}) = 0$ and $\text{Cov}(\tilde{\alpha}_i, \tilde{\psi}_i) = 0$.  

\footnote{Incorporating arrival of exogenous public information (as in the first part) is straightforward, but it makes
the description and notations of the model more complicated.}

\footnote{As in the timing game, we can assume either that the analysts get their signals at the beginning of the
quarter, or alternatively just before they publish their forecast.}
The analysts sequentially publish their forecasts of the forthcoming earnings of a firm.

As shown in the timing game, the realizations of the private signals and the bias parameters of the analysts do not influence the timing of their forecasts. Hence, one cannot make inference from the observed time of the forecasts about the realized private signal of an analyst. Since all the other parameters used in the model are common knowledge, we have all the information needed in order to calculate the investors beliefs. It means that regardless of the question of how the order of forecasters is determined (endogenously or exogenously), the following is needed in order to calculate investors’ beliefs: the precision of the analysts’ private signals, the distribution of the analysts’ bias parameters and the analysts’ cost of a forecast error ($\beta_i$).

I assume w.l.o.g. that the analysts forecast in an increasing order of their index. At time zero, prior to the first forecast, the prior beliefs about the reported earnings of the firm is $\tilde{\pi}_0 \sim N(\mu_{\pi_0}, \sigma^2_{\pi_0})$ and are denoted by $H_0$.

I first solve the optimization problem of the first analyst ($i = 1$) and the subsequent investors beliefs, showing that investors’ expectations after the first analyst’s forecast are normally distributed (with adjusted parameters). Next, I show that the optimization problem of the second analyst is qualitatively identical to that of the first analyst. The same argument holds for each of the following forecasters. The mean according to the investors’ beliefs after the $i^{th}$ forecast will be presented as a weighted average of all the forecasts up to that point and the prior mean. I present a simple algorithm for calculating the mean and variance of investors beliefs following every analyst’s forecast.

5.2 The forecast of the first analyst and the following investors’ beliefs

I derive the optimal forecast of the first analyst given the realization of his private signal and his bias parameter.

The utility function of the first analyst, who forecasts at time $t_1$ is:

$$EU^1 = \alpha_1 \left( \pi^F - E(\pi \mid \psi_1, H_0) \right) - \beta_1 E \left( (\pi^F - \pi)^2 \mid \psi_1, H_0 \right) - f(t_1).$$

The FOC in respect to the analyst’s forecast is:

$$\alpha_1 - 2\beta_1 E \left[ (\pi^F - \pi) \mid \psi_1, H_0 \right] = 0$$

\footnote{Note that while computing investors’ beliefs, the time of every forecast is given. This means that given all the known parameters, the last term in the analyst’s utility function - $f(t)$ is a given constant.}
The SOC for maximum is satisfied. From equation (2) we know that the first analyst’s forecast as a function of his signal and his bias parameter is:

\[
\pi_1^F = \frac{\alpha_1}{2\beta_1} + \frac{\sigma^2_{\infty}}{\sigma^2_{\pi_0} + \sigma^2_{\xi_1}} \mu_{\pi_0} + \frac{\sigma^2_{\pi_0}}{\sigma^2_{\pi_0} + \sigma^2_{\xi_1}} \psi_1
\]

Note that the bias in the analyst’s forecast is independent of the realized signal and equals \(\frac{\alpha_1}{2\beta_1}\).

The ex-ante mean and variance of the analyst’s forecast (which I will use later in order to derive investors’ beliefs) are:

\[
E(\pi_1^F) \equiv \mu_{\pi_1^F} = \frac{1}{2\beta_1} \mu_{\alpha_1} + \mu_{\pi_0} \tag{5}
\]

\[
Var(\pi_1^F) \equiv \sigma^2_{\pi_1^F} = \frac{1}{(2\beta_1)^2} \sigma^2_{\alpha_1} + \frac{\sigma^2_{\pi_0}}{\sigma^2_{\pi_0} + \sigma^2_{\xi_1}} \sigma^2_{\pi_0} \tag{6}
\]

Each analyst learns the realization of two random variables (\(\alpha_i\) and \(\psi_i\)) and publishes his forecasts. Given that \(\alpha_i\) is not observable, a forecast of an analyst does not fully reveal his private signal. The joint distribution of the firm’s earnings and the analyst’s forecast is bivariate normal (for a proof see Appendix 4), hence, the investors’ conditional expectation of the firm’s earnings - \(\mu_{\pi_1}\) is linear in the analyst’s forecast. That is:

\[
\mu_{\pi_1} \equiv E(\pi|H_1) = E(\pi|\pi_1^F, H_0) = \mu_{\pi_0} + \frac{Cov(\pi, \pi_1^F)}{Var(\pi_1^F)} (\pi_1^F - \mu_{\pi_1^F}) \tag{6}
\]

The variance of investors’ beliefs, conditional on the forecast of the first analyst - denoted by \(\sigma^2_{\pi_1}\),(which is the variance of the investors beliefs prior to the forecast of the second analyst) is:

\[
\sigma^2_{\pi_1} = Var[\pi|H_1] = Var[\pi|\pi_1^F] = \sigma^2_{\pi_0} \left(1 - \left(\rho_{\pi_0, \pi_1^F}\right)^2\right)
\]

where,

\[
\rho_{\pi_0, \pi_1^F} = \frac{Cov(\pi_0, \pi_1^F)}{\sigma_{\pi_0} \sigma_{\pi_1^F}} = \frac{\sigma^2_{\pi_0}}{\sigma^2_{\pi_0} + \sigma^2_{\xi_1}} \sigma_{\pi_0}
\]

Substituting into the conditional variance and rearranging we get:
\[ \sigma_{\pi_1}^2 = \sigma_{\pi_0}^2 \left( 1 - \frac{\left( \frac{\sigma_{\pi_0}^2}{\sigma_{\pi_0}^2 + \sigma_{\pi_1}^2} \right)^2 \sigma_{\pi_0}^2}{\frac{1}{(2\beta)} \sigma_{\alpha_i}^2 + \frac{\sigma_{\pi_0}^2}{\sigma_{\pi_0}^2 + \sigma_{\pi_1}^2} \sigma_{\pi_1}^2} \right) \]

\[ = \sigma_{\pi_0}^2 \left( 1 - \frac{\frac{1}{(2\beta)} \sigma_{\alpha_i}^2 + \frac{\sigma_{\pi_0}^2}{\sigma_{\pi_0}^2 + \sigma_{\pi_1}^2} \sigma_{\pi_1}^2}{\frac{1}{(2\beta)} \sigma_{\alpha_i}^2 + \frac{\sigma_{\pi_0}^2}{\sigma_{\pi_0}^2 + \sigma_{\pi_1}^2} \sigma_{\pi_1}^2} \right) \]

\[ \text{(7)} \]

5.3 The forecast of the \(i^{th}\) analyst and the following investors’ beliefs

This section generalizes the forecast of the first analyst and the following investors’ beliefs into the case of the \(i^{th}\) analyst’s and the following investors’ beliefs.

As shown before, the joint distribution of the prior beliefs and the first analyst’s forecast is bivariate normal, hence the investors’ (as well as the other analysts’) beliefs conditional on the first analyst’s forecast are normally distributed with a mean of \(E(\pi|\pi_1^F, H_0) = E(\pi|H_1)\) and a variance of \(\sigma_{\pi_1}^2 = \text{Var}[\pi|H_1]\). When the second analyst is about to forecast, all the public information is incorporated in the parameters of the normal distribution of investors’ beliefs following the first forecast. Qualitatively, the second analyst faces the same situation that the first analyst had faced. Since the optimal forecast of the second analyst is qualitatively similar to the first analyst’s forecast - that is linear in his private signal and in the mean of investors beliefs, his forecast generates a normally distributed investors’ beliefs.\(^{30}\) This can be iteratively generalized as follows:

**Conclusion 1** For each analyst \(i = 1, \ldots, N\), the investors’ beliefs immediately prior to his forecast are normally distributed with a mean of \(\mu_{\pi_{i-1}} = E(\pi|H_{i-1})\) and a variance of \(\sigma_{\pi_{i-1}}^2 = \text{Var}[\pi|H_{i-1}]\)

In order to calculate the mean and the variance of the investors’ beliefs following the forecast of the second analyst, it is sufficient to know his forecast and the distribution of the investors beliefs immediately prior to his forecast. As I show ahead, this holds for every analyst \(i = 1, \ldots, N\), which will enable to derive the consistent investors beliefs following the forecast of every analyst, by using a simple iterative algorithm. Each iteration, will calculate a simple conditional

\(^{30}\)Since \(\bar{\alpha}_i\) is independent of both \(\psi_i\) and \(\alpha_j\) (for \(i \neq j\)), the forecasts of the former analysts do not convey any additional information to analyst \(i\) relative to the normal distribution of the investors’ beliefs immediately prior to his forecast.
distribution of a bivariate normal distribution. The importance of this iterative algorithm is
not only the intuitive simplicity, but also feasibility of the computations. The standard way for
deriving the investors beliefs would be to calculate the joint distribution of the prior beliefs and
all the forecasts published so far. Even today, the constraint of the ”computational power” of
computers becomes binding very quickly. After the forecast of the fourth analyst, for example,
the standard way would calculate a 5-Variate normal distribution, which most computers may
handle, but as the number of forecasts further increases, the computation complexity increases
exponentially, and explodes very quickly.

Next, for each analyst \( i = 2, \ldots, N \) I derive the following: the investors’ beliefs prior to his
forecast (mean and variance), his equilibrium forecast, and the expectation of the investors
beliefs immediately after his forecast.

The forecast of the second analyst equals his expectation plus a constant, and is given by:

\[
\pi_F^2 = \frac{\alpha_2}{2\beta_2} + E[\pi|\psi_2, H_1] = \frac{\alpha_2}{2\beta_2} + \left( \frac{\sigma_{\pi_0}^2 - \sigma_{\pi_1}^2 + \sigma_{\pi_2}^2}{\sigma_{\pi_0}^2 + \sigma_{\pi_2}^2} \right) \mu_{\pi_1} + \frac{\sigma_{\pi_1}^2}{\sigma_{\pi_0}^2 + \sigma_{\pi_2}^2} \psi_2
\]

Note that if the precisions of the private signals of the first and second analysts are equal, then
the relative weight that the second analyst gives to his private signal \( \frac{\sigma_{\pi_1}^2}{\sigma_{\pi_0}^2 + \sigma_{\pi_2}^2} \) is lower than
the weight given by the first analyst (which was \( \frac{\sigma_{\pi_0}^2}{\sigma_{\pi_0}^2 + \sigma_{\pi_1}^2} \)). This result is very intuitive since the
prior beliefs before the second forecast are more precise than the beliefs before the first forecast.

The ex-ante mean and variance of the second analyst’s forecast are calculated similar to
equation (5).\(^{31}\) The conditional expectation of the investors following the forecast of the second
analyst is:

\[
E(\pi|H_2) = E(\pi|H_1) + \frac{Cov(\pi_1, \pi_F^2)}{Var(\pi_F^2)} (\pi_F^2 - \mu_{\pi_F^2})
\]

\[
= \mu_{\pi_1} + \frac{\sigma_{\pi_1}^2 \sigma_{\pi_2}^2}{\sigma_{\pi_0}^2 + \sigma_{\pi_2}^2} \left( \mu_F^2 - \left( \frac{\mu_{\pi_2}}{2\beta_2} + \mu_{\pi_1} \right) \right)
\]

The investors’ conditional variance is:

\[
Var(\pi|H_2) = Var(\pi|H_1) \left( 1 - \rho_{\pi_1,\pi_2} \right)
\]

\(^{31}\) \( E(\pi_F^2) = \frac{\mu_{\pi_2}}{2\beta_2} + \mu_{\pi_1} \) and \( Var(\pi_F^2) = \frac{1}{(2\beta_2)^2} \sigma_{\pi_2}^2 + \frac{\sigma_{\pi_1}^2}{\sigma_{\pi_0}^2 + \sigma_{\pi_2}^2} \sigma_{\pi_1}^2 \)
where $\rho^2_{\pi_1, \pi_2}^F$ is the correlation coefficient which equals:

$$\rho^2_{\pi_1, \pi_2}^F = \frac{\text{Cov}(\pi_1, \pi_2^F)}{\sigma_{\pi_1} \sigma_{\pi_2}} = \frac{\frac{\sigma^2_{\pi_1}}{\sigma^2_{\pi_0} + \sigma^2_{\pi_2}}}{\sigma_{\pi_1}} \sqrt{\frac{1}{(2\beta_i)^2}\sigma^2_{\alpha_2} + \frac{\sigma^2_{\pi_1}}{\sigma^2_{\pi_0} + \sigma^2_{\pi_2}} \sigma^2_{\pi_1}} = \frac{\frac{\sigma^2_{\pi_1}}{\sigma^2_{\pi_0} + \sigma^2_{\pi_2}} \sigma_{\pi_1}}{\sigma_{\pi_1}} \sqrt{\frac{1}{(2\beta_i)^2}\sigma^2_{\alpha_2} + \frac{\sigma^2_{\pi_1}}{\sigma^2_{\pi_0} + \sigma^2_{\pi_2}} \sigma^2_{\pi_1}}$$

By using exactly the same method, one can calculate the investors’ beliefs following every forecast. The general case for the investors’ beliefs following the forecast of analyst $i$ is described in the following Proposition.

**Proposition 4** The expectation and variance of the firm’s earnings according to the investors’ beliefs, following the forecast of the $i$th analyst are:

$$\mu_{\pi_i} = \mu_{\pi_{i-1}} + \frac{\frac{\sigma^2_{\pi_{i-1}}}{\sigma^2_{\pi_0} + \sigma^2_{\pi_i}} \sigma^2_{\pi_{i-1}}}{\frac{1}{(2\beta_i)^2}\sigma^2_{\alpha_2} + \frac{\sigma^2_{\pi_{i-1}}}{\sigma^2_{\pi_0} + \sigma^2_{\pi_i}} \sigma^2_{\pi_{i-1}}} \left( \pi_i^F - \mu_{\alpha_1} - \mu_{\pi_{i-1}} \right)$$

$$\text{Var}(\pi|H_i) = \text{Var}(\pi|H_{i-1}) \left( 1 - \left( \frac{\frac{\sigma^2_{\pi_{i-1}}}{\sigma^2_{\pi_0} + \sigma^2_{\pi_i}} \sigma^2_{\pi_{i-1}}}{\frac{1}{(2\beta_i)^2}\sigma^2_{\alpha_2} + \frac{\sigma^2_{\pi_{i-1}}}{\sigma^2_{\pi_0} + \sigma^2_{\pi_i}} \sigma^2_{\pi_{i-1}}} \right)^2 \right)$$

where the forecast of the $i$th analyst and its ex-ante (unconditional) expectation and variance are:

$$\pi_i^F = \frac{\alpha_i}{2\beta_i} + \left( \frac{\sigma^2_{\pi_0} - \sigma^2_{\pi_{i-1}} + \sigma^2_{\xi_i}}{\sigma^2_{\pi_0} + \sigma^2_{\xi_i}} \right) \mu_{\pi_{i-1}} + \frac{\sigma^2_{\pi_{i-1}}}{\sigma^2_{\pi_0} + \sigma^2_{\xi_i}} \psi_i$$

$$E(\pi_i^F) = \frac{\mu_{\alpha_1}}{2\beta_i} + \mu_{\pi_{i-1}}$$

$$\text{Var}(\pi_i^F) = \frac{1}{(2\beta_i)^2} \sigma^2_{\alpha_i} + \left( \frac{\sigma^2_{\pi_{i-1}}}{\sigma^2_{\pi_0} + \sigma^2_{\xi_i}} \right) \sigma^2_{\pi_{i-1}}$$

In order to derive investors’ beliefs after the forecast of analyst $i$, one should iteratively use the above proposition. After the forecast of analyst $i$, the prior beliefs immediately prior to the forecast of analyst $i + 1$ are formed using equation (8). In order to calculate investors’ beliefs following the forecast of analyst $j$, the above should be implemented from $i = 1$ to $i = j$ iteratively. Since the mean of investors’ beliefs following the forecast of analyst $i$ is a weighted average (linear combination) of the forecast of analyst $i$ and the mean of investors’ beliefs immediately prior to his forecast, we get the following conclusion:
Conclusion 2  the mean of investors beliefs is a weighted average of all the forecasts published so far and the mean of the prior beliefs.

There are three factors that influence the weight given to an analyst’s forecast while calculating the mean of investors’ beliefs. The first is the precision of his private signal: higher precision of the private signal induces higher weight. The second factor is the precision of the investors’ beliefs about the analyst’s bias parameter ($\alpha_i$). While these beliefs do not influence the analyst’s forecast (for a given realization of $\alpha_i$), the more precise investors beliefs are, the more they can learn from the analyst’s forecast about his private signal, hence the investors will give it a higher weight in the mean of their beliefs. The third factor is the precision of the investors’ beliefs immediately prior to the analyst’s forecast: More precise investors’ beliefs induces lower weight to the forecast of analyst $i$. The direction of the influence of the above three factors can easily be obtained from equation (8).

From equation (8) it is also easy to see that as the precision of investors’ beliefs regarding the analysts’ biases goes to infinity (for all $i : \sigma_{\xi_i}^2 \to 0$), the investors expectations converge to the common knowledge case, where the analysts’ forecasts fully reveal their private signals. In this case, the forecast of the last analyst is a sufficient statistic for the mean of investors’ beliefs.\textsuperscript{32} Note that even if the bias of the analyst is common knowledge, in order to calculate the variance of the investors beliefs following his forecast, one should use either the suggested algorithm or the joint distribution of the analysts private signals and biases.

Another intuitive result is that in the case where the private signals of the analysts are i.i.d. and also the bias parameters of the analysts are i.i.d., the weight given to an analyst’s forecast is monotonically increasing in his index (the last analyst gets the highest weight). This indicates that the average of all analysts’ forecasts (“consensus”), which is widely used as a proxy for investors’ beliefs does not represents the investors’ beliefs according to the proposed model. Moreover, the variance of the analysts’ forecasts, which is used as a proxy for the variance of the investors’ beliefs does not coincide with the model. Using the consensus of analysts’ forecast ignores the fact that the forecast of analyst $i$ incorporates the public information conveyed by all the former forecasts, and hence, the effective weight that is given to an early forecast is higher than the weight given to a later one. This claim has been empirically reported by O’Brien (1988) and Brown (1991) who suggest the existence of a trade-off between recency and aggregation.

\textsuperscript{32}If the bias of analyst $j$ is known to investor, then his forecast is a sufficient statistic to the mean of investors’ beliefs following his forecast, regardless of whether the biases of analysts $i = 1, \ldots, j - 1$ are common knowledge.
In this paper, I do not discuss the ways to estimate the following parameters: the precision of the analyst’s private signal; the distribution of the analyst’s bias; and his reputation cost. But, given the estimates of the above (derived from historical data of each analyst for each stock he covers), it is straightforward to calculate the investors’ beliefs by iteratively using proposition 4.

5.4 Consensus Adjustment for Endogenous Forecasting Timing

The question whether the order and the timing of the analysts’ forecasts is endogenously or exogenously determined is an empirical one. If all the parameters are known, then, while deriving investors beliefs, it is irrelevant whether the timings of the analysts’ forecasts is determined strategically - endogenously, or exogenously. The model assumes that both the precision of the private signals and the parameters of the utility functions are common knowledge. But those parameters are not observable, and investors have to estimate them. If we assume that analysts behave according to the suggested timing model, than the time of an analyst’s forecast contain information that can be used in order to have more precise estimation of the above parameters. By increasing the precision of the estimates, investors’ beliefs about the firm’s earnings become more precise as well. If the empirical test will find that using the observable forecasts’ timing can improve the estimation of investors’ beliefs, it will have a considerable contribution to the financial research as well as to investment decision (We check this hypothesis empirically elsewhere).

6 Conclusions

The paper proposes a theoretical model for the order and timing of analysts’ earnings forecasts and the investors beliefs. The question that generated the paper is whether the order and timing of analysts’ forecasts is determined exogenously, as implicitly assumed in much of the literature, or whether analysts choose the timing of their forecasts strategically. The answer to this question may yield important insights into the behavior of sell-side analysts, a subject that received much recent attention. Understanding the information contained in the order of analysts’ forecasts may help investors and academics alike to decipher the informational content of these forecasts. Ignoring this information, which characterizes most of the empirical literature that uses analysts’ “consensus” to proxy for investor beliefs; frequently leads to inconsistent estimates. The goal of this paper is to analyze the optimal timing strategies for the analysts,
and to derive the consistent beliefs of rational investors. I follow the literature in assuming that analysts care foremost about the accuracy of their forecasts, but in some cases may have an incentive to bias their forecasts. I also add an additional feature by assuming that the compensation of the analyst declines in the precision of the investors’ beliefs about earnings at the time of the forecast. Thus, an early-reporting analyst is compensated more for a given forecast than the later-reporting one. I further assume that a continuous stream of information from many sources arrives during the quarter and affects the investors’ beliefs about the firm’s earnings. The analyst faces the trade-off between earlier, but less precise forecast, versus later, but more precise forecast. Since Investors do not necessarily learn the actual bias of the analysts, I assume that they only know the distribution from which the analysts’ bias is drawn.

The paper introduces a strategic game between two analysts, under which: more precise private signal, less precise investors’ beliefs regarding the analysts’ bias and lower error cost will all induce earlier forecast. The suggested equilibrium is the unique pure strategies subgame perfect equilibrium. The equilibrium has two patterns: either each analyst forecasts at his unconstrained optimal time (as if he is the only analyst that covers the firm); or that the forecasts cluster in time - one analyst forecasts immediately after the other. Extensive inflow of information (around an event) endogenously generates more forecasts around that period. As to investors beliefs, I claim that the widely used ”expectation consensus” which is calculated as the average of all forecasts, in general does not reflect the rational expectations. I suggest an iterative algorithm that uses the prior beliefs and all the analysts’ forecasts, in order to calculate the expected earnings and the variance of the firm’s earnings according to all the available public information. I show that the mean of the investors beliefs is a weighted average of all the analysts’ forecasts and the prior mean. In the particular case where all the analysts are identical, the weight given to a later forecast is higher than the weight given to an earlier one. Given the parameters of the analyst’s utility functions, the precisions of the analysts’ private signals, and the precision of investors beliefs about the analysts’ bias, the investors beliefs are independent of whether the time of the forecast is endogenous or exogenous. But since those parameters are not observable, the information contained in the observed timing can produce more accurate estimates of the parameters. This in turn, will increase the precision of investors’ beliefs about the firm’s earnings. If an empirical test of the suggested investors’ beliefs will find that they indeed improve the predictions of the firm’s earnings, it will have a considerable contribution both for financial research and for investment decisions. The question whether the timing of forecasts is endogenous or exogenous is an empirical one. The proposed
model can facilitates its empirical examination.
References


Appendix

Appendix 1. - The one analyst case - general $g(t)$

**Optimal Forecast**

The first order condition in respect to the forecast (at time $t$) is

$$\frac{\partial U_i}{\partial \pi_t^F} = \alpha_i - 2\beta E \left( (\pi_t^F - \pi) | \psi_i, I_t \right) = 0.$$  

The second order condition is $\frac{\partial^2 U_i}{\partial (\pi_t^F)^2} = -2\beta < 0$. Hence, for every given timing $t$, in which the analyst choose to report, his optimal report will be:

$$\pi_t^F = \frac{\alpha_i}{2\beta} + E(\pi | \psi)$$

$$= \frac{\alpha_i}{2\beta} + \mu_{\pi_t} + \frac{\sigma_{\pi_t}^2}{\sigma_{\pi_t}^2 + \sigma_{\psi}^2} (\psi - \mu_\psi)$$

where $\mu_{\pi_t}$ is the public expectation at time $t$ (which may be different than $\mu_{\pi_0}$), and $\mu_\psi$ is the expected private signal given all the public information at time $t$.

Substituting the forecast of the analyst into his utility function yields:

$$EU_t^i = E \left[ \alpha_i (\pi_t^F - \pi) - \beta \left( \frac{\alpha_i}{2\beta} + E(\pi | \psi) - \pi \right)^2 - \gamma g(t) | \psi_i, I_t \right]$$

$$= \frac{\alpha_i^2}{2\beta} - \beta \left( \frac{\alpha_i}{2\beta} \right)^2 - \beta E \left[ (E(\pi | \psi) - \pi)^2 | \psi_i, I_t \right] - 0 - \gamma g(t)$$

$$= \frac{\alpha_i^2}{4\beta} - \frac{\beta}{f(t) + f_s} - \gamma g(t)$$

**Optimal Forecasting Time**

In order to find the optimal forecasting timing, I derive analyst’s expected utility in respect to the precision of the public beliefs.

The FOC in respect to the optimal forecasting timing is:

$$\frac{\partial EU_t^i}{\partial f(t)} = \frac{\partial}{\partial f(t)} \left[ \alpha_i^2 \frac{4\beta}{2\beta} - \beta \frac{f(t)}{f(t) + f_s} - \gamma g(t) \right]$$

$$= \frac{\beta}{(f(t) + f_s)^2} - \gamma \frac{\partial g(t)}{\partial f(t)} = 0$$

The SOC is:

$$\frac{\partial^2 EU_t^i}{\partial f(t)^2} = \frac{\partial}{\partial f(t)} \left[ \beta \frac{(f(t) + f_s)^2}{(f(t) + f_s)^2} - \gamma \frac{\partial g(t)}{\partial f(t)} \right]$$

$$= -\beta \frac{\partial^2 g(t)}{(f(t) + f_s)^3} - \gamma \frac{\partial g(t)}{\partial f(t)}$$
A necessary (and sufficient except for inflection points) condition for internal utility maximizer is that for the timing where the FOC holds, the following holds: \( \frac{\partial^2 g(t)}{\partial f(t)^2} \geq -\frac{\beta}{\gamma(f(t)+f_s)} \).\(^{33}\)

**Corollary 3** If \( f_s > \sqrt{\frac{\beta}{\gamma(0)}} \) then the analyst expected utility decreases in time, and he will forecast at \( t = 0 \).

For \( f_S < \sqrt{\frac{\beta}{\gamma(0)}} \), the unconstrained optimal forecasting timing of the analyst is

\[
f^*(t) = \sqrt{\frac{\beta}{\gamma \partial g(t)/\partial f(t)}} - f_S,
\]

imposing the constraint of the public beliefs precision, we get:

- If \( f(t)^* \leq f(0) \) then the analyst will forecast at \( t = 0 \).
- If \( f(0) \leq f(t)^* \leq f(T) \) then the analyst will forecast at the time at which \( f(t) = f(t)^* \).
- If \( f(t)^* > f(T) \) then the analyst will forecast at time \( T \).

**Remark 3** The optimal forecasting timing is independent of the realized signal and the value of \( \alpha_i \). The reason is that changes in \( \alpha_i \) linearly changes the bias in the analyst’s forecast. All else equal, as can be seen from equation (3) it contributes a constant to the analyst’s expected utility. The optimal forecast does depend on the ratio \( \frac{\beta}{\gamma} \); an increase this ratio implies higher cost of a forecast error, hence the analyst will wait longer.

---

**Appendix 2 - Public precision increase after the analyst’s forecast**

In this part I show the following:

A. First that in the case where the analyst’s bias is common knowledge, the increase in the public beliefs precision after his forecast equals the precision of his private signal- \( f_S \).

B. Second, for the stochastic analyst’s bias (which is not common knowledge), I derive the increase in the precision of the public beliefs after the analyst’s forecast (as a function of the public precision prior to his forecast, the precision of investors beliefs upon the analyst’s bias and the precision of the analyst’s private signal.

C. The increase in the public precision due to the analyst’s forecast is decreasing in the public precision prior to the analyst’s forecast.

\(^{33}\)This condition indicates that for \( t \in R \) there is an internal utility maximizer. But we are interested in a positive and finite support of \( t \), hence the internal solution relevant to our case is for a smaller set of the functions \( g(t) \).
D. when $f_{S_i}$ increases by a marginal unit $\Delta f_{S_i}$ will increase by less than a marginal unit.

2.A. The common knowledge case

Let assume that the bias parameter $\alpha_i$ of the analyst is common knowledge, and that the precision of public beliefs before the analyst’s forecast is $f(t)$. Since the analyst’s forecast fully reveals his signal, the posterior public beliefs precision equal the posterior precision of the analyst and is: (using the conditional variance formula)

$$Var (\pi | \psi) = \sigma^2_\pi (1 - \rho^2_{\pi \psi}) = \sigma^2_\pi \left( 1 - \left( \frac{\sigma^2_\pi}{\sigma^2_\pi + \sigma^2_\varepsilon} \right)^2 \right)$$

$$= \sigma^2_\pi \left( 1 - \frac{\sigma^2_\pi}{\sigma^2_\pi + \sigma^2_\varepsilon} \right) = \frac{\sigma^2_\varepsilon}{\sigma^2_\pi + \sigma^2_\varepsilon} \sigma^2_{\pi \varepsilon}$$

$$Var (\pi | \pi^F) = \sigma^2_\pi (1 - \rho^2_{\pi \pi^F}) = \sigma^2_\pi \left( 1 - \frac{\sigma^2_\pi}{\sigma^2_\pi + \sigma^2_\varepsilon} \right)$$

$$= \frac{\sigma^2_\varepsilon}{\sigma^2_\pi + \sigma^2_\varepsilon} \sigma^2_{\pi \varepsilon}$$

given the analyst’s private signal. The posterior precision denoted by $f^+(t)$ is hence

$$f^+(t) = \frac{1}{Var (\pi | \psi)} \frac{\sigma^2_\pi + \sigma^2_\varepsilon}{\sigma^2_\varepsilon \sigma^2_{\pi \varepsilon}} = \frac{1}{f(t)} \frac{f_S + f(t)}{f(t)} = \frac{f_S + f(t)}{f_S} = f_S + f(t)$$

2.B. The change in the precision of public’s beliefs - stochastic case

Reminding that

$$\pi^F_t = \frac{\alpha_i}{2\beta} + E (\pi_t | \psi) = \frac{\alpha_i}{2\beta} + \mu_{\pi_t} + \frac{\sigma^2_{\pi t}}{\sigma^2_{\pi t} + \sigma^2_\varepsilon} (\psi - \mu_\psi)$$

$$\mu_{\pi^F_t} = \frac{\mu_{\alpha_i}}{2\beta} + \mu_{\pi_t}$$

$$Var (\pi^F_t) = \frac{1}{(2\beta)^2} \sigma^2_\alpha + \left( \frac{\sigma^2_{\pi t}}{\sigma^2_{\pi t} + \sigma^2_\varepsilon} \right)^2 \left( \sigma^2_{\pi t} + \sigma^2_\varepsilon \right) = \frac{1}{(2\beta)^2} \sigma^2_\alpha + \frac{\sigma^2_{\pi t} + \sigma^2_{\pi t}}{\sigma^2_{\pi t} + \sigma^2_\varepsilon} \sigma^2_{\pi t}$$

$$Cov (\pi_t, \pi^F_t) = \sigma^2_{\pi t} \sigma^2_{\pi t}$$
and

\[ E(\pi_t|\pi_t^F) = \mu_{\pi_t} + \frac{Cov(\pi_t, \pi_t^F)}{Var(\pi_t^F)} (\pi_t^F - \mu_{\pi_t}) = \]

\[ = \mu_{\pi_t} + \frac{1}{(2\beta)^2} \sigma_{\pi_t}^2 \left( \frac{\sigma_{\pi_t}^2}{\sigma_{\pi_t}^2 + \sigma_{\pi_t}^2} \right) \left( \pi_t^F - \frac{\mu_{\pi_t}}{2\beta} - \mu_{\pi_t} \right) \]

\[ = \frac{1}{(2\beta)^2} \sigma_{\pi_t}^2 \left( \frac{\sigma_{\pi_t}^2}{\sigma_{\pi_t}^2 + \sigma_{\pi_t}^2} \right) \mu_{\pi_t} - \frac{1}{(2\beta)^2} \sigma_{\pi_t}^2 \left( \frac{\sigma_{\pi_t}^2}{\sigma_{\pi_t}^2 + \sigma_{\pi_t}^2} \right) \mu_{\pi_t} \]

\[ + \frac{1}{(2\beta)^2} \sigma_{\pi_t}^2 \left( \frac{\sigma_{\pi_t}^2}{\sigma_{\pi_t}^2 + \sigma_{\pi_t}^2} \right) \pi_t^F \]

\[ \rho_{\pi_t}^2 = \frac{\left( \frac{\sigma_{\pi_t}^2}{\sigma_{\pi_t}^2 + \sigma_{\pi_t}^2} \right)^2}{\sigma_{\pi_t}^2 \left( \frac{\sigma_{\pi_t}^2}{\sigma_{\pi_t}^2 + \sigma_{\pi_t}^2} \right)^2} = \left( \frac{\sigma_{\pi_t}^2}{\sigma_{\pi_t}^2 + \sigma_{\pi_t}^2} \right)^2 \sigma_{\pi_t}^2 \]

Using the conditional variance formula I get:

\[ Var(\pi_t|\pi_t^F) = \sigma_{\pi_t}^2 \left( 1 - \rho_{\pi_t}^2 \right) \]

\[ = \sigma_{\pi_t}^2 \left( 1 - \frac{\sigma_{\pi_t}^2 \sigma_{\pi_t}^2 \sigma_{\pi_t}^2 \sigma_{\pi_t}^2}{\sigma_{\pi_t}^2 + \sigma_{\pi_t}^2} \right) \]

2.C. The change in the conditional precision

Claim 4 The increase in the public precision due to the analyst’s forecast is decreasing in the public precision prior to the analyst’s forecast.

Proof. Keeping the precision of the analyst’s signal and the precision of public beliefs about the analyst’s bias constant, it is sufficient to show that

\[ \frac{\partial}{\partial Var(\pi)} \left( \frac{1}{Var(\pi)} - \frac{1}{Var(\pi|\pi^F)} \right) < 0 \]

Rewriting \( \frac{1}{Var(\pi|\pi^F)} - \frac{1}{Var(\pi)} \) I get

\[ \frac{1}{Var(\pi|\pi^F)} - \frac{1}{Var(\pi)} = \left( \frac{1}{\sigma_{\pi_t}^2} \left( 1 - \frac{\sigma_{\pi_t}^2 \sigma_{\pi_t}^2 \sigma_{\pi_t}^2 \sigma_{\pi_t}^2}{\sigma_{\pi_t}^2 + \sigma_{\pi_t}^2} \right) \right) - \frac{1}{\sigma_{\pi_t}^2} \]

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Denoting $\sigma_x^2 \equiv x$, $\sigma_e^2 \equiv y$, $\frac{1}{(2\beta)} \sigma_a^2 \equiv z$, and differentiating the above in respect to $x$ we get:

\[
\frac{\partial}{\partial x} \left( \frac{1}{x} \left(1 - \frac{(\frac{x}{x+y})^2 x}{z+(\frac{x}{x+y})^2} \right) - \frac{1}{x} \right) = 2xy \frac{x+y}{(zx^2 + 2xyz + zy^2 + yx^2)^2} > 0
\]

hence we get that the increase in the precision of public beliefs due to the analyst’s forecast increases as the prior variance increases, that is the increase in the public beliefs’ precision due to the analyst’s forecast decreases as the public precision prior to the analyst’s forecast increases.

QED

Claim 5 For all $b > 0$ we have

\[
0 > \frac{\partial}{\partial b} \left( \frac{1}{\frac{1}{b} \left(1 - \frac{(\frac{1}{b})^2}{z+(\frac{1}{b})^2} \right) - b} \right) > -1
\]

Proof. Denote: $w = \frac{1}{y}$, $v = \frac{1}{z}$. Then I have to prove:

\[
\frac{\partial}{\partial b} \left( \frac{1}{\frac{1}{b} \left(1 - \frac{(\frac{1}{b})^2}{z+(\frac{1}{b})^2} \right) - b} \right) > -1
\]

Straightforward differentiation and collecting terms yields that we need to prove:

\[
-\frac{2(w + b)w^2 v}{(w^2 + 2bw + b^2 + wv)^2} < -1
\]

or equivalently:

\[
2(w + b)w^2 v < (w^2 + 2bw + b^2 + wv)^2
\]
However:

\[
(w^2 + 2bw + b^2 + wv)^2 - 2(w + b)w^2v \\
= ((w + b)^2 + wv)^2 - 2(w + b)w^2v \\
= (w + b)^4 + w^2v^2 + 2(w + b)^2wv - 2(w + b)w^2v \\
= (w + b)^4 + w^2v^2 + 2wvb(w + b) > 0
\]

as required. ■

2.D. If \( f_S \) increases by a marginal unit, then \( \Delta f_S \) will increase by less than a marginal unit.

The increase in the precision of public beliefs due to the forecast of the analyst is:

\[
\Delta f_s = \frac{1}{\text{Var}(\pi | \pi^F)} - \frac{1}{\text{Var}(\pi)} = \left( \frac{1}{\sigma^2_{\pi}} \left( 1 - \frac{1}{\frac{1}{(2\beta)^2} \sigma^2_{\alpha} + \frac{1}{\sigma^2_{\pi} + \sigma^2_{\zeta}} \sigma^2_{\zeta}} \right) - \frac{1}{\sigma^2_{\pi}} \right)
\]

In order that the proof of the common knowledge case will hold for this case as well, it is sufficient to show that when \( f_S \) increases by a marginal unit \( \Delta f_S \) will increase by less than a marginal unit.

Denoting \( \sigma^2_{\pi} \equiv x \), \( \sigma^2_{\zeta} \equiv y \), \( v \equiv \frac{1}{y} \equiv f_{S_i} \), \( \frac{1}{(2\beta)^2} \sigma^2_{\alpha} \equiv z \), and differentiating the above in respect to \( x \) we get:

\[
\frac{\partial \Delta f_{S_i}}{\partial f_{S_i}} = \frac{\partial \Delta f_{S_i}}{\partial v} = \frac{\partial}{\partial v} \left( \frac{1}{x} \left( 1 - \frac{1}{\frac{1}{(2\beta)^2} \sigma^2_{\alpha} + \frac{1}{\sigma^2_{\pi} + \sigma^2_{\zeta}} \sigma^2_{\zeta}} \right) - \frac{1}{\sigma^2_{\pi}} \right) = \frac{x^2v}{(x^2v^2z + 2v) + x^2v^2}
\]

It is sufficient to show that

\[
\frac{x^2v}{(x^2v^2z + 2v) + x^2v^2} < 1 \\
\frac{x^2v (2v + 2z + x^2v)}{x^2v (2v + 2z + x^2v)} < (x^2v^2z + 2v + z + x^2v)^2
\]
Which is equivalent to

\[ x^2v (2z xv + 2z + x^2v) - (x^2v^2z + 2z xv + z + x^2v)^2 < 0 \]

rewriting the expression yields

\[ -2zx^3v^2 - x^4v^4z^2 - 4x^3v^3z^2 - 6x^2v^2z^2 - 2x^4v^3z - 4z^2xv - z^2 < 0 \]

Which holds for all \( z, x, v > 0 \). Hence

\[ 0 < \frac{\partial \Delta f_{S_i}}{\partial f_{S_i}} < 1 \]

QED

**Appendix 3 - Proof of Corollary 2**

Assume that \( f_{S_i} \) increases by \( \Delta f_{S_i} \), then the unconstrained optimal forecasting timing of analyst \( i \) decreases by \( \Delta f_{S_i} \). On the other hand, the "forecasting interval" of analyst \( j \) which is of "size" \( f_{S_i} \) increases by \( \Delta f_{S_i} \). This in turn, will decrease \( f^j_L \), but by less than \( \Delta f_{S_i} \). If before the increase of \( f_{S_i} \) analyst \( i \) was the first forecaster, then after the increase in \( f_{S_i} \) he will still forecast first at \( \text{Min} \{ f^i(t)^*, f^j_L \} \). Note that his forecast is in an earlier timing and lower precision of public beliefs. If before the increase in \( f_{S_i} \) analyst \( i \) was the second forecaster, then after the increase both orders are possible. If analyst \( j \) will still forecast first, it will be at lower or equal precision of investors’ beliefs. In this case analyst \( i \) will either forecast immediately after analyst \( j \) where the precision of investors beliefs will be higher then at the original case, or else he will forecast at his unconstrained optimum which is in lower investors’ beliefs precision than before the change. If analyst \( i \) will forecast first, than it will be at lower precision of investors beliefs, and analyst \( j \) will forecast at higher precision of investors beliefs than before the change.

**Appendix 4 - The joint distribution of the firm earnings and the analyst’s forecast is bivariate normal.**

A 2-dimensional random variable \( X \) with mean vector \( \mu \) and covariance matrix \( \Sigma \) is said to have a bivariate normal distribution if the distribution of \( c'X \) is (univariate) \( N(c'\mu, c'\Sigma c) \) for all real vector \( c \).

In our case, it should be shown that for all vectors \( [c_1, c_2] \) of real numbers, the (univariate) random variable
\( X = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} \pi \\ \pi_1 \end{bmatrix} \)

is normally distributed with a mean of \( c\mu \) and a variance of \( c\Sigma c' \).

\[
X = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} \frac{\alpha_1}{2\beta_1} + \frac{\alpha_2}{\sigma^2_{\pi_0} + \sigma^2_{\pi_1}} \mu_{\pi_0} + \frac{\sigma^2_{\pi_0}}{\sigma^2_{\pi_0} + \sigma^2_{\pi_1}} (\pi + \varepsilon_1) \\
\pi \\
\frac{\alpha_1}{2\beta_1} + \frac{\alpha_2}{\sigma^2_{\pi_0} + \sigma^2_{\pi_1}} \mu_{\pi_0} + \frac{\sigma^2_{\pi_0}}{\sigma^2_{\pi_0} + \sigma^2_{\pi_1}} \varepsilon_1 \end{bmatrix}
\]

\[
= \left( c_1 + c_2 \frac{\sigma^2_{\pi_0}}{\sigma^2_{\pi_0} + \sigma^2_{\pi_1}} \right) \pi + c_2 \left( \frac{\alpha_1}{2\beta_1} + \frac{\alpha_2}{\sigma^2_{\pi_0} + \sigma^2_{\pi_1}} \mu_{\pi_0} + \frac{\sigma^2_{\pi_0}}{\sigma^2_{\pi_0} + \sigma^2_{\pi_1}} \varepsilon_1 \right)
\]

Note that \( \pi, \alpha_1, \) and \( \varepsilon_1 \) are all independent and distributed according to a normal distribution.

For all vectors \( \begin{bmatrix} c_1 & c_2 \end{bmatrix} \) the above random variable \( (X) \) is a sum of three independent random variables and a constant - and hence is normally distributed.