Investment Kernels, Accounting Data and Intrinsic Value

Moritz Hiemann*
Maureen McNichols†
Stefan Reichelstein

Preliminary and incomplete – Caveat emptor

* Columbia University. Email: mh3338@columbia.edu.
† Stanford University. Email: fmnich@stanford.edu; reichelstein@stanford.edu.
1. **Introduction**

The valuation of firms and their profitability is a central topic in finance, accounting and economics. The literature has evolved from a focus on dividends as a valuation attribute to cash flows and more recently to book value and earnings. Our paper continues the evolution toward expressing value as a function of attributes that pertain directly to the value-creation process. Specifically, we formulate a model that expresses a firm’s intrinsic value as a function of its investment history and its future investment opportunities.

The basic building block of our model is what we term the investment kernel. For a given level of current investment, it describes the stream of expected future cash flows. Firms are assumed to choose their level of investment optimally in each period up to the point where the net-present value of the last dollar invested is zero. To capture the notion of a firms’ life cycle (or the life cycle of its products), we allow for the investment kernel to vary over time, possibly in a non-monotonic fashion.

Investment opportunity sets are, of course, unobservable in practice because accounting data do not reveal the present value of expected cash flows from current investments. However, firms report their current investment spending, for example, in the form of capital expenditures, R&D and advertising. Furthermore, the earnings and cash flows reported in each period provide information about the economic profitability of past investments. By restricting attention to a parametric class of investment kernels, we can use the time-series of firm-specific accounting information to estimate a particular firm’s investment kernel. This allows us to recover the mapping from investment to payoffs and thereby make inferences about intrinsic value, profitability and the firm’s effective cost of capital.

Our analysis offers several contributions to the literature on valuation, investment and profitability. First, our framework allows for an understanding of the pattern of value generation at a more primitive level than valuation based on outcomes such as cash flows or distributions of dividends. Our paper models the relation between value and the information inherent in the history of a firm’s investment decisions to incorporate more information about the value-generation process in estimates of intrinsic value. It explicitly considers the life cycle of a firm’s investments, and incorporates the information in the relation between profits and investments over the firm’s life cycle.
Second, the model requires no priors about how risk or other attributes should affect a firm’s cost of capital. However, any such relationships between the cost of capital and attributes of the firm should manifest themselves if the model is estimated. Hence, the model can potentially serve as a validation tool for asset pricing theories.

Third, our paper has a number of applications, including valuation, inferring the cost of capital, and estimating Tobin’s q. Specifically, with regard to inferring cost of capital, our model permits estimation of the cost of capital without reference to market price, which can be useful in estimating the cost of capital for divisions or segments of a firm or for firms that are not publicly traded. The model also permits estimation of Tobin’s q without reference to market price, and thus may be applied to firms without public equity.

The valuation framework we develop has several advantages relative to the dividend discount, discounted cash flow and residual income approaches. It has long been recognized that dividend forecasts are subject to substantial uncertainty, as dividends depend on firm’s payout rates, future earnings and capital structure choices. Furthermore, dividends distributed over a finite horizon have no necessary relation to firm value. By moving to free cash flows as the valuation attribute, valuation can abstract from forecasting dividends, share repurchases and capital structure choices. However, while cash flows have moved valuation closer to the earnings process than dividends, the cash flows over a finite horizon may have little relation to the value added over that period.

More recently, the residual income valuation literature has moved the focus to earnings and book value, which aim explicitly to capture value added over a period and resources invested to generate earnings (Ohlson, 1995; Feltham and Ohlson, 1995, 1996; Ohlson and Jüttner-Nauroth, 2005; Ohlson and Gao, 2006). The focus on profitability of investment and growth is therefore on attributes closer to the creation of value to than cash flows. However, a firm’s actual investing decisions are not fully reflected in this framework. By modeling the relation between value and the information inherent in the history of a firm’s investment decisions, our framework incorporates more information about the value-generation process in estimates of intrinsic value.

Our model approach, which regards investments as drivers of value creation, is related to some recent studies in accounting and management that focus on capacity investments, for instance, Rogerson (2008, 2011), Rajan and Reichelstein (2009) and Dutta and Reichelstein (2010).

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1 See for example Penman (2010) for discussion of the relation between cash flows and value additions over a period. (p. 122)
In these studies, firms make overlapping capacity investments, the cumulative effect of which is the productive capacity that is available for producing output in any given period. This framework has proven useful to examine the following issues: (i) accrual accounting rules sufficient for equity valuation based on current financial statements (Nezlobin, 2012), (ii) the mapping from the market-to-book ratio to Tobin’s q (McNichols et al., 2014) and (iii) the sensitivity of the price-earnings ratio to underlying variables such as past-and expected future investment growth (Nezlobin et al., 2014). We note that the formulation in our model framework is more generic insofar as the investment opportunities in any given period are not dependent on investments undertaken in the past. At the same time, our estimation approach relies on the sequence of observed investment decisions to infer to infer the parameters of a firm’s investment kernel.

The paper develops a methodology to apply the valuation framework empirically, by structuring of a series of equations based on the framework to estimate model parameters and intrinsic value of sample companies. The estimation approach generates a number of promising findings. First, the explanatory power of the estimating equations is substantial, with an average $R^2$ between 34% and 55%, depending on the specification. Second, the estimates of the cost of capital are reasonable, with a mean of approximately 12.5% for a sample that is weighted toward smaller firms, relative to the median firm in Compustat. Third, the parameter $\alpha$ reflects the ratio of the intrinsic value of an investment kernel to its cost, in the spirit of a project-level Tobin’s q. This parameter has a mean of 1.41 and median of 1.25. Furthermore, the mean $\alpha$ across industries is consistent with priors about the profitability of investment in those industries, with $\alpha$’s for airlines and utilities close to 1, and substantially higher $\alpha$’s for business services and retail apparel. Fourth, the model generates intrinsic value estimates that have substantial explanatory power for actual market values, with a correlation of 68%. While we have yet to conduct more structured tests to validate the model, such as direct comparisons to alternative valuation approaches, we are encouraged by the reasonableness of the intrinsic value estimates the framework has produced to date.

The layout of the paper is as follows. Section 2 describes the general theory and key assumptions underlying the model. Section 3 characterizes the econometric specification, and section 4

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2 This model framework dates back to Arrow (1964). See also Abel and Eberly (2011).
3 In the capacity models of Nezlobin (2012), McNichols et al. (2014) and Nezlobin et al. (2014) the observations on past investments are important to determine the firm’s current capital stock and the replacement value of current assets.
describes the estimation approach. Section 5 presents the empirical results to date and section 6 concludes.

2. Model

Consider a firm that makes investments continuously through time and, simultaneously, collects the payoffs from its past investments. An investment is defined as a current outflow of resources that generates future resource inflows. For simplicity and ease of exposition, these resource flows are assumed to be settled in cash immediately, so that investment spending and payoffs correspond to cash flows. Investing and collecting payoffs are the only activities the firm engages in, and any excess of current payoffs over investment spending is paid out as dividends or, in the reverse case, any shortfall is covered by raising additional capital. The firm is assumed to have access to frictionless capital markets.

The objective in this model is the derivation of an investment kernel \( v(\cdot) \) that describes the evolution of the firm’s investments and its cash flows from a parsimonious set of assumptions. In particular, let \( v(s, t, l) \) denote the present value as of time \( t \) of the cash flows to be received after time \( t + s \) from an investment in the amount of \( l \) made at time \( t \), where \( t, l \in \mathbb{R} \) and \( s \in \mathbb{R}^+ \). In other words, \( t \) is the origination time of the investment and \( s \) is the amount of time that has elapsed since \( t \). Naturally, when \( s = 0 \), \( v \) is the present value of all future cash flows to be received from the investment. The present value reflected in \( v \) is calculated by applying a discount rate \( r \), which is assumed to be time-invariant and thus reflects the firm’s long-term cost of capital. For tractability, \( v \) is further assumed to be continuously differentiable in all its arguments and have a finite value in the sense that

\[
\lim_{s \to \infty} e^{sr} v(s, t, l) = 0
\]  

(1)

for all \( t \) and \( l \), i.e., the present value of any investment’s remaining cash flows will dissipate eventually as the investment ages.

Knowledge of the investment kernel \( v \) is sufficient to compute the firm’s intrinsic value. In particular, the intrinsic value \( V \) of the firm at any time \( t \) is the sum of the present value of the remaining future cash flows from its past investments and the net present value of its future investment opportunities, or
\[ V(t) = \int_{-\infty}^{t} e^{(t-u)r} v(t-u, u, I(u)) \, du + \int_{t}^{\infty} e^{(t-u)r} \left( v(0, u, I(u)) - I(u) \right) \, du \] (2)

where \( I(\cdot) \) is the firm’s investment plan as a function of time.\(^4\) Like individual investment values, the firm’s intrinsic value is assumed to be finite in the sense that

\[ \lim_{t \to \infty} v(s, t, I) = 0 \] (3)

for all \( s \) and \( I.\(^5\) \) The return to an investor in the firm at time \( t \) has two components: the net capital distribution (or contribution) \( D \), and the change in market value. If the firm distributes its current cash flow as dividends and raises new capital in the amount of \( I \), \( D \) becomes

\[ D(t) = -I(t) - \int_{-\infty}^{t} e^{(t-u)r} v_s(t-u, u, I(u)) \, du \]

where \(-e^{sr} v_s(s, t, I)\) is the cash flow at time \( t+s \) of an investment made at time \( t.\(^6\) \) If the firm is priced at its intrinsic value, the return to an investor is therefore

\[ D(t) + V_e(t) = rV(t) \]

for all \( t \).

The analysis in this paper is not concerned with agency conflicts or other frictions. The firm invests in order to maximize the net present value of future cash flows at all times. The investment plan \( I(t) \) is therefore chosen to solve

\[ \max_{I(t)} \left( v(0, t, I(t)) - I(t) \right) \]

If investment projects are implemented in order of profitability, \( v \) is concave in \( I \) for a profit-maximizing firm. Then the point-wise first-order condition

\[ v_I(0, t, I) = 1 \]

\(^4\) The chosen setup with \(-\infty\) as the lower bound of the timeline is for mathematical convenience. One might argue that the lower bound should instead be \( t = 0 \), the time at which the firm was founded, but the investments on \((-\infty, 0)\) can readily be interpreted as the equivalent of a point-mass investment at time \( t = 0 \) that reflects the benefits from the firm’s startup capital. An additional benefit of the setup with \( t \in (-\infty, \infty) \) is that the model can be estimated without knowledge of the firm’s actual age.

\(^5\) A weaker condition would be \( \lim_{t \to \infty} e^{-tr} v(s, t, I) = 0 \), i.e., only the present value of future investment opportunities needs to vanish. However, this would theoretically permit the firm to grow forever, potentially at a rate higher than the growth in the economy overall, which is impossible in reality.

\(^6\) The firm’s cash flow from a given investment is implicitly defined by

\[ v(s, t, I) = \int_{t+s}^{\infty} e^{(t-u)r} c(t-u, t, I) \, du \]

where \( c(s, t, I) \) is the cash flow rate at time \( t+s \) from an investment made at time \( t \). Since the equality must hold for all \( s \), one obtains

\[ c(s, t, I) = -e^{sr} v_s(s, t, I) \]

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for each point in time $t$ yields a unique optimal investment plan $I^*(t)$, which the firm is hereafter assumed to implement. For analytical convenience, it will be assumed that $v_t(0, t, 0) > 1$ for all $t$, so that the firm will always invest at least a small amount. The firm’s effective investment kernel, i.e., the present value of the investments the firm actually implements, can then be written as

$$v(s, t) \equiv v(s, t, I^*(t))$$

The full function $v(s, t, l)$ for all $l$, including investments $l \neq I^*$ that the firm does not implement, will be referred to as the general investment kernel.

Narrowing down the functional form of $v$ requires additional structure. Rather than imposing a functional form on $v$ directly, the following analysis will recover $v$ by specifying the relationship between current investment opportunities and the payoffs currently received from past investments. The core idea is that the investment spending of one firm, hereafter referred to as the customer, must be revenue to another firm, hereafter called the supplier. The supplier’s revenue, in turn, is the payoff from its own past investments. Formally, one thus obtains the relationship

$$v^S_S(t, u, u) du = I^C(t)$$

where the superscripts $S$ and $C$ stand for supplier and customer, respectively. The transactions between customer and supplier create a link between investment at time $t$, which indirectly reflects the present value of the related future cash flows, and the aggregate cash flows that all past investments generate at time $t$.

It is important to note here that (4) implies no particular form of causality. It is possible that the supplier’s past investments have created current investment opportunities for the customer. Alternatively, the supplier could have invested in the past in anticipation of the customer’s current investment spending. Under either interpretation, the supplier chooses its investments first and, since $v$ is concave in investment, has a unique optimal investment amount at each time $t$, in response to which the customer also faces a unique optimal investment choice. The equilibrium outcome is thus a pair of optimal investment plans.

Equation (4) implies that the customer’s optimal investment can be decomposed into increments, each of which can be linked to the current payoff of one of the supplier’s past investments. Define $\Delta I^C(s, t)$ as the amount by which an investment by the supplier at time $t$ increases the customer’s investment $s$ periods later. Then (4) becomes
\[- \int_{-\infty}^{t} e^{(t-u)r} v_s^S(t-u, u) \, du = \int_{-\infty}^{t} \Delta I^C(t-u, u) \, du = I^C(t) \] (5)

where
\[-e^{(t-u)r} v_s^S(t-u, u) = \Delta I^C(t-u, u)\]
for all \(u\) and \(t\). At the same time \(t\), the supplier makes a new investment with present value
\[v^S(0, t) = -\int_{t}^{\infty} v_s^S(u-t, t) \, du = \int_{t}^{\infty} e^{(u-t)r} \Delta I^C(u-t, t) \, du\]
where
\[-v_s^S(u-t, t) = e^{(u-t)r} \Delta I^C(u-t, t)\]
for all \(u\) and \(t\).

The interdependence of \(v^S\) and \(I^C\) cannot be entirely arbitrary. Investment opportunities for the customer cannot be created by, or otherwise be linked to, investments by the supplier that have not yet occurred. In other words, the customer’s investment can only be reflective of investment opportunities the supplier has actually taken advantage of. If, hypothetically, the supplier had stopped investing at some point in time \(u < t\), the customer’s investment at time \(t\) could only depend on what is left at \(t\) of the investment opportunity existing as of time \(u\) (because \(v^S\) and \(I^C\) are linked, regardless of the direction of causality). Specifically, the customer’s investment at time \(t\), if the supplier has invested through time \(u \leq t\), should be a function of \(e^{(t-u)r} v^S(t-u, u)\), i.e., the present value, as of time \(t\), of the future cash flows left to be collected after \(t\) from the supplier’s investment opportunity as of time \(u\).

The preceding argument implies that investment is a function of both \(t\) and \(u\). In equilibrium, however, the supplier invests at all times (because \(v_t(0, t, 0) > 1\) for all \(t\)), so that the customer’s investment becomes observationally equivalent to a function of \(t\) only. In other words, the customer’s investment function can be written as
\[I^C(t) = I^C(t-u, u) \mid_{u=t}\]
or, to reflect explicitly the rationale of the preceding paragraph,
\[I^C(t) = I^C(t-u, u, e^{(t-u)r} v^S(t-u, u)) \mid_{u=t}\]
for all \(t\). However, not much is gained by this reasoning while \(I^C\) and \(v^S\) are still arbitrary. Proceeding with the analysis requires specification of the relationship between \(I^C\) and \(v^S\).
**Assumption 1** (Linear value chain). For all $t$ and $u \leq t$, 

$$I^C(t-u,u) = \theta e^{(t-u)r} v^S(t-u,u)$$

for some constant $\theta$.

Assumption 1 imposes a linear relationship on the mapping between the customer’s investment and the present value of the supplier’s investment opportunities. An immediate implication is that the supplier’s revenue stays in constant proportion to the present value of its own contemporaneous investments. It should be noted, however, that this assumption places no restrictions on the customer’s cash flow distribution.\(^7\) One should also observe that $\theta > 1$ for any $r > 0$. The following result provides the general solution for the investment kernel $v$ in this setting. Since all firms act in both supplier and customer role, labeling firms as either customer or supplier is no longer important, so superscripts are omitted for notational simplicity from hereon.

**Proposition 1.** Under the linear value chain assumption, the firm’s effective investment kernel is 

$$v(s,t) = e^{\frac{(s-(\theta-1)t)}{\theta^2+(\theta-1)^2}} f(\theta s + (\theta - 1)t)$$

where $f(\cdot)$ is any univariate function such that $v$ obeys the boundary conditions (1) and (3).

In terms of economic meaning, $\theta$ indicates how quickly firms reap the benefits from their investment opportunity set over time. Firms with higher $\theta$ collect the payoffs from their investments faster, or, put differently, their investments have a shorter economic life. By contrast, the payoffs from investments by firms with low $\theta$ take longer to materialize. Proposition 1 also implies that firms with higher $\theta$ move through their life cycle at a faster pace.

The importance of Proposition 1 lies in dimension reduction: while the solution still requires selecting an arbitrary function $f$, the latter only depends on one variable rather than two. In order to characterize the evolution of the firm over time with respect to both $s$ and $t$, it now suffices to choose the investment kernel conditional on either $s$ or $t$. For example, if one were to specify the distribution of cash flows for an investment made at some fixed time $t$, the evolution of $v$ across all $t$ would follow immediately. Alternatively, one could obtain the firm’s cash flow distribution by letting $f$ describe the trajectory of the firm’s total revenue over time.

\(^7\) Assumption 1 only specifies the present value at the time the customer makes the investment. The cash flow distribution describes how this present value evolves subsequently and is therefore not restricted by Assumption 1.
Unless otherwise indicated, Assumption 1 and hence Proposition 1 will continue to apply to the analysis hereafter. Nonetheless, it should be noted that one can generalize Proposition 1 to scenarios in which the customer’s investment is not a constant multiple $\theta$ of the supplier’s investment kernel but the proportion varies over time by some function $\theta(t)$. The investment kernel is then implicitly characterized by

$$(\theta - 1)v_s - \theta v_t + (r\theta + \theta')v = 0$$

An exact solution can be obtained by specifying the functional form of $\theta(t)$.

The preceding discussion still leaves the question of the firm’s economic profitability, i.e., the question how $v$ relates to the optimal investment amount $I^*$ and, more generally, how $v$ depends on investment for $I \neq I^*$. Assumption 1 imposes no restrictions on this relationship because it only relates the firm’s investment to its supplier’s investment kernel, not to the firm’s own investment kernel. As a function of $I^*$, $v$ can therefore be any mapping of the form

$$v(0, t) = \alpha(t, I^*(t))$$

for some arbitrary profit function $\alpha(\cdot)$. The dependence of $\alpha$ on $s$ is not immediately relevant for the following discussion and is therefore suppressed for the moment.

Recovering the general investment kernel $v(s, t, I)$, for $I \neq I^*$, from (6) is not generally possible, but the linear value chain assumption can be applied to obtain a solution in the following setting. If Assumption 1 holds and one were to observe a constant ratio between the firm’s investment and its revenue over time, it must be that

$$v(0, t) = \alpha I^*(t)$$

for some constant $\alpha > 1$, i.e., the firm’s profitability is independent of the level of optimal investment. As the following proposition shows, one can recover the general investment kernel $v(s, t, I)$ in this case for a class of separated solution functions of the form

$$v(0, t, I) = f(I)g(I^*(t))$$

for all $I$ and $t$.

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8 Empirically, investment and revenue indeed appear to stay in roughly constant proportion, at least in mature firms.
Proposition 2. If \( v \) is separable in \( I \) and \( I^* \) and \( v(0,t) = \alpha I^*(t) \), the general investment kernel takes the form
\[
\nu(0,t,I) = \alpha \frac{1}{\alpha} I^*(t)^{\frac{\alpha-1}{\alpha}}
\]
for all \( t \).

Proposition 2 provides an illustration how one can infer the mapping from investment to payoffs even if one only observes the firm’s path of optimal investment choices.\(^9\) On the other hand, one could also attempt to test empirically how well the exponential form fits observed data, for example, if one were able to observe exogenous disturbances to the investment process, but a cautionary note is in order. It is tempting to use Proposition 2 to validate the linear value chain assumption, but validating the exponential form of \( v \) empirically by some econometric means does not imply Assumption 1. While (7) would indeed hold in this case, Assumption 1 makes a more general statement about the relationship between \( v^S \) and \( I^C \) outside the observed equilibrium in which both firms invest at all \( t \). In other words, Assumption 1 implies (7) if revenue and investment stay in constant proportion over time, but the converse is not true.

One can now reintroduce \( s \) to the general investment kernel. Given Proposition 1 and the linearity in (7), optimal investment is
\[
I^*(t) = \frac{1}{\alpha} \frac{\nu(0,t)}{\nu(0,0)} e^{\frac{\theta^2 t}{(\theta-1)^2}} f((\theta - 1)t)
\]
and so
\[
\nu(s,t,I) = \left( \frac{\alpha I}{\nu(0,t)} \right)^{\frac{1}{\alpha}} \nu(s,t)
\]
for all \( s \) and \( t \). Subsequent empirical tests of the model will be based on (8).

3. Econometric Specification

Estimating the solution function in Proposition 1 via firms’ financial report data requires parameterization of \( f \). As noted above, choosing \( f \) is equivalent to imposing an initial value condi-

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\(^9\) This result can be extended to profit functions of the slightly more general form \( v(0,t) = \alpha(f^*(t)) \), which yields an implicit characterization of \( f \) and \( g \) as
\[
f = c g^{\frac{1}{\alpha-1}} e^{\frac{\alpha}{(\alpha-1)^2} \ln g}
\]
but analytical solutions are generally not obtainable in this case.
tion after fixing either \( s \) or \( t \). In the following analysis, \( f \) will be specified conditional on \( t = 0 \), i.e., an assumption will be made about the distribution of cash flows from an individual investment. In particular, the cash flow distribution is assumed to follow the notion of a product life cycle: cash flow starts out low initially as the new product created by the investment is introduced, increases as the market for the product develops until it reaches maturity, and then declines as the product becomes obsolete.

To reflect the notion of a product life cycle, cumulative remaining cash flows will be modeled by a logistic growth curve of the form
\[
e^{sr} v(s, 0) = e^{(\frac{1-\theta}{\sigma} + s + sr)fr} f(\theta s) = \frac{\beta_0}{1 + e^{\beta_1 + \beta_2 \theta s}}
\]
where \( \beta_0, \beta_2 > 0 \). The logistic function is commonly used, e.g., to model the growth of populations or the diffusion process of new products. Then
\[
f(u) = \frac{\beta_0}{1 + e^{\beta_1 + \beta_2 u} e^{\theta (\theta - 1) u} \sigma}
\]
and hence
\[
v(s, t) = \frac{\beta_0}{1 + e^{\beta_1 + \beta_2 (\theta s + (\theta - 1) t)} e^{\frac{t}{\sigma} - s}}
\]
for all \( s \) and \( t \). Boundary condition (3) is thus met for \( r < \beta_2 (\theta - 1) \). The resulting distribution of cash flows takes the form
\[
-e^{sr} v_s(s, t) = \frac{\beta_0 (\beta_2 - r) e^{\beta_1 + \beta_2 (\theta s + (\theta - 1) t)} e^{\frac{t}{\sigma}} - r \beta_0 e^{\frac{t}{\sigma}}}{(1 + e^{\beta_1 + \beta_2 (\theta s + (\theta - 1) t)})^2}
\]
which closely resembles a Hubbert curve. Figure 1 shows a graphical illustration.

It has been assumed so far that the firm’s cash flow consists solely of cash receipts, but in practice, firms incur contemporaneous cash expenditures in order to operate their assets, including inventory or raw material purchases, labor costs, rent, utility charges and others. One may think of these items as variable unit costs. For estimation purposes, it will be convenient to model these expenditures separately from the investment cost \( I \). If the firm incurs these operating costs at a constant rate per unit sold and hence earns a profit margin \( m \) on its sales, its cash flow only represents \( m \) times the revenue. Then (4) becomes
\[
- \int_{-\infty}^{t} e^{(t-u)r} v_s(t-u, u) du = ml^c(t)
\]
which can be accommodated readily by replacing \( \theta \) with \( m\theta \) in all previous results.\(^{10}\)

Another component of the firm’s optimization problem to be considered in an empirical specification is income taxes. Let \( h \) denote the tax rate and assume that all of the firm’s cash inflows and outflows are taxed as incurred.\(^{11}\) The firm’s investment cost \( I \) is also tax-deductible, and so the firm maximizes the net present value of its after-tax cash flows while considering the benefits of tax-deductible investment costs.\(^{12}\) Considering variable profit margins and income taxes changes the investment kernel to

\[
v(s, t) = \frac{(1 - h)\beta_0 e^{(\xi - s)\tau}}{1 + e^{\beta_1 + \beta_2(\eta s + (\eta - 1)\tau)}}
\]

where \( \eta = (1 - h)m\theta \). The firm’s optimal pre-tax investment is now given by

\[
I^*(t) = \frac{v(0, t)}{(1 - h)\alpha}
\]

and the condition \( \theta > 1 \) becomes \( \eta > 1 \).

Further, it has been assumed to this point that all resource inflows and outflows are in the form of cash. In reality, investments also tend to require some amount of working capital, such as inventory, receivables, and payables, throughout the duration of the project in order to maintain the firm’s business operations. These working capital accounts reflect timing differences between the economic resource flow and its conversion to cash. The firm’s total cost of maintaining working capital over the course of an investment project is therefore

\[10\text{ One might observe that the customer’s investment spending } I^C \text{ previously used to motivate Assumption 1 should now represent not only the customer’s investment spending but also its current operating expenditures, such as inventory purchases from the supplier. The linear structure employed so far readily permits this interpretation.}

\[11\text{ The model presented here captures total firm value, and hence the discount rate } r \text{ reflects all forms of financing, including the tax benefits of debt. Taxes are therefore applied before the deduction of interest costs.}

\[12\text{ For simplicity, the estimation in this paper is conducted under the assumption that all investment costs are tax-deductible immediately. In practice, taxation rules vary by investment type. For example, research and development expenditures are usually deductible as incurred, while the tax depreciation of capital assets often follows a type of declining-balance method. To refine the model, one could therefore set the depreciation rate at time } t + s \text{ of one unit of investment in the form of capital expenditures made at time } t \text{ at}

\[-b_k(s) = \frac{\gamma}{T} \left(1 + \int_t^{t+s} b_k(u - t) du\right) \iff -b_k(s) = \frac{\gamma}{T} e^{-\frac{T}{T}}\]

where \( \gamma \) is the declining balance factor, \( T \) is the useful life under tax depreciation rules, and \( b(s) \) is the book value of one unit of investment \( s \) periods after the investment was made, i.e., \( b(0) \equiv 0 \). (This simplified representation is chosen for analytical convenience. Typically, the declining balance method is switched to the straight-line method in later periods, when the latter yields a higher depreciation charge.) The firm’s after-tax investment cost would then equal \( I \) multiplied by the tax factor

\[(1 - \rho)(1 - h) + \rho \frac{(1 - h)\gamma + Tr}{\gamma + Tr} \]

where \( \rho \) is the proportion of total investment comprised of capital expenditures.
\[
w(0, t) + \int_t^\infty e^{(t-u)r}w_s(u - t, t) \, du = \int_t^\infty re^{(t-u)r}w(u - t, t) \, du 
\]  
(12)

where \(w(s, t)\) is the required working capital balance at time \(t + s\) for an investment made at time \(t\). In other words, the firm begins a project by committing an initial amount \(w(0, t)\), e.g., by making upfront purchases of inventory, and adjusting this amount by \(w_s\) over time.

There are two ways to interpret the role of working capital in this model. First, one can think of changes in working capital accounts as part of the firm’s cash flow and hence as a component of \(v\). Second, one can think of the total cost of working capital as a component of the firm’s investment cost \(I\), in which case \(-e^{sr}v_s\) becomes equivalent to accounting income, with all non-cash earnings components removed. The estimation in this paper will follow the second approach, but one should note that this only affects the interpretation of \(v\). The firm’s intrinsic value is the same regardless.

Many working capital accounts, e.g., accounts receivable and inventory, tend to vary with the firm’s sales volume. The working capital balance related to a particular investment will therefore be modeled as a constant proportion \(\bar{w}\) of the firm’s current cash flow, or

\[
w(s, t) = -\bar{w}e^{sr}v_s(s, t) 
\]  
(13)

The firm’s cost of maintaining working capital is therefore

\[
\int_t^\infty re^{(t-u)r}w(u - t, t) \, du = -\int_t^\infty r\bar{w}v_s(u - t, t) \, du = r\bar{w}v(0, t) = r\bar{w}a_t \cdot (t) 
\]  
(14)

for any given investment project. At each point in time, the working capital balance for all past investment projects is thus

\[
\int_{-\infty}^t \bar{w}e^{(t-u)r}v_s(t - u, u) \, du = \bar{w}v(0, t) 
\]  
(15)

The intrinsic value of the firm thus becomes

\[
V(t) = \int_{-\infty}^t (1 + r\bar{w})e^{(t-u)r}v(t - u, u, I(u)) \, du \]

\[
+ \int_t^\infty e^{(t-u)r} \left( v(0, u, I(u)) - I(u) \right) \, du 
\]  
(16)

where the first term now also reflects the firm’s benefit of having committed working capital resources to its ongoing investment projects already.
4. Estimation

The investment kernel \( v \) is unobservable in practice, but several quantities implied by the model are available in the form of accounting data. For example, financial statements provide information about the current payoffs of firms’ past investments and also contain a number of components of the firm’s investment spending, such as capital expenditures and research and development costs. The purpose of this section is to relate the constructs developed in the preceding section to these accounting data.

The discussion so far has assumed a deterministic evolution of the firm but firms clearly operate in a stochastic environment in reality. To accommodate this feature, the model will be augmented by a multiplicative error structure of the form \( e^\varepsilon \). The stochastic analogue of the investment kernel \( v \) is then given by

\[
\tilde{v}(s, t) = \frac{(1 - h)\beta_0e^{(1/\gamma)s + \varepsilon}}{1 + e^{\beta_1 + \beta_2(s + (1-\eta)t)}}
\]

The firm’s reported investment for the fiscal year \([t-1, t]\) can thus be written as

\[
INV(t) = \int_{t-1}^{t} (1 - \bar{w}\alpha)\tilde{I}(u) \, du = \int_{t-1}^{t} \frac{1 - \bar{w}\alpha}{(1 - h)\alpha} \tilde{v}(0, u) \, du \tag{17}
\]

in view of the linear relationship between \( \tilde{I} \) and \( \varepsilon \) in (7). The left-hand side of (17), \( INV \), is the firm’s actual investment spending during the fiscal year ended at time \( t \) and is computed as

\[
INV(t) \equiv CAPX(t) + XRD(t) + XAD(t) + AQC(t)
\]

where \( CAPX \) is the firm’s capital expenditures, \( XRD \) denotes research and development expenses, \( XAD \) denotes advertising expenses, and \( AQC \) is the cash amount spent to acquire other firms. As a caveat, one should note that \( INV \) is an imperfect estimate of investment in two respects. First, firms likely make investments not captured by the four data items used, so that \( INV \) might understate actual investment. On the other hand, \( XRD \) likely contains depreciation of capital assets used in research and development projects, whose acquisition cost has already been captured by capital expenditures in prior periods. The amount of \( XRD \) is therefore adjusted by removing

\[\text{An additive, mean-zero error term could permit illogical outcomes such as negative investment.}\]

\[\text{Technically, the error terms in } \tilde{I} \text{ and } \varepsilon \text{ may differ, but, for notational simplicity, } \varepsilon \text{ will be used to denote all error terms unless distinction between different sources of error or the possible correlation between error terms is needed for expositional purposes.}\]

\[\text{It is assumed that } AQC \text{ captures most of the firm’s acquisition activity, but one should note that } AQC \text{ does not reflect non-cash acquisitions, captures only the price paid for equity, not for total assets, and may include amounts paid for assets that do not constitute investments in the sense of the model.}\]
the estimated depreciation component, based on the ratio of XRD to total operating expenses. Finally, the subtraction of \( r \alpha \) removes the working capital component of the investment cost, which is not reflected in INV.

The firm’s cash flow, excluding changes in working capital accounts, is modeled as the cash component of the firm’s accounting income from operations. In particular, cash flow is estimated via the firm’s earnings before interest, tax, depreciation and amortization, and research and development and advertising expenses. Research and development and advertising are removed because they are modeled as part of the firm’s investment spending, INV. Total income over the year is therefore

\[
EBITDA(t) + XRD(t) + XAD(t) = \int_{t-1}^{t} m \theta \tilde{v}(0, u) \, du
\]

for the year ended at time \( t \), where EBITDA is earnings before interest, tax, depreciation and amortization. As discussed in the preceding section, changes in the firm’s working capital account balances, which are technically part of the firm’s operating cash flow, are treated as part of the firm’s investment cost in this model, as reflected in (17).

Direct estimation of this income number is problematic under a multiplicative error structure because this structure would require income to be positive at all times while firms often incur losses in reality. To circumvent the problem, the estimation is performed on revenues instead. In view of (11), the firm’s reported revenue, REV, in period \([t - 1, t]\) is expected to equal investment payoffs divided by the after-tax profit margin \((1 - h)m\), or

\[
REV(t) = \int_{t-1}^{t} \frac{\eta}{(1 - h)m} \tilde{v}(0, u) \, du = \int_{t-1}^{t} \theta \tilde{v}(0, u) \, du
\]

where the profit margin \( m \) is computed as

\[
m = \frac{\sum_t (REV(t) - EXP(t))}{\sum_t REV(t)} \tag{19}
\]

and

\[
EXP \equiv REV - XRD - XAD - EBITDA
\]

is the firm’s operating expenses. The tax rate \( h \) is set at 35% for all estimations.

To improve the efficiency of the estimation, fixed asset balances are used as auxiliary data items in addition to investment and income. Gross property, plant and equipment is estimated as the accumulation of past capital expenditures, or
\[ PPEGT(t) = \int_{t-T}^{t} \frac{(1 - r \bar{w} \alpha) \rho}{(1 - h) \alpha} \tilde{v}(0, u) \, du \]  

where \( T \) is the useful life assumed for accounting purposes and \( \rho \) is the amount of capital expenditures as a fraction of \( INV \).\(^{16}\) The net carrying amount of property, plant and equipment then becomes

\[ PPENT(t) = \int_{t-T}^{t} \frac{(1 - r \bar{w} \alpha)(u - t + T) \rho}{(1 - h) \alpha T} \tilde{v}(0, u) \, du \]  

if the firm applies the commonly used straight-line depreciation method.\(^{17}\) The useful life \( T \) and the proportion of capital expenditures \( \rho \) are computed directly as

\[
T = \frac{\sum_t PPEGT(t)}{\sum_t DFXA(t)} \tag{22}
\]

and

\[
\rho = \frac{\sum_t CAPX(t)}{\sum_t INV(t)} \tag{23}
\]

where \( DFXA \) is the depreciation expense on fixed assets.\(^{18}\) The error term in \( \tilde{v} \) in the above equations can reflect both measurement error in the firm’s investment spending as well as measurement error related to depreciation policies. For example, the firm may actually apply different useful lives or depreciation methods across asset classes.

The working capital proportion \( \bar{w} \) is estimated from current assets and liabilities. Specifically, working capital is the net amount of current operating assets and current operating liabilities, or

\[ WC(t) = INV + RECT + XPP - AP - TXP - XACC - DRC - DRLT \]

which consists of inventories, \( INV \), receivables, \( RECT \), prepaid expenses, \( XPP \), accounts payable, \( AP \), taxes payable, \( TXP \), accrued expenses, \( XACC \), and current (\( DRC \)) and long-term (\( DRLT \)) deferred revenue. Then the working capital rate \( \bar{w} \) can be computed as

\[
\bar{w} = \frac{\sum_t WC(t)}{\sum_t (REVT(t) - EXP(t))} \tag{24}
\]

An equation for intrinsic value can be added for firms whose market value is available. Market value is estimated as the sum of the stock market value of the firm’s equity and the book value.

\(^{16}\) The proportions of capital expenditures, research and development costs, and advertising expenses indeed generally appear to remain stable over time for a given firm. Hence, assuming a constant value for \( \rho \) seems justifiable.

\(^{17}\) The expressions for \( PPEGT \) and \( PPENT \) assume that no assets are disposed of prior to the end of their useful life.

\(^{18}\) For years in which the \( DFXA \) data item is unavailable, depreciation on fixed assets is estimated by subtracting the amortization expense on intangible assets from the firm’s total depreciation and amortization expense.
ue of all liabilities not included in working capital. Under the assumption that market value is a reasonable approximation of the firm’s intrinsic value, one obtains

\[ MV(t) = \tilde{V}(t) \]  

(25)

where

\[ MV = CSHO \cdot prcc_{f} + LT - AP - TXP - XACC - DRC - DRLT - CH - IVST \]

is the sum of the equity market value, given by the product of the number of shares outstanding \( CSHO \) and the stock price \( prcc_{f} \), and the total liabilities, \( LT \), less the working capital liability items and the firm’s cash, \( CH \), and short-term investments, \( IVST \). The latter two items are excluded because the model describes the firm’s operating activities, not its financial investments. The intrinsic value \( \tilde{V} \) is the stochastic analogue of \( V \), as defined by equation (16).

The parameters \( m, \bar{w}, T \) and \( \rho \) reflect simple ratios of financial statement items and are therefore computed directly, as indicated in (19), (22), (23) and (24). The remaining parameters are estimated from the system of equations (17), (18), (20) and (21). The market value equation (25) is added for some tests. The estimation is performed at the firm level by a non-linear least-squares approach that minimizes the criterion function

\[ Q = \sum_{i=1}^{n} \varepsilon_{i}^{'} \cdot \varepsilon_{i} \]

where \( \varepsilon_{i} \) is the vector of errors in period \( i \) from (17), (18), (20), (21) and, if applicable, (25), and \( n \) is the number of reporting periods available for the firm. Given the multiplicative error structure with \( \varepsilon \), all equations are estimated after taking logarithms. The process generating \( \varepsilon \) is assumed to be stationary and ergodic, in which case consistent estimates of the parameters in \( \nu \) can be obtained.\(^{19}\)

It is worth distinguishing two interpretations of the error term \( \varepsilon \). First, \( \varepsilon \) could constitute measurement error introduced by the accounting process because the accounting records and procedures do not capture all variables accurately. In this case, the error would be generated once per reporting period and would not affect the firm’s investment opportunities. Alternatively, \( \varepsilon \) could arise from random fluctuations in the firm’s optimal investment \( I^{*} \) over time. If the firm’s management observes this random fluctuation before making its investment decision, the error in

\(^{19}\) An example of a stationary process in continuous time is the Ornstein-Uhlenbeck process.
would carry over to the payoffs and, on average, increase \( \nu \).\(^{20}\) The firm’s reported investment, revenues and expenses are likely to reflect some combination of these two interpretations of \( \varepsilon \), but their actual weights are unobservable.

In its deterministic form, the model implies that the firm should continue as a going concern indefinitely and never be liquidated because of business failure.\(^{21}\) In the presence of the stochastic component \( e^{\varepsilon} \), however, large negative realizations of \( \varepsilon \) may imply that terminating the business is preferable to continuation if \( \varepsilon \) is sufficiently persistent across time. The estimation in this paper does not consider the value of this termination option.\(^{22}\)

When the firm’s market value is unavailable, identification of \( \theta \), \( \alpha \), \( \beta_0 \), \( \beta_1 \) \( \beta_2 \) and \( r \) is not possible from the remaining data items because the number of parameters to be estimated exceeds the number of coefficients by one. However, the estimation results imply boundaries on the possible parameter values. First, no firm with investment opportunities yielding only negative net present values, i.e., firms with \( \alpha < 1 \), should be in business, which implies a lower bound of \( \alpha = 1 \). Further, the coefficients \( \beta_0 \eta \), \( \beta_0 \left( \frac{1}{\alpha} - r \bar{w} \right) \), \( \frac{r}{\eta} \), and \( \beta_2 (\eta - 1) \) are given by the estimation, and so fixing any parameter requires adjusting the remaining parameters such that these coefficients remain constant. Then the requirement that \( \eta > 1 \) implies that, after some substituting, \( \alpha < \bar{\alpha} \eta \), where \( \bar{\alpha} \eta \) is the estimated product of \( \alpha \) and \( \eta \), which is identified. In other words, if one were to fix one of the parameter values in order to address the identification problem, the choice must be such that \( \alpha \in (1, \bar{\alpha} \eta) \). Lower and upper bounds can similarly be obtained for the remaining parameters. In particular, the discount rate must be such that \( r \in \left( \frac{\bar{r}}{\eta}, \bar{r} \bar{\alpha} \right) \), where \( \bar{r} \) and \( \bar{r} \bar{\alpha} \) are defined analogously to \( \bar{\alpha} \eta \).

The estimation is performed via numerical optimization over the available time series of accounting data for each firm individually, subject to the boundary constraint in (3). The simulated

\(^{20}\) Stationarity implies that the firm’s expected investment in a given reporting period is

\[
E \left( \ln \left( \int_{t-1}^{t} I^* (u) \, du \right) \right) = E \left( \ln \left( \int_{t-1}^{t} I^* (u) \, du \right) \right) + E \left( \ln \left( \int_{t-1}^{t} e^{\varepsilon} \, du \right) \right)
\]

under the stationary distribution. If \( \varepsilon \) affects the firm’s investment opportunities in each period and were to follow a normal distribution, it would, on average, increase \( I^* \) by an amount bounded above by

\[
E \left( \ln \left( \int_{t-1}^{t} e^{\varepsilon} \, du \right) \right) \leq \ln \left( E \left( \int_{t-1}^{t} e^{\varepsilon} \, du \right) \right) = \frac{\sigma^2}{2}
\]

by Jensen’s inequality, where \( \sigma^2 \) is the stationary variance of \( \varepsilon \) on a unit interval.

\(^{21}\) Bankruptcy is still possible in this setting but should result in a reorganization rather than in a liquidation.

\(^{22}\) An analytical solution for the optimal termination strategy and hence the value of the termination option is generally not obtainable.
annealing algorithm is applied to a set of different vectors of initial values of the model parameters. To refine the outcome, the results of each optimization are used as starting values in a second round of optimization via the Nelder-Mead algorithm.\textsuperscript{23} The result yielding the lowest value of $Q$ is retained. The estimation is performed in two variants: first, with firms’ observable market value included in the system of equations; and, second, with market values excluded.

5. Empirical Results

Samples are selected from all firms in Compustat with at least 5 years of revenue and investment data and with aggregate positive profits over the observable period, i.e., $m > 0$, and with working capital balances, relative to their profits, of $\bar{w} \in (-10,10)$\textsuperscript{24} As defined in (19), (22), (23) and (24), the values of $m$, $\bar{w}$, $T$ and $\rho$ are computed directly from Compustat data ratios and are treated as fixed inputs in the estimation of $\theta$, $\alpha$, $\beta_0$, $\beta_1 \beta_2$ and $r$. Their mean (median) values are $0.18$ ($0.14$) for $m$, $1.17$ ($0.80$) for $\bar{w}$, $15.2$ ($13.8$) for $T$, and $0.73$ ($0.85$) for $\rho$.

The first estimation is conducted with firms’ observable market values included in the system of equations, via (25). Only firms with at least one year with an observable market value are included in the sample, which yields a sample size of 2,837 firms. To avoid distortion of the results by economically implausible values, firms with an estimated value of $\alpha < 0.3$ or $\alpha > 10$ are removed from the results, which reduces the sample by 12.1% to 2,495 firms\textsuperscript{25}. Table 1 shows descriptive statistics. The mean (median) cost of capital $r$ is estimated at 14.4% (12.5%). In interpreting this result, it is important to note that $r$ reflects a long-term discount rate for the firm’s operational activities. Firms’ actual cost of capital in practice, if measured at a point in time, is likely modified by contemporaneous market conditions and by firms’ non-operating activities. The latter includes financial investments, which are not part of the model in this paper. Figure 1 shows a histogram of the distribution of $r$. The mean (median) economic profitability parameter $\alpha$, which is related to Tobin’s $q$, is 1.43 (1.17) and suggests that, over the course of their economic useful lives, firms’ investments add value of 43% (17%) in excess over the investment cost. A histogram of the distribution of $\alpha$ is shown in Figure 3. As predicted by the model, most sample firms have values of $\alpha > 1$, and only 18% of firms have values of $\alpha < 0.95$ even before

\textsuperscript{23} Simulated annealing is a stochastic optimization approach and does therefore not return exact optima.
\textsuperscript{24} The model in this paper describes firms that recover their investment costs through their future profits. Hence, the behavior of firms with $m < 0$ cannot be rationalized in the context of the model.
\textsuperscript{25} Choosing different elimination thresholds does not have a substantive impact on results.
firms with $\alpha < 0.3$ are eliminated. The adjusted mean (median) $R^2$-value in this estimation is 0.34 (0.42).

Table 2 reports estimation results by industry, defined according to two-digit SIC codes. The most economically profitable industries in the sample, by mean and median $\alpha$, include leather products, clothing stores, business services, and the motion picture industry, while utilities, several branches of retail and wholesale, oil and gas exploration, and airlines rank among the least profitable ones. Industries with the highest cost of capital include healthcare, mining, rubber and plastics products, motion pictures, and specialty manufacturing branches (such as toys, jewelry and musical instruments). Industries with a low estimated cost of capital include utilities, textiles, and ground freight. One should note, however, that the number of firms per industry is small in several cases and may be sensitive to the inclusion or exclusion of individual firms.

The second estimation is performed with firms’ observed market values excluded from the system of estimated equations. As noted in the preceding section, parameters can only be identified in terms of ranges in this case. In order to compute firms’ estimated intrinsic values, the midpoint between the theoretical minimum of 1 and the maximum feasible value of $\alpha$, as implied by the estimation results, is used. The estimation is performed on a sample of 2,137 firms. Of these, 782, or 37%, yield estimation results such that the maximum feasible value of $\alpha$ falls below the theoretical minimum of 1. These observations are eliminated from the reported results. In addition, firms with an estimated intrinsic value $V$ such that $\ln(V) > 11$ or $\ln(V) < -1$ are removed as outliers because these extreme estimates produce very large estimation errors in almost all cases. After these eliminations, the sample is reduced to 1,157 firms. Based on the midpoints of the feasible ranges, the mean (median) value of $r$ is 18.2% (13.7%) and the mean (median) value of $\alpha$ is 1.41 (1.25), which in both cases is reasonably close to the values obtained in the first estimation when observable market values are included.

Figure 3 shows a scatter plot of the estimated intrinsic value, $\ln(V)$, and firms’ observed market values, $\ln(MV)$, in logarithmic scale. The correlation between the two variables is 0.68. The interpretation of this result is subject to several caveats. First, further validation is necessary to assess how this result compares to alternative approaches to estimate intrinsic value. More importantly, the correlation only captures the relationship between two estimates of the underlying construct, the firm’s actual intrinsic value. A high correlation between $V$ and the observed mar-
The relatively large number of estimation results where the maximum value of $\alpha$ is below 1 warrants further investigation. Typical characteristics of these firms are: a short history of data in Compustat, profits insufficient to cover investment costs over the data series available, and a disproportionately high rate of eventual elimination from Compustat because of bankruptcy or liquidation. Thus, the model appears to have only limited ability to describe the behavior of firms whose investment payoffs are collected with a delay and that, in view of the short data histories of these firms, are at a relatively early stage of their life cycle. A possible avenue to address the problem is to replace the linearity condition in Assumption 1 by a time-dependent evolution of $\theta$ such that firms could have lower revenue collections, relative to investment spending, in earlier years.

6. Conclusion

This paper develops a model of a firm’s intrinsic value as a function of its past and future investment opportunities, which determine the firm’s investment policy and its future cash flows. The basic building block, termed in the investment kernel, is derived from an assumption about the relationship between the value of current investment opportunities and the cash flow rate of past investments. When estimated with data from financial accounting reports, the model permits insights into intrinsic value, cost of capital and Tobin’s q at the firm and industry level. Preliminary results suggest that the model provides reasonable estimates of these three constructs.

The model can be used to address a variety of research questions. Intrinsic value can be estimated for firms that do not have observable market values or for individual segments of businesses, and this estimation can be conducted solely with accounting data and does not require an exogenously determined cost of capital. Alternatively, the model can be used to determine a firm’s implied cost of capital. Further, one can apply the model to infer the economic profitability of firms in different industries and thereby test their degree of competitiveness. It may also be of interest to examine the evolution of economic profitability in various industries over time. Finally, the model might also prove useful in investigating problems of optimal investment.
References


Appendix

**Proof of Proposition 1.** Assumption 1 implies that

\[ \theta e^{(t-u)r} v^S(t-u,u) \bigg|_{u=t} = \int_{-\infty}^{t} \frac{d}{du} (\theta e^{(t-u)r} v^S(t-u,u)) \, du = \int_{-\infty}^{t} \Delta l^c(t-u,u) \, du = I^c(t) \]

and hence

\[ \Delta l^c(t-u,u) = \frac{d}{du} (\theta e^{(t-u)r} v^S(t-u,u)) = -\theta e^{(t-u)r} \left( v^S_\tau (t-u,u) - v^S_\tau (t-u,u) + rv^S(t-u,u) \right) \]

(A1)

Then equation (4) becomes

\[ -\int_{-\infty}^{t} e^{(t-u)r} v^S_\tau (t-u,u) \, du = \int_{-\infty}^{t} \frac{d}{du} (\theta e^{(t-u)r} v^S(t-u,u)) \, du \]

and so

\[ e^{(t-u)r} v^S_\tau (t-u,u) = \theta e^{(t-u)r} \left( v^S_\tau (t-u,u) - v^S_\tau (t-u,u) + rv^S(t-u,u) \right) \]

must hold for all \( u \) and \( t \). The solution function \( v(s,t) \), with superscripts omitted for simplicity hereafter, must therefore solve the first-order linear PDE

\[ (\theta - 1)v_s - \theta v_t + r\theta v = 0 \]

for all \( s \) and \( t \). Its general solution is

\[ v(s,t) = e^{(\theta t - (\theta - 1)s)\theta r} f(\theta s + (\theta - 1)t) \]

for any univariate function \( f(\cdot) \).

**Proof of Proposition 2.** The separated solution

\[ v(0,t,l) = f(l)g(l^*(t)) \]

combined with the linearity condition

\[ v(0,t) \equiv v(0,t,l) \bigg|_{l=l^*} = \alpha l^*(t) \]

implies that

\[ \frac{d}{dl^*} v(0,t,l) \bigg|_{l=l^*} = f'(l) \bigg|_{l=l^*} g(l^*(t)) + f(l) \bigg|_{l=l^*} g'(l^*(t)) = \alpha \]

By definition, \( l^* \) is the unique value such that
$$f'(l)|_{l=l^*} g(l^*(t)) = 1$$

and hence

$$f(l)|_{l=l^*} g'(l^*(t)) = \alpha - 1$$

Since (__) and (__) must hold for any \( l^* \),

$$\frac{f'}{f} = \frac{1}{\alpha - 1} \frac{g'}{g}$$

with solution

$$f = C g^{\frac{1}{\alpha - 1}}$$

for some constant \( C \). Then

$$\alpha l^*(t) = C g^{\frac{\alpha}{\alpha - 1}}$$

and so

$$v(0, t, l) = \alpha l^*(t)^{\frac{\alpha - 1}{\alpha}}$$

for all \( t \) and \( l \).
Table 1. Descriptive statistics of estimated model parameters, with observable market values included in the estimation.

<table>
<thead>
<tr>
<th></th>
<th>median</th>
<th>mean</th>
<th>min</th>
<th>max</th>
<th>stddev</th>
</tr>
</thead>
<tbody>
<tr>
<td>θ</td>
<td>13.016</td>
<td>22.067</td>
<td>1.885</td>
<td>840.737</td>
<td>40.924</td>
</tr>
<tr>
<td>β₀</td>
<td>2.591</td>
<td>109582.138</td>
<td>0.010</td>
<td>83796675.082</td>
<td>2371379.814</td>
</tr>
<tr>
<td>β₁</td>
<td>-20.534</td>
<td>-882.309</td>
<td>16.314</td>
<td>16800.708</td>
<td>241413.620</td>
</tr>
<tr>
<td>β₂</td>
<td>3.183</td>
<td>45202.130</td>
<td>0.000</td>
<td>4318314.433</td>
<td>241413.620</td>
</tr>
<tr>
<td>α</td>
<td>1.175</td>
<td>1.431</td>
<td>0.305</td>
<td>8.785</td>
<td>0.825</td>
</tr>
<tr>
<td>r</td>
<td>0.125</td>
<td>0.144</td>
<td>-0.686</td>
<td>1.223</td>
<td>0.130</td>
</tr>
</tbody>
</table>

n = 2495
Table 2. Economic profitability, $\alpha$, and cost of capital, $r$, by two-digit SIC industry code, with observable market values included in the estimation. Only industries with 10 or more firms are included in the table.

<table>
<thead>
<tr>
<th>SIC code (2-digit) Industry</th>
<th>n</th>
<th>$\alpha$ mean</th>
<th>$\alpha$ median</th>
<th>$r$ mean</th>
<th>$r$ median</th>
</tr>
</thead>
<tbody>
<tr>
<td>31 Leather, footwear</td>
<td>11</td>
<td>1.595</td>
<td>1.627</td>
<td>0.130</td>
<td>0.117</td>
</tr>
<tr>
<td>56 Retail - clothing</td>
<td>25</td>
<td>1.751</td>
<td>1.467</td>
<td>0.166</td>
<td>0.142</td>
</tr>
<tr>
<td>34 Metal products</td>
<td>64</td>
<td>1.459</td>
<td>1.426</td>
<td>0.139</td>
<td>0.134</td>
</tr>
<tr>
<td>27 Publishing</td>
<td>43</td>
<td>1.551</td>
<td>1.371</td>
<td>0.159</td>
<td>0.144</td>
</tr>
<tr>
<td>30 Rubber and plastics products</td>
<td>35</td>
<td>1.374</td>
<td>1.361</td>
<td>0.158</td>
<td>0.157</td>
</tr>
<tr>
<td>58 Retail - eating places</td>
<td>42</td>
<td>1.470</td>
<td>1.360</td>
<td>0.155</td>
<td>0.138</td>
</tr>
<tr>
<td>32 Glass and cement</td>
<td>22</td>
<td>1.281</td>
<td>1.330</td>
<td>0.103</td>
<td>0.099</td>
</tr>
<tr>
<td>67 Patents, royalty traders</td>
<td>20</td>
<td>1.789</td>
<td>1.326</td>
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<td>0.126</td>
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<td>73 Business services</td>
<td>180</td>
<td>1.640</td>
<td>1.318</td>
<td>0.163</td>
<td>0.137</td>
</tr>
<tr>
<td>22 Textiles</td>
<td>24</td>
<td>1.601</td>
<td>1.317</td>
<td>0.067</td>
<td>0.077</td>
</tr>
<tr>
<td>87 Services - consulting and research</td>
<td>33</td>
<td>1.586</td>
<td>1.284</td>
<td>0.150</td>
<td>0.146</td>
</tr>
<tr>
<td>78 Motion pictures</td>
<td>16</td>
<td>2.170</td>
<td>1.283</td>
<td>0.184</td>
<td>0.140</td>
</tr>
<tr>
<td>54 Retail - grocery/convenience</td>
<td>27</td>
<td>1.362</td>
<td>1.264</td>
<td>0.154</td>
<td>0.138</td>
</tr>
<tr>
<td>16 Construction - heavy, water</td>
<td>10</td>
<td>1.865</td>
<td>1.260</td>
<td>0.188</td>
<td>0.139</td>
</tr>
<tr>
<td>42 Ground freight</td>
<td>25</td>
<td>1.284</td>
<td>1.247</td>
<td>0.088</td>
<td>0.096</td>
</tr>
<tr>
<td>33 Steel and nonferrous metals</td>
<td>52</td>
<td>1.502</td>
<td>1.246</td>
<td>0.109</td>
<td>0.091</td>
</tr>
<tr>
<td>23 Apparel</td>
<td>34</td>
<td>1.461</td>
<td>1.243</td>
<td>0.167</td>
<td>0.117</td>
</tr>
<tr>
<td>70 Hotels</td>
<td>17</td>
<td>1.459</td>
<td>1.242</td>
<td>0.115</td>
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</tr>
<tr>
<td>48 Telecommunications</td>
<td>67</td>
<td>1.491</td>
<td>1.241</td>
<td>0.181</td>
<td>0.132</td>
</tr>
<tr>
<td>20 Food products</td>
<td>65</td>
<td>1.355</td>
<td>1.239</td>
<td>0.116</td>
<td>0.119</td>
</tr>
<tr>
<td>52 Retail - building materials</td>
<td>12</td>
<td>1.401</td>
<td>1.224</td>
<td>0.103</td>
<td>0.082</td>
</tr>
<tr>
<td>26 Paper</td>
<td>41</td>
<td>1.255</td>
<td>1.221</td>
<td>0.126</td>
<td>0.107</td>
</tr>
<tr>
<td>28 Chemicals</td>
<td>143</td>
<td>1.553</td>
<td>1.220</td>
<td>0.155</td>
<td>0.144</td>
</tr>
<tr>
<td>59 Retail - specialty</td>
<td>47</td>
<td>1.356</td>
<td>1.205</td>
<td>0.126</td>
<td>0.125</td>
</tr>
<tr>
<td>10 Mining</td>
<td>29</td>
<td>1.279</td>
<td>1.197</td>
<td>0.173</td>
<td>0.152</td>
</tr>
<tr>
<td>80 Health services</td>
<td>54</td>
<td>1.509</td>
<td>1.185</td>
<td>0.196</td>
<td>0.168</td>
</tr>
<tr>
<td>25 Furniture</td>
<td>28</td>
<td>1.398</td>
<td>1.181</td>
<td>0.142</td>
<td>0.126</td>
</tr>
<tr>
<td>36 Electrical equipment</td>
<td>151</td>
<td>1.327</td>
<td>1.172</td>
<td>0.140</td>
<td>0.139</td>
</tr>
</tbody>
</table>
Table 2 (continued). Economic profitability, $\alpha$, and cost of capital, $r$, by two-digit SIC industry code, with observable market values included in the estimation. Only industries with 10 or more firms are included in the table.

<table>
<thead>
<tr>
<th>SIC code (2-digit) Industry</th>
<th>$n$</th>
<th>$\alpha$ mean</th>
<th>$\alpha$ median</th>
<th>$r$ mean</th>
<th>$r$ median</th>
</tr>
</thead>
<tbody>
<tr>
<td>37 Motor vehicles</td>
<td>61</td>
<td>1.410</td>
<td>1.168</td>
<td>0.123</td>
<td>0.112</td>
</tr>
<tr>
<td>38 Industrial and medical instruments</td>
<td>130</td>
<td>1.254</td>
<td>1.162</td>
<td>0.141</td>
<td>0.121</td>
</tr>
<tr>
<td>35 Machinery and equipment</td>
<td>147</td>
<td>1.338</td>
<td>1.152</td>
<td>0.158</td>
<td>0.130</td>
</tr>
<tr>
<td>24 Lumber, home building</td>
<td>20</td>
<td>1.212</td>
<td>1.133</td>
<td>0.108</td>
<td>0.103</td>
</tr>
<tr>
<td>39 Manufacturing - specialty</td>
<td>29</td>
<td>1.295</td>
<td>1.124</td>
<td>0.164</td>
<td>0.146</td>
</tr>
<tr>
<td>79 Recreation services</td>
<td>35</td>
<td>1.347</td>
<td>1.121</td>
<td>0.125</td>
<td>0.097</td>
</tr>
<tr>
<td>40 Railroads</td>
<td>11</td>
<td>1.378</td>
<td>1.116</td>
<td>0.102</td>
<td>0.081</td>
</tr>
<tr>
<td>53 Retail - department stores</td>
<td>33</td>
<td>1.156</td>
<td>1.102</td>
<td>0.105</td>
<td>0.106</td>
</tr>
<tr>
<td>50 Wholesale - durables</td>
<td>59</td>
<td>1.299</td>
<td>1.094</td>
<td>0.133</td>
<td>0.125</td>
</tr>
<tr>
<td>29 Petroleum refining</td>
<td>14</td>
<td>1.257</td>
<td>1.092</td>
<td>0.112</td>
<td>0.121</td>
</tr>
<tr>
<td>49 Utilities</td>
<td>102</td>
<td>1.186</td>
<td>1.061</td>
<td>0.123</td>
<td>0.090</td>
</tr>
<tr>
<td>55 Retail - automotive</td>
<td>10</td>
<td>1.117</td>
<td>1.053</td>
<td>0.148</td>
<td>0.160</td>
</tr>
<tr>
<td>45 Airlines</td>
<td>16</td>
<td>1.312</td>
<td>1.051</td>
<td>0.142</td>
<td>0.141</td>
</tr>
<tr>
<td>13 Oil and gas exploration</td>
<td>96</td>
<td>1.227</td>
<td>1.048</td>
<td>0.142</td>
<td>0.146</td>
</tr>
<tr>
<td>44 Water transportation</td>
<td>10</td>
<td>1.262</td>
<td>1.043</td>
<td>0.130</td>
<td>0.143</td>
</tr>
<tr>
<td>57 Retail - furniture, electronics</td>
<td>15</td>
<td>1.192</td>
<td>1.011</td>
<td>0.150</td>
<td>0.111</td>
</tr>
<tr>
<td>51 Wholesale - non-durables</td>
<td>45</td>
<td>1.078</td>
<td>1.006</td>
<td>0.141</td>
<td>0.119</td>
</tr>
</tbody>
</table>
Table 3. Descriptive statistics for the upper and lower bounds and the midpoint values of \( r \) and \( \alpha \) when firms’ observed market values are not included in the estimation.

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>median</th>
<th>stdev</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r ) minimum</td>
<td>0.134</td>
<td>0.109</td>
<td>0.102</td>
</tr>
<tr>
<td>midpoint</td>
<td>0.182</td>
<td>0.137</td>
<td>0.181</td>
</tr>
<tr>
<td>maximum</td>
<td>0.229</td>
<td>0.165</td>
<td>0.298</td>
</tr>
<tr>
<td>( \alpha ) minimum</td>
<td>1.000</td>
<td>1.000</td>
<td>0.000</td>
</tr>
<tr>
<td>midpoint</td>
<td>1.405</td>
<td>1.250</td>
<td>0.958</td>
</tr>
<tr>
<td>maximum</td>
<td>1.811</td>
<td>1.501</td>
<td>1.916</td>
</tr>
</tbody>
</table>

\( n = 1157 \)
Figure 1. Illustration of the cash flow, \(-e^{sr}v_s\), from an investment as a function of the time \(s\) that has elapsed since the investment was made.
Figure 2. Histogram of the discount rate $r$ when estimated with firms’ observed market values.
Figure 3. Histogram of the economic profitability parameter $\alpha$ when estimated with firms’ observed market values.
**Figure 3.** Scatter plot, on a logarithmic scale, of firms’ observed market values, $\ln(MV)$, and their estimated intrinsic values, $\ln(V)$, where $V$ is estimated without observed market value as an input.