Greed, *Fear*, *and Rushes*[*]

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Abstract

We develop a continuum player unified timing game that also rationalizes sudden mass movements. Payoffs in the game are a logsubmodular function of time and the stopping quantile. Time captures the payoff-relevant *fundamental* — payoffs "ripen", peak at a "harvest time", and then "rot". Capturing the strategic interaction, payoffs are single-peaked in the *quantile*. We distinguish between *greed* and *fear* — i.e., the hunger for greater rewards from outlasting others, and the fear of missing out on early rewards.

Three local timing games can arise in equilibrium: a war of attrition, a slow pre-emption game, and a pre-emptive rush. With greed, the harvest time precedes an accelerating war of attrition ending in a rush, whereas with fear, an inefficiently early rush precedes a slowing pre-emption game ending at the harvest time. The theory yields predictions for the rush size and time, as well as stopping intensity and duration when fundamentals or the strategic structure change.

Our theory subsumes a wealth of timing games with large numbers of players, yielding old and new predictions. For instance, matching rushes, bank runs, social tipping points happen before fundamentals indicate, and asset sales rushes occur after. We also explain how (a) "unraveling" in matching markets depends on early matching stigma and market thinness; (b) asset sales rushes reflect liquidity and relative compensation; (c) illiquid bank loans yields complete bank runs before incomplete ones.

[*]This supersedes a primitive early version of this paper by Andreas and Lones, growing out of joint work in Andreas' 2004 PhD thesis, that assumed multiplicative payoffs. It was presented at the 2008 Econometric Society Summer Meetings at Northwestern and the 2009 ASSA meetings. Most modeling and results, and all of the analysis and exposition now reflects joint work of Axel and Lones since 2012. It has profited from comments in seminar presentations at Wisconsin, Western Ontario, and Melbourne.

Contents

1 Introduction

"Natura non facit saltus." — Leibniz, Linnaeus, Darwin, and Marshall

Mass rushes periodically grip many economic landscapes — such as fraternity "rush week"; the "unraveling" rushes of young doctors seeking hospital internships; the bubble-bursting sales rushes ending asset price run-ups; land rushes for newly-opening territory; bank runs by fearful depositors; and whites flight from a racially tipping neighborhood. These settings are so far removed from one another that they have been studied in completely disparate fields of economics. Yet by stepping back from their institutional details, a unified theory of all timing games emerges for large numbers of players. This theory explains not only the size and timing of aggregate jumps, but also the more commonly studied speed of entry into the timing game.

Timing games have so far been applied in settings with identified players, like industrial organization. But rushes reflect the coordinated timing decisions of a large mass of people. But a large number of players is a more apt description of almost all cited examples above. This paper introduces a greatly simplified class of timing games flexible enough for all of these economic environments. We thus assume an anonymous continuum of homogeneous players, ensuring that no player enjoys any market power; this also dispenses with any strategic uncertainty. We characterize the Nash equilibrium of a simultaneous-move ("silent") timing game (Karlin, 1959) — thereby also ignoring dynamics, or any learning about exogenous uncertainty.

We venture that payoffs reflect a dichotomy — they are functions of the stopping time and *quantile*. Time intuitively proxies for a slowly-evolving payoff-relevant *fundamental*, while the quantile embodies strategic effects. A mixed strategy is then a stopping time distribution function on the positive reals. When it is continuous, there is *gradual play*, and players slowly stop; *rushes* occur when a positive mass of players suddenly stop, and the cdf jumps up. We assume that payoffs are separately hump-shaped in time and single-peaked in the quantile. This ensures a unique optimal *harvest time* for any quantile, when the fundamental peaks, and a unique optimal *peak quantile* for any time, when stopping is most strategically advantageous.

Standard timing games assume a monotone increasing or decreasing quantile response, so that the first or last mover has an advantage over *all* other quantiles. In our more general class of games, the peak quantile may be interior. The game exhibits *greed* if the last mover has an advantage over the *average* quantile, and *fear* if the first mover does. So the war of attrition is an extreme case of greed, with later payoffs more attractive than average, and similarly pre-emption games are extreme fear. We find that every game either exhibits greed or fear or neither.

Mixed strategies require constant payoffs in equilibrium, balancing quantile and fundamental payoff forces. Two opposing flavors of timing games have been studied. A *war of attrition* entails gradual play in which the passage of time is fundamentally harmful and strategically beneficial. The reverse holds in a *pre-emption game* — the strategic and exogenous delay incentives oppose, balancing the marginal costs and benefits of the passage of time. When the payoffs continuously rise and then fall both in time (fundamentals) and quantile, this opposition is impossible with purely gradual play: To sustain indifference at all times, a mass of consumers must stop at the same time (Proposition 2). Apropos our lead quotation, despite a continuously evolving world, both from fundamental and strategic perspectives, *aggregate behavior must jump.* Only in the well-studied case with a monotone quantile response can equilibrium involve gradual play for all quantiles (Proposition 1). But even in this case, an early rush must happen whenever the gains of immediately stopping as an early quantile dominates the peak fundamentals payoff growth. We call the extreme case with early rushes either *alarm* or —- if the rush includes everyone — *panic*.

In characterizing all equilibria, Proposition 2 notes that equilibrium play cannot straddle the harvest time. Rather, equilibria are either *early* or *late* — transpiring entirely before or entirely after the harvest time. Absent fear, slow wars of attrition start at the harvest time and are followed by rushes. Absent greed, initial rushes are followed by slow pre-emption games ending at the harvest time. So *fear-driven rushes occur inefficiently early, and rushes sparked by greed occur late.* With neither greed nor fear, both types of equilibria arise. This yields a useful corollary insight — that the rush occurs long before fundamentals peaks in a pre-emption equilibrium, and long after fundamentals peaks in a war of attrition equilibrium.

We prove that there is at most one *inaction phase* between the rush and gradual play can emerge in equilibria, when there is no entry; however, a simple and tractable trembling refinement that prunes such inaction phases. An equilibrium is immune to uncertainty about the accuracy of the time clock iff there are no inaction phases — namely, it is a *safe equilibrium*. Proposition 3 also asserts that there is a unique such safe equilibrium given greed, and a different unique such safe equilibrium given fear — and so at most two altogether. Each admits easy computation: the rush size equates the rush payoff and the adjacent quantile payoff, and then the constant payoff requirement fixes the timing and duration of gradual play.

Proposition 4 then characterizes gradual play. Specifically, when fundamental payoffs are log-concave in time, any pre-emption game gradually slows to zero after the early rush, whereas any war of attrition accelerates from zero towards its rush crescendo. In other words, stopping rates wax and wane, respectively, after and before the harvest time. Our dichotomy therefore allows for the identification of timing games from data on stopping rates.

A strength of our paper is the ease with which comparative statics predictions can be derived using our graphical apparatus, since one need only see how the two equilibrium curves shift. But our model yields a rich array of monotone comparative statics. We first consider a change in fundamentals. Proposition 5 builds on a monotone ratio shift in the fundamental payoffs that delays the harvest time. While the delay naturally postpones all activity, it has some surprising implications. Despite the heightened monetary stakes, stopping rates during any war of attrition phase attenuate, and the terminal rush swells. On the other hand, stopping rates during any pre-emption game intensify, and the initial rush shrinks. So an inverse relation between stopping rates and rush size emerges, with higher stopping rates in gradual play accompanying smaller rushes.

We next explore the effect of a monotone strategic shift. A logsupermodular payoff shift towards later quantiles spans our class of games, as extreme *fear* transitions into extreme *greed*. Proposition 6 reports how as greed rises in the war of attrition equilibrium, or fear rises in the pre-emption equilibrium, the gradual play phase lengthens; but in each case stopping rates fall and the rush shrinks. Perhaps counterintuitively, the rush is smaller and farther from the harvest time the more strategically intense is the game (Figure 7).

One might worry that unmodeled dynamic consideration are essential for comparative statics analysis. Fortunately, when we express economic applications in our strategic structure, we agree with established results in existing timing models, but offer a wealth of new testable insights.

We start with a famous and well-documented timing game that arises in matching contexts. We create a reduced form model incorporating economic considerations found in Roth and Xing (1994). All told, fear rises when hiring firms face a thinner market, while greed increases in the stigma of early matching. Firms place weight on learning about caliber of the applicants. We find that matching rushes occur inefficiently early provided the early matching stigma is not too great. By assuming that stigma reflects recent matching outcomes, our model delivers the matching unravelling without appeal to a slow tatonnement process (Niederle and Roth, 2004).

Next, consider two common market forces behind the sales rushes ending asset bubbles: a desire for liquidity fosters fear, while a concern for relative performance engenders greed. Abreu and Brunnermeier (2003) ignores relative performance, and so finds a pre-emption game with no rush before the harvest time. In their model, the bubble bursts when rational sales exceed a threshold; also like them, we find that the bubble burst is larger and later with lower interest rates. Yet by conventional wisdom, the NASDAQ bubble burst in March 2000 occurred after fundamentals peaked. Our model speaks to this puzzle, since a sales rush after the harvest time happens with strong enough relative compensation.

We conclude by exploring timing insights about bank runs. Inspired by the two period model of Diamond and Dybvig (1983), in our simple continuous time model, a run occurs when too many depositors inefficiently withdraw before the harvest time. But since withdrawing later is never strictly better, payoffs monotonically fall in the quantile. Then by Proposition 1, either a slow pre-emption game arises or a rush occurs immediately (alarm or panic). We find a non-monotonicity: as bank fundamentals worsen, a complete panic occurs, but then with even worse fundamentals, only a partial rush occurs; however, adding a reserve requirement lessens its chance.

Literature Review. Maynard Smith (1974) developed the war of attrition to model animal conflicts. Its biggest impact in economics owes to the all-pay auction literature (for instance, Krishna and Morgan (1997)). The economic study of pre-emption games dates back at least to Fudenberg, Gilbert, Stiglitz, and Tirole (1983) and Fudenberg and Tirole (1985). Recently, Brunnermeier and Morgan (2010) and Anderson, Friedman, and Oprea (2010) have experimentally tested it. Park and Smith (2008) explored a finite player game with rank-dependent payoffs: rushes and wars of attrition alternated, but slow pre-emption games were impossible. Ours may be the first timing game with all three timing game equilibria.

All results not immediately argued are proven in the appendix.

2 Model

A continuum of identical risk neutral players $[0,1]$ each chooses a mixture over stopping times τ on $[0,\infty)$, where $\tau \leq t$ with chance $Q(t)$ on $\mathbb{R}_+$. Either choices are made irrevocably at time-0, or players do not observe each other over time, maybe since the game transpires quickly.

By Lebesgue's decomposition theorem, the *quantile function* $Q(t)$ is the sum of atoms and absolutely continuous portions. If Q is absolutely continuous at t, then the *stopping payoff* is $u(t,q)$. But if t is an *atom* of Q, so that $p = Q(t) > Q(t-) = q$, then stopping at t earns $\int_q^p u(t,x)dx/(p-q)$, i.e. the average quantile payoff for the player mass $p-q$ stopping at time t.

For fixed q, payoffs u are quasi-concave in t, strictly rising from $t=0$ ("ripening") until an ideal *harvest time* $t^*(q)$, and then strictly falling ('rotting"). For fixed t, payoffs u are either monotone or log-concave in q, with unique interior *peak quantile* $q^*(t)$. We embed strategic interactions by assuming the payoff function $u(t,q)$ is log-submodular, subsuming the multiplicative special case.[1] Since higher q corresponds to stochastically earlier stopping by the population of agents, this yields proportional complementarity — the proportional gains to postponing until a later quantile are larger, or the proportional losses smaller, the earlier is the stopping time. In-

[1]Almost all of our results only require the weaker complementary condition that $u(t,q)$ be *quasi-submodular*, so that $u(t_L,q_L) \geq (>) u(t_H,q_L)$ implies $u(t_L,q_H) \geq (>) u(t_H,q_H)$, for all $t_H \geq t_L$ and $q_H \geq q_L$.

deed, the harvest time $t^*(q)$ is a decreasing function of q, while the peak quantile function $q^*(t)$ is decreasing in t. To ensure players stop in finite time, we assume waiting forever is dominated by stopping at $t^*(0)$:

$$\lim_{t\to\infty} u(t, q^*(t)) < u(t^*(0), 0). \tag{1}$$

Formally, let $T(p)$ be inverse to the quantile function $Q(T(p)) = p$. Then the payoff to $\tau = t$ equals the Radon Nikodym derivative $w(t) = dW/dQ$, given the running integral

$$W(t) = \int_0^{Q(t)} u(T(p), p)dp$$

A *Nash equilibrium* is a cdf Q whose support contains only maximizers of the function $w(t)$.[2]

We assume agents can perfectly time their actions. But one might venture that even the best timing technology admits error. If so, agents may be wary of equilibria in which tiny timing mistakes incur payoff losses. In Proposition 3, we will see that concern for such timing mistakes leads to *safe equilibria*, namely, equilibria in which the quantile function Q is an atom at time $t = 0$, or a cdf whose support is a non-empty time interval, or the union of both possibilities.

3 The Tradeoff Between Time and Quantile

At any point in its support, a cdf Q is either absolutely continuous or jumps, i.e. $Q(t) > Q(t-)$. This corresponds to slow entry, that we call *gradual play*, or a *rush*, where a positive mass stops at a time-t atom. Since homogeneous players obviously earn the same Nash payoff, say $\bar{w}$, indifference must prevail whenever play is gradual on an interval:

$$w(t) = u(t, Q(t)) = \bar{w} \tag{2}$$

Since u is C^2, the *stopping rate* Q' exists and is C^1 during any gradual play phase; it obeys the differential equation (2):

$$u_q(t, Q(t))Q'(t) + u_t(t, Q(t)) = 0 \tag{3}$$

Because of (3), whenever Q is absolutely continuous, it is also differentiable in time. The stopping rate is given by the marginal rate of substitution, i.e. $Q'(t) = -u_t/u_q$, and since $Q'(t) > 0$, the slope signs u_q and u_t must be mismatched throughout any gradual play phase. So two possible timing game phases are possible on any interval:

[2] Absent any strategic or exogenous uncertainty, Nash equilibrium of the silent timing game is solvable in open-loop strategies (Fudenberg and Levine, 1988).

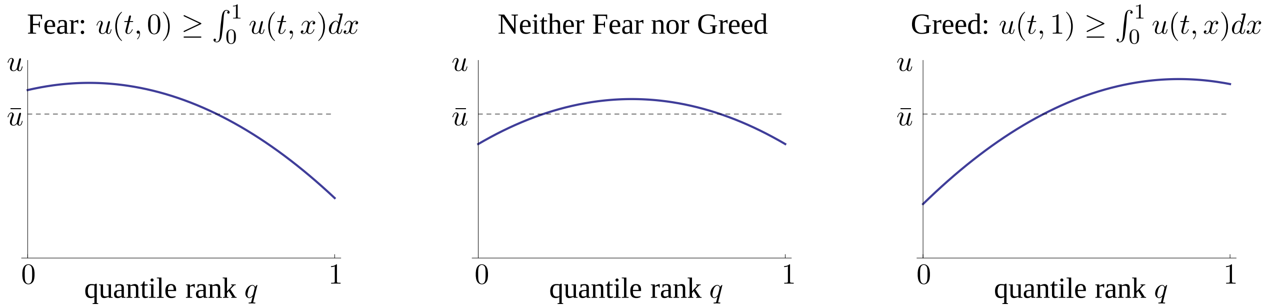

Figure 1: **Fear and Greed.** Payoffs — depicted here for a fixed time t — cannot exhibit greed and fear, with first and last quantile factors above average ($\bar{u}$), but might exhibit neither (middle).

- *Pre-emption game:* a connected interval of gradual play on which $u_t > 0 > u_q$, so that the passage of time is fundamentally beneficial but strategically costly.
- *War of Attrition:* a connected interval of gradual play on which $u_t < 0 < u_q$, so that the passage of time is fundamentally harmful but strategically beneficial.

Parsing equilibrium behavior also requires us to make global comparisons of strategic effects and time effects. Let $V_0(t,q) \equiv q^{-1} \int_0^q u(t,x)dx$ be the *running average payoff function*. Let us write that *fundamental growth dominates strategic quantile effects when*:

$$\max_q V_0(0,q) \leq u(t^*(1),1) \tag{4}$$

When (4) fails, stopping as an early quantile dominates waiting until the harvest time, if one is last. There are then two mutually exclusive possibilities: *alarm* when $V_0(0,1) < u(t^*(1),1) < \max_q V_0(0,q)$, and *panic* when the harvest time payoff is even lower: $u(t^*(1),1) \leq V_0(0,1)$.

Analysis of rushes instead requires comparisons of average and extreme quantile payoffs. A "fear of missing out" is a notion that is backward-looking in quantiles. To capture this idea, we say that there *fear at time* t if $u(t,0) \geq \int_0^1 u(t,x)dx$. Fear therefore rises in the gains of pre-empting others. By contrast, greed is forward-looking in quantiles. Greed rises in the gains of outlasting others. We analogously say that there is *greed at time* t if the last quantile payoff exceeds the average, namely if $u(t,1) \geq \int_0^1 u(t,x)dx$. Both inequalities are tight if the stopping payoff $u(t,q)$ is constant in q, and strict fear and greed correspond to strict inequalities (see Figure 1). *Greed and fear at* t *are mutually exclusive*, as u is single-peaked in q. Respectively, these might be thought of as last mover and first mover advantage.

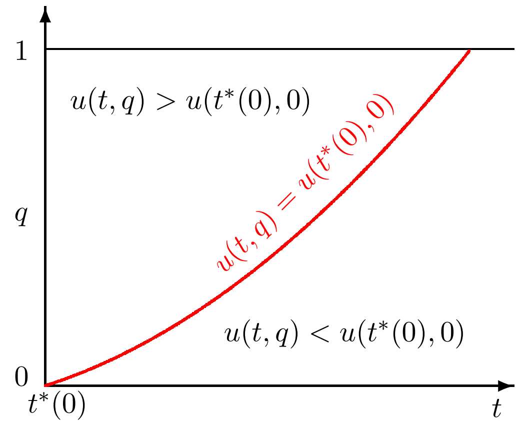

Figure 2: **The Equilibrium for Monotone Increasing Payoffs.** We depict the unique equilibrium when $u_q > 0$, a War of Attrition defined by the indifference curve $u(t,q) = u(t^*(0),0)$.

4 Monotone Payoffs in Quantile

We focus first on the standard timing game when payoffs are monotone in the quantile. When the stopping payoff is increasing in q, rushes are impossible, since stopping immediately after a rush yields a higher payoff than stopping in the rush. Thus, the only possibility is a War of Attrition engulfing all quantiles. In fact, gradual play must begin at time $t^*(0) > 0$. For if gradual plays start at $t > t^*(0)$, then quantile 0 would profitably deviate to $t^*(0)$, whereas if it starts at $t < t^*(0)$, then the required differential equation (3) is impossible — for $u_t(t,0) > 0$ and $u_q(t,0) > 0$. Since gradual play begins at $t^*(0)$, the Nash payoff is $u(t^*(0),0)$, and so $Q(t)$ satisfies:

$$u(t,Q(t)) = u(t^*(0),0) \tag{5}$$

Figure 2 illustrates the resulting time-quantile indifference curve defining the War of Attrition.

Similarly, when $u_q < 0$, any gradual play interval must end at $t^*(1)$, implying an equilibrium value $u(t^*(1),1)$. For if gradual play ends at $t < t^*(1)$ then quantile $q = 1$ benefits from deviating to $t^*(1)$. And if gradual play ends at $t_1 > t^*(1)$, then we have $u_t(t,1) < 0$ for all $t^*(1) < t < t_1$, but since $u_q < 0$ this violates the differential equation (3). Thus, during gradual play Q must satisfy:

$$u(t,Q(t)) = u(t^*(1),1) \tag{6}$$

When $u_q < 0$, a rush at time $t > 0$ is impossible, since pre-empting it dominates stopping in the rush. Since $u(t^*(1),1)$ is the Nash equilibrium payoff, inequality (4) rules out a time zero rush. Note that in the monotone quantile case $u_q < 0$, the left side of (4) reduces to $u(0,0)$. Absent a rush, the only possibility in this case is a pre-emption game for all quantiles satisfying (6), as in the left panel of Figure 3.

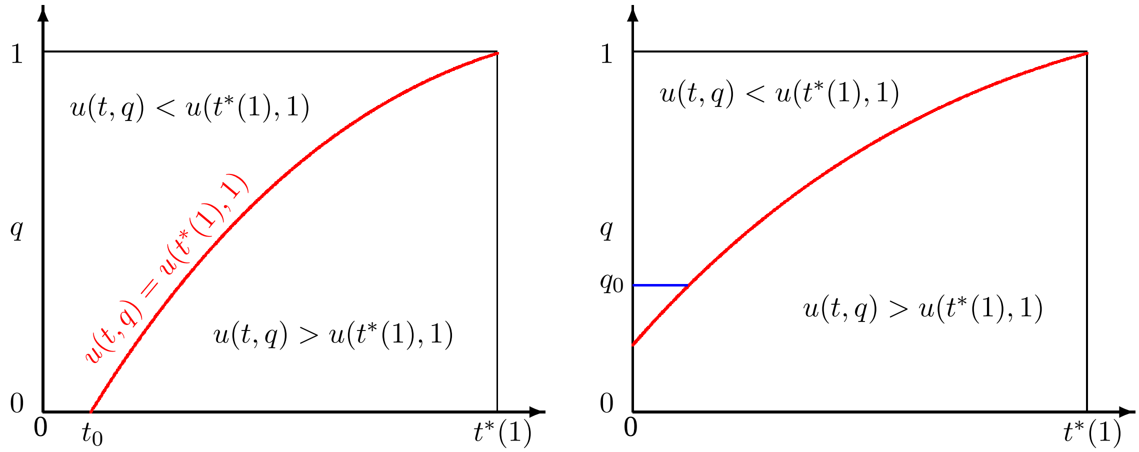

Figure 3: **Equilibria with Payoffs Decreasing in Quantiles.** In both panels, we assume $u_q < 0$. The indifference curves $u(t,q) = \bar{u}$ are upward at times $t < t^*(1)$. When $u(0,0) \le u(t^*(1),1)$, as in the left panel, the red indifference curve $u(t,q) = u(t^*(1),1)$ defines the unique equilibrium. In the right panel, $u(0,0) > u(t^*(1),1)$, as with alarm and panic. In this case, the indifference curve $u(t,q) = u(t^*(1),1)$ intersects the q-axis at $q > 0$, implying there cannot be gradual play for all quantiles. Given alarm, the equilibrium involves a rush of size q_0 at $t = 0$, followed by a period of inaction along the blue line, and then gradual play along the red indifference curve.

Panic rules out all but a unit mass rush at time zero, since it implies $V_0(0,q) > u(t^*(1),1)$ for all q; any equilibrium with gradual play has Nash payoff $u(t^*(1),1)$. Given alarm, stopping immediately for payoff $u(0,0)$ dominates stopping during gradual play for payoff $u(t^*(1),1)$, and thus a rush must occur. But given $V_0(0,1) < u(t^*(1),1)$, any such rush must be of size $q_0 < 1$, and the indifference condition $V_0(0,q_0) = u(t^*(1),1)$ then pins down the rush size. But $V_0(0,q_0) > u(0,q_0)$ given $u_q < 0$, so that stopping just after the time zero rush affords a strictly lower payoff than stopping in the rush. This forces an *inaction phase* — a time interval $[t_1,t_2]$, where $0 < Q(t_1) = Q(t_2) < 1$, as seen in the right panel of Figure 3.

Appendix C completes the proof of the following result when u is strictly monotone in q.

Proposition 1 *Assume the stopping payoff is strictly monotone in quantiles. There is a unique equilibrium. If $u_q > 0$, a war of attrition with all quantiles starts at $t^*(0)$. When $u_q < 0$, either:*
(a) *with neither alarm nor panic there is a pre-emption game for all quantiles ending at $t^*(1)$;*
(b) *with alarm there is a time-0 rush of size q_0 obeying $V_0(0,q_0) = u(t^*(1),1)$, followed by an inaction phase, and then a pre-emption game ending at $t^*(1)$;*
(c) *with panic there is a unit mass rush at time $t = 0$.*

5 Interior Single-Peaked Payoffs in Quantile Lead to Rushes

We have just seen that alarm or panic lead to an initial rush. With an interior peak quantile, two other types of rushes are possible: A *terminal rush* happens when $0 < Q(t-) = q < 1 = Q(t)$, so that all quantiles in $[q, 1]$ rush at the same time t, collecting terminal rush payoff $V_1(q,t) \equiv (1-q)^{-1}\int_q^1 u(t,x)dx$. A *unit mass rush* occurs when $0 = Q(t-) < 1 = Q(t)$, so that all quantiles $[0, 1]$ rush at the same time t. The latter is a pure strategy Nash equilibrium.

Proposition 2 *If payoffs are non-monotone in quantile, only three types of equilibria can occur:*
(a) *A* pre-emption equilibrium*: an initial rush followed by an uninterrupted pre-emption phase ending at harvest time $t^*(1)$ if and only if there is not greed at time $t^*(1)$.*
(b) *A* war of attrition equilibrium*: a terminal rush preceded by an uninterrupted war of attrition phase starting at harvest time $t^*(0)$ if and only if there is not fear at time $t^*(0)$ and no panic.*
(c) *A unit rush at times in an open interval around t^* absent greed at $t^*(1)$ and absent fear at $t^*(0)$. Unit mass rushes cannot occur at any positive time with strict greed or strict fear.*

Notice that if strict fear obtains for all time t, Proposition 2 predicts a pre-emption equilibrium, while strict greed for all t implies a war of attrition.

There is exactly one rush with an interior peak quantile. For purely gradual play requires that early quantiles (for which $u_q > 0$) stop later (when $u_t < 0$) and later quantiles (for which $u_q < 0$) stop earlier (when $u_t > 0$). On the hand, we cannot have more than one rush, since — as we show in Lemma D.1 — all rushes must include the quantile peak, for otherwise, either pre-empting or "post-empting" the rush would yield a higher payoff. Still, Propositions 1 and 2 share common elements: equilibria include at most one rush and at most one time interval of gradual play, either a *pre-emption phase* ending at $t^*(1)$ or a *war of attrition phase* ending at $t^*(0)$.

The same payoff $u(t^*(1), 1)$ obtains for any pre-emption equilibrium; likewise, the same payoff $u(t^*(0), 0)$ obtains for any war of attrition. Nonetheless, there is an amazingly rich array of equilibria. For example, by varying the timing and size of the terminal rush, along with the length of the inaction phase separating gradual play from the terminal rush, one can generally construct a continuum of war of attrition equilibria whenever one exists. We explore the interplay between rush size and timing across gradual play equilibria of each type in Appendix A.

We introduce a natural refinement for timing games that prunes the equilibrium set. Let $w(t; Q) \equiv u(t, Q(t))$ be the payoff to stopping at time $t \geq 0$ given cdf Q. The ε*-secure payoff* at t is:

$$w_\varepsilon(t; Q) \equiv \inf_{s\geq 0}\{w(s; Q)|t-\varepsilon \leq s \leq t+\varepsilon\}$$

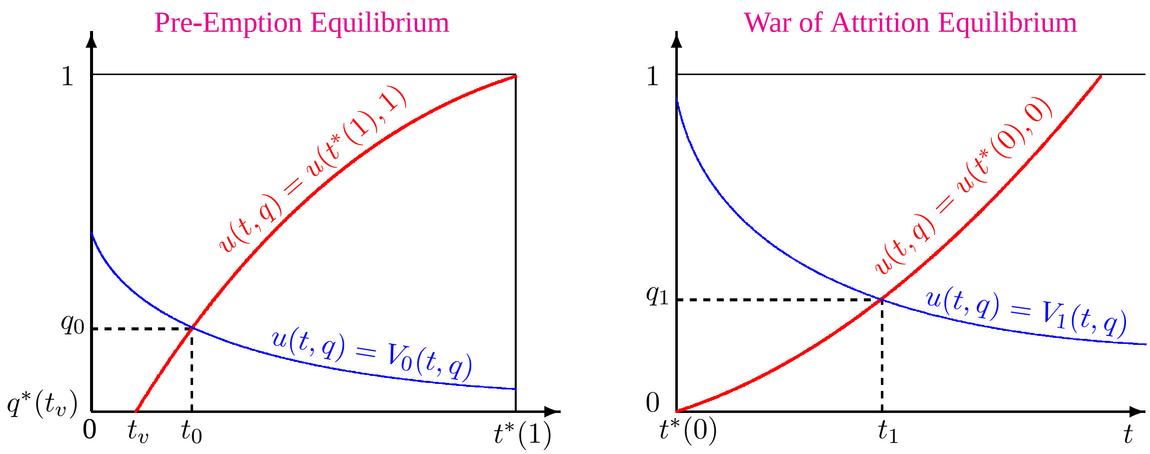

Figure 4: **Safe Equilibria with Non-Monotone Payoffs.** The left graph depicts the pre-emption equilibrium without inaction phases in Proposition 3. A partial initial rush q_0 at time t_0 occurs where the upward sloping gradual play locus (6) intersects the downward sloping peak rush locus (7). Gradual play in the pre-emption phase then follows the gradual play locus on $(t_0, t^*(1))$. In the safe war of attrition equilibrium, gradual play begins at $t^*(0)$, following the upward sloping gradual play locus (5), ending in a terminal rush of quantiles $[q_1, 1]$ at time t_1 where the loci cross.

This is the minmax payoff in the richer model when individuals have access to two different ε-accurate timing technologies: One clock never runs late, and one never runs early. An equilibrium Q is *secure* if $w_\varepsilon(t; Q) = w(t; Q)$ for all t in the support of Q, and all small enough $\varepsilon > 0$. These are the only Nash equilibria that are robust to small timing mistakes.

Proposition 3 (a) *An equilibrium is safe if and only if it is secure.* (b) *There is a unique secure war of attrition equilibrium if there is no fear at harvest time $t^*(0)$. Given no greed at harvest time $t^*(1)$, there exists a unique secure equilibrium:* (i) *with neither alarm nor panic, an initial rush followed immediately by a pre-emption game;* (ii) *with alarm, a $t = 0$ rush followed by a period of inaction and then a pre-emption game; or* (iii) *with panic, a unit mass rush at $t = 0$.*

Here we see that panic and alarm have the same implications as in the monotone decreasing, $u_q < 0$, case analyzed by Proposition 1. In contrast, when neither panic nor alarm obtains, secure equilibria must have both a rush and a gradual play phase and no inaction: (i) an initial rush at $0 < t_0 < t^*(1)$, followed by a pre-emption phase on $[t_0, t^*(1)]$; or (ii) a war of attrition phase on $[t^*(0), t_1]$ ending in a terminal rush at t_1. In each case, the safe equilibrium is fully described by two key curves (Figure 4) that together constitute the quantile function:

- *The gradual play locus,* either (5) or (6) by the same logic as in the monotone case, is upward sloping since, respectively, u_q and u_t have opposite signs in gradual play.

- The *peak rush locus* $i = 0, 1$ is described by $q_i(t) \in \arg\max_q V_i(t, q)$, and so indifference between payoffs in the rush and in adjacent gradual play ("marginal equals average"):

$$u(t, q_i(t)) = V_i(t, q_i(t)) \tag{7}$$

Each is well defined and *decreasing* by log-submodularity of $u(t, q)$, as Lemma D.4 shows.

6 Stopping Rates in Gradual Play

Having found that play consists of rushes and gradual play, we now characterize behavior during the latter. First, we can immediately deduce that *stopping rates start at zero in a war of attrition and end at zero in a pre-emption game.* To see why, recall that wars of attrition begin at $t^*(0)$ and pre-emption games end at $t^*(1)$, by Proposition 2. Since $u_t(t^*(q), q) = 0$ defines the harvest time, the first term of the gradual play equation (3) vanishes at the start of a war of attrition and end of a pre-emption game. It follows at once that $Q'(t^*(0)) = 0$ and $Q'(t^*(1)) = 0$ in these two cases, since $u_q \neq 0$ at the two quantile extremes $q = 0, 1$.

Now we characterize whether stopping rates are increasing or decreasing in each phase of gradual play. Differentiating the differential equation (3) yields:

$$Q'' = -\frac{1}{u_q}\left[u_{tt} + 2u_{qt}Q' + u_{qq}(Q')^2\right]$$

Replacing the marginal rate of substitution $Q' = -u_t/u_q$ implied by (3), we find:

$$Q'' = \frac{1}{u_q^3}\left[2u_{qt}u_qu_t - u_{tt}u_q^2 - u_{qq}u_t^2\right] \tag{8}$$

The bracketed expression is positive when u is log-concave in t,[3] implying $Q'' \gtrless 0 \Leftrightarrow u_q \gtrless 0$. Payoff log-concavity is only used here, but yields a sharp characterization of gradual play:

Proposition 4 (Stopping during Gradual Play) *Assume the payoff function is log-concave in t. In any gradual play phase, the stopping rate $Q'(t)$ is strictly increasing in time from zero during a war of attrition phase, and decreasing down to zero during a pre-emption game phase.*

This result applies to all equilibria in Propositions 1 and 2 — even with inaction phases. We see that war of attrition equilibria are described by waxing exits, possibly climaxing in a rush,

[3] Since $u_qu_t < 0$, the log-submodularity inequality $u_{qt}u \leq u_qu_t$ can be reformulated as $2u_{qt}u_qu_t \geq 2u_q^2u_t^2/u$, or (★). But log-concavity in t and q implies $u_t^2 \geq u_{tt}u$ and $u_q^2 \geq u_{qq}u$, and so $2u_q^2u_t^2/u \geq u_{tt}u_q^2 + u_{qq}u_t^2$. Combining this with (★), we find $2u_{qt}u_qu_t \geq u_{tt}u_q^2 + u_{qq}u_t^2$.

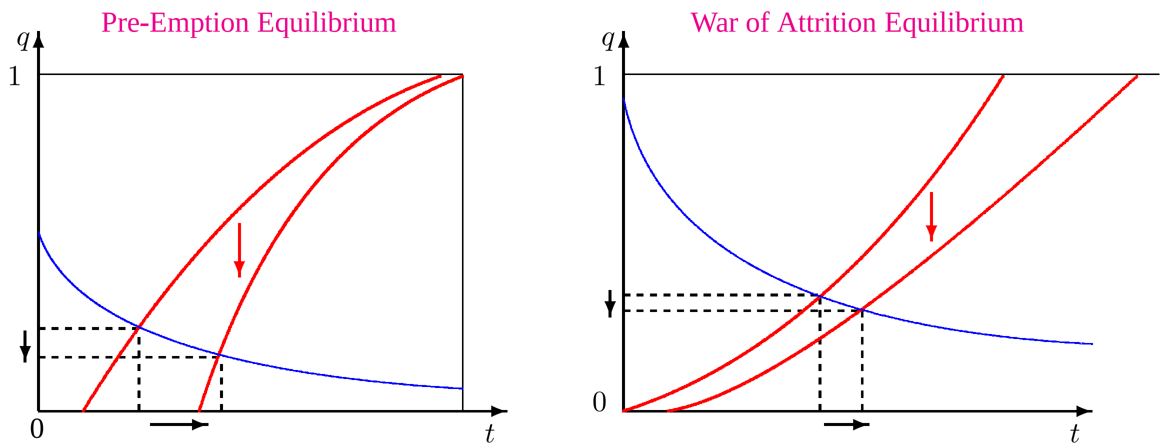

Figure 5: **Harvest Time Delay.** The gradual play locus shifts down in ϕ. In a pre-emption game (left): A smaller initial rush occurs later and stopping rates rise during gradual play. In a war of attrition: A larger terminal rush occurs later, while stopping rates fall during gradual play.

whereas pre-emption equilibria may begin with a rush, but conclude with waning gradual play. So wars of attrition intensify whereas pre-emption games taper off. Figure 4 reflects these facts, since the stopping indifference curve is (i) concave after the initial rush during any pre-emption equilibrium, and (ii) convex prior to the terminal rush during any war of attrition equilibrium.

7 Joint Predictions about Gradual Play and Rushes

We now explore how our equilibria evolve as our two key aspects of the economic environment change: (a) fundamentals adjust so as to advance or postpone the harvest time, or (b) the strategic interaction alters to change quantile rewards, increasing fear or greed.

First consider an acceleration in the rate of change of the stopping payoff in time — essentially, speeding up the passage of time. To this end, let us parameterize the stopping payoff by ϕ, so that $u(t, q|\phi)$ is log-supermodular in (t, ϕ) and log-modular in (q, ϕ). In this way, ϕ scales up the impact of time changes, but leaves unaffected the role of quantiles. An increase in ϕ is a *harvest delay*, since $t^*(q|\phi)$ increases. This change affects gradual play and rushes.

Proposition 5 (Fundamental Changes) *Consider safe equilibria with a harvest delay. Stopping shifts stochastically later. In a war of attrition, stopping rates fall, and a larger terminal rush occurs later. In a pre-emption game, stopping rates rise, and a smaller initial rush happens later.*

Observe that a harvest time delay postpones all activity, but nevertheless intensifies stopping rates during a pre-emption game. Also, an inverse relation between stopping rates and rush size emerges, with higher stopping rates during gradual play associated to smaller rushes.

PROOF: We give the arguments for the safe pre-emption equilibrium, assuming an interior peak quantile (the safe war of attrition proof is similar). Figure 5 depicts the logic for each case, using our two key equilibrium curves. The same gradual play conditions apply for the monotone u case covered by Proposition 1. Our proof below is complete except for the case of a monotone quantile function with alarm (completed in the proof of Lemma E.2).

Since the marginal payoff u is log-modular in (t, ϕ) so is the average. So the maximum $q_0(t) \in \arg\max_q V_0(t, q|\phi)$ is constant in ϕ. Rewrite the pre-emption gradual play locus (6) as:

$$\frac{u(t, Q(t)|\phi)}{u(t, 1|\phi)} = \frac{u(t^*(1|\phi), 1|\phi)}{u(t, 1|\phi)} \tag{9}$$

The LHS of (9) falls in Q since $u_q < 0$ during a pre-emption game, and is constant in ϕ, by log-modularity of u in (q, ϕ). Differentiating the RHS in ϕ and using the Envelope Theorem:

$$\frac{u_\phi(t^*(1|\phi), 1|\phi)}{u(t^*(1|\phi), 1|\phi)} - \frac{u_\phi(t, 1|\phi)}{u(t, 1|\phi)} > 0$$

since u is log-supermodular in (t, ϕ) and $t < t^*(1|\phi)$ during a pre-emption game. Since the RHS of (9) increases in ϕ and the LHS decreases in Q, the gradual play locus $Q(t)$ obeys $\partial Q/\partial \phi < 0$.

Next, rewrite the differential equation (3) for the quantile function during gradual play as:

$$Q'(t) = -\frac{u_t(t, Q(t)|\phi)}{u(t, Q(t)|\phi)} \frac{u(t, Q(t)|\phi)}{u_q(t, Q(t)|\phi)} \tag{10}$$

Differentiating in ϕ, and using log-modularity in (q, ϕ), yields:

$$\frac{\partial Q'(t)}{\partial \phi} = -\left[\left(\frac{\partial [u_t/u]}{\partial \phi} + \frac{\partial [u_t/u]}{\partial Q} \frac{\partial Q}{\partial \phi} \right) \frac{u}{u_q} + \frac{u_t}{u} \left(\frac{\partial [u/u_q]}{\partial Q} \frac{\partial Q}{\partial \phi} + \frac{\partial [u/u_q]}{\partial \phi} \right) \right] > 0$$

The first term in brackets is negative. Indeed, $\partial[u_t/u]/\partial\phi > 0$ by log-supermodularity in (t, ϕ), $\partial[u_t/u]/\partial Q < 0$ by log-submodularity in (t, q), $\partial Q/\partial\phi < 0$ as shown above, and $u_q < 0$ during a pre-emption game. To see that the second term is also negative, note that $u_t > 0$ during a pre-emption game, $\partial[u/u_q]/\partial Q \geq 0$ by log-concavity of $u(q, t)$ in q. $\square$

Next consider pure changes in quantile preferences, by smoothly indexing the stopping payoff by $\gamma \in \mathbb{R}$, so that $u(t, q|\gamma)$ is log-supermodular in (q, γ) and log-modular in (t, γ). In other words, increases in γ inflate the relative return to a quantile delay. Given this log-supermodular structure, the quantile peak $q^*(t|\gamma)$ rises in γ. Define the cardinal measures of greed and fear at time t as

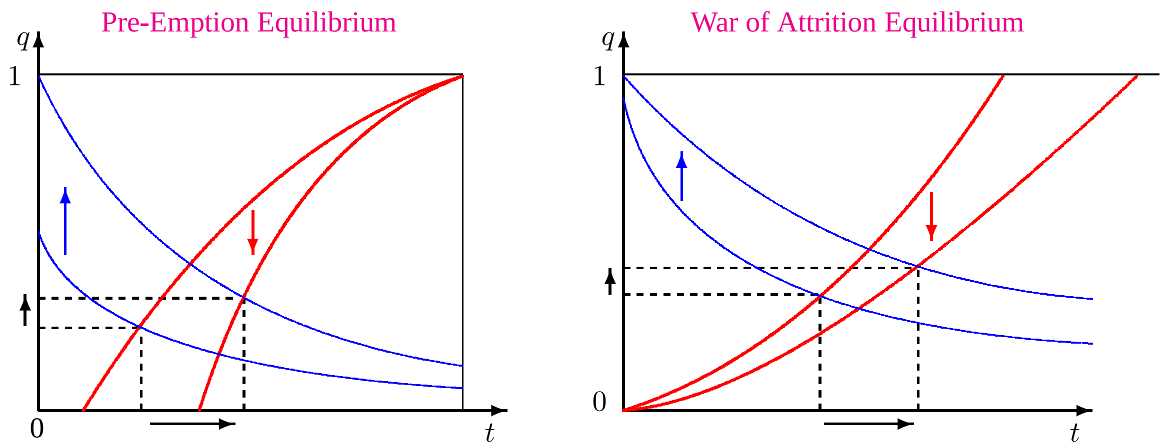

Figure 6: **Monotone Quantile Payoff Changes.** An increase in greed (or a decrease in fear) shifts the gradual play locus down and the locus equating the payoff in the rush to the adjacent gradual play payoff up. In the safe pre-emption equilibrium (left): Larger rushes occur later and stopping rates *rise* on shorter pre-emption games. In the safe war of attrition equilibrium: Smaller rushes occur later and stopping rates *fall* during longer wars of attrition.

the ratios $u(t,1|\gamma)/\int_0^1 u(t,x|\gamma)dx$ and $u(t,0|\gamma)/\int_0^1 u(t,x|\gamma)dx$, respectively. Easily, greed rises and fear falls in γ: Payoffs shift towards later ranks as γ rises; this relatively diminishes potential losses of pre-emption, and relatively inflates the potential gains from later ranks.

Proposition 6 (Quantile Changes) *In the safe war of attrition equilibrium, as greed increases, gradual play lengthens, stopping rates fall, the stopping distribution shifts later, and terminal rush shrinks. In the safe pre-emption equilibrium, as fear rises, stopping rates fall and the stopping distribution shifts earlier; also:* (a) *without alarm the pre-emption game lengthens and the initial rush shrinks,* (b) *with alarm, the pre-emption game shortens and the initial rush grows.*

PROOF: Since the stopping payoff is log-modular in (t,γ), the harvest time $t^*(0)$ at which a war of attrition begins and the harvest time $t^*(1)$ at which a pre-emption game ends are both constant in γ. As in the Proof of Proposition 5, we focus on the pre-emption case and delay the final steps for a monotone quantile function with alarm to Lemma E.2.

Define $\mathbb{I}(q,x) \equiv q^{-1}$ for $x \leq q$ and 0 otherwise, and thus $V_0(t,q|\gamma) = \int_0^1 \mathbb{I}(q,x)u(t,x|\gamma)dx$. Easily, $\mathbb{I}$ is log-supermodular in (q,x), and so the product $\mathbb{I}(\cdot)u(\cdot)$ is log-supermodular in (q,x,γ). Thus, V_0 is log-supermodular in (q,γ) since log-supermodularity is preserved by integration by Karlin and Rinott (1980). So the peak rush locus $q_0(t) = \arg\max_q V_0(t,q|\gamma)$ rises in γ.

Rewrite the gradual play locus (6) as:

$$\frac{u(t,Q(t)|\gamma)}{u(t,1|\gamma)} = \frac{u(t^*(1),1|\gamma)}{u(t,1|\gamma)} \tag{11}$$

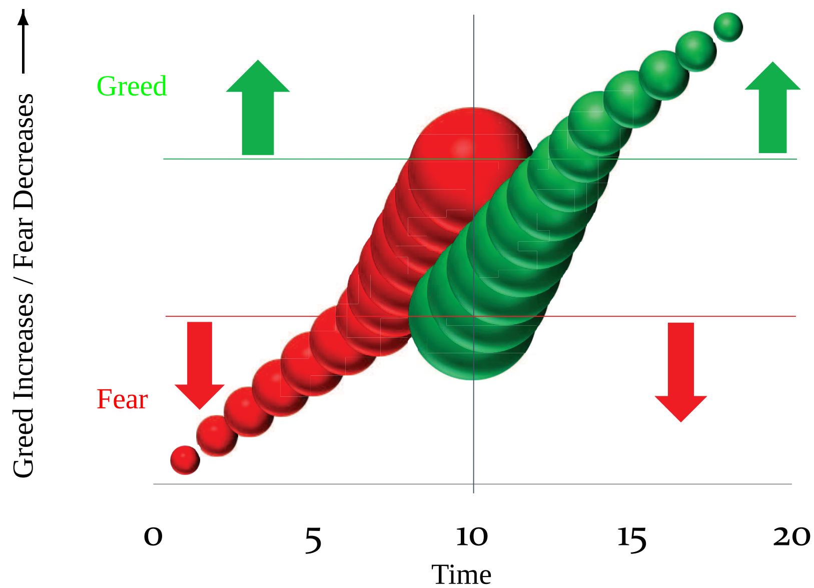

Figure 7: **Rush Size and Timing.** Centered at rush times, circles are proportional to rush sizes. As fear lessens (upward), the unique safe pre-emption equilibrium involves a larger initial rush, closer to the harvest time ($t^* = 10$), followed by a shorter pre-emption phase. As greed rises (upward), the unique safe war of attrition equilibrium involves a longer war of attrition, followed by a smaller terminal rush.

The RHS is constant in γ since $u(t,q|\gamma)$ log-modular in (t,γ), and t^* is constant in γ. Meanwhile the LHS falls in γ since u is log-supermodular in (q,γ), and falls in Q since $u_q < 0$ during a pre-emption game. Altogether, the gradual play locus satisfies $\partial Q/\partial\gamma < 0$.

Differentiate an analogous differential equation for gradual play (10), substituting γ for ϕ:

$$\frac{\partial Q'}{\partial \gamma} = -\left[\frac{\partial [u_t/u]}{\partial Q}\frac{\partial Q}{\partial \gamma}\frac{u}{u_q} + \frac{u_t}{u}\left(\frac{\partial [u/u_q]}{\partial \gamma} + \frac{\partial [u/u_q]}{\partial Q}\frac{\partial Q}{\partial \gamma}\right)\right] > 0$$

by log-modularity in (t,γ). The first term in brackets is negative, since $\partial[u_t/u]/\partial Q < 0$ by log-submodularity in (t,q), $\partial Q/\partial\gamma < 0$ as shown above, and $u_q < 0$ in a pre-emption game. The second term is also negative: $\partial[u/u_q]/\partial\gamma < 0$ by log-supermodularity in (q,γ), $\partial[u/u_q]/\partial Q > 0$ by log-concavity in q, $\partial Q/\partial\gamma < 0$, and $u_t > 0$ in a pre-emption game.

All told, an increase in γ: (i) has no effect on the harvest time; (ii) shifts the gradual play locus (6) down and makes it steeper; and (iii) shifts the peak rush locus (7) up (see Figure 6). Finally, Appendix E establishes that the the peak rush locus determines changes in rush size. $\square$

Now consider a more general monotone shift. Specifically, assume $u(t,q|\varphi)$ log-supermodular in (t,φ) and (q,φ). Then, an increase in φ is a *co-monotone delay*, since the harvest time t^* and the peak quantile q^* both increase in φ. Intuitively, an increase in φ intensifies the game, by proportionally increasing the payoffs in time and quantile space. The insights from Proposition 5 and 6 carry over to this more general change.

Corollary 1 *Consider safe equilibria with a monotone delay. Stopping shifts stochastically later. Stopping rates fall in a war of attrition and rise in a pre-emption game. Given alarm, the time zero initial rush shrinks.*

8 Economic Applications Distilled from the Literature

We conclude by showing how our theory speaks to some famous and well-studied timing games. Our aim is to highlight how these diverse applications fit our common structure, abstracting from details that are surely important features of each example.

8.1 The Rush to Match

We now consider assignment rushes. As with the entry-level gastroenterology labor market in Niederle and Roth (2004) [NR2004], we assume that early matching costs include the "loss of planning flexibility", whereas the penalty for late matching owes to market thinness. For a cost of early matching, we simply follow Avery, et al. (2001) who allude to the condemnation of early match agreements. So we posit a negative stigma to early matching relative to peers.

For a model of this, assume an equal mass of two worker varieties, A and B, each with a continuum of uniformly distributed abilities $\alpha \in [0,1]$. Firms have an equal chance of needing a type A or B. For simplicity, we assume the payoff of hiring the wrong type is zero, and that each firm learns its need at fixed exponential arrival rate $\delta > 0$. Thus, the chance that a firm chooses the right type if it waits until time t to hire is $e^{-\delta t}/2 + \int_0^t \delta e^{-\delta s} ds = 1 - e^{-\delta t}/2$.[4] Assume that an ability α worker of the right type yields flow payoff α, discounted at rate r. Thus, the present value of hiring the right type of ability α worker at time t is $(\alpha/r)e^{-rt}$.

Consider the quantile effect. Assume an initial ratio $2\theta \in (0,2)$ of firms to workers (market *tightness*). If a firm chooses before knowing its type, it naturally selects each type with equal chance; thus, the best remaining worker after quantile q of firms has already chosen is $1-\theta q$. We also assume a *stigma* $\sigma(q)$, with payoffs from early matching multiplicatively scaled by $1-\sigma(q)$, where $1 > \sigma(0) \geq \sigma(1) = 0$, and $\sigma' < 0$. All told, the payoff is multiplicative in time and quantile concerns:

$$u(t,q) \equiv r^{-1}(1-\sigma(q))(1-\theta q)\left(1-e^{-\delta t}/2\right)e^{-rt} \tag{12}$$

[4] We assume firms unilaterally choose the start date t. One can model worker preferences over start dates by simply assuming the actual start date T is stochastic with a distribution $F(T|t)$.

This payoff is log-concave in t, and initially increasing provided the learning effect is strong enough ($\delta > r$). This stopping payoff is concave in quantile q if σ is convex.

This matching payoff (12) is multiplicative in time and quantile, and therefore always exhibits greed, or fear, or neither. Specifically, there is fear whenever $\int_0^1 (1-\sigma(x))(1-\theta x)dx \leq 1-\sigma(0)$, i.e. when the stigma σ of early matching is low relative to the firm demand (tightness) θ. In this case, Proposition 2 predicts a pre-emption equilibrium, with an initial rush followed by a gradual play phase, and Proposition 4 asserts a waning matching rate, since payoffs are log-concave in time. Likewise, there is greed whenever $\int_0^1 (1-\sigma(x))(1-\theta x)dx \leq 1-\theta$. This holds when the stigma σ of early matching is high relative to the firm demand θ. Here, Proposition 2 predicts a war of attrition equilibrium, namely, gradual play culminating in a terminal rush, and Proposition 4 asserts that matching rates start low and rise towards that rush. When neither inequality holds, neither fear nor greed obtains, and so both types of gradual play as well as unit mass rushes are equilibria, by Proposition 4.

For an application, NR2004 chronicle the gastroenterology market. The offer distribution in their reported years (see their Figure 1) is roughly consistent with the pattern we predict for a pre-emption equilibrium as in the left panel of our Figure 4 — i.e., a rush and then gradual play. NR2004 highlight how the offer distribution advances in time ("unravelling") between 2003 and 2005, and propose that an increase in the relative demand for fellows precipitated this shift. Proposition 6 replicates this offer distribution shift. Specifically, assume the market exhibits fear, owing to early matching stigma. Since the match payoff (12) is log-submodular in (q, θ), fear rises in market tightness θ. So the rush for workers occurs earlier by Proposition 6, and is followed by a longer gradual play phase (left panel of Figure 6). This predicted shift is consistent with the observed change in match timing reported in Figure 1 of NR04.[5]

Next consider comparative statics in the interest rate r. Since the matching payoff is log-submodular in (t, r), lower interest rates entail a harvest time delay; therefore, the matching distribution shifts later by Proposition 5. In the case of a pre-emption equilibrium, smaller initial rushes occur later and matching is more intense as r falls, whereas for a war of attrition equilibrium, larger terminal rushes occur later, and stopping rates fall. Observe that in neither case is it true that more patience leads to less frantic matching. For in a pre-emption game, the gradual play is more frenzied, while in the war of attrition game, the rushes are larger. By contrast, in a sorority rush environment (Mongell and Roth, 1991), the extreme urgency corresponds to a high

[5]One can reconcile a tatonnement process playing out over several years, by assuming that early matching in the current year leads to lower stigma in the next year. Specifically, if the ratio $(1-\sigma(x))/(1-\sigma(y))$ for $x < y$, falls in response to earlier matching in the previous year, then a natural feedback mechanism emerges. The initial increase in θ stochastically advances match timing, further increasing fear; the rush to match occurs earlier in each year.

interest rate. Given a low stigma of early matching and a tight market (for the best sororities), this matching market exhibits fear and not greed; therefore, we have a pre-emption game, for which we predict a large and early initial rush, followed by a casual gradual play as stragglers match.

8.2 The Rush to Sell in a Bubble

We wish to parallel Abreu and Brunnermeier (2003) [AB2003], only dispensing with asymmetric information. A continuum of investors each owns a unit of the asset and chooses the time t to sell. Let $Q(t)$ be the fraction of investors that have sold by time t. There is common knowledge among these investors that the asset price is a bubble. As long as the bubble persists, the asset price $p(t|\xi)$ rises deterministically and smoothly in time t, but once the bubble bursts, the price drops to the fundamental value, which we normalize to 0.

The bubble explodes once $Q(t)$ exceeds a threshold $\kappa(t+t_0)$, where t_0 is a random variable with log-concave cdf F common across investors: Investors know the length of the "fuse" κ, but do not know how long the fuse had been lit before they became aware of the bubble at time 0. We assume that κ is log-concave, with $\kappa'(t+t_0)<0$ and $\lim_{t\to\infty}\kappa(t)=0$.[6] In other words, the burst chance is the probability $1-F(\tau(q,t))$ that $\kappa(t+t_0)\leq q$, where $\tau(q,t)$ uniquely satisfies $\kappa(t+\tau(q,t))\equiv q$. The expected stopping price, $F(\tau(q,t))p(t|\xi)$, is decreasing in q.[7]

Unlike AB, we allow for an interior peak quantile by admitting relative performance concerns. Indeed, most market activity on the NYSE owes to institutional investors acting on behalf of others, who are often paid for their performance relative to their peers. This imposes an extra cost to leaving a growing bubble early relative to other investors. For a simple model of this peer effect, scale stopping payoffs by $1+\rho q$, where $\rho\geq 0$ measures *relative performance concern*.[8] All told, the payoff from stopping at time t as quantile q is:

$$u(t,q)\equiv(1+\rho q)F(\tau(q,t))p(t|\xi) \tag{13}$$

In Appendix F, we argue that this payoff is log-submodular in (t,q), and log-concave in t and q. With $\rho=0$, the stopping payoff (13) is then monotonically decreasing in the quantile, and Proposition 1 predicts either a pre-emption game for all quantiles, or a pre-emption game

[6] By contrast, AB assume a constant function κ, but that the bubble eventually bursts exogenously even with no investor sales. Moreover, absent AB's asymmetric information of t_0, if the threshold κ were constant in time, players could perfectly infer the burst time $Q(t_\kappa)=\kappa$, and so strictly gain by stopping before t_κ.

[7] A rising price is tempered by the bursting chance in financial bubble models (Brunnermeier and Nagel, 2004).

[8] When a fund does well relative to its peers, it often experiences cash inflows (Berk and Green, 2004). In particular, Brunnermeier and Nagel (2004) document that during the tech bubble of 1998-2000, funds that rode the bubble longer experienced higher net inflow and earned higher profits than funds that sold significantly earlier.

preceded by a time $t = 0$ rush, or a unit mass rush at $t = 0$. But with a relatively strong concern for performance, the stopping payoff initially rises in the quantile, and thus the peak quantile q^* is interior. In this case, there is fear at time $t^*(0)$ for $\rho > 0$ small enough, and thus an initial rush followed by a pre-emption game with waning stopping rates, by Propositions 2 and 4. On the other hand, for high enough ρ, the stopping payoff exhibits greed at time $t^*(1)$, and we therefore predict an intensifying war of attrition, climaxing in a late rush, as seen in Griffin, Harris, and Topaloglu (2011). Finally, intermediate ranges of relative performance concern allow for two safe equilibria: either a war of attrition or a pre-emption game.

Turning to our comparative statics predictions in the fundamentals, recall that as long as the bubble survives, the price is $p(t|\xi)$. Since it is log-supermodular in (t, ξ), if ξ rises, then so does the rate p_t/p at which the bubble grows, and thus there is a harvest time delay. This stochastically postpones sales, by Proposition 5, and so not only does the bubble inflate faster, but it also lasts longer, since the selling pressure diminishes. Both findings are consistent with the comparative static derived in AB2003 that lower interest rates lead to stochastically later sales and a higher undiscounted bubble price. To see this, simply write our present value price as $p(t|\xi) = e^{\xi t}\hat{p}(t)$, i.e. let $\xi = -r$ and let $\hat{p}$ be their undiscounted price. Then the discounted price is log-submodular in (t, r) and so sales delay with a lower interest rate, leading to a higher undiscounted price.

For a similar quantile comparative static, AB2003 assume the bubble deterministically grows until the rational trader sales exceed a fixed threshold $\kappa > 0$. In their comparative static, they show that if κ increases, then bubbles last stochastically longer, and the price crashes are larger. Consider this exercise in our model. Assume any two quantiles $q_2 > q_1$. We found in §8.2 that the bubble survival chance $F(\tau(q, t))$ is log-submodular in (q, t), so that $F(\tau(q_2, t))/F(\tau(q_1, t))$ falls in t. Since the threshold $\kappa(t)$ is a falling function of time, lower t is equivalent to an upward shift in the κ function. Altogether, an upward shift in our κ function increases the bubble survival odds ratio $F(\tau(q_2, t))/F(\tau(q_1, t))$. In other words, the stopping payoff (13) is log-supermodular in q and κ. Proposition 6 then asserts that sales stochastically delay when κ rises. So the bubble bursts stochastically later, and the price drop is stochastically larger, as in AB2003.[9]

Shleifer and Vishny (1997) also allow for relative performance concerns. Specifically, in their model arbitrageurs rely on investors for funds, and the terms of trade depend on recent arbitrageur performance; positive early performance increases later profitability. In their basic model, the more sensitive are investors to recent arbitrageur performance, the more asset prices diverge

[9]Shleifer and Vishny (1997) find a qualitatively similar result in a model with noise traders, arbitrageurs, and investors. Their prices diverge from true values, and this divergence increases in the noise trader preference shocks. Such shocks act like increased κ in our model, as prices grow less responsive to rational trades, and in both cases, we predict a larger gap between price and fundamentals.

from true values. This parallels the comparative statics for relative performance concerns ρ in our model. Since our payoff (13) is log-supermodular in q and relative performance concerns ρ, thus increases in ρ have the same qualitative effects as increases in κ: bubbles last stochastically longer and thus prices rise stochastically higher prior to the bubble bursting.

8.3 Bank Runs

Bank runs are among the most fabled of rushes in economics. In the benchmark model of Diamond and Dybvig (1983) [DD], these arise because banks make illiquid loans or investments, but simultaneously offer liquid demandable deposits to individual savers. So if they try to withdraw their funds at once, a bank might be unable to honor all demands. In their elegant model, savers deposit money into a bank in period 0. Some consumers are unexpectedly struck by liquidity needs in period 1, and withdraw their money plus an endogenous positive return. In an efficient Nash equilibrium, all other depositors leave their money untouched until period 2, whereupon the bank finally realizes a fixed positive net return. But an inefficient equilibrium also exists, in which all depositors withdraw in period 1 in a bank run that over-exhausts the bank savings, since the bank is forced to liquidate loans, and forego the positive return.[10]

We adapt the withdrawal timing game, abstracting from optimal deposit contract design.[11] Given our homogeneous agent model, we ignore private liquidity shocks. A unit continuum of players $[0, 1]$ have deposited their money in a bank. The bank divides deposits between a safe and a risky asset, subject to the constraint that at least fraction R be held in the safe asset as reserves. The safe asset has log-concave discounted expected value $p(t)$, satisfying $p(0) = 1$, $p'(0) > 0$ and $\lim_{t\to\infty} p(t) = 0$. The present value of the risky asset is $p(t)(1-\zeta)$, where the proportional shock $\zeta \leq 1$ has twice differentiable cdf $H(\zeta|t)$ that is log-concave in ζ and t and log-supermodular in (ζ, t). To balance the risk, we assume this shock has positive expected value: $E[-\zeta] > 0$.

As long as the bank is solvent, depositors can withdraw $\alpha p(t)$, where the *payout rate* $\alpha < 1$, i.e. the bank makes profit $(1 - \alpha)p(t)$ on safe reserves. Since the expected return on the risky asset exceeds the safe return, the profit maximizing bank will hold the minimum fraction R in the safe asset, while fraction $1 - R$ will be invested in the risky project. Altogether, the bank will pay depositors as long as total withdrawals $\alpha q p(t)$ fall short of total bank assets $p(t)(1 - \zeta(1 - R))$, i.e. as long as $\zeta \leq (1 - \alpha q)/(1 - R)$. The stopping payoff to withdrawal at time t as quantile q is:

$$u(t, q) = H\left((1 - \alpha q)/(1 - R)|t\right) \alpha p(t) \tag{14}$$

[10] As DD admit, with a deposit choice in the first period, if depositors rationally anticipate a run, they avoid it.

[11] Thadden (1998) has shown that the ex ante efficient contract is not possible in a continuous time version of DD.

Clearly u is decreasing in q and log-concave in t, and log-submodular in (t, q), since $H(\zeta|t)$ log-supermodular in (ζ, t).

Since the stopping payoff (14) is weakly falling in q, bank runs occur immediately or not at all, by Proposition 1, in the spirit of Diamond and Dybvig (1983) [DD]. But unlike in DD, Proposition 1 predicts a unique equilibrium that may or may not entail a bank run. Specifically, a bank run is avoided *iff* fundamentals $p(t^*(1))$ are strong enough, since (4) is equivalent to:

$$u(t^*(1), 1) = H((1-\alpha)/(1-R)|t^*(1))p(t^*(1)) \geq u(0,0) = H(1/(1-R)|0) = 1 \tag{15}$$

Notice how bank runs do not occur with a sufficiently high reserve ratio or low payout rate. When (15) is violated, the size of the rush depends on the harvest time payoff $u(t^*(1), 1)$. When the harvest time payoff is low enough, panic obtains and all depositors run. For intermediate harvest time payoffs, there is alarm. In this case, Proposition 1 (b) fixes the size q_0 of the initial run via:

$$q_0^{-1} \int_0^{q_0} H((1-\alpha x)/(1-R)|0)dx = H((1-\alpha)/(1-R)|t^*(1))p(t^*(1)) \tag{16}$$

Since the left hand side of (16) falls in q_0, the run shrinks in the peak asset value $p(t^*(1))$ or in the hazard rate H'/H of risky returns.

There is a log-submodular payoff interaction between the payout α and both time and quantiles, as Appendix F proves. Hence, Corollary 1 predicts three consequences of a higher payout rate: withdrawals shift stochastically earlier, the bank run grows (with alarm), and withdrawal rates fall during any pre-emption phase. Next consider changes in the reserve ratio. The stopping payoff is log-supermodular in (t, R), since $H(\zeta|t)$ log-supermodular, and log-supermodular in (q, R) provided the elasticity $\zeta H'(\zeta|t)/H(\zeta|t)$ is weakly falling in ζ (proven in Appendix F). Corollary 1 then predicts that a reserve ratio increase shifts the distribution of withdrawals later, shrinks the bank run, and increases the withdrawal rate during any pre-emption phase.[12]

A Equilibria with Inaction Phases

Given an interior peak quantile, Propositions 5 and 6 yield covariate predictions across safe equilibria. We now consider covariate predictions across all equilibria with gradual play.

[12] An increase in the reserve ratio increases the probability of being paid at the harvest time, but it also increases the probability of being paid in any early run. Log-concavity of H is necessary, but not sufficient, for the former effect to dominate: This requires our stronger monotone elasticity condition.

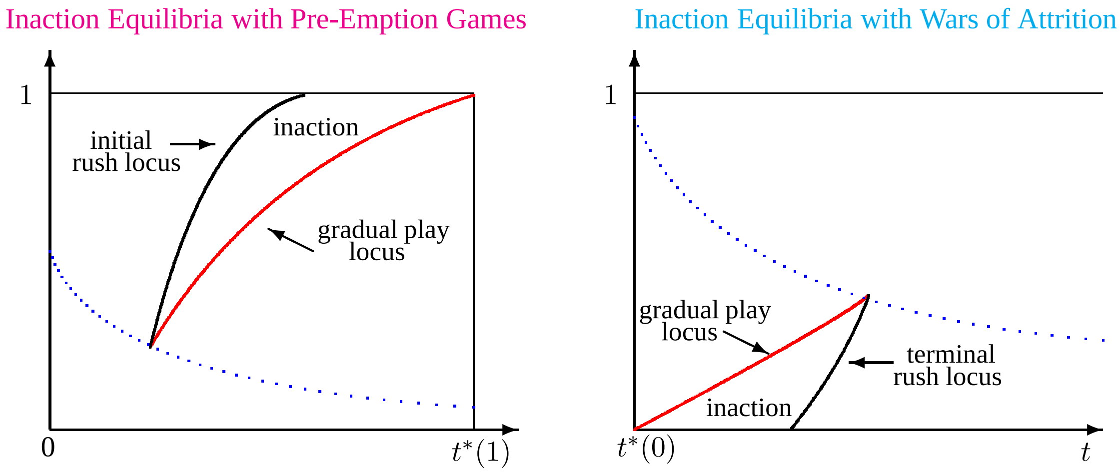

Figure 8: **Inaction Equilibria.** The left graph illustrates all equilibria with pre-emption games: an initial rush occurs on the initial rush locus, followed by no stopping during an inaction phase; then a pre-emption game follows the gradual play locus. The right graph illustrates all equilibria with wars of attrition: beginning at $t^*(0)$ gradual play follows the indicated gradual play locus; then allow no stopping during an inaction phase that ends in a terminal rush with the rush time and lowest quantile included in the rush given by the terminal rush locus. In each case the dotted line equates the rush payoff with the adjacent gradual play payoff.

Proposition 7 (Changing Inaction Phases) *Assume the inaction phase lengthens. In equilibria with pre-emption games, the quantile function during gradual play is identical, and larger rushes occur later and the gradual play shrinks. In equilibria with a war of attrition, the stopping cdf during gradual play is identical, and larger rushes occur earlier and the gradual play shrinks.*

Behavior is consistent across all early (or late) equilibria with phases of inaction: Inaction phases inflate rush size and shorten the gradual play phase. Figure 8 illustrates the set of equilibria. These covariate predictions of rush size, timing, and gradual play length coincide with Proposition 6.

PROOF STEP 1: RUSH LOCI. By Proposition 2, gradual play ends at $t^*(1)$ following an initial rush, while any terminal rush must be preceded by gradual play beginning at $t^*(0)$. Equality between the rush and gradual play payoff fixes the set of allowable rush times and rush sizes:

$$\text{Initial Rushes:} \quad V_0(t,q) = u(t^*(1),1) \quad \text{and} \quad \text{Terminal Rushes:} \quad V_1(t,q) = u(t^*(0),0) \quad (17)$$

PROOF STEP 2: RUSH SIZE/TIME COVARIANCE. Differentiating the initial rush locus (17):

$$q^{-1}\left[u(t,q) - V_0(t,q)\right] dq + q^{-1}\left[\int_0^q u_t(t,x)dx\right] dt = 0 \quad (18)$$

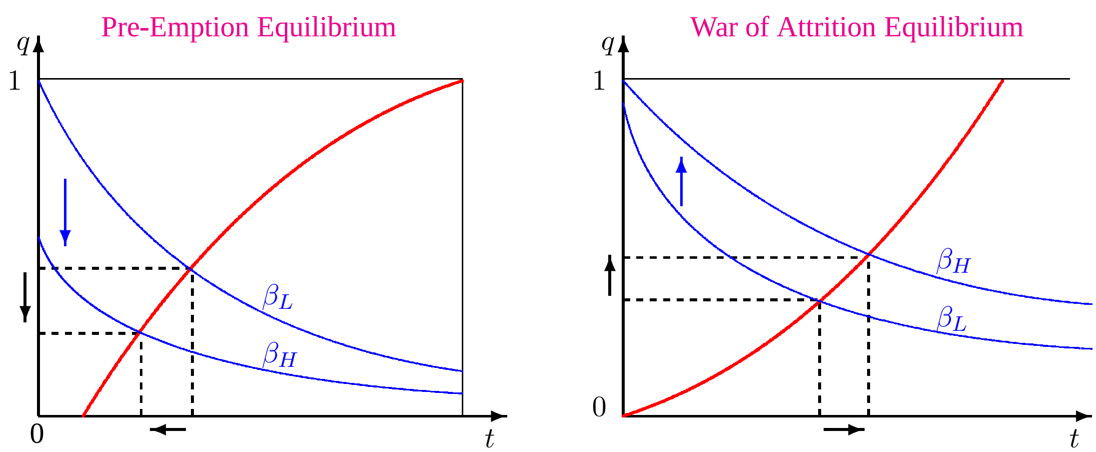

Figure 9: **Geometric Payoff Transformations.** Assume a payoff transformation $u(t,q)^\beta$. The (thick) gradual play locus is constant in β, while the (thin) peak rush locus shifts down in β for a pre-emption equilibrium (left graph) and up in β for a war of attrition equilibrium.

Since gradual play must end at $t^*(1)$, the initial rush must take place before $t^*(1)$. Given t^* non increasing, we must have $u_t(t,x) > 0$ for all x in any initial rush, implying $\int_0^q u_t(t,x)dx > 0$ and $u(t,q)$ rising during the inaction phase. Given $u(t,q)$ rising during an inaction phase, we must have $V_0(t,q) > u(t,q)$ for any initial rush followed by inaction, else stopping during the inaction phase would yield a strictly higher payoff then stopping in the rush.Altogether, for any initial rush followed by an inaction phase we may rearrange (18) to discover:

$$dq/dt = \frac{\int_0^q u_t(t,x)dx}{V_0(t,q) - u(t,q)} > 0$$

Similar reasoning establishes that the terminal rush locus is increasing. □

B Geometric Payoff Transformations

We have formulated greed and fear as descriptions of the strategic environment, in terms of quantile preference. An alternative heuristic use of greed and fear imagines them instead as descriptions of individual risk preference. For example, as a convex or concave transformations of the stopping payoff. Given a deterministic stopping payoff, such transformations correspond to increases or decreases in risk aversion. But, as we have seen, some applications demand a stopping payoff that is itself an expected payoff. As shown in Klibanoff, Marinacci, and Mukerji (2005) concave transformations of expected payoffs correspond to ambiguity aversion.

Proposition 8 *Assume a geometric transformation of payoffs* $u(t,q)^\beta$, *where* $\beta > 0$. *Assume* β

rises. Then rushes shrink, advancing in time in a pre-emption equilibrium, and delaying in a war of attrition equilibrium. The quantile function is unchanged during gradual play.

A comparison to Proposition 6 is instructive. One might think that greater risk (ambiguity) aversion corresponds to more fear. We see instead that concave geometric transformations act like decreases in fear for pre-emption equilibria, and decreases in greed for war of attrition equilibria. Our notions of greed and fear are therefore observationally distinct from risk preference.

PROOF: Consider the transformation $v(t,q) \equiv f(u(t,q))$ with $f' > 0$. Then $v_t = f'u_t$ and $v_q = f'u_q$ and $v_{tq} = f''u_t u_q + f'u_{tq}$, and thus $v_t v_q - v v_{tq} = [(f')^2 - ff'']u_t u_q - ff'u_{tq}$ yields

$$v_t v_q - v v_{tq} \;=\; [(f')^2 - ff'' - ff'/u]u_t u_q - ff'[u_{tq} - u_t u_q/u] \tag{19}$$

The term $u_t u_q$ changes sign. But if $(f')^2 - ff'' - ff'/u = 0$, which requires our geometric form $f(u) = cu^{\beta}$, then (19) is surely nonnegative if $u_t u_q \geq u u_{tq}$, when $c, \beta > 0$. So the proposed transformation preserves log-submodularity. Log-concavity is proven similarly.

To see how the peak rush locus behaves, we focus on a pre-emption equilibrium. Clearly, $f' > 0$ ensures an unchanged gradual play locus (6). Now consider the peak rush locus (7). Given any convex transformation f, Jensen's inequality implies:

$$f(u(t,q_0)) \;=\; f(V_0(q_0,t)) \equiv f\left(q_0^{-1}\int_0^{q_0} u(t,x)dx\right) \leq q_0^{-1}\int_0^{q_0} f(u(t,x))dx$$

So to restore equality, the peak rush locus $q_0(t)$ must decrease. Now consider two geometric transformations with $\beta_H > \beta_L$, i.e. $u^{\beta_H} = (u^{\beta_L})^{\beta_H/\beta_L}$. □

C Monotone Stopping Payoff: Final Steps for Proposition 1

FINAL STEPS FOR $u_q > 0$. The text established that equilibrium must involve gradual play for all quantiles beginning at $t^*(0)$, satisfying (5). We claim that this uniquely pins down $Q(t)$. Indeed, $u_t(t,q) < 0$ for all $t \geq t^*(0)$, while $\lim_{t\to\infty} u(t,q) < u(t^*(0),0) < u(t^*(0),q)$ for all $q > 0$ by $u_q > 0$ and (1). Thus, by continuity there is a unique and finite $t(q) > t^*(0)$ satisfying (5) for all $q \in (0,1]$. Since $u_q > 0$, $u_t(t(q),q) < 0$, and u is c^2, $t'(q) > 0$ by the Implicit Function Theorem. But then $t(q)$ is invertible with an increasing inverse, $Q(t)$, the unique gradual play cdf. To see that this $Q(t)$ is an equilibrium, observe that no agent can gain by pre-empting gradual play, since $t^*(0)$ maximizes $u(t,0)$. Further, since t^* is decreasing, we have $u_t(t,1) < 0$ for all $t \geq t^*(0)$, thus no agent can gain by delaying until after the war of attrition ends. □

UNIQUE RUSH SIZE WITH ALARM. The initial rush q_0 must satisfy indifference between the rush payoff and gradual play payoff: $V_0(0, q_0) = u(t^*(1), 1)$. Since $V_0(0, q)$ is continuous and strictly decreasing in q with endpoints $V_0(0, 1) < u(t^*(1), 1)$ and $\lim_{q \to 0} V_0(0, q) = u(0, 0) > u(t^*(1), 1)$, a unique $q_0 \in (0, 1)$ exists. □

UNIQUE GRADUAL PLAY SOLUTION FOR $u_q < 0$. Let q_0 be the unique rush given alarm or instead set $q_0 = 0$ with no alarm or panic. In either case, there exists a unique $t(q) < t^*(1)$ satisfying (6) for all $q \in (q_0, 1)$. Indeed, u is continuous, $u_t > 0$ for all $t < t^*(1)$ (since t^* is non increasing), with endpoints $u(t^*(1), q) > u(t^*(1), 1)$ for all $q < 1$ (by $u_q < 0$) and $u(0, q) < u(t^*(1), 1)$, where this last inequality follows from $u(0, q) < V_0(0, q_0) = u(t^*(1), 1)$ (given alarm) and $u(0, q) < u(0, 0) \leq u(t^*(1), 1)$ (with no alarm or panic). Since $u_q < 0$, $u_t(t(q), q) > 0$, and u is c^2, $t'(q) > 0$ by the Implicit Function Theorem. But then $t(q)$ is invertible with an increasing inverse, $Q(t)$, the unique gradual play cdf. □

D Interior Peak Quantile: Proofs of Propositions 2 and 3

Lemmas D.1–D.5 yield Proposition 2, while Lemmas D.1, D.3, and D.4 complete Proposition 3.

Lemma D.1 *When payoffs are non-monotone in quantile, equilibrium involves: an initial rush and then an uninterrupted pre-emption phase ending at $t^*(1)$, a terminal rush preceded by an uninterrupted war of attrition phase starting at $t^*(0)$, or a unit mass rush.*

STEP 1: RUSH NECESSITY. Assume gradual play for all Q, starting at t_0. Since $q^*(t_0) > 0$, we have $u_q(t_0, 0) > 0 \Rightarrow u_t(t_0, 0) < 0$ (by (3)), i.e. $t_0 > t^*(0)$. But then by t^* non increasing, $u_t(t, Q(t)) < 0$ on the support of Q, and by (3) $u_q(t, Q(t)) > 0$, i.e. $Q(t) < q^*(t)$ for all t. Since q^* non increasing, we have $Q(t) < q^*(t) \leq q^*(1) < 1$ for all $t \in \text{supp}(Q)$, which is impossible.

STEP 2: AT MOST ONE RUSH. Assume rushes at t_1 and $t_2 > t_1$. Since u strictly falls after the quantile peak, we must have $Q(t_2-) < q^*(t_2)$ else players can strictly gain from preempting the rush. Likewise, to avoid a strict gain from post-empting the rush at t_1 we need $Q(t_1) > q^*(t_1)$. Altogether, $q^*(t_1) < Q(t_1) \leq Q(t_2-) < q^*(t_2)$, which violates q^* weakly decreasing.

STEP 3: STOPPING ENDS IN A RUSH OR WITH GRADUAL PLAY AT $t^*(1)$. Assume gradual play ends at $t \neq t^*(1)$. If $t < t^*(1)$ then quantile 1 benefits from deviating to $t^*(1)$. If instead, $t > t^*(1)$, then we have $u_t(t, 1) < 0 \Rightarrow u_q(t, 1) > 0$, which violates $q^* < 1$.

STEP 4: STOPPING BEGINS IN A RUSH OR WITH GRADUAL PLAY AT $t^*(0)$. On the contrary, assume gradual play begins at $t \neq t^*(0)$. If $t > t^*(0)$ then quantile 0 profits by deviating to $t^*(0)$. If instead, $t < t^*(0)$, then $u_t(t, 0) > 0 \Rightarrow u_q(t, 0) < 0$, violating $q^* > 0$.

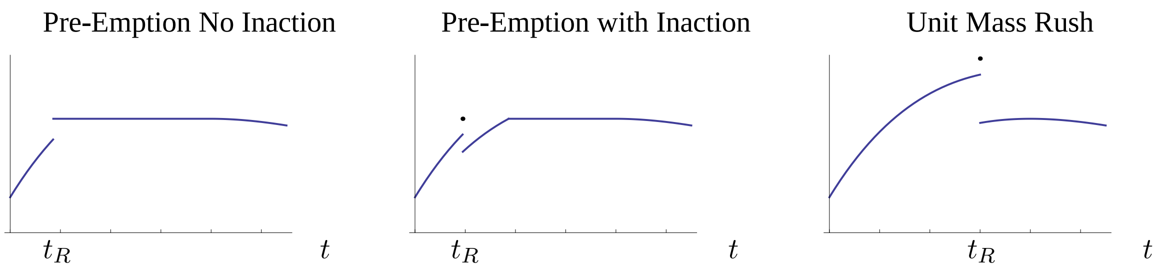

Figure 10: **Equilibrium Payoffs.** We graph equilibrium payoffs as a function of stopping time. On the left is the unique safe pre-emption equilibrium with a flat payoff on an interval. In the middle, a pre-emption equilibrium with inaction. On the right a unit rush equilibrium.

STEP 5: NO INTERIOR QUANTILE RUSH. Assume a rush at $t > 0$, i.e. $0 < Q(t-) < Q(t) < 1$. By Step 2, all other quantiles must stop in gradual play. Then by Steps 3 and 4 we must have $Q(t^*(0)) = 0$ and $Q(t^*(1)) = 1$, but this violates Q weakly increasing and t^* weakly decreasing.

STEP 6: ONLY ONE GRADUAL PLAY PHASE. Assume gradual play on $[t_1, t_2] < [t_3, t_4]$, satisfying $Q(t_3) = Q(t_4)$ (WLOG by Step 5). By steps 1, 2, and 5, stopping must begin or end in a rush. If stopping ends in a rush, then by Step 4, $t_1 \geq t^*(0)$, and since t^* is non-increasing, $u_t(t, Q(t)) < 0$ for all $t > t_1$. But then $u(t_2, Q(t_2)) > u(t_3, Q(t_3))$, contradicting optimal stopping at t_2 and t_3. If instead, stopping begins in a rush, then by step 3, $t_4 \leq t^*(1)$, and since t^* is non-increasing, we must have $u_t(t, Q(t)) > 0$ for all $t < t_4$. But then $u(t_2, Q(t_2)) < u(t_3, Q(t_3))$, again contradicting stopping at t_2 and t_3. $\square$

Lemma D.2 *Greed at $t^*(1)$ rules out pre-emption, while fear at $t^*(0)$ rules out wars of attrition.*

By Lemma D.1 any pre-emption phase must end at $t^*(1)$, implying Nash payoff $\bar{w} = u(t^*(1), 1)$. Also, since t^* is non-increasing, $u_t(t, q) > 0$ for all $(t, q) < (t^*(1), 1)$; and thus, $\bar{w}$ is strictly below the average payoff at $t^*(1) : \int_0^1 u(t^*(1), x)dx$. Altogether, $\bar{w} = u(t^*(1), 1) < \int_0^1 u(t^*(1), x)dx$, which contradicts greed at $t^*(1)$. By similar logic, fear at $t^*(0)$ is inconsistent with the Nash payoff $u(t^*(0), 0)$ in any war of attrition starting. $\square$

Lemma D.3 *An equilibrium Q is secure iff its support is a non-empty interval or the union of a $t = 0$ rush and a non-etmpty interval, and thus an equilibrium is safe if and only if it is secure.*[13]

STEP 1: SAFE EQUILIBRIA. Trivially, any rush at $t = 0$ is secure. Now, assume an interval $[t_a, t_b]$ of gradual play with constant stopping payoff $\hat{\pi}$. So for any $\varepsilon' < (t_b - t_a)/2$ and any

[13]More strongly, equilibria with inaction phases or unit atoms violate an even weaker refinement. Let us call an equilibrium ε-secure if $|w(t; Q) - w_\varepsilon(t; Q)| = O(\varepsilon)$. Then we find that (a) unit mass rushes are generically not ε-secure, and (b) there is at most one ε-secure pre-emption (or war of attrition) equilibrium with an inaction phase.

$t \in [t_a, t_b]$, one of the two intervals $[t, t+\varepsilon')$ or $(t-\varepsilon', t]$ will be contained in $[t_a, t_b]$ and thus obtain payoff $\hat{\pi}$. Clearly security is maintained by adding a rush with payoff $\hat{\pi}$ at t_a or t_b.

STEP 2: UNIT RUSHES AND INACTION EQUILIBRIA. Assume an equilibrium with initial rush of size $\hat{q} \in (0,1]$ at time $\hat{t} \in (0, t^*(1)]$. Since this is an equilibrium, $V_0(\hat{t}, \hat{q}) \geq u(\hat{t}, 0)$. Altogether, $\inf_{s\in(\hat{t}-\varepsilon,\hat{t}]} w(s;Q) = \inf_{s\in(\hat{t}-\varepsilon,\hat{t}]} u(s,0) < V_0(\hat{t},\hat{q}) = w(\hat{t};Q)$ for all $\varepsilon \in (0,\hat{t})$, where the strict inequality follows from $u_t(t,q) > 0$ for all $t < \hat{t} \leq t^*(1) \leq t^*(q)$.

Now consider an interval following the rush $[\hat{t}, \hat{t}+\varepsilon)$. If, $\hat{q} < 1$, gradual play follows the rush after delay $\Delta > 0$, and $V_0(\hat{t},\hat{q}) = u(\hat{t}+\Delta, \hat{q})$. But, since $t+\Delta < t^*(1)$ we have $u_t(t,\hat{q}) > 0$ during the delay, and $w(t;Q) = u(t,\hat{q}) < V_0(\hat{t},\hat{q})$ for all $t \in (\hat{t}, \hat{t}+\Delta)$. Thus, $\inf_{s\in[\hat{t},\hat{t}+\varepsilon)} w(s;Q) < w(\hat{t};Q)$ for all $\varepsilon \in (0,\Delta)$. Now assume $\hat{q} = 1$ and consider the two cases $\hat{t} < t^*(1)$ and $\hat{t} = t^*(1)$. If $\hat{t} < t^*(1)$, then $V_0(\hat{t},1) > u(\hat{t},1)$, else stopping at $t^*(1)$ is strictly optimal. But then by continuity, there exists $\delta > 0$ such that $w(\hat{t};Q) = V_0(\hat{t},1) > u(t,1) = w(t;Q)$ for all $t \in (\hat{t}, \hat{t}+\delta)$. If $\hat{t} = t^*$, equilibrium requires the weaker condition $V_0(t^*(1),1) \geq u(t^*(1),1)$, but then we have $u_t(t,1) < 0$ for all $t > \hat{t}$; and so, $w(\hat{t};Q) = V_0(\hat{t},1) > u(t,1) = w(t;Q)$ for all $t > \hat{t}$.

We have established the result for any equilibrium with $Q(t^*(1)) = 1$. The analysis for an equilibrium transpiring after $t^*(1)$ is similar. □

Lemma D.4 *A pre-emption (war of attrition) equilibrium exists given no greed at $t^*(1)$ (no fear at $t^*(0)$). A unique secure equilibrium exists given greed at $t^*(1)$, while a unique secure war of attrition exists given no fear at harvest time $t^*(0)$.*

STEP 0: FEAR AND GREED OBEY SINGLE CROSSING. Since $u(t,y)/u(t,x)$ is non-increasing in t for all $y \geq x$, fear and greed obey single crossing. Specifically, fear at time t_L implies fear obtains at all later times $t \geq t_L$, while greed at time t_H implies greed at all $t \leq t_H$.

STEP 1: THE PEAK RUSH LOCUS (7). The marginal and average coincide at a local max (min) of the average when the marginal is decreasing (increasing), since:

$$\frac{\partial V_0(t,q)}{\partial q} = \frac{u(t,q) - V_0(t,q)}{q} \quad \Rightarrow \quad q\frac{\partial^2 V_0(t,q)}{\partial q^2} = u_q(t,q) - 2\frac{\partial V_0(t,q)}{\partial q} \tag{20}$$

Since u is log-concave with $q^*(t) \in (0,1)$, $u_q(t,0) > 0$, and the average is initially rising:

$$\lim_{q\to 0} \frac{\partial V_0(t,q)}{\partial q} = \lim_{q\to 0} \frac{u(t,q) - V_0(t,q)}{q} = \lim_{q\to 0} \left[u_q(t,q) - \frac{\partial V_0(t,q)}{\partial q}\right] \Rightarrow 2\lim_{q\to 0} \frac{\partial V_0(t,q)}{\partial q} = u_q(t,0)$$

which we have evaluated using L'Hopital, since $\lim_{q\to 0} V_0(t,q) = u(t,0) > 0$.

Now, let $q_0(t) > 0$ be the smallest solution to (7). Since the average is initially rising, $u(t,q) > V_0(t,q)$ for q sufficiently small and q_0 must be a local max of V_0 with $u_q(t,q_0) < 0$ (by (20)), implying $q_0(t) > q^*(t)$. Given u is log-concave, $u_q(t,q) < 0$ for all $q \geq q_0(t) > q^*(t)$; and so by (20), no $q \geq q_0$ can be a local min for V_0. Further, by continuity, a solution $q_0(t) < 1$ exists by the Intermediate Value Theorem, *iff* $u(t,1) < V_0(t,1)$ (i.e. no greed at t), and is continuous by the Implicit Function Theorem. Altogether, a solution: Exists *iff* no greed at t, is unique and continuous when it exists, exceeds $q^*(t)$, and is necessarily a global max for V_0.

To see that q_0 is non-increasing, define $\mathbb{I}(q,x) \equiv q^{-1}$ for $x \leq q$ and 0 otherwise, and $\ell \equiv t^*(1) - t$, and thus $V_0(t^*(1)-\ell, q) = \int_0^1 \mathbb{I}(q,x)u(t^*(1)-\ell,x)dx$. Easily, $\mathbb{I}$ is log-supermodular in (q,x), and so the product $\mathbb{I}(\cdot)u(\cdot)$ is log-supermodular in (q,x,ℓ). Thus, V_0 is log-supermodular in (q,ℓ) since log-supermodularity is preserved by integration by Karlin and Rinott (1980). So the peak rush locus $q_0(t^*(1)-\ell) = \arg\max_q V_0(t^*(1)-t,q)$ rises in ℓ, i.e. falls in t.

STEP 2: THE GRADUAL PLAY LOCUS (6) IS INCREASING. We claim the solution $q_I(t)$ to (6) is unique on $[0,t^*(1)] \times [q^*(t),1]$. First, consider the case when $u(0,q^*(0)) \leq u(t^*(1),1)$. By $q^* \in (0,1)$, we have $u(t^*(1), q^*(t^*(1))) > u(t^*(1),1)$, while the continuous function $u(t,q^*(t))$ is increasing in t for $t \leq t^*(1)$ (by $u_t(t,q) > 0$). Thus, there exists a unique $t_v \in (0,t^*(1))$ such that $u(t_v,q^*(t_v)) \equiv u(t^*(1),1)$ and $u(t,q^*(t)) > u(t^*(1),1)$ for all $t \in (t_v,t^*(1)]$. Further, by definition $u(t,1) < u(t^*(1),1)$ and $u_t(t,q) > 0$ for all $t < t^*(1) \leq t^*(q)$. Altogether, there exists a unique $q_I(t) \in (q^*(t),1)$ solving (6), for $t \in (t_v,t^*(1))$. For the reverse inequality, $u(0,q^*(0)) > u(t^*(1),1)$, holds then $u(t,q^*(t)) \geq u(0,q^*(0)) > u(t^*(1),1) > u(t,1)$, for all $t \leq t^*(1)$. Again by $u_t > 0$, there exists unique $q_I(t) \in (q^*(t),1]$ satisfying (6) for all $t \leq t^*(1)$. In either case, $q_I(t) \geq q^*(t)$, so that $u_q(t,q_I(t)) < 0 < u_t(t,q_I(t))$, while u is C^2, so that $q_I'(t) > 0$ by the Implicit Function Theorem.

STEP 3: UNIQUE SECURE EQUILIBRIUM. Steps 1 and 2 showed that absent greed $q_0(t)$ is well defined, continuous, and decreasing, while q_I is well defined, continuous and increasing with $q_I(t^*(1)) = 1 > q_0(t^*(1))$. First assume inequality (4), is violated, which along with Step 1 yields $u(0,q_0(0)) > u(t^*(1),1)$, and thus, $q_0(0) < q_I(0)$. Given q_0 decreasing and q_I increasing, there is no $t_0 \leq t^*(1)$ satisfying $q_0(t_0) = q_I(t_0)$. By Lemma D.3, the secure pre-emption equilibrium must involve a rush at $t = 0$, with the size of the rush determined as in Proposition 1 (b) and (c), i.e. a unit rush given panic or the unique rush of size q_R satisfying $V_0(0,q_R) = u(t^*(1),1)$ given alarm. With alarm, inaction follows the rush, until the the unique time $t_1 < t^*(1)$, satisfying $u(t_1,q_R) = u(t^*(1),1)$, followed by gradual play on $(t_1,t^*(1)]$ obeying $Q(t) = q_I(t)$.

Assume instead that inequality (4) obtains. In this case, $q_I(t_v) = q^*(t_v) < q_0(t_v)$, and there exists a unique solution $(q_0,t_0) \in (q^*(t_v),1) \times (0,t^*(1))$ satisfying $q_0 = q_I(t_0) = q_0(t_0)$. The unique secure pre-emption equilibrium involves an initial rush size and time (t_0, q_0), followed by gradual play on $(t_0, t^*(1)]$ obeying $Q(t) = q_I(t)$. $\square$

Lemma D.5 *Given no greed at $t^*(1)$ and no fear at $t^*(0)$, there exist $\underline{T}, \bar{T}$ satisfying $0 \leq \underline{T} < t^*(1) \leq t^*(0) < \bar{T}$, such that a unit rush at any $t \in [\underline{T}, \bar{T}]$ is an equilibrium. Unit mass rushes cannot occur at any positive time with strict greed or strict fear.*

STEP 1: NO UNIT RUSHES WITH STRICT FEAR OR GREED Assume a unit mass rush at time $t_R > 0$. Given strict greed at time t_R, i.e. $u(1, t_R) > V_0(1, t_R)$, post-empting the rush is strictly better than stopping in the rush. Likewise strict fear at time $t_R > 0$, i.e. $u(0, t_R) > V_0(1, t_R)$ implies pre-empting the rush results is a strict improvement.

STEP 2: THE UNIT RUSH TIME INTERVAL $[\underline{T}, \bar{T}]$. Since fear satisfies strict single crossing (Lemma D.4 Step 0) no fear at $t^*(0)$ implies, $\exists \bar{T}_1 > t^*(0)$ such that no fear obtains for all $t \leq \bar{T}_1$. Likewise no greed at $t^*(1)$, implies $\exists \underline{T}_1 < t^*(1)$ such that no greed holds for all $t \geq \underline{T}_1$. Since $t^*(1) \leq t^*(0)$, we have $\underline{T}_1 < \bar{T}$; thus, both no fear and no greed hold on $[\underline{T}_1, \bar{T}_1]$. Given t^* non-increasing, $V_0(t, 1)$ is strictly increasing for $t < t^*(1)$. Thus, no greed at $t^*(1)$, implies $\exists \underline{T}_2 \in [0, t^*(1))$, such that $u(t^*(1), 1) \leq V_0(t, 1)$ for all $t \in [\underline{T}_2, t^*(1)]$. Likewise, $V_0(t, 1)$ is strictly decreasing for all $t \geq t^*(0)$ with $\lim_{t\to\infty} V_0(t, 1) < u(t^*(0), 0)$ (by inequality (1)). Thus, if no fear obtains at $t^*(0)$, $\exists \bar{T}_2 > t^*(0)$, such that $V_0(t, 1) \geq u(t^*(0), 0)$ for all $t \in [t^*(0), \bar{T}_2]$. Finally, let $\underline{T} = \max\{\underline{T}_1, \underline{T}_2\}$ and $\bar{T} = \min\{\bar{T}_1, \bar{T}_2\}$.

STEP 3: VERIFICATION OF UNIT RUSH EQUILIBRIA. If a unit mass rush occurs at $t_R \in [\underline{T}, t^*(1))$, then the stopping payoff rises until the rush, and no fear at t_R, i.e. $u(t_R, 0) \leq V_0(t_R, 1)$ rules out pre-empting the rush. The stopping payoff rises on $(t_R, t^*(1)]$ and then falls: Stopping in the rush is at least as good as stopping after the rush *iff* $u(t^*(1), 1) \leq V_0(t_R, 1)$, which holds for all $t_R \in [\underline{T}, t^*(1)]$ by Step 2. If, instead, $t_R \in [t^*(1), t^*(0)]$, then as long as we have no fear at t_R, $u(t_R, 0) \leq V_0(t_R, 1)$ pre-empting the rush is not optimal, while no greed at t_R, $u(t_R, 1) \leq V_0(t_R, 1)$ ensures post-empting the rush is not optimal. Finally, If $t_R > t^*(0)$, the stopping payoff rises until $t^*(0)$ and falls between $t^*(0)$ and the rush. Thus, $u(t^*(0), 0) \leq V_0(t_R, 1)$ rules out pre-empting the rush, which holds for all $t \in (t^*(0), \bar{T}]$ by Step 2. The stopping payoff falls after the rush; and so, no greed at t_R, $u(t_R, 1) \leq V_0(t_R, 1)$ rules post-empting the rush. $\square$

E Comparative Statics: Propositions 5 and 6

Lemma E.1 *Given an interior peak quantile, the initial rush in the safe pre-emption equilibrium increases in γ, while the terminal rush in the safe war of attrition equilibrium shrinks in γ.*

We present the proof for the safe pre-emption equilibrium. The logic for the safe war of attrition equilibrium is symmetric.

STEP 1: PRELIMINARIES. First we claim that whenever $V_0(t,q|\gamma) \geq u(t,q|\gamma)$, we have:

$$q^{-1}\int_0^q u_t(t,x|\gamma)dx \geq u_t(t,q|\gamma) \tag{21}$$

Indeed, using $u(t,q|\gamma)$ log-submodular in (t,q):

$$q^{-1}\int_0^q \frac{u_t(t,x|\gamma)}{u(t,q|\gamma)}dx = q^{-1}\int_0^q \frac{u_t(t,x|\gamma)}{u(t,x|\gamma)}\frac{u(t,x|\gamma)}{u(t,q|\gamma)}dx \geq \frac{u_t(t,q|\gamma)}{u(t,q|\gamma)}q^{-1}\int_0^q \frac{u(t,x|\gamma)}{u(t,q|\gamma)}dx = \frac{u_t(t,q|\gamma)}{u(t,q|\gamma)}$$

Define $g(t,q|\gamma) \equiv u(t,q|\gamma)/u(t,1|\gamma)$, $G(t,q|\gamma) \equiv q^{-1}\int_0^q g(t,x|\gamma)$, and $h(t) \equiv u(t^*(1),1|\gamma)/u(t,1|\gamma)$, to write the gradual play locus (6) and peak rush locus (7) as:

$$\text{Gradual Play Locus: } g(t,q|\gamma) = h(t) \quad \text{and} \quad \text{Peak Rush Locus: } g(t,q|\gamma) = G(t,q|\gamma) \tag{22}$$

Log-modularity in (t,γ) yields h constant in γ, while $u_t > 0$ in a pre-emption game implies $h' < 0$. By log-submodularity in (t,q) and log-supermodularity in (q,γ), $g_t > 0$ and $g_\gamma < 0$.

STEP 2: $G_t \geq g_t$ AND $-G_\gamma > -g_\gamma$ FOR ALL (t,q) SATISFYING (22). Now evaluate:

$$\begin{aligned}
G_t - g_t &= q^{-1}\int_0^q \left[\frac{u_t(t,x|\gamma)}{u(t,1|\gamma)} - \frac{u(t,x|\gamma)u_t(t,1|\gamma)}{u(t,1|\gamma)^2}\right]dx - \left[\frac{u_t(t,q|\gamma)}{u(t,1|\gamma)} - \frac{u(t,q|\gamma)u_t(t,1|\gamma)}{u(t,1|\gamma)^2}\right] \\
&\geq -q^{-1}\int_0^q \left[\frac{u(t,x|\gamma)u_t(t,1|\gamma)}{u(t,1|\gamma)^2}\right]dx + \frac{u(t,q|\gamma)u_t(t,1|\gamma)}{u(t,1|\gamma)^2} \qquad \text{by (21)} \\
&= \frac{u_t(t,1|\gamma)}{u(t,1|\gamma)^2}\left[u(t,q|\gamma) - q^{-1}\int_0^q u(t,x|\gamma)dx\right] = 0 \qquad \text{by (22)}
\end{aligned}$$

Since u is strictly log-supermodular in (γ,q) symmetric steps establish that $-G_\gamma > -g_\gamma$.

STEP 3: A DIFFERENCE IN DERIVATIVES. We have shown that the gradual play locus is upward sloping and shifts down in γ, while the peak rush locus is downward sloping and shifts up in γ. We now finish the proof that the initial rush rises in γ by establishing that starting from any (t,q,γ) satisfying (22) and holding q fixed, the change $dt/d\gamma$ in the gradual play locus (22) is smaller than the $dt/d\gamma$ in the peak rush locus (22). Evaluating both derivatives, this entails:

$$\frac{g_\gamma(t,q|\gamma) - G_\gamma(t,q|\gamma)}{G_t(t,q|\gamma) - g_t(t,q|\gamma)} > \frac{-g_\gamma(t,q|\gamma)}{g_t(t,q|\gamma) - h'(t)}$$

Since $h', g_t > 0$ and $g_\gamma < 0$ (as shown in Step 1), it is sufficient to show:

$$\frac{g_\gamma - G_\gamma}{G_t - g_t} > \frac{-g_\gamma}{g_t} \quad \Leftrightarrow \quad g_t(g_\gamma - G_\gamma) > -g_\gamma(G_t - g_t) \quad \Leftrightarrow \quad -G_\gamma g_t > -g_\gamma G_t$$

Which follows from $G_t \geq g_t > 0$ and $-G_\gamma > -g_\gamma > 0$ as established in Steps 1 and 2. □

Lemma E.2 *Assume alarm. Then the initial rush at $t = 0$ shrinks in ϕ and γ.*

PROOF: The initial rush of quantiles $[0, q]$ obeys: Given alarm, the initial rush occurs at $t = 0$ and obeys:

$$q^{-1} \int_0^q \frac{u(0, x|\phi)}{u(0, 1|\phi)} dx = \frac{u(t^*(1|\phi), 1|\phi)}{u(0, 1|\phi)} \tag{23}$$

Since the initial rush includes the peak quantile, the LHS of (23) falls in q, while log-supermodularity of u in (q, ϕ) implies the LHS falls in ϕ. The RHS rises in ϕ by the same logic used to establish that the RHS of (9) is increasing in ϕ. Altogether, the initial rush q_0 satisfies $\partial q_0/\partial\phi < 0$. □

F Omitted Proofs for Examples

Claim 1 *The payoff $u(t, q) \equiv (1 + \rho q)F(\tau(q, t))p(t|\xi)$ in (13) is log-submodular in (t, q), and log-concave in t and q.*

Proof: Differentiating $\kappa(t + \tau(q, t)) \equiv q$ gives $\kappa'(t + \tau(q, t))(1 + \tau_t(q, t)) = 0$ and $\kappa'(t + \tau(q, t))\tau_q(q, t) = 1$. So, $\tau_t \equiv -1$ and $\tau_q < 0$ given $\kappa' < 0$. Hence, $\tau_{tq} = 0$, $\tau_{tt} = 0$ and $\tau_{qq} = -(\kappa''/\kappa')(\tau_q)^2$. Thus,

$$\frac{\partial^2 \log(F(\tau(q, t)))}{\partial t \partial q} F(\tau(q, t))^2 = \left[FF'' - (F')^2\right] \tau_t \tau_q + FF'\tau_{tq} = \left[FF'' - (F')^2\right] \tau_t \tau_q \leq 0$$

Twice differentiating $\log(F(\tau(q, t)))$ in t likewise yields $[FF'' - (F')^2]/F^2 \leq 0$. Similarly,

$$\partial^2 \log(F(\tau(q, t)))/\partial q^2 = (\tau_q)^2[FF'' - (F')^2 - (\kappa''/\kappa')FF']/F^2 \leq 0$$

where $-\kappa''/\kappa' \leq 0$ follows since κ is decreasing and log-concave. □

Claim 2 *The bank run payoff (14) is log-submodular in (q, α). This payoff is also log-supermodular in (q, R) provided $\zeta H'(\zeta|t)/H(\zeta|t)$ is weakly falling in ζ.*

PROOF: log-supermodularity of H in (ζ, t), while log-concavity of H in ζ implies u log-submodular in (q, α):

$$\frac{\partial^2 \log(H(\cdot))}{\partial q \partial \alpha} H^2(1-R)^2 = \alpha q \left[HH'' - (H')^2 \right] - (1-R)HH' \leq 0$$

For u log-supermodular in (q, R), we need:

$$\frac{\partial^2 \log(H(\cdot))}{\partial q \partial R} H(\cdot)^2(1-R)^2 = \left(\frac{1-\alpha q}{1-R} \right) \left((H')^2 - HH'' \right) - HH' \geq 0 \tag{24}$$

i.e. $x(H'(x)^2 - H(x)H''(x)) - H(x)H'(x) \geq 0$, namely, distributions with a weakly falling elasticity $xH'(x)/H(x)$. $\square$

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