Competitive Advertising and Pricing*

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Abstract

We consider an oligopoly model in which each firm chooses not only its price but also its advertising strategy regarding how much product information to provide. Unlike most previous studies on advertising, we impose no structural restriction on feasible advertising content, so that each firm can freely disclose or conceal any information. We provide a general and complete characterization of the equilibrium advertising content, which illustrates how competition shapes firms' advertising incentives. We also explore the economic consequences of competitive advertising and investigate how a firm's advertising decision interacts with its pricing decision.

JEL Classification Numbers: D43, L11, L13, L15, M37.

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1 Introduction

One of the fundamental questions in the economics of advertising is how much, and what, product information a firm should provide to consumers.[1] Offering more product information enables the seller to more effectively price-discriminate consumers or be more aggressive in pricing. However, it comes at the cost of losing some consumers who do not find the revealed product characteristics

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[1] We restrict attention to this unbiased information transmission role of advertising. However, the literature has identified several other roles. See Bagwell (2007) and Renault (2015) for thorough overviews of the literature.

appealing. This essential trade-off has been extensively studied in the monopoly context.[2] A common insight in the literature is that in the absence of varying advertising costs, a monopolist wishes to provide either no information or full information: the former enables her to serve all consumers without charging too low a price, while the latter allows her to extract the most from high-value consumers.

In this paper, we study firms' incentives to provide product information in an oligopoly environment. We employ the canonical random-utility discrete-choice framework of Perloff and Salop (1985): there are $n(\geq 2)$ firms that engage in Bertrand competition, and each consumer's value for each product is independently and identically drawn from a certain distribution F. We extend the basic framework to allow for strategic advertising by the firms: each firm chooses how much information to provide about its own product (equivalently, the precision of the information that consumers receive about their valuations of the product).[3] Therefore, the firms compete not only on price but also through (informative) advertising.

Unlike most classical studies on advertising, we impose no structural restriction on the set of feasible advertising strategies and allow each firm to disclose or conceal any information.[4] This can be interpreted as each firm having access to numerous advertising channels and/or fine-grained information on various attributes of its product and, therefore, being able to adjust its advertising content without any restriction. As is well known in the literature on information design (see, e.g., Kolotilin, 2018; Roesler and Szentes, 2017), with risk-neutral consumers—who are concerned only with their expected values of the products—this fully flexible advertising can be modeled as each firm being able to choose any mean-preserving contraction (MPC hereafter) G_i of F.[5] Given G_i, it is as if each consumer's (expected) value of product i is independently and identically drawn according to G_i. In this regard, our model can be interpreted as one in which the distribution of consumers' values—exogenously given in Perloff and Salop (1985)—is endogenously determined by the firms' advertising choices.

We address the following three sets of research questions in sequence. (1) *Advertising content*

[2] Lewis and Sappington (1994) is a seminal contribution to this literature. Subsequent important works include Che (1996) (return policies as a means of facilitating consumer experimentation), Ottaviani and Prat (2001) (the value of revealing a signal associated with the buyer's private signal), Anderson and Renault (2006) (optimal advertising for search goods), Johnson and Myatt (2006) (U-shaped profit function based on the rotation order), and Roesler and Szentes (2017) (buyer-optimal signal for experience goods). See also Boleslavsky et al. (2017), who study the interaction between advertising ("demonstration") and pricing in the entry game context (where consumers' values for the incumbent's product are known, while their idiosyncratic values for the entrant's product are unknown).

[3] We do not allow for "comparative advertising," in which a firm not only controls its own product information but can also provide information about rivals' products. See Anderson and Renault (2009) for an important contribution on comparative advertising.

[4] See Anderson and Renault (2006) for an earlier application of this general formulation to advertising.

[5] Intuitively, each firm can induce a degenerate distribution by revealing no product information and $G_i = F$ by revealing all product information. It obtains any distribution between these two extremes by selectively disclosing information but cannot induce a more dispersed distribution than F.

under competition: How does competition shape an individual firm's advertising content? Under what conditions will each firm choose to provide full product information and, if it elects not to, what information will it conceal/disclose? (2) *Economic consequences of competitive advertising*: What are the effects of strategic advertising on market prices and welfare? Would a policy that requires the firms to provide full information always benefit consumers? (3) *Interaction between pricing and advertising*: How does a firm's advertising decision interact with its pricing decision? Should a firm provide more or less product information when it charges a higher price?

For the first set of questions, we provide a general characterization of the firms' (symmetric pure-price) equilibrium advertising strategy. Specifically, we consider a restricted game, termed the advertising-only game, in which the firms compete by choosing their advertising strategies given symmetric prices. We show that the game has a unique symmetric equilibrium. Importantly, the equilibrium advertising strategy, G^*, is fully characterized by the following two statistical properties: (i) $(G^*)^{n-1}$ is convex (i.e., has non-decreasing density) over its support, and (ii) G^* alternates between the full-information region (in which $G^*(v) = F(v)$) and the $(n-1)$-linear MPC region (in which $(G^*)^{n-1}$ is linear and G^* is an MPC of F).

Our general characterization allows us to answer all specific questions about equilibrium advertising content under competition. For example, the firms provide full product information (i.e., $G^* = F$) if and only if F^{n-1} is convex. If F^{n-1} is concave, each firm partially conceals information in a way that makes $(G^*)^{n-1}$ a uniform distribution. If F^{n-1} has single-peaked density, the equilibrium advertising strategy combines above two structures: G^* equals F up to a certain threshold v^*, and $(G^*)^{n-1}$ is linear above v^*. If the number of firms increases, then each firm reveals more product information, because F^{n-1} becomes more convex in n.

To understand the two properties of G^*, first observe that product information is intimately linked to the dispersion of consumers' expected values: if a firm provides more information, then some consumers find the product more while others find it less appealing, leading to a more dispersed distribution. This indicates that a firm's optimal advertising problem reduces to whether it wishes to spread or contract the induced distribution. Note that whereas a firm can contract its distribution as much as it wishes (up to a degenerate distribution), it faces the MPC constraint when spreading the distribution: a firm cannot induce a more dispersed distribution than F. The two properties of G^* derive from this difference and the symmetric equilibrium requirement. In equilibrium, the density function of $(G^*)^{n-1}$ (which represents the distribution of consumers' best alternatives to an individual product) can be strictly increasing because of the MPC constraint. However, it can never be strictly decreasing anywhere because if it could, a firm would find it profitable to contract its distribution. The specific alternating structure stems from the fact that $(G^*)^{n-1}$ can be strictly convex only when the MPC constraint binds (i.e., $G^*(v) = F(v)$).

For the second set of questions, we explicitly derive a unique equilibrium price, denoted by

p^*, under G^* and compare the market outcome to that of the full information benchmark (i.e., the equilibrium outcome in the model where $G_i = F$ for all i). Strategic advertising always reduces social surplus as it allows the firm to conceal certain information. However, we find that the effects of competitive advertising on the equilibrium price and consumer surplus are ambiguous in general. Competitive advertising induces a less dispersed distribution than F, which tends to lower the equilibrium price. However, the equilibrium price depends on the entire shape of the distribution, not just on its overall dispersion. Multiple distributional effects of competitive advertising make the price effect ambiguous, which in turn leads to an ambiguous result on consumer surplus.

For the last set of questions, we characterize an individual firm's optimal advertising strategy that corresponds to each price while assuming that all other firms play the equilibrium strategy (p^*, G^*). We find that a firm's two instruments—pricing and advertising—are substitutes: if a firm wishes to charge a higher price, then it should provide more product information. Intuitively, if a firm offers a very low price, then a consumer will purchase its product as long as its expected value is not too low relative to other products. In this case, it is optimal for the firm to conceal all product information, thereby maximizing the number of average-value consumers. Conversely, if it charges a very high price, then it can serve only sufficiently loyal consumers (who value the product considerably more than other products). In this case, it is optimal to provide all product information, thereby maximizing the number of such loyal consumers.

We use this characterization of optimal advertising for each price to establish the existence of symmetric pure-price equilibrium in our model. Note that a firm can make a compound deviation in which it deviates both in price and advertising (i.e., $p_i \neq p^*$ and $G_i \neq G^*$). Therefore, the unique (symmetric pure-price) equilibrium candidate (p^*, G^*) is an equilibrium if and only if no compound deviation is profitable. This implies that an equilibrium existence argument for the Perloff-Salop model does not directly apply to our model because a compound deviation is more profitable than a price-only deviation.[6] Nevertheless, we show that a common regularity condition ensures the existence of equilibrium: if the underlying density function f is log-concave, then (p^*, G^*) is indeed an equilibrium in our full model.

Related Literature

As explained above, the interaction between advertising and pricing has been extensively studied in the monopoly context but not in oligopoly settings. To the best of our knowledge, the only exception is Ivanov (2013). He also adopts the Perloff-Salop framework but considers a case in which all

[6]The existence of pure-price equilibrium is a non-trivial problem even in the Perloff-Salop model. The main difficulty lies in the fact that each firm's best response depends on the shape of the entire distribution and, therefore, may not be sufficiently well behaved unless some strong regularity is introduced into F. See Caplin and Nalebuff (1991) and Quint (2014) for some important contributions.

feasible advertising strategies are rotation-ordered in the sense of Johnson and Myatt (2006). Due to his modeling choice, he does not provide a general characterization of the equilibrium advertising strategy. Instead, he focuses on demonstrating that a full-information equilibrium (in which all the firms choose the highest advertising strategy in rotation order) exists if there are sufficiently many firms. This paper supplements his work in multiple ways. In particular, we provide a more comprehensive equilibrium analysis (covering partial information equilibria), demonstrate that his main economic conclusion holds even if each firm faces no restriction on advertising content, and study the interaction between a firm's advertising and pricing decisions.

Given a fixed price, our advertising-only game can be interpreted as a game of competitive information disclosure/provision (with multiple senders and a single receiver). There are two strands in that literature, one in which the senders have access to the same information or the full state of nature (e.g., Gentzkow and Kamenica, 2017; Li and Norman, 2018; Perego and Yuksel, 2018) and another in which each sender controls only his own information (e.g., Ostrovsky and Schwarz, 2010; Boleslavsky and Cotton, 2015, 2018; Au and Kawai, 2018a,b). Our model belongs to the latter category because each firm provides information only about its own product.

Ostrovsky and Schwarz (2010) provide a similar characterization to ours for equilibrium information disclosure (advertising). In particular, they also show that the equilibrium distribution (which corresponds to $(G^*)^{n-1}$ in our model) must be weakly convex and explain how the equilibrium distribution is related to the primitive distribution (which corresponds to F). The main difference from our advertising-only game is that they consider a market environment with many (negligible) heterogeneous senders (schools), which requires a distinct analysis from ours and also leads them to study different economic questions. For example, our important comparative statics exercise regarding the number of firms n cannot be addressed in their model.

All other papers in the same category study a fully strategic setting (with a finite number of senders) but consider discrete distributions. Our analysis with continuous distributions has three important advantages. First, the relationship between F and G^* is more clear and explicit when F is continuous. For example, if F is discrete, then G^* can never coincide with F, while if F is continuous, then $G^* = F$ if and only if F^{n-1} is convex (see Corollary 1). Second, it makes the equilibrium structure (i.e., the essential properties of G^*) more transparent. In particular, if F is continuous, it follows easily that if $(G^*)^{n-1}$ is not linear around v, then $G^*(v) = F(v)$. If F is discrete, this local coincidence is not possible, which greatly complicates the necessary analysis (see Section 4 in Au and Kawai, 2018b). Finally, it allows us to link our work to the large discrete-choice literature, most of which consider continuous distributions.

We significantly benefit from recent technical developments in the literature on information design.[7] In particular, we make extensive use of a verification technique in Dworczak and Martini

[7] As is well known, the elegant concavification method in Aumann and Maschler (1995) and

(2019) (DM hereafter).[8] They consider a general programming problem in which the sender's (indirect) payoff depends only on the expected value (state) she induces, represented by a function $u(v)$, and show that to evaluate the optimality of a signal (equivalently, an MPC G_i of F), it suffices to check whether there exists a convex function $\phi(v)$ that touches $u(v)$ only on the support of G_i (see Theorem 2 in Section 3). As illustrated by DM with a series of examples and shown by our subsequent analysis, this result permits a simple geometric analysis for a variety of information design problems. Our analysis differs from DM in mainly two ways. First, we search for Nash equilibrium in a strategic environment. Therefore, each firm's programming problem, which is exogenously given in DM, is endogenously determined in our model. Second, in our model, an individual firm chooses not only its advertising strategy but also its price. The latter also affects the firm's payoff function $u(v)$, and therefore, DM's result does not apply directly. We address this problem by finding an optimal signal for each price and verifying whether any of those (compound) deviations is profitable. DM's result applies to the former but not to the latter.

The remainder of this paper is organized as follows. We introduce the formal model in Section 2. In Section 3, we study the advertising-only game, the characterization of which yields a unique equilibrium advertising strategy candidate G^*. In Section 4, we derive the unique equilibrium price candidate p^* given G^* and compare it to that of the full-information benchmark. In Section 5, we illustrate how a firm's advertising decision interacts with its pricing decision and provide a sufficient condition for equilibrium existence. In Section 6, we conclude by discussing two important modeling assumptions in our model. All the proofs not included in the main text are relegated to the appendix, unless noted otherwise.

2 The Model

There are $n(\geq 2)$ ex ante homogeneous firms and a unit mass of risk-neutral consumers. Each firm supplies a product with marginal cost normalized to zero. The firms' products are horizontally differentiated: each consumer's *true value* for each product is drawn according to the distribution function F independently and identically across the products and consumers. We assume that F has finite mean μ_F and continuously differentiable density f with $supp(F) = [\underline{v}, \overline{v}]$, where both

Kamenica and Gentzkow (2011), which is powerful for binary-state problems, becomes less useful when there are many types (in particular, when the state space is continuous). The fundamental problem is that it is not possible to visualize the sender's (information designer's) value function with respect to the posterior belief when the type space is large. Recently, several researchers have developed methods that can be used to solve for an optimal signal in such a problem. As anticipated by Kamenica and Gentzkow (2011), the analysis becomes significantly more tractable if the receiver's optimal action depends only on the expected state, and considerable progress has been made for that class of problems. As explained in Section 2, each firm's optimal advertising problem in our model belongs to that class.

[8] Other relevant contributions include Gentzkow and Kamenica (2016), Ivanov (2015), Kolotilin et al. (2017), and Yamashita (2018).

$\underline{v} = -\infty$ and $\overline{v} = \infty$ are allowed. We also make a few mild technical assumptions: both f and f' are bounded, and f has a finite number of peaks and a log-concave upper tail if $\overline{v} = \infty$.

The firms simultaneously choose their price p_i and advertising strategy. Each firm is unrestricted in its choice of advertising content, which conveys information to consumers about their true values. In other words, each firm is allowed to disclose, or conceal, any product information it wishes. As is well known in the literature on information design, this can be formalized as follows: each firm can choose any set of realizations S_i and any joint distribution function H_i over $[\underline{v}, \overline{v}] \times S_i$. Each consumer observes H_i and draws $s_i \in S_i$ according to H_i before making her purchase decision.

Exploiting consumers' risk neutrality, we make use of a more tractable formulation of the advertising strategy. A consumer's purchase decision depends only on her (interim) expected value, $E[v_i|s_i] = \int v_i dF(v_i|s_i)$. This means that only the distribution of consumers' expected values, denoted by G_i, is payoff-relevant to the firms. Therefore, there is no loss of generality in assuming that each firm directly chooses G_i, instead of a signal structure (S_i, H_i). Clearly, the underlying distribution F constrains the set of feasible distributions G_i. As is well known, the precise restriction is that G_i should be an MPC of F: there exists a joint distribution H_i that gives rises to a distribution G_i if and only if G_i is an MPC of F. In what follows, we directly refer to G_i as firm i's advertising strategy.

The timing of the game is as follows. First, the firms simultaneously choose their price p_i and advertising strategy G_i. Second, for each product i, consumers independently draw their expected value v_i according to G_i. Consumers' expected values are also independent across products. Finally, each consumer decides which product to purchase based on prices and her expected values. We assume that each consumer must purchase one of the products.[9] This implies that a consumer purchases product i if and only if $v_i - p_i > v_j - p_j$ for all $j \neq i$.[10]

We analyze symmetric pure-price equilibria of this market game. Since consumers' optimal purchase decisions are straightforward, we focus on Nash equilibria played by the firms. Let $D(p_i, G_i, p, G)$ denote firm i's demand (that is, the measure of consumers who purchase product i) when firm i's strategy is (p_i, G_i), while all other firms adopt strategy (p, G). Given consumers' optimal choice rule,[11]

$$D(p_i, G_i, p, G) = Pr\{v_i - p_i > v_j - p, \forall j \neq i\} = \int G(v_i - p_i + p^*)^{n-1} dG_i(v_i). \qquad (1)$$

[9] This "full market coverage" assumption is common in the random-utility discrete-choice literature. See, e.g., Perloff and Salop (1985), Caplin and Nalebuff (1991), and Zhou (2017). We explain how to relax this assumption in Section 6.1.

[10] As shown below, in any equilibrium in this paper, the measure of consumers who are indifferent among multiple products is negligible. Therefore, for notational and expositional simplicity, we ignore them throughout the paper.

[11] Note that our derivation of $D(p_i, G_i, p, G)$ does not account for the possibility of atoms in G. This significantly simplifies the notation but incurs no loss of generality because in equilibrium, G has no atom.

A tuple (p, G) is a symmetric pure-price equilibrium (henceforth, simply an equilibrium) if (p, G) is a solution to the following programming problem:

$$\max_{(p_i, G_i)} \pi(p_i, G_i, p, G) \equiv p_i D(p_i, G_i, p, G),$$

subject to the constraint that G_i is an MPC of F.

3 Competitive Advertising

In this section, we assume that all firms have chosen the same price $p(> 0)$ and analyze their strategic interaction only in terms of advertising. In other words, we shut down the firms' pricing decisions and consider a game in which each firm maximizes its demand $D(p, G_i, p, G)$ only through its choice of advertising strategy G_i. Since this can be interpreted as an independent game regarding the firms' competitive information revelation, we refer to it as the *advertising-only game*.

3.1 Main Characterization

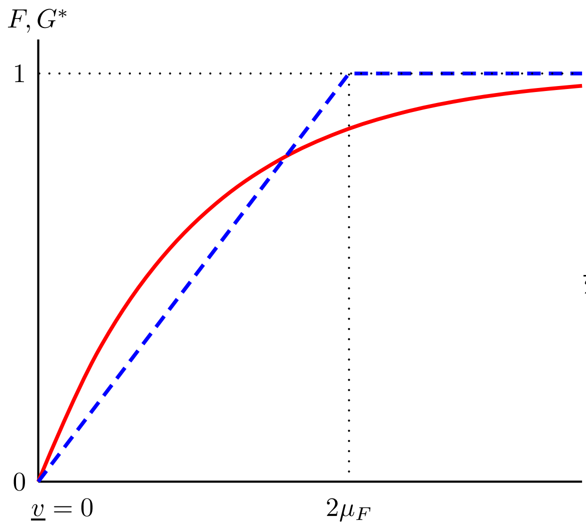
We begin by defining two statistical concepts that play a crucial role in the subsequent analysis.

Definition 1 *A distribution function G_1 is an* $(n-1)$-linear MPC of G_2 over the interval $[v_1, v_2]$ *if (i) G_1^{n-1} is linear over $[v_1, \min\{v_2, \max supp(G_1)\}]$ and (ii) G_1 is an MPC of G_2 over the interval $[v_1, v_2]$.*

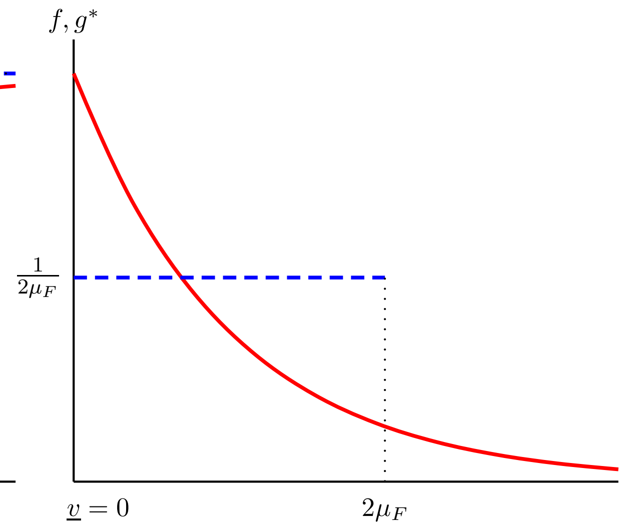
To understand this definition, see Figure 1, which depicts an $(n-1)$-linear MPC of an exponential distribution F with $n = 2$. In this case, $G^* = U[0, 2\mu_F]$ is an $(n-1)$-linear MPC of F over $[0, \infty)$: (i) $(G^*)^{n-1} = G^*$ is linear over $[0, 2\mu_F]$, and (ii) G^* is an MPC of F over $[0, \infty)$ because $\int v dG^*(v) = \mu_F$ and G^* crosses F once from below. This example is unique in that the relevant range covers the entire support. See Figures 2 and 3 for $(n-1)$-linear MPCs over a smaller interval than the support.

Definition 2 *G is an* alternating $(n-1)$-linear MPC *of F if there exists a monotone finite partition $\mathcal{P}^* \equiv \{w_0 = \underline{v}, v_0, w_1, v_1, ...w_m, v_m = \overline{v}\}$ such that (i) $G = F$ over the interval $[w_k, v_k]$ for each $k = 0, \dots, m$, and (ii) G is an $(n-1)$-linear MPC of F over the interval $[v_k, w_{k+1}]$ for each $k = 0, \dots, m-1$.*

This definition builds upon Definition 1. While an $(n-1)$-linear MPC is a local property over a specific interval, an alternating $(n-1)$-linear MPC is a global property that regulates the behavior of G over the entire support. To understand this definition better, consider Figure 2, again



(a) distribution: $F(v)$ (solid) and $G^*(v)$ (dashed)



(b) density: $f(v)$ (solid) and $g^*(v)$ (dashed)

Figure 1: In this figure, F is an exponential distribution with parameter λ (i.e., $F(v) = 1 - e^{-\lambda v}$ for all $v \geq 0$), while G^* is a uniform distribution over $[0, 2\mu_F]$.

with $n = 2$. In this case, G^* is an $(n-1)$-linear MPC of F over the interval $[v^*, \overline{v}]$, which is independent of whether $G^* = F$ below v^*. However, since $G^*(v) = F(v)$ for all $v \leq v^*$, G^* is an alternating $(n-1)$-linear MPC of F.[12]

The following theorem—the main result of this section—shows that there exists a unique equilibrium in the advertising-only game and, more importantly, identifies two necessary and sufficient statistical properties of the equilibrium advertising strategy.

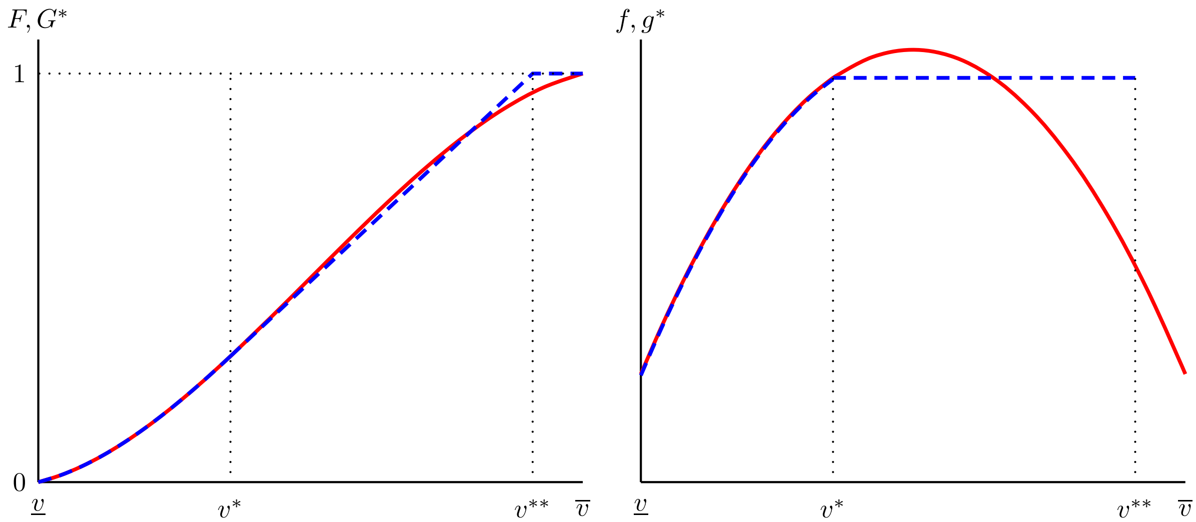
Theorem 1 *Given F, let G^* be a unique distribution such that (i) $(G^*)^{n-1}$ is convex over its support, and (ii) G^* is an alternating $(n-1)$-linear MPC of F. Then, it is the unique equilibrium in the advertising-only game that the firms advertise according to G^*.*

To understand this general result, we apply it to four prominent classes of distribution functions. The first two (Corollaries 1 and 2) clarify and highlight the roles of convexity and $(n-1)$-linear MPC, while the last two (Corollaries 3 and 4) explain how the alternating structure between full information and the $(n-1)$-linear region emerges. We begin by considering the case in which F^{n-1} is convex (that is, F^{n-1} has increasing density).

Corollary 1 (Convex F^{n-1}) *If F^{n-1} is convex, then it is the unique equilibrium in the advertising-only game that the firms provide full information (that is, $G^* = F$).*

Suppose that all other firms provide full product information (i.e., $G_j = F$ for all $j \neq i$).

[12] See also Figure 3, in which G^* is an $(n-1)$-linear MPC of F over $[\underline{v}, v^*]$ and also an alternating $(n-1)$-linear MPC of F.



(a) distribution: $F(v)$ (solid) and $G^*(v)$ (dashed) (b) density: $f(v)$ (solid) and $g^*(v)$ (dashed)

Figure 2: In this figure, $F(v) = -\frac{4}{3}v^3 + 2v^2 + \frac{1}{3}v$ over $[0, 1]$, which gives rise to a single-peaked density function f. G^* coincides with F below v^* and is linear above v^*.

Given symmetric prices, a consumer with expected value v_i purchases product i with probability $F(v_i)^{n-1}$. Therefore, firm i's demand is given by

$$D(p, G_i, p, F) = \int F(v_i)^{n-1} dG_i(v_i).$$

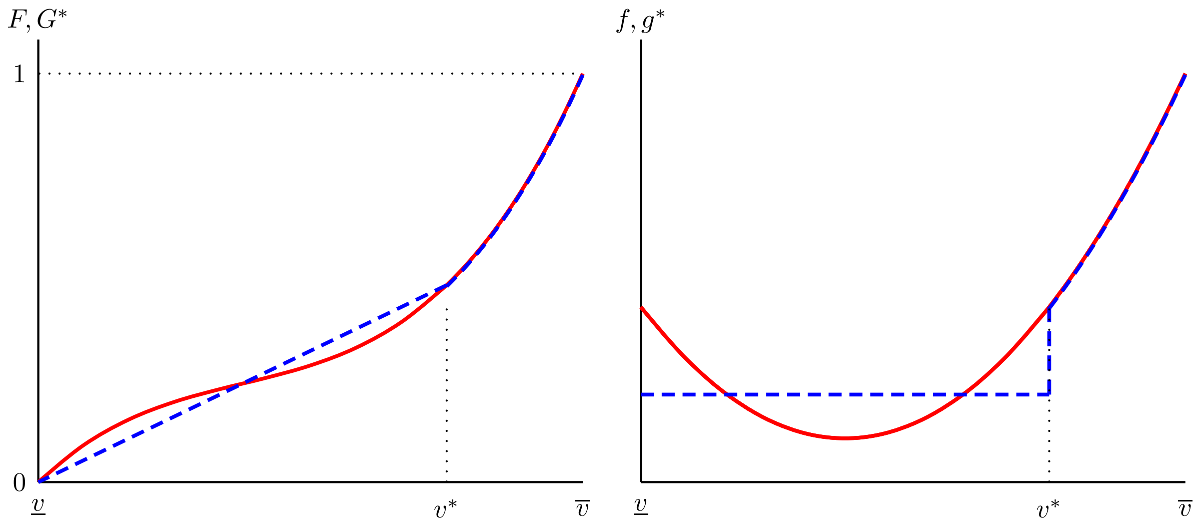
Observe that if F^{n-1} is convex then, by Jensen's inequality, the more dispersed G_i is, the higher firm i's demand is. This suggests that at the optimal solution, the MPC constraint should bind, which is the case when $G_i = F$.

For an economic interpretation, notice that $\prod_{j \neq i} G_j = F^{n-1}$ represents the distribution of consumers' best alternatives to product i. If it has increasing density, that means that firm i's *marginal* returns to inducing a higher expected value increases in v_i. In this case, it is more profitable for firm i to provide more product information, thereby giving high expected values to a subset of consumers, than to suppress product information, thereby yielding a mediocre expected value for all consumers.

By applying the same logic, if F^{n-1} is concave, then full product information is no longer an equilibrium: given $G_j = F$ for all $j \neq i$, firm i's marginal returns to inducing a higher expected value decrease in v_i. Therefore, it is optimal for firm i to reveal no information. The next corollary characterizes the unique equilibrium in this case. See Figure 1 for a visualization of this result.

Corollary 2 (Concave F) *If $n = 2$ and F is concave with (normalized) $\underline{v} = 0$,*[13] *then it is the*

[13]The result and the argument apply unchanged even if $n > 2$, provided that F^{n-1} is concave. However, for F^{n-1}



(a) distribution: $F(v)$ (solid) and $G^*(v)$ (dashed) (b) density: $f(v)$ (solid) and $g^*(v)$ (dashed)

Figure 3: In this figure, $F(v) = \frac{16}{7}v^3 - \frac{18}{7}v^2 + \frac{9}{7}v$ over $[0,1]$, which yields a U-shaped density function f. G^* is linear over $[\underline{v}, v^*]$ and coincides with F above v^*.

unique equilibrium in the advertising-only game that both firms advertise according to $G^* = U[0, 2\mu_F]$.

Suppose that F is concave, and consider firm i's optimal response when firm j provides full information about its product. Opposite to the previous convex case, firm i's profit is higher when G_i is *less* dispersed. In fact, it is optimal for firm i to put all mass on one point μ_F. Given firm i's response, however, firm j has no incentive to provide full information: one profitable deviation is to induce a binary distribution, with one point arbitrarily close to $\underline{v}$ and the other slightly above μ_F. This in turn induces firm i to change its response, leading to a matching-pennies type of competition. In this case, an equilibrium typically consists of players being indifferent over a set of actions. In our model, this means that each firm should be indifferent between spreading and contracting its own distribution. Then, via Jensen's inequality, the uniform distribution, which neutralizes the effects of spreading and contracting, emerges as a unique equilibrium solution.

Corollaries 1 and 2 show that if F^{n-1} has monotone density, then G^* takes one of the two structures over the entire support: $G^* = F$ or $(n-1)$-linear MPC. The next two corollaries show that if F^{n-1} has non-monotone density, G^* combines the above two structures in a natural way.

Corollary 3 (Single-peaked density) *Suppose that F^{n-1} has single-peaked density. Then, it is the unique equilibrium in the advertising-only game that for some $v^* \in [\underline{v}, \overline{v}]$, each firm fully reveals information for $v < v^*$ and advertises according to the $(n-1)$-linear MPC for $v \geq v^*$. If $v^* > \underline{v}$, then G^* is smooth at $v = v^*$.*

to be concave, f must be unbounded at $\underline{v}$: $(F(\underline{v})^{n-1})' = (n-1)F(\underline{v})^{n-2}f(\underline{v}) = 0$ for any finite $f(\underline{v})$.

Corollary 4 (U-shaped density) *Suppose that $n = 2$ and f is U-shaped.*[14] *Then, it is the unique equilibrium in the advertising-only game that for some $v^* \in (\underline{v}, \overline{v}]$, each firm advertises according to the $(n-1)$-linear MPC over $[\underline{v}, v^*]$ and fully reveals information for $v \geq v^*$.*

If F^{n-1} is initially convex and then concave, then G^* initially coincides with F and then takes the $(n-1)$-linear structure (see Figure 2).[15] On the contrary, if F^{n-1} is initially concave and then convex, then G^* initially features the $(n-1)$-linear structure and then follows F (see Figure 3). Intuitively, a firm has an incentive to provide full information over the region where its marginal returns to raising v_i increase in v_i, as in Corollary 1. If the marginal returns decrease in v_i, then the $(n-1)$-linear structure arises as an equilibrium outcome, as in Corollary 2. Therefore, the structure of G^* follows the concave-convex structure of F^{n-1} (i.e., the increasing-decreasing pattern of the density function of F^{n-1}).

We note that in both cases, the full-information region can be degenerate, in which case the equilibrium advertising strategy has a globally linear structure as in Corollary 2. For example, if we slightly modify the exponential distribution in Figure 1 such that f is single-peaked but $f(\underline{v})$ is still greater than the dashed line in the right panel ($1/(2\mu_F)$), then G^* remains globally linear. Similarly, in Figure 3, if f is symmetric around μ_F, then $G^* = U[\underline{v}, \overline{v}]$.

If F^{n-1} takes an even more complicated shape, then the above simple structures may not be sufficient. Nevertheless, Theorem 1 argues that the unique equilibrium advertising strategy G^* always alternates between full information and the $(n-1)$-linear MPC.

3.2 Proof of Theorem 1

We now provide a formal proof for Theorem 1. Recall that G^* is defined to be a distribution that satisfies two statistical properties. Our proof proceeds in three steps. First, we verify that G^* is indeed a symmetric equilibrium strategy (i.e., G^* is a firm's best response if all other firms play G^*) in the advertising-only game. Then, we demonstrate that the two properties of G^* are necessary for any equilibrium. Finally, we prove that given any F, there exists a unique distribution G^* that satisfies the two properties. All details not reported in this subsection can be found in the appendix.

3.2.1 Equilibrium Verification

Suppose that all other firms advertise according to G^*. Then, firm i faces the following constrained maximization problem:

$$\max_{G_i} \int G^*(v_i)^{n-1} dG_i(v_i),$$

[14] The restriction to the duopoly case (i.e., $n = 2$) is imposed for the same reason as for Corollary 2.

[15] The fact that G^* is smooth at $v^*(> \underline{v})$ is a straightforward implication of Theorem 1 because the convexity of $(G^*)^{n-1}$ requires that $f(v^*) \leq g^*(v^*)$, while $f(v^*) \geq g^*(v^*)$ is necessary for G^* to be an MPC of F.

subject to the constraint that G_i is an MPC of F. Our first goal is to prove that $G_i = G^*$ is an optimal solution to this problem. This is a non-trivial problem because firm i's choice set consists of all MPCs of F, and therefore, classical constrained optimization techniques do not apply. Nevertheless, some recent technical developments in the information design literature are applicable to our problem. In particular, the following result obtained by DM—which provides a tractable method to verify the optimality of a distribution in a class of Bayesian persuasion problems—significantly simplifies equilibrium verification in our model.

Theorem 2 (Dworczak and Martini, 2019) *Suppose that F is a distribution function with $supp(F) = [\underline{v}, \overline{v}]$. Consider the following programming problem:*

$$\max_{G} \int_{\underline{v}}^{\overline{v}} u(x) dG(x)$$

subject to the constraint that G is an MPC of F. A distribution G is a solution to the problem if there exists a convex function $\phi : [\underline{v}, \overline{v}] \to \mathcal{R}$ such that (i) $\phi(x) \geq u(x)$ for all $x \in [\underline{v}, \overline{v}]$, (ii) $supp(G) \subset \{x \in [\underline{v}, \overline{v}] : u(x) = \phi(x)\}$, and (iii) $\int_{\underline{v}}^{\overline{v}} \phi(x) dG(x) = \int_{\underline{v}}^{\overline{v}} \phi(x) dF(x)$.

To apply this result to our problem, let $\underline{v}^*$ and $\overline{v}^*$ denote the lower and the upper bounds of $supp(G^*)$, respectively. Note that $\underline{v}^* \geq \underline{v}$ and $\overline{v}^* \leq \overline{v}$ because G^* is an MPC of F. Define a function $\phi : [\underline{v}, \overline{v}] \to \mathcal{R}_+$ as

$$\phi(v) = \begin{cases} G^*(v)^{n-1}, & \text{if } v \in [\underline{v}, \overline{v}^*], \\ \alpha(v - \overline{v}^*) + 1, & \text{if } v \in (\overline{v}^*, \overline{v}], \end{cases}$$

where

$$\alpha \equiv \limsup_{v \to \overline{v}^*-} \frac{G^*(\overline{v}^*)^{n-1} - G^*(v)^{n-1}}{\overline{v}^* - v}. \tag{2}$$

The function ϕ is well defined. If $\overline{v}^* = \overline{v}$, then $\phi(v) = G^*(v)^{n-1}$ over the entire domain. If $\overline{v}^* < \overline{v}$, then G^* must end with an $(n-1)$-linear MPC, in which case $\phi(v)$ simply extends that linear portion above $\overline{v}^*$.

By definition, $\phi(v) \geq G^*(v)^{n-1}$ for all v, which holds with equality for all $v \in supp(G^*) = [\underline{v}^*, \overline{v}^*]$. Since this immediately implies (i) and (ii) in Theorem 2, it suffices to show that

$$\int_{\underline{v}}^{\overline{v}} \phi(v) dG^*(v) = \int_{\underline{v}}^{\overline{v}} \phi(v) dF(v).$$

Since $G^*(v) = F(v)$ whenever $v \in [w_k, v_k]$ for some $k = 0, ..., m$, we have

$$\int_{\underline{v}}^{\overline{v}} \phi(v)dG^*(v) - \int_{\underline{v}}^{\overline{v}} \phi(v)dF(v) = \sum_{k=0}^{m-1} \left(\int_{v_k}^{w_{k+1}} \phi(v)dG^*(v) - \int_{v_k}^{w_{k+1}} \phi(v)dF(v) \right),$$

The desired result then follows from the fact that for each $k = 0, ..., m-1$,

$$\int_{v_k}^{w_{k+1}} \phi(v)dG^*(v) = \int_{v_k}^{w_{k+1}} \phi(v)dF(v).$$

This crucial equality holds because $\phi(v)$ is linear over $[v_k, w_{k+1}]$ and G^* is an MPC of F over the same interval. Intuitively, the firm is "risk neutral" over $[v_k, w_{k+1}]$ and, therefore, indifferent between F and its MPC G^* over the interval.

3.2.2 Necessity of G^*

Now we demonstrate that if G is a symmetric equilibrium in the advertising-only game, then it must satisfy the two properties in Theorem 1.

Lemma 1 *In equilibrium, G^{n-1} must be convex over its support.*

For the intuition, suppose that G^{n-1} is strictly concave around v. In this case, as explained in the intuition for Corollary 2, it is optimal for firm i to concentrate all local mass on one point. This, however, violates the symmetry requirement that G must be a best response to itself. Economically, a firm can always conceal its product information as much as it wishes and, therefore, faces no limit in contracting G_i. By contrast, due to the MPC constraint, a firm cannot induce a more dispersed distribution than F. Therefore, in equilibrium, a firm's marginal returns to raising v_i (i.e., the density function of $(G^*)^{n-1}$) may increase (insofar as the MPC constraint binds) but can never strictly decrease in v_i (otherwise, the firm would exploit it by providing less information).

One immediate but important corollary of Lemma 1 is that equilibrium G must have a convex support, that is, $supp(G) = [\underline{v}^*, \overline{v}^*]$ for some $\underline{v}^*, \overline{v}^* \in [\underline{v}, \overline{v}]$. This is because G^{n-1}, which is also a distribution function, can be convex only when its support is convex and $supp(G) = supp(G^{n-1})$.

Lemma 2 *In equilibrium, G must be an alternating $(n-1)$-linear MPC of F.*

A key to this result is that, given that G^{n-1} is convex, it is weakly beneficial for firm i to spread G_i as much as possible. In particular, since G is an MPC of F, by Jensen's inequality,

$$\int G^{n-1}dG \leq \int G^{n-1}dF,$$

provided that G has the same support as F.[16] However, in equilibrium, firm i should weakly prefer G to F, which is always feasible. This implies that the above inequality must hold with equality. Since G^{n-1} is convex, the equality can hold either when $G = F$ or when G^{n-1} is linear. This is the fundamental reason that G^* always takes this particular alternating structure.

3.2.3 Existence and Uniqueness of G^*

We complete the proof of Theorem 1 by demonstrating that for each F, there exists a unique G^*. Our existence proof is constructive and, therefore, can also be used to analyze specific examples.

Preliminaries for existence. For each $\tilde{v} \in [\underline{v}, \overline{v}]$ and $a \in \mathcal{R}_+$, let $H_{\tilde{v},a}$ denote the distribution function that coincides with F below $\tilde{v}$ and is $(n-1)$-linear above $\tilde{v}$ with slope a. Formally,

$$H_{\tilde{v},a}(v)^{n-1} = \begin{cases} F(v)^{n-1}, & \text{if } v \leq \tilde{v}, \\ \min\{a(v-\tilde{v}) + F(\tilde{v})^{n-1}, 1\}, & \text{if } v \in (\tilde{v}, \overline{v}), \\ 1, & \text{if } v = \overline{v}. \end{cases}$$

Note that, unlike F, $H_{\tilde{v},a}$ may place positive mass on $\overline{v}$ (if $a(\overline{v}-\tilde{v}) > 1 - F(\tilde{v})^{n-1}$) or reach 1 before $\overline{v}$ (if $a(\overline{v}-\tilde{v}) < 1 - F(\tilde{v})^{n-1}$). For examples, see the dashed and dotted lines in Figure 4.

Given $H_{\tilde{v},a}$, we define the following function:

$$W_{\tilde{v},a}(v) \equiv \int_{\underline{v}}^{v} (F(x) - H_{\tilde{v},a}(x))dx.$$

As is well known, $H_{\tilde{v},a}$ is an MPC of F over $[v_1, v_2]$ if and only if $W_{\tilde{v},a}(v_1) = W_{\tilde{v},a}(v_2) = 0$ and $W_{\tilde{v},a}(v) \geq 0$ for all $v \in [v_1, v_2]$ (see Section 6.D in Mas-Colell et al., 1995). We use this result to check the MPC constraint for G^*.

Consider a distribution $H_{\tilde{v},(F(\tilde{v})^{n-1})'}$, the $(n-1)$-linear portion of which is tangent to F^{n-1} at $v = \tilde{v}$. By construction, $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) = 0$ for all $v \leq \tilde{v}$. For $v > \tilde{v}$, we distinguish between the following two cases:

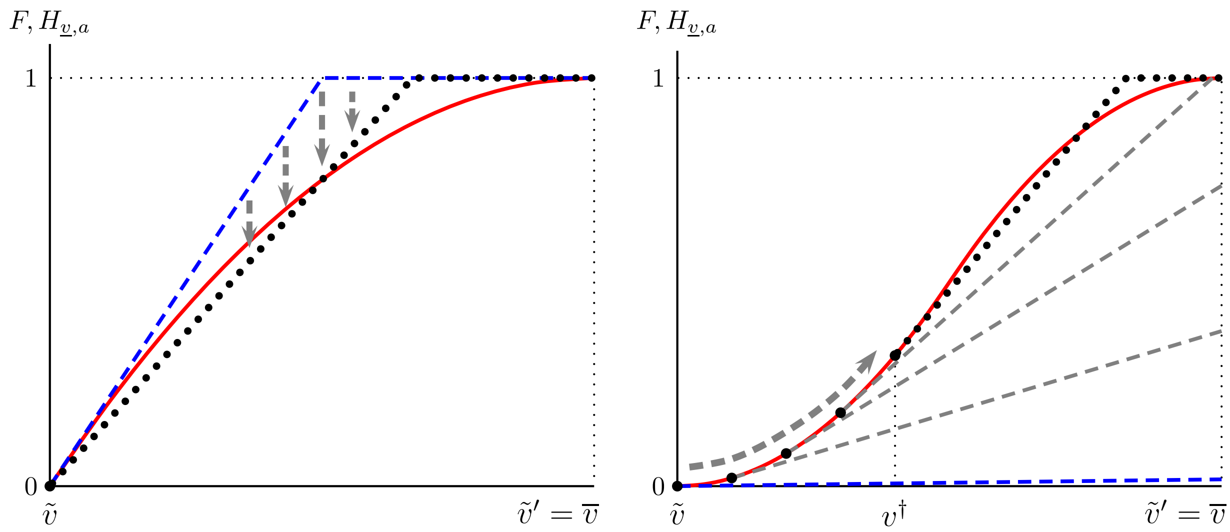
(i) $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(\tilde{v}') \leq 0$ for some $v \in (\tilde{v}, \overline{v}]$.

(ii) $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) > 0$ for all $v \in (\tilde{v}, \overline{v}]$.

The following lemmas illustrate how to find an alternating $(n-1)$-linear MPC of F in each case.

Lemma 3 *If $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) \leq 0$ for some $v \in (\tilde{v}, \overline{v})$, then there exist $a^* \in (0, (F(\tilde{v})^{n-1})']$ and $\tilde{v}' \in (\tilde{v}, \overline{v}]$ such that $H_{\tilde{v},a^*}$ is an MPC of F over $[\tilde{v}, \tilde{v}']$.*

[16] The same-support assumption is only for expositional simplicity and fails in general (see Figure 1). In the proof of Lemma 2 in the appendix, we show that a similar argument applies even to the case in which $supp(G) \neq supp(F)$.



(a) An illustration of Lemma 3 (b) An illustration of Lemma 4

Figure 4: This figure visualizes the logic behind Lemmas 3 and 4. In both panels, $\tilde{v} = \underline{v} = 0$ and $\overline{v} = 1$. The distribution function used for the left panel is $F(v) = -v^2 + 2v$, while that for the right panel is $F(v) = 2v^2$ for $v \in [0, 1/2]$ and $F(v) = -2(v-1)^2 + 1$ for $v \in (1/2, 1]$.

Lemma 4 *If $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) > 0$ for all $v \in (\tilde{v}, \overline{v})$, then either F^{n-1} is convex over $[\tilde{v}, \overline{v}]$, or there exist $v^\dagger, \tilde{v}' \in (\tilde{v}, \overline{v}]$ such that $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ is an MPC of F over $[\tilde{v}, \tilde{v}']$.*

Figure 4 visualizes the arguments in Lemmas 3 and 4 (for the case in which $n = 2$). In the left panel, F (solid red) is concave. Therefore, $H_{\tilde{v},(F(\tilde{v})^{n-1})'}$ (dashed blue) is uniformly above F, and thus, it is not an MPC of F. In this case, we can find an $(n-1)$-linear MPC of F (dotted black) simply by reducing the slope of $H^{n-1}_{\tilde{v},a}$ (that is, rotating down $H^{n-1}_{\tilde{v},a}$).

In the right panel, $H_{\tilde{v},(F(\tilde{v})^{n-1})'}$ is uniformly below F. Then, simply increasing the slope of $H^{n-1}_{\tilde{v},a}$ does not work, as doing so yields $W_{\tilde{v},a}(\tilde{v} + \varepsilon) < 0$ for small ε, violating the MPC constraint. In this case, one can move $\tilde{v}$ to the right, which reduces the gap between F and $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ (observe how the dashed gray line shifts as $v^\dagger$ rises). If F^{n-1} is convex above $\tilde{v}$, then $W_{v^\dagger,(F(v^\dagger)^{n-1})'}$ remains bounded above zero for all values of $v^\dagger < \overline{v}$, in which case the only possible alternating $(n-1)$-linear MPC is F itself. Otherwise, Lemma 4 states that there exist $v^\dagger \in (\tilde{v}, \overline{v})$ and $\tilde{v}' \in (v^\dagger, \overline{v}]$ such that $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ is an alternating $(n-1)$-linear MPC of F over $[\tilde{v}, \tilde{v}']$.

Construction of G^*. Using $H_{\tilde{v},a}$ for different values of $\tilde{v}$ and a as building blocks, we recursively construct G^* in the *forward* direction. We begin by setting $\tilde{v} = \underline{v}$. Then, we construct G^* in the following three cases:

(1) $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(\tilde{v}') \leq 0$ for some $v \in (\tilde{v}, \overline{v}]$: in this case, we set $G^*(v) = H_{\tilde{v},a^*}(v)$ for $v \in [\tilde{v}, \tilde{v}']$, where a^* and $\tilde{v}'$ are defined in Lemma 3. If there exist multiple solutions of $(a^*, \tilde{v}')$,

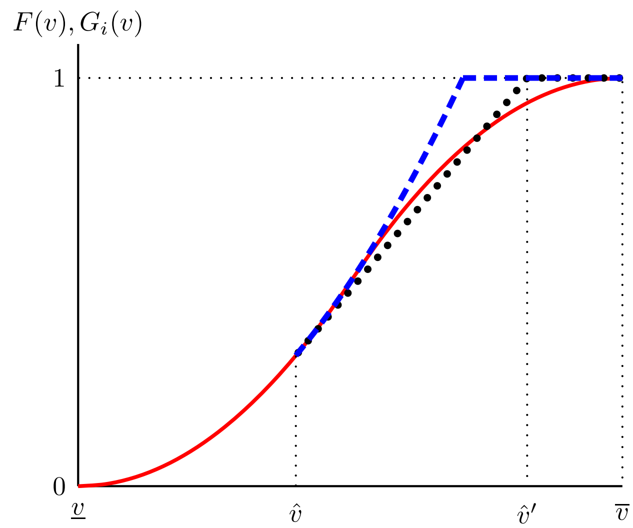


Figure 5: This figure visualizes the argument for the uniqueness of G^*. The underlying distribution used for this figure (solid red curve) is $F(v) = 2v^2$ if $v \in [0, 1/2]$ and $F(v) = -2(v-1)^2 + 1$ if $v \in (1/2, 1]$. The black dotted curve represents G_1, while the blue dashed curve depicts G_2, when $G_1(\hat{v}+\varepsilon) < G_2(\hat{v}+\varepsilon)$ for ε sufficiently small.

we select the *smallest* a^*.

(2) $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) > 0$ for all $v \in (\tilde{v}, \overline{v})$, and F is not convex over $[\tilde{v}, \overline{v}]$: in this case, we set $G^*(v) = H_{v^\dagger,(F(v^\dagger)^{n-1})'}(v)$ for $v \in [\tilde{v}, \tilde{v}']$, where $v^\dagger$ and $\tilde{v}'$ are defined in Lemma 4. If there exist multiple solutions of $(v^\dagger, \tilde{v}')$, then we select the *smallest* $v^\dagger$.

(3) F is convex over $[\tilde{v}, \overline{v}]$: in this case, we set $G^* = F$ for $v \in [\tilde{v}, \overline{v}]$, which completes the construction.

In cases (1) and (2), the construction is complete if $\tilde{v}' = \overline{v}$. Otherwise, we construct G^* for the next *block* by setting $\tilde{v} = \tilde{v}'$ and repeating the same process.

This construction clearly guarantees that G^* has the alternating $(n-1)$-linear MPC structure. For the convexity of $(G^*)^{n-1}$, note that we choose the smallest a^* in case (1) and the smallest $v^\dagger$ in case (2). This ensures that the slope of G^* at the beginning of the next block is no less than the slope at the end of the previous block, leading to the global convexity of $(G^*)^{n-1}$.

Uniqueness of G^*. It remains to show that G^* is unique, that is, there exists only one distribution that satisfies the properties in Theorem 1. We provide our main proof idea here, relegating the complete proof to the appendix.

Suppose that there are two distinct distributions, G_1 and G_2, that satisfy the properties in Theorem 1. Letting $\hat{v}$ denote the first point at which G_1 and G_2 diverge, the two distribution functions have the relationship depicted in Figure 5. In particular, $G_1(v) \leq G_2(v)$ for any $v \geq \hat{v}$, and therefore, G_1 dominates G_2 in the sense of first-order stochastic dominance over a certain interval. This is primarily because each G_i^{n-1} is either linear or strictly convex, and therefore, if $G_2(\hat{v}+\varepsilon) > G_1(\hat{v}+\varepsilon)$, then it must hold at least over the interval where G_1^{n-1} is linear. This, however, implies that at least one of them is not an MPC of F.

3.3 The Effect of Competition

Theorem 1 illustrates how competition influences firms' advertising incentives and, therefore, shapes equilibrium advertising content. A natural and important question in this context is how the intensity of competition affects advertising content, in particular, whether more intense competition causes firms to provide more product information. Ivanov (2013) addresses the same economic question but allows for only a restricted set of advertising strategies that can be ranked in the sense of rotation order (Johnson and Myatt, 2006). He finds that when there are sufficiently many firms, each firm chooses to provide as much product information as possible. Our results below show that the same economic conclusion can be drawn even with no restriction on feasible advertising content.

Proposition 1 *Let G_n^* be the equilibrium advertising strategy with n firms. For any $v \in (\underline{v}, \overline{v})$, there exists $n(v)$ such that if $n \geq n(v)$ then $G_n^*(v) = F(v)$.*

Technically, this result is due to the fact that F^{n-1} becomes more convex as n increases: in the proof in the appendix, we show that if $\overline{v} < \infty$, then F^{n-1} is necessarily globally convex when n is sufficiently large. The argument for the case in which $\overline{v} = \infty$ is more subtle but again relies on the fact that F^{n-1} becomes more convex as n increases. Economically, this is precisely the effect of competition. When there are many competitors, the probability of each firm's winning a consumer is low. In that case, it is better for a firm to provide all product information and, therefore, serve at least a few loyal consumers than to give compromised values to consumers by concealing some product information and, therefore, lose them all.

With some regularity, we can obtain a clearer and stronger result. Specifically, suppose that the density function of F^{n-1} is singled-peaked for any n.[17] Then, by Corollary 3, $(G^*)^{n-1}$ has at most one linear portion and, therefore, can be characterized by one cutoff v^* such that $G^*(v) = F(v)$

[17] This assumption is satisfied by many canonical distribution functions. In particular, this assumption holds whenever f is log-concave because

$$(F^{n-1})'' = (n-1)\{(n-2)F^{n-3}f^2 + F^{n-2}f'\} = (n-1)F^{n-2}f\left((n-2)\frac{f}{F} + \frac{f'}{f}\right),$$

and if f is log-concave, then both f/F and f'/f decrease.

if $v \leq v^*$, while $(G^*)^{n-1}$ is linear over $[v^*, \overline{v}^*]$. The following result shows that in this case, the full-information region $[\underline{v}, v^*]$ always expands in n and converges to $[\underline{v}, \overline{v}]$ as n tends to infinity.

Proposition 2 *Let G_n^* be the equilibrium advertising strategy with n firms. Suppose that the density function of F^{n-1} is single-peaked for any $n \geq 2$. For each n, let v_n^* denote the cutoff such that $G_n^*(v) = F(v)$ if $v \leq v_n^*$ and $(G_n^*)^{n-1}$ is linear above v_n^*. Then, $v_n^* \leq v_{n+1}^*$ for any $n \geq 2$ and $\lim_{n\to\infty} v_n^* = \overline{v}$.*

4 Competitive Pricing

In this section, we investigate the effects of competitive advertising on the equilibrium price and welfare. We take the equilibrium advertising strategy G^* characterized in Theorem 1 as given and determine the market price p^*. In addition, we compare p^* to the market price in the full-information benchmark (that is, in the model in which $G_i = F$ for all i).

4.1 Equilibrium Price

Given the symmetric advertising strategy G^*, our problem reduces to the random-utility discrete choice model of Perloff and Salop (1985). Assuming that all other firms charge p^*, a consumer purchases product i if and only if $v_i - p_i > v_j - p^*$ for all $j \neq i$. Therefore, an individual firm's optimal pricing problem is given by

$$\max_{p_i} \pi(p_i, G^*, p^*, G^*) = p_i D(p_i, G^*, p^*, G^*) = p_i \int G^*(v_i - p_i + p^*)^{n-1} dG^*(v_i).$$

The firm's first-order condition can be rearranged as follows:

$$\frac{1}{p^*} = -\frac{\partial D(p_i, G^*, p^*, G^*)/\partial p_i|_{p_i=p^*}}{D(p^*, G^*, p^*, G^*)}. \tag{3}$$

This is a well-known optimal pricing formula, which states that the optimal price (markup) is inversely related to the proportion of marginal consumers $(\partial D(p_i, G^*, p^*, G^*)/\partial p_i)$ among those who purchase product i $(D(p_i, G^*, p^*, G^*))$. Intuitively, when there are more consumers on the margin, firm i faces a stronger incentive to capture them by lowering its price.

Imposing the symmetry requirement (that p^* must also be firm i's optimal price) and using the fact that $D(p^*, G^*, p^*, G^*) = 1/n$, we arrive at the following result.

Proposition 3 *In equilibrium, each firm charges*

$$p^* = -\frac{1/n}{\partial D(p_i, G^*, p^*, G^*)/\partial p_i|_{p_i=p^*}} = \frac{1}{n(n-1)\int (G^*)^{n-2} g^* dG^*}. \tag{4}$$

Proposition 3 provides a unique equilibrium price candidate: if a symmetric pure-price equilibrium exists, then the equilibrium price must be equal to p^* in equation (4). Note, however, that we have not yet established the existence of equilibrium. We address this issue and provide a sufficient condition for equilibrium existence in Section 5.

The following example illustrates how Proposition 3 can be combined with Theorem 1 to determine the unique equilibrium candidate (p^*, G^*).

Example 1 (Exponential distribution) *Suppose that $F(v) = 1 - e^{-\lambda v}$, where $\lambda > 0$, $\underline{v} = 0$, and $\overline{v} = \infty$. Note that $\mu_F = 1/\lambda$.*

If $n = 2$, then $F^{n-1} = F$ is concave. Then, Corollary 2 implies that $G^* = U[0, 2\mu_F] = U[0, 2/\lambda]$. It is then straightforward to calculate that

$$p^* = \frac{1}{2\int_0^{2/\lambda} \left(\frac{1}{2\lambda}\right)^2 dv} = \frac{1}{\lambda}.$$

If $n > 2$, then the density function of F^{n-1} is single-peaked. Therefore, by Corollary 3, there exists $v^* (> 0)$ such that $G^*(v) = F(v)$ if $v \leq v^*$ and $(G^*)^{n-1}$ is linear above v^*. Since G^* is an MPC of F,

$$E_F[v|v \geq v^*] = E_{G^*}[v|v \geq v^*] = \int_{v^*}^{\overline{v}^*} v \frac{dG^*(v)}{1 - G(v^*)}.$$

Applying the closed-form solution for G^* over $[v^*, \overline{v}^*]$ and $f(v) = \lambda e^{-\lambda v}$, the equality reduces to

$$n(n-1)F(v^*)^{n-2}(1 - F(v^*)) = \frac{1 - F(v^*)^n}{1 - F(v^*)} - nF(v^*)^{n-1}.$$

Combining this with the formula in Proposition 3, it can be shown that

$$\frac{1}{p^*} = \lambda(F(v^*)^n + nF(v^*)^{n-1}(1 - F(v^*)) + n(n-1)F(v^*)^{n-2}(1 - F(v^*))^2) = \lambda.$$

4.2 Price and Welfare under Competitive Advertising

Does providing more product information always benefit consumers? The answer is not trivial because, as shown in Proposition 3, the market price responds to the amount of information provided (revealed). To systematically analyze this problem, we first compare the equilibrium price p^* to the equilibrium price under full product information and then discuss the welfare effects of competitive advertising.

Price effects of competitive advertising. Let p^F denote the equilibrium price under full information (i.e., $G_i = F$ for all i). By the same logic as for Proposition 3, p^F is given by

$$p^F = -\frac{1/n}{\partial D(p_i, F, p^*, F)/\partial p_i|_{p_i=p^*}} = \frac{1}{n(n-1)\int F^{n-2} f dF}. \tag{5}$$

It seems likely that competitive advertising leads to a lower price than that in the full-information benchmark, that is, $p^* \leq p^F$. In the Perloff-Salop framework, product differentiation enables firms to charge positive markups: note that if F is degenerate, then the problem reduces to standard Bertrand competition, and therefore, $p^F = 0$. Combining this with the fact that G^* is an MPC of F, it is plausible that $p^* \leq p^F$.

The above intuition is not complete, however, because mean-preserving spread (contraction) is not an appropriate measure of product differentiation (preference diversity) in this context: Perloff and Salop (1985) note that p^F may or may not increase when F changes in terms of mean-preserving spread.[18] While their result does not directly apply to our model—the alternating $(n-1)$-linear structure is only a *particular* type of MPC—we also find that the price effect of competitive advertising is ambiguous, as demonstrated by the following example.

Example 2 *Suppose that $n = 2$. In this case, by equations (4) and (5),*

$$p^* < p^F \Leftrightarrow \int f^2 dv < \int (g^*)^2 dv.$$

(a) Decreasing density: Suppose that $f(v) = \frac{2}{3} - \frac{2}{9}v$ for $v \in [0, 3]$. In this case, $\int f^2 dv = 4/9$. In addition, since $f(v)$ decreases in v, by Corollary 2, $G^* = U[0, 2]$. Therefore,

$$\int (g^*)^2 dv = \frac{1}{2} > \frac{4}{9} = \int f^2 dv \Rightarrow p^* < p^F.$$

(b) U-shaped density: Suppose that f is U-shaped over $[0, 2]$ and symmetric around 1. In this case, $\int f^2 dv > 1/2$.[19] From the symmetric U shape of f, it is straightforward that

[18]Two recent studies, Zhou (2017) and Choi et al. (2018), show that dispersive order (which requires the quantile function of one distribution to be steeper than that of the other distribution at every interior quantile) provides a clear prediction about the change in p^F. However, F and G^* cannot be ranked in terms of dispersive order.

[19]By the Cauchy-Schwartz inequality, whenever f is not constant,

$$\int_0^2 f(v)^2 dv = \int_0^2 f(v)^2 dv \cdot \int_0^2 \frac{1}{2} dv > \left(\int_0^2 f(v) \frac{1}{\sqrt{2}} dv \right)^2 = \left(\frac{1}{\sqrt{2}} \int_0^2 f(v) dv \right)^2 = \frac{1}{2}.$$

$G^* = U[0,2]$. *Therefore,*

$$\int (g^*)^2 dv = \frac{1}{2} < \int f^2 dv \Rightarrow p^* > p^F.$$

To understand this ambiguity result, observe that G^* differs from F in two ways: (i) a smaller support (because G^* is an MPC of F) and (ii) flatter density (because g^* is uniform whenever $G^* \neq F$). Due to the convexity of the square function, these two have opposing effects on the value of $\int (g^*)^2 dv$ relative to $\int f^2 dv$. The former raises $\int (g^*)^2 dv$ because $f(a)^2 + f(b)^2 \leq (f(a)+f(b))^2$. On the contrary, the latter reduces $\int (g^*)^2 dv$ because $f(a)^2 + f(b)^2 \geq 2((f(a)+f(b))/2)^2$. In Case (b), G^* has the same support as F, and thus only the latter effect is present, leading to $p^* > p^F$. In Case (a), both effects are operative, but the former effect is stronger than the latter, which leads to a lower price under competitive advertising.

Welfare effects of competitive advertising. In our model, a consumer purchases product i if and only if $v_i > v_j$ for all $j \neq i$, where v_1, ..., v_n are independently and identically drawn according to G^*. Therefore, social surplus is given by

$$SS_n(G^*) \equiv E_{G^*}[\max\{v_1, \ldots, v_n\}] = \int v dG^*(v)^n.$$

Similarly, since each consumer purchases one of the products, consumer surplus is given by

$$CS_n(p^*, G^*) \equiv SS_n(G^*) - p^*.$$

Let $SS_n(F)$ and $CS_n(p^F, F)$ denote the corresponding values for the full-information benchmark.

The following result implies that competitive advertising always reduces social surplus relative to full information.

Lemma 5 *Suppose that G_1 is an MPC of G_2. Then, for any n, $SS_n(G_1) \leq SS_n(G_2)$.*

Intuitively, having more product information enables consumers to be more likely to choose the best product and, therefore, always contributes to social welfare.

By contrast, the effect of competitive advertising on consumer surplus is ambiguous in general. This is precisely because of the ambiguous price effect (see Example 2). In particular, competitive advertising, which suppresses some product information, can contribute to consumer welfare if p^* is significantly lower than p^F. It can be shown that in Case (a) of Example 2, consumer surplus is indeed higher under competitive advertising. This suggests that if a third-party imposes a policy that requires firms to provide full information, then social surplus unambiguously increases, but consumer surplus may or may not increase, depending on the shape of F.

5 Advertising and Pricing: Equilibrium Existence

Theorem 1 pins down the equilibrium advertising strategy G^* by considering the advertising-only game with a fixed symmetric price. By contrast, Proposition 3 determines the equilibrium price p^* by solving a firm's optimal pricing problem given G^*. In our market game, however, a firm can make a compound deviation in which it chooses $p_i \neq p^*$ *and* $G_i \neq G^*$. In this section, we study such compound deviations. The analysis not only illustrates how a firm's pricing decision interacts with its advertising decision but is also necessary to establish equilibrium existence in our full model.

5.1 Optimal Advertising for Non-equilibrium Prices

We begin by studying how an individual firm's optimal advertising strategy depends on its price. Specifically, we characterize firm i's optimal advertising strategy G_i^* when the firm chooses p_i and all other firms play (p^*, G^*). Formally, for each p_i, we search for G_i^* that solves

$$\max_{G_i} \int_{\underline{v}}^{\overline{v}} G^*(v - p_i + p^*)^{n-1} dG_i(v), \tag{6}$$

subject to the constraint that G_i is an MPC of F.

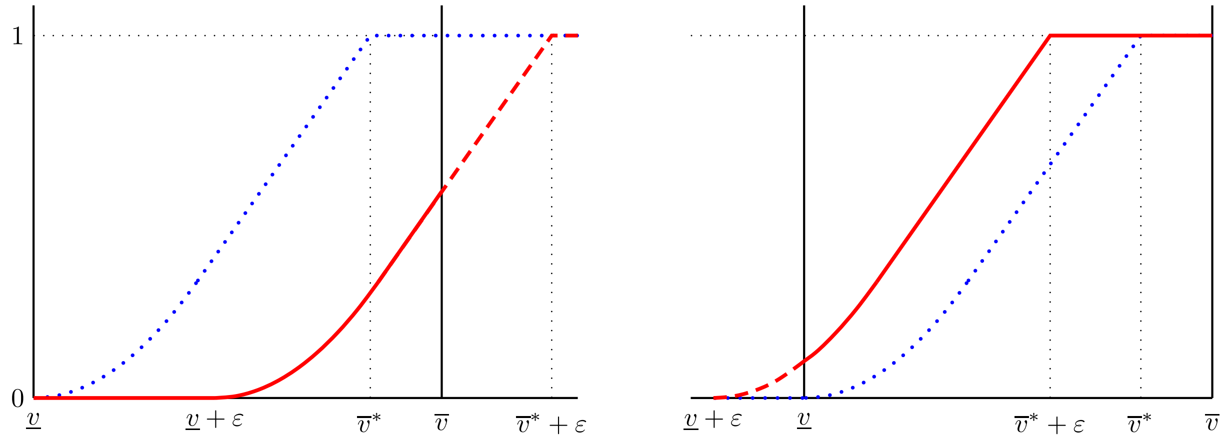
The following proposition provides a full characterization of G_i^*. We note that the advertising strategy reported below is not necessarily a unique optimal strategy, but it is one with a particularly simple structure.

Proposition 4 *Suppose that all other firms play (p^*, G^*). For each $p_i > 0$, an optimal advertising strategy G_i^* for firm i is given as follows:*

1. *If $p_i - p^* \geq \overline{v} - \overline{v}^*$, then $G_i^* = F$.*
2. *If $p_i - p^* \leq \mu_F - \overline{v}^*$, then $G_i^* = \delta_{\mu_F}$.*
3. *If $p_i - p^* \in (\mu_F - \overline{v}^*, \overline{v} - \overline{v}^*)$, then*

$$G_i^*(v) = \begin{cases} F(v), & \text{if } v \leq \psi, \\ F(\psi), & \text{if } v \in (\psi, \overline{v}^* + p_i - p^*), \\ 1, & \text{if } v \geq \overline{v}^* + p_i - p^*, \end{cases}$$

where ψ is the value such that $E_F[v | v \geq \psi] = \overline{v}^* + p_i - p^*$.



(a) Case 1 ($\varepsilon > \overline{v} - \overline{v}^*$) (b) Case 3 ($\varepsilon \in (\mu_F - \overline{v}^*, \overline{v} - \overline{v}^*)$)

Figure 6: An illustration of Proposition 4. The dotted blue lines represent firm i's value function $G^*(v)^{n-1}$ when firm i posts the equilibrium price $p_i = p^*$. The red lines depict firm i's shifted value function $G^*(v-\epsilon)^{n-1}$, where $\varepsilon = p_i - p^*$. Note that only the values of $G^*(v-\varepsilon)^{n-1}$ for $v \in supp(F) = [\underline{v}, \overline{v}]$ (solid red) affect the firm's advertising choice.

To understand this result, first note that charging $p_i \neq p^*$ shifts firm i's value function $G^*(v - p_i + p^*)^{n-1}$ rightward (if $p_i > p^*$) or leftward (if $p_i < p^*$): see Figure 6 for the illustration of each case. Then, as in Section 3, the optimal advertising strategy is determined by the concave-convex structure of the shifted value function.

If p_i is sufficiently large (Case 1 in Proposition 4), then a consumer purchases product i only when she values it more than the other products by at least $p_i - p^*$. As shown in the left panel of Figure 6, this generates a large rightward jump of $G^*(v - p_i + p^*)^{n-1}$. This makes the resulting value function $G^*(v-p_i+p^*)^{n-1}$ convex over $supp(F) = [\underline{v}, \overline{v}]$ (solid red line). Then, by the same argument as for Corollary 1, it is optimal for firm i to reveal all product information ($G_i^* = F$). Intuitively, if firm i charges a significantly higher price than the other firms, it can serve only sufficiently loyal consumers (who value product i considerably more than the other products). Providing full product information maximizes the number of such loyal consumers and, therefore, is an optimal advertising strategy for the firm.

On the contrary, if p_i is sufficiently low (Case 2), then it is optimal for firm i to provide no product information (i.e., G_i^* places all mass on μ_F). Intuitively, if firm i offers a sufficiently lower price than the other firms, then a consumer would purchase product i even if she had an average value μ_F for product i and a very high value $\overline{v}^*$ for another product. If so, clearly, it is optimal for firm i to maximize the number of average-value consumers by concealing all product information.

When p_i is neither too high nor too low (Case 3), the optimal advertising strategy combines the above two structures: there exists a threshold ψ such that the firm reveals all information below ψ but pools all values above ψ. The right panel of Figure 6 shows that the shifted value function is neither convex nor concave over $[\underline{v}, \overline{v}]$. By applying Theorem 2, it can be shown that it is optimal to pool all high values at $\overline{v}^* + p_i - p^*$ and provide full information for consumers below a certain threshold. This structure is optimal because the former pooling maximizes the number of consumers who purchase product i with probability 1, while the latter revelation allows the firm to capture at least some consumers with low values.

Note that the optimal advertising strategy moves continuously from no information to full information: as p_i increases, the full-information region $[\underline{v}, \psi]$ expands, while the pooling region $[\psi, \overline{v}]$ shrinks (Case 3). When p_i is sufficiently small, the full-information region is degenerate, and thus no information is optimal (Case 1). On the contrary, if p_i is sufficiently large, then the pooling region vanishes, and thus full information is optimal (Case 2).

5.2 Equilibrium Existence

As discussed above, (p^*, G^*) is an equilibrium if and only if no compound deviation is profitable. Proposition 4 significantly reduces the technical burden of proving the existence of equilibrium because it suffices to verify whether any deviation (p_i, G_i^*) in Proposition 4 is profitable. In other words, since G_i^* dominates any other advertising strategy G_i given p_i, the unique equilibrium candidate (p^*, G^*) is an equilibrium if and only if

$$p^* \in \arg\max_{p_i} \pi(p_i, G_i^*, p^*, G^*) \equiv p_i D(p_i, G_i^*, p^*, G^*).$$

Exploiting this observation, we provide a sufficient condition for equilibrium existence, which is reported in the following result.

Theorem 3 *If f is log-concave, then (p^*, G^*) is a unique symmetric pure-price equilibrium.*

It is well known that the log-concavity of the density function—which is satisfied by various canonical distributions—imposes sufficient regularity on the behavior of the distribution function and, therefore, allows us to obtain general (i.e., not distribution-specific) results in many economic problems (see Bagnoli and Bergstrom, 2005). In particular, it is a sufficient condition that ensures the existence of pure-price equilibrium in the standard Perloff-Salop model (see Caplin and Nalebuff, 1991; Quint, 2014).

While Theorem 3 also utilizes several desirable properties of log-concave density, it is not a direct implication of the existing results in the literature. Among others, a firm faces a stronger incentive to deviate in our model (where it can make a compound deviation) than in the standard Perloff-Salop model (where it can adjust only its price). This implies that a sufficient condition for the standard Perloff-Salop model is not necessarily sufficient for our model and that Theorem 3 requires a separate proof.

5.2.1 Proof of Theorem 3

Since a similar argument can be applied to other related problems, we explain our proof of Theorem 3 in some detail. Our proof consists of the following two steps.

Step 1: Defining a pseudo-demand function. Let $\tilde{D}(p_i)$ denote firm i's demand when it plays (p_i, G_i^*) while all other firms play (p^*, G^*) (i.e., $\tilde{D}(p_i) \equiv D(p_i, G_i^*, p^*, G^*)$). As explained above, by Proposition 4, it suffices to show that

$$\pi(p^*, G^*, p^*, G^*) = p^* D(p^*, G^*, p^*, G^*) \geq \pi(p_i, G_i^*, p^*, G^*) = p_i \tilde{D}(p_i) \text{ for any } p_i.$$

Although this problem is much simpler than the original problem (with all compound deviations), it is still not sufficiently tractable because the structure of G_i^* depends on p_i. We solve this issue by considering another function that is closely related to, but has a simpler structure than, G_i^*.

Define a *pseudo-value* function $\varphi : \mathcal{R} \to \mathcal{R}_+$ as follows:

$$\varphi(v) = \begin{cases} G^*(v)^{n-1}, & \text{if } v \leq \overline{v}^*, \\ 1 + (n-1)g^*(\overline{v}^*)(v - \overline{v}^*), & \text{if } v > \overline{v}^*. \end{cases}$$

In other words, φ follows $G^*(v)^{n-1}$ until $\tilde{v}^*$ (the upper bound of $supp(G^*)$) and then linearly extends it above $\tilde{v}^*$ (see Figure 7). If f is log-concave, then this function is well defined: F^{n-1} has single-peaked density for any n (see footnote 17). Therefore, there exists $v^* \in [\underline{v}, \overline{v}]$ such that $G^*(v) = F(v)$ for $v \leq v^*$ and $(G^*)^{n-1}$ is linear for $v > v^*$ (see Corollary 3). This function is constructed such that the slope of $\varphi(v)$ above $\overline{v}^*$, $(n-1)g^*(\overline{v}^*) = \lim_{v \to \overline{v}^*-}(G^*(v)^{n-1})'$, coincides with the slope of the $(n-1)$-linear region.

There are two important differences between $\varphi(v)$ and $(G^*(v))^{n-1}$. First, $\varphi(v)$ does not represent a probability because $\varphi(v) > 1$ if $v > \overline{v}^*$. Second, and more importantly, whereas $(G^*(v))^{n-1}$ is convex only over $(-\infty, \overline{v}^*]$ (i.e., not above $\overline{v}^*$),[20] $\varphi(v)$ is convex over $\mathcal{R}$ because of its smooth linear extension of $(G^*(v))^{n-1}$ above $\overline{v}^*$ (see Figure 7). As shown below, this second property provides necessary tractability for the subsequent analysis.

Given $\varphi(v)$, we define a *pseudo-demand* function $\widehat{D}(p_i)$ as follows:

$$\widehat{D}(p_i) \equiv \int \varphi(v - p_i + p^*) dF(v).$$

In other words, firm i's pseudo-demand is the measure of consumers that firm i would serve if it

[20]Note that the structure of G^* requires that $\overline{v}^* < \infty$, because the $(n-1)$-linear region $[v^*, \overline{v}^*]$ cannot extend indefinitely.

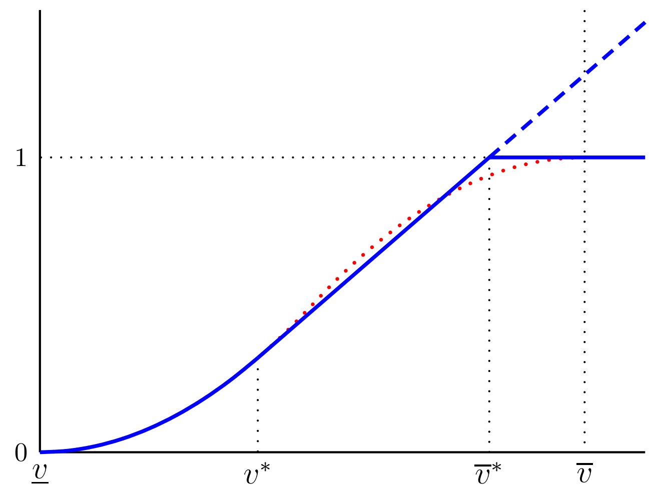


Figure 7: An example of $\varphi(v)$ in a duopoly case ($n = 2$). The figure depicts $F(v)$ (dotted red), $G^*(v)$ (solid blue), and the extended portion of $\varphi(v)$ (dashed blue).

faced a pseudo-value function $\varphi(v - p_i + p^*)$ (instead of the real value function $G^*(v - p_i + p^*)^{n-1}$) and used the fully informative advertising strategy $G_i = F$ (instead of the optimal advertising strategy G_i^* associated with p_i). Notice that the former property raises the firm's demand (because $\varphi(v) \geq G^*(v)^{n-1}$ for any v, with strict inequality for $v > \overline{v}^*$), while the latter property does the opposite. The following lemma shows that the overall effect is always positive.

Lemma 6 $\widehat{D}(p_i) \geq \tilde{D}(p_i)$ *for any* p_i, *which holds with equality when* $p_i = p^*$.

To understand this result, recall that $\varphi(v)$ is convex over $\mathcal{R}$. This implies that firm i's *shifted* value function $\varphi(v - p_i + p^*)$ is also globally convex, regardless of p_i. Then, as a straightforward application of Theorem 2 (or using the same argument as for Corollary 1), $G_i = F$ is firm i's optimal advertising strategy for any p_i. This means that the advertising strategy disadvantage (whereby firm i is restricted to use $G_i = F$) is costless to the firm, and therefore, the firm's pseudo-demand cannot be smaller than its real demand. The equality result when $p_i = p^*$ stems from the fact that $G^*(v)^{n-1}$ is convex over its support, and therefore, the extended convex part of $\varphi(v)$ does not benefit the firm (as long as it uses an optimal advertising strategy G^*).

Step 2: Proving the optimality of p^*. Recall that our goal is to demonstrate that no compound deviation (p_i, G_i^*) is profitable for a firm, which is equivalent to $\tilde{D}(p_i)$ being maximized at p^*. We prove this result by showing that p^* maximizes $p_i\widehat{D}(p_i)$. This is sufficient because combining it with Lemma 6 yields

$$p^*\tilde{D}(p^*) = p^*\widehat{D}(p^*) \geq p_i\widehat{D}(p_i) \geq p_i\tilde{D}(p_i) \text{ for any } p_i.$$

The analysis is possible because, unlike $\tilde{D}(p_i)$, $\widehat{D}(p_i)$ exhibits the following regularity.

Lemma 7 *If f is log-concave, then $\widehat{D}(p_i)$ is log-concave.*

As is common in the literature (e.g., Caplin and Nalebuff, 1991; Quint, 2014), we obtain this result by applying Prékopa's theorem (1971), which effectively states that log-concavity is preserved under integration. The log-concavity of f implies the same property for $\varphi(v)$ (although not for $G^*(v)^{n-1}$), which in turn ensures the log-concavity of $\widehat{D}(p_i)$ via Prékopa's theorem.

Given Lemma 7, a firm's pseudo-profit function $p_i\widehat{D}(p_i)$ is log-concave, that is, $\log(p_i) + \log(\widehat{D}(p_i))$ is concave. This implies that for the optimality of p^*, it suffices to show that p^* satisfies the following first-order condition:

$$\frac{1}{p^*} + \frac{\widehat{D}'(p^*)}{\widehat{D}(p^*)} = 0 \Leftrightarrow \frac{1}{p^*} = -\frac{\widehat{D}'(p^*)}{\widehat{D}(p^*)}. \tag{7}$$

By Lemma 6, $\widehat{D}(p^*) = D(p^*, G^*, p^*, G^*) = 1/n$. The desired result then follows from the following result, which states that a firm's ability to adjust its advertising strategy has no first-order effect on its demand around p^* (that is, a firm's optimal compound deviation is only as profitable as its price-only deviation around p^*), which is intuitive given all the results presented thus far.

Lemma 8 *If f is log-concave, then*

$$\widehat{D}'(p^*) = \frac{\partial D(p_i, G^*, p^*, G^*)}{\partial p_i}\Big|_{p_i=p^*}.$$

6 Discussion

We conclude by discussing two important assumptions we have maintained thus far.

6.1 Outside Option for Consumers

One of the most significant assumptions of our main model is that consumers do not have an option of not purchasing any product. Clearly, this assumption is innocuous if $\underline{v}$ is sufficiently large (such that $\underline{v} - p^*$ exceeds consumers' outside option). In this subsection, we explain how our main analysis can be modified when the assumption binds.

To make the discussion concise and informative, we focus on the advertising-only game, in which the firms choose G_i to maximize their demand given p^*. Other necessary results can be generalized as in our main model.[21] We normalize the value of each consumer's outside option to 0 and assume that $\underline{v} < p^* < \overline{v}$: the former inequality implies that the outside option for consumers is binding (some consumers do not purchase any product under F), while the latter ensures that it is not too restrictive (nonzero demand under F).

To understand how the presence of the outside option affects competitive advertising, suppose that all firms play G^* in Theorem 1, as if consumers had no outside option. Then, consumers with the best expected value less than p^* do not purchase any product. This means that an individual firm faces a *discontinuous* value function, which remains at zero for $v < p^*$ and then jumps to $G^*(p^*)^{n-1}$ at $v = p^*$. Therefore, firm i's optimal advertising problem is now given as

$$\max_{G_i} \int 1_{\{v \geq p^*\}} G^*(v)^{n-1} dG_i(v),$$

subject to the constraint that G_i is an MPC of F.

Clearly, $G_i = G^*$ does not solve the above problem: given the discontinuous value function, firm i's optimal advertising strategy must pool some values and put an atom on $v = p^*$. This, however, cannot be part of an equilibrium because it would provide an incentive for the other firms to also deviate. Note that this issue is similar to the one that arises when F^{n-1} is concave (Corollary 2). The solution to this problem is also similar: the equilibrium advertising strategy must have the $(n-1)$-linear MPC structure. However, there are a few crucial differences, as formally reported below. The proof of the following proposition is in the Online Appendix.

Proposition 5 *Suppose that consumers have the option of not purchasing any product. Given $p^* \in (\underline{v}, \overline{v})$, there exists an equilibrium in the advertising-only game in which the firms play G^{**}, where G^{**} is an MPC of F that satisfies the following properties: there exist $v^\dagger \in (\underline{v}, p^*)$, $v^{\dagger\dagger} \in (v^\dagger, \overline{v})$, and $\beta > 0$ such that*

(i) if $v \leq v^{\dagger\dagger}$ then

$$G^{**}(v)^{n-1} = \begin{cases} \min\{F(v)^{n-1}, F(v^\dagger)^{n-1}\}, & \text{if } v \leq p^*, \\ F(v^\dagger)^{n-1} + \beta(v - p^*), & \text{if } v \in (p^*, v^{\dagger\dagger}] \end{cases}$$

where

$$F(v^\dagger)^{n-1} + \beta(v^\dagger - p^*) = 0 \Leftrightarrow \beta \equiv \frac{F(v^\dagger)^{n-1}}{p^* - v^\dagger},$$

[21] One important difference is that G^* and p^* need to be simultaneously determined in the presence of an outside option for consumers. In the advertising-only game of our main model, each firm's demand is independent of p^*, which allows us to first identify G^* (Section 3) and then p^* (Section 4). With an outside option for consumers, p^* affects each firm's demand and, therefore, equilibrium advertising G^*. This implies that p^* is now characterized as a fixed point such that G^*, which can be obtained given p^*, must yield p^* as an equilibrium of the pricing game.

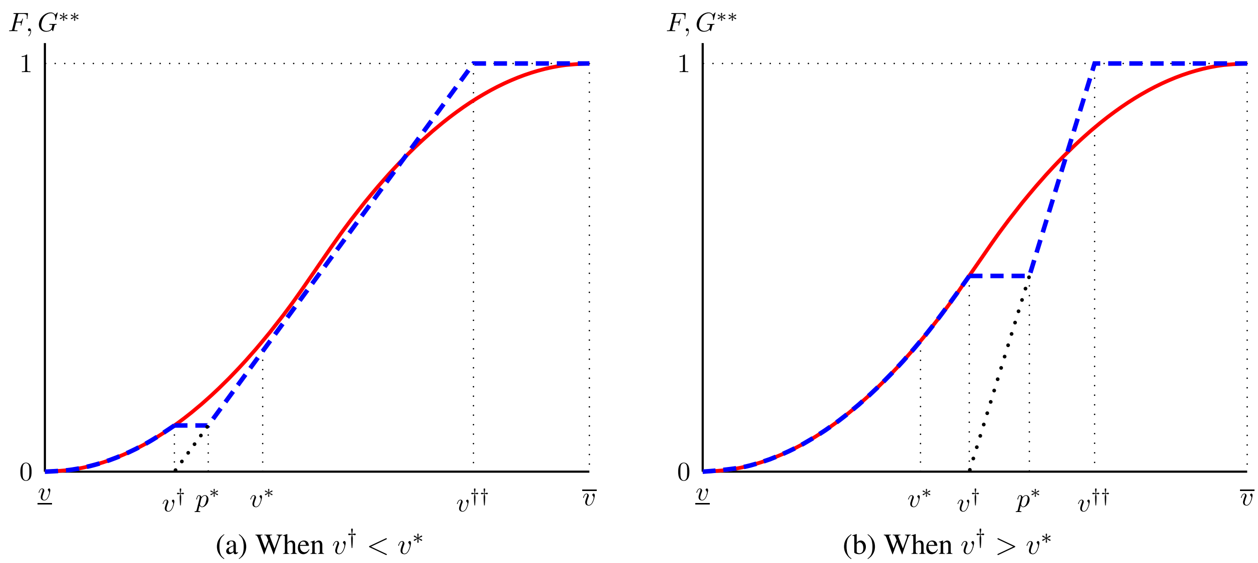


Figure 8: An illustration of Proposition 5. The distribution function F (solid red curve) is identical to that in the right panel of Figure 4. The blue dashed curve represents G^{**} in each case. v^* denotes the starting point of the $(n-1)$-linear region of G^* in the absence of an outside option for consumers.

(ii) G^{**} *is an MPC of* F *over* $[v^\dagger, v^{\dagger\dagger}]$*, and*

(iii) G^{**} *features the same properties as* G^* *in Theorem 1 above* $v^{\dagger\dagger}$.

Figure 8 depicts G^{**} for the case in which $n = 2$ and F is single-peaked (such that G^* has a full-information region $[\underline{v}, v^*)$ and an $(n-1)$-linear MPC region $[v^*, \overline{v}^*]$). Whether p^* is below v^* (left panel) or above v^* (right panel), G^{**} does not place any mass on $[v^\dagger, p^*]$, which reflects an individual firm's incentive to ensure an expected value of at least p^*. Unlike G^*, $(G^{**})^{n-1}$ is not convex over its support, as clearly shown in Figure 8. However, it is still convex above p^*, which is sufficient for the firms' incentives.

Importantly, $(G^{**})^{n-1}$ has the same (linear) slope at p^* as the line that connects $(v^\dagger, 0)$ and $(p^*, F(v^\dagger)^{n-1})$. In other words, if $(G^{**})^{n-1}$ is linearly extended below p^*, then it must meet $(v^\dagger, 0)$ (see the black dotted line in each panel). To understand why this is necessary, suppose that $(G^{**})^{n-1}$ is flatter than the line between $(v^\dagger, 0)$ and $(p^*, F(v^\dagger)^{n-1})$. In this case, as when $(G^*)^{n-1}$ is concave, it is profitable for a firm to concentrate local mass. On the contrary, if $(G^{**})^{n-1}$ is steeper than the line between $(v^\dagger, 0)$ and $(p^*, F(v^\dagger)^{n-1})$, then it is profitable for a firm to reveal more product information. The collinearity is necessary and sufficient for neither to be profitable.

6.2 Asymmetric Firms

We have considered an environment in which the firms are ex ante symmetric and focused on symmetric equilibria. Both are clearly restrictive, but it is technically beyond the scope of this paper to relax them. We illustrate the extent to which our analysis thus far applies to the case of asymmetric firms, whether they are ex ante asymmetric or behave asymmetrically in the symmetric environment.[22]

Let F_i denote the underlying distribution and G_i^* denote the equilibrium distribution (advertising strategy) for firm i. We also let $\underline{v}_i^*$ and $\overline{v}_i^*$ denote the lower and upper bound, respectively, of the support of G_i^*. Finally, we let p_i^* denote firm i's equilibrium price and $\mathbf{p}^*$ denote the equilibrium price vector (i.e., $\mathbf{p}^* \equiv (p_1^*, ..., p_n^*)$). Given the G_j^*s, $\mathbf{p}^*$ can be derived as in Section 5. The only difference is that the firms no longer equally divide the market, and therefore, the equilibrium prices cannot take a simple form, as in Proposition 3.

Consider a consumer who has expected value v_i for product i. She purchases product i if and only if $v_i - p_i^* > v_j - p_j^*$ and, therefore, with probability

$$Q_i(v, \mathbf{p}^*) \equiv \prod_{j \neq i} G_j^*(v_i - p_i^* + p_j^*).$$

Note that in the symmetric equilibrium, $Q_i(v, \mathbf{p}^*)$ reduces to $G^*(v)^{n-1}$. Then, by the same argument as for Theorem 1, $Q_i(\cdot, \mathbf{p}^*)$ must be convex over $[\underline{v}_i, \overline{v}_i]$: if $Q_i(\cdot, \mathbf{p}^*)$ is locally concave, then firm i would put mass on one point, which would trigger the other firms to adjust their strategies.

It also holds that firm i's equilibrium advertising strategy takes an alternating structure between full information ($G_i^* = F$) and risk neutrality ($Q_i(\cdot, \mathbf{p}^*)$ is linear): as explained in Section 3.2, this is necessary for firm i to either not be able to adjust its strategy (i.e., binding MPC constraint) or be indifferent over both spread and contraction. In symmetric equilibrium, $Q_i(v, \mathbf{p}^*) = G^*(v)^{n-1}$, which enables us to determine G^* when $G^*(v) \neq F(v)$ and, therefore, complete the characterization. With asymmetric firms, however, this is clearly not sufficient, and unfortunately, we are not aware of how to pin down each G_i^* from this linear result on $Q_i(\cdot, \mathbf{p}^*)$. Note also that, similarly to Section 6.1, $\mathbf{p}^*$ affects each firm's optimal advertising strategy G_i^*, which introduces further complication.

[22]Even in the standard Perloff-Salop framework, firm asymmetry significantly complicates the analysis. Nevertheless, considerable progress has been made. See, among others, Quint (2014), who provides a sufficient condition for the existence and uniqueness of pure-price equilibrium for a general (asymmetric) environment and reports various comparative statics results. A key technical innovation is to recast the pricing problem as a supermodular game. The technique does not apply (at least directly) to our problem, where each firm chooses not only its price but also its advertising strategy with full flexibility.

Appendix: Omitted Proofs

Proof of Lemma 1. Suppose that G^{n-1} is not convex. Then, there exist v_1, v_2, v_3, and ε (sufficiently small) such that $v_1 < v_2 < v_3$, $(v_1 - \varepsilon, v_1) \cup (v_3, v_3 + \varepsilon) \subset supp(G)$, and

$$G(v_2)^{n-1} > (1-\alpha)G(v_1)^{n-1} + \alpha G(v_3)^{n-1}, \tag{8}$$

where

$$\alpha \equiv \frac{v_2 - v_1}{v_3 - v_1} \in (0, 1).$$

Then, there exist $\delta \in (0, \varepsilon)$ and $\delta' \in (0, \varepsilon)$ such that

$$v_2 = \frac{\int_{v_1-\varepsilon}^{v_1-\varepsilon+\delta} v dG(v) + \int_{v_3+\varepsilon-\delta'}^{v_3+\varepsilon} v dG(v)}{G(v_1-\varepsilon+\delta) - G(v_1-\varepsilon) + G(v_3+\varepsilon) - G(v_3+\varepsilon-\delta')}.$$

For notational convenience, let $\Delta \equiv G(v_1-\varepsilon+\delta) - G(v_1-\varepsilon)$ and $\Delta' \equiv G(v_3+\varepsilon) - G(v_3+\varepsilon-\delta')$.

Consider the following alternative distribution function G_i:

$$G_i(v) = \begin{cases} G(v), & \text{if } v < v_1 - \varepsilon, \\ G(v_1 - \varepsilon), & \text{if } v \in [v_1 - \varepsilon, v_1 - \varepsilon + \delta], \\ G(v) - \Delta, & \text{if } v \in [v_1 - \varepsilon + \delta, v_2), \\ G(v) + \Delta', & \text{if } v \in [v_2, v_3 + \varepsilon - \delta'), \\ G(v_3 + \varepsilon), & \text{if } v \in [v_3 + \varepsilon - \delta', v_3 + \varepsilon), \\ G(v), & \text{if } v \geq v_3 + \varepsilon. \end{cases}$$

By construction, G_i is an MPC of G and, therefore, an MPC of F. In addition,

$$\int G^{n-1} dG_i - \int G^{n-1} dG = (\Delta + \Delta')G(v_2)^{n-1} - \left(\int_{v_1-\varepsilon}^{v_1-\varepsilon+\delta} G^{n-1} dG + \int_{v_3+\varepsilon-\delta'}^{v_3+\varepsilon} G^{n-1} dG \right) > 0,$$

where the inequality is due to (8). This proves that if G^{n-1} is not convex, then G cannot be an individual firm's best response to G^{n-1}. ■

Proof of Lemma 2. Define a function $W : [\underline{v}, \overline{v}] \to \mathcal{R}$ as follows:

$$W(v) = \int_{\underline{v}}^{v} (F(v) - G(v)) dv.$$

Since G is an MPC of F, $W(v) \geq 0$ for any v. We show that whenever there is an interval over which $W(v) > 0$, G^{n-1} must be linear over the interval. Since $W(v) = 0$ over an interval only when $F(v) = G(v)$ over the same interval, this proves the alternating structure in the lemma.

Fix an interval $(v_1, v_2) \in [\underline{v}, \overline{v}]$ such that $W(v_1) = W(v_2) = 0$ and $W(v) > 0$ for all $v \in (v_1, v_2)$. There are two cases to consider, depending on whether $v_2 \leq \overline{v}^*$.

(i) Suppose that $v_2 \leq \overline{v}^*$. In this case, consider the following alternative advertising strategy:

$$G_i(v) = \begin{cases} G(v), & \text{if } v < v_1, \\ F(v), & \text{if } v \in [v_1, v_2), \\ G(v), & \text{if } v \geq v_2. \end{cases}$$

In other words, G_i coincides with F on (v_1, v_2) and follows G elsewhere. By construction, G_i is (still) an MPC of F and, therefore, feasible. Since G must be a best response to itself, we must have

$$\int G(v)^{n-1} dG_i(v) \leq \int G(v)^{n-1} dG(v) \Leftrightarrow \int_{v_1}^{v_2} G(v)^{n-1} dG_i(v) \leq \int_{v_1}^{v_2} G(v)^{n-1} dG(v).$$

Now note that, since $v_2 \leq \overline{v}^*$, G^{n-1} is convex over (v_1, v_2). Combining this with the fact that G_i is a mean-preserving spread of G yields

$$\int G(v)^{n-1} dG_i(v) \geq \int G(v)^{n-1} dG(v) \Leftrightarrow \int_{v_1}^{v_2} G(v)^{n-1} dG_i(v) \geq \int_{v_1}^{v_2} G(v)^{n-1} dG(v).$$

It follows that

$$\int_{v_1}^{v_2} G(v)^{n-1} dG_i(v) = \int_{v_1}^{v_2} G(v)^{n-1} dG(v).$$

The desired result follows from the fact that, as $G_i = F \neq G$ on (v_1, v_2), this equality can hold only when G^{n-1} is linear over (v_1, v_2).

(ii) Now, suppose that $v_2 > \overline{v}^*$. Since G is an MPC of F, this case arises only when $v_2 = \overline{v}$. Let $\tilde{v}$ be the value such that

$$E_F[v|v \geq \tilde{v}] \equiv \int_{\tilde{v}}^{\overline{v}} v \frac{dF(v)}{1 - F(\tilde{v})} = \overline{v}^*.$$

$\tilde{v}$ is well defined, because $E_F[v|v \geq \tilde{v}]$ is strictly increasing in $\tilde{v}$, $E_F[v|v \geq v_1] = E_G[v|v \geq v_1] < \tilde{v}^*$ (recall that G is an MPC of F), and $E_F[v|v \geq \tilde{v}^*] > \tilde{v}^*$. Then, consider the following alternative advertising strategy:

$$G_i(v) = \begin{cases} G(v), & \text{if } v < v_1, \\ F(v), & \text{if } v \in [v_1, \tilde{v}), \\ F(\tilde{v}), & \text{if } v \in (\tilde{v}, \overline{v}^*), \\ 1, & \text{if } v \geq \overline{v}^*. \end{cases}$$

In other words, G_i coincides with G below v_1, follows F until $\tilde{v}$, and then places all remaining mass on $\tilde{v}^*$. By construction, G_i is an MPC of F and, therefore, feasible. Moreover, G_i is an MPS of G: by construction (and since G is an MPC of F above v_1),

$$E_{G_i}[v|v \geq \tilde{v}] = E_F[v|v \geq \tilde{v}] = E_{G_i}[v|v \geq \tilde{v}].$$

In addition, also by construction, G_i crosses G only once from above. Given these (G_i is an MPC of F and an MPS of G), the argument given for the case in which $v_2 \leq \overline{v}^*$ applies effectively unchanged: by the optimality of G over G_i and the convexity of G^{n-1}, it must hold that

$$\int_{v_1}^{v_2} G(v)^{n-1} dG_i(v) = \int_{v_1}^{v_2} G(v)^{n-1} dG(v).$$

If $G \neq F$ (which implies $G \neq G_i$), then this can hold only when G^{n-1} is linear over $(v_1, \overline{v}^*)$. ■

Proof of Lemma 3. Suppose that there exists $v \in (\tilde{v}, \overline{v})$ such that $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) \leq 0$. If $a = 0$, then F stays uniformly above $H_{\tilde{v},a}$ and, therefore, $W_{\tilde{v},a} > 0$ for all v. In addition, for each $v \in (\tilde{v}, \overline{v}]$, $W_{\tilde{v},a}(v)$ is continuously and strictly increasing in a. Therefore, there always exists $a^* \in (0, (F(\tilde{v})^{n-1})')$ and $\overline{v}' \in (\tilde{v}, \overline{v}]$ such that $W_{\tilde{v},a^*}(v) \geq 0$ for all $v \in (\tilde{v}, \tilde{v}')$ and $W_{\tilde{v},a^*}(\tilde{v}') = 0$. It follows that $H_{\tilde{v},a^*}$ is an $(n-1)$-linear MPC of F over $[\tilde{v}, \tilde{v}']$. ■

Proof of Theorem 1. Given the results in Section 3.2 and above, it suffices to show that G^* is unique. Suppose that there exist two distribution functions, G_1 and G_2, that satisfy the properties in Theorem 1. Let $\hat{v}$ denote the lowest point at which G_1 and G_2 diverge, that is, $\hat{v} \equiv \inf\{v : G_1(v) \neq G_2(v)\}$ (see Figure 5). Without loss of generality, we assume that $G_1(v) < G_2(v)$ for v sufficiently close to $\hat{v}$. For notational simplicity, let

$$W_i(v) = \int_{\underline{v}}^{v} (F(x) - G_i(x))dx \text{ for } i = 1, 2.$$

Note that, since G_i is an MPC of F, $W_i(v) \geq 0$ for all v and $i = 1, 2$.

We make two observations about the G_is. First, it must be that $G_i(\hat{v}) = F(\hat{v})$: otherwise, G_1^{n-1} and G_2^{n-1} must be linear with different slopes around $\hat{v}$, which violates the definition of $\hat{v}$. Then, obviously, $W_1(\hat{v}) = W_2(\hat{v}) = 0$. Second, G_1^{n-1} must be linear over $(\hat{v}, \hat{v} + \varepsilon)$ for some ε: otherwise, $G_1(v) = F(v)$ for $v \in (\hat{v}, \hat{v} + \varepsilon)$, in which case $G_2(v) > F(v)$, and therefore, $W_2(\hat{v} + \varepsilon) = \int_{\hat{v}}^{\hat{v}+\varepsilon} (F(x) - G_2(x))dx < 0$, violating the MPC constraint. Let $\hat{v}'$ denote the end point of the $(n-1)$-linear MPC region (that is, $\hat{v}' \equiv \sup\{v : G_1^{n-1} \text{ is linear over } [\hat{v}, v]\}$).

Now observe that, since G_2^{n-1} is convex over its support, it must be that $G_2(v) > G_1(v)$ for all $v \in (\hat{v}, \hat{v}')$ (see Figure 5). We complete the proof by showing that this dominance leads to

$W_2(v) < 0$ for some v. If G_1 does not end on $\hat{v}'$ (that is, $G_1(\hat{v}') < 1$), then G_1 must be an $(n-1)$-linear MPC of F over $[\hat{v}, \hat{v}']$. In this case, $W_1(\hat{v}') = \int_{\hat{v}}^{\hat{v}'}(F(x) - G_1(x))dx = 0$. However, then,

$$W_2(\hat{v}') = \int_{\hat{v}}^{\hat{v}'}(F(x) - G_2(x))dx = \int_{\hat{v}}^{\hat{v}'}(G_1 - G_2(x))dx < 0.$$

If G_1 ends on $\hat{v}'$ (that is, $G_1(\hat{v}') = 1$), then G_1 is an $(n-1)$-linear MPC of F over $[\hat{v}, \overline{v}]$. In this case, similar to the previous case,

$$W_2(\overline{v}) = \int_{\hat{v}}^{\overline{v}}(F(x) - G_2(x))dx = \int_{\hat{v}}^{\overline{v}}(G_1 - G_2(x))dx < 0.$$

■

Proof of Lemma 4. Suppose that F^{n-1} is convex over $[\tilde{v}, \overline{v}]$. In this case, for any $v^\dagger \in [\tilde{v}, \overline{v}]$, $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ is uniformly below F. Therefore, there cannot exist an $(n-1)$-linear MPC of F above $\tilde{v}$. Now suppose that F^{n-1} is not convex. In this case, one can always find $v' \in (\tilde{v}, \overline{v})$ and $v \in (v', \overline{v}]$ such that $H_{v',(F(v')^{n-1})'}(v) < 0$, because if v' belongs to a concave region of F^{n-1}, then $H_{v',(F(v')^{n-1})'}$ remains above F around v'. Now, let $v^\dagger$ denote the infimum among such v's. Clearly, $v^\dagger \in (\tilde{v}, \overline{v})$. In addition, since $W_{\tilde{v},(F(\tilde{v})^{n-1})'}(v) > 0$ for all $v \in (\tilde{v}, \overline{v}]$, there must exist $\tilde{v}' \in (v^\dagger, \overline{v}]$ such that $W_{v^\dagger,(F(v^\dagger)^{n-1})'}(v) \geq 0$ for all $v \in [v^\dagger, \tilde{v}']$, and $W_{v^\dagger,(F(v^\dagger)^{n-1})'}(v^\dagger) = W_{v^\dagger,(F(v^\dagger)^{n-1})'}(\tilde{v}') = 0$. This implies that $H_{v^\dagger,(F(v^\dagger)^{n-1})'}$ is a $(n-1)$-linear MPC of F over $[v^\dagger, \overline{v}']$. ■

Proof of Proposition 1. We first consider the case in which $\overline{v} < \infty$. Let $\varepsilon \equiv \min\{f(v) : v \in [\underline{v}, \overline{v}]\}$ and $M \equiv \max\{|f'(v)| : v \in [\underline{v}, \overline{v}]\}$. Under the maintained technical assumptions on F (namely that $f(v) > 0$ for all $v \in [\underline{v}, \overline{v}]$ and $f'(v)$ is bounded), $\varepsilon > 0$ and $M < \infty$. Then, for any $v \in [\underline{v}, \overline{v}]$,

$$\left(F^{n-1}\right)'' = (n-1)F^{n-3}\left((n-2)f^2 + Ff'\right) \geq (n-1)F^{n-3}((n-2)\varepsilon^2 - M), \tag{9}$$

which is positive everywhere for n sufficiently large. Then, by Corollary 1, $G^* = F$.

Now consider the case in which $\overline{v} = \infty$. Given F, let G_n^* denote the equilibrium advertising strategy when there are n firms. We first observe that under our maintained technical assumptions, if n is sufficiently large, then the density function of F^{n-1} has exactly one peak (when $\overline{v} = \infty$): let v' denote the point from which f is log-concave. Then, for any n, $(F^{n-1})'$ has only one peak above v' (see footnote 17). In addition, for the same reason as for the case with $\overline{v} < \infty$, F^{n-1} is necessarily convex ($(F^{n-1})'$ is increasing) over $[\underline{v}, v']$. Combining these findings with the fact that $\overline{v} = \infty$, it follows that $(F^{n-1})'$ has exactly one peak.

Given the above observation, for any n sufficiently large, the equilibrium is characterized by v_n^* such that $G_n^*(v) = F(v)$ if $v \leq v_n^*$ and $(G_n^*)^{n-1}$ is linear above v_n^* (see Corollary 3). For each

n, we let $\overline{v}_n^*$ denote the upper bound of the support of G_n^* (i.e., $G_n^*(\overline{v}_n^*) = 1$). Then,

$$G_n^*(v)^{n-1} = F(v_n^*)^{n-1} + \left(F(v_n^*)^{n-1}\right)'(v - v_n^*) \text{ whenever } v \in [v_n^*, \overline{v}_n^*].$$

This explicit solution for G_n^* follows from the fact that $v_n^* > \underline{v}$ (because $(F(\underline{v})^{n-1})' = (n-1)F(\underline{v})^{n-2}f(\underline{v}) = 0$ whenever $n \geq 3$), and therefore, the slope of $(G_n^*)^{n-1}$ must be identical to $(F^{n-1})'$ at v_n^*: MPC implies that $g_n^*(v_n^*) \leq f(v_n^*)$, while the convexity of $(G_n^*)^{n-1}$ implies that

$$(n-1)G_n^*(v_n^*)^{n-2}g_n^*(v_n^*) \geq \lim_{v \to v_n^*+}(n-1)G_n^*(v)^{n-2}g_n^*(v) = (n-1)F(v_n^*)^{n-2}f(v_n^*).$$

Now, using the explicit $(n-1)$-linear solution for G_n^*, it can be shown that

$$\begin{aligned}
\int_{v_n^*}^{\overline{v}_n^*} v dG_n^*(v) &= \overline{v}_n^* - v_n^* G_n^*(v_n^*) - \frac{1 - F(v_n^*)^n}{nF(v_n^*)^{n-2}f(v_n^*)} \\
&= \frac{1 - F(v_n^*)^{n-1}}{(n-1)F(v_n^*)^{n-2}f(v_n^*)} + v_n^*(1 - F(v_n^*)) - \frac{1 - F(v_n^*)^n}{nF(v_n^*)^{n-2}f(v_n^*)} \\
&= \frac{1 - nF(v_n^*)^{n-1} + (n-1)F(v_n^*)^n}{n(n-1)F(v_n^*)^{n-2}f(v_n^*)} + v_n^*(1 - F(v_n^*)).
\end{aligned}$$

For the desired result, it suffices to show that v_n^* grows unboundedly as n tends to infinity. Assume, toward a contradiction, that there exist a subsequent $\{n_k\}$ and $\widehat{v}(< \infty)$ such that for any n_k, $v_n^* \leq \widehat{v}$. If so, all $n_k F(v_{n_k})^{n_k-1}$, $(n_k - 1)F(v_{n_k})^{n_k}$, and $n(n-1)F(v_{n_k})^{n_k-2}$ converge to 0 as n_k tends to infinity. This implies that for n_k sufficiently large,

$$\int_{v_{n_k}^*}^{\overline{v}_{n_k}^*} v dG_{n_k}^*(v) > \int_{v_{n_k}^*}^{\overline{v}_{n_k}^*} v dF(v),$$

which contradicts the fact that, as G_n^* is an MPC of F, $\int_{v_n^*}^{\overline{v}_n^*} v dG_n^*(v) = \int_{v_n^*}^{\overline{v}_n^*} v dF(v)$ for all n. ■

Proof of Proposition 2. It suffices to prove the monotonicity result ($v_n^* \leq v_{n+1}^*$ for any n), because the asymptotic result ($\lim_{n\to\infty} v_n^* = \overline{v}$) follows from Proposition 1 and the monotonicity result.

Since f is log-concave, for any $n \geq 2$, there exist v_n^* and $\overline{v}_n^*$ such that $G_n^*(\overline{v}_n^*) = 1$ and G_n^* is $(n-1)$-linear over $[v_n^*, \overline{v}_n^*]$. If $v_2^* = \underline{v}$, then it is trivial that $v_2^* \leq v_3^*$. In addition, for any $n \geq 3$, $v_n^* > \underline{v}$, because $(F(\underline{v})^{n-1})' = (n-1)F(\underline{v})^{n-2}f(\underline{v}) = 0$. Therefore, without loss of generality, we assume that $v_n^* > \underline{v}$ for any $n \geq 2$. Then, for the same argument as in the proof of Proposition 1, G_n^* is given by

$$G_n^*(v)^{n-1} = \begin{cases} F(v)^{n-1}, & \text{if } v \leq v_n^*, \\ F(v_n^*)^{n-1} + (F(v_n^*)^{n-1})'(v - v_n^*), & \text{if } v \in [v_n^*, \overline{v}_n^*] \end{cases}$$

Let $\widehat{v}_n$ denote the point at which $(F^{n-1})' = (n-1)F^{n-2}f$ is maximized. Since $(n-2)f^2+Ff'$ is increasing in n, $\widehat{v}_n \le \widehat{v}_{n+1}$ for any $n \ge 2$. Since the linear portion can start only in the region where $(n-1)F^{n-2}f$ is increasing (see Figure 2), it is clear that $v_n^* \in (\underline{v}, \widehat{v}_n]$. Now, for each $\tilde{v} \in [\underline{v}, \widehat{v}_n]$, let $G_{\tilde{v},n}$ denote the distribution function defined as follows:

$$G_{v',n}(v)^{n-1} = \begin{cases} F(v)^{n-1}, & \text{if } v \le v', \\ \min\{F(v')^{n-1} + (F(v')^{n-1})'(v-v'), 1\}, & \text{if } v > v'. \end{cases}$$

In words, $G_{v',n}(v)$ is a distribution function that is $(n-1)$-linear above v' (with the slope equal to $(F(v')^{n-1})'$. We also let $\overline{v}'$ denote the upper bound of the support of $G_{v',n}$.

We use the following two claims to establish the desired result.

Claim 1 *Given v', $E_{G_{v',n}}[v|v \ge v']$ increases in n.*

Proof. Observe that

$$\begin{aligned} E_{G_{v',n}}[v|v \ge v'] &= \int_{v'}^{\overline{v}'} v \frac{dG_{v',n}(v)}{1-G_{v',n}(v')} = \frac{\overline{v}' - v'G_{v',n}(v')}{1-G_{v',n}(v')} - \int_{v'}^{\overline{v}'} G_{v',n}(v) \frac{dv}{1-G_{v',n}(v')} \\ &= v' + \frac{1}{1-G_{v',n}(v')}\left(\overline{v}' - v' - \frac{n-1}{n}\frac{1-G_{v',n}(v')^n}{(F(v')^{n-1})'}\right) \\ &= v' + \frac{1}{(F(v')^{n-1})'}\left(\sum_{k=1}^{n-1} G_{v',n}(v')^{k-1} - \frac{n-1}{n}\sum_{k=1}^{n} G_{v',n}(v')^{k-1}\right) \\ &= v' + \frac{\sum_{k=1}^{n-1} F(v')^{k-1} - (n-1)F(v')^{n-1}}{n(n-1)F(v')^{n-2})f(v')}. \end{aligned}$$

Therefore, it suffices to show that

$$\frac{\sum_{k=1}^{n-1} F(v')^{k-1} - (n-1)F(v')^{n-1}}{n(n-1)F(v')^{n-2}f(v')} < \frac{\sum_{k=1}^{n} F(v')^{k-1} - nF(v')^{n}}{(n+1)nF(v')^{n-1}f(v')},$$

which is equivalent to

$$(n-1)(1+F(v')^n) > 2\sum_{k=1}^{n-1} F(v')^k.$$

Note that the inequality holds strictly if $v' = \underline{v}$, while it holds with equality if $v' = \overline{v}$. The stated claim then follows from

$$\begin{aligned} \frac{d}{dF(v')}\left((n-1)(1+F(v')^n) - 2\sum_{k=1}^{n-1} F(v')^k\right) &= n(n-1)F(v')^{n-2} - 2\sum_{k=1}^{n-1} kF(v')^{k-1} \\ &\le F(v')^{n-2}\left(n(n-1) - 2\sum_{k=1}^{n-2} k\right) = 0. \end{aligned}$$

■

Claim 2 *If $E_F[v|v \geq v'] \leq E_{G_{v',n}}[v|v \geq v']$, then $v_n^* \geq v'$.*

Proof. Since $(F^{n-1})' = (n-1)F^{n-2}f$ is increasing over $v \in (\underline{v}, \widehat{v}_n)$, if v' increases, then $G_{v',n}$ decreases in the sense of first-order stochastic dominance: note that for any v, $G_{v',n}(v)$ increases in v'. This implies that $E_F[v] - E_{G_{v',n}}[v]$ is strictly decreasing in v'. Since $G_{v_n^*,n}$ is an MPC of F, it must be that $E_F[v] - E_{G_{v_n^*,n}}[v] = 0$. Therefore, if $E_F[v] \geq E_{G_{v',n}}[v]$, then it is necessarily the case that $v_n^* \geq v'$. The desired result then follows from the fact that, by construction, $G_{v',n}(v) = F(v)$ if $v \leq v'$, and therefore,

$$E_F[v] - E_{G_{v',n}}[v] = (1 - F(v'))(E_F[v|v \geq v'] - E_{G_{v',n}}[v|v \geq v']).$$

■

Fix n, and identify the corresponding equilibrium cutoff v_n^* such that $E_{G_{v_n^*,n}}[v|v \geq v_n^*] = E_F[v|v \geq v_n^*]$. By Claim 1, if n increases, then $E_{G_{v_n^*,n}}[v|v \geq v_n^*] \geq E_F[v|v \geq v_n^*]$. Then, by Claim 2, $v_{n+1}^* \geq v_n^*$. ■

Proof of Lemma 5. Suppose that $X_{i,1}$, ..., and $X_{i,n}$ are identically and independently drawn according to G_i for each $i = 1, 2$. Then, since $SS_n(G_i) = E[\max\{X_{1,1}, ..., X_{1,n}\}]$, it suffices to show that

$$E[\max\{X_{1,1}, ..., X_{1,n}\}] \leq E[\max\{X_{2,1}, ..., X_{2,n}\}].$$

Since G_1 is an MPC of G_2 and $\max\{y, X_n\}$ is a convex function of X_n for any y,

$$\begin{aligned} E[\max\{X_{2,1}, ..., X_{2,n-1}, X_{1,n}\}] &= E_{X_{2,1},...,X_{2,n-1}}[E[\max\{\max\{X_{2,1}, ..., X_{2,n-1}\}, X_{1,n}\}]] \\ &\leq E_{X_{2,1},...,X_{2,n-1}}[E[\max\{\max\{X_{2,1}, ..., X_{2,n-1}\}, X_{2,n}\}]] \\ &= E[\max\{X_{2,1}, ..., X_{2,n-1}, X_{2,n}\}]. \end{aligned}$$

Repeating the above argument, we have

$$\begin{aligned} E[\max\{X_{1,1}, ..., X_{1,n}\}] &\leq E[\max\{X_{2,1}, X_{1,2}, ..., X_{1,n}\}] \\ &\leq E[\max\{X_{2,1}, X_{2,2}, X_{1,3}, ..., X_{1,n}\}] \\ &\leq ... \\ &\leq E[\max\{X_{2,1}, ..., X_{2,n-1}, X_{1,n}\}] \leq E[\max\{X_{2,1}, ..., X_{2,n-1}, X_{2,n}\}]. \end{aligned}$$

■

Proof of Proposition 4. For each case, we apply Theorem 2 to establish the optimality of the given advertising strategy. Since p_i is fixed, we assume, without loss of generality, that firm i wishes to maximize its demand $D(p_i, G_i, p^*, G^*)$ by choosing G_i.

(i) $p_i - p^* \geq \overline{v} - \overline{v}^*$: In this case, $G^*(v - p_i + p^*)^{n-1}$ is convex over $[\underline{v}, \overline{v}]$ because $G^*(v - p_i + p^*)^{n-1} = 0$ if $v \leq \underline{v} + p_i - p^*$ and then follows $(G^*)^{n-1}$ until $\overline{v}(\leq \overline{v}^* + p_i - p^*)$. Thus, by Theorem 2, $G_i^* = F$: it suffices to set $\varphi(v) = G^*(v - p_i + p^*)^{n-1}$ for any $v \in [\underline{v}, \overline{v}]$. With $G_i^* = F$, the necessary conditions are trivially satisfied.

(ii) $p_i - p^* \leq \mu_F - \overline{v}^*$: In this case, it is trivially optimal for firm i not to provide any information because all consumers then purchase product i.

(iii) $p_i - p^* \in (\mu_F - \overline{v}^*, \overline{v} - \overline{v}^*)$: For this case, consider the following convex function:

$$\varphi(v) = \begin{cases} G^*(v - p_i + p^*)^{n-1}, & \text{if } v \leq \psi, \\ G^*(\psi - p_i + p^*)^{n-1} + \frac{1 - G^*(\psi - p_i + p^*)^{n-1}}{\overline{v}^* - p_i + p^* - \psi}(v - \psi), & \text{if } v > \psi. \end{cases}$$

Since $G^*(v - p_i + p^*)^{n-1}$ is convex over $[\underline{v}, \overline{v}^* + p_i - p^*]$, φ, which is created by extending $G^*(v + p_i - p^*)^{n-1}$ linearly above ψ, is convex over $[\underline{v}, \overline{v}]$. The desired result follows from the fact that the other properties in Theorem 2 also hold: by construction, $\varphi(v) \geq G^*(v + p_i - p^*)^{n-1}$ for any $v \in [\underline{v}, \overline{v}]$ and $supp(G_i^*) = [\underline{v}, \psi] \cup \{\overline{v}^* + p_i - p^*\} = \{v \in [\underline{v}, \overline{v}] : G^*(v + p_i - p^*)^{n-1} = \varphi(v)\}$. In addition,

$$\int_{\underline{v}}^{\overline{v}} \varphi(v) dG_i^*(v) - \int_{\underline{v}}^{\overline{v}} \varphi(v) dF(v) = \int_{\psi}^{\overline{v}} \varphi(v) dG_i^*(v) - \int_{\psi}^{\overline{v}} \varphi(v) dF(v) = 0,$$

where the first equality is because, by construction, $G_i^* = F$ if $v \leq \psi$, and the second equality is because G_i^* is an MPC of F and φ is linear over $[\psi, \overline{v}]$. ■

Proof of Lemma 6. The first part of the Lemma is proven in the main text. For the second part, first observe that, by construction, $\varphi(v)$ is linear whenever $G^*(v) \neq F(v)$ (that is, for $v \in [v^*, \overline{v}]$). Therefore,

$$\widehat{D}(p^*) = \int_{\underline{v}}^{\overline{v}} \varphi(v) dF(v) = \int_{\underline{v}}^{\overline{v}} \varphi(v) dG^*(v).$$

In addition, since φ coincides with $(G^*)^{n-1}$ over $supp(G^*) = [\underline{v}, \overline{v}^*]$, we have

$$\int_{\underline{v}}^{\overline{v}} \varphi(v) dG^*(v) = \int_{\underline{v}}^{\overline{v}^*} G^*(v)^{n-1} dG^*(v) = \tilde{D}(p^*),$$

which completes the proof. ■

Proof of Lemma 7. If both $f(v)$ and $\varphi(v - p_i + p^*)$ are log-concave in v and p_i, then, by Prékopa's theorem (1971), the pseudo-demand function $\widehat{D}(p_i)$ is log-concave because it is the convolution of

φ and f:

$$\widehat{D}(p_i) = \int \varphi(v - p_i + p^*) f(v) dv.$$

If $\varphi(v)$ is log-concave in v, then $\varphi(v - p_i + p^*)$ is log-concave in both v and p_i. Therefore, it suffices to establish the log-concavity of $\varphi(v)$. If $v < v^*$, then

$$\frac{\varphi'(v)}{\varphi(v)} = \frac{(n-1)F(v)^{n-2} f(v)}{F(v)^{n-1}} = (n-1)\frac{f(v)}{F(v)}.$$

Since F is log-concave (which is implied by the log-concavity of f), φ'/φ is decreasing. If $v > v^*$, then $\phi(v)$ is linear and, therefore, clearly log-concave. The desired result then follows from the fact that φ is smooth, which ensures that φ'/φ is continuous around v^*. ■

Proof of Lemma 8. Consider $p_i = p^* + \varepsilon$. Then, the relevant demand functions are given by

$$\widehat{D}(p^* + \varepsilon) = \int \varphi(v - \varepsilon) dF(v) \text{ and } D(p^* + \varepsilon, G^*, p^*, G^*) = \int G^*(v - \varepsilon)^{n-1} dG^*(v).$$

Define $\Delta(\varepsilon) \equiv \widehat{D}(p^* + \varepsilon) - D(p^* + \varepsilon, G^*, p^*, G^*)$. Observe that $\Delta(0) = 0$ by Lemma 6. Then, it suffices to show that

$$\lim_{\varepsilon \to 0+} \Delta'(\varepsilon) = \lim_{\varepsilon \to 0-} \Delta'(\varepsilon) = 0.$$

Recall that if f is log-concave, then there exists $v^* \in [\underline{v}, \overline{v}]$ such that G^* coincides with F for $v < v^*$ and $(G^*)^{n-1}$ is linear for $v \geq v^*$. We analyze each of the following three cases: (1) $v^* = \overline{v}$, (2) $v^* = \underline{v}$, and (3) $v^* \in (\underline{v}, \overline{v})$.

Case 1: $v^* = \overline{v}$. In this case, F^{n-1} is globally convex, and thus,

$$G^*(v) = \begin{cases} F(v) & \text{if } v \leq \overline{v}, \\ 1 & \text{if } v > \overline{v}, \end{cases} \qquad \varphi(v) = \begin{cases} F(v) & \text{if } v \leq \overline{v}, \\ 1 + \hat{\beta}(v - \overline{v}) & \text{if } v > \overline{v}, \end{cases}$$

where $\hat{\beta} = (n-1)f(\overline{v})$. The case with upward price deviations ($\varepsilon > 0$) is trivial: since $\varphi(v - \varepsilon) = G^*(v - \varepsilon)^{n-1}$ for all $v \in [\underline{v}, \overline{v}]$, $\Delta(\varepsilon) = 0$ for any $\varepsilon > 0$.

Now, consider the case in which $\varepsilon < 0$. In this case, the two demand functions differ only over $[\overline{v} + \varepsilon, \overline{v}]$. Formally,

$$\widehat{D}(p^* + \varepsilon) = \int_{\underline{v}}^{\overline{v}+\varepsilon} F(v - \varepsilon)^{n-1} dF(v) + \int_{\overline{v}+\varepsilon}^{\overline{v}} (1 + \hat{\beta}(v - \varepsilon - \overline{v})) dF(v),$$

$$D(p^* + \varepsilon, G^*, p^*, G^*) = \int_{\underline{v}}^{\overline{v}+\varepsilon} F(v - \varepsilon)^{n-1} dF(v) + \int_{\overline{v}+\varepsilon}^{\overline{v}} 1 dF(v).$$

Therefore,

$$\Delta(\varepsilon) = \int_{\overline{v}+\varepsilon}^{\overline{v}} \hat{\beta}(v - \epsilon - \overline{v})dF(v),$$

and

$$\Delta'(\varepsilon) = \int_{\overline{v}+\varepsilon}^{\overline{v}} \hat{\beta}dF(v) = -\hat{\beta}(F(\overline{v}) - F(\overline{v}+\varepsilon)),$$

which goes to zero as $\varepsilon \to 0$.

Case 2: $v^* = \underline{v}$**.** Recall that this case arises only when $n = 2$ (because $(F(\underline{v})^{n-1})' = 0$ if $n > 2$). Therefore, G^* and φ are given by

$$G^*(v) = \begin{cases} 0 & \text{if } v \leq \underline{v}, \\ \hat{\gamma}(v - \underline{v}) & \text{if } v \in (\underline{v}, \overline{v}^*], \\ 1 & \text{if } v > \overline{v}^*, \end{cases} \qquad \varphi(v) = \begin{cases} 0 & \text{if } v \leq \underline{v}, \\ \hat{\gamma}(v - \underline{v}) & \text{if } v > \underline{v}, \end{cases}$$

where $\hat{\gamma} = 1/(\overline{v}^* - \underline{v})$ and $\overline{v}^* = 2\mu_F - \underline{v}$.

Consider an upward price deviation (i.e., $\varepsilon > 0$). Then, the demand functions are

$$\widehat{D}(p^* + \varepsilon) = \int_{\underline{v}+\varepsilon}^{\overline{v}} \hat{\gamma}(v - (\underline{v}+\varepsilon))dF(v) = \hat{\gamma}\left[\mu_F - \int_{\underline{v}}^{\underline{v}+\varepsilon} vf(v)dv - (\underline{v}+\varepsilon)(1 - F(\underline{v}+\varepsilon))\right],$$

and

$$D(p^*+\varepsilon, G^*, p^*, G^*) = \int_{\underline{v}+\varepsilon}^{\overline{v}^*} \hat{\gamma}(v-(\underline{v}+\varepsilon))dG^*(v) = \hat{\gamma}^2 \int_{\underline{v}+\varepsilon}^{\overline{v}^*} (v-(\underline{v}+\varepsilon))dv = \frac{\hat{\gamma}^2(\overline{v}^* - (\underline{v}+\varepsilon))^2}{2}.$$

A straightforward calculation yields

$$\Delta'(\varepsilon) = \hat{\gamma}\left(-(1 - F(\underline{v}+\varepsilon)) + \frac{\overline{v}^* - (\underline{v}+\varepsilon)}{\overline{v}^* - \underline{v}}\right),$$

which converges to zero as $\varepsilon \to 0$.

Next, consider a downward price deviation ($\varepsilon < 0$). In this case, the two demand functions are

$$\widehat{D}(p^* + \varepsilon) = \int_{\underline{v}}^{\overline{v}} \hat{\gamma}(v - (\underline{v}+\varepsilon))dF(v) = \frac{1}{2} - \hat{\gamma}\varepsilon,$$

and

$$D(p^*+\varepsilon, G^*, p^*, G^*) = \int_{\underline{v}}^{\overline{v}^*+\varepsilon} \hat{\gamma}(v-(\underline{v}+\varepsilon))dG^*(v) + \int_{\overline{v}^*+\varepsilon}^{\overline{v}^*} 1dG^*(v)$$
$$= \hat{\gamma}^2\left(\frac{(\overline{v}^*+\varepsilon)^2-\underline{v}^2}{2} - (\underline{v}+\varepsilon)((\overline{v}^*+\varepsilon)-\underline{v})\right) + \hat{\gamma}\varepsilon.$$

Therefore,

$$\Delta'(\varepsilon) = -\hat{\gamma} + \hat{\gamma}^2((\overline{v}^*+\varepsilon) - (\overline{v}^*+\varepsilon-\underline{v}) - (\underline{v}+\varepsilon)) + \hat{\gamma} = \hat{\gamma}^2\varepsilon,$$

which converges to zero as $\varepsilon \to 0$.

Case 3: $v^* \in (\underline{v}, \overline{v})$**.** For this case, we prove the lemma by showing that

$$\widehat{D}'(p^*) = \frac{d\tilde{D}(p_i)}{dp_i}|_{p_i=p^*} = \frac{\partial D(p_i, G^*, p^*, G^*)}{\partial p_i}|_{p_i=p^*}. \tag{10}$$

Note the difference between the second and the third derivatives: $\tilde{D}(p_i)$ is firm i's demand function when it deviates to $p_i \neq p^*$ *and* chooses the corresponding optimal advertising strategy G_i^*; $D(p_i, G^*, p^*, G^*)$ is firm i's demand function when it deviates only in its price.

The next two claims prove each of the above two equalities.

Claim 3 *There exists $\delta > 0$ such that $\widehat{D}(p_i) = \tilde{D}(p_i)$ for any $p_i \in (p^*-\delta, p^*+\overline{v}-\overline{v}^*)$. Therefore, $\widehat{D}'(p^*) = \frac{d\tilde{D}(p_i)}{dp_i}|_{p_i=p^*}$.*

Proof. Notice that in this case, $G^*(v-p_i+p^*)^{n-1}$ is convex over $[\underline{v}, v^*+p_i-p^*]$ and flat over $[v^*+p_i-p^*, \overline{v}]$. Then, via Theorem 2, the following advertising strategy is optimal for firm i (note that this is different from that in Proposition 4, but there may exist multiple optimal advertising strategies):

$$G_i^*(v) = \begin{cases} F(v), & \text{if } v \leq v^*+p_i-p^*, \\ F(v^*+p_i-p^*), & \text{if } v \in [v^*+p_i-p^*, \mu_F(v^*+p_i-p^*)), \\ 1, & \text{if } v \geq \mu_F(v^*+p_i-p^*). \end{cases}$$

$$
\begin{aligned}
&D(p_i, G_i^*, p^*, G^*) = \int G^*(v-\varepsilon)^{n-1} dG_i^* \\
&= \int_{\underline{v}}^{v^*+\varepsilon} G^*(v-\varepsilon)^{n-1} dF(v) + (1 - F(v^*+\varepsilon)) G^*(\mu_F(v^*+\varepsilon) - \varepsilon)^{n-1} \\
&= \int_{\underline{v}}^{v^*+\varepsilon} \varphi(v-\varepsilon) dF(v) + \int_{v^*+\varepsilon}^{\overline{v}} \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(v - (v^*+\varepsilon)) \right) dF(v) \\
&= \int_{\underline{v}}^{v^*+\varepsilon} \varphi(v-\varepsilon) dF(v) + \int_{v^*+\varepsilon}^{\overline{v}} \varphi(v-\varepsilon) dF(v) = \widehat{D}(p_i).
\end{aligned}
$$

■

Claim 4 *Let G_i^* denote a firm's optimal advertising strategy corresponding to p_i. Then,*

$$
\frac{dD(p_i, G_i^*, p^*, G^*)}{dp_i}|_{p_i=p^*} = \frac{\partial D(p_i, G^*, p^*, G^*)}{\partial p_i}|_{p_i=p^*},
$$

Proof. Recall from the proof of Claim 3 that for p_i around p^*, an optimal advertising strategy is given by

$$
G_i^*(v) = \begin{cases} F(v), & \text{if } v \le v^* + p_i - p^*, \\ F(v^* + p_i - p^*), & \text{if } v \in [v^* + p_i - p^*, \mu_F(v^* + p_i - p^*)), \\ 1, & \text{if } v \ge \mu_F(v^* + p_i - p^*). \end{cases}
$$

Therefore, letting $\varepsilon = p_i - p^*$ and using the structure of G^*,

$$
\begin{aligned}
D(p_i, G_i^*, p^*, G^*) &= \int_{\underline{v}}^{v^*+\varepsilon} G^*(v-\varepsilon)^{n-1} dF(v) + (1 - F(v^*+\varepsilon)) G^*(\mu_F(v^*+\varepsilon) - \varepsilon)^{n-1} \\
&= \int_{\underline{v}}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dF(v) + \int_{v^*+\varepsilon}^{\overline{v}} \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(v - \varepsilon - v^*) \right) dF(v),
\end{aligned}
$$

while

$$
\begin{aligned}
&D(p_i, G^*, p^*, G^*) = \int_{\underline{v}}^{v^*} G^*(v-\varepsilon)^{n-1} dF(v) + \int_{v^*}^{\overline{v}^*} G^*(v-\varepsilon)^{n-1} dG^*(v) \\
&= \int_{\underline{v}}^{v^*} F(v-\varepsilon)^{n-1} dF(v) + \int_{v^*}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dG^*(v) + \int_{v^*+\varepsilon}^{\overline{v}^*} \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(v-\varepsilon-v^*) \right) dG^*(v) \\
&= \int_{\underline{v}}^{v^*} F(v-\varepsilon)^{n-1} dF(v) + (F(v^*)^{n-1})' \int_{v^*}^{v^*+\varepsilon} \left(F(v-\varepsilon)^{n-1} - F(v^*)^{n-1} - (F(v^*)^{n-1})'(v - (v^*+\varepsilon)) \right) dv \\
&\quad + \int_{v^*}^{\overline{v}} \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(v-\varepsilon-v^*) \right) dF(v)
\end{aligned}
$$

where the second equality is due to the fact that $\int_{v^*}^{\overline{v}^*} vdG^*(v) = \int_{v^*}^{\overline{v}} vdF(v)$. Therefore,

$$\begin{aligned}
&D(p_i, G_i^*, p^*, G^*) - D(p_i, G^*, p^*, G^*) \\
&= \int_{v^*}^{v^*+\varepsilon} F(v-\varepsilon)^{n-1} dF(v) - \int_{v^*}^{v^*+\varepsilon} \left(F(v^*)^{n-1} + (F(v^*)^{n-1})'(v-\varepsilon-v^*)\right) dF(v) \\
&\quad - (F(v^*)^{n-1})' \int_{v^*}^{v^*+\varepsilon} \left(F(v-\varepsilon)^{n-1} - F(v^*)^{n-1} - (F(v^*)^{n-1})'(v-(v^*+\varepsilon))\right) dv
\end{aligned}$$

It is straightforward to see that the derivative of this different with respect to ε at 0 (that is, $p_i = p^*$) is equal to 0. ■

Combining the above two claims demonstrates that the equality in (10) indeed holds, providing the desired result. ■

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Online Appendix: Proof of Proposition 5

In this Online Appendix, we provide the proof of Proposition 5, which characterizes an equilibrium in the advertising-only game in which consumers have an outside option.

Equilibrium verification First, we show that G^{**} is a best response to itself and, therefore, indeed an equilibrium. Given that all other firms play G^{**}, an individual firm faces the following advertising problem:

$$\max_{G} \int_{p^*}^{\overline{v}} G^{**}(v)^{n-1} dG(v), \qquad s.t.\ G \text{ is an MPC of } F.$$

To utilize Theorem 2, let $u(v) = 0$ if $v \le p^*$ and $u(v) = G^{**}(v)^{n-1}$ if $v > p^*$. In addition, define $\phi : [\underline{v}, \overline{v}] \to \mathcal{R}$ as follows:

$$\phi(v) = \begin{cases} 0 & \text{if } v \in [\underline{v}, v^{\dagger}), \\ F(v^{\dagger})^{n-1} + \beta(v - p^*) & \text{if } v \in [v^{\dagger}, p^*), \\ u(v) = G^{**}(v)^{n-1} & \text{if } v \in [p^*, \overline{v}^*], \\ \alpha(v - \overline{v}^*) + 1 & \text{if } v \in (\overline{v}^*, \overline{v}], \end{cases}$$

where $\overline{v}^*$ is the upper bound of support of G^{**} and α is as defined in equation (2). By the structure of $(G^{**})^{n-1}$, it is clear that ϕ is convex over $[\underline{v}, \overline{v}]$ and $\phi(v) \ge u(v)$ for all v. In addition, $\phi(v) = u(v)$ whenever $v \in supp(G^{**}) = [\underline{v}, v^{\dagger}] \cup [p^*, \overline{v}^*]$. Therefore, it suffices to show that

$$\int_{\underline{v}}^{\overline{v}} \phi(v) dG^{**}(v) = \int_{\underline{v}}^{\overline{v}} \phi(v) dF(v).$$

Since $G^{**}(v) = F(v)$ if $v \in [\underline{v}, v^{\dagger}]$,

$$\begin{aligned} &\int_{\underline{v}}^{\overline{v}} \phi(v) dG^{**}(v) - \int_{\underline{v}}^{\overline{v}} \phi(v) dF(v) \\ &= \left(\int_{v^{\dagger}}^{v^{\dagger\dagger}} \phi(v) dG^{**}(v) - \int_{v^{\dagger}}^{v^{\dagger\dagger}} \phi(v) dF(v) \right) + \left(\int_{v^{\dagger\dagger}}^{\overline{v}} \phi(v) dG^{**}(v) - \int_{v^{\dagger\dagger}}^{\overline{v}} \phi(v) dF(v) \right). \end{aligned}$$

The terms inside the first set of parentheses are equal to 0 because ϕ is linear and G^{**} is an MPC of F over $[v^{\dagger}, v^{\dagger\dagger}]$. Those within the second are also equal to 0 because G^{**} is an MPC of F above $v^{\dagger\dagger}$, with the same alternating structure as in Theorem 1.

Second, since G^{**} features the same properties as G^* in Theorem 1 above $v^{\dagger\dagger}$, the proof of

Theorem 2 implies that the second term on the right-hand side must be zero. Then, since $\phi(v)$ is linear over $[v^\dagger, v^{\dagger\dagger}]$ and G^{**} is an MPC over F over the same interval, the first term on the right-hand side must also be zero, leading to the desired result.

Equilibrium existence Now, we prove that there always exists an equilibrium by showing that there exists a unique pair of $v^\dagger \in (\underline{v}, p^*)$ and $\beta > 0$ such that the corresponding G^{**} satisfies the properties in the proposition.

Similar to the existence proof of Theorem 1 (Section 3.2.3), we use a constructive method to prove existence. For any $v' \in (\underline{v}, p^*)$ and $b \geq 0$, define a function $H(v; v', b)$ as follows:

$$H(v; v', b)^{n-1} = \begin{cases} F(v)^{n-1} & \text{if } v \in [\underline{v}, v'), \\ F(v')^{n-1} & \text{if } v \in [v', p^*), \\ \min\{F(v')^{n-1} + b(v - p^*), 1\} & \text{if } v \in [p^*, \overline{v}]. \end{cases}$$

Moreover, define

$$W(v; v', b) = \int_{\underline{v}}^{v} (F(v) - H(v; v', b))dv.$$

Recall that $\mu_F(a) = \mathbb{E}_F[v|v \geq a]$, which is continuous in a and strictly increasing over $[\underline{v}, \overline{v}]$. The following lemma states a technical result that we utilize in this proof.

Lemma 9 $\lim_{b\to\infty} W(\overline{v}; v', b) < 0$ *if and only if* $\mu_F(v') > p^*$.

Proof.

$$\begin{aligned}
\lim_{b\to\infty} W(\overline{v}; v', b) &= \lim_{b\to\infty} \int_{v'}^{\overline{v}} (F(v) - H(v; v', b))dv \\
&= \int_{v'}^{\overline{v}} F(v)dv - \left(\int_{v'}^{p^*} F(v')dv + \int_{p^*}^{\overline{v}} 1dv \right) \\
&= (\overline{v} - v'F(v')) - \int_{v'}^{\overline{v}} vdF(v) - (F(v')(p^* - v') + (\overline{v} - p^*)) \\
&= (1 - F(v'))(p^* - \mu_F(v')).
\end{aligned}$$

■

Define $\underline{v}^* = \min\{\nu \in [\underline{v}, p^*) : \mu_F(\nu) \geq p^*\}$. Observe that $\underline{v}^*$ always exists since $\mu_F(p^*) > p^*$. Throughout the next three claims, we show that there exist a unique $v^\dagger \in (\underline{v}^*, p^*)$ and $\beta > 0$ such that $H(v; v^\dagger, \beta)$ is the equilibrium G^{**} for $v \in [\underline{v}, v^{\dagger\dagger}]$.

The first claim states that for any $v' \in (\underline{v}^*, p^*)$, there exists a unique $\tilde{b}(v')$ such that $H(v; v', \tilde{b}(v'))$ can be used for the construction of G^{**}.

Claim 5 *For each $v' \in (\underline{v}^*, p^*)$, there exists a unique $\tilde{b}(v') \in (0, \infty)$ such that (a) $W(v; v', \tilde{b}(v')) \geq 0$ for all $v \in [\underline{v}, \overline{v}]$ and (b) $W(\hat{v}^*; v', \tilde{b}(v')) = 0$ for some $\hat{v}^* \in (p^*, \overline{v}]$.*

Proof. Observe that $W(v; v', b) = 0$ for any $v \in [\underline{v}, v']$ and $W(v; v', b) < 0$ for any $v \in (v', p^*]$. Moreover, observe that for any $v \in (p^*, \overline{v}]$, $W(v; v', b)$ is continuous and strictly decreases in b. Since $W(v; v', b = 0) > 0$ for any $v \in (p^*, \overline{v}]$ (since F is strictly increasing in v) and $\lim_{b\to\infty} W(\overline{v}; v', b) < 0$ (from Lemma 9 and the fact that $\mu_F(v') > p^*$ for any $v' > \underline{v}^*$), by the Intermediate Value Theorem, there must exist a unique $\tilde{b}(v') \in (0, \infty)$ that satisfies the conditions in the claim. ■

We now find the unique $v^\dagger$ such that $H(v; v^\dagger, \tilde{b}(v^\dagger))$ becomes a part of the equilibrium G^{**}. For each $v' \in (\underline{v}^*, p^*)$, define $\kappa(v')$ such that it satisfies

$$F(v')^{n-1} + \tilde{b}(v')(\kappa(v') - p^*) = 0.$$

Observe that if $\kappa(v^\dagger) = v^\dagger$ for some $v^\dagger \in (\underline{v}^*, p^*)$, then $v^\dagger$ and $\beta = \tilde{b}(v^\dagger)$ would satisfy the equation in condition (i) of Proposition 5. Solving for $\kappa(v')$ yields

$$\kappa(v') \equiv p^* - \frac{F(v')^{n-1}}{\tilde{b}(v')}. \tag{11}$$

Using the next two claims, we show that there exists a unique $v^\dagger \in (\underline{v}^*, p^*)$ such that $\kappa(v^\dagger) = v^\dagger$.

Claim 6 *$\kappa(v')$ is continuous and decreasing in v'.*

Proof. Since $F(v')$ is continuous and increasing in v', from (11), it suffices to show that $\tilde{b}(v')$ is continuous and decreasing in v'. The continuity of $\tilde{b}(v')$ is naturally derived from the continuity of $W(v; v', b)$ in both v' and b. For monotonicity, suppose to the contrary that there exists $v', v'' \in (\underline{v}^*, p^*)$ $(v' < v'')$ such that $\tilde{b}(v') < \tilde{b}(v'')$. Let $\hat{\nu} \in (p^*, \overline{v}]$ be such that $W(\hat{\nu}; v', \tilde{b}(v')) = 0$.

However, from the definition of $H(v; v', b)$, it must be that $H(v; v', \tilde{b}(v')) \leq H(v; v'', \tilde{b}(v''))$ for all $v \in [\underline{v}, \hat{\nu})$ with strict inequality holding at least for $v \in (v', p^*)$. Therefore, it must be that $W(\hat{\nu}; v'', \tilde{b}(v'')) < 0$, contradicting to the definition of $\tilde{b}(v'')$. ■

Claim 7 *(a) $\lim_{v'\to\underline{v}^*} \kappa(v') = p^*$ and (b) $\lim_{v'\to p^*} \kappa(v') < p^*$.*

Proof. (a) Suppose that $\mathbb{E}_F[v] \geq p^*$, which implies that $\underline{v}^* = \underline{v}$. Then it must be that $\lim_{v'\to\underline{v}^*} \tilde{b}(v') > 0$, as $W(v; \underline{v}, b = 0) > 0$ for any $v > (p^*, \overline{v}]$. Since $\lim_{v'\to\underline{v}} F(v')^{n-1} = 0$, we have the desired result.

Now, suppose that $\mathbb{E}_F[v] < p^*$. Then, it must be that $\underline{v}^* > \underline{v}$ and $\mu_F(\underline{v}^*) = p^*$. We claim that $\lim_{v' \to \underline{v}^*} \tilde{b}(v') = \infty$. By Lemma 9, $\mu_F(\underline{v}^*) = p^*$ implies that $\lim_{b \to \infty} W(\overline{v}; \underline{v}^*, b) = 0$. Since $\lim_{b \to \infty} H(v; v^*, b) = 1$ for any $v \in (p^*, \overline{v}]$, it follows that for any $v \in (p^*, \overline{v})$, $\lim_{b \to \infty} W(v; \underline{v}^*, b) > 0$. Then, for any *finite* b and $v \in (p^*, \overline{v}]$, we have

$$\begin{aligned}\lim_{v' \to \underline{v}^*} W(v; v', b) &= W(v; \underline{v}^*, b) \\ &> \lim_{b \to \infty} W(v; \underline{v}^*, b) > 0,\end{aligned}$$

since $W(\cdot; v', b)$ is continuous in v' and strictly decreasing in b. Therefore, it must be that $\lim_{v' \to \underline{v}^*} \tilde{b}(v') = \infty$, which implies that $\lim_{v' \to \underline{v}^*} \kappa(v') = p^*$.

(b) It must be that $\lim_{v' \to p^*} \tilde{b}(v') > 0$, since $W(v; v', b = 0) > 0$ for any $v \in (p^*, \overline{v}]$. Therefore, from (11), $\lim_{v' \to p^*} \kappa(v') < p^*$. ■

Claims 6 and 7, and the Intermediate Value Theorem together imply that there exists a unique $v^\dagger \in (\underline{v}, p^*)$ such that $\kappa(v^\dagger) = v^\dagger$.

Construction of G^{**}. Let $v^{\dagger\dagger} = \max\{v > p^* : W(v; v^\dagger, \tilde{b}(v^\dagger)) = 0\}$. Observe from Claim 5 that $v^{\dagger\dagger}$ always exists and that $v^{\dagger\dagger} \leq \overline{v}$. For $v \in [\underline{v}, v^{\dagger\dagger}]$, construct G^{**} as

$$G^{**}(v)^{n-1} = \begin{cases} F(v)^{n-1} & \text{if } v < v^\dagger, \\ F(v^\dagger)^{n-1} & \text{if } v \in [v^\dagger, p^*), \\ \min\{F(v^\dagger)^{n-1} + \tilde{b}(v^\dagger)(v - p^*), 1\} & \text{if } v \in [p^*, v^{\dagger\dagger}]. \end{cases}$$

If $v^{\dagger\dagger} = \overline{v}$, then the construction of G^{**} is complete. If $v^{\dagger\dagger} < \overline{v}$, construct G^{**} for $v \in (v^{\dagger\dagger}, \overline{v}]$ using a method identical to that used in the main model (Subsection 3.2.3). By the construction of $v^\dagger$ and $\tilde{b}(v^\dagger)$, it is straightforward to verify that G^{**} satisfies the properties in Proposition 5.