A New Look at Local Expected Utility*

S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci
Department of Decision Sciences and IGIER, Università Bocconi

February 22, 2014

Abstract

We revisit the classical local Expected Utility analysis of Machina [30] and show which is its global behavioral foundation, as an illustration, we compute the local utilities for the Prospect Theory model.
JEL classification: D81

Keywords: Local utilities, Risk aversion, Multi–utility representation, Prospect theory

1 Introduction

The seminal paper of Machina [30] showed that the global risk aversion analysis carried on for the Expected Utility (briefly, EU) model naturally extends to a local risk aversion analysis in the case of non-EU binary relations $\succsim$ that can be represented via a Frechet differentiable utility function V. In the non-EU realm the role of the "single global" utility of the standard setting is taken by "multiple local" utilities, which, for example, are all concave if and only if $\succsim$ is averse to Mean Preserving Spreads. Despite the recognized importance of the local approach, the *global* role of these *local* utilities, their preferential counterpart, remained unexplained. In this paper, we show how the set of local utilities has a natural interpretation in terms of behavior: it represents the largest part of $\succsim$ that is consistent with the EU axioms.

More formally, we study binary relations $\succsim$ on $\mathcal{D}$: the space of probability distributions on a closed interval I. We interpret $\succsim$ as capturing the preferences of a Decision Maker (briefly, DM) in a situation of choice under risk and we assume that $\succsim$ can be represented by a continuous utility function V. Given $\succsim$, we consider the auxiliary binary relation $\succsim^*$ defined by

$$F \succsim^* G \iff \lambda F + (1-\lambda) H \succsim \lambda G + (1-\lambda) H \qquad \forall \lambda \in (0,1], \forall H \in \mathcal{D}.$$

We interpret $F \succsim^* G$ as saying that the DM is sure that F is weakly better than G. In fact, no matter how F is mixed/hedged with a third prospect H, the mixture with F dominates the mixture with G. In Lemma 1, we show that $\succsim^*$ satisfies all the assumptions of EU, with the potential exception of completeness, and it is the largest subrelation of $\succsim$ with these properties. In other words, $\succsim^*$ collects the largest portion of $\succsim$ which is consistent with the EU paradigm. If V is further assumed to be differentiable, then our Theorem 1 shows that

$$F \succsim^* G \iff \int_I u(x)\, dF(x) \geq \int_I u(x)\, dG(x) \qquad \forall u \in \nabla V(\mathcal{D}) \tag{1}$$

where $\nabla V(\mathcal{D})$ is the collection of all derivatives of V, that is, of all Machina's local utilities. In this way, we are able to formalize the idea that individually each local utility models a local expected utility behavior of $\succsim$, as in Machina [30], but jointly all local utilities characterize a global expected utility feature of $\succsim$.

Beyond the main theorem, our other results investigate the properties of $\succsim^*$ and, especially, the ones related to consistency with stochastic orders. The results we obtain are of both conceptual and technical

*We thank Luigi Montrucchio, as well as the participants of the HEC-Paris workshop "Decision Making Under Uncertainty and Beyond", for useful comments. We gratefully acknowledge the financial support of MIUR (PRIN grant 20103S5RN3_005) and of the AXA Research Fund.

interest. For example, Proposition 2 should clarify why outside the EU model the dominant notion of risk aversion is aversion to MPSs. In fact, even without any differentiability hypothesis, this is equivalent to require standard risk aversion *but* only in terms of the binary relation $\succsim^*$: the EU part of the DM's ranking. From a technical point of view, (a) our results are just in terms of Gateaux derivatives whereas Machina [30] required the more stronger notion of Frechet differentiability and (b) they provide a unifying framework for some of the results in the literature.

To better see point (b), consider the following example. In the Finance literature (see Arditti [2], Tsiang [37], Kraus and Litzenberger [28]), it has often been argued that Expected Utility DMs should exhibit a preference toward (positive) skewness in conjunction with risk aversion and a preference for more money rather than less. In the EU model, this behavioral condition is equivalent to the DM's Bernoulli utility function being increasing, concave, and with convex marginal utility. This is also equivalent (see Whitmore [41]) to impose that, if F dominates G with respect to third order stochastic dominance, then F should be preferred to G. Outside the EU realm, preference toward skewness can thus be modeled by assuming that $\succsim$ is consistent with third order stochastic dominance. Our Proposition 1 yields that this is equivalent to impose that $\succsim^*$, the EU part of the DM's rankings, is consistent with third order stochastic dominance, and under differentiability, that each local utility is increasing, concave, and with convex derivative. In Proposition 1 (see also Proposition 2), we show that a result of this kind can be stated for *any* integral stochastic order and in particular also if we dispense with the differentiability assumption. This latter generalization is important since it applies to models for which differentiability is not always granted, for example the Betweenness model of Dekel [17] and Chew [12], but the representation of $\succsim^*$ is connected to the representation of $\succsim$ (see Example 2). In this way, we can apply this result to obtain as corollaries some of the existing results in the literature: Machina [30, Theorem 1], Dekel [17, Properties 1 and 2], Chew [12, Theorem 5], Chew, Epstein, Segal [13, Theorem 3], and Chew and Nishimura [15, Lemma 1 and Corollary 1]. A similar conclusion applies for some other models of choice under risk. Indeed, we can obtain as corollaries Chew [11, Corollary 6], Cerreia-Vioglio, Dillenberger, and Ortoleva [6, Theorem 3], Maccheroni [29, Section 5], and Chatterjee and Krishna [10, Theorem 4.2 and Proposition 4.3].

Finally, in order to exemplify the tractability of this approach we compute the local utilities for the Prospect Theory model and characterize risk aversion and preference for skewness within this model. In particular, Corollary 2 shows that the Prospect Theory model is incompatible with third order stochastic dominance (preference for skewness).

1.1 Related literature

Ambiguity The derived binary relation $\succsim^*$ is the risk counterpart of the revealed unambiguous preference relation introduced by Ghirardato, Maccheroni, and Marinacci [22] in a setting of decision under ambiguity and for invariant biseparable preferences $\succsim$ (see also Nehring [32]). Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi [8] study $\succsim^*$ for the more general class of rational preferences $\succsim$. Several differential characterizations of $\succsim^*$ have been proposed. The first one can be found in Ghirardato, Maccheroni, and Marinacci [22]. A direct extension of this result appears in Ghirardato and Siniscalchi [23], who develop in an ambiguity setup a local analysis close to the spirit of Machina [30]. Finally, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [7] provide an alternative differential characterization and, inter alia, also characterize $\succsim^*$ for several ambiguity averse models.

Risk In a context of choice under risk, $\succsim^*$ was first studied by Cerreia-Vioglio [5] for convex preferences $\succsim$. This derived binary relation plays a central role in Cerreia-Vioglio, Dillenberger, and Ortoleva [6], where preferences that satisfy the Negative Certainty Independence axiom of Dillenberger [18] are represented as minima of certainty equivalents. Preferences satisfying Negative Certainty Independence are typically non-differentiable as proved by Dillenberger [18] and as suggested by the representation. To the best of our knowledge, the present paper is the first one providing a differential characterization of $\succsim^*$.

Gateaux derivatives The notion of Gateaux derivative that we use is due to von Mises [31, p. 323]. It has been widely used in Statistics since Hampel [24] for the study of robustness (see Huber [25, pp. 34-40] and Fernholz [21]). It was adopted in Risk Theory by Chew, Karni, and Safra [14] (see also Wang [40]). The paper of Chew, Karni, and Safra [14] is also connected to ours since their Theorem 1 and Corollary 2 are related to our Proposition 3 and Corollary 3. Nevertheless, they restrict themselves to the Rank Dependent Utility model and, more importantly, they do not study the global aspect of the differential approach. In Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio [9] we study this notion of differentiability in an abstract setting.

2 Preliminaries

2.1 Notation and mathematical preliminaries

Let $\mathcal{D} = \mathcal{D}(I)$ be the set of all cumulative distribution functions on a (possibly unbounded) closed interval I of $\mathbb{R}$. We denote by F, G, and H generic elements of $\mathcal{D}$, and by x, y, and z generic elements of I. Given an element $x \in I$, we denote by G_x the distribution that yields x with probability 1. We endow $\mathcal{D}$ with the topology of weak convergence.[1] We denote by $C_b(I)$ the set of all bounded and continuous functions on I. If I is bounded, then $C_b(I)$ is the space $C(I)$ of all continuous functions on I. We endow $C_b(I)$ with the topology induced by the supnorm.

Given a function $V : \mathcal{D} \to \mathbb{R}$, we say that V is Gateaux differentiable at F if and only if there exists a function $u_F \in C_b(I)$ such that for each $G \in \mathcal{D}$

$$\lim_{\theta \downarrow 0} \frac{V((1-\theta)F + \theta G) - V(F)}{\theta} = \int_I u_F(x)\, d(G-F)(x). \tag{2}$$

We say that u_F is a *local utility* function. The function V is Gateaux differentiable if and only if it is differentiable at each $F \in \mathcal{D}$ and we denote by $\nabla V : \mathcal{D} \rightrightarrows C_b(I)$ the derivative correspondence.[2] Finally, we define

$$\operatorname{range} \nabla V = \bigcup_{F \in \mathcal{D}} \nabla V(F) = \{u_F \in C_b(I) : F \in \mathcal{D}\}.$$

The set $\operatorname{range} \nabla V$ is the collection of all local utilities of V.

Remark 1 Since Hampel [24], in Statistics the function $IC_{V,F} : I \to \mathbb{R}$ defined by the limit

$$IC_{V,F}(x) = \lim_{\theta \downarrow 0} \frac{V((1-\theta)F + \theta G_x) - V(F)}{\theta} \qquad \forall x \in I \tag{3}$$

is known as the *influence curve* of V at F. As observed, if $u_F \in \nabla V(F)$, then $\nabla V(F) = \{u_F + k : k \in \mathbb{R}\}$. In particular, the element ψ_F of $\nabla V(F)$ given by $\psi_F = u_F - \int_I u_F(x)\, dF(x) \in C_b(I)$ coincides with the influence curve of V at F; that is, $\psi_F = IC_{V,F}$. Moreover,

$$\lim_{\theta \downarrow 0} \frac{V((1-\theta)F + \theta G) - V(F)}{\theta} = \int_I \psi_F(x)\, dG(x) \qquad \forall G \in \mathcal{D}. \tag{4}$$

Consider $x_0 \in I$. By (3), $\psi_F(x_0)$ describes the impact on V of an infinitesimal change in the probability assigned to x_0 in F. For this reason we can call $\psi_F(x_0)$ the *marginal risk utility* of x_0 at F.

This interpretation is best understood when a finitely supported distribution

$$F = p_{x_0} G_{x_0} + p_{x_1} G_{x_1} + \cdots + p_{x_n} G_{x_n}$$

is considered. It holds

$$\begin{aligned} (1-\theta)F + \theta G_{x_0} &= (p_{x_0} - \theta p_{x_0} + \theta) G_{x_0} + \sum_{i=1}^{n} (1-\theta) p_{x_i} G_{x_i} \\ &= (p_{x_0} + \theta(p_{x_1} + \cdots + p_{x_n})) G_{x_0} + (p_{x_1} - \theta p_{x_1}) G_{x_1} + \cdots + (p_{x_n} - \theta p_{x_n}) G_{x_n}. \end{aligned}$$

Therefore, the difference quotient

$$\frac{V((1-\theta)F + \theta G_{x_0}) - V(F)}{\theta}$$

[1] See Appendix A for a formal definition of the topology of weak convergence and other technical details.

[2] In this setting Gateaux derivatives, that is local utilities, as defined in (2), are unique only up to a constant. Thus, the derivative $\nabla V : \mathcal{D} \rightrightarrows C_b(I)$ defined by $F \mapsto \{u_F + k\}_{k \in \mathbb{R}}$ is a correspondence where u_F is a function that satisfies (2).

is the (average) additional utility gained from, at the same time, augmenting the probability of x_0 by θ percent and reducing the probability of all other outcomes in a proportional way. Gateaux differentiability at F, as expressed by (4), states that the marginal risk contribution of G to utility at F can be regarded as the average, with respect to G, of the marginal risk utility contributions of the single outcomes. $\triangle$

We conclude by introducing a last mathematical object. If I is bounded, given a set $\mathcal{U} \subseteq C(I)$, we denote by $\langle \mathcal{U} \rangle$ the set

$$\langle \mathcal{U} \rangle = \mathrm{cl}\left(\mathrm{cone}\mathcal{U} + \{\lambda 1_I\}_{\lambda \in \mathbb{R}}\right)$$

where cone$\mathcal{U}$ is the smallest convex cone containing $\mathcal{U}$ and cl is the supnorm closure.

2.2 Decision theoretic preliminaries

The object of our study is a binary relation $\succsim$ defined on $\mathcal{D}$.[3] A function $V : \mathcal{D} \to \mathbb{R}$ is said to represent $\succsim$ or to be a utility function for $\succsim$ if and only if for each $F, G \in \mathcal{D}$

$$F \succsim G \Longleftrightarrow V(F) \geq V(G).$$

Given $F \in \mathcal{D}$, we denote by $e(F)$ its expected value. We interpret $\succsim$ as representing the DM's preferences. The axiomatic properties on $\succsim$ we discuss and use in this paper are few and classic. We next list them for completeness.

Preorder The relation $\succsim$ is reflexive and transitive.

Weak Order The relation $\succsim$ is complete and transitive.

Continuity For each pair of convergent sequences $\{F_n\}$ and $\{G_n\}$ in $\mathcal{D}$,

$$F_n \succsim G_n \quad \forall n \implies \lim_n F_n \succsim \lim_n G_n.$$

Independence For each $F, G, H \in \mathcal{D}$ and for each $\lambda \in (0,1)$

$$F \succsim G \implies \lambda F + (1-\lambda) H \succsim \lambda G + (1-\lambda) H.$$

Throughout the paper, we will consider binary relations $\succsim$ that can be represented by a continuous utility function V. It is well known (see Debreu [16]) that, in our setting, this is equivalent to assume that $\succsim$ satisfies Weak Order and Continuity.

3 Main result

3.1 A key relation

In this section, we will show that the set of local utilities captures the part of the DM's preferences which are globally Expected Utility and not only, as shown by Machina [30] and Chew, Karni, and Safra [14], a local behavior which can be reconciled with the Expected Utility model. This part of the DM's preferences is a subrelation of $\succsim$ and so it is not specific to the utility function V which represents it. In order to do so, we define an auxiliary binary relation $\succsim^*$. Given the binary relation $\succsim$ on $\mathcal{D}$, $\succsim^*$ is defined to be such that

$$F \succsim^* G \iff \lambda F + (1-\lambda) H \succsim \lambda G + (1-\lambda) H \qquad \forall \lambda \in (0,1], \forall H \in \mathcal{D}.$$

Formally, $\succsim^*$ is a subrelation of $\succsim$. We interpret this derived binary relation as capturing the rankings for which the DM is sure. For, no matter how F is mixed with a third prospect H, the mixture with F dominates the same mixture with F replaced by G. This binary relation is the risk counterpart of the revealed unambiguous preference relation studied by Ghirardato, Maccheroni, and Marinacci [22] in a setting of decision under ambiguity. We next derive $\succsim^*$ for two well known risk models.

[3] We denote by $\succ$ and $\sim$, respectively, the asymmetric and the symmetric parts of $\succsim$.

Example 1 (Mixture Symmetry) Consider $I = [m, M]$ and let $\phi : I \times I \to \mathbb{R}$ be a symmetric and continuous function. Chew, Epstein, and Segal [13], inter alia, study a class of binary relations on $\mathcal{D}$ represented by a utility function $V : \mathcal{D} \to \mathbb{R}$ defined by

$$V(F) = \int_I \int_I \phi(x, y)\, dF(x)\, dF(y) \qquad \forall F \in \mathcal{D}.$$

The key assumption satisfied by these binary relations is Mixture Symmetry.[4] They also show that V is continuous and Gateaux differentiable in the sense of (2). On the other hand, by applying the definition of $\succsim^*$, it can be proved (see Appendix B) that

$$F \succsim^* G \iff \int_I \phi(x, y)\, dF(x) \geq \int_I \phi(x, y)\, dG(x) \qquad \forall y \in I. \tag{5}$$

▲

Example 2 (Betweenness) Consider $I = [m, M]$ and let $\psi : I \times [0,1] \to \mathbb{R}$ be a continuous function in both components which is strictly increasing in the first component and such that $\psi(M, y) - 1 = 0 = \psi(m, y)$ for all $y \in [0, 1]$. Dekel [17] (see also Chew [12]) studies a class of binary relations on $\mathcal{D}$ represented (implicitly) by a continuous utility function $V : \mathcal{D} \to [0, 1]$ where, for each $F \in \mathcal{D}$, $V(F)$ is the unique number such that

$$\int_I \psi(x, V(F))\, dF(x) = V(F). \tag{6}$$

The key assumption satisfied by these binary relations is Betweenness.[5] A priori (see Wang [40]), V might not be Gateaux differentiable. By applying the definition, it can be proved (see Appendix B) that

$$F \succsim^* G \iff \int_I \psi(x, y)\, dF(x) \geq \int_I \psi(x, y)\, dG(x) \qquad \forall y \in [0, 1]. \tag{7}$$

▲

The first lemma lists some properties of $\succsim^*$.

Lemma 1 *Let $\succsim$ be a binary relation represented by a continuous utility function V. The following statements are true:*

(i) $\succsim^*$ is a preorder that satisfies Continuity and Independence;

(ii) If I is bounded, then there exists a set $\mathcal{U}^* \subseteq C(I)$ such that

$$F \succsim^* G \iff \int_I u(x)\, dF(x) \geq \int_I u(x)\, dG(x) \qquad \forall u \in \mathcal{U}^*; \tag{8}$$

(iii) $\succsim$ is consistent with $\succsim^*$;[6]

(iv) If $\succsim$ is consistent with a binary relation that satisfies Independence, $\succsim^*$ is also consistent with it.

The first point shows that $\succsim^*$ satisfies all the Expected Utility axioms with the potential exception of completeness. The second point provides a characterization of $\succsim^*$ when I is bounded (it follows from the main result of Dubra, Maccheroni, and Ok [19]). The third point shows that $\succsim^*$ is a subrelation of $\succsim$, thus capturing a part of the rankings expressed by the DM. The last point implies that $\succsim^*$ is the largest subrelation of $\succsim$ that satisfies the Expected Utility axioms with the potential exception of completeness, thus supporting the interpretation that $\succsim^*$ summarizes the rankings for which the DM behaves like a standard Expected Utility agent. Point (iv) actually yields more, in fact it implies consistency of $\succsim^*$ with any binary relation that satisfies *just* independence. This is important in connection with the Mean Preserving Spread relation (see Section 4.1).

---

[4] See [13] for a definition of Mixture Symmetry.

[5] See [17] for a definition of Betweenness.

[6] That is, for each $F, G \in \mathcal{D}$ if $F \succsim^* G$, then $F \succsim G$.

3.2 Main result

In the next theorem, our main result, we show that the set of all local utilities represents $\succsim^*$.[7]

Theorem 1 *If $\succsim$ is a binary relation represented by a continuous and Gateaux differentiable utility function V, then*

$$F \succsim^* G \iff \int_I u(x)\, dF(x) \geq \int_I u(x)\, dG(x) \qquad \forall u \in \operatorname{range} \nabla V.$$

Local utilities thus capture both local and global behavior that is consistent with expected utility. In particular, individually each of them models a local expected utility behavior of $\succsim$, as Machina [30] emphasized, but jointly they characterize a global expected utility feature of $\succsim$, as our result shows. The next result is a simple consequence of the theorem and shows that larger sets of local utilities characterize "less" expected utility preferences.

Corollary 1 *Let $\succsim_1$ and $\succsim_2$ be two binary relations that satisfy the hypotheses of the previous theorem and let I be bounded. The following statements are equivalent:*

(i) $F \succsim_2^* G$ *implies* $F \succsim_1^* G$*;*

(ii) $\langle \operatorname{range} \nabla V_1 \rangle \subseteq \langle \operatorname{range} \nabla V_2 \rangle$.

4 Integral stochastic orders

In the rest of the paper, to better compare our results with the literature and to avoid technicalities, we confine ourselves to the case $I = [m, M]$.[8] A binary relation $\hat{\succsim}$ on $\mathcal{D}$ is an integral stochastic order if and only if there exists a set $\mathcal{U}^{\wedge} \subseteq C(I)$ such that

$$F \hat{\succsim} G \iff \int_I u(x)\, dF(x) \geq \int_I u(x)\, dG(x) \qquad \forall u \in \mathcal{U}^{\wedge}.$$

First order stochastic dominance is an integral stochastic order with $\mathcal{U}^{\wedge}$ being the set of all increasing functions in $C(I)$. Similarly, second order stochastic dominance is an integral stochastic order with $\mathcal{U}^{\wedge}$ being the set of all increasing and concave functions in $C(I)$. Finally, the concave order is an integral stochastic order with $\mathcal{U}^{\wedge}$ being the set of all concave functions in $C(I)$.[9]

Proposition 1 *Let $\succsim$ be a binary relation represented by a continuous utility function V and let I be bounded. Given an integral stochastic order $\hat{\succsim}$, the following statements are equivalent:*

(i) $\succsim$ *is consistent with* $\hat{\succsim}$*;*

(ii) $\succsim^*$ *is consistent with* $\hat{\succsim}$*;*

(iii) $\mathcal{U}^* \subseteq \langle \mathcal{U}^{\wedge} \rangle$.

If, in addition, V is Gateaux differentiable, then they are also equivalent to:

(iv) $\operatorname{range} \nabla V \subseteq \langle \mathcal{U}^{\wedge} \rangle$.

As a result, under differentiability, $\succsim$ is consistent with:

1. first order stochastic dominance if and only if all local utilities are increasing;
2. second order stochastic dominance if and only if all local utilities are increasing and concave;

[7] Note that in Theorem 1 the interval I is not required to be bounded. Thus, we cannot rely on Dubra, Maccheroni, and Ok [19] to represent $\succsim^*$, as shown by Evren [20].

[8] Though for concreteness we consider closed intervals, our main results (Lemma 1, Theorem 1, Corollary 1, and Proposition 1) actually hold in metric spaces (compact when the intervals are required to be bounded).

[9] The concave order is connected to aversion to mean preserving spreads and risk aversion (see Section 4.1).

3. third order stochastic dominance if and only if all local utilities are increasing, concave, and have convex derivative on (m, M);[10]

4. the concave order if and only if all local utilities are concave.

Given Examples 1 and 2, the above proposition generalizes Machina [30, Theorem 1], Chew, Epstein, and Segal [13, Theorem 3], and Dekel [17, Property 1] (see also Chew [12, Theorem 5]). In this way, it provides a unifying framework for this type of results. The closest existing results, due to Chew and Nishimura [15, Lemma 1 and Corollary 1], essentially show that if range $\nabla V \subseteq \mathcal{U}^{\hat{}}$, then $\succsim$ is consistent with $\hat{\succsim}$. The improvement of Proposition 1 is twofold. First, our condition (iv) is weaker as well as necessary and sufficient.[11] Second, our result also has consequences when the underlying preference is not represented by a differentiable V (see Example 2).

4.1 Risk aversion

To discuss (absolute) risk aversion, we have to first give a definition of risk aversion. Outside the realm of Expected Utility, we have two competing notions: classic risk aversion and aversion to Mean Preserving Spreads. The second notion requires the definition of Mean Preserving Spread (henceforth, MPS). We start by providing the more general notion of Simple Compensated Spread, first introduced by Machina [30], for a binary relation $\succsim$. The notion of MPS will be a particular case.

Given F and G in $\mathcal{D}$ and $\succsim$ on $\mathcal{D}$, we say that G is a Simple Compensated Spread (henceforth, SCS) of F for $\succsim$ if and only if $F \sim G$ and there exists $z \in [m, M]$ such that

$$\begin{cases} F(x) \leq G(x) & \forall x \in [m, z) \\ F(x) \geq G(x) & \forall x \in [z, M] \end{cases} . \tag{9}$$

As anticipated, G is a MPS of F, written $F \succsim_{MPS} G$, if and only if $e(F) = e(G)$ and there exists $z \in [m, M]$ such that (9) holds.[12]

Given a binary relation $\succsim$ on $\mathcal{D}$, we say that $\succsim$ is

1. *risk averse* if and only if $G_{e(F)} \succsim F$ for all $F \in \mathcal{D}$;

2. *MPS averse* if and only if $\succsim$ is consistent with $\succsim_{MPS}$.[13]

Proposition 2 *Let $\succsim$ be a binary relation represented by a continuous utility function V and let I be bounded. The following statements are equivalent:*

(i) $\succsim$ is consistent with the concave order;

(ii) $\succsim$ is MPS averse;

(iii) $\succsim^*$ is MPS averse;

(iv) $\succsim^*$ is risk averse.

If, in addition, V is Gateaux differentiable, then they are also equivalent to:

(v) Each $u \in \text{range}\, \nabla V$ is concave.

Each of the previous conditions imply that $\succsim$ is risk averse, but the converse is false.

[10] Recall that consistency with respect to third order stochastic dominance is connected to preference for (positive) skewness and in the EU model is actually equivalent to it.

[11] For example, in order to derive consistency with third order stochastic dominance Chew and Nishimura [15, Corollary 1] require range ∇V to be contained in the set $\mathcal{U}^{\hat{}}$ of all continuous functions that have decreasing first derivative and increasing second derivative. On the other hand, the weaker condition range $\nabla V \subseteq \langle \mathcal{U}^{\hat{}} \rangle$ is necessary *and* sufficient and the latter set consists of all continuous functions that are increasing, concave, and have convex derivative.

[12] We could have also opted for the standard definition of Mean Preserving Spread for distributions with finite support (see Rothschild and Stiglitz [35]). The following analysis would be unchanged. Also, note that $\succsim_{MPS}$ is a binary relation on $\mathcal{D}$.

[13] That is, $F \succsim_{MPS} G$ implies $F \succsim G$.

Since it is not transitive, $\succsim_{MPS}$ is not an integral stochastic order, and so Proposition 2 is not an immediate corollary of Proposition 1.

Like [14], given two binary relations $\succsim_1$ and $\succsim_2$ on $\mathcal{D}$, we say that $\succsim_1$ is more risk averse than $\succsim_2$ if and only if whenever G is a SCS of F for $\succsim_2$, then $F \succsim_1 G$.

Proposition 3 *Let $\succsim_1$ be a binary relation represented by a continuous and Gateaux differentiable utility function V and let $\succsim_2$ be an Expected Utility binary relation with continuous and strictly increasing Bernoulli utility function v. The following statements are equivalent:*

(i) $\succsim_1$ is more risk averse than $\succsim_2$;

(ii) Each $u \in \operatorname{range} \nabla V$ is a concave transformation of v.

If, in addition, $\succsim_1$ is consistent with first order stochastic dominance, then they are also equivalent to:

(iii) Each $u \in \operatorname{range} \nabla V$ is an increasing and concave transformation of v.

Propositions 2 and 3 provide an alternative proof and a generalization to Machina [30, Theorems 3 and 4].[14] The contribution of our results is both conceptual and technical. From a conceptual point of view, Proposition 2 should clarify why outside the Expected Utility model the dominant notion of risk aversion is aversion to MPSs. In fact, even without any differentiability hypothesis, this is equivalent to require standard risk aversion *but* only in terms of the binary relation $\succsim^*$: the Expected Utility part of the DM's ranking.

From a technical point of view our results, which are just in terms of Gateaux derivatives rather than Frechet, provide a unifying framework for some of the results in the literature, and further highlight the strict connection between integral stochastic orders and local utilities. To see this latter fact, assume $\succsim_2$ is an Expected Utility binary relation with continuous and strictly increasing Bernoulli utility function v. Without loss of generality, assume that $v(m) = m$ and $v(M) = M$. Then, G is a SCS of F for $\succsim_2$ *only if*

$$\int_{[m,M]} u(v(x))\, dF(x) \geq \int_{[m,M]} u(v(x))\, dG(x) \text{ for all concave } u \in C([m,M]). \tag{10}$$

In particular, G is a MPS of F only if F and G satisfy (10) with v equal to the identity. To see the unifying feature of Proposition 2, it is enough to note how, given Example 1, it yields [13, Theorem 3], while given Example 2, it improves [17, Property 2]. After a minor technical specification, it also delivers [6, Theorem 3].

5 Prospect theory and stochastic dominance

In Prospect Theory (see Wakker [39] for a textbook introduction) the utility function $V : \mathcal{D} \to \mathbb{R}$ that represents $\succsim$ is given by

$$V(F) = \int_0^M w(1 - F(x))\, dv(x) - \int_m^0 \tilde{w}(F(x))\, dv(x) \tag{11}$$

where $I = [m, M]$, with $m \leq 0 \leq M$, $v : I \to \mathbb{R}$ is a continuous and strictly increasing function such that $v(0) = 0$, and $w, \tilde{w} : [0,1] \to [0,1]$ are strictly increasing and onto functions.

This well known model has been proposed by Tversky and Kahneman [38] to extend the scope of the classic analysis of Kahneman and Tversky [26]. Two special cases (see [38, p. 302]) are noteworthy:

(i) $\tilde{w} = w$ is the original specification considered in Kahneman and Tversky [26].

(ii) $\tilde{w} = \bar{w}$, where $\bar{w} : [0,1] \to \mathbb{R}$ is given by $\bar{w}(p) = 1 - w(1-p)$ for all $p \in [0,1]$, is the Rank Dependent Utility model (see Quiggin [33]).

[14] See also Chew, Karni, and Safra [14] for similar results concerning the Rank Dependent Utility model.

Chew, Karni, and Safra [14] computed local utilities for the Rank Dependent Utility case. Next we compute them for the general Prospect Theory model.[15]

Proposition 4 *If $w, \tilde{w} : [0,1] \to [0,1]$ are continuously differentiable, then the Prospect Theory utility function (11) is Gateaux differentiable, with*

$$u_F(x) = \int_{[m,x]} \left[w'(1 - F(y)) 1_{[0,M]}(y) + \tilde{w}'(F(y)) 1_{[m,0)}(y) \right] dv(y).$$

The final result of the paper shows how, under mild differentiability assumptions, the Prospect Theory model is incompatible with third order stochastic dominance. In Finance, this latter property has been used to model preference for skewness (as discussed in Introduction).

Corollary 2 *Let w, $\tilde{w}$, and v be continuously differentiable and $m < 0 < M$. The following statements are equivalent:*

(i) $\succsim$ is consistent with third order stochastic dominance;

(ii) $w(p) = p = \tilde{w}(p)$ for all $p \in [0,1]$ and v is increasing, concave, and has convex derivative on (m, M).

This result builds on our previous results, as well as on the proof of the characterization of aversion to MPSs in the Prospect Theory model. This latter characterization was first proved by Schmidt and Zank [36] and by Chew, Karni, and Safra [14] for the Rank Dependent Utility model. In Appendix B we restate this result and provide a novel proof which hinges on the shape of the local utility functions (Proposition 4) and their concavity (Proposition 2).

A Distributions and integrals

We denote a closed interval by I. Let $m, M \in \mathbb{R}$ be such that $M > m$. We next formally define the set $\mathcal{D}(I)$. We have four possible cases:

1. $\mathcal{D}((-\infty,\infty)) = \left\{ F \in \mathbb{R}^{\mathbb{R}} : F \text{ is increasing, right continuous}, \lim_{t\to-\infty} F(t) = 0, \lim_{t\to+\infty} F(t) = 1 \right\}$;

2. $\mathcal{D}([m,\infty)) = \{ F \in \mathcal{D}(-\infty,\infty) : F(y) = 0 \text{ for all } y < m \}$;

3. $\mathcal{D}((-\infty,M]) = \{ F \in \mathcal{D}(-\infty,\infty) : F(y) = 1 \text{ for all } y \geq M \}$;

4. $\mathcal{D}([m,M]) = \mathcal{D}([m,\infty)) \cap \mathcal{D}((-\infty,M])$.

Next, we define two other important sets:

1. $\Delta_I(\mathbb{R})$, the set of all Borel probability measures with support I;

2. $\Delta(I)$, the set of all Borel probability measures on I.

Given $\mathcal{D}(I)$, we endow it with the topology of weak convergence: given $\{F_n\} \subseteq \mathcal{D}(I)$ and $F \in \mathcal{D}(I)$ we have that $\lim_n F_n = F$ if and only if $\lim_n F_n(x) = F(x)$ for all $x \in (-\infty,\infty)$ which is a continuity point of F (see Billingsley [3, p. 327]).

Given I bounded and $\Delta(I)$, we endow the latter set with the weak* topology: given $\{\mu_n\} \subseteq \Delta(I)$ and $\mu \in \Delta(I)$ we have that $\lim_n \mu_n = \mu$ if and only if $\lim_n \int_I f d\mu_n = \int_I f d\mu$ for all $f \in C(I)$.

Next, we define two maps $T : \mathcal{D}(I) \to \Delta_I(\mathbb{R})$ and $P : \Delta_I(\mathbb{R}) \to \Delta(I)$. T is such that $T(F)$ is the unique measure on the real line, denoted by $\hat{\mu}_F$, such that $\hat{\mu}_F((a,b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$. By [3, Theorem 12.4], T is well defined. It is immediate to see that this map is affine. On the other hand, P is such that $\mu = P(\hat{\mu})$ is the measure $\hat{\mu}$ restricted to I, that is, $P(\hat{\mu})(B) = \hat{\mu}(B \cap I)$ for all Borel sets B of I. It is immediate to see that P is well defined and affine. Note that $P \circ T : \mathcal{D}(I) \to \Delta(I)$ is a map that associates to each distribution $F \in \mathcal{D}(I)$ a unique probability measure denoted by μ_F in $\Delta(I)$. If I is bounded, then $P \circ T$ is an affine homeomorphism.

[15] Recall that local utilities at a point F are unique only up to an additive constant. In Proposition 4, the local utility u_F has been computed by further imposing that $u_F(m) = 0$.

Given $u \in C_b(I)$, we denote the Lebesgue-Stieltjes integral $\int_I u d\mu_F$ by $\int_I u(x) dF(x)$. If I is equal to $[m, M]$, then its relation with the Riemann-Stieltjes integral is such that:

$$\int_{[m,M]} u(x) dF(x) = \int_{[m,M]} u d\mu_F = u(m) F(m) + \int_m^M u(x) dF(x),$$

where the first equality is by definition and the second one is a well known fact. Note that the last integral is a Riemann-Stieltjes integral. Often, to differentiate a Lebesgue-Stieltjes integral from a Riemann-Stieltjes integral, we will denote the first one by $\int_{[m,M]} u(x) dF(x)$ and the second one by $\int_m^M u(x) dF(x)$. Finally, given $u \in C_b(I)$, we denote by $\int_I u(x) d(G - F)(x)$ the difference $\int_I u(x) dG(x) - \int_I u(x) dF(x)$.

B Proofs and related analysis

The proof of Lemma 1 is basically contained in [5], the difference being that here the setting are distribution functions rather than probability measures. We report it for the sake of completeness.

Proof of Lemma 1. (i) and (iii). Trivially, we have that $\succsim^*$ is a preorder. Next, consider $\{F_n\}, \{G_n\} \subseteq \mathcal{D}$ such that $F_n \to F \in \mathcal{D}$, $G_n \to G \in \mathcal{D}$, and $F_n \succsim^* G_n$ for all $n \in \mathbb{N}$. Fix $H \in \mathcal{D}$ and $\lambda \in (0,1]$. It follows that $\lambda F_n + (1-\lambda) H \succsim \lambda G_n + (1-\lambda) H$ for all $n \in \mathbb{N}$. Since $\succsim$ satisfies Continuity and $\lambda F_n + (1-\lambda) H \to \lambda F + (1-\lambda) H$ and $\lambda G_n + (1-\lambda) H \to \lambda G + (1-\lambda) H$, this implies that $\lambda F + (1-\lambda) H \succsim \lambda G + (1-\lambda) H$. Since $H \in \mathcal{D}$ and $\lambda \in (0,1]$ were arbitrarily chosen, we can conclude that $F \succsim^* G$. Next, consider $F, G, H \in \mathcal{D}$. Assume that $F \succsim^* G$ and $\lambda \in (0,1)$. It follows that

$$\begin{aligned}
\mu(\lambda F + (1-\lambda) H) + (1-\mu) H' &= (\mu\lambda) F + (1-\mu\lambda) \left[\frac{\mu(1-\lambda)}{1-\mu\lambda} H + \frac{1-\mu}{1-\mu\lambda} H'\right] \\
&\succsim (\mu\lambda) G + (1-\mu\lambda) \left[\frac{\mu(1-\lambda)}{1-\mu\lambda} H + \frac{1-\mu}{1-\mu\lambda} H'\right] \\
&= \mu(\lambda G + (1-\lambda) H) + (1-\mu) H' \qquad \forall \mu \in (0,1], \forall H' \in \mathcal{D},
\end{aligned}$$

proving that $\lambda F + (1-\lambda) H \succsim^* \lambda G + (1-\lambda) H$. Thus, $\succsim^*$ satisfies Independence. Finally, by definition of $\succsim^*$, (iii) trivially follows.

(ii). Define $S = P \circ T$. Define also $\succsim^\circ$ on $\Delta(I)$ by $\mu \succsim^\circ \nu$ if and only if $S^{-1}(\mu) \succsim^* S^{-1}(\nu)$. By [19] and given the properties of S and $\succsim^*$, it follows that there exists a set $\mathcal{U}^* \subseteq C(I)$ such that

$$\mu \succsim^\circ \nu \iff \int_I u d\mu \geq \int_I u d\nu \qquad \forall u \in \mathcal{U}^*.$$

Thus, we can conclude that

$$\begin{aligned}
F \succsim^* G \iff S(F) \succsim^\circ S(G) \iff \mu_F \succsim^\circ \mu_G \iff \int_I u d\mu_F \geq \int_I u d\mu_G \quad \forall u \in \mathcal{U}^* \\
\iff \int_I u(x) dF(x) \geq \int_I u(x) dG(x) \quad \forall u \in \mathcal{U}^*.
\end{aligned}$$

(iv) Let $\succsim$ be consistent with $\hat{\succsim}$ and let $\hat{\succsim}$ satisfy Independence. Assume that $F \hat{\succsim} G$. Since $\hat{\succsim}$ satisfies Independence, it follows that $\lambda F + (1-\lambda) H \hat{\succsim} \lambda G + (1-\lambda) H$ for all $\lambda \in (0,1]$ and for all $H \in \mathcal{D}$. Since $\succsim$ is consistent with $\hat{\succsim}$, it follows that $\lambda F + (1-\lambda) H \succsim \lambda G + (1-\lambda) H$ for all $\lambda \in (0,1]$ and for all $H \in \mathcal{D}$, that is, $F \succsim^* G$. ■

Next, we give a version of the Mean Value Theorem. Given our framework and since the notion of differentiability we are using is a notion of Gateaux differentiability which involves just one side derivatives and a particular domain, this result is not obvious even though the proof is rather simple.

Proposition 5 *If $V : \mathcal{D} \to \mathbb{R}$ is continuous and Gateaux differentiable, then for each $F, G \in \mathcal{D}$ there exists $t \in (0,1)$ such that*

$$V(F) - V(G) = \int_I u_{F_t}(x) dF(x) - \int_I u_{F_t}(x) dG(x).$$

Proof. See Cerreia-Vioglio, Maccheroni, Marinacci, Montrucchio [9]. ■

Proof of Theorem 1. Define $\hat{\succsim}$ by

$$F \hat{\succsim} G \iff \int_I u(x)\, dF(x) \geq \int_I u(x)\, dG(x) \qquad \forall u \in \text{range}\, \nabla V.$$

We next show that $\hat{\succsim}$ coincides to $\succsim^*$. Consider F and G in $\mathcal{D}$. Assume that $F \hat{\succsim} G$. By Proposition 5 and since $F \hat{\succsim} G$, we have that there exists $t \in (0,1)$ such that

$$V(F) - V(G) = \int_I u_{F_t}(x)\, dF(x) - \int_I u_{F_t}(x)\, dG(x) \geq 0,$$

yielding that $F \succsim G$. By Lemma 1 and since $\hat{\succsim}$ satisfies Independence, it follows that $F \succsim^* G$. Viceversa, assume that $F \succsim^* G$. Consider $H \in \mathcal{D}$. By definition of $\succsim^*$ and since V represents $\succsim$, we have that $V((1-\theta)H + \theta F) \geq V((1-\theta)H + \theta G)$ for all $\theta \in (0,1]$. This implies that

$$\frac{V((1-\theta)H + \theta F) - V(H)}{\theta} \geq \frac{V((1-\theta)H + \theta G) - V(H)}{\theta}.$$

By passing to the limit and since V is Gateaux differentiable, it follows that $\int_I u_H(x)\, dF(x) - \int_I u_H(x)\, dH(x) \geq \int_I u_H(x)\, dG(x) - \int_I u_H(x)\, dH(x)$. Since H was arbitrarily chosen, we can conclude that $F \hat{\succsim} G$. ■

Proof of Corollary 1. The statement follows by standard separation techniques and Theorem 1 (see [19], [5], and [6]). ■

Derivation of Equation 5. Define $W : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ by

$$W(F,G) = \int_{[m,M]} \int_{[m,M]} \phi(x,y)\, dF(x)\, dG(y) \qquad \forall (F,G) \in \mathcal{D} \times \mathcal{D}.$$

It is immediate to check that W is affine in both components, $W(F,G) = W(G,F)$ for all $(F,G) \in \mathcal{D} \times \mathcal{D}$, and $V(F) = W(F,F)$ for all $F \in \mathcal{D}$. We start by observing two facts:

(a) Fix $F, H \in \mathcal{D}$. If we define $F_\gamma = \gamma F + (1-\gamma) H$ for all $\gamma \in (0,1]$, then

$$\begin{aligned} V(F_\gamma) &= W(F_\gamma, F_\gamma) = \gamma W(F, F_\gamma) + (1-\gamma) W(H, F_\gamma) \\ &= \gamma^2 W(F,F) + \gamma(1-\gamma) W(F,H) + (1-\gamma)\gamma W(H,F) + (1-\gamma)^2 W(H,H) \\ &= \gamma^2 W(F,F) + 2(1-\gamma)\gamma W(F,H) + (1-\gamma)^2 W(H,H). \end{aligned}$$

(b) Fix $F, G \in \mathcal{D}$. If $\int_{[m,M]} \phi(x,y)\, dF(x) \geq \int_{[m,M]} \phi(x,y)\, dG(x)$ for all $y \in [m,M]$, then for each $H \in \mathcal{D}$

$$W(F,H) = \int_{[m,M]} \int_{[m,M]} \phi(x,y)\, dF(x)\, dH(y) \geq \int_{[m,M]} \int_{[m,M]} \phi(x,y)\, dG(x)\, dH(y) = W(G,H).$$

In particular, since H was arbitrarily chosen, we have that

$$V(F) = W(F,F) \geq W(G,F) = W(F,G) \geq W(G,G) = V(G).$$

Next, by facts (a) and (b), observe that

$$\begin{aligned} F \succsim^* G &\iff \lambda F + (1-\lambda) H \succsim \lambda G + (1-\lambda) H \qquad \forall \lambda \in (0,1], \forall H \in \mathcal{D} \\ &\iff V(\lambda F + (1-\lambda) H) - V(\lambda G + (1-\lambda) H) \geq 0 \qquad \forall \lambda \in (0,1], \forall H \in \mathcal{D} \\ &\iff \lambda^2 (V(F) - V(G)) + 2\lambda(1-\lambda)(W(F,H) - W(G,H)) \geq 0 \qquad \forall \lambda \in (0,1], \forall H \in \mathcal{D} \\ &\iff \lambda (V(F) - V(G)) + 2(1-\lambda)(W(F,H) - W(G,H)) \geq 0 \qquad \forall \lambda \in (0,1], \forall H \in \mathcal{D} \\ &\iff V(F) \geq V(G) \text{ and } W(F,H) - W(G,H) \geq 0 \qquad \forall H \in \mathcal{D} \\ &\iff V(F) \geq V(G) \text{ and } \int_{[m,M]} \phi(x,y)\, dF(x) \geq \int_{[m,M]} \phi(x,y)\, dG(x) \qquad \forall y \in [m,M] \\ &\iff \int_{[m,M]} \phi(x,y)\, dF(x) \geq \int_{[m,M]} \phi(x,y)\, dG(x) \qquad \forall y \in [m,M], \end{aligned}$$

proving the statement. ■

Derivation of Equation 7. We proceed by Steps. Nevertheless, before starting we introduce a mathematical object and discuss some of its properties. Define $K : \mathcal{D} \times [0,1] \to \mathbb{R}$ by

$$K(F,y) = \int_{[m,M]} \psi(x,y)\, dF(x) \qquad \forall y \in [0,1], \forall F \in \mathcal{D}.$$

It is immediate to see that K is affine with respect to the first component. Note that for each $y \in [0,1]$ and for each $F \in \mathcal{D}$

$$\begin{aligned} K(F, V(F)) &= \int_{[m,M]} \psi(x, V(F))\, dF(x) = V(F) = V(F)\psi(M,y) + (1-V(F))\psi(m,y) \\ &= \int_{[m,M]} \psi(\cdot, y)\, d(V(F)G_M + (1-V(F))G_m) = K(V(F)G_M + (1-V(F))G_m, y). \end{aligned}$$

We also define $\hat{\succsim}$ on $\mathcal{D}$ by

$$F \hat{\succsim} G \iff \int_{[m,M]} \psi(x,y)\, dF(x) \geq \int_{[m,M]} \psi(x,y)\, dG(x) \qquad \forall y \in [0,1]. \tag{12}$$

Note that $F \hat{\succsim} G$ if and only if $K(F,y) \geq K(G,y)$ for all $y \in [0,1]$.

Step 1. If $F \hat{\succsim} G$ then $F \succsim G$. In particular, if $F \hat{\succsim} G$ then $F \succsim^* G$.

Proof of the Step. Consider $F, G \in \mathcal{D}$. By contradiction, assume that $K(F,y) \geq K(G,y)$ for all $y \in [0,1]$ and $V(G) > V(F)$. By assumption and since $\succsim$ is defined as in Example 2, note that

$$K(F, V(G)) \geq K(G, V(G)) = V(G). \tag{13}$$

On the other hand, by working hypothesis, we have $V(G) > V(F)$. By the initial part of the proof, it follows that

$$\begin{aligned} V(G) > V(F) &= K(V(F)G_M + (1-V(F))G_m, V(G)) \\ &= K(V(F)G_M + (1-V(F))G_m, V(F)) = V(F) = K(F, V(F)). \end{aligned}$$

In particular, this yields that

$$K(V(F)G_M + (1-V(F))G_m, V(F)) = V(F) = K(F, V(F)) \tag{14}$$

and

$$V(G) > K(V(F)G_M + (1-V(F))G_m, V(G)). \tag{15}$$

Define $H = V(F)G_M + (1-V(F))G_m$. By (13) and (15) and since K is affine with respect to the first component, it follows that there exists $\lambda \in (0,1]$ such that

$$K(\lambda F + (1-\lambda)H, V(G)) = V(G).$$

By equation (6), we can conclude that $V(\lambda F + (1-\lambda)H) = V(G)$. By (14) and equation (6), we have that $V(F) = V(H)$, in particular $V(F) = V(\lambda F + (1-\lambda)H) = V(H)$. We can conclude that $V(G) > V(F) = V(\lambda F + (1-\lambda)H) = V(G)$, a contradiction. Thus, we showed that if $F \hat{\succsim} G$ then $F \succsim G$. By Lemma 1, it follows that if $F \hat{\succsim} G$ then $F \succsim^* G$, proving the step. □

Step 2. If $F \succsim^* G$ then $K(F,y) \geq K(G,y)$ for all $y \in (0,1)$.

Proof of the Step. Consider $F, G \in \mathcal{D}$. By contradiction, assume that $F \succsim^* G$ and that there exists $\bar{y} \in (0,1)$ such that $K(F,\bar{y}) < K(G,\bar{y})$. Then, there exists $\lambda \in (0,1]$ and $y \in [m,M]$ such that $V(\lambda F + (1-\lambda)G_y) = \bar{y}$.[16] It follows that

$$\begin{aligned} \bar{y} &= K(\lambda F + (1-\lambda)G_y, \bar{y}) = \lambda K(F,\bar{y}) + (1-\lambda)K(G_y,\bar{y}) \\ &< \lambda K(G,\bar{y}) + (1-\lambda)K(G_y,\bar{y}) = K(\lambda G + (1-\lambda)G_y, \bar{y}). \end{aligned}$$

[16] If $V(F) \geq \bar{y} > 0 = V(G_m)$ then $y = m$ and if $V(F) < \bar{y} < 1 = V(G_M)$ then $y = M$. The existence of λ is then granted by the continuity of V.

Define $H_1 = \lambda F + (1-\lambda) G_y$ and $H_2 = \lambda G + (1-\lambda) G_y$ so that $\bar{y} = V(H_1)$. In particular, we have that

$$V(H_1) < K(H_2, V(H_1)). \tag{16}$$

By Lemma 1 and since $F \succsim^* G$, it follows that $H_1 \succsim^* H_2$, yielding that $V(H_1) \geq V(H_2)$. Define $H_3 = V(H_2) G_M + (1 - V(H_2)) G_m$. It is immediate to see that $V(H_2) = V(H_3)$. On the other hand, by the initial part of the proof and (16), we have that

$$K(H_3, V(H_1)) = K(H_3, V(H_2)) = V(H_2) \leq V(H_1) < K(H_2, V(H_1)).$$

By (16) and since K is affine with respect to the first component, we have that there exists $\gamma \in [0,1)$ such that

$$K(\gamma H_2 + (1-\gamma) H_3, V(H_1)) = V(H_1).$$

It follows that $V(\gamma H_2 + (1-\gamma) H_3) = V(H_1)$. By equation (6) and since $V(H_2) = V(H_3)$, this yields that

$$V(H_2) = V(\gamma H_2 + (1-\gamma) H_3) = V(H_1).$$

We can then conclude that $V(H_2) = V(H_1)$, that is, $V(H_1) = V(H_2) = K(H_2, V(H_2)) = K(H_2, V(H_1))$, a contradiction with (16). □

Step 3. If $F \succsim^* G$ then $K(F, y) \geq K(G, y)$ for all $y \in [0,1]$.

Proof of the Step. By Step 2, we have that if $F \succsim^* G$ then $K(F, y) \geq K(G, y)$ for all $y \in (0,1)$. We are left to show the same inequality holds when $y \in \{0,1\}$. Consider $\{y_n\} \in (0,1)$ such that $y_n \to \bar{y}$ where $\bar{y}$ is either 0 or 1. Note that $|\psi(x, y_n)| \leq 1$ for all $n \in \mathbb{N}$ and for all $x \in [m, M]$. Since ψ is continuous in both components, we have that $\psi(x, y_n) \to \psi(x, \bar{y})$ for all $x \in [m, M]$. By the Lebesgue Dominated Convergence Theorem (see [1, Theorem 11.21]) and Step 2, we have that

$$\begin{aligned} K(F, \bar{y}) &= \int_{[m,M]} \psi(x, \bar{y})\, dF(x) = \lim_n \int_{[m,M]} \psi(x, y_n)\, dF(x) = \lim_n K(F, y_n) \\ &\geq \lim_n K(G, y_n) = \lim_n \int_{[m,M]} \psi(x, y_n)\, dG(x) = \int_{[m,M]} \psi(x, \bar{y})\, dG(x) = K(G, \bar{y}), \end{aligned}$$

proving the statement. □

Given (12), the statement follows by Steps 1–3. ■

Proof of Proposition 1. (i) implies (ii). Since $\hat{\succsim}$ is an integral stochastic order, it satisfies Independence. By Lemma 1 and since $\succsim$ is consistent with $\hat{\succsim}$, $\succsim^*$ is consistent with $\hat{\succsim}$.

(ii) implies (iii). By point (b) of Lemma 1 and since I is bounded, the statement follows by standard duality techniques (see [19] and [5]).

(iii) implies (i). Note that

$$\begin{aligned} F \hat{\succsim} G &\implies \int_I u(x)\, dF(x) \geq \int_I u(x)\, dG(x) \quad \forall u \in \langle \mathcal{U}^{\hat{}} \rangle \implies \int_I u(x)\, dF(x) \geq \int_I u(x)\, dG(x) \quad \forall u \in \mathcal{U}^* \\ &\implies F \succsim^* G \implies F \succsim G. \end{aligned}$$

proving the implication.

We just showed that (i), (ii), and (iii) are equivalent. Now assume that V is also Gateaux differentiable. By Theorem 1, it follows that $\mathcal{U}^*$ can be chosen to be range ∇V. By the same proof of (ii) implies (iii), this yields that (ii) implies (iv). For the same reason and the same proof of (iii) implies (i), (iv) implies (i). ■

Proof of Proposition 2. Before starting consider also this condition

(v)′ Each $u \in \mathcal{U}^*$ is concave.

(i) implies (ii). It is well known that if $F \succsim_{MPS} G$, then $\int_I u(x)\, dF(x) \geq \int_I u(x)\, dG(x)$ for all concave $u \in C(I)$. Since $\succsim$ is consistent with the concave order, it follows that $F \succsim G$.

(ii) implies (iii). By definition of MPS and since $\succsim$ is MPS averse, note that

$$\begin{aligned} F \succsim_{MPS} G &\implies \lambda F + (1-\lambda) H \succsim_{MPS} \lambda G + (1-\lambda) H \qquad \forall \lambda \in (0,1], \forall H \in \mathcal{D} \\ &\implies \lambda F + (1-\lambda) H \succsim \lambda G + (1-\lambda) H \qquad \forall \lambda \in (0,1], \forall H \in \mathcal{D} \implies F \succsim^* G, \end{aligned}$$

proving that $\succsim^*$ is MPS averse.

(iii) implies (iv). Since $G_{e(F)} \succsim_{MPS} F$ for all $F \in \mathcal{D}$, it follows that $G_{e(F)} \succsim^* F$ for all $F \in \mathcal{D}$.

(iv) implies (v)$'$. Since I is bounded and by Lemma 1, we have that there exists a set $\mathcal{U}^* \subseteq C(I)$ that represents $\succsim^*$ as in (8). Pick $x, y \in I$. Consider $F = \frac{1}{2}G_x + \frac{1}{2}G_y$. By assumption, it follows that $G_{\frac{1}{2}x+\frac{1}{2}y} \succsim^* F$. We can conclude that $u\left(\frac{1}{2}x + \frac{1}{2}y\right) \geq \frac{1}{2}u(x) + \frac{1}{2}u(y)$ for all $u \in \mathcal{U}^*$, that is, each $u \in \mathcal{U}^*$ is concave.

(v)$'$ implies (i). By Proposition 1 and since each $u \in \mathcal{U}^*$ is concave, the statement follows.

We just showed that (i), (ii), (iii), (iv) and (v)$'$ are equivalent. Now assume that V is also Gateaux differentiable. By Theorem 1, it follows that $\mathcal{U}^*$ can be chosen to be range ∇V. By the same proof of (iv) implies (v)$'$, this yields that (iv) implies (v). For the same reason and the same proof of (v)$'$ implies (i), (v) implies (i). ■

Proof of Proposition 3. Without loss of generality assume that $v \in C([m, M])$ is normalized, that is, $v(m) = m$ and $v(M) = M$. Define $\bar{V} : \mathcal{D} \to \mathbb{R}$ by

$$\bar{V}(F) = \int_{[m,M]} v(x)\, dF(x) \qquad \forall F \in \mathcal{D}.$$

Consider F and G in $\mathcal{D}$. Assume that G is a SCS of F for $\succsim_2$, we denote it by $F \succsim_{SCS} G$. Recall that $F \succsim_{SCS} G$ if and only if $\bar{V}(F) = \bar{V}(G)$ and there exists $z \in [m, M]$ such that

$$\begin{cases} F(x) \leq G(x) & \forall x \in [m, z) \\ F(x) \geq G(x) & \forall x \in [z, M] \end{cases}.$$

(i) implies (ii). By definition of SCS and since $\succsim_1$ is more risk averse than $\succsim_2$ and $\succsim_2$ is Expected Utility, note that

$$\begin{aligned} F \succsim_{SCS} G &\implies \lambda F + (1-\lambda) H \succsim_{SCS} \lambda G + (1-\lambda) H \qquad \forall \lambda \in (0,1], \forall H \in \mathcal{D} \\ &\implies \lambda F + (1-\lambda) H \succsim_1 \lambda G + (1-\lambda) H \qquad \forall \lambda \in (0,1], \forall H \in \mathcal{D} \implies F \succsim_1^* G, \end{aligned}$$

proving that $\succsim_1^*$ is more risk averse than $\succsim_2$. Consider $F \in \mathcal{D}$. It is immediate to see that $G_{v^{-1}(\bar{V}(F))} \in \mathcal{D}$. Next, note that $G_{v^{-1}(\bar{V}(F))} \succsim_{SCS} F$ for all $F \in \mathcal{D}$. Consider $y_1, y_2 \in [m, M] = v([m, M])$. There exists $x_1, x_2 \in [m, M]$ such that $v(x_i) = y_i$ for $i \in \{1, 2\}$. Define $F = \frac{1}{2}G_{x_1} + \frac{1}{2}G_{x_2}$ and $\bar{y} = v^{-1}\left(\bar{V}(F)\right) = v^{-1}\left(\frac{1}{2}v(x_1) + \frac{1}{2}v(x_2)\right)$. We thus have that $G_{\bar{y}} \succsim_{SCS} F$ and so $G_{\bar{y}} \succsim_1^* F$. For each $u \in$ range ∇V define $f_u = u \circ v^{-1} \in C([m, M])$. By Theorem 1 and since $G_{\bar{y}} \succsim_1^* F$, we have that for each $u \in$ range ∇V

$$\begin{aligned} f_u\left(\frac{1}{2}y_1 + \frac{1}{2}y_2\right) &= u \circ v^{-1}\left(\frac{1}{2}y_1 + \frac{1}{2}y_2\right) = u\left(v^{-1}\left(\frac{1}{2}v(x_1) + \frac{1}{2}v(x_2)\right)\right) = u(\bar{y}) \\ &= \int_{[m,M]} u(x)\, dG_{\bar{y}}(x) \geq \int_{[m,M]} u(x)\, dF(x) = \frac{1}{2}u(x_1) + \frac{1}{2}u(x_2) \\ &= \frac{1}{2}u\left(v^{-1}(v(x_1))\right) + \frac{1}{2}u\left(v^{-1}(v(x_2))\right) = \frac{1}{2}u\left(v^{-1}(y_1)\right) + \frac{1}{2}u\left(v^{-1}(y_2)\right) \\ &= \frac{1}{2}f_u(y_1) + \frac{1}{2}f_u(y_2), \end{aligned}$$

proving that f_u is concave and $u = f_u \circ v$.

(ii) implies (i). Consider $F, G \in \mathcal{D}$ and $u \in$ range ∇V. By Theorem 1 and since each $u \in$ range ∇V is a concave transformation of v, if $F \succsim_{SCS} G$, then $\int_I u(x)\, dF(x) \geq \int_I u(x)\, dG(x)$ for all $u \in$ range ∇V which, in turn, implies that $F \succsim G$, proving the statement.

We just showed that (i) and (ii) are equivalent. Now assume that $\succsim_1$ is also consistent with first order stochastic dominance. By Proposition 1, it follows that each $u \in$ range ∇V is also increasing. By the same proof of (i) implies (ii), we have that f_u is also increasing and this yields that (i) implies (iii). Trivially, (iii) implies (ii). ■

Consider V defined as in (11). We first report a simple property.

Lemma 2 *$V : \mathcal{D} \to \mathbb{R}$ is continuous.*

Proof of Proposition 4. We want to compute the Gateaux derivative of V at F in direction $G-F$, that is,

$$\lim_{\theta\downarrow 0}\frac{V\left(\left(1-\theta\right)F+\theta G\right)-V\left(F\right)}{\theta}\qquad\forall F,G\in\mathcal{D}.\tag{17}$$

The computation is simplified by the observation that for each function $f:[m,M]\to\mathbb{R}$ of bounded variation, the Riemann-Stieltjes integral $\int_m^M f\left(x\right)dv\left(x\right)$ coincides with the Lebesgue-Stieltjes integral $\int_{[m,M]}fdv$ of f with respect to the Borel measure induced on $[m,M]$ by any continuous and increasing extension of v to $\mathbb{R}$. Set $H=G-F$, and note that, provided the limit in (17) exists, it is equal to

$$\begin{aligned}
&=\lim_{\theta\downarrow 0}\frac{V\left(F+\theta\left(G-F\right)\right)-V\left(F\right)}{\theta}\\
&=\lim_{\theta\downarrow 0}\frac{\int_{[0,M]}w\left(1-F-\theta H\right)dv-\int_{[m,0]}\tilde{w}\left(F+\theta H\right)dv-\int_{[0,M]}w\left(1-F\right)dv+\int_{[m,0]}\tilde{w}\left(F\right)dv}{\theta}\\
&=\lim_{\theta\downarrow 0}\int_{[0,M]}\frac{w\left(1-F\left(x\right)-\theta H\left(x\right)\right)-w\left(1-F\left(x\right)\right)}{\theta}dv\left(x\right)-\int_{[m,0]}\frac{\tilde{w}\left(F\left(x\right)+\theta H\left(x\right)\right)-\tilde{w}\left(F\left(x\right)\right)}{\theta}dv\left(x\right).
\end{aligned}$$

For each $x\in[m,M]$, we have that

- if $x\in[0,M]$ and $H\left(x\right)\neq 0$, then

  $$\begin{aligned}
  \lim_{\theta\downarrow 0}\frac{w\left(1-F\left(x\right)-\theta H\left(x\right)\right)-w\left(1-F\left(x\right)\right)}{\theta}&=\lim_{\theta\downarrow 0}\frac{w\left(1-F\left(x\right)-\theta H\left(x\right)\right)-w\left(1-F\left(x\right)\right)}{-\theta H\left(x\right)}\left(-H\left(x\right)\right)\\
  &=-w'\left(1-F\left(x\right)\right)H\left(x\right)
  \end{aligned}$$

  and the same holds when $H\left(x\right)=0$;

- if $x\in[m,0]$ and $H\left(x\right)\neq 0$, then

  $$\lim_{\theta\downarrow 0}\frac{\tilde{w}\left(F\left(x\right)+\theta H\left(x\right)\right)-\tilde{w}\left(F\left(x\right)\right)}{\theta}=\lim_{\theta\downarrow 0}\frac{\tilde{w}\left(F\left(x\right)+\theta H\left(x\right)\right)-\tilde{w}\left(F\left(x\right)\right)}{\theta H\left(x\right)}H\left(x\right)=\tilde{w}'\left(F\left(x\right)\right)H\left(x\right)$$

  and the same holds when $H\left(x\right)=0$.

Continuous differentiability of w and $\tilde{w}$ implies their Lipschitzianity so that, for each $x\in[m,M]$ and each $\theta\in(0,1)$,

$$\left|\frac{w\left(1-F\left(x\right)-\theta H\left(x\right)\right)-w\left(1-F\left(x\right)\right)}{\theta}\right|\leq\frac{L_w\left|1-F\left(x\right)-\theta\left(G\left(x\right)-F\left(x\right)\right)-\left(1-F\left(x\right)\right)\right|}{\theta}\leq L_w$$

and

$$\left|\frac{\tilde{w}\left(F\left(x\right)+\theta H\left(x\right)\right)-\tilde{w}\left(F\left(x\right)\right)}{\theta}\right|\leq\frac{L_{\tilde{w}}\left|F\left(x\right)+\theta\left(G\left(x\right)-F\left(x\right)\right)-F\left(x\right)\right|}{\theta}\leq L_{\tilde{w}}.$$

Therefore the Dominated Convergence Theorem applied to each sequence $\theta_n\to 0^+$ yields that

$$\begin{aligned}
\lim_{\theta\downarrow 0}\frac{V\left(F+\theta\left(G-F\right)\right)-V\left(F\right)}{\theta}&=\int_{[0,M]}-w'\left(1-F\left(x\right)\right)\left(G\left(x\right)-F\left(x\right)\right)dv\left(x\right)\\
&\quad-\int_{[m,0]}\tilde{w}'\left(F\left(x\right)\right)\left(G\left(x\right)-F\left(x\right)\right)dv\left(x\right).
\end{aligned}$$

Now define

$$\phi_F\left(x\right)=\begin{cases}w'\left(1-F\left(x\right)\right) & x\in[0,M]\\ \tilde{w}'\left(F\left(x\right)\right) & x\in[m,0)\end{cases}\tag{18}$$

and note that ϕ_F is bounded and Borel measurable on $[m,M]$ with

$$\lim_{\theta\downarrow 0}\frac{V\left(F+\theta\left(G-F\right)\right)-V\left(F\right)}{\theta}=-\int_{[m,M]}\left(G\left(x\right)-F\left(x\right)\right)\phi_F\left(x\right)dv\left(x\right).$$

Setting "$du_F = \phi_F dv$", or more precisely $u_F(x) = \int_{[m,x]} \phi_F dv$ for all $x \in [m, M]$, it is not difficult to show that $u_F \in C([m, M])$ and

$$\begin{aligned}
\int_{[m,M]} (G(x) - F(x)) \phi_F(x) dv(x) &= \int_m^M (G - F) du_F \\
&= (G(M) - F(M)) u_F(M) - (G(m) - F(m)) u_F(m) - \left(\int_m^M u_F dG - \int_m^M u_F dF \right) \\
&= -\left(\int_{[m,M]} u_F dG - \int_{[m,M]} u_F dF \right) = -\int_{[m,M]} u_F d(G - F)
\end{aligned}$$

where the second equality follows by integration by parts, the third by $G(M) = F(M) = 1$ and $u_F(m) = 0$, and the last one by definition, proving the statement. ■

Corollary 3 *Let w, $\tilde{w}$, and v be continuously differentiable and $m < 0 < M$. The following statements are equivalent:*

(i) $\succsim$ is MPS averse;

(ii) w is convex, $\tilde{w}$ is concave, v is concave, and

$$\sup_{p \in (0,1)} \frac{w'(1-p)}{\tilde{w}'(p)} \leq 1.$$

Proof. Before starting note that for all $F \in \mathcal{D} \cap C_b((-\infty, \infty))$

$$\begin{aligned}
u'_F(x) &= w'(1 - F(x)) v'(x) \qquad \forall x \in (0, M) \\
u'_F(x) &= \tilde{w}'(F(x)) v'(x) \qquad \forall x \in (m, 0) \\
(u_F)'_+(0) &= w'(1 - F(0)) v'(0) \\
(u_F)'_-(0) &= \tilde{w}'(F(0)) v'(0)
\end{aligned}$$

Given $\bar{x}, \bar{y} \in (m, M)$ such that $\bar{x} > \bar{y}$, consider $G, H \in \mathcal{D}$ defined by

$$G(x) = \begin{cases} 1 & x > M \\ \frac{x-\bar{x}}{M-\bar{x}} & x \in [\bar{x}, M] \\ 0 & x < \bar{x} \end{cases} \qquad \forall x \in \mathbb{R}$$

and

$$H(x) = \begin{cases} 1 & x > \bar{y} \\ \frac{x-m}{\bar{y}-m} & x \in [m, \bar{y}] \\ 0 & x < m \end{cases} \qquad \forall x \in \mathbb{R}.$$

Given $p \in (0,1)$, define $H_p = pH + (1-p)G$. Finally, since $\tilde{w}$ and w are strictly increasing and continuously differentiable, there exist $p, \tilde{p} \in (0,1)$ such that $\tilde{w}'(\tilde{p}) > 0$ and $w'(p) > 0$.

(i) implies (ii). By Proposition 4 and Proposition 2, we have that for each $F \in \mathcal{D}$ a local utility u_F exists and it is concave. Given the initial part of the proof, we have that $(u_F)'_+$ and $(u_F)'_-$ exists on (m, M) and are decreasing. Next, observe that:

1. If $\bar{x}, \bar{y} \in (m, 0]$ are such that $\bar{x} > \bar{y}$, then

$$\tilde{w}'(\tilde{p}) v'(\bar{x}) = \tilde{w}'(H_{\tilde{p}}(\bar{x})) v'(\bar{x}) = (u_{H_{\tilde{p}}})'_-(\bar{x}) \leq (u_{H_{\tilde{p}}})'_-(\bar{y}) = \tilde{w}'(H_{\tilde{p}}(\bar{y})) v'(\bar{y}) = \tilde{w}'(\tilde{p}) v'(\bar{y}),$$

proving that $v'(\bar{x}) \leq v'(\bar{y})$. If $\bar{x}, \bar{y} \in [0, M)$ are such that $\bar{x} > \bar{y}$, then

$$\begin{aligned}
w'(p) v'(\bar{x}) &= w'(1 - H_{1-p}(\bar{x})) v'(\bar{x}) = (u_{H_{1-p}})'_+(\bar{x}) \\
&\leq (u_{H_{1-p}})'_+(\bar{y}) = w'(1 - H_{1-p}(\bar{y})) v'(\bar{y}) = w'(p) v'(\bar{y}),
\end{aligned}$$

proving that $v'(\bar{x}) \leq v'(\bar{y})$. If $\bar{x} > \bar{y}$ and $\bar{x} \geq 0 \geq \bar{y}$, then $v'(\bar{x}) \leq v'(0) \leq v'(\bar{y})$. We can conclude that v' is decreasing and so v is concave. In particular, since v is strictly increasing, this implies that $v' > 0$ on (m, M).

2. Consider $p, q \in (0,1)$ such that $p > q$. Consider $\bar{x}, \bar{y} \in (0, M)$ such that $\bar{x} > \bar{y}$. Define $\hat{G} \in \mathcal{D}$ to be such that $\hat{G} = qG_{\bar{y}} + (p-q) G_{\bar{x}} + (1-p) G_M$. It follows that there exists a sequence $\{F_n\} \subseteq \mathcal{D} \cap C_b((-\infty, \infty))$ such that $F_n(x) \to \hat{G}(x)$ for all $x \in [m, M]$. We can conclude that for each $n \in \mathbb{N}$

$$w'(1 - F_n(\bar{x})) v'(\bar{x}) = u'_{F_n}(\bar{x}) \leq u'_{F_n}(\bar{y}) = w'(1 - F_n(\bar{y})) v'(\bar{y}).$$

By passing to the limit, we have that

$$w'(1-p) v'(\bar{x}) = w'\left(1 - \hat{G}(\bar{x})\right) v'(\bar{x}) \leq w'\left(1 - \hat{G}(\bar{y})\right) v'(\bar{y}) = w'(1-q) v'(\bar{y}).$$

Since $\bar{x}$ and $\bar{y}$ were arbitrarily chosen and v' is continuous, we have that $w'(1-p) v'(\bar{x}) \leq w'(1-q) v'(\bar{x})$. Since $v'(\bar{x}) > 0$ and p and q were arbitrarily chosen in $(0,1)$, we can conclude that

$$p > q \implies 1 - q > 1 - p \implies w'(1 - (1-q)) v'(\bar{x}) \leq w'(1 - (1-p)) v'(\bar{x}) \implies w'(q) \leq w'(p).$$

Thus, we have that w' is increasing and so w is convex.

3. Consider $p, q \in (0,1)$ such that $p > q$. Consider $\bar{x}, \bar{y} \in (m, 0)$ such that $\bar{x} > \bar{y}$. Define $\hat{G} \in \mathcal{D}$ to be such that $\hat{G} = qG_{\bar{y}} + (p-q) G_{\bar{x}} + (1-p) G_0$. It follows that there exists a sequence $\{F_n\} \subseteq \mathcal{D} \cap C_b((-\infty, \infty))$ such that $F_n(x) \to \hat{G}(x)$ for all $x \in [m, M]$. We can conclude that for each $n \in \mathbb{N}$

$$\tilde{w}'(F_n(\bar{x})) v'(\bar{x}) = u'_{F_n}(\bar{x}) \leq u'_{F_n}(\bar{y}) = \tilde{w}'(F_n(\bar{y})) v'(\bar{y}).$$

By passing to the limit, we have that

$$\tilde{w}'(p) v'(\bar{x}) = \tilde{w}'\left(\hat{G}(\bar{x})\right) v'(\bar{x}) \leq \tilde{w}'\left(\hat{G}(\bar{y})\right) v'(\bar{y}) = w'(q) v'(\bar{y}).$$

Since $\bar{x}$ and $\bar{y}$ were arbitrarily chosen and v' is continuous, we have that $\tilde{w}'(p) v'(\bar{x}) \leq \tilde{w}'(q) v'(\bar{x})$. Since $v'(\bar{x}) > 0$ and p and q were arbitrarily chosen in $(0,1)$, we can conclude that $p > q$ implies $\tilde{w}'(p) \leq \tilde{w}'(q)$. Thus, we have that $\tilde{w}'$ is decreasing and so $\tilde{w}$ is concave. In particular, since $\tilde{w}$ is strictly increasing, this implies that $\tilde{w}' > 0$ on $(0,1)$.

4. Fix $p \in (0,1)$. Consider $F_p = pG_0 + (1-p) G_M$. It follows that there exists a sequence $\{F_n\} \subseteq \mathcal{D} \cap C_b((-\infty, \infty))$ such that $F_n(x) \to F_p(x)$ for all $x \in [m, M]$. We can conclude that

$$\tilde{w}'(F_n(0)) v'(0) = (u_{F_n})'_-(0) \geq (u_{F_n})'_+(0) = w'(1 - F_n(0)) v'(0).$$

By passing to the limit and since $v'(0) > 0$, we can conclude that $\tilde{w}'(p) \geq w'(1-p)$. Since $\tilde{w}' > 0$ on $(0,1)$ and p was arbitrarily chosen, we have that $\sup_{p \in (0,1)} \frac{w'(1-p)}{\tilde{w}'(p)} \leq 1$.

(ii) implies (i). Fix $F \in \mathcal{D}$. It is routine to show that (ii) implies that the function

$$f = \left(w'(1-F) 1_{[0,M]} + \tilde{w}'(F) 1_{[m,0)}\right) v'$$

is decreasing. By Proposition 4 and [34, Theorem 24.2] and since $u_F(x) = \int_{[m,x]} f(y)\, dy$ for all $x \in [m, M]$, if follows that u_F is concave. By Proposition 2 and since F was arbitrarily chosen, it follows that $\succsim$ is MPS averse. ■

Proof of Corollary 2. Before starting recall that F dominates G with respect to third order stochastic dominance if and only if for each $u \in C([m, M])$ which is increasing, concave, and with convex derivative in (m, M)[17]

$$\int_{[m,M]} u(x)\, dF(x) \geq \int_{[m,M]} u(x)\, dG(x).$$

(i) implies (ii). Note that if $\succsim$ is consistent with third order stochastic dominance, then it is consistent with second order stochastic dominance. Thus, it is MPS averse. It follows that we can apply all the facts

[17]See Border [4], Ziegler [42, pp. 562–563], and Karlin and Studden [27, Chapter 11, Theorem 2.1]. Moreover, by [34, Theorem 10.8 and Theorem 24.5], we can obtain that the set of all $u \in C([m, M])$ that are increasing, concave, and with convex derivative on (m, M) is a closed convex cone containing the constant functions.

contained in the proof of Corollary 3. By Proposition 1, we also have that each local utility u_F is increasing, concave, and with convex derivative in (m, M). Fix $p \in (0,1)$. Consider $F_p = pG_0 + (1-p) G_M$. It follows that there exists a sequence $\{F_n\} \subseteq \mathcal{D} \cap C_b((-\infty,\infty))$ such that $F_n(x) \to F_p(x)$ for all $x \in [m, M]$. We can conclude that

$$\tilde{w}'(F_n(0)) v'(0) = (u_{F_n})'_-(0) = (u_{F_n})'_+(0) = w'(1 - F_n(0)) v'(0).$$

By passing to the limit and since $v'(0) > 0$, we can conclude that $\tilde{w}'(p) = w'(1-p)$. Since p was arbitrarily chosen and $\tilde{w}'$ and w' are continuous, it follows that $\tilde{w}(p) = 1 - w(1-p)$ for all $p \in [0,1]$. Consider G_0. Since each u_F is differentiable in (m, M), we have that

$$u'_{G_0}(x) = (u_{G_0})'_+(x) = w'(1 - G_0(x)) v'(x) \qquad \forall x \in (m, M).$$

Since each u'_F is convex on (m, M), it follows that u'_F is continuous on (m, M). Consider $\{x_n\}_{n\in\mathbb{N}} \subseteq (m, 0)$ and $\{y_n\}_{n\in\mathbb{N}} \subseteq (0, M)$ such that $\lim_n x_n = \lim_n y_n = 0$. Thus, we can conclude that

$$\begin{aligned} w'(1) v'(0) &= \lim_n w'(1) v'(x_n) = \lim_n w'(1 - G_0(x_n)) v'(x_n) \\ &= \lim_n u'_{G_0}(x_n) = u'_{G_0}(0) = \lim_n u'_{G_0}(y_n) \\ &= \lim_n w'(1 - G_0(y_n)) v'(y_n) = \lim_n w'(0) v'(y_n) = w'(0) v'(0). \end{aligned}$$

Since $v'(0) > 0$ and w' is increasing, it follows that w' is constant. This implies that $w(p) = p$ for all $p \in [0,1]$. We can conclude that $\tilde{w} = w$ and that $\succsim$ is an Expected Utility preference. It trivially follows that v, other than being increasing, is also concave and with convex derivative on (m, M).

(ii) implies (i). It is a well known fact.

References

[1] C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis*, 3rd ed., Springer, New York, 2006.

[2] F. D. Arditti, Risk and the required return on equity, *Journal of Finance*, 22, 19–36, 1967.

[3] P. Billingsley, *Probability and measure*, 3rd ed., Wiley, New York, 1995.

[4] K. C. Border, Functional analytic tools for expected utility theory, in C. D. Aliprantis, K. C. Border, and W. A. J. Luxemburg (Eds.), *Positive operators, Riesz spaces, and Economics*, Springer, New York, 1991.

[5] S. Cerreia–Vioglio, Maxmin expected utility on a subjective state space: convex preferences under risk, mimeo, 2009.

[6] S. Cerreia–Vioglio, D. Dillenberger, and P. Ortoleva, Cautious expected utility and the certainty effect, mimeo, 2013.

[7] S. Cerreia–Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio, Uncertainty averse preferences, *Journal of Economic Theory*, 146, 1275–1330, 2011.

[8] S. Cerreia–Vioglio, P. Ghirardato, F. Maccheroni, M. Marinacci, and M. Siniscalchi, Rational preferences under ambiguity, *Economic Theory*, 48, 341–375, 2011.

[9] S. Cerreia–Vioglio, F. Maccheroni, M. Marinacci, L. Montrucchio, Affine Gateaux differentials and the von Mises statistical calculus, mimeo, 2014.

[10] K. Chatterjee and R. V. Krishna, A nonsmooth approach to nonexpected utility theory under risk, *Mathematical Social Sciences*, 62, 166–175, 2011.

[11] S. H. Chew, A generalization of the quasilinear mean with applications to the measurement of income inequality and decision theory resolving the Allais paradox, *Econometrica*, 51, 1065–1092, 1983.

[12] S. H. Chew, Axiomatic utility theories with the betweenness property, *Annals of Operations Research*, 19, 273–298, 1989.

[13] S. H. Chew, L. Epstein, and U. Segal, Mixture symmetry and quadratic utility, *Econometrica*, 59, 139–163, 1991.

[14] S. H. Chew, E. Karni, and Z. Safra, Risk aversion in the theory of expected utility with rank dependent probabilities, *Journal of Economic Theory*, 42, 370–381, 1987.

[15] S. H. Chew and N. Nishimura, Differentiability, comparative statics, and non-expected utility preferences, *Journal of Economic Theory*, 56, 294–312, 1992.

[16] G. Debreu, Continuity properties of Paretian utility, *International Economic Review*, 5, 285–293, 1964.

[17] E. Dekel, An axiomatic characterization of preferences under uncertainty: weakening the independence axiom, *Journal of Economic Theory*, 40, 304–318, 1986.

[18] D. Dillenberger, Preferences for one-shot resolution of uncertainty and Allais-type behavior, *Econometrica*, 78, 1973–2004, 2010.

[19] J. Dubra, F. Maccheroni, and E. A. Ok, Expected utility theory without the completeness axiom, *Journal of Economic Theory*, 115, 118–133, 2004.

[20] O. Evren, On the existence of expected multi-utility representations, *Economic Theory*, 35, 575–592, 2008.

[21] L. T. Fernholz, *Von Mises calculus for statistical functionals*, Springer, New York, 1983.

[22] P. Ghirardato, F. Maccheroni, and M. Marinacci, Differentiating ambiguity and ambiguity attitude, *Journal of Economic Theory*, 118, 133–173, 2004.

[23] P. Ghirardato and M. Siniscalchi, Ambiguity in the small and in the large, *Econometrica*, 80, 2827–2847, 2012.

[24] F. R. Hampel, The influence curve and its role in robust estimation, *Journal of the American Statistical Association*, 69, 383–393, 1974.

[25] P. J. Huber, *Robust statistics*, Wiley, New York, 1981.

[26] D. Kahneman and A. Tversky, Prospect theory: an analysis of decision under risk, *Econometrica*, 47, 263–291, 1979.

[27] S. J. Karlin and W. J. Studden, Tchebycheff systems: with applications in analysis and statistics, Wiley, New York, 1966.

[28] A. Kraus and R. H. Litzenberger, Skewness preference and the valuation of risk assets, *Journal of Finance*, 31, 1085–1100, 1976.

[29] F. Maccheroni, Maxmin under risk, *Economic Theory*, 19, 823–831, 2002.

[30] M. J. Machina, "Expected utility" analysis without the independence axiom, *Econometrica*, 50, 277–323, 1982.

[31] R. von Mises, On the asymptotic distribution of differentiable statistical functions, *Annals of Mathematical Statistics,* 18, 309–348, 1947.

[32] K. Nehring, Imprecise probabilistic beliefs as a context for decision-making under ambiguity, *Journal of Economic Theory*, 144, 1054–1091, 2009.

[33] J. Quiggin, A theory of anticipated utility, *Journal of Economic Behavior and Organization,* 3, 323–343, 1982.

[34] R. T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton, 1970.

[35] M. Rothschild and J. E. Stiglitz, Increasing risk: I. A definition, *Journal of Economic Theory*, 2, 225–243, 1970.

[36] U. Schmidt and H. Zank, Risk aversion in cumulative prospect theory, *Management Science*, 54, 208–216, 2008.

[37] S. C. Tsiang, The rationale of the mean-standard deviation analysis, skewness preference, and the demand for money, *American Economic Review*, 62, 354–371, 1972.

[38] A. Tversky and D. Kahneman, Advances in prospect theory: cumulative representation of uncertainty, *Journal of Risk and uncertainty,* 5, 297–323, 1992.

[39] P. Wakker, *Prospect theory: for risk and ambiguity*, Cambridge University Press, Cambridge, 2010.

[40] T. Wang, L_p–Frechet differentiable preference and "local utility" analysis, *Journal of Economic Theory*, 61, 139–159, 1993.

[41] G. A. Whitmore, Third-degree stochastic dominance, *American Economic Review*, 60, 457–459, 1970.

[42] Z. Ziegler, Generalized convexity cones, *Pacific Journal of Mathematics*, 17, 561–580, 1966.