Common Belief Foundations of Global Games*

Stephen Morris
Princeton University

Hyun Song Shin
Bank of International Settlements

Muhamet Yildiz
M.I.T.

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Abstract

Complete information games—i.e., games where there is common certainty of payoffs—often have multiple rationalizable actions. What happens if the common certainty of payoffs assumption is relaxed? Consider a two player game where a player's payoff depends on a payoff parameter. Define a player's "rank belief" as the probability he assigns to his payoff parameter being higher than his opponent's. We show that there is a unique rationalizable action played when there is *common certainty of rank beliefs*, and we argue this is the driving force behind selection results in the global games literature. We consider what happens if players' payoff parameters are derived from common and idiosyncratic terms with fat tailed distributions, and we approach complete information by shrinking the distribution of both terms. There are two cases. If the common component has a thicker tails, then there is common certainty of rank beliefs, and thus uniqueness and equilibrium selection. If the idiosyncratic terms has thicker tails, then there is *approximate common certainty of payoffs*, and thus multiple rationalizable actions.

1 Introduction

Complete information games often have many equilibria. Even when they have a single equilibrium, they often have many actions that are rationalizable, and are therefore consistent with common

*This paper incorporates material from a working paper of the same title circulated in 2009 (Morris and Shin (2009)). We are grateful for comments from seminar participants at Northwestern on this iteration of the project.

certainty of rationality. The inability of theory to make a prediction is problematic for economic applications of game theory.

Carlsson and van Damme (1993) suggested a natural perturbation of complete information that gives rise to a unique rationalizable equilibrium for each player. They introduced the idea of "global games"—where any payoffs of the game are possible and each player observes the true payoffs of the game with a small amount of noise. They showed—for the case of two player two action games—that as the noise about payoffs become small, there is a unique equilibrium; moreover, the perturbation selects a particular equilibrium (the *risk dominant* one) of the underlying game. This result has since been generalized in a number of directions and widely used in applications.[1] When the global game approach can be applied to more general games, it can be used to derive unique predictions in settings where the underlying complete information game has multiple equilibria, making it possible to carry out comparative static and policy analysis. An informal argument that has been made for this approach is that the "complete information"—or common certainty of payoffs—assumption is an unrealistic one that makes it easier to support multiplicity, so the natural perturbation underlying global games captures the more realistic case.

However, the global game selection result uses a particular form of perturbation away from "complete information." Complete information entails the assumption that a player is certain of the payoffs of the game, certain that other players are certain, and so on. Weinstein and Yildiz (2007) consider more general perturbations, saying that a situation is close to a complete information game if players are almost certain that payoffs are close to those complete information game payoffs, almost certain that other players are almost certain that payoffs are close to those payoffs, and so on. Formally, they consider closeness in the product topology on the universal belief space. They show that for *any* action which is rationalizable for a player in a complete information game, there exists a nearby type of that player in the product topology for whom there is a unique rationalizable action. Thus by considering a richer but also intuitive class of perturbations, they replicate the global game uniqueness result but reverse the selection result.

In this paper, we identify what is the driving force behind global game uniqueness and selection results. In particular, we do not want to take literally the (implicit) assumption in global games that there is common certainty among the players of a technology which generates (conditionally

[1] Morris and Shin (1998) analyzed a global game with a continuum of players making binary choices, and this case has been studied in a number of later applications. See Morris and Shin (2003b) for an early survey of some theory and applications of global games. Frankel, Morris, and Pauzner (2003) study global game selection in general games with strategic complementarities.

independent) noisy signals observed by the players. Rather, we want to argue that global game perturbations are a metaphor, or a convenient modelling device, for a more general intuitive class of relaxations of common certainty which must be stronger than the product topology perturbations of Weinstein and Yildiz (2007), and we want to characterize and analyze the key property of that more general class.

Our analysis is carried out for in a two player, binary action game. Each player must decide whether to "invest" or "not invest". Payoffs are given by the following matrix:

	invest	not invest
invest	x_1, x_2	$x_1 - 1, 0$
not invest	$0, x_2 - 1$	$0, 0$

(1)

Each player i knows his own payoff parameter, or return to investment, x_i but may not know the other player's payoff parameter. There are strategic complementarities, because a player has a loss of 1 if the other player does not invest. If x_1 and x_2 are both in the interval $[0, 1]$, then there are multiple equilibria, both invest and both not invest, under complete information. In the symmetric case, with $x_1 = x_2 = x$, the *risk dominant* equilibrium in this game is the equilibrium that has each player choose a best response to a 50/50 distribution over his opponent's action. Thus both invest is the risk dominant equilibrium if $x > \frac{1}{2}$.

If x_1 and x_2 are both in the interval $[0, 1]$, both actions remain rationalizable for player i if there is approximate common certainty of payoffs. There is approximate common certainty of an event if everyone beliefs it with probability close to 1, everyone believes with probability 1 that everyone believes it with probability close to 1, and so on. This is a well known sufficient condition for multiple rationalizable actions, going back to Monderer and Samet (1989). But it is a strong condition. A key concept to understand uniqueness in this situation is a player's "rank belief" - that is, his belief about whether he has the higher payoff parameter (so he has rank 1) or the lower payoff parameter (so he has rank 2). A player has the *uniform rank belief* if he assigns probability $\frac{1}{2}$ to having rank 1. If there is *common certainty of uniform rank beliefs*, then an player has a unique rationalizable action. In particular, action invest is uniquely rationalizable for a player only if it is risk dominant. To see why this is true, let x^{**} be the smallest payoff parameter such that invest is uniquely rationalizable whenever there is common certainty of uniform rank beliefs for a player with payoff parameter greater than or equal to x^{**}. If x^{**} were strictly greater than $\frac{1}{2}$, a player with payoff parameter close to x^{**} (and common certainty of uniform rank beliefs) would assign probability close to $\frac{1}{2}$ to his opponent investing and would therefore have a strict incentive to invest. Thus we would have a contradiction. Thus x^{**} must equal $\frac{1}{2}$. This is the standard global game argument. But because it is expressed for general situations - and, in particular, without reference to noisy signals or one dimensional type spaces - it provides a primitive common belief foundation for global game selection. One of our main results will be a formalization of an appropriate weakening of this sufficient condition (*approximate* common certainty of *approximately* uniform rank beliefs).

A second contribution of this paper is to identify new and insightful conditions under which either approximate common certainty of payoffs and or approximate common certainty of approximately uniform rank beliefs will emerge. Like the classical global games literature, we will consider the case where players have one dimensional types that reflect both common and idiosyncratic components, and consider sequences approaching complete information. Unlike the classical global games literature, we consider the case where the common and idiosyncratic components have distributions with "fat tails," i.e., with greater than exponential densities. Fat tails generate intuitive and insightful higher-order beliefs as complete information is approached. In particular, close to the complete information limit, if a player observes a payoff parameter which is surprisingly different from his prior expectation, does he attribute the difference to the common component or the idiosyncratic component? Under fat tails, the most likely explanation is that either one or the other but not both had an extreme realization. If the idiosyncratic component has a fatter tail, then a player attributes the difference to the idiosyncratic component. He thinks that the common component was close to its mean, and thus expects that his opponent's payoff type is closer to that mean. Thus as we approach the complete information limit, there is approximate common certainty of payoffs. In that case, there will be multiple rationalizable actions whenever the there is multiple equilibria under complete information. But if the common component has a fatter tail, then he thinks that an unusual realization of the common component accounts for the deviation, and thus he expects his opponent's payoff type to be close to his. He finds it equally likely that either player has the higher signal. Thus as we approach the complete information limit, there is common certainty of uniform rank beliefs. This leads to unique selection of risk-dominant equilibrium as in the global games literature. Thus global game uniqueness and selection is tied to whether the players have a strong private information in the sense that the idisyncratic component has a thinner tail.

We can contrast these results to existing results in the global games literature. The leading exercise in the global games literature is to consider a fixed prior about payoffs and have players observe conditionally independent noise about payoffs (Carlsson and van Damme (1993), Morris and Shin (2003b) and Frankel, Morris, and Pauzner (2003)). As the noise shrinks to zero, there is a unique equilibrium. The arguments for these results rely on uniform convergence of conditional probability properties which, in the case of symmetric settings, reduces to (variants of) common certainty of rank beliefs. Another leading exercise is to examine what happens if there is normally distributed common payoff parameter, and players observe the prior mean (a public signal) and also a conditionally independent normal private signal (Morris and Shin (2001) and Morris and Shin (2003a)), and allow the variances of both the prior and the noisy private signals to converge to zero. In this case, there is a unique equilibrium if and only if the noise in the private signal shrinks sufficiently fast relative to the public signal. Intuitively, relatively fast convergence of private noise gives common certainty of rank beliefs in the limit while relatively slow convergence gives approximate common certainty of payoffs.[2] The critical condition turns out to be that the variance of the private signal must shrink faster than the square of the variance of the public signal. However, this rate condition is special to the normal distribution and our results highlight quite different properties that determine limit behavior in general. We find the fat tail result more insightful and robust than the normal distribution insights which two of us have highlighted in earlier work. This paper is the first to examine what happens to higher-order beliefs or global game results without normality when both the prior and private signals become concentrated.

These results shed light on when global game models can offer useful insights in applications. The one dimensional normal model suggested the simple message that there will be unique rationalizable outcomes if there is enough private information (measured by its precision) relative to the amount of public information. This suggests that we can assess the relevance of the uniqueness prediction by focussing on whether private signals have less variance than public signals (see, e.g., Svennson (2006)); and it suggests that uniqueness results will be less relevant in settings where information is being endogenously and publicly generated (see, e.g., Atkeson (2001) and Angeletos and Werning (2006)). Going beyond the one dimensional normal model (where precision is the unique parameter characterizing the shape of the distribution) suggests more nuanced conclusions. Our fat tail results highlight the importance of the shape of the tail of the distribution for determining the relevant properties of higher-order beliefs. And, more generally, the common belief foundations results focus attention on the properties of higher-order beliefs that matter for global game results rather than the conditionally independent noisy signal story that generates them.

[2] As we will show in the body of the paper, common certainty of rank beliefs in the limit never arises in the normal case, but a weaker condition - common certainty of rank beliefs conditional on undominated actions - is satisfied and sufficient for limit uniqueness.

We present our results in the context of the simple, two player, strategic setting described above, maintaining the common prior assumption. We do this in order to focus on the common belief foundations rather than the details of the strategic setting. But we can then describe how our results can be mapped back to more general settings considered in the global games literature.

Under the common prior assumption, if there is common certainty that players' beliefs about their rank is $r \in [0,1]$, we must have $r = \frac{1}{2}$, i.e., what we refer to as uniform rank beliefs. However, if the common prior assumption is dropped, common certainty of rank beliefs continues to lead to uniqueness and selection, but the selected action now depends on the rank belief r. This observation provides global game foundations for the results of Izmalkov and Yildiz (2010).

We state our results for two player two action games, but they extend straightforwardly to many player two action games. Morris and Shin (2003b) highlight that in many player symmetric games, the selected equilibrium is the *Laplacian* one: each player is choosing a best response to a uniform conjecture over the number of his opponents choosing each action. This selection is key in the vast majority of applied papers. Common certainty of rank beliefs in the N player case corresponds to assigning probability $\frac{1}{N}$ to exactly n players having higher payoff states, for each $n = 0, 1, ..., N-1$, and is a sufficient condition for the global game selection of the Laplacian action.

Most of the global games literature focusses on a "common values" case, where there is a common payoff parameter of interest and each player observes it with idiosyncratic private noise. Our analysis concerns a "private value" case, where the idiosyncratic shock is payoff relevant to the players. Results in the private value case are close to those in the common value case (see Morris and Shin (2005) for some theory and Argenziano (2008) for an application of private value global games). Our analysis covers the private values case. Analogous results could be obtained for the common value case, but at the expense of significantly more complicated statements to account for the small amount of payoff uncertainty. Another convenient property of the game we consider is the additive separability between a component of payoffs that depends on others' actions and a component about which there is uncertainty. This gives a sharp separation between the role of strategic uncertainty and payoff uncertain. Again, this provides sharper results although qualitative similar results would go through without this assumption.

However, the global game selection results reported in the paper rely on symmetric payoffs; in particular, it is required for common certainty of approximately uniform rank beliefs to be the relevant sufficient condition for unique rationalizable actions. Higher-order belief foundations can be provided for asymmetric global game results, but they are qualitatively different from those provided for symmetric games here.

We present a general characterization of rationalizable behavior in section 2, reporting sufficient conditions, in general type spaces, for multiple rationalizable outcomes based on approximate common certainty of payoffs and for unique rationalizable outcomes based on approximate common certainty of approximately uniform rank beliefs. In section 3, we introduce and analyze the one dimensional model generated by common and idiosyncratic payoff shocks with fat tails. In section 4, we discuss extensions to more general strategic settings and the relation to the global games literature.

2 General Type Spaces

2.1 Higher Order Beliefs

There are two players, 1 and 2. Let T_1 and T_2 be the sets of types for players 1 and 2, respectively. A mapping $x_i : T_i \to \mathbb{R}$ describes a payoff parameter of interest to player i and a mapping $\pi_i : T_i \to \Delta(T_j)$ describes player i's beliefs about the other player. We assume that T_i is endowed with a topology and Borel sigma-algebra. We make a couple of minimal continuity assumptions: x_i and π_i are continuous, and the pre-image $x_i^{-1}([a,b])$ of every compact interval $[a,b]$ is sequentially compact. This type space can be arbitrarily rich, and in particular can encode any beliefs and higher-order beliefs, and thus our results apply if the type space is the universal (private value) belief space of Mertens and Zamir (1985).[3]

We start with describing the belief and common belief operators, as in Monderer and Samet (1989). The state space is $T = T_1 \times T_2$. An *event* is a subset of T. An event is *simple* if $E = E_1 \times E_2$ where $E_i \subseteq T_i$. For our game theoretic analysis, we will be interested in events that are simple and compact and we restrict attention to such events in the analysis that follows. For any such simple and compact event E, we write E_1 and E_2 for the compact projections of E onto T_1 and T_2, respectively. Now, for probability p_i, write $B_i^{p_i}(E)$ for the set of states where player i believes E

[3]Mertens and Zamir (1985) constructed a space that encodes all beliefs and higher order beliefs about a common state space. We maintain the assumption that the state space is a pair of payoff types of the players, where each player knows his own payoff type. It is a simple adaption to the classic construction to build in this restriction, see for example Heifetz and Neeman (2006). The classical construction assumes a compact state space. We need to allow the state space to be $\mathbb{R}^2$ but instead impose sequential compactness.

with probability at least p_i:

$$B_i^{p_i}(E) = \{(t_1, t_2) | t_i \in E_i \text{ and } \pi_i(E_j|t_i) \geq p_i\}.$$

For a pair of probabilities (p_1, p_2), say that event E is (p_1, p_2)-*believed* if each player i believes event E with probability at least p_i. Writing $B_*^{p_1,p_2}(E)$ for the set of states where E is (p_1, p_2)-believed, we have:

$$B_*^{p_1,p_2}(E) = B_1^{p_1}(E) \cap B_2^{p_2}(E).$$

Say that there is *common* (p_1, p_2)-*belief* of event E if it is (p_1, p_2)-believed, it is (p_1, p_2)-believed that it is (p_1, p_2)-believed, and so on. We write $C^{p_1,p_2}(E)$ for set of states at which E is common (p_1, p_2)-belief. Thus

$$C^{p_1,p_2}(E) = \bigcap_{n\geq 1} [B_*^{p_1,p_2}]^n(E)$$

An event is (p_1, p_2)-evident if it is (p_1, p_2)-believed whenever it is true. Generalizing a characterization of common knowledge by Aumann (1976), Monderer and Samet (1989) provides the following useful characterization.

Lemma 1 (Monderer and Samet (1989)) *Event E is common (p_1, p_2)-belief if and only if there exists a (p_1, p_2)-evident event F with $F \subseteq B_*^{p_1,p_2}(F)$.*

In our formulation, we make the belief operators type dependent, and the above properties generalize immediately to this case. For any $f_i : T_i \to \mathbb{R}$, we say that event E is f_i-*believed by type* t_i *of player* i if he believes it with probability at least $f_i(t_i)$:

$$B_i^{f_i}(E) = \{(t_1, t_2) | t_i \in E_i \text{ and } \pi_i(E_j|t_i) \geq f_i(t_i)\}.$$

Clearly we can make richer statements about beliefs and higher-order beliefs in this language. We will continue to write $B_i^{p_i}(E)$ for the original p_i-belief operator, where p_i is now understood as the constant function of types taking the value p_i. Note that we allow f_i to take values below 0 and above 1. This convention gives a special role to the events $B_i^{f_i}(\varnothing)$ and $B_i^{f_i}(T)$, since a player always believes an event with probability at least 0 and never believes an event with probability greater than 1. Thus

$$\begin{aligned} B_i^{f_i}(\varnothing) &= \{(t_1, t_2) | f_i(t_i) \leq 0\} \\ B_i^{f_i}(T) &= \{(t_1, t_2) | f_i(t_i) \leq 1\}. \end{aligned}$$

These operators behave just like the type-independent ones. In particular, writing $B_*^{f_1,f_2}(E)$ for the set of states where E is (f_1, f_2)-believed, we have:

$$B_*^{f_1,f_2}(E) = B_1^{f_1}(E) \cap B_2^{f_2}(E).$$

Say that there is *common* (f_1, f_2)*-belief* of event E if it is (f_1, f_2)-believed, it is (f_1, f_2)-believed that it is (f_1, f_2)-believed, and so on. We write $C^{f_1,f_2}(E)$ for set of states E is common (f_1, f_2)-belief. An event is (f_1, f_2)*-evident* if it is (f_1, f_2)-believed whenever it is true:

$$C^{f_1,f_2}(E) = \bigcap_{n\geq 1} \left[B_*^{f_1,f_2}\right]^n (E)$$

The Monderer and Samet (1989) characterization immediately generalizes to our case.

Lemma 2 *Event E is common (f_1, f_2)-belief if and only if there exists a (f_1, f_2)-evident event F with $F \subseteq B_*^{f_1,f_2}(F)$.*

2.2 Common-Belief Characterization of Rationalizability

In the baseline model, we consider the following action space and the payoff function:

	invest	not invest
invest	x_1, x_2	$x_1 - 1, 0$
not invest	$0, x_2 - 1$	$0, 0$

Each x_i is assumed to be a continuous function of type t_i. Player i knows his own payoff parameter $x_i(t_i)$ but does not necessarily know the other player's payoff parameter $x_j(t_j)$. Moreover, he gets a return $x_i(t_i)$ if he invests but faces a penalty 1 if the other player does not invest. Hence, he only wants to invest if the probability he assigns to his opponent investing is at least $1 - x_i(t_i)$. We now define rationalizability in the context of this game (it corresponds to standard general definitions). Say that an action is $(k+1)$*th level rationalizable* if it is a best response to kth level rationalizable play of his opponent; and say that any action is 0th level rationalizable. Write R_i^k for the set of types of player i for whom action invest is kth-level rationalizable and let $R_i^0 = T_i$.

We start with carefully describing the set R_i^1 of types for whom invest is 1st-level rationalizable in terms of our type-dependent belief operators. On the one hand, for any type t_i of player i, action invest is 1st-level rationalizable for t_i if and only if $x_i(t_i) \geq 0$; this is in response to the belief that his opponent invests with probability 1. On the other hand, since player i always assigns probability 1 to T, he assigns probability at least $1 - x_i(t_i)$ to T if and only if $x_i(t_i) \geq 0$. Thus,

$$R_i^1 = B_i^{1-x_i}(T).$$

That is, the types for which invest is first-level rationalizable coincide with those in $B_i^{1-x_i}(T)$.

Now, action invest is 2nd-level rationalizable for type t_i of player i if, in addition, he assigns probability at least $1 - x_i(t_i)$ to $x_j(t_j) \geq 0$; thus

$$R_i^2 = B_i^{1-x_i}\left(B_*^{1-x_1,1-x_2}(T)\right).$$

More generally, action invest is $(k+1)$th-level rationalizable for a type t_i if he $(1 - x_i(t_i))$-believes that T is kth-order $(1 - x_1, 1 - x_2)$-believed:

$$R_i^{k+1} = B_i^{1-x_i}\left(\left[B_*^{1-x_1,1-x_2}\right]^k(T)\right).$$

Action invest is rationalizable if it is kth level rationalizable for all k. Thus, action invest is rationalizable for both players exactly if T is common $(1 - x_1, 1 - x_2)$-believed:

$$R^\infty = C^{1-x_1,1-x_2}(T). \tag{2}$$

By a symmetric argument, action Not invest is rationalizable if T is common (x_1, x_2)-belief. The next result states this characterization.

Proposition 1 *Action invest is rationalizable for type t_i if and only if*

$$t_i \in B_i^{1-x_i}\left(C^{1-x_1,1-x_2}(T)\right);$$

action Not invest is rationalizable for type t_i if and only if

$$t_i \in B_i^{x_i}\left(C^{x_1,x_2}(T)\right).$$

It is useful to note that, since $R_i^1 = B_i^{1-x_i}(T)$, $C^{1-x_1,1-x_2}(T)$ corresponds to a high common belief in the event that action invest is rational. Hence, each part of the proposition states that *an action a_i is rationalizable for a type t_i if and only if t_i assigns sufficiently high probability on a sufficiently high common belief in the event that a_i is rational.* That is, he finds it sufficiently likely that the action is rational, finds it sufficiently likely that the other player finds it sufficiently likely that the action is rational,..., ad infinitum. The key innovation that yields such a simple characterization is taking the threshold for the sufficiency type dependent, depending on the payoff functions of the types throughout.

We report a many-player extension of this characterization in section 4.3.

2.3 Risk-Dominant Selection and Multiplicity

Our focus in this paper is on when both actions are rationalizable for both players and when one action is uniquely rationalizable for both players—without loss of generality, we focus on uniqueness of action invest. Building on the characterization in the previous section, we present intuitive sufficient conditions for each case. Our first result characterizes the cases with multiplicity and uniqueness, as an immediate corollary to the characterization in the previous section.

Corollary 1 *Both actions are rationalizable for both players if and only if*

$$(t_1, t_2) \in C^{x_1,x_2}(T) \cap C^{1-x_1,1-x_2}(T).$$

Invest is the uniquely rationalizable action for a type t_i if and only if

$$t_i \in B_i^{1-x_i}\left(C^{1-x_1,1-x_2}(T)\right) \backslash B_i^{x_i}\left(C^{x_1,x_2}(T)\right).$$

The first part characterizes the cases with multiple rationalizable solutions. It states that both actions are rationalizable if and only if there is sufficiently high common belief that both actions are rational for both players. The second part characterizes the cases in which invest is the only rationalizable solution. It states that invest is uniquely rationalizable if and only if there is sufficiently high common belief in rationality of invest (i.e., $t_i \in B_i^{1-x_i}\left(C^{1-x_1,1-x_2}(T)\right)$) but there is not sufficiently high common belief in rationality of Not invest (i.e., $t_i \notin B_i^{x_i}\left(C^{x_1,x_2}(T)\right)$). Once again, we obtain such simple and straightforward characterizations by taking the threshold for sufficiency type-dependent.

While such characterizations are useful conceptually, they may not be practical. In the rest of this section, we provide simple tractable sufficient conditions for multiplicity and uniqueness. We start with a result for multiplicity:

Proposition 2 *Both actions are rationalizable whenever there is approximate common certainty that payoffs support multiple strict equilibria, i.e., for some $\varepsilon \in [0, 1/2]$,*

$$(t_1, t_2) \in C^{1-\varepsilon,1-\varepsilon}(M_\varepsilon)$$

where

$$M_\varepsilon = \left\{(t_1', t_2') \,|\, \varepsilon \leq x_i(t_i') \leq 1-\varepsilon \text{ for both } i\right\}.$$

Proof. By definition, both actions are 0th-level rationalizable throughout $C^{1-\varepsilon,1-\varepsilon}(M_\varepsilon)$. Towards an induction, assume that both actions are kth-level rationalizable throughout $C^{1-\varepsilon,1-\varepsilon}(M_\varepsilon)$. Now, for any type $t_i \in B_i^{1-\varepsilon}(C^{1-\varepsilon,1-\varepsilon}(M_\varepsilon))$ and any action a, consider a belief of type t_i that puts probability 1 on a kth-level rationalizable strategy of the other player j such that each $t_j \in B_j^{1-\varepsilon}(C^{1-\varepsilon,1-\varepsilon}(M_\varepsilon))$ plays the action a. Observe that $\varepsilon \leq x_i(t_i) \leq 1-\varepsilon$ and that t_i assigns at least probability $1-\varepsilon$ on the other player playing action a. Hence, a is a best response to that belief for t_i, showing that a is $(k+1)$th-level rationalizable for t_i. Therefore, both actions are $(k+1)$th-level rationalizable for both players throughout $C^{1-\varepsilon,1-\varepsilon}(M_\varepsilon)$. ■

Proposition 2 states that both actions are rationalizable whenever there is common $(1-\varepsilon, 1-\varepsilon)$-belief that both actions would have been ε-dominant Nash equilibrium under complete information. This follows a key observation in the robustness literature, going back to Monderer and Samet (1989) which states that any p-dominant equilibrium of a game can be extended to a larger type space in which the original game is $(1-p)$-evident. Under additional technical conditions (e.g. when T is finite), one can use the standard techniques in the robustness literature to construct Bayesian Nash equilibria on T so that (invest, invest) is played throughout $C^{1-\varepsilon,1-\varepsilon}(M_\varepsilon)$ in one equilibrium and (Not invest, Not invest) is played throughout $C^{1-\varepsilon,1-\varepsilon}(M_\varepsilon)$ in another equilibrium. Here, we dispense with those technical conditions by using the weaker solution concept of rationalizability.

We next turn to establishing sufficient conditions under which there is a unique rationalizable action. Our key concept will be approximate uniformity of "rank beliefs", which we now define. We will write $\overline{r}_i(t_i)$ for the probability that a player assigns to his payoff parameter being greater than or equal to that of the other player, and $\underline{r}_i(t_i)$ for the probability that it is strictly greater. We will refer to these expressions as "rank beliefs" as they reflect the player's belief about his rank if the players are ordered by the payoff parameter. Formally, we define

$$\begin{aligned} \overline{r}_i(t_i) &= \pi_i(\{t_j | x_j(t_j) \leq x_i(t_i)\} | t_i), \\ \underline{r}_i(t_i) &= \pi_i(\{t_j | x_j(t_j) < x_i(t_i)\} | t_i), \end{aligned}$$

and

$$r_i(t_i) = (\overline{r}_i(t_i) + \underline{r}_i(t_i))/2.$$

We refer to $\overline{r}_i(t_i)$, $\underline{r}_i(t_i)$, and $r_i(t_i)$ as the *upper rank belief*, the *lower rank belief* and the *rank belief* of type t_i, respectively. When $x_j(\cdot)$ is atomless according to type t_i, all these beliefs coincide: $\overline{r}_i(t_i) = \underline{r}_i(t_i) = r_i(t_i)$. We define upper and lower rank beliefs separately in order to deal with point masses. Such point messes may arise, for example, under complete information.

Rank belief of a type t_i is uniform if he finds it equally likely that either player's value is higher. Formally, we say that rank belief of a type t_i is *ε-uniform* if

$$\frac{1}{2} - \varepsilon \leq \underline{r}_i\left(t_i\right) \leq \overline{r}_i\left(t_i\right) \leq \frac{1}{2} + \varepsilon.$$

We write URB_ε for the set of type profiles (t_1, t_2) where both players have ε-uniform rank beliefs:

$$URB_\varepsilon = \left\{ (t_1, t_2) \left| \frac{1}{2} - \varepsilon \leq \underline{r}_i\left(t_i\right) \leq \overline{r}_i\left(t_i\right) \leq \frac{1}{2} + \varepsilon \text{ for each } i \right. \right\}.$$

We say that rank beliefs are *approximately uniform* if they are ε-uniform for some $\varepsilon \geq 0$.

Our second concept is a strict version of risk dominance. We say that action invest is *ε-strictly risk dominant* for t_i if

$$x_i\left(t_i\right) \geq \frac{1}{2} + \varepsilon.$$

We write SRD_ε for the set of type profiles for which invest is ε-strictly risk dominant for each player, so

$$SRD_\varepsilon = \left\{ (t_1, t_2) \left| x_i\left(t_i\right) \geq \frac{1}{2} + \varepsilon \text{ for each } i \right. \right\}.$$

We say that action invest is *strictly risk dominant* if invest is *ε-strictly risk dominant* for some $\varepsilon \geq 0$.

Proposition 3 *Invest is the uniquely rationalizable action for both players if it is ε-strictly risk dominant for both players and there is common $(1-\varepsilon)$-belief of ε-uniform rank beliefs for some $\varepsilon \geq 0$, i.e., if*

$$(t_1, t_2) \in SRD_\varepsilon \cap C^{1-\varepsilon, 1-\varepsilon}\left(URB_\varepsilon\right).$$

We require only common $(1-\varepsilon)$-belief, not common certainty, of ε-uniform rank beliefs for this sufficient condition. But note that in cases where we use this result—to prove results in the fat tails model and to explain the existing global games literature—we actually have common certainty.

Proposition 3 provides a useful sufficient condition for uniqueness, identifying common features of the uniqueness results in the global games literature. It states that invest is uniquely rationalizable if it is strictly risk-dominant and it is common knowledge that the rank beliefs are approximately uniform. Observe that our result establish this result without explicitly assuming some of the critical features of global games, such as existence of dominance regions. Moreover, it allows arbitrary type spaces with minimal continuity and compactness properties. In the rest of this section, we will further discuss our result and present its proof.

Although our game lacks the monotonicity assumptions in global games, we can use rank beliefs to obtain bounds on rationalizable strategies. To this end, we define two cutoffs x^* and x^{**} with $0 \leq x^* \leq x^{**} \leq 1$:

$$\begin{aligned} x^* &= \inf \{z | z = x_i(t_i) \geq \underline{r}_i(t_i) \text{ for some } i \text{ and } t_i\} \\ x^{**} &= \sup \{z | z = x_i(t_i) \leq \overline{r}_i(t_i) \text{ for some } i \text{ and } t_i\}. \end{aligned}$$

Here, x^* is the lowest value for which any type of either player has that value and has a lower lower-rank belief; and x^{**} is the highest value for which any type of any player has that value and a higher upper-rank belief. The next lemma establishes that invest cannot be rationalizable when the value is lower than x^*, and Not invest cannot be rationalizable when the value exceeds x^{**}.

Lemma 3 *Invest is uniquely rationalizable for any t_i with $x_i(t_i) > x^{**}$ and Not invest is uniquely rationalizable for any t_i with $x_i(t_i) < x^*$.*

Proof. We will show that invest is not rationalizable when $x_i(t_i) < x^*$. In particular, let

$$\hat{x}_i \equiv \inf \{x_i(t_i) | t_i \in R_i^\infty\}$$

and assume without loss that

$$\widehat{x}_1 \leq \widehat{x}_2. \tag{3}$$

By definition, there exists a sequence $(t_{1,m})$ of types $t_{1,m} \in R_1^\infty$ such that $x_1(t_{1,m}) \in [\hat{x}_1, \hat{x}_1 + 1]$ for each m and $x_1(t_{1,m}) \to \hat{x}$. Since $x_1^{-1}([\hat{x}_1, \hat{x}_1 + 1])$ is sequentially compact, there then exists a convergent subsequence with some limit $\hat{t}_1$. Since x_1 is continuous and $x_1(t_{1,m}) \to \hat{x}_1$, we have

$$\hat{x}_1 = x_1\left(\hat{t}_1\right). \tag{4}$$

But since $t_{1,m} \in R_1^\infty$, we also have $x_1(t_{1,m}) \geq 1 - \pi_1(R_2^\infty | t_{1,m})$ for all m. By continuity of x_1 and π_1, we thus have

$$x_1\left(\hat{t}_1\right) \geq 1 - \pi_1\left(R_2^\infty | \hat{t}_1\right). \tag{5}$$

Therefore,

$$\begin{aligned} \hat{x}_1 &= x_1\left(\hat{t}_1\right), \text{ by (4)} \\ &\geq 1 - \pi_1\left(R_2^\infty | \hat{t}_1\right), \text{ by (5)} \\ &\geq 1 - \pi_1\left(\{t_2 | x_2(t_2) \geq \hat{x}_2\} | \hat{t}_1\right), \text{ by definition of } \widehat{x}_2 \\ &\geq 1 - \pi_1\left(\{t_2 | x_2(t_2) \geq \hat{x}_1\} | \hat{t}_1\right), \text{ by (3)} \\ &= \pi_1\left(\left\{t_1 | x_2(t_2) < x_1\left(\hat{t}_1\right)\right\} | \hat{t}_1\right) \\ &= \underline{r}_1\left(\hat{t}_1\right), \end{aligned}$$

showing that

$$\widehat{x}_2 \geq \widehat{x}_1 \geq x^*.$$

A symmetric argument establishes that not invest is not rationalizable if $x_i(t_i) > x^{**}$. ■

When $x^{**} = 1$ (or $x^* = 0$), Lemma 3 is vacuous, stating that invest is uniquely rationalizable when it is dominant. When $x^{**} < 1$, Lemma 3 establishes that invest remains uniquely rationalizable throughout the interval $(x^{**}, 1)$. This is similar to the main result of Carlsson and Van Damme (1993). In their result, if one can connect at type t_i to a type $\bar{t}_i$ at which invest is a dominant action, via a continuous path along which invest is either risk-dominant or dominant, then invest is uniquely rationalizable at t_i. Here, we do not explicitly assume the existence of any type with a dominant action and make only minimal assumptions about the structure of the type space. Nonetheless, when $x^{**} < 1$, each type t_i with $x^{**} < x_i(t_i) < 1$ assigns a substantial probability (i.e. a probability greater than $1 - x_i(t_i)$) on a set of types t_j whose values are higher than that of t_i and assign a substantial probability to yet another set of types with similar properties. Under our compactness assumption, this chain leads to types who assign a substantial probability on types for which invest is a dominant action. Such a chain forms a contagion path. Note that the types in the chain have type-dependent thresholds as in Proposition 1.

Proof of Proposition 3. Proposition 3 immediately follows from Lemma 3. Under common $(1-\varepsilon, 1-\varepsilon)$-belief of URB_ε, we clearly have $x^{**} \leq 1/2 + 2\varepsilon$. Hence, whenever $(t_1, t_2) \in SRD_\varepsilon \cap C^{1-\varepsilon,1-\varepsilon}(URB_\varepsilon)$, we have $x_i(t_i) > x^{**}$ and invest is uniquely rationalizable (by Lemma 3) for both players. ■

While Proposition 3 will be the focus on our analysis in this paper, to relate our work to the existing literature, we will also note a weaker sufficient condition:

Corollary 2 (Uniqueness II) *Invest is the uniquely rationalizable action for both players if it is strictly risk dominant for both and there is approximate common certainty that each player has approximately uniform rank beliefs when his action is strictly risk dominant but not dominant, i.e., for some* $\varepsilon \in [0, 1]$,

$$(t_1, t_2) \subseteq C^{1-\varepsilon,1-\varepsilon}\left(\left\{(t_1', t_2') \;\middle|\; \begin{array}{l} x_i'(t_i') \geq \frac{1}{2} + \varepsilon \textit{ and} \\ x_i'(t_i') \in \left[\frac{1}{2} + \varepsilon, 1\right] \Rightarrow \frac{1}{2} - \varepsilon \leq \underline{r}_i(t_i') \leq \overline{r}_i(t_i') \leq \frac{1}{2} + \varepsilon \\ \textit{for both } i \end{array}\right\}\right).$$

Note that this sufficient condition merges the uniform rank beliefs and strict risk dominance properties into a single condition. We simply drop the restriction on rank beliefs when a player has a dominant strategy. This will clearly not matter for strategic results.

3 One Dimensional Type Space Model

We now restrict our attention to a particular class of symmetric "global game" type spaces. Players' types are one dimensional and drawn according to a common prior. In particular,

$$t_i = \tau\eta + \sigma e_i$$

where $\tau, \sigma \in (0,1)$; η is a random variable with symmetric, positive, Lipschitz-continuous density function g (with corresponding cumulative distribution function G); and random variables (e_1, e_2) are independently and identically distributed with symmetric, positive, Lipschitz-continuous density function f (and corresponding cumulative distribution function F). Assume that there exist $\bar{e} > 0$ and $\bar{\eta} > 0$ such that f and g are monotonically decreasing on $[\bar{e}, \infty)$ and $[\bar{\eta}, \infty)$, respectively. For some $y \in \mathbb{R}$, we assume that

$$x_1(t_1) = y + t_1 \text{ and } x_2(t_2) = y + t_2.$$

The functions f and g, and the parameters y, σ and τ are all common knowledge. Thus each (f, g, y, σ, τ) describes a type space.

The interpretation of the model is that y is a "public signal", or prior mean, of the players' types. The noise in their types—i.e., the difference between their values and the prior mean, is the sum of a common shock η and conditionally independent shock e_i. The size of these common and idiosyncratic shocks are parameterized by τ and σ respectively. An equivalent interpretation—in keeping with global games literature—is that there is an unknown "true state" θ drawn according to density $g\left(\frac{\theta - y}{\tau}\right)$ and the players observe signals that are equal to true state plus conditionally independent noise. A player's type is then the difference between this signal of θ and prior mean of θ.

We will be concerned about players' beliefs about other players' payoffs. We will write $r_{\sigma,\tau}(t)$ for the probability that type t assigns to the other player having a lower type, and thus a lower value, than him; so

$$r_{\sigma,\tau}(t) = \int_{\eta=-\infty}^{\infty} F\left(\frac{t - \tau\eta}{\sigma}\right) g(\eta)\, d\eta.$$

As in the general model, we refer to $r_{\sigma,\tau}(t)$ as the *rank belief* of t. Note that $r_{\sigma,\tau}$ is continuous, as we have a density; also, $r_{\sigma,\tau}(t)$ is independent of the prior mean y. (Since we have density, upper and lower rank beliefs coincide.)

We will be interested in what happens when σ and τ are both small; in particular, some results will focus on sequences $(\sigma_n, \tau_n) \to (0,0)$. Thus we will be considering a sequence where there is very little uncertainty about payoffs and, in particular, types converge in the product topology to a complete information model, where the payoffs are common knowledge. We will ultimately be interested certain limit properties of strategic behavior in such sequences.

Definition 1 *A sequence* $(\sigma_n, \tau_n) \to (0,0)$ *is said to satisfy* limit multiplicity *if, for any* $0 < y < 1$ *and any* $\varepsilon > 0$*, there exists* $\overline{n}$ *such that, for all* $n \geq \overline{n}$*, both actions are rationalizable for both players whenever* $x_i(t_i) \in [\varepsilon, 1-\varepsilon]$ *for both* i.

Definition 2 *A sequence* $(\sigma_n, \tau_n) \to (0,0)$ *is said to satisfy* limit uniqueness *if, for any* $y \in \mathbb{R}$ *and any* $\varepsilon > 0$*, there exists* $\overline{n}$ *such that, for all* $n \geq \overline{n}$*, action invest is uniquely rationalizable for both players whenever* $x_i(t_i) \geq \frac{1}{2} + \varepsilon$ *for both* i.

In the remainder of this section, we will maintain the assumption that distributions f and g have well-behaved "fat tails". In the next section, we will consider the case where f and g are standard normal distributions, a special case that has been much studied in the literature.

3.1 Fat Tails

In the remainder of this section, we will assume that both common and idiosyncratic components are drawn from fat-tailed distributions. Fat tailed distributions arise naturally in situations of model uncertainty, as discussed in Acemoglu, Chernozhukov, and Yildiz (2006). When we are interested in approximating complete information, tail properties are central to understanding higher-order beliefs. Assuming away mass at the tails is potentially very misleading. We discuss further issues in using fat tailed distributions following some formal definitions.

A function $L : \mathbb{R} \to \mathbb{R}_+$ is said to have *slow-varying tails* if

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1$$

for all $\lambda > 0$. Convergence is uniform over compact intervals of λ. We assume that f and g have *regularly-varying tails*, so that

$$\begin{aligned} f(e) &= A L_f(e)\, e^{-\alpha} \\ g(\eta) &= B L_g(\eta)\, \eta^{-\beta} \end{aligned}$$

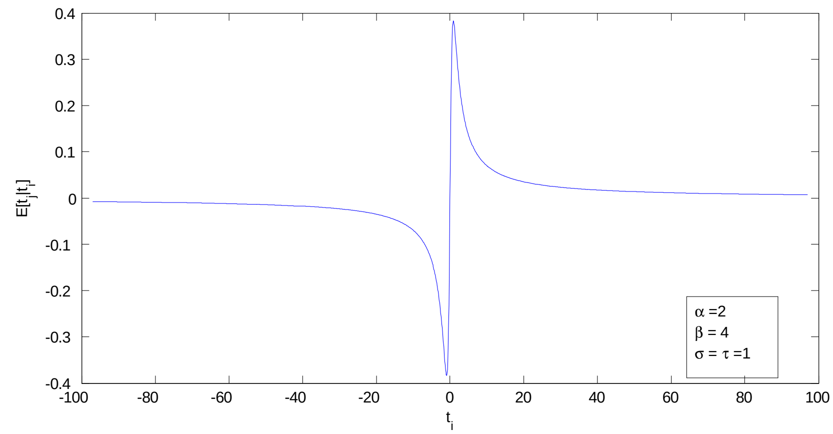

Figure 1: Expectation of the other player's type conditional on ones own when the idiosyncratic components have thicker tails than the common component (i.e., $\alpha < \beta$).

for some $\alpha, \beta > 1$, $A, B > 0$, and for some slowly-varying functions L_f and L_g. A particular example of regularly varying distributions is Pareto distribution where f and g are constant up to some $\bar{e}$ and $\bar{\eta}$, respectively, and $f(e)$ and $g(\eta)$ are proportional to $e^{-\alpha}$ and $\eta^{-\beta}$, respectively, thereafter. We will use Pareto distribution in our numerical examples. Throughout the paper, when we refer to fat-tailed distributions, we are assuming the stronger property that distributions have regularly-varying tails.

By the generalized central limit theorem, the limit of averages can be approximated by a Normal or a Pareto distribution, depending on whether the underlying distribution has finite or infinite variance.

Belief updating under fat tails is quite different from belief updating under normal distribution. Under the normal distribution, a player updates his belief by taking a weighted average of his signals and his prior. This updating is well behaved and gives rise to many monotonicity properties. A player's expectation of his own and the other player's payoff type are increasing in his own signal. A monotone likelihood ratio property is satisfied. And so on.

These properties are not preserved by belief updating under fat tails. However, the structure of updating under fat tails has its own psychological and economic intuition, and we believe it to be very relevant for economic applications. Suppose that a player observes the signal $t_i = \eta + e_i$. What is his expectation of the signal of the other player $t_j = \eta + e_j$. If the signal t_i is close to its prior mean of 0, the player's expectation of η, and thus of t_j, will be - roughly speaking - an average of his prior 0 and his signal t_i, as in the normal case. But if signal t_i is a long way from 0, the player will think it most likely that either η or e_i, but not both, were different from 0. Which one he thinks most likely depends on the shape of the tails.

This is illustrated in Figure 1 for the case of Pareto distributions, where the e_i are drawn from the Pareto distribution with tail parameter $\beta = 4$ and η is drawn from a Pareto distribution with a thinner tail $\alpha = 2$. For small positive t_i, he updates his expectation of η and thus his expectation of the other player's type t_j. So his expectation of t_j is increasing in t_i for a while. But if he observes a substantially larger t_i, he then questions whether his private signal is valuable, attributing a substantial part of this deviation to e_i. This leads to a lower expectation about t_j than the expectation he would have after observing a lower signal t_i (as in the decreasing part of the graph). In the limit, when he observes an extremely positive signal, he discredits his private signal altogether as a meaningless noise and sticks to his prior, expecting $E[t_j|t_i] = 0$. This is a behaviorally intuitive form of updating which is also rational. While such non-monotone behavior prevents us from using many existing techniques in the global games literature, the above updating behavior under regularly-varying tails provide us other means that lead to sharper characterizations in the final analysis. We next report our results about the updating and higher-order beliefs; we will then combine these results with our general results to obtain sharp characterizations about the strategic behavior afterwards.

3.2 Higher-Order Beliefs

For any given (σ_n, τ_n), we can consider the beliefs of a player observing signal $t = \tau_n \eta + \sigma_n e$ about the common component of his signal $\tau_n \eta$. Belief about other variables will then be derived from this. We write $\mu_n(\cdot|t)$ for the probability distribution of $\tau_n \eta$ conditional on (σ_n, τ_n, t).[4] Now if we fix a sequence $(\sigma_n, \tau_n) \to (0, 0)$, we will be interested in characterizing the limit of $\mu_n(\cdot|t)$ as $n \to \infty$.

[4]The probability of any interval $(\tau_n \eta_1, \tau_n \eta_2)$ under $\mu_n(\cdot|t)$ is

$$\mu_n((\tau_n \eta_1, \tau_n \eta_2)|t_i) = I_n(t_i - \tau_n \eta_1, t_i - \tau_n \eta_2) / I_n(-\infty, \infty)$$

where

$$I_n(z_1, z_2) = \int_{z_1}^{z_2} f(z/\sigma_n)\, g\left(\frac{t_i - z}{\tau_n}\right) dz.$$

Towards this goal, we define

$$q_n(t) = \frac{\sigma_n g(t/\tau_n)}{\sigma_n g(t/\tau_n) + \tau_n f(t/\sigma_n)},$$

and assume that $q_n(t)$ converges to some $q(t) \in [0,1]$ for each t, and that the convergence is uniform over $[\underline{t}, \overline{t}]$ for any $\overline{t} > \underline{t} > 0$. Under regularly-varying tails, it is straightforward to show that

$$q_n(t) \cong \frac{1}{1 + \rho_n t^{\beta-\alpha} A/B} \text{ and } q(t) = \frac{1}{1 + \rho t^{\beta-\alpha} A/B}$$

where

$$\rho_n = \frac{\sigma_n^{\alpha-1}}{\tau_n^{\beta-1}} \frac{L_f(1/\sigma_n)}{L_g(1/\tau_n)} \text{ and } \rho = \lim_{n\to\infty} \rho_n.$$

In the case of Pareto distribution, we simply have $\rho_n(t) = \sigma_n^{\alpha-1}/\tau_n^{\beta-1}$ allowing a tractable analysis based on the tail indices α and β of idosyncratic and common components, respectively. In general, except for knife-edge cases, ρ is either 0 or ∞, and $q(t)$ is either 1 or 0, independent of t. Finally, we define our notion of convergence for type-dependent probability distributions as follows. For any sequence of type-dependent probability distributions $P_n(\cdot|\cdot)$ and any $P(\cdot|\cdot)$, where $P_n(\cdot|t_i)$ and $P(\cdot|t_i)$ are probability distributions over the same type space for each t_i, we say that $P_n(\cdot|\cdot)$ *converges to $P(\cdot|\cdot)$ uniformly over compact sets of types* if $\int h dP_n(\cdot|t_i) \to \int h dP(\cdot|t_i)$ uniformly over types $t_i \in [\underline{t}, \overline{t}]$ for any bounded continuous function h and any compact interval $[\underline{t}, \overline{t}]$.

The following result establishes the implication of fat tails that, in the limit, players rely only on one of the public and the private signals, ignoring the other.

Lemma 4 *As $n \to \infty$, the beliefs $\mu_n(\cdot|\cdot)$ about the common shock $\tau_n \eta$ converge to $\mu_\infty(\cdot|\cdot)$ uniformly over compact sets of types where*

$$\mu_\infty(0|t_i) = 1 - q(t_i) \text{ and } \mu_\infty(t_i|t_i) = q(t_i)$$

for each $t_i \neq 0$ and $\mu_\infty(0|0) = 1$.

Thus, in the limit $(\sigma_n, \tau_n) \to (0,0)$, each individual puts positive probability only on two points, entertaining only two possible theories. He puts probability $q(t_i)$ on $(\eta = 0, \sigma_n e_i = t_i)$, attributing all of the variation in his type t_i from 0 to the noise e_i in his individual information. He puts probability $1 - q(t_i)$ on $(\tau_n \eta = t_i, e_i = 0)$, attributing all of the variation in his type t_i from 0 to the common shock. Consequently, the players have stark limiting beliefs regarding their relative ranking. When the tails of the idiosyncratic components are fater than that of the common component ($\alpha < \beta$), we have $q(t_i) = 1$, and he ends up attributing all the deviation to his individual information, maintaining the belief that the other player's type is around 0, as in (the extremes of) Figure 1. When the tails of the idiosyncratic components are thinner than that of the common component ($\alpha > \beta$), we have $q(t_i) = 0$, and he ends up attributing all the deviation to the common shock, maintaining the belief that the other player's type is around his own, as in (the extremes of) Figure 2 below.

Identifying the limiting rank beliefs requires more care because, conditional on t_i, the limit beliefs regarding t_j are somewhat complicated. In the limit, by Lemma 4, player i assigns probability $1 - q(t_i)$ on $\tau\eta = 0$ and probability $q(t_i)$ on $\tau\eta = t_i$. But in the limit case of $\tau\eta = t_j = t_i$, the relative ranking of t_j and t_i are not determined from Lemma 4 because the limit CDF of t_j conditional on $\tau\eta$ is discontinuous at $\tau\eta$.[5] After all, probability of $t_j = t_i$ is zero. The next result establishes the limiting rank beliefs, defined as

$$r_\infty(t_i) \equiv \lim_{n\to\infty} r_{\sigma_n,\tau_n}(t_i) = \lim_{n\to\infty} \Pr(t_j \le t_i | t_i, \sigma_n, \tau_n).$$

The result shows that when the noise is small so that t_i is near $\tau\eta$, the player assigns probability $1/2$ on the other player's signal being lower than his own.

Lemma 5 *For any $\bar{t} > \underline{t} > 0$, as $n \to \infty$, $r_{\sigma_n,\tau_n}(t_i)$ uniformly converges to*

$$r_\infty(t) = \begin{cases} 1 - q(t)/2 & \text{if } t > 0 \\ q(t)/2 & \text{if } t < 0 \end{cases}$$

over $\left[-\bar{t}, -\underline{t}\right] \cup \left[\underline{t}, \bar{t}\right]$.

The limiting rank-beliefs have an interesting form. When $\rho \equiv \lim_{n\to\infty} \frac{\sigma_n^{\alpha-1}}{\tau_n^{\beta-1}} \frac{L_f(1/\sigma_n)}{L_g(1/\tau_n)} = \infty$, we have $q(t) \equiv 0$, and the limiting rank belief is a step function at $t = 0$. The player nearly attributes all of the variation in his signal to the noise in his own private information and keeps believing that t_j is around zero—and that θ and x_j are around y. Since both players do so for all types, it is then easy to obtain a high common belief in the event that both players assign high probability on θ being around y. The next corollary establishes this formally.

Corollary 3 *If $\rho = \infty$ and $\underline{x} < y < \overline{x}$, then, whenever payoffs are in $[\underline{x}, \overline{x}]$, there is limit approximate common certainty that payoffs are in $[\underline{x}, \overline{x}]$, i.e., for any $\varepsilon > 0$, there exists $\overline{n}$ such that, for all $n \ge \overline{n}$,*

$$[\underline{x} - y, \overline{x} - y]^2 \subseteq C^{1-\varepsilon,1-\varepsilon}\left([\underline{x} - y, \overline{x} - y]^2\right)$$

[5] With our formulation, the rank beliefs is 1/2 in the limit.

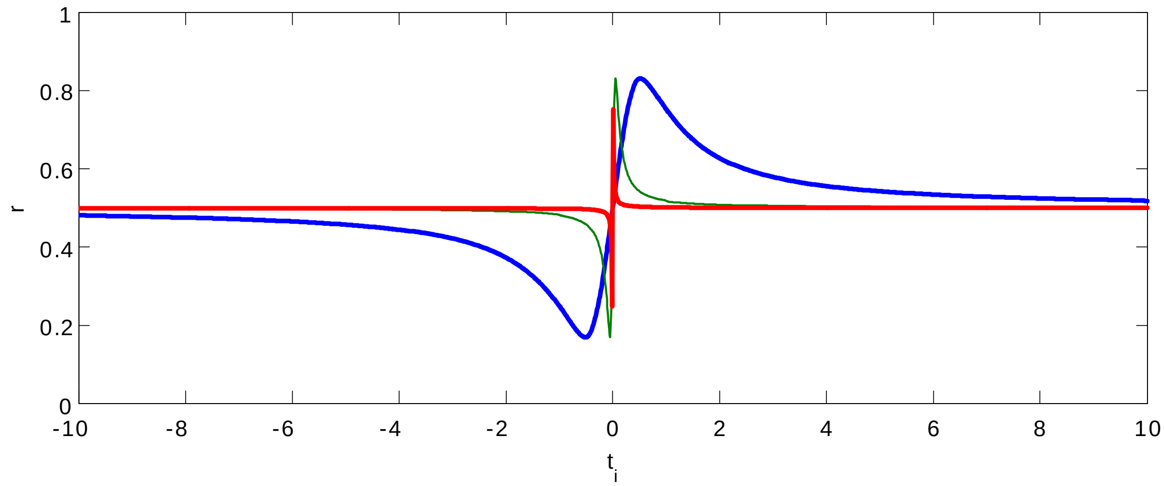

Figure 2: Rank beliefs under $\alpha = 4 > 2 = \beta$ for $\sigma = \tau = 1$, $\sigma = \tau = 0.1$, and $\sigma = \tau = 0.01$.

in the (σ_n, τ_n) type space, where $[\underline{x} - y, \overline{x} - y]^2$ is the set of type profiles under which the payoffs are in $[\underline{x}, \overline{x}]$.

Proof. Since $\mu_n(\cdot|t_i) \to \mu_\infty(\cdot|t_i)$ uniformly over $t_i \in [\underline{x} - y, \overline{x} - y]$ (by Lemma 4), and since $\sigma_n \to 0$, there exists $\bar{n}$ such that, for all $n \geq \bar{n}$ and for all $t_i \in [\underline{x} - y, \overline{x} - y]$, the probability that $t_j \equiv \tau_n \eta + \sigma_n e_j \in [\underline{x} - y, \overline{x} - y]$ is at least $\mu_\infty(0|t_i) - \varepsilon$. But since $\rho = \infty$, we have $\mu_\infty(0|t_i) = 1$. Therefore, for all $n \geq \bar{n}$, all types $t_i \in [\underline{x} - y, \overline{x} - y]$ assign at least probability $1 - \varepsilon$ on $t_j \in [\underline{x} - y, \overline{x} - y]$, as desired. ■

That is, when the common component has thinner tails than the idiosyncratic component (e.g., $\alpha < \beta$ and $\sigma_n/\tau_n \gg 0$ so that $\rho = \infty$), the event that the payoffs are around their ex-ante mean is a $(1-\varepsilon, 1-\varepsilon)$-evident event: whenever it happens, the players assign at least probability $1 - \varepsilon$ on that event. This will further imply that the equilibria of the complete information game extend to the (σ_n, τ_n) type space.

On the other hand, if $\alpha > \beta$ and σ_n/τ_n is bounded so that the common component has fater tails than the idiosyncratic component, then $\rho = 0$ and $q(t_i) = 1$ everywhere. Our lemma then implies that the rank beliefs converge to $r_\infty(t_i) = \frac{1}{2}$ on compact intervals of t_i that exclude 0. However, this does not deliver common certainty of approximately uniform rank beliefs, because we do not have uniform convergence around zero. The problem is vividly illustrated in Figure 2. Although $r_{\sigma,\tau}$ converges to $1/2$ pointwise everywhere, $r_{\sigma,\tau}$ traces the entire interval $[0.2, 0.8]$ around $t_i = 0$ no matter how small (σ, τ) is. (This is because when one fixes the ratio of σ/τ, reducing the noise

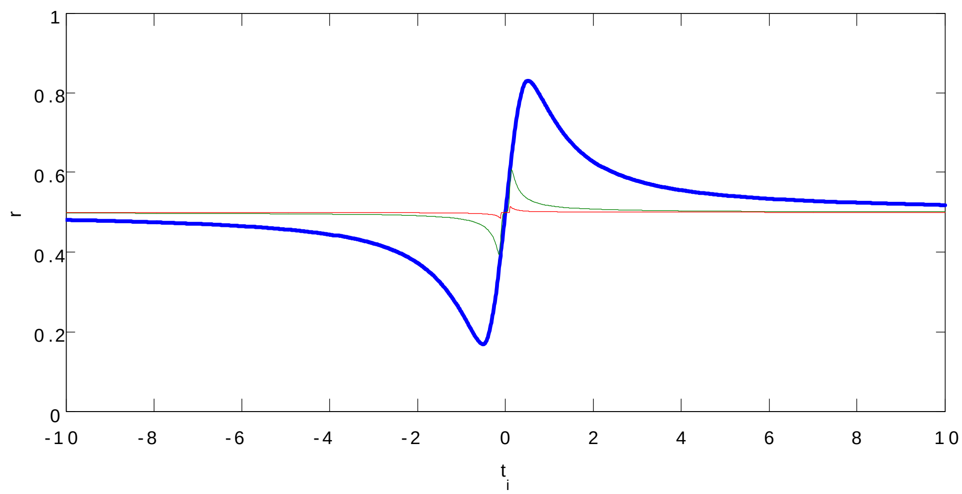

Figure 3: Rank beliefs for $\sigma = 1, 0.1, 0.001$ and $(\alpha, \beta, \tau) = (4, 2, 1)$.

is equivalent to shrinking t_i around 0, i.e., $r_{a\sigma,a\tau}(t_i) = r_{\sigma,\tau}(t_i/a)$.) To obtain common certainty of approximately uniform rank beliefs, we need approximately uniform rank beliefs for types around 0, and for this it is necessary to have σ small relative to τ. However, it turns out that if we have both these types of conditions, we can prove results that hold for both large and small (σ, τ). In that case, $r_{\sigma,\tau}$ converges to $1/2$ uniformly everywhere as σ/τ gets small, as illustrated in Figure 3. This is formally established in the next proposition, which is the main finding of this section.

Proposition 4 *If $\alpha > \beta + 1$ and σ/τ is sufficiently small, then there is common certainty of approximately uniform rank beliefs, i.e., for any $\varepsilon > 0$, there exists $\bar{\xi} > 0$ such that whenever $\sigma/\tau < \bar{\xi}$,*

$$\frac{1}{2} - \varepsilon < r_{\sigma,\tau}(t) < \frac{1}{2} + \varepsilon \qquad (\forall t).$$

This theorem establishes common certainty of approximately uniform rank beliefs when σ/τ is sufficiently small.

Proof. Writing $\xi = \sigma/\tau$, note that $r_{\sigma,\tau}(t) = r_{\xi,1}(t/\tau)$. Hence, it suffices to show that there exists $\bar{\xi}$ such that $r_{\xi,1}(t) < 1/2 + \varepsilon$ for all $t \geq 0$ whenever $\xi < \bar{\xi}$; throughout the proof, we will assume that $\xi \ll 1/2$ is sufficiently small. All the other bounds follow from symmetry. To this end, we first observe that, since g has regularly varying tails, there exists t_0 such that for all $t > t' \geq t_0$,

$$(t/t')^{-\beta-1} / (1 + \varepsilon/2) \leq \frac{g(t)}{g(t')} \leq (1 + \varepsilon/2)(t/t')^{-\beta+1}. \tag{6}$$

We write

$$\tilde{e}(t) = \max\{t, \delta/\xi^{\gamma}\} \geq \bar{e} \tag{7}$$

for some large δ and for some $\gamma \in (1/(\alpha - \beta), 1)$. We suppress the dependence on t in the sequel. We observe that

$$r_{y,\xi,1}(t) \leq (I_1 + I_2)/I_3$$

where

$$\begin{aligned}
I_1 &= \int_{-\xi\tilde{e}}^{\xi\tilde{e}} f(z/\xi) F(z/\xi) g(t-z)\, dz \leq \frac{1}{2}(F(\tilde{e}) - F(-\tilde{e}))\, \xi g^{M,\xi}(t), \\
I_2 &= \int_{z \notin (-\xi\tilde{e}, \xi\tilde{e})} f(z/\xi) F(z/\xi) g(t-z)\, dz \leq f(\tilde{e}), \\
I_3 &= \int_{-\xi\tilde{e}}^{\xi\tilde{e}} f(z/\xi) g(t-z)\, dz \geq (F(\tilde{e}) - F(-\tilde{e}))\, \xi g^{m,\xi}(t)
\end{aligned}$$

and

$$g^{M,\xi}(t) = \max_{z \in [t-\xi\tilde{e}, t+\xi\tilde{e}]} g(t) \text{ and } g^{m,\xi}(t) = \min_{z \in [t-\xi\tilde{e}, t+\xi\tilde{e}]} g(t).$$

Combining the above inequalities, we conclude that

$$r_{y,\xi,1}(t) \leq \frac{1}{2}\frac{g^{M,\xi}(t)}{g^{m,\xi}(t)} + \frac{f(\tilde{e})}{(F(\tilde{e}) - F(-\tilde{e}))\, \xi g^{m,\xi}(t)}. \tag{8}$$

Now, note that, since $\alpha > \beta + 1$, there exists δ such that

$$\frac{f(z/\xi^{\gamma})}{(F(z/\xi^{\gamma}) - F(-z/\xi^{\gamma}))\, \xi g(2z/\xi^{\gamma})} < \varepsilon/2 \qquad \forall z \geq \delta.$$

We pick such a δ in definition of $\tilde{e}$. Then, using the fact that $g^{m,\xi}(t) \geq g(t + \tilde{e}(t)) \geq g(2\tilde{e}(t))$, one can easily check that

$$\frac{f(\tilde{e}(t))}{(F(\tilde{e}(t)) - F(-\tilde{e}(t)))\, \xi g^{m,\xi}(t)} < \varepsilon/2 \qquad \forall t. \tag{9}$$

In order to find an upper bound for $\frac{1}{2}g^{M,\xi}(t)/g^{m,\xi}(t)$, consider only $\xi \ll (t_0/\delta)^{1/(1-\gamma)}$, so that $\tilde{e}(t_0) = t_0$. Then, for any $t > t_0/(1-\xi)$, we have $t - \xi\tilde{e} = (1-\xi)t > t_0$, and hence, by (6),

$$\frac{g^{M,\xi}(t)}{g^{m,\xi}(t)} \leq (1 + \varepsilon/2)\left(\frac{t + \xi\tilde{e}}{t - \xi\tilde{e}}\right)^{\beta+1} = (1 + \varepsilon/2)\left(\frac{1+\xi}{1-\xi}\right)^{\beta+1}. \tag{10}$$

On the other hand, since g is Lipschitz continuous and positive, there also exists $\kappa > 0$ such that

$$g(t') \leq g(t)(1 + (t' - t)\kappa)$$

for any $t, t' \in [0, 2t_0]$. (Note that κ is independent or ξ.) Hence, for any $t \in [0, t_0/(1-\xi)]$, we also have

$$\frac{g^{M,\xi}(t)}{g^{m,\xi}(t)} \leq \frac{1+\kappa\xi\tilde{e}}{1-\kappa\xi\tilde{e}} \leq \frac{1+\kappa\max\left\{\xi t_0/(1-\xi), \xi^{1-\gamma}\delta\right\}}{1-\kappa\max\left\{\xi t_0/(1-\xi), \xi^{1-\gamma}\delta\right\}}. \tag{11}$$

Substituting (9-11) in (8), we obtain

$$r_{y,\xi,1}(t) \leq \frac{1}{2}\max\left\{\frac{1+\kappa\max\left\{\xi t_0/(1-\xi), \xi^{1-\gamma}\delta\right\}}{1-\kappa\max\left\{\xi t_0/(1-\xi), \xi^{1-\gamma}\delta\right\}}, (1+\varepsilon/2)\left(\frac{1+\xi}{1-\xi}\right)^{\beta+1}\right\} + \varepsilon/2.$$

Observe that the right-hand side is independent of t and converges to $1/2+3\varepsilon/4$ as $\xi \to 0$. Therefore, there exists $\bar{\xi} > 0$ such that for all $\xi \in (0, \bar{\xi})$,

$$r_{y,\xi,1}(t) < \frac{1}{2}(1+\varepsilon) + \varepsilon/2 = \frac{1}{2} + \varepsilon \qquad \left(\forall t \geq 0, \forall \xi < \bar{\xi}\right).$$

■

Note that the bound in Proposition 4 depends only on ε and the density functions f and g; when the ratio σ/τ is below the bound, the rank beliefs will be uniformly within ε neighborhood of $1/2$ regardless of how large the noise parameters σ and τ are, or what the realized types are. Of course, this also true when $(\sigma_n, \tau_n) \to (0,0)$:

Corollary 4 *Assume $\alpha > \beta + 1$ and $\lim_{n\to\infty} \frac{\sigma_n}{\tau_n} = 0$. Then, there is limit common certainty of approximately uniform rank beliefs, i.e., for any $\varepsilon > 0$, there exists $\overline{n}$ such that if $n \geq \overline{n}$,*

$$\frac{1}{2} - \varepsilon < r_{\sigma_n,\tau_n}(t) < \frac{1}{2} + \varepsilon \qquad (\forall t)$$

in the (σ_n, τ_n) type space.

3.3 Multiplicity and Uniqueness of Rationalizable Actions

We can now immediately apply our results about higher-order beliefs in the fat-tails model to the general strategic analysis in Section 2.2 to derive strategic results in the fat-tails model. In particular, we conclude that there is multiplicity whenever the common component has a thinner tail than the idiosyncratic components, and conversely, the risk-dominant action is uniquely rationalizable whenever the idiosyncratic components have thinner tails. Building on Proposition 2 and Corollary 3, our first result establishes limit multiplicity in the former case.

Corollary 5 *If $\rho = \lim_{n\to\infty} \frac{\sigma_n^{\alpha-1}}{\tau_n^{\beta-1}} \frac{L_f(1/\sigma_n)}{L_g(1/\tau_n)} = \infty$, then there is limit multiplicity, i.e., for any $0 < y < 1$ and $\varepsilon > 0$, there exists $\overline{n}$ such that, in every (σ_n, τ_n) type space with $n \geq \overline{n}$, both actions are rationalizable for both players whenever $x_i(t_i) \in [\varepsilon, 1-\varepsilon]$ for both i.*

Proof. Let $\rho = \infty$ and take any $\varepsilon > 0$. Assume without loss of generality that $\varepsilon < y < 1-\varepsilon$. Then, Corollary 3 establishes that the event M_ε that the payoffs are within the interval $[\varepsilon, 1-\varepsilon]$ is $(1-\varepsilon, 1-\varepsilon)$-evident whenever n is sufficiently large. Then, for any such n, Proposition 2 establishes that both actions are rationalizable throughout M_ε. ■

The multiplicity here is illustrated in Figure 4. When $(\sigma, \tau) = (0.1, 0.1)$, the rank beliefs are as in the flatter curve. For $y = 1/2$, this curve intersect the value function x_i at three points when $y = 0.5$, one at $x^* \simeq 0$, one at $x^{**} \simeq 1$, and one in the middle. Each of the extremal intersections corresponds to a symmetric equilibrium. In one equilibrium, players invest iff their values exceed x^*, and in another equilibrium they invest iff their values exceed x^{**}, yielding multiple solutions for types with values in (x^*, x^{**}) as in the corollary. For this value of (σ, τ), such multiplicity is supported for an interval $(\underline{y}, \bar{y})$, where $\underline{y}$ is slightly above 0.2 and $\bar{y}$ is slightly below 0.8. For values $y = 0.2$ and $y = 0.8$, we have a unique rationalizable action everywhere except for the type corresponding to the unique intersection (by Lemma 3). When the noise reduces to $(\sigma, \tau) = (0.1, 0.1)$, the rank beliefs become as in the steeper curve. Now, a larger set values for y leads to multiple equilibria, including $y = 0.2$ and $y = 0.8$, as the corresponding value functions intersect the new curve at multiple points. In the limit $(\sigma, \tau) \to (0, 0)$, the curve becomes the step function at $t_i = 0$, yielding multiple equilibria for all values of $y \in (0, 1)$.

The key insight for this result is as follows. When the players' private information is weaker than the public information, each player attributes all of the deviation in his type from the ex-ante mean 0 to the noise in his private information and keeps believing that the other player's payoff is near the ex-ante mean y. Since this is true for all types of both players, such high confidence is maintained throughout the players' higher-order beliefs, resulting in a high common belief that the payoffs are near the ex-ante mean y whenever this is indeed the case. Then, the rationalizable behavior under the complete information about payoffs around y extend to the case of incomplete information with "approximate common knowledge" about the payoffs. Since both actions are rationalizable under complete information, both actions remain rationalizable under the latter case of incomplete information.

When the players' private information is stronger than their public information (e.g. the tail index α of the idiosyncratic component is larger than the tail index β of the common component

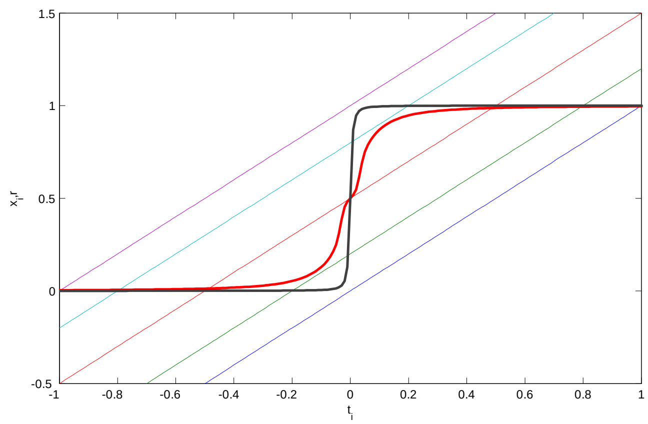

Figure 4: The rank beliefs for $(\sigma, \tau) = (0.1, 0.1)$ and $(\sigma, \tau) = (0.01, 0.01)$ under $(\alpha, \beta) = (2, 4)$. The straight lines corresponds to x_i for $y = 0, 0.2, 0.5, 0.8, 1$.

and σ_n/τ_n is bounded), the opposite happens. Now, each player attributes all of the deviation in his type from the ex-ante mean 0 to the noise in the public information and comes to believe that the other players' payoff is similar to his own, having equal probability of being higher or lower than his own. In such a model, one cannot maintain any high level of common belief regarding the *whereabouts of the payoffs*. Instead, one maintains a common certainty about the players' rank beliefs being approximately 1/2, as established by Proposition 4. But, such common certainty of the rank beliefs leads to selection of the risk-dominant action as the unique rationalizable action, as established by our general result Proposition 3. Since Proposition 4 applies more generally, such a uniqueness is maintained away from the limit as well as in the limit. Our next result formally establishes this as a corollary to the above results.

Corollary 6 *If $\alpha > \beta + 1$ and σ/τ is sufficiently small, then invest is the unique rationalizable action of both players if it is strictly risk dominant, i.e., for any $\varepsilon > 0$, there exists $\bar{\xi} > 0$ such that whenever $\sigma/\tau < \bar{\xi}$, invest is uniquely rationalizable for both players if $x_i(t_i) \geq \frac{1}{2} + \varepsilon$ for both i.*

Building on Proposition 3, Corollary 6 establishes that the risk dominant action is also uniquely rationalizable whenever the ratio σ/τ of the individual noise size to the common noise size is below a threshold regardless of how large σ and τ are. This establishes the unique selection of the risk-dominant action when the players private information is stronger than their public information (i.e. when $\alpha > \beta + 1$ and σ/τ is small) generally. In particular, this also implies such a unique selection in the limit, a result that can also be obtained by combining Proposition 3 and Corollary 4.

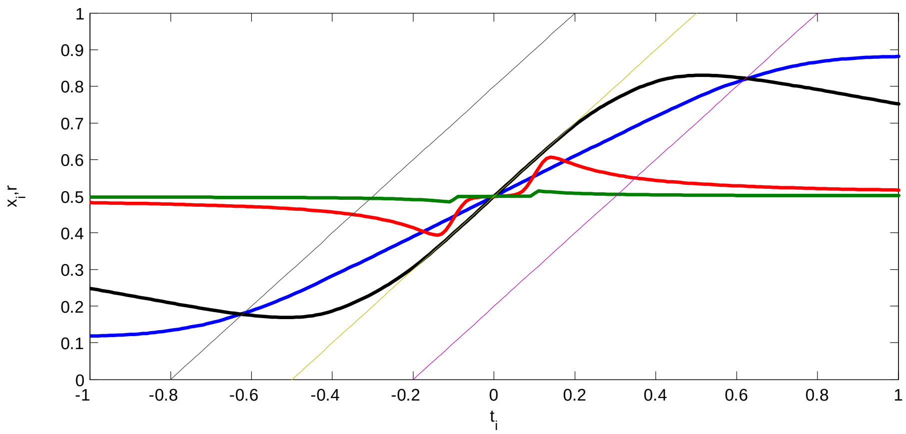

Figure 5: Rank beliefs for $\sigma = 2, 1, 0.1$, and 0.01 under $(\alpha, \beta, \tau) = (4, 2, 1)$; the maximum deviation from $1/2$ is decreasing in σ. The straight lines correspond to x_i for $y = 0.2, .5, 0.8$.

Corollary 7 *If $\alpha > \beta + 1$ and $\lim_{n\to\infty} \frac{\sigma_n}{\tau_n} = 0$, then there is limit uniqueness, i.e., for any $\varepsilon > 0$, there exists $\overline{n}$ such that, if $n \geq \overline{n}$, invest is uniquely rationalizable for both players if $x_i(t_i) \geq \frac{1}{2} + \varepsilon$ for both i, in the (σ_n, τ_n) type space.*

The unique selection above is illustrated in Figure 5. As σ/τ decreases, the rank belief $r_{\sigma,\tau}$ uniformly converges to $1/2$, restricting the x values corresponding to an intersection of the rank belief $r_{\sigma,\tau}$ and x_i to a smaller ε neighborhood of $1/2$. This ensures the unique selection of strictly risk-dominant actions, *uniformly* over the possible payoff functions x_i and the sizes of the idiosyncratic and common shock (σ and τ). For example, the selection here does not depend on the value of y.

Several remarks are in order. First, there can be multiple equilibria for a given payoff function; for example, $r_{1,1}$ intersects $x_i = t_i + 1/2$ at three points, but the intersections are all contained in $(-0.2, 0.2)$, allowing the selection of 0.2-strict risk dominant actions. Second, one can switch back and forth between multiplicity and unqueness as σ/τ decreases; for example, we have multiplicity under $(\sigma, \tau) = (1, 1)$ but unique selection under $(\sigma, \tau) \in \{(2, 1), (0.1, 1), (0.01, 1)\}$. Finally, under any (σ, τ) with unique selection, we could still have multiple equilibria by considering another payoff function, e.g., $x_i = at_i + 1/2$ for sufficiently small a, allowing a wide range of types with multiple actions. But all those types will also have values that are near the risk-dominance cutoff $1/2$, and we will ultimately be able to select strictly risk dominant action as the unique rationalizable solution whenever such an action exists.

4 Discussion: Extensions and Relation to The Literature

4.1 Thin Tails and The Normal Case

Our analysis in section 3 was limited to fat tailed distributions. What happens if we allowed thin tailed distributions, where tail densities converge to zero at an exponential rate?

Some extensions are straightforward. If the common component has a fat tailed distribution and the idiosyncratic component has a thin tailed distribution, then there will be common certainty of uniform rank beliefs in the limit. Conversely, if the common component has a thin tailed distribution and the idiosyncratic component has a fat tailed distribution, then there will be approximate common certainty of payoffs in the limit. Of course, a distribution with bounded support is a special case of a thin tailed distribution.

We do not have general results for the case where both distributions have thin tails However, a leading case from the global games literature corresponds to assuming a particular thin-tailed distribution - the normal distribution - for both distributions. In this section, we will review the well known analysis of this case to highlight how it fits into the framework of this paper.

Thus we will consider the one dimensional model in the previous section but under the assumption that η and e_i have standard normal distributions and demonstrate how known results for this problem are implied by the common belief foundations described above. In this case, rank beliefs can be characterized in closed form, and this characterization has been extensively used in the existing literature, e.g., Morris and Shin (2001) and Morris and Shin (2003a).

Player i will believe that $\tau\eta$ is distributed with mean

$$\frac{\tau^2}{\sigma^2+\tau^2}t_i$$

and variance

$$\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}$$

and so t_j is distributed with mean

$$\frac{\tau^2}{\sigma^2+\tau^2}t_i$$

and variance

$$\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}+\sigma^2=\frac{\sigma^2\left(\sigma^2+2\tau^2\right)}{\sigma^2+\tau^2}.$$

Thus the player assigns probability

$$\Phi\left(\sqrt{\frac{\sigma^2+\tau^2}{\sigma^2\left(\sigma^2+2\tau^2\right)}}\frac{\sigma^2}{\sigma^2+\tau^2}t_i\right)=\Phi\left(\sqrt{\frac{\sigma^2}{\left(\sigma^2+2\tau^2\right)\left(\sigma^2+\tau^2\right)}}t_i\right)$$

to the other player having a lower signal. Thus

$$r_{\sigma,\tau}(t) = \Phi\left(\sqrt{\frac{\sigma^2}{(\sigma^2+2\tau^2)(\sigma^2+\tau^2)}}t\right).$$

What can we say about higher-order beliefs in this case? Observe that, as $\sigma_n \to 0$ and $\tau_n \to 0$, $\sqrt{\frac{\sigma_n^2}{(\sigma_n^2+2\tau_n^2)(\sigma_n^2+\tau_n^2)}}$ converges to ∞ if $\frac{\sigma_n}{\tau_n^2} \to \infty$ and converges to 0 if $\frac{\sigma_n}{\tau_n^2} \to 0$. Thus if $\frac{\sigma_n}{\tau_n^2} \to \infty$, we obtain approximate common certainty of payoffs and we have limit multiplicity as defined in Definition 1. If $\frac{\sigma_n}{\tau_n^2} \to 0$, we do have pointwise convergence to uniform rank beliefs, so that $r_{\sigma_n,\tau_n}(t) \to \frac{1}{2}$ for each t. But this convergence is not uniform in the tails, so that for any n, $r_{\sigma_n,\tau_n}(t) \to 1$ as $t \to \infty$ and $r_{\sigma_n,\tau_n}(t) \to 0$ as $t \to -\infty$. Thus there is never approximate common certainty of approximately uniform rank beliefs. But for any ε, if we look at the set of t where rank beliefs are within ε of $\frac{1}{2}$, we have—by construction—that the marginal type puts a high probability (greater than $\frac{1}{2} - \varepsilon$) to his opponent having a type outside the set. However, it is enough for global game results that there is common certainty that rank beliefs are close $\frac{1}{2}$ when players do not have a dominant strategy. Thus, the weaker sufficient condition of Lemma 2 for limit uniqueness is satisfied and we have limit uniqueness as defined in Definition 2.

Thus we find that the fat tailed limit results deliver sharper sufficient conditions for limit uniqueness while the well known normal results achieve limit uniqueness via a less clean common belief foundation.

4.2 The Common Prior Assumption

We focussed on the case where the common prior assumption holds. Under the common prior assumption, if there is common certainty that players have rank belief p, i.e., they assign probability p to having the higher rank, then it must be the case that $p = \frac{1}{2}$. On the other hand, without the common prior assumption, common certainty of alternative rank beliefs is possible. Our uniqueness results extend immediately to common certainty of non-uniform rank beliefs, although the nature of global game selection is then different.

Specifically, define RB_ε^p for the set of type profiles (t_1, t_2) where both players have ε-rank belief p:

$$RB_\varepsilon^p = \{(t_1,t_2) \,|\, p - \varepsilon \le \underline{r}_i(t_i) \le \overline{r}_i(t_i) \le p + \varepsilon \text{ for each } i\}.$$

Say that action invest is *p-dominant* for t_i if

$$x_i(t_i) \ge p.$$

We write D^p for the set of type profiles for which invest is p-dominant for each player, so

$$D^p = \{(t_1, t_2) \,|x_i (t_i) \geq p \text{ for each } i\}.$$

Proposition 5 *Invest is the uniquely rationalizable action for both players if it is p-dominant for both players and there is common $(1-\varepsilon)$-belief of ε-rank belief $p+2\varepsilon$, i.e., if*

$$(t_1, t_2) \in D^p \cap C^{1-\varepsilon, 1-\varepsilon} \left(RB_{\varepsilon}^{p+2\varepsilon} \right).$$

This observation, suitably generalized to many players, is the driving force behind the results of Izmalkov and Yildiz (2010).

4.3 Many Players

We focussed for simplicity on the case of two players. The extension of the results of this paper to N players— maintaining the symmetry and separability of payoffs assumptions in this paper— is straightforward, and we describe this extension in this section.

Suppose that each player i had a payoff type x_i, the payoff to not investing was 0, and the payoff to investing was $x_i - 1 + \psi(l)$ where l is the proportion of other players investing and $\psi : [0,1] \to [0,1]$ is increasing with $\psi(0) = 0$ and $\psi(1) = 1$. In the special case of two players, these payoffs reduce to those studied in this paper.

We can define generalized belief operators for this case:

$$B_i^{f_i}(E) = \left\{ t \left| t_i \in E_i \text{ and } \sum_{n=0}^{N-1} \psi\left(\frac{n}{N-1}\right) \pi_i \left(\{t_{-i} \,|\# \{j | t_j \in E_j\} = n\} \,|t_i \right) \geq f_i(t_i) \right. \right\}.$$

For a vector of type dependent probability functions $f = (f_1, .., f_N)$, we can define *f-belief* and common f-belief operators as before,

$$B_*^f(E) = \cap_{i=1}^N B_i^{f_i}(E).$$

$$C^f(E) = \cap_{n=1}^{\infty} \left[B^f\right]^n (E).$$

and analogous fixed point characterizations will hold. Now Proposition 1 will continue to hold as stated for these modified operators.

This extension of proposition 1 gives a sufficient condition for a uniquely rationalizable action in symmetric games. It is also possible to give generalized belief operator characterizations of rationalizable actions in more general games, see Morris and Shin (2009).

We can now generalize the limit uniqueness sufficient conditions. A player's rank belief now gives the probability that he assigns to his payoff type being ranked kth for each k, and we can define corresponding upper and lower rank beliefs. Thus

$$\begin{aligned} \overline{r}_i\left(k|t_i\right) &= \pi_i\left(\#\left\{t_j|x_j\left(t_j\right) \leq x_i\left(t_i\right)\right\} = k|t_i\right), \\ \underline{r}_i\left(k|t_i\right) &= \pi_i\left(\#\left\{t_j|x_j\left(t_j\right) < x_i\left(t_i\right)\right\} = k|t_i\right), \end{aligned}$$

We say that rank belief of a type t_i is ε-*uniform* if

$$\frac{1}{N} - \varepsilon \leq \underline{r}_i\left(k|t_i\right) \leq \overline{r}_i\left(k|t_i\right) \leq \frac{1}{N} + \varepsilon.$$

for each $k = 1, .., N$. We write URB_ε for the set of type profiles t where both players have ε-uniform rank beliefs:

$$URB_\varepsilon = \left\{ t \left| \frac{1}{N} - \varepsilon \leq \underline{r}_i\left(n|t_i\right) \leq \overline{r}_i\left(n|t_i\right) \leq \frac{1}{N} + \varepsilon \text{ for each } i \right. \right\}.$$

We say that rank beliefs are *approximately uniform* if they are ε-uniform for some $\varepsilon \geq 0$.

Morris and Shin (2003b) noted that the global game selection in this case was the "Laplacian action", corresponding to a uniform belief over the proportion of opponents investing. Our second concept is a strict version of the Laplacian property. We say that action invest is ε-*Laplacian* for t_i if

$$x_i\left(t_i\right) \geq 1 - \frac{1}{N}\sum_{j=0}^{N-1}\psi\left(\frac{j}{N-1}\right) + \varepsilon.$$

We write L_ε for the set of type profiles for which invest is ε-strictly Laplacian for each player, so

$$L_\varepsilon = \left\{ t \left| x_i\left(t_i\right) \geq 1 - \frac{1}{N}\sum_{j=0}^{N-1}\psi\left(\frac{j}{N-1}\right) + \varepsilon \text{ for each } i \right. \right\}.$$

Proposition 6 *Invest is the uniquely rationalizable action for both players if it is ε-Laplacian for both players and there is common $(1-\varepsilon)$-belief of ε-uniform rank beliefs for some $\varepsilon \geq 0$, i.e., if*

$$t \in L_\varepsilon \cap C^{1-\varepsilon}\left(URB_\varepsilon\right).$$

Finally, if each of the N players had a type $t_i = \tau\eta + \sigma e_i$, where each e_i was independently distributed according to f, then the fat tail arguments can be easily adapted to show that there is approximate common certainty of payoffs and thus limit multiplicity if $\lim_{n\to\infty} \frac{\sigma_n^{\alpha-1}}{\tau_n^{\beta-1}} \frac{L_f(1/\sigma_n)}{L_g(1/\tau_n)} = \infty$, and common certainty of approximately uniform rank beliefs and thus limit uniqueness if $\lim_{n\to\infty} \frac{\sigma_n^{\alpha-1}}{\tau_n^{\beta-1}} \frac{L_f(1/\sigma_n)}{L_g(1/\tau_n)} \to 0$.

Two papers that use explicit statements about higher-order beliefs to give sufficient conditions for unique rationalizable actions are Morris and Shin (2012) and Morris (2014). Morris and Shin (2012) consider a particular coordination problem that arises among uninformed investors deciding whether to trade in a market with adverse selection. They say that there is "market confidence" if statements of the form "most players think that..." (k times) that the losses from informed trade are sufficiently small are true. This corresponds to a many player example of the higher-order belief characterization of Proposition 1. Morris (2014) discusses higher-order beliefs sufficient conditions for coordination when there is synchronization problem. Oyama and Takahashi (2013) use generalized belief operators as tool to prove results about robustness to incomplete information games in the sense of Kajii and Morris (1997).

4.4 Private Values and Separable Payoffs

We simplified the analysis by focussing on a private value global game rather than the more common value case. The alternative "common value" case would correspond to the case where player i's payoff is given by $y + \tau\eta$, i.e., the prior mean plus the common shock, but he still observes a signal $t_i = \tau\eta + \sigma e_i$ which contains an idiosyncratic (but now payoff irrelevant) noise. Much of the literature on global games has focussed on the common value case (e.g., Carlsson and van Damme (1993), Morris and Shin (1998) and Frankel, Morris, and Pauzner (2003)). However, the private value case (Morris and Shin (2005) and Argenziano (2008)) is as tractable and more relevant for many applications.

One complication of extending the analysis to the common value case is that, with regularly varying tails, the underlying payoff functions will no longer satisfy increasing differences, so that higher types of a player have a greater incentive to invest given the other players' action. In particular, in the fat tails case, the distribution of $\theta = y + \tau\eta$ conditional on $t_i = \tau\eta + \sigma e_i$ need not be monotone with respect to t_i.

An additional simplifying assumption we made was that we assumed that payoffs were additively separable between a component that depended on the opponent's action and a component that depended on an unknown payoff state. This additive separability allows for a tighter description of the connection

We also exploited the fact that there was additive separability between how payoffs depended on the payoff parameters and how they depended on others' actions. Extensions are possible here also, but are messy and involve tedious continuity arguments. See Morris and Shin (2009).

4.5 Asymmetric Payoffs

Like much of the applied literature, we focussed on a game which was symmetric across players. However, global game results go through with asymmetric games. One can state analogous higher-order belief properties driving global game results (relating to translation invariance) but they are not as clean. The issue is discussed in Morris and Shin (2009).

4.6 Global Games Literature

Let us briefly summarize how the results described in this note relate to the existing global games literature.

The classical exercise in the global games literature initiated by Carlsson and van Damme (1993) is to consider what happens if we fix a prior over payoff relevant states and let the size of the noise in players' conditionally independent signals of the state converge to zero. A key step in such arguments (e.g., in Carlsson and van Damme (1993) and Frankel, Morris, and Pauzner (2003)) is the assumption that the prior distribution is smooth which ensures that conditional probabilities converge uniformly, or uniformly over a compact interval. In the symmetric case, globally uniform convergence of conditional probabilities gives common certainty of rank beliefs. However, as highlighted by lemma 2, it is enough to have common certainty of rank beliefs on a compact interval including states where players actions are undominated. In this sense, we highlight in this note (using a special case) the properties of higher-order beliefs that drive results in the symmetric case.

Another exercise in the global games literature is to assume what both the prior and private signals are normally distributed and see what happens when both prior distribution and distribution of noise in private signals shrink to zero (Morris and Shin (2001) and Morris and Shin (2003a)). We discussed this case in section 4.1. In this case, one can explicitly compute rank beliefs and see how they evolve as the distributions shrink to zero at different rates. However, this paper has highlighted the fact that these results are special. Focussing on the most interesting case of fat tailed distributions, we have identified distinct properties that give limit uniqueness and limit multiplicity depending on the tail properties of the distributions.

In this sense, our results generalize some key insights in the existing literature.

Our results generalize those in the existing literature because we do not exploit monotonicity properties of the type space and thus the game is not supermodular. This is true both with general type spaces, or with one dimensional type spaces with fat-tailed distributions. The role of an order structure on types in the existing literature. In the analysis of Carlsson and van Damme (1993), Morris and Shin (2003b) and Frankel, Morris, and Pauzner (2003), there is monotonicity with respect to types in the limit (as noise goes to zero) and continuity arguments are used to provide results in the (not necessarily) monotonic type space away from the limit. In the normal models of Morris and Shin (2001) and Morris and Shin (2003a), monotonicity away from the limit is implied by the normal distribution. In highlighting the connection between general supermodular games and global games, van Zandt and Vives (2007) impose monotonicity away from the limit in general games. Mathevet (2010) imposes a stochastic dominance property even in the limit to provide global game results via a contraction argument. By contrast, results in this paper are proved without any order structure on types.

A Omitted Proofs

Proof of Lemma 4. Take any $\varepsilon > 0$. We will find $\bar{n} < \infty$ such that for all $n > \bar{n}$ and $t \in [0, \bar{t}]$,[6] $\mu_n(\cdot|y,t)$ puts at most probability ε on $(-\infty, y-\varepsilon)$ and $(y+\varepsilon, \infty)$, for which $\mu_\infty(\cdot|y,t)$ assigns zero probability. Furthermore, when $t > 2\varepsilon$, $\mu_n(\cdot|y,t)$ puts at most probabilities $1 - q(t) + \varepsilon$, ε, and $q(t) + \varepsilon$ on sets $[y-\varepsilon, y+\varepsilon]$, $[y+\varepsilon, y+t-\varepsilon]$, and $[y+t-\varepsilon, y+t+\varepsilon]$, respectively. Consequently $\mu_n(\cdot|y,t)$ puts at least probabilities $1 - q(t) - 4\varepsilon$ and $q(t) - 4\varepsilon$ on sets $[y-\varepsilon, y+\varepsilon]$ and $[y+t-\varepsilon, y+t+\varepsilon]$, respectively, for which $\mu_\infty(\cdot|y,t)$ assigns probabilities $1-q(t)$ and $q(t)$, respectively.

Now, $F(\varepsilon/\sigma_n)$ and $G(\varepsilon/\tau_n)$ converge to 1, while $F(-\varepsilon/\sigma_n) = 1 - F(\varepsilon/\sigma_n)$ and $G(-\varepsilon/\tau_n) = 1 - G(\varepsilon/\tau_n)$ converge to 0. Hence, there exists $\bar{n}_1 < \infty$ such that

$$\varepsilon > \max\left\{\frac{F(-\varepsilon/\sigma_n)}{F(\varepsilon/\sigma_n) - F(-\varepsilon/\sigma_n)}, \frac{G(-\varepsilon/\tau_n)}{G(\varepsilon/\tau_n) - G(-\varepsilon/\tau_n)}\right\} \tag{12}$$

for all $n > \bar{n}_1$.

Consider any $t \geq 0$ and any $n > \bar{n}_1$. We start by showing that $\mu_n(\cdot|y,t)$ puts at most probability ε on

[6] Since f and g are symmetric, it suffices to consider the case $z \geq 0$.

$(-\infty, y-\varepsilon)$:

$$\begin{aligned}\mu_n(\theta < y-\varepsilon|y,t) &\leq I_n(t+\varepsilon,\infty)/I_n(t-\varepsilon,t+\varepsilon)\\ &\leq \frac{\tau_n G(-\varepsilon/\tau_n) f((t+\varepsilon)/\sigma_n)}{\tau_n(G(\varepsilon/\tau_n)-G(-\varepsilon/\tau_n)) f((t+\varepsilon)/\sigma_n)}\\ &= \frac{G(-\varepsilon/\tau_n)}{G(\varepsilon/\tau_n)-G(-\varepsilon/\tau_n)} < \varepsilon.\end{aligned}$$

Here, the first inequality is by definition and the monotonicity of I_n, while the last inequality is by (12). To see the second inequality, note that

$$I_n(t+\varepsilon,\infty) = \int_{-\infty}^{-\varepsilon/\tau_n} f((t-\tau_n\eta)/\sigma_n)\, g(\eta)\, \tau_n d\eta \leq \tau_n G(-\varepsilon/\tau_n) f((t+\varepsilon)/\sigma_n),$$

and (similarly)

$$I_n(t-\varepsilon,t+\varepsilon) \geq (G(\varepsilon/\tau_n)-G(-\varepsilon/\tau_n))\, \tau_n f\left(\frac{t+\varepsilon}{\sigma_n}\right).$$

Using symmetric arguments, we next conclude that $\mu_n(\cdot|y,t)$ puts at most probability ε on $(y+t+\varepsilon,\infty)$:

$$\begin{aligned}\mu_n(\theta > y+t+\varepsilon|y,t) &\leq I_n(-\infty,-\varepsilon)/I_n(-\varepsilon,\varepsilon)\\ &\leq \frac{\sigma_n F(-\varepsilon/\sigma_n)\, g((t+\varepsilon)/\tau_n)}{\sigma_n(F(\varepsilon/\sigma_n)-F(-\varepsilon/\sigma_n))\, g((t+\varepsilon)/\tau_n)} < \varepsilon.\end{aligned}$$

When $t \leq 2\varepsilon$, this is sufficient because it shows that both $\mu_n(\cdot|y,t)$ and $\mu_\infty(\cdot|y,t)$ put at least probability $1-2\varepsilon$ on $[y-3\varepsilon, y+3\varepsilon]$.

Now consider any $t > 2\varepsilon$. For this case, we will further show that $\mu_n(\cdot|y,t)$ and $\mu_\infty(\cdot|y,t)$ assign similar probabilities on sets $[y-\varepsilon,y+\varepsilon]$, $[y+\varepsilon,y+t-\varepsilon]$, and $[y+t-\varepsilon,y+t+\varepsilon]$. Firstly, for any $n > \bar{n}_1$,

$$\begin{aligned}\mu_n(y+\varepsilon < \theta < y+t-\varepsilon|y,t) &\leq I_n(\varepsilon,t-\varepsilon)/I_n(-\varepsilon,\varepsilon)\\ &\leq \frac{\sigma_n(1-F(\varepsilon/\sigma_n))\, g(\varepsilon/\tau_n)}{\sigma_n(F(\varepsilon/\sigma_n)-F(-\varepsilon/\sigma_n))\, g((t+\varepsilon)/\tau_n)}\\ &= \frac{F(-\varepsilon/\sigma_n)}{F(\varepsilon/\sigma_n)-F(-\varepsilon/\sigma_n)}\frac{g(\varepsilon/\tau_n)}{g((t+\varepsilon)/\tau_n)}\\ &\leq \frac{F(-\varepsilon/\sigma_n)}{F(\varepsilon/\sigma_n)-F(-\varepsilon/\sigma_n)}\frac{g(\varepsilon/\tau_n)}{g((\bar{t}+\varepsilon)/\tau_n)}.\end{aligned}$$

Here, the equality is by $1-F(\varepsilon/\sigma_n) = F(-\varepsilon/\sigma_n)$, and the last inequality is by monotonicity of g. To see the inequality on the second line, note that

$$I_n(\varepsilon,t-\varepsilon) = \int_{\varepsilon}^{t-\varepsilon} f(z/\sigma_n)\, g((t-z)/\tau_n)\, dz \leq g(\varepsilon/\tau_n)\int_{\varepsilon}^{t-\varepsilon} f(z/\sigma_n)\, dz;$$

the lower bound for $I_n(-\varepsilon,\varepsilon)$ has been established in the previous case. Now, since g has regularly-varying tails, $g(\varepsilon/\tau_n)/g((\bar{t}+\varepsilon)/\tau_n) \to (\varepsilon/(\bar{t}+\varepsilon))^{-\beta} < \infty$. Hence, there exists $\bar{n}_2 > \bar{n}_1$ such that

$$\frac{F(-\varepsilon/\sigma_n)}{F(\varepsilon/\sigma_n)-F(-\varepsilon/\sigma_n)}\frac{g(\varepsilon/\tau_n)}{g((\bar{t}+\varepsilon)/\tau_n)} < \varepsilon \qquad (\forall n > \bar{n}_2).$$

Therefore,

$$\mu_n\left(y+\varepsilon<\theta<y+t-\varepsilon|y,t\right)<\varepsilon \qquad \left(\forall n>\bar{n}_2, t\in[2\varepsilon,\bar{t}]\right).$$

(Note that $\bar{n}_2$ does not depend on t.)

Bounding the probabilities of the sets $[y-\varepsilon,y+\varepsilon]$ and $[y+t-\varepsilon,y+t+\varepsilon]$ requires more subtle arguments. To this end, we construct sequences $e_n=\sigma_n^{-\gamma}$ and $\eta_n=\tau_n^{-\gamma}$ for some γ with $\max\{1/\alpha,1/\beta\}<\gamma<1$. Since $(\sigma_n,\tau_n)\to(0,0)$, one can take $e_n>\bar{e}$ and $\eta_n>\bar{\eta}$ without loss of generality. Moreover, $\sigma_n e_n\to 0$, $\tau_n\eta_n\to 0$, $e_n\to\infty$, and $\eta_n\to\infty$, and we consider only $\sigma_n e_n<\varepsilon$ and $\tau_n\eta_n<\varepsilon$. Now,

$$\mu_n\left(y+t-\varepsilon<\theta<y+t+\varepsilon|y,t\right)=I_n\left(-\varepsilon,\varepsilon\right)/I_n\left(-\infty,\infty\right).$$

Towards finding an upper bound, we first obtain the following tighter lower bound for $I_n\left(-\infty,\infty\right)$:

$$\begin{aligned} I_n\left(-\infty,\infty\right) &> I_n\left(-\sigma_n e_n,\sigma_n e_n\right)+I_n\left(t-\tau_n\eta_n,t+\tau_n\eta_n\right)\\ &\geq \left(F\left(e_n\right)-F\left(-e_n\right)\right)\sigma_n g\left(\frac{t+\sigma_n e_n}{\tau_n}\right)+\left(G\left(\eta_n\right)-G\left(-\eta_n\right)\right)\tau_n f\left(\frac{t+\tau_n\eta_n}{\sigma_n}\right)\equiv D_n\left(t\right).\end{aligned}$$

We next find the following upper bounds:

$$\begin{aligned} I_n\left(-\varepsilon,-\sigma_n e_n\right) &\leq \left(F\left(-e_n\right)-F\left(-\varepsilon/\sigma_n\right)\right)\sigma_n g\left(\frac{t+\sigma_n e_n}{\tau_n}\right)\leq F\left(-e_n\right)\sigma_n g\left(\frac{t+\sigma_n e_n}{\tau_n}\right);\\ I_n\left(\sigma_n e_n,\varepsilon\right) &\leq F\left(-e_n\right)\sigma_n g\left(\frac{t-\varepsilon}{\tau_n}\right);\\ I_n\left(-\sigma_n e_n,\sigma_n e_n\right) &\leq \left(F\left(e_n\right)-F\left(-e_n\right)\right)\sigma_n g\left(\frac{t-\sigma_n e_n}{\tau_n}\right).\end{aligned}$$

Combining with the lower bound above, we further observe that

$$\begin{aligned} I_n\left(-\varepsilon,-\sigma_n e_n\right)/D_n\left(t\right) &\leq \frac{F\left(-e_n\right)}{F\left(e_n\right)-F\left(-e_n\right)};\\ I_n\left(\sigma_n e_n,\varepsilon\right)/D_n\left(t\right) &\leq \frac{F\left(-e_n\right)}{F\left(e_n\right)-F\left(-e_n\right)}\frac{g\left(\frac{t-\varepsilon}{\tau_n}\right)}{g\left(\frac{t+\sigma_n e_n}{\tau_n}\right)};\\ I_n\left(-\sigma_n e_n,\sigma_n e_n\right)/D_n\left(t\right) &\leq \frac{\left(F\left(e_n\right)-F\left(-e_n\right)\right)\sigma_n g\left(\frac{t-\sigma_n e_n}{\tau_n}\right)}{\left(F\left(e_n\right)-F\left(-e_n\right)\right)\sigma_n g\left(\frac{t+\sigma_n e_n}{\tau_n}\right)+\left(G\left(\eta_n\right)-G\left(-\eta_n\right)\right)\tau_n f\left(\frac{t+\tau_n\eta_n}{\sigma_n}\right)}.\end{aligned}$$

Now, clearly, the upper bound for $I_n\left(-\varepsilon,-\sigma_n e_n\right)/D_n\left(t\right)$ converges to 0. Since $g\left(\frac{t-\varepsilon}{\tau_n}\right)/g\left(\frac{t+\sigma_n e_n}{\tau_n}\right)\to\left(\frac{t-\varepsilon}{t}\right)^{-\beta}<2^{\beta}$, the upper bound for $I_n\left(\sigma_n e_n,\varepsilon\right)/D_n\left(t\right)$ also goes to 0. Finally, since $g\left(\frac{t-\sigma_n e_n}{\tau_n}\right)/g\left(\frac{t}{\tau_n}\right)$, $g\left(\frac{t+\sigma_n e_n}{\tau_n}\right)/g\left(\frac{t}{\tau_n}\right)$, and $f\left(\frac{t+\tau_n\eta_n}{\sigma_n}\right)/f\left(\frac{t}{\sigma_n}\right)$ all uniformly converge to 1, the upper bound for $I_n\left(-\sigma_n e_n,\sigma_n e_n\right)/D_n$ uniformly converges to $q\left(t\right)$. Here, all the limits are taken uniformly over $t\in[2\varepsilon,\bar{t}]$, and hence there exists $\bar{n}_3$ such that

$$\mu_n\left(y+t-\varepsilon<\theta<y+t+\varepsilon|y,t\right)\leq\frac{I_n\left(-\varepsilon,\varepsilon\right)}{D_n\left(t\right)}<q\left(t\right)+\varepsilon \qquad \left(\forall n>\bar{n}_3, t\in[2\varepsilon,\bar{t}]\right).$$

It is important to observe that $\bar{n}_3$ does not depend on t. Similarly, one can find $\bar{n}_4$ such that

$$\mu_n \left(y - \varepsilon < \theta < y + \varepsilon | y, t\right) < 1 - q\left(t\right) + \varepsilon \qquad \left(\forall n > \bar{n}_3, t \in [2\varepsilon, \bar{t}]\right).$$

Setting $\bar{n} = \max\left\{\bar{n}_1, \bar{n}_2, \bar{n}_3, \bar{n}_4\right\}$, one concludes the proof. ■

Proof of Lemma 5. For every event Θ, define $r_{n,\Theta} : \mathbb{R} \to [0,1]$ by

$$r_{n,\Theta}\left(t\right) \equiv \int_{\Theta} F\left(\left(y + t - \theta\right)/\sigma_n\right) d\mu_n\left(\theta | y, t\right).$$

Note that $\Pr\left(x_j < x_i | x_i, \sigma_n, \tau_n, \theta\right) = F\left(\left(x_i - \theta\right)/\sigma_n\right)$, and hence

$$r_{y,\sigma_n,\tau_n}\left(t\right) = \int F\left(\left(y + t - \theta\right)/\sigma_n\right) d\mu_n\left(\theta | y, t\right) = r_{n,(-\infty,\infty)}\left(t\right).$$

Towards computing the limit of $r_{n,(-\infty,\infty)}$, we fix $\varepsilon \in (0, \underline{z})$ and partition $(-\infty, \infty)$ into intervals

$$\begin{aligned} \Theta_1 &= \mathbb{R} \backslash \left(y + t - \varepsilon, y + t + \varepsilon\right), \\ \Theta_{2,n} &= \left(y + t - \varepsilon, y + t - \sigma_n e_n\right) \cup \left(y + t + \sigma_n e_n, y + t + \varepsilon\right), \\ \Theta_{3,n} &= \left[y + t - \sigma_n e_n, y + t + \sigma_n e_n\right], \end{aligned}$$

using the notation in the proof of Lemma 4.

We start with Θ_1. Define $F_\infty\left(\theta | y, t\right) \equiv \lim_n F\left(\left(y + t - \theta\right)/\sigma_n\right)$ and observe that it is 1 when $y + t > \theta$ and 0 when $y + t < \theta$. Since $F_\infty\left(\theta | y, t\right)$ is continuous on Θ_1,

$$\lim_{n\to\infty} r_{n,\Theta_1}\left(t\right) = \int_{\Theta_1} F_\infty\left(\theta | y, t\right) d\mu_\infty\left(\theta | y, t\right) = F_\infty\left(y | y, t\right)\left(1 - q\left(t\right)\right).$$

For $\Theta_{2,n}$, since $F\left(\cdot | y, t\right) \in [0,1]$, we write

$$0 \leq r_{n,\Theta_{2,n}}\left(t\right) \leq \mu_n\left(\Theta_{2,n} | y, t\right)$$

and observe from the proof of Lemma 4 that $\mu_n\left(\Theta_{2,n} | y, t\right)$ uniformly converges to zero. Finally, we obtain bounds for $r_{n,\Theta_{3,n}}\left(t\right)$. For the upper bound, we write

$$\begin{aligned} r_{n,\Theta_{3,n}}\left(t\right) &\leq \int_{-\sigma_n e_n}^{\sigma_n e_n} F\left(z/\sigma_n\right) f\left(z/\sigma_n\right) g\left(\left(t - z\right)/\tau_n\right) dz / D_n\left(t\right) \\ &\leq g\left(\frac{t - \sigma_n e_n}{\tau_n}\right) \int_{-\sigma_n e_n}^{\sigma_n e_n} F\left(z/\sigma_n\right) f\left(z/\sigma_n\right) dz / D_n\left(t\right) \\ &= \frac{1}{2}\left(F\left(e_n\right) - F\left(-e_n\right)\right) g\left(\frac{t - \sigma_n e_n}{\tau_n}\right) / D_n\left(t\right). \end{aligned}$$

Similarly, for the lower bound, we write

$$r_{n,\Theta_{3,n}}\left(t\right) \geq \frac{1}{2}\left(F\left(e_n\right) - F\left(-e_n\right)\right) g\left(\frac{t + \sigma_n e_n}{\tau_n}\right) / D_n\left(t\right).$$

As in the proof of Lemma 4, both bounds uniformly converge to $q(t)/2$. Therefore,

$$r_{y,\sigma_n,\tau_n}(t) = r_{n,\Theta_1}(t) + r_{n,\Theta_{2,n}}(t) + r_{n,\Theta_{3,n}}(t)$$

uniformly converges to

$$\begin{aligned} r_y(t) &= F_\infty(y|y,t)(1-q(t)) + q(t)/2 \\ &= \begin{cases} 1-q(t)/2 & \text{if } t>0 \\ q(t)/2 & \text{if } t<0. \end{cases} \end{aligned}$$

■

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