Timing Decisions in Organizations: Communication and Authority in a Dynamic Environment*

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Abstract

This paper develops a theory of how organizations make timing decisions. We consider a problem in which an uninformed principal decides when to exercise an option and needs to rely on the information of an informed but biased agent. This problem is common in firms: examples include headquarters deciding when to shut down an underperforming division, drill an oil well, or launch a new product. We first study centralized decision-making, when the principal retains authority and communicates with the agent. We show that equilibria are different from those in the static "cheap talk" setting. When the agent is biased towards late exercise, full communication of information often occurs, but the decision is inefficiently delayed. Conversely, when the agent favors early exercise, communication is partial, while the decision is either unbiased or delayed. Next, we consider delegation as an alternative to centralized decision-making with communication. If the agent favors late exercise, delegation is always weakly inferior. If the agent favors early exercise, delegation is optimal if the bias is low. Thus, it is never optimal to delegate decisions such as plant closures, but may be optimal to delegate decisions such as product launches.

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1 Introduction

Many decisions in organizations deal with the optimal timing of taking a certain action. Because information in organizations is dispersed, the decision-maker needs to rely on the information of his better-informed subordinates who, however, may have conflicting preferences. Consider the following two examples of such settings. 1) In a typical hierarchical firm, top executives may be less informed than the product manager about the optimal timing of the launch of a new product. It would not be surprising for an empire-building product manager to be biased in favor of an earlier launch. 2) The CEO of a multinational corporation is contemplating when to shut down a plant in a struggling economic region. While the local plant manager is better informed about the prospects of the plant, he may be biased towards a later shutdown due to personal costs of relocation.

These examples share a common theme. An uninformed principal faces an optimal stopping-time problem (when to exercise a real option). An agent is better informed than the principal but is biased towards earlier or later option exercise. In this paper, we study how organizations make timing decisions in such a setting. We first examine the effectiveness of centralized decision-making, where the principal retains formal authority over the decision and gets information via communication with the agent ("cheap talk"). We next compare this with decentralized decision-making, where the principal delegates the decision to the agent, and study the optimal allocation of authority. Since most decisions that organizations make can be delayed and thus have option-like features, our analysis of pure timing decisions is relevant for organizational design more generally.

We show that the economics underlying this problem are quite different from those when the decision is static rather than dynamic, and the decision variable is scale of the action rather than a stopping time. In particular, there is a large asymmetry in the equilibrium properties of communication and decision-making and the optimal allocation of authority depending on the direction of the agent's bias. In the first example above, the agent is biased towards early exercise, while in the second example above, the agent is biased towards late exercise. Unlike in the static problem (e.g., Crawford and Sobel, 1982, and Dessein, 2002), the results for these two cases are not mirror images of each other. For example, within our framework, there is no benefit from delegating decisions for which the agent favors late exercise, such as plant closures, as opposed to decisions for which the agent favors early exercise, such as product launches.

Our setting combines the framework of real option exercise problems with the framework of cheap talk communication between an agent and a principal. The principal must decide when to exercise an option whose payoff depends on an unknown parameter. The agent knows the parameter, but the agent's payoff from exercise differs from the principal's due to a bias. If the principal retains formal authority over the decision, he relies on communication with the agent: At any point in time, the agent sends a message to the principal about whether or not to exercise the option. Conditional on the received message and the history of the game, the principal chooses whether to exercise or wait. Importantly, not exercising today provides an option to get advice in the future. In equilibrium, the agent's communication strategy and the principal's exercise decisions are mutually optimal, and the principal rationally updates his beliefs about the agent's private information. In most of the paper, we look for stationary equilibria in this setting.

We show that when the agent is biased towards late exercise and the bias is not too high, there is often an equilibrium with full revelation of information. However, the equilibrium timing of the decision always involves delay relative to the principal's preferences. This is different from the static cheap talk setting of Crawford and Sobel (1982), where information is only partially revealed but the decision is conditionally optimal from the principal's standpoint. In contrast, when the agent is biased towards early exercise, all equilibria have a partition structure and thus feature incomplete revelation of information. Conditional on this incomplete information, the equilibrium exercise times are either unbiased or delayed from the principal's standpoint, despite the agent's bias towards early exercise.

The intuition for these strikingly different results for the two directions of the agent's bias lies in the nature of time as a decision variable. While the principal always has the choice to exercise at a point later than the present, he cannot do the reverse, i.e., exercise at a point earlier than the present. If the agent is biased towards late exercise, she can withhold information and reveal it later, exactly at the point where she finds it optimal to exercise the option. When the agent with a late exercise bias recommends to exercise, the principal learns that it is too late to do so and is tempted to go back in time and exercise the option in the past. This, however, is not feasible, and hence the principal finds it optimal to follow the agent's recommendation. Knowing that, the agent communicates honestly, but communication occurs with delay. When the principal chooses whether to wait for the agent's recommendation to exercise, he trades off the value of information against the cost of delay. In our stationary setting, the principal always finds it optimal to wait for the agent's recommendation provided that the bias is not too big, and hence full revelation of information occurs. In our non-stationary setting, the principal waits for the agent's recommendation up to a certain cutoff, and hence full revelation of information occurs up to a cutoff. Conversely, if the agent is biased towards early exercise, she does not benefit from withholding information, but when she discloses it, the principal can always postpone exercise if it is not in his best interest. Thus, only partial information revelation is possible.

These results have implications for the informativeness and timeliness of option exercise decisions in organizations where the principal has formal authority. First, other things equal, the agent's information is likely to explain more variation in the timing of option exercise for decisions with a late exercise bias (e.g., shutting down a plant) than for decisions with an early exercise bias (e.g., launching a new product or making an acquisition). Second, decisions with a late exercise bias are always delayed relative to the optimal exercise time from the principal's perspective. In contrast, the timing of decisions with an early exercise bias is on average unbiased or delayed.

The asymmetric nature of time also has important implications for the optimal allocation of authority in organizations. In particular, we examine the principal's choice between delegating decision-making rights to the agent and retaining authority and communicating with the agent – the problem studied by Dessein (2002) in the context of static decisions. We show that if the agent is biased towards late exercise, as in the case of a plant closure, the principal is always weakly better off keeping formal authority and communicating with the agent, rather than delegating the decision to the agent. This preference is strict in our non-stationary setting. This result is different from the result for static decisions, where delegation is optimal if the agent's bias is sufficiently small (Dessein, 2002). Intuitively, the inability to go back in time and act on the information before it is received allows the principal to commit to follow the recommendation of the agent, i.e., to exercise exactly when the agent recommends to exercise. This commitment ability makes communication sufficiently effective, so that delegation has no further benefit. In fact, we show that the communication equilibrium in this case coincides with the solution under the optimal contract with commitment, and hence the ability to commit to any decision rule does not improve the principal's payoff.

In contrast, if the agent is biased towards early exercise, as in the case of a product launch, delegation is optimal for the principal if the agent's bias is not too high. Intuitively, in this case, if the agent recommends to exercise the option at her most preferred time, the principal is tempted to delay the decision. Unlike changing past decisions, changing future decisions is possible, and hence time does not have valuable built-in commitment. Thus, communication is not as efficient as in the case of a late exercise bias. As a consequence, delegation can now be optimal because it allows for more effective use of the agent's private information. The trade-off between information and bias suggests that delegation is superior when the agent's bias is sufficiently small, similar to the argument for static decisions (Dessein, 2002).

We next allow the principal to time the delegation decision strategically, i.e., to choose the optimal timing of delegating authority to the agent. When the agent is biased towards late exercise, the principal finds it optimal to retain authority forever: The principal's built-in commitment power due to his inability to go back in time makes communication effective and eliminates the need for delegation. In contrast, when the agent is biased towards early exercise, the principal finds it optimal to delegate authority to the agent at some point in time, and delegation occurs later when the agent's bias is higher. This result further emphasizes that the direction of the agent's bias is the main driver of the allocation of authority for timing decisions. This is different from static decisions, like choosing the scale of the project, where the key drivers of the allocation of authority are the magnitude of the agent's bias and the importance of her private information.

We also study the comparative statics of the communication equilibrium with respect to the parameters of the stochastic environment. We show that in settings in which the agent is biased towards early exercise, an increase in volatility or in the growth rate of the option payoff, as well as a decrease in the discount rate, lead to less information being revealed in equilibrium. Intuitively, these changes increase the value of the option to delay exercise and thereby effectively increase the conflict of interest between the principal and the agent with an early exercise bias. Finally, we show that given the same absolute bias, the principal is better off with an agent who is biased towards late exercise.

The paper proceeds as follows. The remainder of this section discusses the related literature. Section 2 describes the setup of the model and solves for the benchmark case of full information. Section 3 provides the analysis of the main model of communication under asymmetric information. Section 4 examines the delegation problem. Section 5 considers comparative statics and other implications. Section 6 shows the robustness of the results to several alternative formulations of the model. Finally, Section 7 concludes.

Related literature

Our paper is related to the literature that analyzes decision-making in the presence of an informed but biased expert. The seminal paper in this literature is Crawford and Sobel (1982), who consider a "cheap talk" setting where the expert sends a message to the decision-maker and the decision-maker cannot commit to the way he reacts to the expert's messages. Our paper differs from Crawford and Sobel (1982) in that communication between the expert and the decision-maker is dynamic and concerns the timing of option exercise, rather than a static decision, such as choosing the scale of a project. To our knowledge, ours is the first paper that studies the option exercise problem in a cheap talk setting. Surprisingly, even though there is no flow of additional private information to the agent, equilibria differ conceptually from the ones in Crawford and Sobel (1982).

By studying the choice between communication and delegation, our paper contributes to the literature on authority in organizations (e.g., Holmstrom, 1984; Aghion and Tirole, 1997; Dessein, 2002; Alonso and Matouschek, 2008). Gibbons, Matouschek, and Roberts (2013), Bolton and Dewatripont (2013), and Garicano and Rayo (2014) provide comprehensive reviews of this literature. Unlike Crawford and Sobel (1982), where the principal has no commitment power, the papers in this literature allow the principal to have some degree of commitment, although most of them rule out contingent transfers to the agent. Our paper is most closely related to Dessein (2002), who assumes that the principal can commit to delegate full decision-making authority to the agent. Dessein (2002) studies the principal's choice between delegating the decision and communicating with the agent via cheap talk to make the decision himself, and shows that delegation dominates communication if the agent's bias is not too large. Relatedly, Harris and Raviv (2005, 2008) and Chakraborty and Yilmaz (2013) analyze the optimality of delegation in settings with two-sided private information. Alonso, Dessein, and Matouschek (2008, 2014) and Rantakari (2008) compare centralized and decentralized decision-making in a multidivisional organization that faces a trade-off between adapting divisions' decisions to local conditions and coordinating decisions across divisions.[1] Our paper contributes to this literature by studying delegation of timing decisions and showing that unlike in static settings, the optimality of delegation crucially depends on the direction of the agent's bias. In particular, unlike in the static problem, it is never optimal to delegate decisions where the agent has a delay bias.

Other papers in this literature assume that the principal can commit to a decision rule and thus focus on a partial form of delegation: the principal offers the agent a set of decisions from which the agent can choose her preferred one. These papers include Holmstrom (1984), Melumad and Shibano (1991), Alonso and Matouschek (2008), and Goltsman et al. (2009), among others. In Baker, Gibbons, and Murphy (1999) and Alonso and Matouschek (2007), the principal's commitment power arises endogenously through relational contracts. Guo (2014) studies the optimal mechanism without transfers in an experimentation setting where the agent prefers to experiment longer than the principal.[2] The optimal contract in her paper is time-consistent but becomes time-inconsistent if the agent prefers to experiment less than the principal, which is related to the asymmetry of our results in the direction of the agent's bias. Our paper differs from this literature because it focuses on the principal's choice between simple delegation and keeping the control rights and communicating with the agent. We derive the optimal mechanism under commitment as an intermediate result to study the role of delegation in organizations.

Several papers analyze dynamic extensions of Crawford and Sobel (1982). In Sobel (1985), Benabou and Laroque (1992), and Morris (2001), the advisor's preferences are unknown and her messages in prior periods affect her reputation with the decision-maker.[3] Aumann and Hart (2003),

[1] See also Dessein, Garicano, and Gertner (2010) and Friebel and Raith (2010). Dessein and Santos (2006) study the benefits of specialization in the context of a similar trade-off, but do not analyze strategic communication.

[2] Halac, Kartik, and Liu (2013) also analyze optimal dynamic contracts in an experimentation problem, but in a different setting and allowing for transfers.

[3] Ottaviani and Sorensen (2006a,b) study a single-period reputational cheap talk setting, where the expert is concerned about appearing well-informed. Boot, Milbourn, and Thakor (2005) compare delegation and centralization when the agent's reputational concerns can distort her recommendations on whether to accept the project.

Krishna and Morgan (2004), Goltsman et al. (2009), and Golosov et al. (2014) consider settings with persistent private information where the principal actively participates in communication by either sending messages himself or taking an action following each message of the advisor. Our paper differs from this literature because of the dynamic nature of the decision problem: the decision variable is the timing of option exercise, rather than a static variable. The inability to go back in time creates an implicit commitment device for the principal to follow the advisor's recommendations and thereby improves communication, a feature not present in prior literature.

Finally, our paper is related to the literature on option exercise in the presence of agency problems. Grenadier and Wang (2005), Gryglewicz and Hartman-Glaser (2013), and Kruse and Strack (2013) study such settings but assume that the principal can commit to contracts and make contingent transfers to the agent, which makes the problem conceptually different from ours. Several papers study signaling through option exercise.[4] They assume that the decision-maker is informed, while in our setting the decision-maker is uninformed.

2 Model setup

A firm (or an organization, more generally) has a project and needs to decide on the optimal time to implement it. There are two players, the uninformed party (principal, P) and the informed party (agent, A). Both parties are risk-neutral and have the same discount rate $r > 0$. Time is continuous and indexed by $t \in [0, \infty)$. The persistent type θ is drawn and learned by the agent at the initial date $t = 0$. The principal does not know θ. It is common knowledge that θ is a random draw from the uniform distribution over $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$, where $0 \leq \underline{\theta} < \bar{\theta}$. Without loss of generality, we normalize $\bar{\theta} = 1$. For much of the paper, we also assume $\underline{\theta} = 0$.

We start by considering the exercise of a call option. We will refer to it as the option to invest, but it can capture any perpetual American call option, such as the option to go public or the option to launch a new generation of the product. We also extend the analysis to a put option (e.g., if the decision is about shutting down a poorly performing division) and show that the main results continue to hold (see Section 6.2 for details).

The exercise at time t generates the payoff to the principal of $\theta X(t) - I$, where $I > 0$ is the exercise price (the investment cost), and $X(t)$ follows geometric Brownian motion with drift μ and volatility σ:

$$dX(t) = \mu X(t)\, dt + \sigma X(t)\, dB(t), \tag{1}$$

where $\sigma > 0$, $r > \mu$, and $dB(t)$ is the increment of a standard Wiener process. The starting point

[4] Grenadier and Malenko (2011), Morellec and Schuerhoff (2011), Bustamante (2012), Grenadier, Malenko, and Strebulaev (2013).

$X(0)$ is low enough, so that immediate exercise does not happen. Process $X(t)$, $t \geq 0$ is observable by both the principal and the agent. While the agent knows θ, she is biased. Specifically, upon exercise, the agent receives the payoff of $\theta X(t) - I + b$, where $b \neq 0$ is the commonly known bias of the agent. Positive bias $b > 0$ means that the agent is biased in the direction of early exercise: her personal exercise price $(I - b)$ is lower than the principal's (I), so her most preferred timing of exercise is earlier than the principal's for any θ. In contrast, negative bias $b < 0$ means that the agent is biased in the direction of late exercise. These preferences can be viewed as reduced-form implications of an existing revenue-sharing agreement.[5] An alternative way to model the conflict of interest between the agent and the principal is to assume that $b = 0$ but the players discount the future using different discount rates. A bias towards early exercise corresponds to the agent being more impatient than the principal, $r_A > r_P$, and vice versa. We have analyzed the setting with different discount rates and shown that the results are identical to those in the bias setting (see Section 6.1 for details). In our basic model, we focus on the bias setting ($b \neq 0$, $r_A = r_P = r$) to make our setup similar to the cheap talk literature.

The principal has formal authority over when to exercise the option. We adopt an incomplete contracting approach by assuming that the timing of the exercise cannot be contracted upon. Furthermore, the organization is assumed to have a resource, controlled by the principal, which is critical for the implementation of the project. This resource is the reason why the agent cannot implement the project without the principal's approval. First, in Section 3, we consider the advising setting, where the principal has no commitment power and can only rely on informal "cheap talk" communication with the agent. This problem is the option exercise analogue of Crawford and Sobel's (1982) cheap talk model. Then, in Section 4, we relax this assumption by allowing the principal to grant the agent full authority over the exercise of the option. This problem is the option exercise analogue of Dessein's (2002) analysis on authority and communication. As an intermediate result, in Section 4.1, we derive the optimal mechanism if the principal could commit to any decision rule.

As an example, consider an oil-producing firm that owns an oil well and needs to decide on the optimal time to drill it. The publicly observable oil price process is represented by $X(t)$. The top management of the firm has formal authority over the decision to drill. The regional manager has private information about how much oil the well contains (θ), which stems from her local knowledge and prior experience with neighborhood wells. The firm cannot simply sell the oil well to the regional manager because of its resources, such as human capital and existing relationships

[5] For example, suppose that the principal supplies financial capital $\hat{I}$, the agent supplies human capital ("effort") valued at $\hat{e}$, and the principal and the agent hold fractions α_P and α_A of equity of the realized value from the project. Then, at exercise, the principal's (agent's) expected payoff is $\alpha_P \theta X(t) - \hat{I}$ $(\alpha_A \theta X(t) - \hat{e})$. This is analogous to the specification in the model with $I = \frac{\hat{I}}{\alpha_P}$ and $b = \frac{\hat{I}}{\alpha_P} - \frac{\hat{e}}{\alpha_A}$.

with suppliers. Depending on its ability and willingness to delegate, the top management may assign the right to decide on the timing of drilling to the regional manager. If the top management is not willing or unable to commit to delegate, it will decide on the timing of drilling itself.

For now, assume that authority is not contractible. The timing is as follows. At each time t, knowing the state of nature $\theta \in \Theta$ and the history of the game $\mathcal{H}_t$, the agent decides on a message $m(t) \in M$ to send to the principal, where M is a set of messages. At each t, the principal decides whether to exercise the option or not, given $\mathcal{H}_t$ and the current message $m(t)$. If the principal exercises the option, the game ends. If the principal does not exercise the option, the game continues. Because the game ends when the principal exercises the option, we can only consider histories such that the option has not been exercised yet. Then, the history of the game at time t has two components: the sample path of the public state $X(t)$ and the history of messages of the agent. Formally, it is represented by $(\mathcal{H}_t)_{t\geq 0}$, where $\mathcal{H}_t = \{X(s), s \leq t; m(s), s < t\}$. Thus, the strategy m of the agent is a family of functions $(m_t)_{t\geq 0}$ such that for any t function m_t maps the agent's information set at time t into the message she sends to the principal: $m_t : \Theta \times \mathcal{H}_t \to M$. The strategy e of the principal is a family of functions $(e_t)_{t\geq 0}$ such that for any t function e_t maps the principal's information set at time t into the binary exercise decision: $e_t : \mathcal{H}_t \times M \to \{0, 1\}$. Here, $e_t = 1$ stands for "exercise" and $e_t = 0$ stands for "wait." Let $\tau(e) \equiv \inf\{t : e_t = 1\}$ denote the stopping time implied by strategy e of the principal. Finally, let $\mu(\theta|\mathcal{H}_t)$ and $\mu(\theta|\mathcal{H}_t, m(t))$ denote the updated probability that the principal assigns to the type of the agent being θ given the history $\mathcal{H}_t$ before and after getting message $m(t)$, respectively.

Heuristically, the timing of events over an infinitesimal time interval $[t, t + dt]$ prior to option exercise can be described as follows:

1. The nature determines the realization of X_t.

2. The agent sends message $m(t) \in M$ to the principal.

3. The principal decides whether to exercise the option or not. If the option is exercised, the principal obtains the payoff of $\theta X_t - I$, the agent obtains the payoff of $\theta X_t - I + b$, and the game ends. Otherwise, the game continues, and the nature draws $X_{t+dt} = X_t + dX_t$.

This is a dynamic game with observed actions (messages and the exercise decision) and incomplete information (type θ of the agent). We focus on equilibria in pure strategies. The equilibrium concept is Perfect Bayesian Equilibrium in Markov strategies, defined as:

Definition 1. Strategies $m^* = \{m_t^*, t \geq 0\}$ and $e^* = \{e_t^*, t \geq 0\}$, beliefs μ^*, and a message space M constitute a *Perfect Bayesian equilibrium in Markov strategies (PBEM)* if and only if:

1. For every t, $\mathcal{H}_t$, $\theta \in \Theta$, and strategy m,

$$\begin{aligned} & \mathbb{E}\left[e^{-r\tau(e^*)}\left(\theta X\left(\tau\left(e^*\right)\right) - I + b\right)|\mathcal{H}_t, \theta, \mu^*\left(\cdot|\mathcal{H}_t\right), m^*, e^*\right] \\ \geq \; & \mathbb{E}\left[e^{-r\tau(e^*)}\left(\theta X\left(\tau\left(e^*\right)\right) - I + b\right)|\mathcal{H}_t, \theta, \mu^*\left(\cdot|\mathcal{H}_t\right), m, e^*\right]. \end{aligned} \tag{2}$$

2. For every t, $\mathcal{H}_t$, $m(t) \in M$, and strategy e,

$$\begin{aligned} & \mathbb{E}\left[e^{-r\tau(e^*)}\left(\theta X\left(\tau\left(e^*\right)\right) - I\right)|\mathcal{H}_t, \mu^*\left(\cdot|\mathcal{H}_t, m\left(t\right)\right), m^*, e^*\right] \\ \geq \; & \mathbb{E}\left[e^{-r\tau(e)}\left(\theta X\left(\tau\left(e\right)\right) - I\right)|\mathcal{H}_t, \mu^*\left(\cdot|\mathcal{H}_t, m\left(t\right)\right), m^*, e\right]. \end{aligned} \tag{3}$$

3. Bayes' rule is used to update beliefs $\mu^*(\theta|\mathcal{H}_t)$ to $\mu^*(\theta|\mathcal{H}_t, m(t))$ whenever possible: For every $\mathcal{H}_t$ and $m(t) \in M$, if there exists θ such that $m_t^*(\theta, \mathcal{H}_t) = m(t)$, then for all θ

$$\mu^*\left(\theta|\mathcal{H}_t, m\left(t\right)\right) = \frac{\mu^*(\theta|\mathcal{H}_t)\mathbf{1}\{m_t^*(\theta, \mathcal{H}_t) = m\left(t\right)\}}{\int_{\underline{\theta}}^1 \mu^*(\tilde{\theta}|\mathcal{H}_t)\mathbf{1}\{m_t^*(\tilde{\theta}, \mathcal{H}_t) = m\left(t\right)\}d\tilde{\theta}}, \tag{4}$$

where $\mu^*(\theta|\mathcal{H}_0) = \frac{1}{1-\underline{\theta}}$ for $\theta \in \Theta$ and $\mu^*(\theta|\mathcal{H}_0) = 0$ for $\theta \notin \Theta$.

4. For every t, $\mathcal{H}_t$, $\theta \in \Theta$, and $m(t) \in M$,

$$m_t^*\left(\theta, \mathcal{H}_t\right) = m^*\left(\theta, X\left(t\right), \mu^*\left(\cdot|\mathcal{H}_t\right)\right); \tag{5}$$

$$e_t^*\left(\mathcal{H}_t, m\left(t\right)\right) = e^*\left(X\left(t\right), \mu^*\left(\cdot|\mathcal{H}_t, m\left(t\right)\right)\right). \tag{6}$$

The first three conditions, given by (2)–(4), are requirements of the Perfect Bayesian equilibrium. Inequalities (2) require the equilibrium strategy m^* to be sequentially optimal for the agent for any possible history $\mathcal{H}_t$ and type realization θ. Similarly, inequalities (3) require equilibrium strategy e^* to be sequentially optimal for the principal. Equation (4) requires beliefs to be updated according to Bayes' rule. Finally, conditions (5)–(6) are requirements that the equilibrium strategies and the message space are Markov.

Bayes' rule does not apply if the principal observes a message that should not be sent by any type in equilibrium. To restrict beliefs following such off-equilibrium actions, we impose another constraint:

Assumption 1. If, at any point t, the principal's belief $\mu(\theta|\mathcal{H}_t)$ and the observed message $m(t)$ are such that no type that could exist (according to the principal's belief) could possibly send message $m(t)$, then the principal's belief is unchanged: If $\{\theta : m_t^*(\theta, \mathcal{H}_t) = m(t), \mu^*(\theta|\mathcal{H}_t) > 0\} =$

$\emptyset$, then $\mu^*(\theta|\mathcal{H}_t, m(t)) = \mu^*(\theta|\mathcal{H}_t)$.

This assumption is related to a frequently imposed restriction in models with two types that if, at any point, the posterior assigns probability one to a given type, then this belief persists no matter what happens (e.g., Rubinstein, 1985; Halac, 2012). Because our model features a continuum of types, an action that no one was supposed to take may occur off equilibrium even if the belief is not degenerate. As a consequence, we impose a stronger restriction.

Let stopping time $\tau^*(\theta)$ denote the equilibrium exercise time of the option if the type is θ. In almost all standard option exercise models, the optimal exercise strategy for a perpetual American call option is a threshold: It is optimal to exercise the option at the first instant the state process $X(t)$ exceeds some critical level, which depends on the parameters of the environment. It is thus natural to look for equilibria that exhibit a similar property, formally defined as:

Definition 2. An equilibrium is a *threshold-exercise* PBEM if $\tau^*(\theta) = \inf\left\{t \geq 0 | X(t) \geq \bar{X}(\theta)\right\}$ for some $\bar{X}(\theta)$ (possibly infinite), $\theta \in \Theta$.

For any threshold-exercise equilibrium, let $\mathcal{X}$ denote the set of equilibrium exercise thresholds: $\mathcal{X} \equiv \left\{X : \exists \theta \in \Theta \text{ such that } \bar{X}(\theta) = X\right\}$. We next prove two useful auxiliary results that hold in any threshold-exercise PBEM. The first lemma shows that in any threshold-exercise PBEM, the option is exercised weakly later if the agent has less favorable information:

Lemma 1. *$\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$ for any $\theta_1, \theta_2 \in \Theta$ such that $\theta_2 \geq \theta_1$.*

Intuitively, because talk is "cheap," the agent with information θ_1 can adopt the message strategy of the agent with information $\theta_2 > \theta_1$ (and the other way around) at no cost. Thus, between choosing dynamic communication strategies that induce exercise at thresholds $\bar{X}(\theta_1)$ and $\bar{X}(\theta_2)$, the type-θ_1 agent must prefer the former, while the type-θ_2 agent must prefer the latter. This is simultaneously possible only if $\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$.

The second auxiliary result is that it is without loss of generality to reduce the message space significantly. Specifically, the next lemma shows that for any threshold-exercise equilibrium, there exists an equilibrium with a binary message space $M = \{0, 1\}$ and simple equilibrium strategies that implements the same exercise times and hence features the same payoffs of both players:

Lemma 2. *If there exists a threshold-exercise PBEM with thresholds $\bar{X}(\theta)$, then there exists an equivalent threshold-exercise PBEM with the binary message space $M = \{0, 1\}$ and the following* *strategies of the agent and the principal and beliefs of the principal:*

1. *The agent with type θ sends message $m(t) = 1$ if and only if $X(t)$ is greater or equal than threshold $\bar{X}(\theta)$:*

$$\bar{m}_t\left(\theta, X(t), \bar{\mu}(\cdot|\mathcal{H}_t)\right) = \begin{cases} 1, & \text{if } X(t) \geq \bar{X}(\theta), \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

2. *The posterior belief of the principal at any time t is that θ is distributed uniformly over $[\check{\theta}_t, \hat{\theta}_t]$ for some $\check{\theta}_t$ and $\hat{\theta}_t$ (possibly, equal).*

3. *The exercise strategy of the principal as a function of the state process and his beliefs is*

$$\bar{e}_t\left(X(t), \check{\theta}_t, \hat{\theta}_t\right) = \begin{cases} 1, & \text{if } X(t) \geq \check{X}(\check{\theta}_t, \hat{\theta}_t) \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

   for some threshold $\check{X}(\check{\theta}_t, \hat{\theta}_t)$. Function $\check{X}(\check{\theta}_t, \hat{\theta}_t)$ is such that on equilibrium path the option is exercised at the first instant when the agent sends message $m(t) = 1$, i.e., when $X(t)$ hits threshold $\bar{X}(\theta)$ for the first time.

Lemma 2 implies that it is without loss of generality to focus on equilibria of the following simple form. At any time t, the agent can send one of the two messages, 1 or 0. Message $m = 1$ can be interpreted as a recommendation of exercise, while message $m = 0$ can be interpreted as a recommendation of waiting. The agent plays a threshold strategy, recommending exercise if and only if the public state $X(t)$ is above threshold $\bar{X}(\theta)$, which depends on private information θ of the agent. The principal also plays a threshold strategy: the principal who believes that $\theta \in [\check{\theta}_t, \hat{\theta}_t]$ exercises the option if and only if $X(t)$ exceeds some threshold $\check{X}(\check{\theta}_t, \hat{\theta}_t)$. As a consequence of the agent's strategy, there is a set $\mathcal{T}$ of "informative" times, when the agent's message has information content, i.e., it affects the belief of the principal and, in turn, her exercise decision. These are instances when the state process $X(t)$ first passes a new threshold from the set of possible exercise thresholds $\mathcal{X}$. At all other times, the agent's message has no information content and does not lead the principal to update his belief. In equilibrium, each type θ of the agent recommends exercise (sends $m = 1$) at the first time when the state process $X(t)$ passes the threshold $\bar{X}(\theta)$ for the first time, and the principal exercises the option immediately.

Lemma 2 states that if there exists *some* equilibrium with the set of thresholds $\left\{\bar{X}(\theta), \theta \in \Theta\right\}$, then there exists an equilibrium of the above form with the same set of exercise thresholds. The intuition behind this result is that at each time the principal faces a binary decision: to exercise or to wait. Because the information of the agent is important only for the timing of the exercise, one can achieve the same efficiency by choosing the timing of communicating a binary message as through the richness of the message space. Therefore, message spaces that are richer than binary cannot improve the efficiency of decision making.

In what follows, we focus on threshold-exercise PBEM of the form in Lemma 2 and refer to them as simply "equilibria." When $\underline{\theta} = 0$, the problem exhibits stationarity in the following sense. Because the prior distribution of types is uniform over $[0,1]$ and the payoff structure is multiplicative, a time-t sub-game in which the posterior belief of the principal is uniform over $[0,\hat{\theta}]$ is equivalent to the game where the belief is that θ is uniform over $[0,1]$, the true type is $\frac{\theta}{\hat{\theta}}$, and the modified state process is $\tilde{X}(t) = \hat{\theta}X(t)$. Because of this scalability of the game, it is natural to restrict attention to stationary equilibria, which are formally defined as follows:

Definition 3. Suppose that $\underline{\theta} = 0$. A threshold-exercise PBEM (m^*, e^*, μ^*, M) is *stationary* if whenever posterior belief $\mu^*(\cdot|\mathcal{H}_t)$ is uniform over $[0,\hat{\theta}]$ for some $\hat{\theta} \in (0,1)$:

$$m^*(\theta, X(t), \mu^*(\cdot|\mathcal{H}_t)) = m^*\left(\frac{\theta}{\hat{\theta}}, \hat{\theta}X(t), \mu^*(\cdot|\mathcal{H}_0)\right), \quad (9)$$

$$e^*(X(t), \mu^*(\cdot|\mathcal{H}_t, m(t))) = e^*\left(\hat{\theta}X(t), \mu^*(\cdot|\mathcal{H}_0, m(t))\right), \quad (10)$$

for all $\theta \in [0,\hat{\theta}]$.

Condition (9) means that every type $\theta \in [0,\hat{\theta}]$ sends the same message when the public state is $X(t)$ and the posterior is uniform over $[0,\hat{\theta}]$ as type $\frac{\theta}{\hat{\theta}}$ when the public state is $\hat{\theta}X(t)$ and the posterior is uniform over $[0,1]$. Condition (10) means that the exercise strategy of the principal is the same when the public state is $X(t)$ and his belief is that θ is uniform over $[0,\hat{\theta}]$ as when the public state is $\hat{\theta}X(t)$ and his belief is that θ is uniform over $[0,1]$.

From now on, if $\underline{\theta} = 0$, we focus on threshold-exercise PBEM in the form stated in Lemma 2 that are stationary. We refer to these equilibria as *stationary equilibria.*

2.1 Benchmark cases

As benchmarks, we consider two simple settings: one in which the principal knows θ and the other in which the agent has formal authority to exercise the option.

2.1.1 Optimal exercise for the principal

Suppose that the principal knows θ, so communication with the agent is irrelevant. Let $V_P^*(X,\theta)$ denote the value of the option to the principal in this case, if the project's type is θ and the current value of $X(t)$ is X. According to the standard argument (e.g., Dixit and Pindyck, 1994), in the range prior to exercise, $V_P^*(X,\theta)$ solves

$$rV_P^*(X,\theta) = \mu X \frac{\partial V_P^*(X,\theta)}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 V_P^*(X,\theta)}{\partial X^2}. \tag{11}$$

Suppose that type θ exercises the option when $X(t)$ reaches threshold $X_P^*(\theta)$. Then,

$$V_P^*(X_P^*(\theta),\theta) = \theta X_P^*(\theta) - I. \tag{12}$$

Solving the differential equation (11) subject to the boundary condition (12) and condition $V_P^*(0,\theta) = 0$,[6] we obtain

$$V_P^*(X,\theta) = \begin{cases} \left(\frac{X}{X_P^*(\theta)}\right)^\beta (\theta X_P^*(\theta) - I), & \text{if } X \leq X_P^*(\theta) \\ \theta X - I, & \text{if } X > X_P^*(\theta), \end{cases} \tag{13}$$

where

$$\beta = \frac{1}{\sigma^2}\left[-\left(\mu - \frac{\sigma^2}{2}\right) + \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right] > 1 \tag{14}$$

is the positive root of the fundamental quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0$.

The optimal exercise trigger $X_P^*(\theta)$ maximizes the value of the option (13), and is given by

$$X_P^*(\theta) = \frac{\beta}{\beta-1}\frac{I}{\theta}. \tag{15}$$

2.1.2 Optimal exercise for the agent

Suppose that the agent has complete formal authority over when to exercise the option. If $b < I$, then substituting $I - b$ for I in (11)–(15), we obtain that the optimal exercise strategy for the agent is to exercise the option at the first moment when $X(t)$ is greater or equal than the threshold

$$X_A^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}. \tag{16}$$

[6] $V_P^*(0,\theta) = 0$ because $X = 0$ is an absorbing barrier: if the value of the process is zero, it will remain zero forever.

The value of the option to the agent in this case is

$$V_A^*(X,\theta) = \begin{cases} \left(\frac{X}{X_A^*(\theta)}\right)^\beta (\theta X_A^*(\theta) - I + b), & \text{if } X \leq X_A^*(\theta), \\ \theta X - I + b, & \text{if } X > X_A^*(\theta). \end{cases} \tag{17}$$

If $b \geq I$, the optimal exercise strategy for the agent is to exercise the option immediately, and the value of the option to the agent is $\theta X - I + b$.

3 Communication game

By Lemmas 1 and 2, the history of the game at time t on equilibrium path can be summarized by two cutoffs, $\check{\theta}_t$ and $\hat{\theta}_t$. Moreover, before the agent recommends to exercise, the history of the game can be summarized by a single cutoff $\hat{\theta}_t$, where $\hat{\theta}_t \equiv \sup\left\{\theta : \bar{X}(\theta) > \max_{s\leq t} X(s)\right\}$. Indeed, by Lemma 2, on equilibrium path, the principal exercises the option at the first time t with $X(t) \in \mathcal{X}$ at which the agent sends $m(t) = 1$. If the agent has not recommended exercise by time t, the principal infers that the agent's type does not exceed $\hat{\theta}_t$. Therefore, process $\hat{\theta}_t$ summarizes the belief of the principal at time t, provided that he has not deviated from his equilibrium strategy of exercising the option at the first instant $X(s) \in \mathcal{X}$ at which the agent recommends exercise.

Consider the case $\underline{\theta} = 0$, in which the problem becomes stationary. Note that if $b \geq I$, the agent prefers immediate exercise regardless of her type, and hence the principal must exercise the option at his optimal uninformed threshold

$$\bar{X}_u = \frac{\beta}{\beta - 1} 2I. \tag{18}$$

Hence, we focus on the case $b < I$. Using Lemma 1 and the stationarity condition, we conclude that any stationary equilibrium must either have partitioned exercise or continuous exercise, as explained below.

First, if the equilibrium exercise has a partition structure, such that the set of types Θ is partitioned into intervals with each interval inducing exercise at a given threshold, then stationarity implies that the set of partitions must take the form $[\omega, 1]$, $\left[\omega^2, \omega\right]$, ..., $\left[\omega^n, \omega^{n-1}\right]$, $n \in \mathbb{N}$, for some $\omega \in [0, 1)$, where $\mathbb{N}$ is the set of natural numbers. This implies that the set of exercise thresholds $\mathcal{X}$ is given by $\left\{\bar{X}, \frac{\bar{X}}{\omega}, \frac{\bar{X}}{\omega^2}, ..., \frac{\bar{X}}{\omega^n}, ...\right\}$, $n \in \mathbb{N}$, such that if $\theta \in \left(\omega^n, \omega^{n-1}\right)$, the option is exercised at threshold $\frac{\bar{X}}{\omega^{n-1}}$. We refer to an equilibrium of this form as the *ω-equilibrium.*

For ω and $\bar{X}$ to constitute an equilibrium, incentive compatibility conditions for the principal and the agent must hold. Because the problem is stationary, it is sufficient to only consider the incentive compatibility conditions for the game up to reaching the first threshold $\bar{X}$. First, consider the agent's problem. Pair $(\omega, \bar{X})$ satisfies the agent's incentive compatibility if and only if types above ω have incentives to recommend exercise ($m = 1$) at threshold $\bar{X}$ rather than to wait, whereas types below ω have incentives to recommend delay ($m = 0$). From the agent's point of view, the set of possible exercise thresholds is given by $\mathcal{X}$: The agent can induce exercise at any threshold in $\mathcal{X}$ by recommending exercise at the first instant when $X(t)$ reaches a desired point in $\mathcal{X}$, but cannot induce exercise at any point not in $\mathcal{X}$. This implies that the agent's incentive compatibility condition holds if and only if type ω is exactly indifferent between exercising the option at threshold $\bar{X}$ and at threshold $\frac{\bar{X}}{\omega}$:

$$\left(\frac{X(t)}{\bar{X}}\right)^{\beta} \left(\omega \bar{X} + b - I\right) = \left(\frac{X(t)}{\bar{X}/\omega}\right)^{\beta} \left(\omega \frac{\bar{X}}{\omega} + b - I\right), \tag{19}$$

which can be simplified to

$$\omega \bar{X} + b - I = \omega^{\beta} \left(\bar{X} + b - I\right). \tag{20}$$

Indeed, if (19) holds, then $\left(\frac{X(t)}{\bar{X}}\right)^{\beta} \left(\theta \bar{X} + b - I\right) \gtrless \left(\frac{X(t)}{\bar{X}/\omega}\right)^{\beta} \left(\theta \frac{\bar{X}}{\omega} + b - I\right)$ if $\theta \gtrless \omega$. Hence, if type ω is indifferent between exercise at threshold $\bar{X}$ and at threshold $\frac{\bar{X}}{\omega}$, then any higher type strictly prefers recommending exercise at $\bar{X}$, while any lower type strictly prefers recommending delay at $\bar{X}$. By stationarity, if (19) holds, then type ω^2 is indifferent between recommending exercise and recommending delay at threshold $\frac{\bar{X}}{\omega}$, so types in (ω^2, ω) strictly prefer recommending exercise at threshold $\frac{\bar{X}}{\omega}$, and so on. Thus, (19) is the necessary and sufficient condition for the agent's incentive compatibility condition to hold. Equation (20) implies the following relation between the first possible exercise threshold $\bar{X}$ and ω:

$$\bar{X} = Y(\omega) \equiv \frac{\left(1 - \omega^{\beta}\right)(I - b)}{\omega\left(1 - \omega^{\beta - 1}\right)}. \tag{21}$$

Next, consider the principal's problem. For ω and $\bar{X}$ to constitute an equilibrium, the principal must have incentives (1) to exercise the option immediately when she gets recommendation $m = 1$ from the agent at threshold in $\mathcal{X}$; and (2) not to exercise the option before getting the message $m = 1$. We refer to the former (latter) incentive compatibility condition as the *ex-post (ex-ante) incentive compatibility* constraint. Suppose that $X(t)$ reaches threshold $\bar{X}$ for the first time, and the principal receives recommendation $m = 1$ at that instant. By Bayes' rule, the principal updates his beliefs to θ being uniform over $[\omega, 1]$. If the principal exercises immediately, he obtains the expected payoff of $\frac{\omega+1}{2}\bar{X} - I$. If the principal delays, he expects that there will be no further informative communication in the continuation game, given the conjectured equilibrium strategy of the agent. Therefore, upon receiving recommendation $m = 1$ at threshold $\bar{X}$, the principal faces the standard perpetual call option exercise problem (e.g., Dixit and Pindyck, 1994) as if the type of the project were $\frac{\omega+1}{2}$. The solution to this problem is immediate exercise if and only if exercising at threshold $\bar{X}$ dominates waiting until $X(t)$ reaches a higher threshold $\hat{X}$ and exercising the option then for any possible $\hat{X} > \bar{X}$:

$$\bar{X} \in \arg\max_{\hat{X} \geq \bar{X}} \left(\frac{\bar{X}}{\hat{X}}\right)^{\beta} \left(\frac{\omega+1}{2}\hat{X} - I\right). \tag{22}$$

Using the fact that the unconditional maximizer of the right-hand side is $\hat{X} = \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$ and that the right-hand side is an inverted U-shaped function of $\hat{X}$, the ex-post incentive compatibility condition for the principal is equivalent to

$$Y(\omega) \geq \frac{\beta}{\beta-1}\frac{2I}{\omega+1}. \tag{23}$$

This condition has a clear intuition. It means that at the moment when the agent recommends the principal to exercise the option, it must be "too late" to delay exercise. If (23) is violated, the principal delays exercise, so the recommendation loses its responsiveness as the principal does not follow it. In contrast, if (23) holds, the principal's optimal response to getting the recommendation $m = 1$ is to exercise the option immediately. As with the incentive compatibility condition of the agent, stationarity implies that if (23) holds, then a similar condition holds for all higher thresholds in $\mathcal{X}$. The fact that constraint (23) is an inequality rather than an equality highlights the built-in asymmetric nature of time. When the agent recommends exercise to the principal, the principal can either exercise immediately or can delay, but cannot go back in time and exercise in the past, even if it is tempting to do so.

Let $V_P\left(X(t), \hat{\theta}_t; \omega\right)$ denote the expected value to the principal in the ω-equilibrium, given that the public state is $X(t)$ and the principal's belief is that θ is uniform over $[0, \hat{\theta}_t]$. In the appendix, we solve for the principal's value in closed form: if $\hat{\theta}_t = 1$,

$$V_P(X, 1; \omega) = \frac{1-\omega}{1-\omega^{\beta+1}} \left(\frac{X}{Y(\omega)}\right)^{\beta} \left(\frac{1}{2}(1+\omega)Y(\omega) - I\right), \tag{24}$$

for any $X \leq Y(\omega)$, where $Y(\omega)$ is given by (21). Using stationarity, (24) can be generalized to any $\hat{\theta}$:

$$V_P\left(X, \hat{\theta}; \omega\right) = V_P\left(\hat{\theta}X, 1; \omega\right) = \frac{1-\omega}{1-\omega^{\beta+1}} \left(\frac{X\hat{\theta}}{Y(\omega)}\right)^{\beta} \left(\frac{1}{2}(1+\omega)Y(\omega) - I\right). \tag{25}$$

The principal's ex-ante incentive compatibility constraint requires that the principal is better off waiting, rather than exercising immediately, at any time prior to receiving message $m = 1$ at $X(t) \in \mathcal{X}$:

$$V_P\left(X(t), \hat{\theta}_t; \omega\right) \geq \frac{\hat{\theta}_t}{2} X(t) - I \tag{26}$$

for any $X(t)$ and $\hat{\theta}_t = \sup\left\{\theta : \bar{X}(\theta) > \max_{s \leq t} X(s)\right\}$. By stationarity, it is sufficient to verify the ex-ante incentive compatibility constraint for $X(t) \leq \bar{X}(1) = Y(\omega)$ and beliefs equal to the prior:

$$V_P(X, 1; \omega) \geq \frac{1}{2} X - I \quad \forall X \leq Y(\omega). \tag{27}$$

This inequality states that at any point up to threshold $Y(\omega)$, the principal is better off waiting than exercising the option. If (27) does not hold for some $X \leq Y(\omega)$, then the principal is better off exercising the option when $X(t)$ reaches X, rather than waiting for informative recommendations from the agent. If (27) holds, then the principal does not exercise the option prior to reaching threshold $Y(\omega)$. By stationarity, if (27) holds, then a similar condition holds for the n^{th} partition for any $n \in \mathbb{N}$, which implies that (27) and (26) are equivalent. To summarize, a ω-equilibrium exists if and only if conditions (21), (23), and (27) are satisfied.

So far, we have considered only partitioned equilibria, which satisfy $\bar{X}(\theta) = \bar{X}(1)$ for any $\theta \in (\omega, 1]$. In addition, there may be equilibria with $\bar{X}(\theta) \neq \bar{X}(1)$ for all $\theta < 1$. Then, by stationarity of the problem, $\bar{X}(\theta) = \bar{X}(1)/\theta$ for any θ. We refer to such equilibria, if they exist, as *equilibria with continuous exercise*, and analyze them below.

3.1 Preference for late exercise

Suppose that the agent is biased in the direction of late exercise, $b < 0$. We start with the stationary case $\underline{\theta} = 0$. First, consider equilibria with continuous exercise. By stationarity, $\mathcal{X} = \{X : X \geq \underline{X}\}$ for some $\underline{X}$. Incentive compatibility of the agent of type θ can be written as

$$\bar{X}(\theta) \in \arg\max_{\hat{X} \geq \underline{X}} \left(\frac{X(t)}{\hat{X}}\right)^{\beta} \left(\theta \hat{X} - I + b\right).$$

It implies that exercise occurs at the agent's most preferred threshold as long as it is above $\underline{X}$:

$$\bar{X}(\theta) = X_A^*(\theta) = \frac{\beta}{\beta - 1} \frac{I - b}{\theta}. \tag{28}$$

Stationarity implies that separation must hold for all types, including $\theta = 1$, which implies that (28) holds for any $\theta \in \Theta$. Hence, $\mathcal{X} = \{X : X \geq X_A^*(1)\}$. This exercise schedule satisfies the ex-post incentive compatibility of the principal. Indeed, since the agent is biased towards delay and recommends exercise at her most preferred threshold, when the agent recommends to exercise, the principal infers that it is already too late and thus does not benefit from delaying exercise even further. Formally, $X_P^*(\theta) < X_A^*(\theta)$.

Consider the ex-ante incentive compatibility condition for the principal. Let $V_P^c(X,\hat{\theta})$ denote the expected value to the principal in the equilibrium with continuous exercise, given that the public state is X and the principal's belief is that θ is uniform over $[0,\hat{\theta}]$. If the agent's type is θ, exercise occurs at threshold $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$, and the principal's payoff upon exercise is $\frac{\beta}{\beta-1}(I-b)-I$. Hence,

$$V_P^c(X,\hat{\theta}) = \int_0^{\hat{\theta}} \frac{1}{\hat{\theta}} X^\beta \left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta} \frac{I-\beta b}{\beta-1} d\theta = \frac{(X\hat{\theta})^\beta}{\beta+1}\left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta}\frac{I-\beta b}{\beta-1}. \tag{29}$$

By stationarity of the problem, it is sufficient to verify the principal's ex-ante incentive compatibility constraint for $\hat{\theta}=1$, which yields

$$V_P^c(X,1) \geq \frac{1}{2}X - I \ \ \forall X \leq X_A^*(1). \tag{30}$$

The proof of Proposition 1 shows that this constraint holds if and only if $b \geq -I$.

Second, consider equilibria with partitioned exercise, characterized by ω. To be an equilibrium, the implied exercise thresholds must satisfy the incentive compatibility conditions of the principal (23) and (27). As the proof of Proposition 1 demonstrates, the principal's ex-post incentive compatibility condition is satisfied for any $\omega \in (0,1)$ when the agent is biased towards late exercise. The principal's ex-ante incentive compatibility condition is satisfied if communication is informative enough, which puts a lower bound on ω, denoted $\underline{\omega} > 0$. The following proposition summarizes the set of all stationary equilibria:[7]

Proposition 1. *Suppose that $b \in (-I,0)$. The set of non-babbling stationary equilibria is given by:*

1. *Equilibrium with continuous exercise. The principal exercises at the first time t at which the agent sends $m=1$, provided that $X(t) \geq X_A^*(1)$ and $X(t) = \max_{s\leq t} X(s)$. The agent of type θ sends message $m=1$ at the first moment when $X(t)$ crosses her most-preferred threshold $X_A^*(\theta)$.*

2. *Equilibria with partitioned exercise (ω-equilibria), indexed by $\omega \in [\underline{\omega},1)$, where $0<\underline{\omega}<1$, and $\underline{\omega}$ is the unique solution to $V_P(X,1;\underline{\omega}) = \left(\frac{X}{\bar{X}_u}\right)^\beta\left(\frac{1}{2}\bar{X}_u - I\right)$, where $\bar{X}_u$ is given by (18). The principal exercises at time t at which $X(t)$ crosses threshold $Y(\omega)$, $\frac{1}{\omega}Y(\omega)$, ... for the*

[7] As always in cheap talk games, there always exists a "babbling" equilibrium, in which the agent's recommendations are uninformative, and the principal exercises at his optimal uninformed threshold, $\frac{\beta}{\beta-1}2I$. We do not consider this equilibrium unless it is the unique equilibrium of the game.

first time, provided that the agent sends message $m = 1$ at that point, where $Y(\omega)$ is given by (21). The principal does not exercise the option at any other time. The agent of type θ sends message $m = 1$ at the first moment when $X(t)$ crosses threshold $Y(\omega)\frac{1}{\omega^n}$, where $n \geq 0$ is such that $\theta \in (\omega^{n+1}, \omega^n)$. There exists a unique equilibrium for each $\omega \in [\underline{\omega}, 1)$.

If $b = -I$, the unique non-babbling stationary equilibrium is the equilibrium with continuous exercise. If $b < -I$, the principal exercises the option at his optimal uninformed threshold $\frac{\beta}{\beta-1}2I$.

Thus, as long as $b > -I$, there exist an infinite number of stationary equilibria: one equilibrium with continuous exercise and infinitely many equilibria with partitioned exercise. Both the equilibrium with continuous exercise and the equilibria with partitioned exercise feature delay relative to the principal's optimal timing given the information available to him at the time of exercise.

Clearly, not all of these equilibria are equally reasonable. It is common in cheap talk games to focus on the equilibrium with the most information revelation, which here corresponds to the equilibrium with continuous exercise.[8] It turns out that the equilibrium with continuous exercise dominates all other possible equilibria in the Pareto sense: it leads to a weakly higher expected payoff for both the principal and all types of the agent. Indeed, in this equilibrium, exercise occurs at the unconstrained optimal time of any type θ of the agent. Therefore, the payoff of any type θ of the agent is higher in the equilibrium with continuous exercise than in any other possible equilibrium. In addition, as Section 4 shows, the exercise strategy implied by the optimal mechanism if the principal could commit to any mechanism, coincides with the exercise strategy in the equilibrium with continuous exercise. Therefore, the principal's expected payoff in this equilibrium exceeds his expected payoff under the exercise rule implied by any other equilibrium. We conclude:

Proposition 2. *The equilibrium with continuous exercise from Proposition 1 dominates all other possible equilibria in the Pareto sense: both the principal's expected payoff and the expected payoff of each type of agent in this equilibrium are higher than in any other equilibrium.*

Using Pareto dominance as a selection criterion, we conclude that there is full revelation of information if the agent's bias is not very large, $b \geq -I$. However, although information is com-

[8] In general, equilibrium selection in cheap-talk games is a delicate issue. Unfortunately, most equilibrium refinements that reduce the set of equilibria in costly signaling games, do not work well in games of costless signaling (i.e., cheap talk). Some formal approaches to equilibrium selection in cheap-talk games are provided by Farrell (1993) and Chen, Kartik, and Sobel (2008).

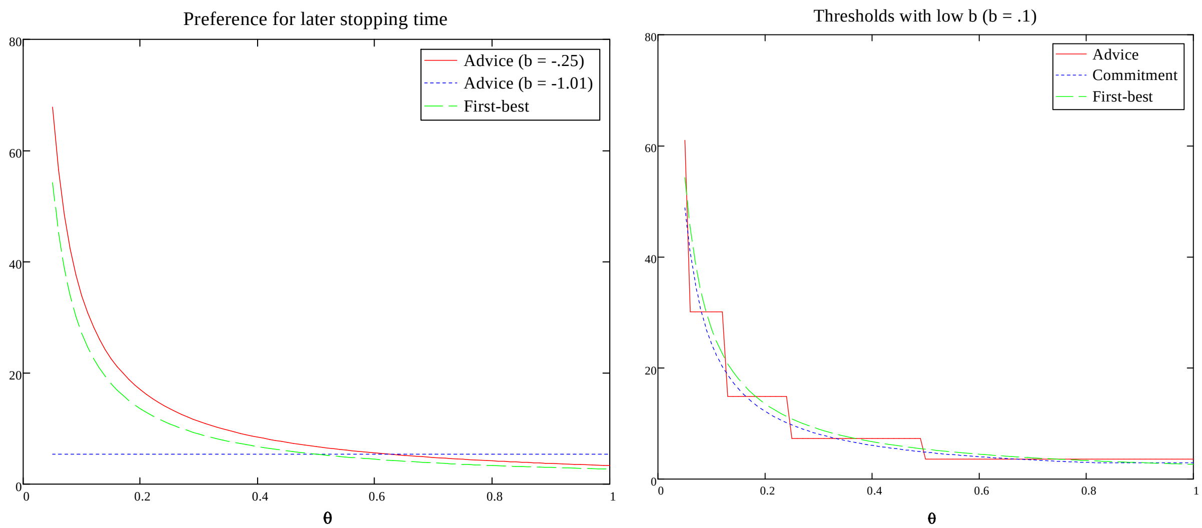

Figure 1. Equilibrium exercise threshold for the case $\underline{\theta} = 0$. The left panel illustrates the equilibrium the case b<0. The right panel illustrates the case b>0.

municated fully in equilibrium, communication and exercise are inefficiently (from the principal's point of view) delayed. Using the terminology of Aghion and Tirole (1997), the equilibrium is characterized by unlimited real authority of the agent, even though the principal has unlimited formal authority. The left panel of Figure 1 illustrates how the equilibrium exercise thresholds depend on the bias and type. If the bias is not too big, there is full revelation of information but delay in option exercise. If the bias is very big, no information is revealed at all, and the principal exercises according to his prior.

Now, consider the non-stationary case of $\underline{\theta} > 0$. In this case, we show that the equilibrium with continuous exercise from the stationary case of $\underline{\theta} = 0$ takes the form of the equilibrium with continuous exercise up to a cutoff:

Proposition 3. *Suppose that $\underline{\theta} > 0$. The equilibrium with continuous exercise from Proposition 2 does not exist. However, if $b \in (-\frac{1-\underline{\theta}}{1+\underline{\theta}} I, 0]$, the equilibrium with continuous exercise up to a cutoff exists. In this equilibrium, there exists a cutoff $\hat{X}$ such that the principal's exercise strategy is: (1) to exercise at the first time t at which the agent sends $m = 1$, provided that $X(t) \in [X_A^*(1), \hat{X}]$ and $X(t) = \max_{s \leq t} X(s)$; (2) to exercise at the first time t at which $X(t) \geq \hat{X}$, regardless of the agent's recommendation. The agent of type θ sends message $m = 1$ at the first moment when $X(t)$ crosses the minimum between her most-preferred threshold $X_A^*(\theta)$*

and $\hat{X}$. Threshold $\hat{X}$ is given by

$$\hat{X} = \frac{\beta}{\beta - 1} \frac{I + b}{\underline{\theta}} = X_A^* \left(\hat{\theta}^* \right),$$

where $\hat{\theta}^* \equiv \left(\frac{I-b}{I+b}\right) \underline{\theta} < 1$. If $b \leq -\frac{1-\underline{\theta}}{1+\underline{\theta}} I$, the principal exercises the option at his optimal uninformed threshold $\frac{\beta}{\beta-1} \frac{2I}{1+\underline{\theta}}$.

The intuition is as follows. At any time, the principal who obtains a recommendation to delay exercise faces the following trade-off. On the one hand, he can wait and see what the agent will recommend in the future. This option leads to more informative exercise because the agent communicates her information to the principal, but has a drawback in that communication and thus exercise will be excessively delayed. On the other hand, the principal can overrule the agent's recommendation and exercise immediately. This option results in less informative exercise, but not in excessive delay. Thus, the principal's trade-off is between the value of information and the cost of excessive delay. When $\underline{\theta} = 0$, the problem is stationary and the trade-off persists over time even though the principal updates his belief about θ: If the agent's bias is not too high ($b > -I$), waiting for the agent's recommendation is strictly better, while if the agent's bias is too high ($b < -I$), waiting for the agent's recommendation is too costly and communication does not happen. However, when $\underline{\theta} > 0$, the problem is non-stationary, and the trade-off between information and delay changes over time. Specifically, as time goes by and the agent continues recommending against exercise, the principal learns that the agent's type is not too high (below $\hat{\theta}_t$ at time t). This results in the shrinkage of the principal's belief about where θ is: The interval $\left[\underline{\theta}, \hat{\theta}_t\right]$ shrinks over time. Because $\underline{\theta} > 0$, the shrinkage of this interval implies that the remaining value of the agent's private information declines over time. At the same time, the cost of waiting for information persists. Once the interval shrinks to $\left[\underline{\theta}, \hat{\theta}^*\right]$, which happens at threshold $\hat{X}$, the remaining value of the agent's private information becomes sufficiently small to make it optimal for the principal to exercise immediately regardless of the agent's recommendation. Figure 2 illustrates this logic.

The comparative statics of the cutoff type $\hat{\theta}^*$ are intuitive. As b decreases, i.e., the conflict of interest gets bigger, $\hat{\theta}^*$ increases and $\hat{X}$ decreases, implying that the principal waits less for the agent's recommendation.

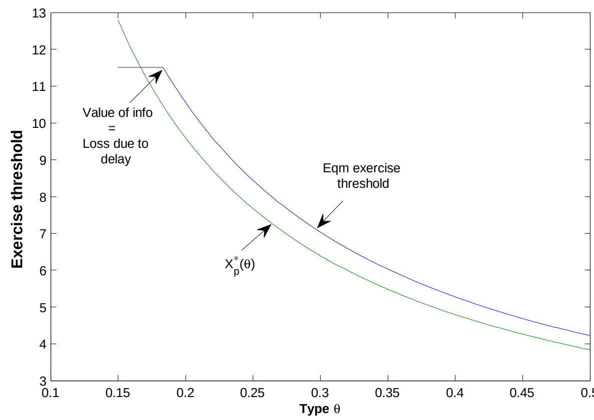

Figure 2. Equilibrium exercise threshold for the case b<0 and $\underline{\theta} > 0$.

3.2 Preference for early exercise

Suppose that $b > 0$, i.e., the agent is biased in the direction of early exercise. We focus on the stationary case $\underline{\theta} = 0$. Because the principal's optimal exercise time is later than the agent's, there is no equilibrium with continuous exercise. Indeed, if the agent follows the strategy of recommending exercise at her most-preferred threshold $X_A^*(\theta)$, the principal infers the agent's type perfectly and prefers delay over immediate exercise upon getting the recommendation to exercise. Knowing this, the agent is tempted to change her recommendation strategy, mimicking a lower type. Thus, no equilibrium with continuous exercise exists in this case.

For ω-equilibrium with partitioned exercise to exist, the expected value $V_P(X, 1; \omega)$ that the principal gets from waiting for recommendations of the agent and the threshold $Y(\omega)$ must satisfy the ex-post and the ex-ante incentive compatibility conditions (23) and (27). First, consider equilibria where the ex-post incentive compatibility condition (23) holds as an equality:

$$Y(\omega) = \frac{\beta}{\beta - 1} \frac{2I}{1 + \omega}. \tag{31}$$

Then, using the expression (21) for $Y(\omega)$, we can find ω as the solution to:

$$\omega = \frac{1}{\frac{\beta}{\beta-1} \frac{1-\omega^{\beta-1}}{1-\omega^{\beta}} \frac{2I}{I-b} - 1}. \tag{32}$$

The next lemma shows that when $b \in (0, I)$, equation (32) has a unique solution, denoted ω^*:

Lemma 3. *Suppose that* $0 < b < I$. *In the range* $[0, 1]$, *equation (32) has a unique solution*

$\omega^* \in (0,1)$, *where* ω^* *decreases in* b, $\lim_{b\to 0}\omega^* = 1$, *and* $\lim_{b\to I}\omega^* = 0$.

Second, consider equilibria where the ex-post incentive compatibility condition (23) holds as a strict inequality, $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$. As we show in the proof of Proposition 4, this is equivalent to $\omega < \omega^*$. The ω-equilibrium, $\omega \leq \omega^*$, will then exist as long as the ex-ante incentive compatibility condition (27) holds as well. The proof of Proposition 4 shows that (27) is satisfied if and only if ω is high enough and is not satisfied for ω close to zero, which puts a lower bound on ω, denoted $\underline{\omega} > 0$. The next proposition summarizes the set of equilibria:

Proposition 4. *Suppose that* $0 < b < I$. *The set of non-babbling equilibria is given by equilibria with partitioned exercise (ω-equilibria), indexed by* $\omega \in [\underline{\omega}, \omega^*]$, *where* $0 < \underline{\omega} < \omega^* < 1$, ω^* *is the unique solution to (32), and* $\underline{\omega}$ *is the unique solution to* $V_P(X, 1; \underline{\omega}) = \left(\frac{X}{\bar{X}_u}\right)^{\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$, *where* $\bar{X}_u$ *is given by (18). The principal exercises at time* t *at which* $X(t)$ *crosses threshold* $Y(\omega)$, $\frac{1}{\omega}Y(\omega)$, *... for the first time, provided that the agent sends message* $m = 1$ *at that point, where* $Y(\omega)$ *is given by (21). The principal does not exercise the option at any other time. The agent of type* θ *sends message* $m = 1$ *at the first moment when* $X(t)$ *crosses threshold* $\frac{1}{\omega^n}Y(\omega)$, *where* $n \geq 0$ *is such that* $\theta \in (\omega^{n+1}, \omega^n)$. *There exists a unique equilibrium for each* $\omega \in [\underline{\omega}, \omega^*]$.

If $b \geq I$, *the principal exercises the option at his optimal uninformed threshold* $\frac{\beta}{\beta-1}2I$.

As in the case of the delay preference, the equilibria can also be ranked in informativeness. The most informative equilibrium is the one with the smallest partitions, i.e., ω^*. In this equilibrium, exercise is unbiased: since the ex-post incentive compatibility condition of the principal holds as an equality, the exercise rule maximizes the principal's payoff given that the agent's type lies in a given partition. In all other equilibria, there is both loss of information and delay in option exercise. Interestingly, delay in option exercise occurs despite the fact that the agent is biased in the direction of exercising too early.

As the following proposition demonstrates, the expected utility of the principal and the ex-ante expected utility of the agent (before the agent's type is realized) is higher in the ω^*-equilibrium than in any other stationary equilibrium with partitioned exercise. Intuitively, this is because the ω^*-equilibrium is both the most informative and features no delay, which is detrimental for both the principal and the agent with a bias towards early exercise.

Proposition 5. *The* ω^**-equilibrium dominates all other equilibria with partitioned exercise in the following sense: both the principal's expected payoff and the ex-ante expected payoff of the agent before the agent's type is realized are higher in the* ω^**-equilibrium than in the* ω*-equilibrium* *for any* $\omega < \omega^*$.

Motivated by this result, we focus on the ω^*-equilibrium in the remainder of the paper.

3.3 The role of dynamic communication

In this section, we highlight the role of dynamic communication by comparing our communication model to a model where communication is restricted to a one-shot interaction at the beginning of the game. Specifically, we analyze the setting of the basic model where instead of communicating with the principal continuously, the agent sends a single message m_0 at time $t = 0$, and there is no subsequent communication. After receiving the message, the principal updates his beliefs about the agent's type and then exercises the option at the optimal threshold given these beliefs.

First, note that for any equilibrium of the static communication game, there exists an equivalent equilibrium of the dynamic communication game where all communication after time 0 is uninformative (babbling), and the principal's beliefs are that the agent's messages are uninformative. However, as we show next, the opposite is not true: many equilibria of the dynamic communication game described by Propositions 1 and 4 do not exist in the static communication game. The following result summarizes our findings.

Proposition 6. *Suppose* $\underline{\theta} = 0$*. If* $b < 0$*, there is no non-babbling stationary equilibrium of the dynamic communication game that is also an equilibrium of the static communication game. If* $b > 0$*, the only non-babbling stationary equilibrium of the dynamic communication game that is also an equilibrium of the static communication game is the one with* $\omega = \omega^*$*, where* ω^* *is given by the unique solution to (32).*

The intuition behind this result is the following. All non-babbling stationary equilibria of the dynamic communication game for $b < 0$ feature delay relative to what the principal's optimal timing of exercise would have been ex ante, given the information he learns in equilibrium. In a dynamic communication game, this delay is feasible because the principal learns information with delay, after his optimal exercise time has passed. However, in a static communication game, this delay cannot be sustained in equilibrium because it would violate the principal's incentive compatibility constraint: since the principal learns all the information at time 0, his exercise decision is always optimal given the available information.[9] By the same argument, the only

[9]Similarly, if $\underline{\theta} > 0$, the equilibrium with continuous exercise up to a cut-off, described in Proposition 3, does not exist in the static communication game either.

sustainable equilibrium of the dynamic communication game for $b > 0$ is the one that features no delay relative to the principal's optimal threshold, i.e., the equilibrium with $\omega = \omega^*$.

Thus, even though information of the agent is persistent, the ability to communicate dynamically makes the analysis rather different from the static cheap talk problem. When the agent has a bias for late exercise, dynamic communication expands the set of equilibria in a way that improves the equilibrium payoffs of both players. Timing the recommendation strategically is helpful for both the principal and the agent, because it ensures that the principal does not overrule the agent's recommendation and thereby makes communication effective.

4 Delegation versus communication

So far, we have made an extreme assumption that the principal has no commitment power at all. In this section, we relax this assumption by allowing the principal to choose between delegating formal authority to exercise the option to the agent and keeping formal authority but playing the communication game analyzed in the previous section. We examine the conditions under which the principal benefits from delegating the decision to the agent.

Formally, we consider the problem studied by Dessein (2002), but focus on stopping time decisions. In an insightful paper, Dessein (2002) shows that delegating the decision to the informed but biased agent dominates keeping the authority and communicating with the agent if the agent's bias is small enough. Importantly, Dessein (2002) considers static decisions. In this section, we show that the choice between delegation and centralization can be quite different for decisions about timing. In particular, we show that if the agent is biased towards late exercise, delegation is always weakly inferior to communication, and is strictly inferior if $\underline{\theta} > 0$. In contrast, if the agent is biased towards early exercise, we obtain a Dessein-like result that delegation dominates communication for small biases and communication dominates delegation for large biases. Because most decisions in organizations can be delayed and thus involve the stopping time component, these results are important for organizational design.

4.1 Optimal mechanism with commitment

To analyze the choice between delegation and communication, it is helpful to derive an auxiliary result: what the optimal mechanism would be if the principal could commit to any decision-making mechanism. By the revelation principle, we can restrict attention to direct revelation mechanisms, i.e., mechanisms in which the message space is $\Theta = [\underline{\theta}, 1]$ and that provide the agent with incentives to report her type θ truthfully. Furthermore, it is easy to show that a mechanism in which exercise occurs not at the first passage time cannot be optimal.

Hence, we can restrict attention to mechanisms in the form $\{\hat{X}(\theta) \geq X(0), \theta \in \Theta\}$: If the agent reports θ, the principal exercises when $X(t)$ passes threshold $\hat{X}(\theta)$ for the first time. Let $\hat{U}_A(\hat{X}, \theta)$ and $\hat{U}_D(\hat{X}, \theta)$ denote the time-zero expected payoffs of the agent and the principal, respectively, when the true state is θ and the exercise occurs at threshold $\hat{X}$:

$$\hat{U}_A(\hat{X}, \theta) \equiv \left(\frac{X(0)}{\hat{X}}\right)^{\beta} \left(\theta \hat{X} - I + b\right), \tag{33}$$

$$\hat{U}_D(\hat{X}, \theta) \equiv \left(\frac{X(0)}{\hat{X}}\right)^{\beta} \left(\theta \hat{X} - I\right). \tag{34}$$

The optimal mechanism maximizes the ex-ante expected payoff to the principal:

$$\max_{\{\hat{X}(\theta), \theta \in \Theta\}} \int_{\underline{\theta}}^{1} \hat{U}_D(\hat{X}(\theta), \theta) \frac{1}{1-\underline{\theta}} d\theta, \tag{35}$$

subject to the incentive compatibility constraint of the agent:

$$\hat{U}_A(\hat{X}(\theta), \theta) \geq \hat{U}_A(\hat{X}\left(\hat{\theta}\right), \theta) \ \forall \theta, \hat{\theta} \in \Theta. \tag{36}$$

The next proposition characterizes the optimal decision-making rule under commitment:

Proposition 7. *The optimal incentive-compatible threshold schedule $\hat{X}(\theta)$, $\theta \in \Theta$, is given by:*

- *If $b \in \left(-\infty, -\frac{1-\underline{\theta}}{1+\underline{\theta}} I\right] \cup \left[\frac{1-\underline{\theta}}{1+\underline{\theta}} I, \infty\right)$, then*

$$\hat{X}(\theta) = \frac{\beta}{\beta - 1} \frac{2I}{\underline{\theta} + 1} \ \forall \theta \in \Theta. \tag{37}$$

- *If $b \in \left[-\frac{1-\underline{\theta}}{1+\underline{\theta}} I, 0\right]$, then*

$$\hat{X}(\theta) = \begin{cases} \frac{\beta}{\beta-1} \frac{I+b}{\underline{\theta}}, & \text{if } \theta \leq \left(\frac{I-b}{I+b}\right) \underline{\theta}; \\ \frac{\beta}{\beta-1} \frac{I-b}{\theta}, & \text{if } \theta \geq \left(\frac{I-b}{I+b}\right) \underline{\theta}. \end{cases} \tag{38}$$

- *If $b \in \left[0, \frac{1-\underline{\theta}}{1+\underline{\theta}} I\right]$, then*

$$\hat{X}(\theta) = \begin{cases} \frac{\beta}{\beta-1} \frac{I-b}{\theta}, & \text{if } \theta \leq \frac{I-b}{I+b}; \\ \frac{\beta}{\beta-1} (I+b), & \text{if } \theta \geq \frac{I-b}{I+b}. \end{cases} \tag{39}$$

The reasoning behind this proposition is similar to the reasoning of why the optimal decision-making rule in the static linear-quadratic model features perfect separation of types up to a cutoff and pooling beyond the cutoff (Melumad and Shibano, 1991; Goltsman et al., 2009). Intuitively, because the agent does not receive additional private information over time and the optimal stopping rule can be summarized by a threshold, the optimal dynamic contract is similar to the optimal contract in a static game with equivalent payoff functions.

By comparing previous results with Proposition 7, it is easy to see that if the agent is biased towards late exercise ($b < 0$), the solutions to the communication problem under no commitment and under commitment coincide for any $\underline{\theta} \geq 0$. This result does not hold when the agent is biased in the direction of early exercise ($b > 0$). In this case, the principal could benefit from commitment power. This asymmetry occurs because of the asymmetric nature of time: one cannot go back in time. Therefore, even without formal commitment power, as time passes, the principal effectively commits not to exercise earlier because she cannot go back in time. This is not true when the decision is not a stopping time, but rather a point in the continuum, such as the scale of a project. The idea that one cannot go back in time highlights the unique characteristic of stopping time decisions.

The next proposition summarizes these results:

Proposition 8. *If $b < 0$, the exercise threshold in the most informative equilibrium of the advising game coincides with the optimal exercise threshold under commitment: $\bar{X}(\theta) = \hat{X}(\theta)$ for all $\theta \in \Theta$. In particular, the equilibrium payoffs of both parties in the advising game coincide with their payoffs under the optimal commitment mechanism. If $b > 0$, the payoff of the principal in the advising game is lower than his payoff under the optimal commitment mechanism.*

From the organizational design perspective, this result implies that investing in commitment power is not important for decisions where the agent wants to delay exercise, as in the case of headquarters seeking a local plant manager's advice on closing the plant. In contrast, investing in commitment power is important for decisions where the agent is biased towards early exercise, such as the decision when to drill an oil well, launch a new product, or make an acquisition.

4.2 Delegation when the agent has a preference for late exercise

It follows from Proposition 8 and the asymmetric nature of the equilibrium in the communication game that implications for delegation are quite different between the "late exercise bias" and the "early exercise bias" cases. First, consider $b < 0$, i.e., the case when the agent is biased in the direction of late exercise. If the principal does not delegate the decision to the agent, the principal and the agent play the communication game. The outcome of this game is either that the option is exercised at the agent's most preferred threshold (if $\underline{\theta} = 0$) or that the option is exercised at the agent's most preferred threshold up to a cutoff (if $\underline{\theta} > 0$). In contrast, if the principal delegates formal authority to exercise the option to the agent, the agent will exercise the option at her most preferred threshold $X_A^*(\theta)$. Clearly, if $\underline{\theta} = 0$, delegation and communication are equivalent. However, if $\underline{\theta} > 0$, they are not equivalent: Not delegating the decision and playing the communication game implements *conditional* delegation (delegation up to a cutoff $\hat{X}$), while delegation implements *unconditional* delegation. By Proposition 8, the principal is strictly better off with the former rather than the latter. This result is summarized in the following proposition.

Proposition 9. *If $b < 0$, i.e., the agent is biased in the direction of late exercise, then the principal prefers retaining control and getting advice from the agent to delegating the exercise decision. The preference is strict if $\underline{\theta} > 0$. If $\underline{\theta} = 0$, retaining control and delegation are equivalent.*

This result contrasts with the opposite implications for static decisions, such as choosing the scale of the project (Dessein, 2002). Dessein (2002) shows that in the leading quadratic-uniform setting of Crawford and Sobel (1982), regardless of the direction of the agent's bias, delegation always dominates communication as long as the agent's bias is not too high so that at least some informative communication is possible. For general payoff functions, Dessein (2002) shows that delegation dominates communication if the agent's bias is sufficiently small. In contrast, we show that if the agent is biased towards late exercise, then regardless of the magnitude of her bias, the principal never wants to delegate decision-making authority to her. Intuitively, the inability to go back in time is useful when the agent has a preference for late exercise because it allows the principal to commit to follow the recommendation of the agent: Even though ex-post, the principal is tempted to revise history and exercise in the past, it is not feasible. This built-in commitment role of time ensures that communication is sufficiently effective so that delegation has no further benefit.[10]

[10] Note also that in our context, option exercise happens with delay even under centralization. This is different from Bolton and Farrell (1990), where centralization helps to avoid inefficient delay caused by coordination problems between competing firms.

4.3 Delegation when the agent has a preference for early exercise

In contrast to the case when the agent has a preference for late exercise, we show that delegation is beneficial if the agent has a preference for early exercise and the bias is low enough. Specifically:

Proposition 10. *Suppose $b > 0$, i.e., the agent is biased in the direction of early exercise, and consider the most informative equilibrium of the advising game, ω^*. There exist $\underline{b}$ and $\bar{b}$, such that the principal's expected value in the ω^*-equilibrium is lower than his expected value in the delegation game if $b < \underline{b}$, and is higher than in the delegation game if $b > \bar{b}$.*

The result that delegation is beneficial when the agent's bias is small enough is similar to the result of Dessein (2002) for static decisions and shows that Dessein's argument extends to stopping time decisions when the agent has a preference for early exercise. Intuitively, the principal faces a trade-off: delegating the decision to the agent leads to early exercise due to the agent's bias but allows to use the agent's information more efficiently. When the agent's bias is small enough, the cost from early exercise is smaller than the cost due to the loss of the agent's information, and hence delegation dominates.

4.4 Optimal timing of delegation

In a dynamic setting, the principal does not need to delegate authority to the agent from the start: he may retain authority for some time and delegate authority to the agent later. In this section, we study whether timing delegation strategically may help the principal. In particular, we consider the following problem: The principal and the agent play the communication game analyzed in Section 3, but at any time, the principal may delegate full decision-making authority to the agent. After this authority is granted, the agent retains it until the end of the game and thus is free to choose her optimal exercise threshold.

According to Proposition 8, if the agent is biased towards late exercise, the advising equilibrium implements the optimal commitment mechanism. Hence, the principal cannot do better with delegation than with keeping authority forever and communicating with the agent, and delegation will never occur. In contrast, when the agent is biased towards early exercise, simply communicating with the agent brings a lower payoff than under the optimal commitment mechanism. However, as the next proposition shows, the principal can implement the optimal mechanism by delegating the decision to the agent at the right time.

Proposition 11. *If $b > 0$, there exists the following equilibrium. The principal delegates au-*

thority to the agent at the first moment when $X(t)$ reaches the threshold $X_d \equiv \min(\frac{\beta(I+b)}{\beta-1}, \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1})$ and does not exercise the option before that. For any θ, the agent sends message $m=0$ at any point before she is granted authority. If $\theta \geq \frac{I-b}{I+b}$, the agent exercises the option immediately after she is given authority, and if $\theta \leq \frac{I-b}{I+b}$, the agent exercises the option when $X(t)$ first reaches her preferred exercise threshold $X_A^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$. The exercise threshold in this equilibrium coincides with the optimal exercise threshold under commitment.

Intuitively, timing delegation strategically ensures that the information of low types ($\theta \leq \frac{I-b}{I+b}$) is used efficiently, and that all types above $\frac{I-b}{I+b}$ exercise immediately at the time of delegation, exactly as under the optimal contract from Proposition 7. The higher is the agent's bias, the later will delegation occur.

Propositions 8 and 11 imply that the direction of the conflict of interest is the key driver of the allocation of control rights for timing decisions. When the agent is biased towards late exercise, the principal should always retain control and rely on communication with the agent. In contrast, when the agent is biased towards early exercise, it is optimal for the principal to delegate the decision to the agent at some point in time.

5 Implications

5.1 Comparative statics

The model delivers interesting comparative statics with respect to the underlying parameters. First, consider the case of an agent with a late exercise bias, $b < 0$. There is full revelation of information, independent of the agent's bias and the underlying parameters of the model μ, σ, and r. The decision is delayed by the factor $\frac{I-b}{I}$, which is also independent of these parameters.

Second, consider the case of an agent biased towards early exercise, $b > 0$. In this case, equilibrium exercise is unbiased, but there is loss of information since $\omega^* < 1$. The comparative statics results are presented in the next proposition and illustrated in Figure 3.

Proposition 12. *Consider the case of an agent biased towards early exercise, $b > 0$. Then, ω^* decreases in b and increases in β, and hence decreases in σ and μ, and increases in r.*

The equilibrium partitioning multiple ω^* is decreasing in the bias of the agent, in line with Crawford and Sobel's (1982) result that less information is revealed if the misalignment of preferences is bigger. More interesting are the comparative statics results in parameters μ, σ, and

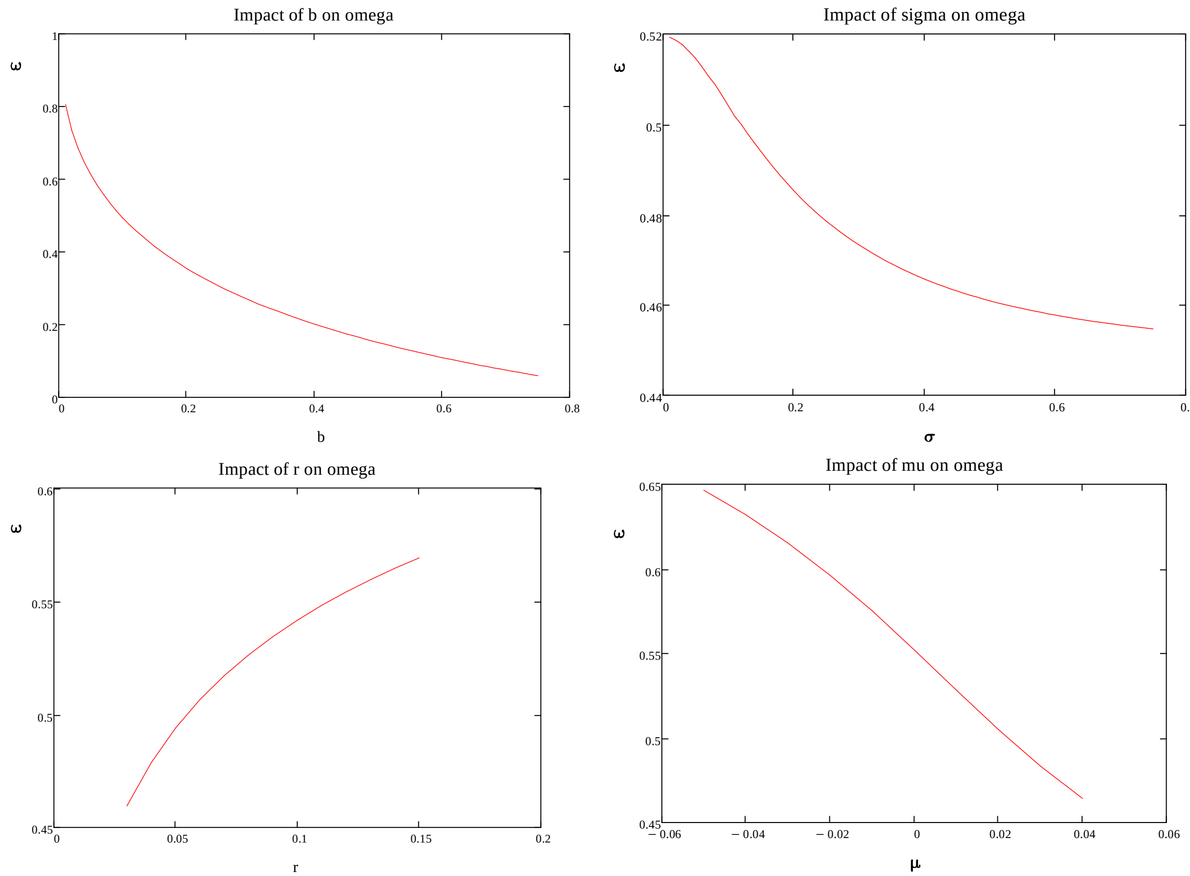

Figure 3. Comparative statics of ω^* in parameters of the model.

r. We find that communication is less efficient when the value of the option to wait is higher. For example, there is less information revelation in equilibrium (ω^* is lower) if the underlying environment is more uncertain (σ is higher). Intuitively, high uncertainty increases the value of the option to delay exercise and thereby effectively increases the conflict of interest between the principal and the agent biased towards early exercise. Similarly, communication is less efficient in lower-interest-rate and higher-growth environments. Figure 3 illustrates these comparative statics.

We next analyze how the principal's choice between delegating the decision to the agent from the start and retaining formal authority and communicating with the agent depends on the underlying parameters. As shown in Section 4.2, delegation is always weakly inferior when the agent is biased towards late exercise. We therefore focus on the case of an early exercise bias, $b > 0$. As Proposition 10 demonstrates, delegation is superior (inferior) to communication if the agent's bias is sufficiently small (large). In numerical analysis, we show that there exists a cutoff $\bar{b}$ such that the principal's value from delegation is higher if and only if $0 < b < \bar{b}$. Figure 4(a) illustrates

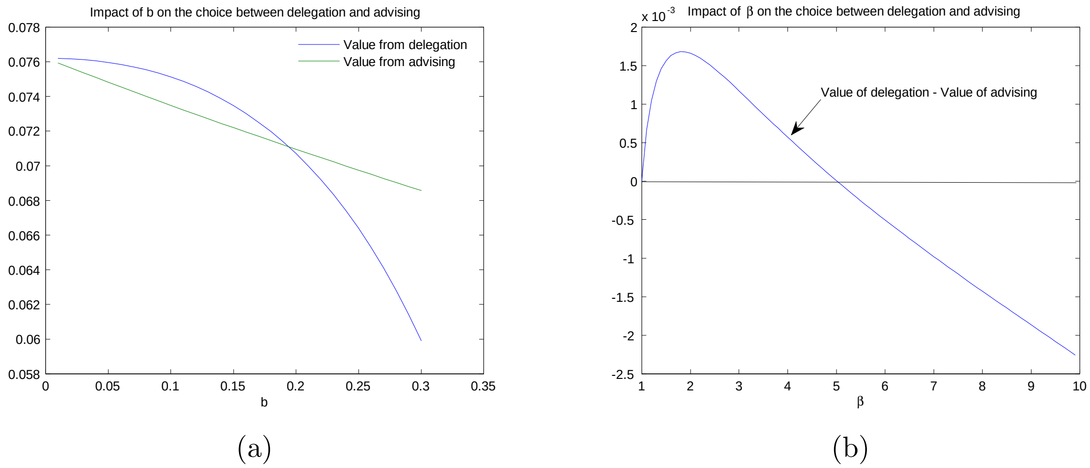

Figure 4. Comparative statics of the delegation decision.

this result for $I = 1$, $r = 0.15$, $\mu = 0.05$, $\sigma = 0.2$.

The comparative statics of the delegation decision in parameters σ, μ, and r are demonstrated in Figure 4(b) for $b = 0.1$ and $I = 1$. The x-axis corresponds to parameter β, given by (14), which increases in r and decreases in σ and μ. The y-axis corresponds to the difference between the principal's value from delegation and his value from advising. The figure shows that delegation dominates advising when β is sufficiently small (σ and μ are high, r is small), but is inferior when β is large. Intuitively, a small β corresponds to a high value of the option to wait. As shown in Proposition 12, communication between the principal and the agent is less efficient when the value of the option to wait is higher. Thus, the principal prefers delegating the decision to the agent over retaining control and communicating with the agent when the option to wait is sufficiently valuable. As β increases, communication becomes more efficient and eventually dominates delegation.

5.2 Strategic choice of the agent

We next show that asymmetry of time has important implications for the strategic choice of an agent. We focus on the stationary case, $\underline{\theta} = 0$. The next proposition shows that if the principal needs to choose between an agent biased towards early exercise and an agent biased towards late exercise with the same (in absolute value) bias, the principal is better off choosing the agent with a late exercise bias, regardless of whether or not the principal has the option to delegate authority to the agent:

Proposition 13. *(i) Let $V_0(b)$ be the expected payoff of the principal at the initial date $t = 0$*

in the most informative equilibrium of the advising game, given that the agent's bias is b. Then, $V_0(-b) \geq V_0(b)$ *for any* $b \geq 0$ *and* $V_0(-b) > V_0(b)$ *for any* $b \in (0, I)$*.*

(ii) Let $\tilde{V}_0(b)$ *be the expected payoff of the principal at the initial date* $t = 0$ *if the principal can choose between delegating authority to the agent and retaining authority and communicating with the agent. Then,* $\tilde{V}_0(-b) \geq \tilde{V}_0(b)$ *for any* $b \geq 0$ *and* $\tilde{V}_0(-b) > \tilde{V}_0(b)$ *for any* $b \in (0, I)$*.*

The first result implies that between the two problems, poor communication but unbiased timing and full communication but delayed timing, the former is a bigger problem. Intuitively, the advising game features built-in commitment power of the principal when the agent is biased towards late exercise, but not when the agent is biased towards early exercise. Because of this, as shown in Proposition 8, the principal's payoff in the advising equilibrium coincides with his payoff in the optimal mechanism for $b < 0$, but is strictly smaller than in the optimal mechanism for $b > 0$. Because the principal's utility in the optimal mechanism only depends on the magnitude of the agent's bias and not on its direction, the principal is better off dealing with an agent biased towards late exercise. Allowing the principal to delegate authority to the agent does not change this result: although delegation can make the principal better off if the agent is biased towards early exercise, it does not allow him to implement the optimal mechanism.

6 Robustness

6.1 Model with different discount rates

In the spirit of the cheap talk literature, in our basic setup, we model the conflict of interest between the agent and the principal by assuming that the agent has a bias b. However, we have shown that our results are similar in an alternative setup, where the conflict of interest arises because the agent and the principal have different discount rates. This section presents the summary of this analysis, and the full analysis is available from the authors upon request.

Consider a setting where the agent's discount rate is r_A, the principal's discount rate is r_P, and both players' payoff from exercise at time t is given by $\theta X(t) - I$. By analogy with the basic model, we can define β_A and β_P, where β_i is the positive root of the fundamental quadratic equation $\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r_i = 0$.

The case where the principal is more impatient than the agent ($r_P > r_A$, or equivalently, $\beta_P > \beta_A$) is similar to the case $b < 0$ in the basic model. We show that if $\underline{\theta} = 0$, then as long as $\beta_A > \frac{2\beta_P}{1+\beta_P}$, there exists an equilibrium with continuous exercise in which exercise occurs at the agent's most preferred exercise threshold $\frac{\beta_A}{\beta_A - 1}\frac{I}{\theta}$. If $\underline{\theta} > 0$, the equilibrium features continuous exercise at the agent's most preferred threshold up to a cutoff.

The case where the agent is more impatient than the principal ($r_P < r_A$) is similar to the case $b > 0$ in the basic model. We show that the equilibrium with continuous exercise does not exist and derive the analog of Proposition 4 for this setting. Specifically, in the most informative stationary equilibrium, the principal's ex-post incentive compatibility constraint is satisfied as an equality, i.e., exercise is unbiased given the principal's information. This equilibrium is characterized by $\tilde{\omega}^* < 1$, which is the unique solution of the equation

$$\frac{\left(1-\omega^{\beta_A}\right) I}{\omega\left(1-\omega^{\beta_A-1}\right)}=\frac{\beta_P}{\beta_P-1} \frac{2 I}{\omega+1}.$$

In addition, for any $\omega \in [\underline{\tilde{\omega}}, \tilde{\omega}^*)$, where $0 < \underline{\tilde{\omega}} < \tilde{\omega}^* < 1$, there exists a unique ω-equilibrium where exercise happens with delay. Equilibria with $\omega < \underline{\tilde{\omega}}$ do not exist because the principal is better off following an uninformed exercise policy.

6.2 Put option

So far, we have assumed that the decision problem is over the timing of exercise of a call option, such as the decision when to invest. In this section, we show that if the decision problem is over the timing of exercise of a put option, such as the decision when to liquidate a project, the analysis and economic insights are similar. The nature of the option, call or put, is irrelevant for the results. What matters is the asymmetric nature of time: Time moves forward and thereby creates a one-sided commitment device for the principal to not overrule recommendations of the agent.

Consider the model of Section 2 with the following change. The exercise of the option leads to the following payoffs of the principal and the agent:

$$\begin{aligned} &\text{P: } \theta I - X(t), \\ &\text{A: } \theta(I+b) - X(t). \end{aligned}$$

As before, θ is the private information of the agent. It is a random draw from a uniform distribution on $[\underline{\theta}, 1]$ and is learned by the agent at the initial date. If $\underline{\theta} = 0$, the model exhibits stationarity. For example, if the decision represents shutting down a project, $I\theta$ corresponds to the salvage value of the project, $b\theta$ corresponds to the agent's private cost (if $b < 0$) or benefit (if $b > 0$) of liquidating the project, and $X(t)$ corresponds to the present value of the cash flows from keeping the project afloat. Private knowledge of θ means that the project manager is more informed about the salvage value of the project than the principal. The solution of this model follows the same structure as the solution of the model with the call option. We summarize our findings below, and the full analysis is available from the authors upon request.

Suppose that we start with a high enough $X(0)$, so that immediate exercise does not happen. If θ were known, the optimal exercise policy of each party would be given by a lower trigger on $X(t)$:

$$\begin{aligned} X_A^{**}(\theta) &= \frac{\gamma}{\gamma+1}(b+I)\theta, \\ X_P^{**}(\theta) &= \frac{\gamma}{\gamma+1}I\theta, \end{aligned}$$

where $-\gamma$ is the negative root of the quadratic equation that defined β. If $b > 0$, then $X_A^{**}(\theta) > X_P^{**}(\theta)$, which implies that the agent's preferred exercise policy under common information is to exercise earlier than the principal. Similarly, if $b < 0$, the agent is biased towards late exercise.

Suppose that $\underline{\theta} = 0$ and consider the communication problem analogous to the one in Section 3. If $b \in \left(-\frac{I}{2}, 0\right)$, there exists an equilibrium with full information revelation: The agent recommends to delay option exercise as long as $X(t)$ exceeds her most preferred exercise threshold $X_A^*(\theta)$ and recommends exercise at the first moment when $X(t)$ hits $X_A^*(\theta)$. As in the case of the call option, upon getting the recommendation to exercise, the principal realizes it is too late and finds it optimal to exercise immediately. Prior to that, the principal finds it optimal to wait because the value of learning θ exceeds the cost of delay. If $b > 0$, such an equilibrium does not exist. The problem exhibits stationarity, and all stationary partition equilibria are of the form $\left\{(\omega, 1), \left(\omega^2, \omega\right), ...\right\}$, where type $\theta \in \left(\omega^n, \omega^{n-1}\right)$, $n \in \mathbb{N}$ recommends exercise at threshold $\omega^{n-1}Y_{put}(\omega)$, where

$$Y_{put}(\omega) = \frac{\omega - \omega^{\gamma+1}}{1-\omega^{\gamma+1}}(I+b).$$

7 Conclusion

This paper studies timing decisions in organizations. We consider a problem in which an uninformed principal must decide when to exercise an option and has to rely on the information of a better-informed but biased agent. Depending on the application, the agent may be biased in favor of late or early exercise. We first analyze centralized decision-making, when the principal retains formal authority and repeatedly communicates with the agent via cheap talk. In contrast to the static cheap talk setting, where the decision variable is scale rather than stopping time, the properties of the equilibria are asymmetric in the direction of the agent's bias. When the agent is biased towards late exercise, there is often full revelation of information but suboptimal delay in option exercise. Conversely, when the agent is biased towards early exercise, there is partial revelation of information, while exercise is either unbiased or delayed. The reason for this asymmetry lies in the asymmetric nature of time as a decision variable: While the principal can always get advice and exercise the option at a later point in time, he cannot go back and exercise the option at an earlier point in time. When the agent is biased towards late exercise, the inability to go back in time creates an implicit commitment device for the principal to follow the agent's recommendation and often allows full information revelation. In contrast, when the agent is biased towards early exercise, time does not have built-in commitment, and only partial information revelation is possible. The analysis has implications for the informativeness and timeliness of option exercise decisions, depending on the direction of the agent's bias and on the parameters of the stochastic environment, such as volatility, growth rate, and discount rate.

We next analyze the optimal allocation of authority for timing decisions by studying the principal's choice between centralized decision-making with communication and delegating the decision to the agent. We show that the optimal choice between delegation and centralization is also asymmetric in the direction of the agent's bias. Delegation is always weakly inferior when the agent is biased towards late exercise, but is optimal when the agent is biased towards early exercise and her bias is not very large. If the principal can time the delegation decision strategically, she finds it optimal to delegate authority to the agent at some point in time if the agent is biased towards early exercise, but always retains authority in the case of a late exercise bias. The principal's benefit from the ability to commit to any decision rule also depends on the direction of the agent's bias. When the agent is biased towards late exercise, the ability to commit to any decision rule does not improve the principal's payoff relative to the advisory setting without commitment. In contrast, when the agent is biased towards early exercise, the principal would benefit from an ability to commit.

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Appendix: Proofs

Proof of Lemma 1. By contradiction, suppose that $\bar{X}(\theta_1) < \bar{X}(\theta_2)$ for some $\theta_2 > \theta_1$. Using the same arguments as in the benchmark case of Section 2.1 (these arguments do not depend on Lemma 1), it is easy to show that if exercise occurs at a cutoff $\bar{X}$ and the current value of $X(t)$ is $X \leq \bar{X}$, then the agent's expected utility is given by $\left(\frac{X}{\bar{X}}\right)^{\beta}\left(\theta\bar{X} - I + b\right)$, where $\beta > 1$ is defined by (14). Hence, because the message strategy of type θ_1 is feasible for type θ_2, the incentive compatibility (IC) of type θ_2 implies:

$$\left(\frac{X(t)}{\bar{X}(\theta_2)}\right)^{\beta}\left(\theta_2\bar{X}(\theta_2) - I + b\right) \geq \left(\frac{X(t)}{\bar{X}(\theta_1)}\right)^{\beta}\left(\theta_2\bar{X}(\theta_1) - I + b\right). \tag{40}$$

Similarly, because the message strategy of type θ_2 is feasible for type θ_1,

$$\left(\frac{X(t)}{\bar{X}(\theta_1)}\right)^{\beta}\left(\theta_1\bar{X}(\theta_1) - I + b\right) \geq \left(\frac{X(t)}{\bar{X}(\theta_2)}\right)^{\beta}\left(\theta_1\bar{X}(\theta_2) - I + b\right). \tag{41}$$

These inequalities imply

$$\begin{aligned}\theta_2\bar{X}(\theta_1)\left(1 - \left(\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}\right)^{\beta-1}\right) &\leq (I-b)\left(1 - \left(\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}\right)^{\beta}\right) \\ &\leq \theta_1\bar{X}(\theta_1)\left(1 - \left(\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}\right)^{\beta-1}\right),\end{aligned} \tag{42}$$

which is a contradiction, because $\theta_2 > \theta_1$ and $\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)} < 1$. Therefore, $\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$ for any $\theta_1, \theta_2 \in \Theta$ such that $\theta_2 \geq \theta_1$. ■

Proof of Lemma 2. Consider a threshold exercise equilibrium E with an arbitrary message space M^* and equilibrium message strategy m^*, in which exercise occurs at stopping time $\tau^*(\theta) = \inf\left\{t \geq 0 | X(t) \geq \bar{X}(\theta)\right\}$ for some set of thresholds $\bar{X}(\theta)$, $\theta \in \Theta$. By Lemma 1, $\bar{X}(\theta)$ is weakly decreasing. Define $\theta_l(X) \equiv \inf\left\{\theta : \bar{X}(\theta) = X\right\}$ and $\theta_h(X) \equiv \sup\left\{\theta : \bar{X}(\theta) = X\right\}$ for any $X \in \mathcal{X}$. We will construct a different equilibrium, denoted by $\bar{E}$, which implements the same equilibrium exercise time $\tau^*(\theta)$ and has the structure specified in the formulation of the lemma. As we will see, it will imply that on the equilibrium path, the principal exercises the option at the first informative time $t \in \mathcal{T}$ at which he receives message $m(t) = 1$, where the set $\mathcal{T}$ of informative times is defined as

$$\mathcal{T} \equiv \left\{t : X(t) = \bar{X} \text{ for some } \bar{X} \in \mathcal{X} \text{ and } X(s) < \bar{X}\ \forall s < t\ \right\},$$

i.e., the set of times when the process $X(t)$ reaches one of the thresholds in X for the first time.

For the collection of strategies (7) and (8) and the corresponding beliefs to be an equilibrium, we need to verify incentive compatibility conditions of the agent and the principal.

1 - Incentive compatibility of the agent.

The IC condition of the agent requires that any type θ is better off sending a message $m(t) = 1$ when $X(t)$ first reaches $\bar{X}(\theta)$ than following any other strategy. By Assumption 1, a deviation to sending $m(t) = 1$ at any $t \notin \mathcal{T}$ does not lead the principal to change his beliefs, and hence, his behavior. Thus, it is without loss of generality to only consider deviations at $t \in \mathcal{T}$. There are two possible deviations: sending $m(t) = 1$ before $X(t)$ first reaches $\bar{X}(\theta)$ and sending $m(t) = 0$ at that moment and following some other strategy after that. Consider the first deviation: the agent of type θ can send $m(t) = 1$ when $X(t)$ hits threshold $\bar{X}(\hat{\theta})$, $\hat{\theta} > \theta_h\left(\bar{X}(\theta)\right)$ for the first time, and then the principal will exercise immediately. Consider the second deviation: if type θ deviates to sending $m(t) = 0$ when $X(t)$ hits threshold $\bar{X}(\theta)$, she can then either send $m(t) = 1$ at one of the future $t \in \mathcal{T}$ or continue sending the message $m(t) = 0$ at any future

$t \in \mathcal{T}$. First, if the agent deviates to sending $m(t) = 1$ at one of the future $t \in \mathcal{T}$, the principal will exercise the option at one of the thresholds $\hat{X} \in \mathcal{X}$, $\hat{X} > \bar{X}(\theta)$. Note that the agent can ensure exercise at any threshold $\hat{X} \in \mathcal{X}$ such that $\hat{X} \geq X(t)$ by adopting the equilibrium message strategy of type $\hat{\theta}$ at which $\bar{X}(\hat{\theta}) = \hat{X}$. Second, if the agent deviates to sending $m(t) = 0$ at all of the future $t \in \mathcal{T}$, there are two cases. If $\bar{X}(\underline{\theta}) = \infty$, the principal will never exercise the option. If $\bar{X}(\underline{\theta}) = \bar{X}_{\max} < \infty$, then the principal's belief when $X(t)$ first reaches $\bar{X}_{\max}$ is that $\theta = \underline{\theta}$, if $\bar{X}(\underline{\theta}) \neq \bar{X}(\theta)\ \forall \theta \neq \underline{\theta}$, or that $\theta \in \left[\underline{\theta}, \theta_h\left(\bar{X}_{\max}\right)\right]$, otherwise. Upon receiving $m(t) = 0$ at this moment, the principal does not change his belief by Assumption 1 and hence exercises the option at $\bar{X}_{\max} = \bar{X}(\underline{\theta})$. Finally, note that the agent cannot induce exercise at $\hat{X} \in \mathcal{X}$ if $\hat{X} < X(t)$: in this case, the principal's belief is that the agent's type is smaller than the type that could induce exercise at $\hat{X}$ and this belief cannot be reversed according to Assumption 1. Combining all possible deviations, at time t, the agent can deviate to exercise at any $\hat{X} \in \mathcal{X}$ as long as $\hat{X} \geq X(t)$. Using the same arguments as in the benchmark case of Section 2.1 (e.g., Dixit and Pindyck, 1994), it is easy to show that the agent's expected utility given exercise at threshold $\bar{X}$ is $\left(\frac{X(t)}{\bar{X}}\right)^{\beta}\left(\theta\bar{X} - I + b\right)$, where $\beta > 1$ is given by (14). Hence, the incentive compatibility condition of the agent is that

$$\left(\frac{X(t)}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta) - I + b\right) \geq \max_{\hat{X} \in \mathcal{X}, \hat{X} \geq X(t)} \left(\frac{X(t)}{\hat{X}}\right)^{\beta}\left(\theta\hat{X} - I + b\right). \tag{43}$$

Let us argue that it holds using the fact that E is an equilibrium. Suppose otherwise. Then, there exists a pair $(\theta, \hat{X})$ with $\hat{X} \in \mathcal{X}$ such that

$$\frac{\theta\bar{X}(\theta) - I + b}{\bar{X}(\theta)^{\beta}} < \frac{\theta\hat{X} - I + b}{\hat{X}^{\beta}}. \tag{44}$$

However, (44) implies that in equilibrium E type θ is better off deviating from the message strategy $m^*(\theta)$ to the message strategy $m^*\left(\tilde{\theta}\right)$ of type $\tilde{\theta}$, where $\tilde{\theta}$ is any type satisfying $\bar{X}\left(\tilde{\theta}\right) = \hat{X}$ (since $\hat{X} \in \mathcal{X}$, at least one such $\tilde{\theta}$ exists). This is impossible, and hence (43) holds. Hence, if the principal plays strategy (8), the agent finds it optimal to play strategy (7).

Given Lemma 1 and the fact that the agent plays (7), the posterior belief of the principal at any time t is that θ is distributed uniformly over $[\check{\theta}_t, \hat{\theta}_t]$ for some $\check{\theta}_t$ and $\hat{\theta}_t$ (possibly, equal).

Next, consider incentive compatibility conditions of the principal. They are comprised of two parts, as evident from (8): we refer to the top line of (8) (exercising immediately when the principal "should" exercise) as the ex-post incentive compatibility condition, and to the bottom line of (8) (not exercising when the principal "should" wait) as the ex-ante incentive compatibility condition.

2 - "Ex-post" incentive compatibility of the principal. First, consider the ex-post incentive compatibility condition: we prove that the principal exercises immediately if the agent sends message $m(t) = 1$ at the first moment when $X(t)$ hits threshold $\hat{X}$ for some $\hat{X} \in \mathcal{X}$ (and sent message $m(t) = 0$ before). Given this message, the principal believes that $\theta \sim Uni[\theta_l(\hat{X}), \theta_h(\hat{X})]$. Because the principal expects the agent to play (7), the principal now expects the agent to send $m(t) = 1$ if $X(t) \geq \hat{X}$, and $m(t) = 0$ otherwise, regardless of $\theta \in [\theta_l(\hat{X}), \theta_h(\hat{X})]$. Hence, the principal does not expect to learn any new information. This implies that the principal's problem is now equivalent to the standard option exercise problem with the option paying off $\frac{\theta_l(\hat{X}) + \theta_h(\hat{X})}{2} X(t)$ upon exercise at time t. Using the same arguments as in the benchmark case of Section 2.1 (e.g., Dixit and Pindyck, 1994), the principal's expected payoff from exercise at threshold $\bar{X}$ is $\left(\frac{X(t)}{\bar{X}}\right)^{\beta}\left(\frac{\theta_l(\hat{X}) + \theta_h(\hat{X})}{2}\bar{X} - I\right)$, which is an inverse U-shaped function with an unconditional maximum at $\frac{\beta}{\beta-1}\frac{2I}{\theta_l(\hat{X}) + \theta_h(\hat{X})}$. Thus, the solution of the problem is to exercise the option immediately if and only if

$$X(t) \geq \frac{\beta}{\beta - 1}\frac{2I}{\theta_l(\hat{X}) + \theta_h(\hat{X})}. \tag{45}$$

Let us show that any threshold $\hat{X} \in \mathcal{X}$ and the corresponding type cutoffs $\theta_l(\hat{X})$ and $\theta_h(\hat{X})$ in equilibrium E satisfy (45). Consider equilibrium E. For the principal to exercise at threshold $\bar{X}(\theta)$, the value that the principal gets upon exercise must be greater or equal than what he gets from delaying the exercise. The value from immediate exercise equals $\mathbb{E}[\theta|\mathcal{H}_t, m(t)]\,\bar{X}(\theta) - I$, where $(\mathcal{H}_t, m(t))$ is any history of the sample path of $X(t)$ and equilibrium messages that leads to exercise at time t at threshold $\bar{X}(\theta)$ in equilibrium E. Because waiting until $X(t)$ hits a threshold $\tilde{X} > \bar{X}(\theta)$ and exercising then is a feasible strategy, the value from delaying exercise is greater or equal than the value from such a deviation, which equals

$$\left(\frac{\bar{X}(\theta)}{\tilde{X}}\right)^{\beta}\left(\mathbb{E}[\theta|\mathcal{H}_t, m(t)]\,\tilde{X} - I\right).$$

Hence, $\bar{X}(\theta)$ must satisfy

$$\bar{X}(\theta) \in \arg\max_{\tilde{X} \geq \bar{X}(\theta)} \left(\frac{\bar{X}(\theta)}{\tilde{X}}\right)^{\beta}\left(\mathbb{E}[\theta|\mathcal{H}_t, m(t)]\,\tilde{X} - I\right).$$

Using the fact that the unconditional maximizer of the right-hand side is $\tilde{X} = \frac{\beta}{\beta-1}\frac{I}{\mathbb{E}[\theta|\mathcal{H}_t, m(t)]}$ and that function $\left(\frac{\bar{X}(\theta)}{\tilde{X}}\right)^{\beta}\left(\mathbb{E}[\theta|\mathcal{H}_t, m_t]\,\tilde{X} - I\right)$ is inverted U-shaped in $\tilde{X}$, this condition can be equivalently re-written as

$$\bar{X}(\theta) \geq \frac{\beta}{\beta-1}\frac{I}{\mathbb{E}[\theta|\mathcal{H}_t, m(t)]},$$

for any history $(\mathcal{H}_t, m(t))$ with $X(t) = \bar{X}(\theta)$ and $m(s) = m_s^*(\mathcal{H}_s, \theta)$ for some $\theta \in [\theta_l(\hat{X}), \theta_h(\hat{X})]$ and $s \leq t$. Let $\mathbb{H}_t^*$ denote the set of such histories. Then,

$$\bar{X}(\theta) \geq \frac{\beta}{\beta-1} \max_{(\mathcal{H}_t, m(t)) \in \mathbb{H}_t^*} \frac{I}{\mathbb{E}[\theta|\mathcal{H}_t, m(t)]},$$

or, equivalently,

$$\begin{aligned}
\frac{\beta}{\beta-1}\frac{I}{\bar{X}(\theta)} &\leq \min_{(\mathcal{H}_t, m(t)) \in \mathbb{H}_t^*} \mathbb{E}[\theta|\mathcal{H}_t, m(t)] \\
&\leq \mathbb{E}\left[\mathbb{E}[\theta|\mathcal{H}_t, m(t)]\,|\theta \in [\theta_l(\hat{X}), \theta_h(\hat{X})], \mathcal{H}_0\right] \\
&= \mathbb{E}\left[\theta|\theta \in [\theta_l(\hat{X}), \theta_h(\hat{X})]\right] = \frac{\theta_l(\hat{X}) + \theta_h(\hat{X})}{2},
\end{aligned}$$

where the inequality follows from the fact that the minimum of a random variable cannot exceed its mean, and the first equality follows from the law of iterated expectations. Therefore, when the principal obtains message $m = 1$ at threshold $\hat{X} \in \mathcal{X}$, his optimal reaction is to exercise immediately. Thus, the ex-post incentive compatibility condition of the principal is satisfied.

3 - "Ex-ante" incentive compatibility of the principal. Finally, consider the ex-ante incentive compatibility condition of the principal stating that the principal is better off waiting following a history $\mathcal{H}_t$ with $m(s) = 0$, $s \leq t$, and $\max_{s \leq t} X(s) < \bar{X}(\underline{\theta})$. Given that the agent follows (8), for any such history $\mathcal{H}_t$, the principal's belief is that $\theta \sim Uni[\underline{\theta}, \theta_l(\hat{X})]$ for some $\hat{X} \in \mathcal{X}$. If the principal exercises immediately, her expected payoff is $\frac{\underline{\theta}+\theta_l(\hat{X})}{2}X(t) - I$. If the principal waits, her expected payoff is

$$\int_{\underline{\theta}}^{\theta_l(\hat{X})} \left(\frac{X(t)}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta) - I\right)\frac{1}{\theta_l(\hat{X}) - \underline{\theta}}d\theta.$$

Suppose that there exists a pair $\hat{X} \in \mathcal{X}$ and $\tilde{X} < \lim_{\theta \uparrow \theta_l(\hat{X})} \bar{X}(\theta)$ such that immediate exercise is optimal when $X(t)=\tilde{X}$:

$$\frac{\underline{\theta}+\theta_l(\hat{X})}{2}\tilde{X}-I>\int_{\underline{\theta}}^{\theta_l(\hat{X})}\left(\frac{\tilde{X}}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta)-I\right)\frac{1}{\theta_l(\hat{X})-\underline{\theta}}d\theta. \tag{46}$$

We can re-write (46) as

$$\mathbb{E}_{\theta}\left[\left(\frac{1}{\tilde{X}}\right)^{\beta}\left(\theta\tilde{X}-I\right)|\theta<\theta_l(\hat{X})\right]>\mathbb{E}_{\theta}\left[\left(\frac{1}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta)-I\right)|\theta<\theta_l(\hat{X})\right]. \tag{47}$$

Let us show that if equilibrium E exists, then (47) must be violated. Consider equilibrium E, any type $\tilde{\theta}<\theta_l(\hat{X})$, time $t<\tau^*\left(\tilde{\theta}\right)$, and any history $(\mathcal{H}_t,m(t))$ such that $X(t)=\tilde{X}$, $\max_{s\le t,s\in\mathcal{T}}X(s)=\hat{X}$, which is consistent with the equilibrium play of type $\tilde{\theta}$, i.e., with $m(s)=m_s^*\left(\tilde{\theta},\mathcal{H}_s\right)\ \forall s\le t$. Let $\mathbb{H}_t^{**}(\tilde{\theta},\tilde{X},\hat{X})$ denote the set of such histories. Because the principal prefers waiting in equilibrium E, the payoff from immediate exercise in equilibrium E cannot exceed the payoff from waiting:

$$\mathbb{E}\left[\theta\tilde{X}-I|\mathcal{H}_t,m(t)\right]\le\mathbb{E}\left[\left(\frac{\tilde{X}}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta)-I\right)|\mathcal{H}_t,m(t)\right],$$

or, equivalently,

$$\mathbb{E}\left[\left(\frac{1}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta)-I\right)-\left(\frac{1}{\tilde{X}}\right)^{\beta}\left(\theta\tilde{X}-I\right)|\mathcal{H}_t,m(t)\right]\ge 0.$$

This inequality must hold for all histories $(\mathcal{H}_t,m(t))\in\mathbb{H}_t^{**}(\tilde{\theta},\tilde{X},\hat{X})$. In any history $(\mathcal{H}_t,m(t))\in\mathbb{H}_t^{**}(\tilde{\theta},\tilde{X},\hat{X})$, the option is never exercised by time t if $\tilde{\theta}<\theta_l(\hat{X})$ and is exercised before time t if $\tilde{\theta}>\theta_l(\hat{X})$. Therefore, conditional on $\tilde{X}$, $\hat{X}$, and $\tilde{\theta}<\theta_l(\hat{X})$, the distribution of $\tilde{\theta}$ is independent of the sample path of $X(s), s\le t$. Fixing $\tilde{X}$ and $\hat{X}$ and integrating over histories $(\mathcal{H}_t,m(t))\in\mathbb{H}_t^{**}(\tilde{\theta},\tilde{X},\hat{X})$ and then over $\tilde{\theta}\in[\underline{\theta},\theta_l(\hat{X}))$, we obtain that

$$\mathbb{E}_{\tilde{\theta}}\left[\mathbb{E}_{(\mathcal{H}_t,m(t))}\left[\left(\frac{1}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta)-I\right)-\left(\frac{1}{\tilde{X}}\right)^{\beta}\left(\theta\tilde{X}-I\right)|(\mathcal{H}_t,m(t))\in\mathbb{H}_t^{**}(\tilde{\theta},\tilde{X},\hat{X})\right]|\tilde{\theta}\in[\underline{\theta},\theta_l(\hat{X})),\hat{X},\tilde{X}\right]$$

must be non-negative. Equivalently,

$$\mathbb{E}_{\theta}\left[\left(\frac{1}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta)-I\right)-\left(\frac{1}{\tilde{X}}\right)^{\beta}\left(\theta\tilde{X}-I\right)|\theta<\theta_l(\hat{X})\right]\ge 0,$$

where we applied the law of iterated expectations and the conditional independence of the sample path of $X(t)$ and the distribution of $\tilde{\theta}$ (conditional on $\tilde{X}$, $\hat{X}$, and $\tilde{\theta}<\theta_l(\hat{X})$). Therefore, (47) cannot hold. Hence, the ex-ante incentive compatibility condition of the principal is also satisfied.

We conclude that if there exists a threshold exercise equilibrium E in which $\tau^*(\theta)=\inf\left\{t\ge 0|X(t)\ge\bar{X}(\theta)\right\}$ for some threshold $\bar{X}(\theta)$, then there exists a threshold exercise equilibrium $\bar{E}$ of the form specified in the lemma, in which the option is exercised at the same time. Finally, let us show that on the equilibrium path, the option is indeed exercised at the first informative time t at which the principal receives message $m(t)=1$. Because any message sent at $t\notin\mathcal{T}$ does not lead to updating of the principal's beliefs and because of the second part of (8), the principal never exercises the option prior to the first informative time $t\in\mathcal{T}$ at which she receives message $m(t)=1$. Consider the first informative time $t\in\mathcal{T}$ at which the principal receives $m(t)=1$. By Bayes' rule, the principal believes that θ is distributed uniformly over $(\theta_l(X(t)), \theta_h(X(t)))$. Equilibrium strategy of the agent (7) implies $X(t) = \bar{X}(\theta)$ $\forall \theta \in (\theta_l(X(t)), \theta_h(X(t)))$. Therefore, in equilibrium the principal exercises the option immediately. ■

Derivation of the principal's value function in the ω-equilibrium, $V_P(X(t), 1; \omega)$. It satisfies

$$rV_P(X, 1; \omega) = \mu X V_{P,X}(X, 1; \omega) + \frac{1}{2}\sigma^2 X^2 V_{P,XX}(X, 1; \omega). \tag{48}$$

The value matching condition is:

$$V_P(Y(\omega), 1; \omega) = \int_\omega^1 (\theta Y(\omega) - I)\, d\theta + \omega V_P(Y(\omega), \omega; \omega). \tag{49}$$

The intuition behind (49) is as follows. With probability $1 - \omega$, θ is above ω. In this case, the agent recommends exercise, and the principal follows the recommendation. The payoff of the principal, given θ, is $\theta Y(\omega) - I$. With probability ω, θ is below ω, so the agent recommends against exercise, and the option is not exercised. The continuation payoff of the principal in this case is $V_P(Y(\omega), \omega; \omega)$. Solving (48) subject to (49), we obtain

$$V_P(X, 1; \omega) = \left(\frac{X}{Y(\omega)}\right)^\beta \left(\int_\omega^1 (\theta Y(\omega) - I)\, d\theta + \omega V_P(Y(\omega), \omega; \omega)\right). \tag{50}$$

By stationarity,

$$V_P(Y(\omega), \omega; \omega) = V_P(\omega Y(\omega), 1; \omega). \tag{51}$$

Evaluating (50) at $X = \omega Y(\omega)$ and using the stationarity condition (51), we obtain:

$$V_P(\omega Y(\omega), 1; \omega) = \omega^\beta \left[\frac{1}{2}\left(1 - \omega^2\right) Y(\omega) - (1 - \omega) I\right] + \omega^{\beta+1} V_P(\omega Y(\omega), 1; \omega). \tag{52}$$

Therefore,

$$V_P(\omega Y(\omega), 1; \omega) = \frac{\omega^\beta (1 - \omega)}{1 - \omega^{\beta+1}} \left[\frac{1}{2}(1 + \omega) Y(\omega) - I\right]. \tag{53}$$

Plugging (53) into (50), we obtain the principal's value function (24). ■

Proof of Proposition 1. Part 1: Existence of equilibrium with continuous exercise. As shown in the text, the principal's ex-post IC is satisfied, and hence we only need to check the ex-ante IC (30). We show that (30) is satisfied if and only if $b \geq -I$. Using (29), (30) is equivalent to

$$\frac{1}{\beta + 1}\left(\frac{\beta}{\beta - 1}(I - b)\right)^{-\beta} \frac{I - \beta b}{\beta - 1} \geq \max_{X \in (0, X_A^*(1)]} X^{-\beta}\left(\frac{1}{2}X - I\right). \tag{54}$$

The function $X^{-\beta}\left(\frac{1}{2}X - I\right)$ is inverse U-shaped with a maximum at $\bar{X}_u \equiv \frac{\beta}{\beta - 1} 2I$, where $\bar{X}_u > X_A^*(1) \Leftrightarrow b > -I$. First, suppose that $b < -I$, and hence $\bar{X}_u < X_A^*(1)$. Then, (54) is equivalent to

$$\frac{1}{\beta + 1}\left(\frac{\beta}{\beta - 1}(I - b)\right)^{-\beta} \frac{I - \beta b}{\beta - 1} \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right) \Leftrightarrow \frac{1}{\beta + 1}(I - b)^{-\beta}(I - \beta b) \geq (2I)^{-\beta} I. \tag{55}$$

Consider $f(b) \equiv (I - b)^{-\beta}(I - \beta b) - (\beta + 1)(2I)^{-\beta} I$. Note that $f(-I) = 0$ and $f'(b) > 0$. Hence, $f(b) \geq 0 \Leftrightarrow b \geq -I$, and hence (54) is violated when $b < -I$.

Second, suppose that $b \geq -I$, and hence (55) is satisfied. Since, in this case, $\bar{X}_u \geq X_A^*(1)$, then

$$\max_{X \in (0, X_A^*(1)]} X^{-\beta} \left(\frac{1}{2} X - I \right) \leq \bar{X}_u^{-\beta} \left(\frac{1}{2} \bar{X}_u - I \right),$$

and hence the inequality (54) follows from the fact that inequality (55) is satisfied.

Part 2. Existence of ω-equilibria. We first show that if $b < 0$, then for any positive $\omega < 1$, the principal's ex-post IC is strictly satisfied, i.e., $Y(\omega) > \frac{\beta}{\beta-1} \frac{2I}{1+\omega}$. Define:

$$G(\omega) = \frac{\left(1 - \omega^{\beta}\right)(I - b)}{\omega \left(1 - \omega^{\beta-1}\right)} - \frac{\beta}{\beta - 1} \frac{2(I - b)}{1 + \omega}.$$

Note that $G(\omega) = \frac{2(I-b)}{1+\omega} g(\omega)$, where

$$g(\omega) = \frac{\left(1 - \omega^{\beta}\right)(1 + \omega)}{2\left(\omega - \omega^{\beta}\right)} - \frac{\beta}{\beta - 1}.$$

We have:

$$\begin{aligned} \lim_{\omega \to 1} g(\omega) &= \lim_{\omega \to 1} \frac{1 - \omega^{\beta} - \beta \omega^{\beta-1}(1 + \omega)}{2\left(1 - \beta \omega^{\beta-1}\right)} - \frac{\beta}{\beta - 1} = 0, \\ g'(\omega) &= \frac{\beta\left(\omega^{\beta-1} - \omega^{\beta+1}\right) + \omega^{2\beta} - 1}{2\left(\omega - \omega^{\beta}\right)^2}, \end{aligned}$$

where the first limit holds by l'Hopital's rule. Denote the numerator of $g'(\omega)$ by $h(\omega) \equiv \omega^{2\beta} - \beta\omega^{\beta+1} + \beta\omega^{\beta-1} - 1$. Function $h(\omega)$ is a generalized polynomial. By an extension of Descartes' Rule of Signs to generalized polynomials (Laguerre, 1883),[11] the number of positive roots of $h(\omega)$, counted with their orders, does not exceed the number of sign changes of coefficients of $h(\omega)$, i.e., three. Because $\omega = 1$ is the root of $h(\omega)$ of order three and $h(0) < 0$, then $h(\omega) < 0$ for all $\omega \in [0, 1)$, and hence $g'(\omega) < 0$ for all $\omega \in [0, 1)$. Combined with $\lim_{\omega \to 1} g(\omega) = 0$, this implies $g(\omega) > 0$ and $G(\omega) > 0$ for all $\omega \in [0, 1)$. Thus,

$$Y(\omega) > \frac{\beta}{\beta - 1} \frac{2(I - b)}{1 + \omega} > \frac{\beta}{\beta - 1} \frac{2I}{1 + \omega},$$

where the second inequality follows from the fact that $b < 0$.

Hence, the ex-post IC condition of the principal is satisfied for any $\omega < 1$, which implies that the ω-equilibrium exists if and only if the ex-ante IC (27) is satisfied, where $V_P(X, 1; \omega)$ is given by (24). Because $X^{-\beta} V_P(X, 1; \omega)$ does not depend on X, we can rewrite (27) as

$$X^{-\beta} V_P(X, 1; \omega) \geq \max_{X \in (0, Y(\omega)]} X^{-\beta} \left(\frac{1}{2} X - I \right). \tag{56}$$

We pin down the range of ω that satisfies this condition in the following steps.

Step 1: *If $b < 0$, $V_P(X, 1; \omega)$ is strictly increasing in ω for any $\omega \in (0, 1)$.*

Because $V_P(X, 1; \omega)$ is proportional to X^{β}, it is enough to prove the statement for $X = 1$. We can

[11] See Theorem 3.1 in Jameson (2006).

re-write $V_P(1,1;\omega)$ as $2^{-\beta} f_1(\omega) f_2(\omega)$, where

$$f_1(\omega) \equiv \frac{(1-\omega)(1+\omega)^{\beta}}{1-\omega^{\beta+1}}, \tag{57}$$
$$f_2(\omega) \equiv \frac{\frac{1}{2}(1+\omega)Y(\omega) - I}{\left(\frac{1}{2}(1+\omega)Y(\omega)\right)^{\beta}}.$$

Since, as shown above, $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ for $\omega < 1$, then $\frac{1}{2}(1+\omega)Y(\omega) > \frac{\beta}{\beta-1}I > I$, and hence $f_2(\omega) > 0$ for $\omega < 1$. Because $f_1(\omega) > 0$ and $f_2(\omega) > 0$ for any $\omega \in (0,\omega^*)$, a sufficient condition for $V_P(1,1;\omega)$ to be increasing is that both $f_1(\omega)$ and $f_2(\omega)$ are increasing for $\omega \in (0,\omega^*)$.

First, consider $f_2(\omega)$. As an auxiliary result, we prove that $(1+\omega)Y(\omega)$ is strictly decreasing in ω. This follows from the fact that

$$\frac{\partial((1+\omega)Y(\omega))}{\partial\omega} = (I-b)\frac{-1+\beta\omega^{\beta-1}-\beta\omega^{\beta+1}+\omega^{2\beta}}{(\omega-\omega^{\beta})^2}$$

and that as shown above, the numerator, $h(\omega)$, is strictly negative for all $\omega \in [0,1)$. Next,

$$f_2'(\omega) = \frac{(\beta-1)(1+\omega)}{4\left(\frac{1}{2}(1+\omega)Y(\omega)\right)^{\beta+1}}\left(\frac{\beta}{\beta-1}\frac{2I}{1+\omega} - Y(\omega)\right)\frac{\partial((1+\omega)Y(\omega))}{\partial\omega}.$$

Because $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ for $\omega < 1$ as shown above, and because $(1+\omega)Y(\omega)$ is strictly decreasing in ω, $f_2'(\omega) > 0$ for any $\omega \in (0,\omega^*)$.

Second, consider $f_1(\omega)$. Note that

$$f_1'(\omega) = \frac{(1+\omega)^{\beta-1}}{1-\omega^{\beta+1}}\frac{\beta-1-(\beta+1)\omega+(\beta+1)\omega^{\beta}-(\beta-1)\omega^{\beta+1}}{1-\omega^{\beta+1}}.$$

Denote the numerator of the second fraction by $d(\omega) \equiv -(\beta-1)\omega^{\beta+1}+(\beta+1)\omega^{\beta}-(\beta+1)\omega+\beta-1$. By an extension of Descartes' Rule of Signs to generalized polynomials, the number of positive roots of $d(\omega)$ does not exceed the number of sign changes of coefficients of $d(\omega)$, i.e., three. It is easy to show that $d(1) = d'(1) = d''(1) = 0$. Hence, $\omega = 1$ is the root of $d(\omega) = 0$ of order three, and $d(\omega)$ does not have roots other than $\omega = 1$. Since $d(0) = \beta - 1 > 0$, this implies that for any $\omega \in (0,1)$, $d(\omega) > 0$. Hence, $f_1'(\omega) > 0$, which completes the proof of this step.

Step 2: $\lim_{\omega\to 1} V_P(X,1;\omega) = V_P^c(X,1)$.

According to (57), $V_P(X,1;\omega) = 2^{-\beta}X^{\beta}f_1(\omega)f_2(\omega)$. By l'Hopital's rule, $\lim_{\omega\to 1} f_1(\omega) = \frac{2^{\beta}}{\beta+1}$, $\lim_{\omega\to 1} Y(\omega) = \frac{\beta}{\beta-1}(I-b)$, and hence

$$\lim_{\omega\to 1} f_1(\omega) = \frac{\frac{\beta}{\beta-1}(I-b)-I}{\left(\frac{\beta}{\beta-1}(I-b)\right)^{\beta}}.$$

Using (29), it is easy to see that $\lim_{\omega\to 1} V_P(X,1;\omega) = 2^{-\beta}X^{\beta}\lim_{\omega\to 1} f_1(\omega)\lim_{\omega\to 1} f_1(\omega) = V_P^c(X,1)$.

Step 3. *Suppose $-I < b < I$. For ω close enough to zero, the ex-ante IC condition (56) does not hold.*

The function $X^{-\beta}\left(\frac{1}{2}X - I\right)$ is inverse U-shaped and has a maximum at $\bar{X}_u \equiv \frac{\beta}{\beta-1}2I$. When ω is close to zero, $Y(\omega) = \frac{(1-\omega^{\beta})(I-b)}{\omega(1-\omega^{\beta-1})} \to +\infty$, and hence $\max_{X\in(0,Y(\omega)]} X^{-\beta}\left(\frac{1}{2}X - I\right) = \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$. Hence, we can rewrite (56) as $X^{-\beta}V_P(X,1;\omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$, and it is easy to show that it is equivalent to

$$\left(\omega-\omega^{\beta}\right)^{\beta-1}H(\omega) \geq (I-b)^{\beta}\left(1-\omega^{\beta+1}\right)\left(1-\omega^{\beta}\right)^{\beta}, \tag{58}$$

where

$$H(\omega) = 2^{\beta-1}\beta^{\beta}\left(\frac{I}{\beta-1}\right)^{\beta-1}(1-\omega)\left(I(1-\omega)\left(1+\omega^{\beta}\right)-b(1+\omega)\left(1-\omega^{\beta}\right)\right).$$

Since $H(0) > 0$, then as ω converges to zero, the left-hand side of (58) converges to zero, while the right-hand side converges to $(I-b)^{\beta} > 0$. Hence, for ω close enough to 0, the ex-ante IC condition is violated.

Step 4. *Suppose $-I < b < I$. Then (56) is satisfied for any $\omega \geq \bar{\omega}$, where $\bar{\omega}$ is the unique solution to $Y(\omega) = \bar{X}_u$. For any $\omega < \bar{\omega}$, (56) is satisfied if and only if $X^{-\beta}V_P(X,1;\omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$.*

Note that for any $b > -I$, $\lim_{\omega\to 1} Y(\omega) = \frac{\beta(I-b)}{\beta-1} < \frac{\beta}{\beta-1}2I = \bar{X}_u$, and hence there exists a unique $\bar{\omega}$ such that $Y(\omega) \leq \bar{X}_u \Leftrightarrow \omega \geq \bar{\omega}$. Hence, (56) becomes

$$X^{-\beta}V_P(X,1;\omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right) \text{ for } \omega \leq \bar{\omega}, \tag{59}$$

$$X^{-\beta}V_P(X,1;\omega) \geq Y(\omega)^{-\beta}\left(\frac{1}{2}Y(\omega) - I\right) \text{ for } \omega \geq \bar{\omega}. \tag{60}$$

Suppose that (60) is satisfied for some $\tilde{\omega} \geq \bar{\omega}$. Because the function $Y(\omega)$ is decreasing, $Y(\tilde{\omega}) \geq Y(\omega)$ for $\omega \geq \tilde{\omega}$. Because the function $X^{-\beta}\left(\frac{1}{2}X - I\right)$ is increasing for $X \leq \bar{X}_u$ and because $Y(\omega) \leq Y(\tilde{\omega}) \leq Y(\bar{\omega}) = \bar{X}_u$ for $\omega \geq \tilde{\omega} \geq \bar{\omega}$, we have $Y(\omega)^{-\beta}\left(\frac{1}{2}Y(\omega) - I\right) \leq Y(\tilde{\omega})^{-\beta}\left(\frac{1}{2}Y(\tilde{\omega}) - I\right)$ for any $\omega \geq \tilde{\omega}$. On the other hand, according to Step 1, $X^{-\beta}V_P(X,1;\tilde{\omega}) \leq X^{-\beta}V_P(X,1;\omega)$ for any $\omega \geq \tilde{\omega}$. Hence, if (60) is satisfied for $\tilde{\omega} \geq \bar{\omega}$, it is also satisfied for any $\omega \in [\tilde{\omega}, 1)$. Hence, to prove that (56) is satisfied for any $\omega \geq \bar{\omega}$, it is sufficient to prove (60) for $\omega = \bar{\omega}$. Using (24) and the fact that $Y(\bar{\omega}) = \bar{X}_u$, (60) for $\omega = \bar{\omega}$ is equivalent to

$$\frac{1-\bar{\omega}}{1-\bar{\omega}^{\beta+1}}\bar{X}_u^{-\beta}\left(\frac{1}{2}(1+\bar{\omega})\bar{X}_u - I\right) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right) \Leftrightarrow \frac{1}{2}\bar{X}_u\left(\frac{1-\bar{\omega}^2}{1-\bar{\omega}^{\beta+1}} - 1\right) \geq I\left(\frac{1-\bar{\omega}}{1-\bar{\omega}^{\beta+1}} - 1\right)$$

$$\Leftrightarrow \frac{1}{2I}\bar{X}_u \leq \frac{\bar{\omega}-\bar{\omega}^{\beta+1}}{\bar{\omega}^2-\bar{\omega}^{\beta+1}} \Leftrightarrow \frac{\beta}{\beta-1} \leq \frac{\bar{\omega}-\bar{\omega}^{\beta+1}}{\bar{\omega}^2-\bar{\omega}^{\beta+1}} \tag{61}$$

Consider the function $g(\omega) \equiv \frac{\omega-\omega^{\beta+1}}{\omega^2-\omega^{\beta+1}}$. Note that $g'(\omega) < 0 \Leftrightarrow h(\omega) \equiv (\beta-1)\omega^{\beta} - \beta\omega^{\beta-1} + 1 > 0$. By an extension of Descartes' Rule of Signs to generalized polynomials (Laguerre, 1883), the number of positive roots of $h(\omega)$, counted with their orders, does not exceed the number of sign changes of coefficients of $h(\omega)$, i.e., two. Since $h(1) = h'(1) = 0$, $h(\omega)$ does not have any roots on $(0,\infty)$ other than 1. Since $h''(1) > 0$, we have $h(\omega) > 0$ for all $\omega \in (0,1)$, and hence $g'(\omega) < 0$. By l'Hopital's rule, $\lim_{\omega\to 1} g(\omega) = \frac{\beta}{\beta-1}$, and hence $\frac{\beta}{\beta-1} \leq g(\omega)$ for any $\omega \in (0,1)$, which proves (61).

Step 5. *Conditions for (56) to hold.*

Combining the four steps above yields the statement of the proposition. First, if $b \leq -I$, then $I-b \geq 2I$, and hence $\lim_{\omega\to 1} Y(\omega) = \frac{\beta(I-b)}{\beta-1} \geq \bar{X}_u$. Since $Y(\omega)$ is decreasing, it implies that $Y(\omega) > \bar{X}_u$ for any $\omega < 1$, and hence (56) is equivalent to

$$X^{-\beta}V_P(X,1;\omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right). \tag{62}$$

According to Steps 1 and 2, for any $\omega < 1$, $V_P(X,1;\omega) < \lim_{\omega\to 1} V_P(X,1;\omega) = V_P^c(X,1)$. As shown in the proof of the equilibrium with continuous exercise above, $X^{-\beta}V_P^c(X,1) \leq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$ for $b \leq -I$, and hence (62) is violated. Hence, there is no ω-equilibrium in this case.

Second, if $0 > b > -I$, then according to Step 4, (56) is satisfied for any $\omega \geq \bar{\omega}$, and for any $\omega \leq \bar{\omega}$ (56) is satisfied if and only if $X^{-\beta}V_P(X,1;\omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$. The left-hand side of this inequality is increasing in ω according to Step 1, while the right-hand side is constant. Hence, if (56) is satisfied for some $\tilde{\omega}$, it is satisfied for any $\omega \geq \tilde{\omega}$. According to Step 3, for ω close to 0, (56) does not hold. Together, this implies that there exists a unique $\underline{\omega} \in (0, \bar{\omega})$ such that the principal's ex-ante IC (56) holds if and only if $\omega \geq \underline{\omega}$, and that $X^{-\beta} V_P(X, 1; \underline{\omega}) = \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$. ■

Proof of Proposition 3. First, we prove that the principal's ex-ante IC constraint is violated in the equilibrium with continuous exercise, and hence such equilibrium does not exist. To show this, we prove that when the current belief of the principal is that $\theta \in \left[\underline{\theta}, \hat{\theta}\right]$, where $\hat{\theta}$ is sufficiently close to $\underline{\theta}$, then the principal is strictly better off exercising immediately at $X_A^*\left(\hat{\theta}\right)$. Indeed, the value from waiting if the current value of $X(t)$ is $X_A^*\left(\hat{\theta}\right)$ is given by

$$\int_{\underline{\theta}}^{\hat{\theta}} \frac{1}{\hat{\theta} - \underline{\theta}} \left(\frac{X_A^*\left(\hat{\theta}\right)}{X_A^*(\theta)}\right)^{\beta} \left(X_A^*(\theta)\theta - I\right) d\theta,$$

while the value from immediate exercise at $X_A^*\left(\hat{\theta}\right)$ is $X_A^*\left(\hat{\theta}\right)\frac{\hat{\theta}+\underline{\theta}}{2} - I$. Hence, we want to show that for a sufficiently small $\hat{\theta}$,

$$\int_{\underline{\theta}}^{\hat{\theta}} X_A^*(\theta)^{-\beta}\left(X_A^*(\theta)\theta - I\right) d\theta < X_A^*\left(\hat{\theta}\right)^{-\beta}\left(\hat{\theta} - \underline{\theta}\right)\left(X_A^*\left(\hat{\theta}\right)\frac{\hat{\theta}+\underline{\theta}}{2} - I\right) = \int_{\underline{\theta}}^{\hat{\theta}} X_A^*\left(\hat{\theta}\right)^{-\beta}\left(X_A^*\left(\hat{\theta}\right)\theta - I\right) d\theta \Leftrightarrow$$
$$\int_{\underline{\theta}}^{\hat{\theta}} \left[f\left(X_A^*(\theta), \theta\right) - f\left(X_A^*\left(\hat{\theta}\right), \theta\right)\right] d\theta < 0, \tag{63}$$

where $f(X, \theta) = X^{-\beta}(X\theta - I)$. We next show that (63) is satisfied for any $\hat{\theta} \in \left(\underline{\theta}, \frac{I-b}{I}\underline{\theta}\right)$. Indeed, for any such $\hat{\theta}$, we have $\hat{\theta} < \frac{I-b}{I}\underline{\theta} \Leftrightarrow X_P^*(\underline{\theta}) < X_A^*\left(\hat{\theta}\right)$, and hence $X_A^*\left(\hat{\theta}\right) > X_P^*(\theta)$ for any $\theta \in \left[\underline{\theta}, \hat{\theta}\right]$. The function $f(X, \theta)$ is inverse U-shaped with a maximum at $X_P^*(\theta)$, and since $X_P^*(\theta) < X_A^*\left(\hat{\theta}\right) < X_A^*(\theta)$ for any $\theta \in \left[\underline{\theta}, \hat{\theta}\right]$, then $f\left(X_A^*(\theta)\right) < f\left(X_A^*\left(\hat{\theta}\right)\right)$, which implies (63).

Next, we consider the existence of the equilibrium with continuous exercise up to a cutoff. First, suppose that $b > -\frac{1-\underline{\theta}}{1+\underline{\theta}}I$. Note that this implies $b > -I$, and hence $b > -\frac{1-\underline{\theta}}{1+\underline{\theta}} \Leftrightarrow b + I > (I - b)\underline{\theta} \Leftrightarrow \hat{\theta}^* \equiv \frac{I-b}{I+b}\underline{\theta} < 1$. Given that the principal plays the strategy stated in the proposition, it is clear that the strategy of any type θ of the agent is incentive compatible. Indeed, for any type $\theta \geq \left(\frac{I-b}{I+b}\right)\underline{\theta}$, exercise occurs at her most preferred time. Therefore, no type $\theta \geq \left(\frac{I-b}{I+b}\right)\underline{\theta}$ can benefit from a deviation. Any type $\theta < \left(\frac{I-b}{I+b}\right)\underline{\theta}$ cannot benefit from a deviation either: the agent would lose from inducing the principal to exercise earlier because she is biased towards late exercise, and it is not feasible for her to induce the principal to exercise later because the principal exercises at threshold $\hat{X}$ regardless of the recommendation.

We next show that the principal's ex-post IC constraint is satisfied. If the agent sends a message to exercise when $X(t) < \hat{X}$, the principal learns the agent's type θ and realizes that it is already too late ($X_P^*(\theta) < X_A^*(\theta)$) and thus does not benefit from delaying exercise even further. If the agent sends a message to exercise when $X(t) = \hat{X}$, the principal infers that $\theta \leq \hat{\theta}^*$ and that he will not learn any additional information by waiting more. Given the belief that $\theta \in \left[\underline{\theta}, \hat{\theta}^*\right]$, the optimal exercise threshold for the principal is given by

$$\frac{\beta}{\beta - 1}\frac{2I}{\underline{\theta} + \hat{\theta}^*} = \frac{\beta}{\beta - 1}\frac{2I}{\underline{\theta} + \left(\frac{I-b}{I+b}\right)\underline{\theta}} = \frac{\beta}{\beta - 1}\frac{I + b}{\underline{\theta}} = \hat{X},$$

and hence the ex-post IC constraint is satisfied.

Next, we consider the principal's ex-ante IC constraint. Let $V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)$ denote the expected value to the principal in the equilibrium with continuous exercise up to a cutoff if the current value of $X(t)$ is X and the current belief is that $\theta\in\left[\underline{\theta},\hat{\theta}\right]$ for some $\hat{\theta}>\hat{\theta}^*$. If the agent's type is $\theta>\hat{\theta}^*$, exercise occurs at threshold $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$, and the principal's payoff upon exercise is $\frac{\beta}{\beta-1}(I-b)-I$. If $\theta<\hat{\theta}^*$, exercise occurs at threshold $\hat{X}$. Hence,

$$\left(\hat{\theta}-\underline{\theta}\right)V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)=X^\beta\int_{\underline{\theta}}^{\hat{\theta}^*}\hat{X}^{-\beta}\left(\theta\hat{X}-I\right)d\theta+X^\beta\int_{\hat{\theta}^*}^{\hat{\theta}}\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right)d\theta.$$

Given belief $\theta\in\left[\underline{\theta},\hat{\theta}\right]$, the principal can either wait and get $V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)$ or exercise immediately and get $X\frac{\underline{\theta}+\hat{\theta}}{2}-I$. The current value of $X(t)$ satisfies $X(t)\leq X_A^*\left(\hat{\theta}\right)$ because otherwise, the principal's belief would not be that $\theta\in\left[\underline{\theta},\hat{\theta}\right]$. Hence, the ex-ante IC condition requires that for any $\hat{\theta}>\hat{\theta}^*$, $V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)\geq X\frac{\underline{\theta}+\hat{\theta}}{2}-I$ for any $X\leq X_A^*\left(\hat{\theta}\right)$. Because $X^{-\beta}V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)$ does not depend on X, this condition is equivalent to

$$X^{-\beta}V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)\geq\max_{X\in(0,X_A^*(\hat{\theta})]}\frac{1}{X^\beta}\left(X\frac{\underline{\theta}+\hat{\theta}}{2}-I\right).\tag{64}$$

The function $\frac{1}{X^\beta}\left(X\frac{\underline{\theta}+\hat{\theta}}{2}-I\right)$ is inverse U-shaped and has an unconditional maximum at $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+\hat{\theta}}$, which is strictly greater than $X_A^*\left(\hat{\theta}\right)$ for any $\hat{\theta}>\frac{I-b}{I+b}\underline{\theta}=\hat{\theta}^*$. Because $X^{-\beta}V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)$ does not depend on X, (64) is equivalent to

$$X^{-\beta}V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)\geq X_A^*\left(\hat{\theta}\right)^{-\beta}\left(X_A^*\left(\hat{\theta}\right)\frac{\underline{\theta}+\hat{\theta}}{2}-I\right).$$

Suppose there exists $\hat{\theta}$ for which the ex-ante IC constraint is violated, i.e.,

$$\int_{\underline{\theta}}^{\hat{\theta}^*}\hat{X}^{-\beta}\left(\theta\hat{X}-I\right)d\theta+\int_{\hat{\theta}^*}^{\hat{\theta}}\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right)d\theta<\left(\hat{\theta}-\underline{\theta}\right)X_A^*\left(\hat{\theta}\right)^{-\beta}\left(X_A^*\left(\hat{\theta}\right)\frac{\underline{\theta}+\hat{\theta}}{2}-I\right).\tag{65}$$

We show that this implies that the contract derived in Proposition 7 cannot be optimal, which is a contradiction. In particular, (65) implies that the contract (38) is dominated by the contract with continuous exercise at $X_A^*(\theta)$ for $\theta\geq\hat{\theta}$ and exercise at $X_A^*\left(\hat{\theta}\right)$ for $\theta\leq\hat{\theta}$. Indeed, the principal's expected utility under the contract (38), divided by $X(0)^\beta$, is

$$\int_{\underline{\theta}}^{\hat{\theta}^*}\hat{X}^{-\beta}\left(\theta\hat{X}-I\right)d\theta+\int_{\hat{\theta}^*}^{1}\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right)d\theta.\tag{66}$$

Similarly, the principal's expected utility under the modified contract (also divided by $X(0)^\beta$), where $\frac{I-b}{I+b}\underline{\theta}$ in (38) is replaced by $\hat{\theta}$ and the cutoff $\frac{\beta}{\beta-1}\frac{I+b}{\underline{\theta}}$ in (38) is replaced by $X_A^*\left(\hat{\theta}\right)=\frac{\beta}{\beta-1}\frac{I-b}{\hat{\theta}}$, is given by

$$\int_{\underline{\theta}}^{\hat{\theta}}X_A^*\left(\hat{\theta}\right)^{-\beta}\left(\theta X_A^*\left(\hat{\theta}\right)-I\right)d\theta+\int_{\hat{\theta}}^{1}\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right)d\theta.\tag{67}$$

Combining (66) and (67), it is easy to see that the contract with continuous exercise up to the cutoff $\hat{\theta}$ dominates the contract (38) if and only if (65) is satisfied. Hence, the ex-ante IC constraint is indeed satisfied.

Finally, consider $b \leq -\frac{1-\theta}{1+\underline{\theta}} I$. According to Proposition 7, if $b \leq -\frac{1-\theta}{1+\underline{\theta}} I$, the optimal contract is characterized by $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$, i.e., the uninformed exercise threshold of the principal. Denote V_u the expected utility of the principal under this contract. Consider any equilibrium of the communication game, and note that the payoff of the principal in this equilibrium cannot be higher than V_u: otherwise, the contract $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$ would not be optimal. It cannot be lower than V_u either: otherwise, the principal would be better off deviating to exercising at $\bar{X} = \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$. Hence, the payoff of the principal in any equilibrium is exactly V_u. According to the proof of Proposition 7, there is a unique exercise policy $\hat{X}(\theta)$ that maximizes the principal's expected utility, and hence any equilibrium must be characterized by the principal exercising at $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$. ■

Proof of Lemma 3. We can rewrite this equation as

$$\omega = \frac{(\beta-1)\left(1-\omega^{\beta}\right)(I-b)}{\beta\left(1-\omega^{\beta-1}\right)2I-(\beta-1)\left(1-\omega^{\beta}\right)(I-b)}, \tag{68}$$

or, equivalently,

$$\frac{2\beta I\left(\omega-\omega^{\beta}\right)+(\beta-1)(I-b)\left(\omega^{\beta+1}-\omega-1+\omega^{\beta}\right)}{\beta\left(1-\omega^{\beta-1}\right)2I-(\beta-1)\left(1-\omega^{\beta}\right)(I-b)}=0. \tag{69}$$

Denote the left-hand side as a function of ω by $l(\omega)$. The denominator of $l(\omega)$, $l_d(\omega)$, is nonnegative on $\omega \in [0,1]$ and equals zero only at $\omega = 1$. This follows from:

$$\begin{aligned} l_d(0) &= 2\beta I-(\beta-1)(I-b)>0, \\ l_d(1) &= 0, \\ l_d'(\omega) &= -2\beta(\beta-1)I\omega^{\beta-2}+\beta(\beta-1)(I-b)\omega^{\beta-1} \\ &= \beta(\beta-1)\omega^{\beta-2}(-2I+\omega(I-b))<0. \end{aligned} \tag{70}$$

Therefore, $l(\omega)=0$ if and only if the numerator of $l(\omega)$, $l_n(\omega)$, equals zero at $\omega \in (0,1)$. Since $b \in (0,I)$, then $l_n(0) = -(\beta-1)(I-b) < 0$,

$$\begin{aligned} l_n'(\omega) &= 2\beta I\left(1-\beta\omega^{\beta-1}\right)+(\beta-1)(I-b)\left((\beta+1)\omega^{\beta}-1+\beta\omega^{\beta-1}\right), \\ l_n''(\omega) &= -2\beta^2(\beta-1)I\omega^{\beta-2}+(\beta-1)(I-b)\left(\beta(\beta+1)\omega^{\beta-1}+\beta(\beta-1)\omega^{\beta-2}\right), \end{aligned}$$

and

$$l_n''(\omega)<0 \Leftrightarrow (I-b)((\beta+1)\omega+\beta-1)<2\beta I \Leftrightarrow \omega<\frac{(\beta+1)I+(\beta-1)b}{(\beta+1)(I-b)}.$$

Since $\frac{(\beta+1)I+(\beta-1)b}{(\beta+1)(I-b)} > 1$, $l_n''(\omega) < 0$ for any $\omega \in [0,1]$. Since $l_n'(0) = 2\beta I - (\beta-1)(I-b) > 0$ and $l_n'(1) = -2\beta(\beta-1)b < 0$, there exists $\hat{\omega} \in (0,1)$ such that $l_n(\omega)$ increases to the left of $\hat{\omega}$ and decreases to the right. Since $\lim_{\omega\to1} l_n(\omega) = 0$, then $l_n(\hat{\omega}) > 0$, and hence $l_n(\omega)$ has a unique root ω^* on $(0,1)$.

Since the function $l_n(\omega)$ increases in b and is strictly increasing at the point ω^*, then ω^* decreases in b. To prove that $\lim_{b\to0}\omega^* = 1$, it is sufficient to prove that for any small $\varepsilon > 0$, there exists $b(\varepsilon) > 0$ such that $l_n(1-\varepsilon) < 0$ for $b < b(\varepsilon)$. Since $l_n(\omega) > 0$ on $(\omega^*, 1)$, this would imply that $\omega^* \in (1-\varepsilon, 1)$, i.e., that ω^* is infinitely close to 1 when b is close to zero. Using the expression for $l_n(\omega)$, $l_n(\omega) < 0$ is equivalent to

$$\frac{2\beta}{\beta-1}\frac{\omega}{\omega+1}\frac{1-\omega^{\beta-1}}{1-\omega^{\beta}}<1-\frac{b}{I}. \tag{71}$$

Denote the left-hand side of (71) by $L(\omega)$. Note that $L(\omega)$ is increasing on $(0,1)$. Indeed, differentiating $L(\omega)$ and simplifying, $L'(\omega) > 0 \Leftrightarrow g(\omega) \equiv 1-\omega^{2\beta}-\beta\omega^{\beta-1}+\beta\omega^{\beta+1} > 0$. The function $g(\omega)$ is decreasing on $(0,1)$ because $g'(\omega)<0 \Leftrightarrow h(\omega)\equiv -2\omega^{\beta+1}-(\beta-1)+(\beta+1)\omega^2<0$, where $h'(\omega)>0$ and $h(1)=0$. Since $g(\omega)$ is decreasing and $g(1)=0$, then, indeed, $g(\omega)>0$ and hence $L'(\omega)>0$ for all $\omega\in(0,1)$. In addition, by l'Hopital's rule, $\lim_{\omega\to 1} L(\omega)=1$. Hence, $L(1-\varepsilon)<1$ for any $\varepsilon>0$, and thus $l_n(1-\varepsilon)<0$ for $b\in[0, I(1-L(1-\varepsilon)))$.

Finally, to prove that $\lim_{b\to I}\omega^*=0$, it is sufficient to prove that for any small $\varepsilon>0$, there exists $b(\varepsilon)$ such that $l_n(\varepsilon)>0$ for $b>b(\varepsilon)$. Since $l_n(0)<0$, this would imply that $\omega^*\in(0,\varepsilon)$ for $b>b(\varepsilon)$, i.e., that ω^* is infinitely close to zero when b is close to I. Based on (71), $l_n(\omega)>0\Leftrightarrow L(\omega)>1-\frac{b}{I}$. Then, for any $\varepsilon>0$, if $b>I(1-L(\varepsilon))$, we get $1-\frac{b}{I}<L(\varepsilon)\Leftrightarrow l_n(\varepsilon)>0$, which completes the proof. ■

Proof of Proposition 4. Since the agent's IC condition is guaranteed by (21), we only have to ensure that the principal's ex-post and ex-ante IC conditions are satisfied. First, we check the principal's ex-post IC condition (23). To see this, we start with proving that $Y(\omega)$ is strictly decreasing in ω for $\omega\in(0,1)$. Note that

$$\frac{\partial Y(\omega)}{\partial\omega}=\frac{(I-b)}{\omega\left(\omega-\omega^\beta\right)^2}\left[-(\beta-1)\omega^{\beta+1}+\beta\omega^\beta-\omega\right],$$

where $\frac{(I-b)}{\omega(\omega-\omega^\beta)^2}>0$. Thus, we need to show that $k(\omega)\equiv-(\beta-1)\omega^{\beta+1}+\beta\omega^\beta-\omega<0$. According to an extension of Descartes' Rule of Signs to generalized polynomials (Laguerre, 1883), the number of positive roots of $k(\omega)=0$, counted with their orders, does not exceed the number of change of signs of its coefficients, i.e., two. Since $k(1)=0$, $k'(1)=0$, and $k''(1)=-\beta(\beta-1)<0$, $\omega=1$ is a root of order two, and there are no other positive roots. Further, $k(0)=0$ and $k'(0)=-1<0$. It follows that $k(0)=k(1)=0$ and $k(\omega)<0$ for all $\omega\in(0,1)$, and hence, indeed, $\frac{\partial Y(\omega)}{\partial\omega}<0$.

Since $\lim_{\omega\to0}Y(\omega)=+\infty$, and $Y(\omega)=\frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ has only one solution $\omega=\omega^*$ according to Lemma 3, it follows that the principal's ex-post IC condition is equivalent to $\omega\le\omega^*$.

Next, we check the principal's ex-ante IC condition (27), which is equivalent to (56), where $V_P(X,1;\omega)$ is given by (24). We pin down the range of ω that satisfies this condition in three steps, which are similar to the steps used in the proof of Proposition 1.

Step 1: *If $b>0$, $V_P(X,1;\omega)$ is strictly increasing in ω for any $\omega\in(0,\omega^*)$.*

The proof of this step is the same as the proof of Step 1 in the proof of Proposition 1 with the only difference: instead of relying on the inequality $Y(\omega)>\frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ for all $\omega\in(0,1)$ as in the proof of Proposition 1 (which holds for $b<0$), we rely on the inequality $Y(\omega)>\frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ for all $\omega\in(0,\omega^*)$, which was proved above.

Step 2: *If $0<b<I$, then the ex-ante IC condition (56) holds as a strict inequality for $\omega=\omega^*$.*

Using (24) and (31), we can rewrite $V_P(X,1;\omega^*)$ as $X^\beta f(\omega^*)$, where

$$f(\omega)\equiv\frac{1-\omega}{1-\omega^{\beta+1}}\left(\frac{\beta}{\beta-1}\frac{2I}{\omega+1}\right)^{-\beta}\frac{I}{\beta-1}.$$

Note that $f(0)=\bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u-I\right)$ and that

$$f'(\omega)>0\Leftrightarrow g(\omega)\equiv-(\beta-1)\omega^{\beta+1}+(\beta+1)\omega^\beta-(\beta+1)\omega+\beta-1>0.$$

By an extension of Descartes' Rule of Signs to generalized polynomials, the number of positive roots of $g(\omega)$, counted with their orders, does not exceed the number of change of signs of its coefficients, i.e., three. Note that $\omega=1$ is the root of $g(\omega)$ of order three: $g(1)=g'(1)=g''(1)=0$, and hence there are no other roots. Since $g(0)=\beta-1>0$, it follows that $g(\omega)>0$ and hence $f'(\omega)>0$ for all $\omega\in[0,1)$. Therefore, $f(\omega)$ is strictly increasing in ω, which implies

$$X^{-\beta}V_P(X,1;\omega^*)=f(\omega^*)>f(0)=\bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u-I\right). \tag{72}$$

Because the function $X^{-\beta}\left(\frac{1}{2}X-I\right)$ achieves its global maximum at the point $\bar{X}_u$, (72) implies that (56) holds as a strict inequality for $\omega=\omega^*$.

Step 3. *Suppose $-I<b<I$. For ω close enough to zero, the ex-ante IC condition (56) does not hold.*

Step 4. *Suppose $-I<b<I$. Then (56) is satisfied for any $\omega\geq\bar{\omega}$, where $\bar{\omega}$ is the unique solution to $Y(\omega)=\bar{X}_u$. For any $\omega<\bar{\omega}$, (56) is satisfied if and only if $X^{-\beta}V_P(X,1;\omega)\geq\bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u-I\right)$.*

The statements of steps 3 and 4 have been proved in the proof of Proposition 1.

Step 5. *Conditions for (56) to hold.*

Combining the four steps above yields the statement of the proposition. Suppose $b<I$. As shown above, the ex-post IC condition holds if and only if $\omega\leq\omega^*$. Recall that $Y(\omega^*)=\frac{\beta}{\beta-1}\frac{2I}{\omega^*+1}<\frac{\beta}{\beta-1}2I=\bar{X}_u$, and hence $\omega^*>\bar{\omega}$. According to Step 4, the ex-ante IC condition (56) is satisfied for any $\omega\geq\bar{\omega}$, and for any $\omega\leq\bar{\omega}$ (56) is satisfied if and only if $X^{-\beta}V_P(X,1;\omega)\geq\bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u-I\right)$. The left-hand side of this inequality is increasing in ω for $\omega\leq\omega^*$ according to Step 1, while the right-hand side is constant. Together, this implies that if (56) is satisfied for some $\tilde{\omega}$, it is satisfied for any $\omega\geq\tilde{\omega}$. According to Step 3, for ω close to 0, (56) does not hold. Hence, there exists a unique $\underline{\omega}\in(0,\bar{\omega})$ such that the principal's ex-ante IC (56) holds if and only if $\omega\geq\underline{\omega}$, and $X^{-\beta}V_P(X,1;\underline{\omega})=\bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u-I\right)$. Because, $\underline{\omega}<\bar{\omega}$ and $\bar{\omega}<\omega^*$, we have $\underline{\omega}<\omega^*$. We conclude that both the ex-post and the ex-ante IC conditions hold if and only if $\omega\in[\underline{\omega},\omega^*]$.

Finally, consider $b\geq I$. In this case, all types of agents want immediate exercise, which implies that the principal must exercise the option at the optimal uninformed threshold $\bar{X}_u=\frac{\beta}{\beta-1}2I$. ■

Proof of Proposition 5. The expected utility of the principal in the ω-equilibrium is $V_P(X,1;\omega)$, given by (24). As shown in Step 1 in the proof of Proposition 4, $V_P(X,1;\omega)$ is strictly increasing in ω for $\omega\in(0,\omega^*)$. Hence, $V_P(X,1;\omega^*)>V_P(X,1;\omega)$ for any $\omega<\omega^*$.

Denote the ex-ante expected utility of the agent (before the agent's type is realized) by $V_A(X,1;\omega)$. Repeating the derivation of the principal's value function $V_P(X,1;\omega)$ in the appendix above, it is easy to see that

$$V_A(X,1;\omega)=\frac{1-\omega}{1-\omega^{\beta+1}}\left(\frac{X}{Y(\omega)}\right)^{\beta}\left(\frac{1}{2}(1+\omega)Y(\omega)-(I-b)\right).$$

The only difference of this expression from the expression for $V_P(X,1;\omega)$ given by (24) is that I in the second bracket of (24) is replaced by $(I-b)$. To prove that $V_A(X,1;\omega^*)>V_A(X,1;\omega)$ for any $\omega<\omega^*$, we prove that $V_A(X,1;\omega)$ is strictly increasing in ω for $\omega\in(0,\omega^*)$. The proof repeats the arguments of Step 1 in the proof of Proposition 4. In particular, we can re-write $V_A(X,1;\omega)$ as $2^{-\beta}X^{\beta}f_1(\omega)\tilde{f}_2(\omega)$, where

$$f_1(\omega)\equiv\frac{(1-\omega)(1+\omega)^{\beta}}{1-\omega^{\beta+1}}\text{ and }\tilde{f}_2(\omega)\equiv\frac{\frac{1}{2}(1+\omega)Y(\omega)-(I-b)}{\left(\frac{1}{2}(1+\omega)Y(\omega)\right)^{\beta}}.$$

As shown in Step 1 in the proof of Proposition 4, $f_1(\omega)>0$ and $f_1'(\omega)>0$. In addition, $\tilde{f}_2(\omega)>0$ because $Y(\omega)>\frac{\beta}{\beta-1}\frac{2I}{1+\omega}>\frac{2(I-b)}{1+\omega}$ for any $\omega<\omega^*$, and $\tilde{f}_2'(\omega)>0$ for the same reasons why $f_2'(\omega)>0$ in the proof of Proposition 4. Hence, $V_A'(X,1;\omega)>0$ for any $\omega\in(0,\omega^*)$, which completes the proof. ■

Proof of Proposition 6. First, consider the case $b<0$. Proposition 1 shows that in the dynamic communication game, there exists an equilibrium with continuous exercise, where for each type θ, the option is exercised at the threshold $X_A^*(\theta)$. No such equilibrium exists in the static communication game. Indeed, continuous exercise requires that the principal perfectly infers the agent's type. However, since the principal gets this information at time 0, he will exercise the option at $X_P^*(\theta)\neq X_A^*(\theta)$.

We next show that no stationary equilibrium with partitioned exercise exists in the static communication game either. To see this, note that for such an equilibrium to exist, the following conditions must hold. First, the boundary type ω must be indifferent between exercise at $\bar{X}$ and at $\frac{\bar{X}}{\omega}$. Repeating the derivations in Section 3, this requires that (21) holds: $\bar{X}=\frac{\left(1-\omega^{\beta}\right)(I-b)}{\omega\left(1-\omega^{\beta-1}\right)}$. Second, given that the exercise threshold $\bar{X}$ is optimally chosen by the principal given the belief that $\theta\in[\omega,1]$, it must satisfy $\bar{X}=\frac{\beta}{\beta-1}\frac{2I}{\omega+1}$. Combining these two equations, ω must be the solution to (32), which can be rewritten as

$$2\beta I\left(\omega-\omega^{\beta}\right)-(\beta-1)(I-b)(1+\omega)\left(1-\omega^{\beta}\right)=0. \tag{73}$$

We next show that the left-hand side of (73) is negative for any $b<0$ and $\omega<1$. Since $b<0$, it is sufficient to prove that

$$\begin{aligned} 2\beta\left(\omega-\omega^{\beta}\right) &< (\beta-1)(1+\omega)\left(1-\omega^{\beta}\right) \Leftrightarrow \\ l_n(\omega) &\equiv 2\beta\left(\omega-\omega^{\beta}\right)+(\beta-1)\left(\omega^{\beta+1}-\omega-1+\omega^{\beta}\right)<0 \end{aligned}$$

It is easy to show that $l_n'(1)=0$ and that $l_n''(\omega)<0 \Leftrightarrow \omega<1$, and hence $l_n'(\omega)>0$ for any $\omega<1$. Since $l_n(1)=0$, then, indeed, $l_n(\omega)<0$ for all $\omega<1$.

Next, consider the case $b>0$. As argued above, for ω-equilibrium to exist in the static communication game, ω must satisfy (32). According to Lemma 3, for $b>0$, this equation has a unique solution, denoted by ω^*. Thus, among equilibria with $\omega\in[\underline{\omega},\omega^*]$, which exist in the dynamic communication game, only equilibrium with $\omega=\omega^*$ is an equilibrium of the static communication game. ■

For Proposition 7, we prove the following lemma, which characterizes the structure of any incentive-compatible decision-making rule and is an analogue of Proposition 1 in Melumad and Shibano (1991) for the payoff specification in our model:

Lemma 4. *An incentive-compatible threshold schedule $\hat{X}(\theta)$, $\theta\in\Theta$ must satisfy the following conditions:*

1. *$\hat{X}(\theta)$ is weakly decreasing in θ.*
2. *If $\hat{X}(\theta)$ is strictly decreasing on (θ_1,θ_2), then*

$$\hat{X}(\theta)=\frac{\beta}{\beta-1}\frac{I-b}{\theta}. \tag{74}$$

3. *If $\hat{X}(\theta)$ is discontinuous at $\hat{\theta}$, then the discontinuity satisfies*

$$\hat{U}_A\left(\hat{X}^-\left(\hat{\theta}\right),\hat{\theta}\right) = \hat{U}_A\left(\hat{X}^+\left(\hat{\theta}\right),\hat{\theta}\right), \tag{75}$$

$$\hat{X}(\theta) = \begin{cases} \hat{X}^-\left(\hat{\theta}\right), & \forall\theta\in\left[\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-(\hat{\theta})},\hat{\theta}\right), \\ \hat{X}^+\left(\hat{\theta}\right), & \forall\theta\in\left(\hat{\theta},\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+(\hat{\theta})}\right], \end{cases} \tag{76}$$

$$\hat{X}\left(\hat{\theta}\right) \in \left\{\hat{X}^-\left(\hat{\theta}\right),\hat{X}^+\left(\hat{\theta}\right)\right\}, \tag{77}$$

where $\hat{X}^-\left(\hat{\theta}\right)\equiv\lim_{\theta\uparrow\hat{\theta}}\hat{X}(\theta)$ and $\hat{X}^+\left(\hat{\theta}\right)\equiv\lim_{\theta\downarrow\hat{\theta}}\hat{X}(\theta)$.

Proof of Lemma 4. Proof of Part 1. The first part of the lemma can be proven by contradiction. Suppose there exist $\theta_1,\theta_2\in\Theta$, $\theta_2>\theta_1$, such that $\hat{X}(\theta_2)>\hat{X}(\theta_1)$. Inequality (36) for $\theta=\theta_1$ and $\hat{\theta}=\theta_2$, $\hat{U}_A\left(\hat{X}(\theta_1),\theta_1\right)\geq\hat{U}_A\left(\hat{X}(\theta_2),\theta_1\right)$, can be written in the integral form:

$$\int_{\hat{X}(\theta_1)}^{\hat{X}(\theta_2)}\left(\frac{X(0)}{\hat{X}}\right)^{\beta}\frac{-(\beta-1)\theta_1\hat{X}+\beta(I-b)}{\hat{X}}d\hat{X}\leq 0. \tag{78}$$

Because $\theta_2 > \theta_1$ and $\beta > 1$, (78) implies

$$\int_{\hat{X}(\theta_1)}^{\hat{X}(\theta_2)} \left(\frac{X(0)}{\hat{X}}\right)^{\beta} \frac{-(\beta-1)\theta_2\hat{X} + \beta(I-b)}{\hat{X}} d\hat{X} < 0, \tag{79}$$

or, equivalently, $\hat{U}_A\left(\hat{X}(\theta_1), \theta_2\right) > \hat{U}_A\left(\hat{X}(\theta_2), \theta_2\right)$. However, this inequality violates (36) for $\theta = \theta_2$ and $\hat{\theta} = \theta_1$: $\hat{U}_A\left(\hat{X}(\theta_2), \theta_2\right) \geq \hat{U}_A\left(\hat{X}(\theta_1), \theta_2\right)$. Therefore, $\hat{X}(\theta)$ is weakly decreasing in θ.

Proof of Part 2. To prove the second part of the lemma, note that $\hat{U}_A\left(\hat{X}, \theta\right)$ is differentiable in θ for all $\hat{X} \in (X(0), \infty)$. Because $\hat{U}_A(\hat{X}, \theta)$ is linear in θ, it satisfies the Lipschitz condition and hence is absolutely continuous in θ for all $\hat{X} \in (X(0), \infty)$. Also,

$$\sup_{\hat{X}\in\mathbf{X}} \left| \frac{\partial \hat{U}_A\left(\hat{X}, \theta\right)}{\partial \theta} \right| = \sup_{\hat{X}\in\mathbf{X}} \left| \left(\frac{X(0)}{\hat{X}}\right)^{\beta} \hat{X} \right|. \tag{80}$$

Hence, $\sup_{\hat{X}\in\mathbf{X}} \left|\frac{\partial \hat{U}_A(\hat{X},\theta)}{\partial\theta}\right|$ is integrable on $\theta \in \Theta$. By the generalized envelope theorem (see Corollary 1 in Milgrom and Segal, 2002), the equilibrium utility of the agent in any mechanism implementing exercise at thresholds $\hat{X}(\theta)$, $\theta \in \Theta$, denoted $V_A(\theta)$, satisfies the integral condition,

$$V_A(\theta) = V_A(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \left(\frac{X(0)}{\hat{X}(s)}\right)^{\beta} \hat{X}(s)\, ds. \tag{81}$$

On the other hand, $V_A(\theta) = \hat{U}_A\left(\hat{X}(\theta), \theta\right)$. At any point θ at which $\hat{X}(\theta)$ is strictly decreasing, we have

$$\begin{aligned} \frac{dV_A(\theta)}{d\theta} &= \frac{d\hat{U}_A\left(\hat{X}(\theta), \theta\right)}{d\theta} \Leftrightarrow \\ \left(\frac{X(0)}{\hat{X}(\theta)}\right)^{\beta} \hat{X}(\theta) &= \left(\frac{X(0)}{\hat{X}(\theta)}\right)^{\beta} \hat{X}(\theta) - \left(\frac{X(0)}{\hat{X}(\theta)}\right)^{\beta} \frac{(\beta-1)\theta\hat{X}(\theta) - \beta(I-b)}{\hat{X}(\theta)} \frac{d\hat{X}(\theta)}{d\theta}. \end{aligned}$$

Because $d\hat{X}(\theta) < 0$, it must be that $(\beta-1)\theta\hat{X}(\theta) - \beta(I-b) = 0$. Thus, $\hat{X}(\theta)$ satisfies (74). This proves the second part of the lemma.

Proof of Part 3. Finally, consider the third part of the lemma. Eq. (75) follows from (76), continuity of $\hat{U}_A(\cdot)$, and incentive compatibility of the contract. Otherwise, for example, if $\hat{U}_A\left(\hat{X}^+(\hat{\theta}), \hat{\theta}\right) > \hat{U}_A\left(\hat{X}^-(\hat{\theta}), \hat{\theta}\right)$, then $\hat{U}_A\left(\hat{X}(\hat{\theta}-\varepsilon), \hat{\theta}-\varepsilon\right) = \hat{U}_A\left(\hat{X}^-(\hat{\theta}), \hat{\theta}-\varepsilon\right) < \hat{U}_A\left(\hat{X}^+(\hat{\theta}), \hat{\theta}-\varepsilon\right)$ for a sufficiently small ε, and hence types close enough to $\hat{\theta}$ from below would benefit from a deviation to $\hat{X}^+(\hat{\theta})$, i.e., from mimicking types slightly above $\hat{\theta}$.

Next, we prove (76). First, note that, (76) is satisfied at the boundaries. Indeed, denote $\theta_1^* \equiv \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-(\hat{\theta})}$ and suppose that $\hat{X}(\theta_1^*) \neq \hat{X}^-(\hat{\theta})$. Then, by the first part of the lemma, $\hat{X}(\theta_1^*) > \hat{X}^-(\hat{\theta})$. Because $\hat{X}^-\left(\hat{\theta}\right) \equiv \lim_{\theta\uparrow\hat{\theta}} \hat{X}(\theta)$, there exists $\varepsilon > 0$ such that $\hat{X}(\theta_1^*) > \hat{X}\left(\hat{\theta}-\varepsilon\right) \geq \hat{X}^-(\hat{\theta})$. Because the function $\hat{U}_A\left(\hat{X}, \theta_1^*\right)$ has a maximum at $\hat{X}^-(\hat{\theta})$ and is strictly decreasing for $\hat{X} > \hat{X}^-(\hat{\theta})$, this would imply $\hat{U}_A\left(\hat{X}(\theta_1^*), \theta_1^*\right) < \hat{U}_A\left(\hat{X}\left(\hat{\theta}-\varepsilon\right), \theta_1^*\right)$, and hence would contradict incentive compatibility for type θ_1^*. The proof for the boundary $\theta_2^* \equiv \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+(\hat{\theta})}$ is similar.

We next prove (76) for interior values of θ. First, suppose that $\hat{X}(\theta) \neq \hat{X}^{-}(\hat{\theta})$ for some $\theta \in \left(\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{-}(\hat{\theta})}, \hat{\theta}\right)$. By part 1 of the lemma, $\hat{X}(\theta) > \hat{X}^{-}(\hat{\theta})$. By incentive compatibility, $\hat{U}_A\left(\hat{X}(\theta), \theta\right) \geq \hat{U}_A\left(\hat{X}^{-}(\hat{\theta}), \theta\right)$, which can be written in the integral form as:

$$\int_{\hat{X}^{-}(\hat{\theta})}^{\hat{X}(\theta)} \left(\frac{X(0)}{\hat{X}}\right)^{\beta} \frac{-(\beta-1)\theta\hat{X} + \beta(I-b)}{\hat{X}} d\hat{X} \geq 0. \tag{82}$$

The function under the integral on the left-hand side is strictly decreasing in θ and the interval $(\hat{X}^{-}\left(\hat{\theta}\right), \hat{X}(\theta))$ is non-empty. Therefore, we can replace θ by $\tilde{\theta} < \theta$ under the integral and get a strict inequality: $\hat{U}_A\left(\hat{X}(\theta), \tilde{\theta}\right) > \hat{U}_A\left(\hat{X}^{-}(\hat{\theta}), \tilde{\theta}\right)$ for every $\tilde{\theta} \in \left[\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{-}(\hat{\theta})}, \theta\right)$. However, this contradicts $\hat{X}^{-}\left(\hat{\theta}\right) = \arg\max_{\hat{X}} \hat{U}_A\left(\hat{X}, \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{-}(\hat{\theta})}\right)$. Second, suppose that $\hat{X}(\theta) \neq \hat{X}^{+}\left(\hat{\theta}\right)$ for some $\theta \in \left(\hat{\theta}, \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{+}(\hat{\theta})}\right)$. By part 1 of the lemma, $\hat{X}(\theta) < \hat{X}^{+}(\hat{\theta})$. By incentive compatibility, $\hat{U}_A\left(\hat{X}(\theta), \theta\right) \geq \hat{U}_A\left(\hat{X}^{+}(\theta), \theta\right)$, which can be written as

$$\int_{\hat{X}(\theta)}^{\hat{X}^{+}(\theta)} \left(\frac{X(0)}{\hat{X}}\right)^{\beta} \frac{-(\beta-1)\theta\hat{X} + \beta(I-b)}{\hat{X}} d\hat{X} \leq 0. \tag{83}$$

The function under the integral on the left-hand side is strictly decreasing in θ and the interval $(\hat{X}(\theta), \hat{X}^{+}(\hat{\theta}))$ is non-empty. Therefore, we can replace θ by $\tilde{\theta} > \theta$ under the integral and get a strict inequality, $\hat{U}_A\left(\hat{X}(\theta), \tilde{\theta}\right) > \hat{U}_A\left(\hat{X}^{+}(\theta), \tilde{\theta}\right)$, for every $\tilde{\theta} \in \left(\theta, \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{+}(\hat{\theta})}\right]$. However, this contradicts $\hat{X}^{+}\left(\hat{\theta}\right) = \arg\max_{\hat{X}} \hat{U}_A\left(\hat{X}, \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{+}(\hat{\theta})}\right)$.

Last, (77) follows from continuity of $\hat{U}_A(\cdot)$ and incentive compatibility of $\hat{X}(\theta)$. Because $\hat{\theta} \in \left(\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{-}(\hat{\theta})}, \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^{+}(\hat{\theta})}\right)$, every policy with thresholds strictly below $\hat{X}^{-}(\hat{\theta})$ or strictly above $\hat{X}^{+}(\hat{\theta})$ is strictly dominated by $\hat{X}^{-}(\hat{\theta})$ and $\hat{X}^{+}(\hat{\theta})$, respectively, and thus cannot be incentive compatible. Suppose that $\hat{X}(\hat{\theta}) \in \left(\hat{X}^{-}(\hat{\theta}), \hat{X}^{+}(\hat{\theta})\right)$. Incentive compatibility and (75) imply $\hat{U}_A\left(\hat{X}(\hat{\theta}), \hat{\theta}\right) \geq \hat{U}_A\left(\hat{X}^{-}(\hat{\theta}), \hat{\theta}\right) = \hat{U}_A\left(\hat{X}^{+}(\hat{\theta}), \hat{\theta}\right)$. Because $\hat{U}_A\left(\hat{X}, \hat{\theta}\right)$ is strictly increasing in $\hat{X}$ for $\hat{X} < \frac{\beta}{\beta-1}\frac{I-b}{\hat{\theta}}$ and strictly decreasing in $\hat{X}$ for $\hat{X} > \frac{\beta}{\beta-1}\frac{I-b}{\hat{\theta}}$, the inequality must be strict: $\hat{U}_A\left(\hat{X}(\hat{\theta}), \hat{\theta}\right) > \hat{U}_A\left(\hat{X}^{-}(\hat{\theta}), \hat{\theta}\right) = \hat{U}_A\left(\hat{X}^{+}(\hat{\theta}), \hat{\theta}\right)$. However, this together with (76) and continuity of $\hat{U}_A(\cdot)$ implies that types close enough to $\hat{\theta}$ benefit from a deviation to threshold $\hat{X}(\hat{\theta})$. Hence, it must be that $\hat{X}(\hat{\theta}) \in \left\{\hat{X}^{-}(\hat{\theta}), \hat{X}^{+}(\hat{\theta})\right\}$. ■

Proof of Proposition 7. We consider the following three cases: 1) $b \geq I$, 2) $b \in (-I, I)$, and 3) $b \leq -I$.

- $b \geq I$.

In this case, all types of agents want to exercise the option immediately. This means that any incentive-compatible contract must be flat. Among flat contracts $\hat{X}(\theta) = \bar{X}$, the one that maximizes the payoff to the principal solves

$$\arg\max_{\bar{X}} \int_{\underline{\theta}}^{1} \frac{\theta\bar{X} - I}{\bar{X}^{\beta}} d\theta = \frac{2\beta}{\beta-1}\frac{I}{1+\underline{\theta}}. \tag{84}$$

- $b \in (-I, I)$

The proof for this case proceeds in two steps. First, we show that the optimal contract cannot have discontinuities. Second, we show that the optimal continuous contract is as specified in the proposition.

By contradiction, suppose that the optimal contract $C = \left\{ \hat{X}(\theta), \theta \in \Theta \right\}$ has a discontinuity at some point $\hat{\theta} \in (\underline{\theta}, 1)$. By Lemma 4, the discontinuity must satisfy (75)–(77). In particular, (76) implies that there exist $\theta_1 < \hat{\theta}$ and $\theta_2 > \hat{\theta}$ such that $\hat{X}(\theta) = X_A^*(\theta_1)$ for $\theta \in \left[\theta_1, \hat{\theta}\right)$ and $\hat{X}(\theta) = X_A^*(\theta_2)$ for $\theta_2 \in \left(\hat{\theta}, \theta_2\right]$. For any $\tilde{\theta}_2 \in (\hat{\theta}, \theta_2]$, consider a contract $C_1 = \left\{ \hat{X}_1(\theta), \theta \in \Theta \right\}$, defined as

$$\hat{X}_1(\theta) = \begin{cases} \hat{X}(\theta), & \text{if } \theta \in [\underline{\theta}, \theta_1] \cup [\theta_2, 1], \\ X_A^*(\theta_1), & \text{if } \theta \in \left[\theta_1, \tilde{\theta}\right), \\ X_A^*\left(\tilde{\theta}_2\right), & \text{if } \theta \in \left(\tilde{\theta}, \tilde{\theta}_2\right], \\ X_A^*(\theta), & \text{if } \theta \in \left(\tilde{\theta}_2, \theta_2\right), \end{cases}$$

where $\tilde{\theta} = \tilde{\theta}\left(\tilde{\theta}_2\right)$ satisfies

$$\frac{\tilde{\theta} X_A^*(\theta_1) - I + b}{X_A^*(\theta_1)^\beta} = \frac{\tilde{\theta} X_A^*\left(\tilde{\theta}_2\right) - I + b}{X_A^*\left(\tilde{\theta}_2\right)^\beta}. \tag{85}$$

Because $X^{-\beta}(\theta X - I + b)$ is maximized at $X_A^*(\theta)$, the function $f(\theta) \equiv \frac{\theta X_A^*(\theta_1) - I + b}{X_A^*(\theta_1)^\beta} - \frac{\theta X_A^*\left(\tilde{\theta}_2\right) - I + b}{X_A^*\left(\tilde{\theta}_2\right)^\beta}$ satisfies $f(\theta_1) > 0 > f\left(\tilde{\theta}_2\right)$, and hence, by continuity of $f(\theta)$, there exists $\tilde{\theta} \in \left(\theta_1, \tilde{\theta}_2\right)$ such that $f\left(\tilde{\theta}\right) = 0$, i.e., (85) is satisfied.

Intuitively, contract C_1 is the same as contract C, except that it substitutes a subset $\left[\tilde{\theta}_2, \theta_2\right]$ of the flat region with a continuous region where $\hat{X}_1(\theta) = \frac{\beta}{\beta - 1} \frac{I - b}{\theta}$. Because contract C is incentive-compatible and $\tilde{\theta}$ satisfies (85), contract C_1 is incentive-compatible too. Below we show that the payoff to the principal from contract C_1 exceeds the payoff to the principal from contract C for $\tilde{\theta}_2$ very close to θ_2. Because $\hat{X}_1(\theta) = \hat{X}(\theta)$ for $\theta \leq \theta_1$ and $\theta \geq \theta_2$, it is enough to restrict attention to the payoff in the range $\theta \in (\theta_1, \theta_2)$. The payoff to the principal from contract C_1 in this range, divided by $X(0)^\beta \frac{1}{1 - \underline{\theta}}$, is

$$\int_{\theta_1}^{\tilde{\theta}(\tilde{\theta}_2)} \frac{\theta X_A^*(\theta_1) - I}{X_A^*(\theta_1)^\beta} d\theta + \int_{\tilde{\theta}(\tilde{\theta}_2)}^{\tilde{\theta}_2} \frac{\theta X_A^*\left(\tilde{\theta}_2\right) - I}{X_A^*\left(\tilde{\theta}_2\right)^\beta} d\theta + \int_{\tilde{\theta}_2}^{\theta_2} \frac{\theta X_A^*(\theta) - I}{X_A^*(\theta)^\beta} d\theta. \tag{86}$$

The derivative of (86) with respect to $\tilde{\theta}_2$, after the application of (85) and Leibniz's integral rule, is

$$\int_{\tilde{\theta}}^{\tilde{\theta}_2} \frac{\beta I - (\beta - 1)\theta X_A^*\left(\tilde{\theta}_2\right)}{X_A^*\left(\tilde{\theta}_2\right)^{\beta+1}} X_A^{*\prime}\left(\tilde{\theta}_2\right) d\theta + b \left(\frac{1}{X_A^*\left(\tilde{\theta}_2\right)^\beta} - \frac{1}{X_A^*(\theta_1)^\beta} \right) \frac{d\tilde{\theta}}{d\tilde{\theta}_2}. \tag{87}$$

Because $X_A^{*\prime}(\theta) = -\frac{X_A^*(\theta)}{\theta}$, the first term of (87) can be simplified to

$$\frac{(\beta-1)\, X_A^*\left(\tilde{\theta}_2\right)\frac{\tilde{\theta}_2^2-\tilde{\theta}^2}{2} - \beta I\left(\tilde{\theta}_2-\tilde{\theta}\right)}{X_A^*\left(\tilde{\theta}_2\right)^{\beta+1}}\frac{X_A^*\left(\tilde{\theta}_2\right)}{\tilde{\theta}_2} \tag{88}$$

$$= (\beta-1)\frac{\tilde{\theta}_2-\tilde{\theta}}{\tilde{\theta}_2}\frac{X_A^*\left(\tilde{\theta}_2\right)\frac{\tilde{\theta}_2+\tilde{\theta}}{2} - \frac{\beta}{\beta-1}I}{X_A^*\left(\tilde{\theta}_2\right)^{\beta}} = \beta\frac{\tilde{\theta}_2-\tilde{\theta}}{\tilde{\theta}_2}X_A^*\left(\tilde{\theta}_2\right)^{-\beta}\left[\frac{I-b}{\tilde{\theta}_2}\frac{\tilde{\theta}_2+\tilde{\theta}}{2} - I\right].$$

From (85),

$$\frac{d\tilde{\theta}}{d\tilde{\theta}_2} = \frac{(\beta-1)\,\tilde{\theta}_2^{\beta-2}\left(\tilde{\theta}-\tilde{\theta}_2\right)}{\left(\frac{\theta_1^\beta}{\theta_1} - \frac{\tilde{\theta}_2^\beta}{\tilde{\theta}_2}\right)}. \tag{89}$$

Using (89) and (85), the second term of (87) can be simplified to

$$\frac{b}{X_A^*\left(\tilde{\theta}_2\right)^\beta}\left(1-\left(\frac{\tilde{\theta}_2}{\theta_1}\right)^{-\beta}\right)\frac{d\tilde{\theta}}{d\tilde{\theta}_2} = \beta\frac{\tilde{\theta}_2-\tilde{\theta}}{\tilde{\theta}_2}X_A^*\left(\tilde{\theta}_2\right)^{-\beta}\left(\frac{\tilde{\theta}}{\tilde{\theta}_2}\right)b. \tag{90}$$

Adding up (88) and (90), the derivative of the principal's payoff with respect to $\tilde{\theta}_2$ is given by $-\beta\frac{(\tilde{\theta}_2-\tilde{\theta})^2}{2\tilde{\theta}_2^2}X_A^*\left(\tilde{\theta}_2\right)^{-\beta}(I+b)$, which is strictly negative for $b > -I$. By the mean value theorem, if $U_P\left(\tilde{\theta}_2\right)$ stands for the expected principal's utility from contract C, then $\frac{U_P\left(\tilde{\theta}_2\right)-U_P(\theta_2)}{\tilde{\theta}_2-\theta_2} = U_P'\left(\hat{\theta}_2\right) < 0$ for some $\hat{\theta}_2 \in \left(\tilde{\theta}_2, \theta_2\right)$, and hence a deviation from contract C to contract C_1 is beneficial for the principal. Hence, contract C cannot be optimal for $b > -I$.

Second, we prove that among continuous contracts satisfying Lemma 4, the one specified in the proposition maximizes the payoff to the principal. By Lemma 4 and continuity of the optimal contract proved above, it is sufficient to restrict attention to contracts that are combinations of, at most, one downward sloping part $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$ and two flat parts: any contract that has at least two disjoint regions with $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$ will exhibit discontinuity. Consider a contract such that $\hat{X}(\theta)$ is flat for $\theta \in [\underline{\theta}, \theta_1]$, is downward-sloping with $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$ for $\theta \in [\theta_1, \theta_2]$, and is again flat for $\theta \in [\theta_2, 1]$, for some $\theta_1 \in [0, \theta_2]$ and $\theta_2 \in [\theta_1, 1]$. This consideration allows for all possible cases, because it can be that $\theta_1 = \underline{\theta}$ and/or $\theta_2 = 1$, or $\theta_1 = \theta_2$). The payoff to the principal, divided by $X(0)^\beta\frac{1}{1-\underline{\theta}}$, is

$$P = \int_{\underline{\theta}}^{\theta_1}\frac{\theta X_A^*(\theta_1) - I}{X_A^*(\theta_1)^\beta}d\theta + \int_{\theta_1}^{\theta_2}\frac{\theta X_A^*(\theta) - I}{X_A^*(\theta)^\beta}d\theta + \int_{\theta_2}^{1}\frac{\theta X_A^*(\theta_2) - I}{X_A^*(\theta_2)^\beta}d\theta, \tag{91}$$

where $X_A^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$ is the exercise threshold most preferred by type θ the agent. The derivative with respect to θ_1 is

$$\frac{\partial P}{\partial \theta_1} = \int_{\underline{\theta}}^{\theta_1}\frac{\beta I - (\beta-1)\,\theta X_A^*(\theta_1)}{X_A^*(\theta_1)^{\beta+1}}X_A^{*\prime}(\theta_1)\,d\theta = -\frac{\beta}{\theta_1 X_A^*(\theta_1)^\beta}\int_{\underline{\theta}}^{\theta_1}\left(I - \theta\frac{I-b}{\theta_1}\right)d\theta$$

$$= -\frac{\beta}{\hat{X}(\theta_1)^\beta}\left[\frac{I+b}{2} - I\frac{\underline{\theta}}{\theta_1} + \left(\frac{\underline{\theta}}{\theta_1}\right)^2\frac{I-b}{2}\right].$$

Since $b \in (-I, I)$, the function $x^2 \frac{I-b}{2} - Ix + \frac{I+b}{2}$ is U-shaped and has two roots, 1 and $\frac{I+b}{I-b}$, which coincide for $b = 0$. If $b \in [0, I)$, this function is strictly positive for $x < 1$ because $\frac{I+b}{I-b} \geq 1$. Hence, $\frac{\partial P}{\partial \theta_1} < 0$ for $\theta_1 > \underline{\theta}$, which implies that (91) is maximized at $\theta_1 = \underline{\theta}$. If $b < 0$, then $\frac{I+b}{I-b} < 1$ and hence $\frac{\partial P}{\partial \theta_1} < 0$ when $\frac{\underline{\theta}}{\theta_1} < \frac{I+b}{I-b}$ or $\frac{\underline{\theta}}{\theta_1} > 1$, and $\frac{\partial P}{\partial \theta_1} > 0$ when $\frac{\underline{\theta}}{\theta_1} \in \left(\frac{I+b}{I-b}, 1\right)$. Because $\frac{\underline{\theta}}{\theta_1} \leq 1$, we conclude that (91) is increasing in θ_1 in the range $\theta_1 < \frac{I-b}{I+b}\underline{\theta}$ and decreasing in θ_1 in the range $\theta_1 > \frac{I-b}{I+b}\underline{\theta}$. Therefore, if $b < 0$, (91) reaches its maximum at $\theta_1 = \min\left\{\frac{I-b}{I+b}\underline{\theta}, 1\right\}$. In particular, the maximum is achieved at $\theta_1 = \frac{I-b}{I+b}\underline{\theta}$ if $b \in [-\frac{1-\underline{\theta}}{1+\underline{\theta}}I, 0)$, and $\theta_1 = 1$ if $b \leq -\frac{1-\underline{\theta}}{1+\underline{\theta}}I$.

Similarly, the derivative of (91) with respect to θ_2 is

$$\begin{aligned} \frac{\partial P}{\partial \theta_2} &= \int_{\theta_2}^{1} \frac{\beta I - (\beta - 1)\theta X_A^*(\theta_2)}{X_A^*(\theta_2)^{\beta+1}} X_A^{*\prime}(\theta_2)\, d\theta = -\frac{\beta}{\theta_2 X_A^*(\theta_2)^{\beta}} \int_{\theta_2}^{1} \left(I - \theta \frac{I-b}{\theta_2}\right) d\theta \qquad (92) \\ &= \frac{\beta(1-\theta_2)}{2\theta_2^2 \hat{X}(\theta_2)^{\beta}} \left(I - b - (I+b)\theta_2\right). \end{aligned}$$

If $b \in (-I, 0]$, then $I - b - (I+b)\theta_2 \geq I - b - (I+b) > 0$, and hence (92) is positive for any $\theta_2 \in [\underline{\theta}, 1)$. Therefore, (91) is maximized at $\theta_2 = 1$. Combined with the conclusion $\theta_1 = 1$ for $b \leq -\frac{1-\underline{\theta}}{1+\underline{\theta}}I$, this implies that if $-I < b \leq -\frac{1-\underline{\theta}}{1+\underline{\theta}}I$, then $\theta_1 = \theta_2 = 1$, i.e., the principal always prefers to "flatten" the contract. Similarly, if $I - b - (I+b)\underline{\theta} \leq 0 \Leftrightarrow b \geq \frac{1-\underline{\theta}}{1+\underline{\theta}}I$, then $I - b - (I+b)\theta_2 \leq I - b - (I+b)\underline{\theta} \leq 0$, and hence (91) is decreasing in θ_2. Hence, (91) is maximized at $\theta_2 = \theta_1$. Combined with $\theta_1 = \underline{\theta}$ for $b \geq 0$, this implies that if $b \geq \frac{1-\underline{\theta}}{1+\underline{\theta}}I$, then $\theta_1 = \theta_2 = \underline{\theta}$, i.e., the principal again always prefers to "flatten" the contract. As shown above, among flat contracts $\hat{X}(\theta) = \bar{X}$, the one that maximizes the payoff to the principal is $\hat{X}(\theta) = \frac{2\beta}{\beta-1}\frac{I}{1+\underline{\theta}}$. This proves the first part of the proposition for the range $b > -I$.

If $b \in [-\frac{1-\underline{\theta}}{1+\underline{\theta}}I, 0]$, then, as shown above, (91) is maximized at $\theta_2 = 1$. Combined with $\theta_1 = \frac{I-b}{I+b}\underline{\theta}$, this proves the second part of the proposition. Finally, if $b \in [0, \frac{1-\underline{\theta}}{1+\underline{\theta}}I]$, then (91) is increasing in θ_2 up to $\frac{I-b}{I+b}$ and decreasing after that. Hence, (91) is maximized at $\theta_2 = \frac{I-b}{I+b}$. Combined with $\theta_1 = \underline{\theta}$, this proves the final part of the proposition.

- $b < -I$

We show that in this case, the optimal contract is flat with $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}\ \forall \theta \in \Theta$. The proof for this case proceeds in two steps. First, we show that the optimal contract cannot have any strictly decreasing regions. Second, we show that any contract with more than one flat region is dominated by a completely flat contract.

We start by showing that the optimal contract cannot have any strictly decreasing regions. Indeed, according to Lemma 4, any strictly decreasing region is characterized by $\hat{X}(\theta) = X_A^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$. Consider θ_1 and θ_2 such that $\hat{X}(\theta) = X_A^*(\theta)$ for $\theta \in [\theta_1, \theta_2]$. For any $\hat{\theta}_2 \in (\theta_1, \theta_2)$, consider a contract $C_2 = \left\{\hat{X}_2(\theta), \theta \in \Theta\right\}$, defined as

$$\hat{X}_2(\theta) = \begin{cases} \hat{X}(\theta), & \text{if } \theta \in [\underline{\theta}, \theta_1] \cup [\theta_2, 1], \\ X_A^*(\theta), & \text{if } \theta \in \left[\theta_1, \hat{\theta}_2\right), \\ X_A^*\left(\hat{\theta}_2\right), & \text{if } \theta \in \left(\hat{\theta}_2, \hat{\theta}\right], \\ X_A^*(\theta_2), & \text{if } \theta \in \left(\hat{\theta}, \theta_2\right), \end{cases}$$

where $\hat{\theta} = \hat{\theta}\left(\hat{\theta}_2\right)$ satisfies

$$\frac{\hat{\theta} X_A^*\left(\hat{\theta}_2\right) - I + b}{X_A^*\left(\hat{\theta}_2\right)^\beta} = \frac{\hat{\theta} X_A^*\left(\theta_2\right) - I + b}{X_A^*\left(\theta_2\right)^\beta}. \tag{93}$$

(such $\hat{\theta}$ always exists and lies between $\hat{\theta}_2$ and $\hat{\theta}_1$ for the same reason as in contract C_1). Intuitively, contract C_2 is the same as contract C, except that it substitutes a subset $\left[\hat{\theta}_2, \theta_2\right]$ of the decreasing region with a piecewise flat region with a discontinuity at $\hat{\theta}$. Because contract C is incentive-compatible and $\hat{\theta}$ satisfies (93), contract C_2 is incentive-compatible too. Below we show that the payoff to the principal from contract C_2 exceeds the payoff to the principal from contract C for $\hat{\theta}_2$ very close to θ_2. Because $\hat{X}_2(\theta) = \hat{X}(\theta)$ for $\theta \leq \theta_1$ and $\theta \geq \theta_2$, it is enough to restrict attention to the payoff in the range $\theta \in (\theta_1, \theta_2)$. The payoff to the principal from contract C_2 in this range, divided by $X(0)^\beta \frac{1}{1-\underline{\theta}}$, is

$$\int_{\theta_1}^{\hat{\theta}_2} \frac{\theta X_A^*(\theta) - I}{X_A^*(\theta)^\beta} d\theta + \int_{\hat{\theta}_2}^{\hat{\theta}(\hat{\theta}_2)} \frac{\theta X_A^*\left(\hat{\theta}_2\right) - I}{X_A^*\left(\hat{\theta}_2\right)^\beta} d\theta + \int_{\hat{\theta}(\hat{\theta}_2)}^{\theta_2} \frac{\theta X_A^*(\theta_2) - I}{X_A^*(\theta_2)^\beta} d\theta. \tag{94}$$

Following the same arguments as for the derivative of (86) with respect to $\tilde{\theta}_2$ above, we can check that the derivative of (94) with respect to $\hat{\theta}_2$ is given by $\beta \frac{\left(\hat{\theta}-\hat{\theta}_2\right)^2}{2\hat{\theta}_2^2} X_A^*\left(\hat{\theta}_2\right)^{-\beta}(I+b)$, which is strictly negative at any point $\hat{\theta}_2 < \theta_2$ for $b < -I$. By the mean value theorem, if $U_P\left(\hat{\theta}_2\right)$ stands for the expected principal's utility from contract C, then $\frac{U_P\left(\hat{\theta}_2\right) - U_P(\theta_2)}{\hat{\theta}_2 - \theta_2} = U_P'\left(\tilde{\theta}_2\right) < 0$ for some $\tilde{\theta}_2 \in \left(\hat{\theta}_2, \theta_2\right)$, and hence a deviation from contract C to contract C_2 is beneficial for the principal. Hence, contract C cannot be optimal for $b < -I$.

This result implies that any optimal contract must consist only of flat regions. We next prove that any contract with more than one flat region is dominated by a contract with one flat region. We present the proof for the contract with two flat regions, and the proof for any finite number of flat regions follows from this proof by induction. Consider a contract with two flat regions: Types $\left[\underline{\theta}, \hat{\theta}\right]$ pick exercise at $\hat{X}_L$, and types $\left[\hat{\theta}, 1\right]$ pick exercise at $\hat{X}_H < \hat{X}_L$. Type $\hat{\theta} \in (\underline{\theta}, 1)$ satisfies

$$\frac{\hat{\theta}\hat{X}_L - I + b}{\hat{X}_L^\beta} = \frac{\hat{\theta}\hat{X}_H - I + b}{\hat{X}_H^\beta}. \tag{95}$$

The principal's expected value is

$$\frac{\hat{\theta} - \underline{\theta}}{1 - \underline{\theta}} \frac{\frac{\underline{\theta}+\hat{\theta}}{2}\hat{X}_L - I}{\hat{X}_L^\beta} + \frac{1 - \hat{\theta}}{1 - \underline{\theta}} \frac{\frac{\hat{\theta}+1}{2}\hat{X}_H - I}{\hat{X}_H^\beta}.$$

Consider an alternative contract with $\hat{X}(\theta) = \hat{X}_H$ for all θ. The difference between the principal's value under this pooling contract and his value under the original contract, divided by $X(0)^\beta$, is given by

$$\begin{aligned} \Delta U &= \int_{\underline{\theta}}^1 \frac{\theta \hat{X}_H - I}{\hat{X}_H^\beta} \frac{d\theta}{1-\underline{\theta}} - \left[\int_{\underline{\theta}}^{\hat{\theta}} \frac{\theta \hat{X}_L - I}{\hat{X}_L^\beta} \frac{d\theta}{1-\underline{\theta}} + \int_{\hat{\theta}}^1 \frac{\theta \hat{X}_H - I}{\hat{X}_H^\beta} \frac{d\theta}{1-\underline{\theta}}\right] = \int_{\underline{\theta}}^{\hat{\theta}} \left(\frac{\theta \hat{X}_H - I}{\hat{X}_H^\beta} - \frac{\theta \hat{X}_L - I}{\hat{X}_L^\beta}\right) \frac{d\theta}{1-\underline{\theta}} \\ &= \frac{\hat{\theta}-\underline{\theta}}{1-\underline{\theta}} \left(\frac{\frac{\hat{\theta}+\underline{\theta}}{2}\hat{X}_H - I}{\hat{X}_H^\beta} - \frac{\frac{\hat{\theta}+\underline{\theta}}{2}\hat{X}_L - I}{\hat{X}_L^\beta}\right) = \frac{\hat{\theta}-\underline{\theta}}{1-\underline{\theta}} \left(\frac{\frac{\hat{\theta}}{2}\hat{X}_H - I}{\hat{X}_H^\beta} - \frac{\frac{\hat{\theta}}{2}\hat{X}_L - I}{\hat{X}_L^\beta} + \frac{\underline{\theta}}{2}\left(\frac{1}{\hat{X}_H^{\beta-1}} - \frac{1}{\hat{X}_L^{\beta-1}}\right)\right) \end{aligned} \tag{96}$$

Using (95) and the fact that $b \leq -I$,

$$\begin{aligned} \frac{\hat{\theta}\hat{X}_H - I}{\hat{X}_H^{\beta}} - \frac{\hat{\theta}\hat{X}_L - I}{\hat{X}_L^{\beta}} &= b\left(\frac{1}{\hat{X}_L^{\beta}} - \frac{1}{\hat{X}_H^{\beta}}\right) \geq I\left(\frac{1}{\hat{X}_H^{\beta}} - \frac{1}{\hat{X}_L^{\beta}}\right) \Leftrightarrow \frac{\hat{\theta}\hat{X}_H - 2I}{\hat{X}_H^{\beta}} - \frac{\hat{\theta}\hat{X}_L - 2I}{\hat{X}_L^{\beta}} \geq 0 \\ &\Leftrightarrow \frac{\frac{\hat{\theta}}{2}\hat{X}_H - I}{\hat{X}_H^{\beta}} - \frac{\frac{\hat{\theta}}{2}\hat{X}_L - I}{\hat{X}_L^{\beta}} \geq 0, \end{aligned}$$

and the inequalities are strict if $b < -I$. Combining this with $\hat{X}_H < \hat{X}_L$ and using (96), implies that $\Delta U \geq 0$ and $\Delta U > 0$ if at least one of $b < -I$ or $\underline{\theta} > 0$ holds. Thus, the contract with two flat regions is dominated by a contract with one flat region. Combining this with (84) completes the proof of this case. ■

Proof of Proposition 10. Let $VD(X, b)$ denote the expected value to the principal under delegation if the current value of $X(t)$ is X. If the decision is delegated to the agent, exercise occurs at threshold $X_A^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$, and the principal's payoff upon exercise is $\frac{\beta}{\beta-1}(I-b) - I$. Hence,

$$VD(X,b) = \int_0^1 X^{\beta}\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b) - I\right)d\theta = \frac{X^{\beta}}{\beta+1}\left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b) - I\right).$$

Let $VA(X, b)$ denote the expected value to the principal in the most informative equilibrium of the advising game if the current value of $X(t)$ is X. Using (24) and (31),

$$VA(X,b) = X^{\beta}\frac{1-\omega^*(b)}{1-\omega^*(b)^{\beta+1}}\left(\frac{\beta}{\beta-1}\frac{2I}{1+\omega^*(b)}\right)^{-\beta}\frac{I}{\beta-1}, \tag{97}$$

where $\omega^*(b)$ is the unique solution to (32), given b. Because X^{β} enters as a multiplicative factor in both $VD(X, b)$ and $VA(X, b)$, it is sufficient to compare $VD(b)$ and $VA(b)$, where $VD(b) \equiv X^{-\beta}VD(X, b)$ and $VA(b) \equiv X^{-\beta}VA(X, b)$.

First, consider the behavior of $VA(b)$ and $VD(b)$ around $b = I$. Note that $\lim_{b\to I}\omega^*(b) = 0$, and hence

$$\lim_{b\to I} VD(b) = -\infty, \tag{98}$$

$$\lim_{b\to I} VA(b) = \left(\frac{\beta}{\beta-1}2I\right)^{-\beta}\frac{I}{\beta-1}. \tag{99}$$

By continuity of $VD(b)$ and $VA(b)$ in b, (98) and (99) imply that there exists $\bar{b} \in (0, I)$, such that for any $b > \bar{b}$, $VA(b) > VD(b)$. In other words, advising dominates delegation if the conflict of interest between the agent and the principal is big enough.

Second, consider the behavior of $VA(b)$ and $VD(b)$ for small but positive b. By l'Hopital's rule,

$$\lim_{b\to 0+} VD(b) = \lim_{b\to 0+} VA(b) = \frac{1}{\beta+1}\left(\frac{\beta}{\beta-1}I\right)^{-\beta}\frac{I}{\beta-1}.$$

Note that

$$VD'(b) = -\frac{\beta b}{(\beta+1)(I-b)}\left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta}$$

In particular, $\lim_{b\to 0+} VD'(b) = 0$ and

$$\lim_{b\to 0} \frac{VD'(b)}{b} = -\frac{\beta^2}{\beta^2-1}\left(\frac{\beta}{\beta-1}I\right)^{-\beta-1}.$$

The derivative of $VA(b)$ with respect to b can be found as

$$VA'(b) = C\frac{d\omega^*(b)}{db}\left[\frac{(1-\omega)(1+\omega)^\beta}{1-\omega^{\beta+1}}\right]'|_{\omega=\omega^*(b)}, \tag{100}$$

where $C = \left(\frac{\beta}{\beta-1}2I\right)^{-\beta}\frac{I}{\beta-1}$. Recall that $\omega^*(b)$ solves (32), which is equivalent to

$$\frac{2I}{I-b}\frac{\beta}{\beta-1} = \left(\frac{1}{\omega}+1\right)\frac{1-\omega^\beta}{1-\omega^{\beta-1}}. \tag{101}$$

Differentiating this equation, we get

$$\frac{2I}{(I-b)^2}\frac{\beta}{\beta-1}db = \frac{-\left(1-\omega^\beta\right)\left(1-\omega^{\beta-1}\right)+(1+\omega)\,\omega\left(-\beta\omega^{\beta-1}\left(1-\omega^{\beta-1}\right)+(\beta-1)\,\omega^{\beta-2}\left(1-\omega^\beta\right)\right)}{\omega^2\left(1-\omega^{\beta-1}\right)^2}d\omega. \tag{102}$$

Because (101) is equivalent to $\frac{1}{I-b} = \frac{1}{2I}\frac{\beta-1}{\beta}\frac{1+\omega}{\omega}\frac{1-\omega^\beta}{1-\omega^{\beta-1}}$, we can rewrite the left-hand side of (102) as

$$2I\frac{\beta}{\beta-1}\left(\frac{1}{2I}\right)^2\left(\frac{\beta-1}{\beta}\right)^2\frac{(1+\omega)^2}{\omega^2}\frac{\left(1-\omega^\beta\right)^2}{\left(1-\omega^{\beta-1}\right)^2}db.$$

Substituting this into (102) and simplifying, we get

$$\frac{d\omega}{db}|_{\omega=\omega^*(b)} = \frac{1}{2I}\frac{\beta-1}{\beta}\frac{(1+\omega)^2\left(1-\omega^\beta\right)^2}{-\left(1-\omega^\beta\right)\left(1-\omega^{\beta-1}\right)+(1+\omega)\,\omega^{\beta-1}\left(-\beta\omega+\beta-1+\omega^\beta\right)}. \tag{103}$$

Plugging (103) and

$$\left[\frac{(1-\omega)(1+\omega)^\beta}{1-\omega^{\beta+1}}\right]' = \frac{(1+\omega)^{\beta-1}}{\left(1-\omega^{\beta+1}\right)^2}\left[(\beta-1)\left(1-\omega^{\beta+1}\right)-(\beta+1)\left(\omega-\omega^\beta\right)\right],$$

into (100), we get

$$VA'(b) = -D\frac{(1+\omega)^{\beta+1}\left(1-\omega^\beta\right)^2\left[(\beta-1)\left(1-\omega^{\beta+1}\right)-(\beta+1)\left(\omega-\omega^\beta\right)\right]}{\left(1-\omega^{\beta+1}\right)^2\left[\left(1-\omega^\beta\right)\left(1-\omega^{\beta-1}\right)-(1+\omega)\,\omega^{\beta-1}\left(-\beta\omega+\beta-1+\omega^\beta\right)\right]},$$

where $D \equiv \frac{C}{2I}\frac{\beta-1}{\beta}$. To find $\lim_{b\to 0}\frac{VA'(b)}{b}$, we express $\frac{1}{b}$ from (101) as

$$\frac{1}{b} = \frac{(\beta-1)(1+\omega)\left(1-\omega^\beta\right)}{I\left[(\beta-1)(1+\omega)\left(1-\omega^\beta\right)-2\beta\omega\left(1-\omega^{\beta-1}\right)\right]},$$

and hence

$$\begin{aligned}\frac{VA'(b)}{b} &= -D\frac{(1+\omega)^{\beta+1}\left(1-\omega^{\beta}\right)^2\left[(\beta-1)\left(1-\omega^{\beta+1}\right)-(\beta+1)\left(\omega-\omega^{\beta}\right)\right]}{(1-\omega^{\beta+1})^2[(1-\omega^{\beta})(1-\omega^{\beta-1})-(1+\omega)\omega^{\beta-1}(-\beta\omega+\beta-1+\omega^{\beta})]}\frac{(\beta-1)(1+\omega)\left(1-\omega^{\beta}\right)}{I[(\beta-1)(1+\omega)(1-\omega^{\beta})-2\beta\omega(1-\omega^{\beta-1})]} \\ &= -\frac{(\beta-1)D}{I}\frac{(1+\omega)^{\beta+2}\left(1-\omega^{\beta}\right)^3}{(1-\omega^{\beta+1})^2[(1-\omega^{\beta})(1-\omega^{\beta-1})-(1+\omega)\omega^{\beta-1}(-\beta\omega+\beta-1+\omega^{\beta})]}.\end{aligned}$$

Hence,

$$\lim_{b\to 0}\frac{VA'(b)}{b} = -\frac{(\beta-1)\,2^{\beta+2}D}{I}\lim_{\omega\to 1}\left[\frac{1-\omega^{\beta}}{1-\omega^{\beta+1}}\right]^2\lim_{\omega\to 1}\left[\frac{1-\omega^{\beta}}{(1-\omega^{\beta})\left(1-\omega^{\beta-1}\right)-(1+\omega)\,\omega^{\beta-1}\left(-\beta\omega+\beta-1+\omega^{\beta}\right)}\right].$$

By l'Hopital's rule, the first limit equals $(\frac{\beta}{\beta+1})^2$, and the second limit equals ∞. Therefore,

$$\lim_{b\to 0}\frac{VA'(b)}{b} = -\infty < \lim_{b\to 0}\frac{VD'(b)}{b} = -\frac{\beta^2}{\beta^2-1}\left(\frac{\beta}{\beta-1}I\right)^{-\beta-1}.$$

By continuity of $VA'(b)$ and $VD'(b)$ for $b>0$, there exists $\underline{b}>0$ such that $VA'(b) < VD'(b)$ for any $b<\underline{b}$. Because $VA(0)=VD(0)$, then

$$VD(b) - VA(b) = \int_0^b \left(VD'(y) - VA'(y)\right)dy > 0$$

for any $b\in(0,\underline{b}]$. In other words, delegation dominates advising when the agent is biased towards early exercise but the bias is low enough. ■

Proof of Proposition 11. Note that the following three inequalities are equivalent: $b \le \frac{1-\underline{\theta}}{1+\underline{\theta}}I \Leftrightarrow \frac{I-b}{I+b} \ge \underline{\theta} \Leftrightarrow \frac{\beta}{\beta-1}(I+b) \le \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$. Hence, there are two cases. If $b<\frac{1-\underline{\theta}}{1+\underline{\theta}}I$, delegation occurs at the threshold $\frac{\beta}{\beta-1}(I+b) = X_A^*\left(\frac{I-b}{I+b}\right)$, where $\frac{I-b}{I+b}>\underline{\theta}$. If $b\ge\frac{1-\underline{\theta}}{1+\underline{\theta}}I$, then $\frac{I-b}{I+b}\le\underline{\theta}$ and delegation occurs at the principal's uninformed exercise threshold $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$.

We need to prove that neither the agent nor the principal wants to deviate from the specified strategies. First, consider the incentives of the agent. Given Assumption 1, sending a message $m=1$ is never beneficial for the agent because it does not change the principal's belief and hence the principal's strategy. Hence, the agent cannot induce exercise before she is given authority. After the agent is given authority, her optimal exercise strategy is to: 1) exercise the option immediately if $b\ge I$, or if $b<I$ and $X_d\ge X_A^*(\theta)$; 2) exercise the option when $X(t)$ first reaches $X_A^*(\theta)$ if $b<I$ and $X_d<X_A^*(\theta)$. Consider two cases. If $0<b<\frac{1-\underline{\theta}}{1+\underline{\theta}}I$ $(\le I)$, then $X_d = X_A^*\left(\frac{I-b}{I+b}\right)$, and hence $X_d<X_A^*(\theta)$ if and only if $\theta<\frac{I-b}{I+b}$. Thus, types below $\frac{I-b}{I+b}$ find it optimal to exercise at $X_A^*(\theta)$ and types above $\frac{I-b}{I+b}$ find it optimal to exercise immediately at X_d, consistent with the equilibrium strategy. Second, if $b\ge\frac{1-\underline{\theta}}{1+\underline{\theta}}I$, the agent finds it optimal to exercise immediately at X_d regardless of her type: if $b\ge I$, this is always the case, and if $\frac{1-\underline{\theta}}{1+\underline{\theta}}I\le b<I$, this is true because $X_A^*(\theta)\le X_A^*(\underline{\theta}) = \frac{\beta}{\beta-1}\frac{I-b}{\underline{\theta}} \le \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1} = X_d$. Since in this case $\frac{I-b}{I+b}\le\underline{\theta}$, this strategy again coincides with the equilibrium strategy. This proves that the agent does not want to deviate.

Next, consider the principal's strategy. The above arguments show that the equilibrium exercise times coincide with the exercise times under the optimal contract in Proposition 7, both for $b<\frac{1-\underline{\theta}}{1+\underline{\theta}}I$ and for $b\ge\frac{1-\underline{\theta}}{1+\underline{\theta}}I$. Hence, the principal's expected utility in this equilibrium coincides with his expected utility in the optimal contract. Consider possible deviations of the principal, taking into account that the agent's messages are uninformative and hence the principal does not learn any new information about θ by waiting. First, the principal can exercise the option himself, before or after $X(t)$ first reaches X_d. Because a contract with such an exercise policy is incentive compatible, the principal's utility from such a deviation cannot exceed his utility under the optimal contract and hence his equilibrium utility. Thus, such a deviation cannot be strictly profitable. Second, the principal can deviate by delegating authority to the agent before or after $X(t)$ first reaches X_d. An agent who receives authority at some point t will exercise the option immediately if $b \geq I$, or if $b < I$ and $X(t) \geq X_A^*(\theta)$, and will exercise the option when $X(t)$ first reaches $X_A^*(\theta)$ otherwise. Because a contract with such an exercise schedule is incentive compatible, the principal's utility from this deviation cannot exceed his utility under the optimal contract and hence his equilibrium utility. This proves that the principal does not want to deviate either. ■

Proof of Proposition 12. The fact that ω^* decreases in b has been proved in the proof of Lemma 3. We next show that ω^* increases in β. From (32), ω^* solves $F(\omega, \beta) = 0$, where

$$F(\omega, \beta) = \frac{\beta}{\beta - 1} \frac{1 - \omega^{\beta-1}}{1 - \omega^{\beta}} \frac{2I}{I - b} - 1 - \frac{1}{\omega}.$$

Denote the unique solution by $\omega^*(\beta)$. Function $F(\omega, \beta)$ is continuously differentiable in both arguments on $\omega \in (0, 1)$, $\beta > 1$. Differentiating $F(\omega^*(\beta), \beta)$ in β:

$$\frac{\partial \omega^*}{\partial \beta} = -\frac{F_\beta(\omega^*(\beta), \beta)}{F_\omega(\omega^*(\beta), \beta)}.$$

Since $F(0, \beta) < 0$, $F(1, \beta) = \frac{2b}{I-b} > 0$, and ω^* is the unique solution of $F(\omega, \beta) = 0$ in $(0, 1)$, we know that $F_\omega(\omega^*(\beta), \beta) > 0$. Hence, it is sufficient to prove that $F_\beta(\omega, \beta) < 0$. Differentiating $F(\omega, \beta)$ with respect to β and reorganizing the terms, we obtain that $F_\beta(\omega, \beta) < 0$ is equivalent to

$$\frac{\left(1 - \omega^{\beta-1}\right)\left(1 - \omega^{\beta}\right)}{\omega^{\beta-1}(1 - \omega)} + \beta(\beta - 1)\ln\omega > 0.$$

Denote the left-hand side as a function of β by $L(\beta)$. Because $L(1) = 0$, a sufficient condition for $L(\beta) > 0$ for any $\beta > 1$ is that $L'(\beta) > 0$ for $\beta > 1$. Differentiating $L(\beta)$:

$$L'(\beta) = \ln\omega\left[-\frac{\omega^{1-\beta} - \omega^{\beta}}{1 - \omega} + 2\beta - 1\right].$$

Because $\ln\omega < 0$ for any $\omega \in (0, 1)$, condition $L'(\beta) > 0$ is equivalent to

$$d(\beta) \equiv \frac{\omega^{1-\beta} - \omega^{\beta}}{1 - \omega} - 2\beta + 1 > 0.$$

Note that $\lim_{\beta \to 1} d(\beta) = 0$ and

$$d'(\beta) = -\left(\omega^{1-\beta} + \omega^{\beta}\right)\frac{\ln\omega}{1 - \omega} - 2 \equiv g(\beta).$$

Note that

$$\begin{aligned} g(\beta) &= g(1) + \int_1^\beta g'(x)\,dx \\ &= -\frac{(1 + \omega)\ln\omega}{1 - \omega} - 2 + \frac{(\ln\omega)^2}{1 - \omega}\int_1^\beta \left(\left(\frac{1}{\omega}\right)^{2x-1} - 1\right)\omega^x dx. \end{aligned} \tag{104}$$

The second term of (104) is positive, because $\left(\frac{1}{\omega}\right)^{2x-1} - 1 > 0$, since $\frac{1}{\omega} > 1$ and $2x - 1 > 1$ for any $x > 1$.

The first term of (104) is positive, because

$$\lim_{\omega\to 1}\left(-\frac{(1+\omega)\ln\omega}{1-\omega}-2\right) = \lim_{\omega\to 1}\left(\ln\omega+\frac{1+\omega}{\omega}\right)-2=0 \tag{105}$$

$$\text{and } \frac{\partial\left(-\frac{(1+\omega)\ln\omega}{1-\omega}-2\right)}{\partial\omega} = \frac{-2\ln\omega-\frac{1}{\omega}+\omega}{(1-\omega)^2}<0, \tag{106}$$

where (105) is by l'Hopital's rule, and (106) is because $\left(-2\ln\omega-\frac{1}{\omega}+\omega\right)'=\frac{(1-\omega)^2}{\omega^2}>0$ and $-2\ln\omega-\frac{1}{\omega}+\omega$ equals zero at $\omega=1$. Therefore, $g(\beta)>0$ and hence $d'(\beta)>0$ for any $\beta>1$, which together with $d(1)=0$ implies $d(\beta)>0$, which in turn implies that $L(\beta)>0$ for any $\beta>1$. Hence, $F_\beta(\omega,\beta)<0$. Therefore, ω^* is strictly increasing in $\beta>1$. Finally, a standard calculation shows that $\frac{\partial\beta}{\partial\sigma}<0$, $\frac{\partial\beta}{\partial\mu}<0$, and $\frac{\partial\beta}{\partial r}>0$. Therefore, ω^* is decreasing in β and μ and increasing in r. ■

Proof of Proposition 13. (i) To prove this proposition, we use the solution for the optimal contract offered by the principal assuming full commitment power (Proposition 7). Suppose that the current value of $X(t)$ is 1, and let $VC(b)$ denote the ex-ante utility of the principal under commitment as a function of b. We start by showing that $VC(b)=VC(-b)$ for any $b>0$. First, consider $b\notin(-I,I)$. In this case, the exercise trigger equals $\frac{2\beta}{\beta-1}I$, regardless of θ and b. Hence, $VC=VC(-b)$ for any $b\geq I$. Second, consider $b\in(-I,I)$. For $b\in(-I,0]$, the exercise trigger for type θ is $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$. Thus, the expected payoff of the principal is:

$$\begin{aligned} VC(b) &= \int_0^1\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right)d\theta \\ &= \frac{(\beta-1)^{\beta-1}}{(\beta(I-b))^{\beta}}\frac{I-\beta b}{\beta+1}=\frac{(\beta-1)^{\beta-1}}{(\beta(I+|b|))^{\beta}}\frac{I+\beta|b|}{\beta+1}. \end{aligned}$$

Next, consider $b\in[0,I)$. The exercise trigger for type θ is $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$ if $\theta\leq\frac{I-b}{I+b}$ and $\frac{\beta}{\beta-1}(I+b)$ otherwise. Thus, the utility of the principal is:

$$VC(b)=\int_0^{\frac{I-b}{I+b}}\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b)-I\right)d\theta+\int_{\frac{I-b}{I+b}}^1\left(\frac{\beta}{\beta-1}(I+b)\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I+b)\theta-I\right)d\theta,$$

which can be shown to be equal to $\frac{(\beta-1)^{\beta-1}}{(\beta(I+b))^{\beta}}\frac{I+\beta b}{\beta+1}$. Thus, $VC(b)=VC(-b)$ for any $b\in[0,I)$. Combining the two cases, we conclude that $VC(b)=VC(-b)$ for any $b\geq 0$.

We now use this property to prove the statement of the proposition. Recall that for $b\in(-I,0)$, the principal's expected utility in the advising equilibrium coincides with his expected utility under commitment. Hence, $V_0(-b)=VC(-b)$ for $b\in(0,I)$. Recall also that the principal strictly benefits from commitment when $b>0$: $V_0(b)<VC(b)$ for $b>0$. Since, as shown above, $VC(b)=VC(-b)$, we conclude that $V_0(-b)=VC(-b)=VC(b)>V_0(b)$ for $b\in(0,I)$.

Next, consider $b\geq I$. Denote by V_u the principal's utility from following the optimal uniformed exercise strategy $\bar{X}_u=\frac{\beta}{\beta-1}2I$. As shown in the proof of Proposition 1, when $b=-I$, the principal's utility in the equilibrium with continuous exercise equals V_u. For $b<-I$, only babbling equilibria exist and hence the principal's utility is again V_u. Similarly, if $b\geq I$, there is no informative equilibrium in the advising game. Hence, the principal exercises the option at the uninformed threshold $\bar{X}_u=\frac{\beta}{\beta-1}2I$, and his utility is also given by V_u. Thus, $V_0(-b)=V_0(b)=V_u$ for any $b\geq I$, which completes the proof.

(ii) Consider $b\in(0,I)$. According to Proposition 9, $\tilde{V}_0(-b)=V_0(-b)$, and hence $\tilde{V}_0(-b)=VC(-b)=VC(b)$. For an agent biased towards early exercise, the exercise schedule under both delegation and communication is different from the exercise schedule in the optimal mechanism of Proposition 7. Hence, regardless of the principal's choice between delegation and retaining authority, $\tilde{V}_0(b) < VC(b) = \tilde{V}_0(-b)$. If $|b| \geq I$, then for any direction of the agent's bias, the principal's utility in the advising equilibrium is V_u, which coincides with his utility in the optimal mechanism. Hence, $\tilde{V}_0(b) = V_u = \tilde{V}_0(-b)$ for $b \geq I$. ■