

Dynamic Aggregation of Heterogeneous Interacting Agents and Network: An Analytical Solution for Agent Based Models[☆]

C. Di Guilmi^{a,*}, M. Gallegati^b, S. Landini^c, J. E. Stiglitz^d

^a*Economics Discipline Group, University of Technology, Sydney, Australia.*

^b*DiSES, Università Politecnica delle Marche, Italy*

^c*IRES Piemonte, Turin, Italy*

^d*Columbia University, New York, USA*

Abstract

One of the consequences of the failure of mainstream theory in convincingly taking into account the heterogeneity of agents and their interaction is the misrepresentation of markets with asymmetric information. This paper presents a credit network model with heterogeneous firms and banks which is solved analytically. The solution is articulated in two steps which involve two different master equations. The first performs the aggregation over a population of firms that are heterogeneous in size and financial conditions in order to identify the number of firms that enter the credit market. The second master equation uses the solution of the first to describe the dynamic evolution of the network structure and in particular of the network degree. The analytical solution provides a system of equations that generates the same outcomes of the numerical solution and can provide some insights on the causal relationships governing the dynamics of boom and bust in the economy as a consequence of the evolution of the concentration in the credit market.

Keywords: heterogeneity, financial fragility, master equation, interaction, dynamic aggregation.

JEL classification: E1, E6.

1. Introduction

The current crisis has carried a growing number of economists to reflect on the state of the economic theory. Even though unreliable economic models

[☆]The authors would like to thank Duncan Foley, Bruce Greenwald, Thomas Lux, the participants to the 37th Eastern Economic Association annual conference, to the seminars at the University of New South Wales, Australian National University and the University of Technology, Sydney. The usual caveats apply.

*Corresponding author: Corrado Di Guilmi. Economics Discipline Group - University of Technology, Sydney - PO Box 123, Broadway, NSW 2007, Australia. Ph.: +61295144746, Fax: +61295147711. E-mail: corrado.diguilmi@uts.edu.au.

have not provoked the crisis, they have not been able to prevent or even to forecast it. In particular, those reductionist models populated by a perfectly rational and fully informed representative agent turned out to be extremely fallacious (Stiglitz, 2009). If information is asymmetric, agents are heterogeneous and, as a consequence, there is interaction among them (Stiglitz and Gallegati, 2011). Interaction, outside completely unique and unrealistic cases, generates non-linearities and we know that “standard” aggregation works only with linear relationships (Fisher, 1987; Hildebrand, 2008).

A method completely different from the mainstream is proposed by the agent based models (: ABM), which is characterized by a bottom-up approach: it builds up from the description of the behaviours of the individuals; their interaction creates an emergent behaviour (with characteristics that we cannot find at the level of the single agents) which feeds back on the agents themselves. The solution for this class of models is usually obtained by numerical simulations. This paper and the companion Di Guilmi et al. (2012b) follow a quite different approach, adopting analytical tools originally developed in statistical mechanics, and subsequently adopted by social disciplines (in economics: Foley, 1994; Aoki, 1996, 2002; Aoki and Yoshikawa, 2006; Di Guilmi, 2008; Landini and Uberti, 2008; Weidlich, 2000). As the economy is populated by a very large number of heterogeneous and interacting agents (: HIA), we cannot know which agent is in which condition at a given time and whether an agent will change its condition, but we can know the probability of a given state of the world.

The basic idea consists of introducing a meso-level of aggregation, obtained by grouping the agents in clusters according to a measurable variable. This state variable is chosen such that the dynamics of the number of individuals in each cluster also defines the evolution of the whole economy¹. The process of reducing the vector of observations of a variable over a population to a single value (i. e. the computation of an average level of output for each dimensional bin) is defined as mean field approximation. The definitions of the mean-field variables and of the probabilities involve some level of interaction among agents².

The subsequent step is the study of the dynamics of the number of agents in each cluster (occupation numbers). The stochastic evolution of the occupation numbers can be modelled by means of the master equation (: ME). It is a stochastic differential equation, which quantifies the evolution through time of the probability of observing a given number of agents in a certain state.

As demonstrated in this paper, the ME is an analytical device which has a potentially revolutionary impact on macroeconomic modelling from three different perspectives. First, it represents a fascinating and effective solution to

¹For example, we can classify firms within an industry into bins according to their number of employees and, accordingly, compute an average production for each dimensional class. The evolution of the relative density of firms in each cluster (i.e. the occupation numbers) then allows us to study the dynamics of the total output of the industry.

²Mean-field theory has been introduced in economics in different models by Brock and Durlauf, who show how mean-field interaction is able to generate a multiplicity of Nash-type equilibria (Brock and Durlauf, 2001).

the problem of performing the aggregation when heterogeneity and nonlinearities are present, an issue which is debated in the literature at least since the introduction of the exact aggregation by Gorman (1953). In fact, the ME is an analytical tool that models the aggregate dynamics by means of a (mechanical) statistical approach, that integrates the heterogeneity by considering a multiplicity of representative states of a system, and implements the interaction (as a mean field interaction) by means of the specification of the *transition rates*. In such a way, the problem of the aggregation of heterogeneous agents is originally solved, without resorting neither to the unrealistic simplifications and assumptions of mainstream theory, nor to the “black box” of computer simulations as in the standard ABM approach. Furthermore this methodology can effectively deal with the issue of agents interaction, by functionally embodying it in the determination of the probabilistic transition rules (the transition rates), which change endogenously in time. It is worth noticing that this approach overcomes another limit of the mainstream modelling, as the equilibrium is no more a fixed point in the space, but a probability distribution (Foley, 1994; Aoki, 1996). Hence, a system can be in equilibrium even if its constitutive elements are not.

The second perspective concerns the outcomes of the solution of ME model. As we show in this paper, the ME approach is able to deal with another issue, which is largely debated in the economic and econometric literature: the decomposition of a time series in trend and fluctuations. More precisely, by adapting one of the methods proposed in Di Guilmi et al. (2012a), we attain a solution of the master equation which is composed by two elements: an ordinary differential equation, which describes the long run trend of the state variable, and a probability density function, which defines the probability of fluctuations around this trend. Hence, by choosing opportunely the state variable, a micro-founded macro-model can endogenously generate the different components of a macroeconomic time series, which can be then analysed separately.

The third perspective concerns the particular application that we propose in this paper. As demonstrated by the physics literature (see Ehrhardt et al., 2006), the ME can be effectively employed to analytically model the dynamic of the topology of a network. In this work, this methodology is applied in economics for analysing the evolution of the degree distribution of a network, further developing and extending Di Guilmi et al. (2012b). Besides slight changes in the model, the main difference with our existing paper is in the refinement of the solution technique, which was previously developed by exogenously determining the size of the biggest bank in order to study the effects of concentration. In the present paper all the components of the solution are endogenously determined within the system. The solution of the ME yields the dynamics of the network degree, providing a synthetic and formal representation of the concentration in the market.

The target of the present paper is chiefly methodological. Nevertheless the analytical solution provides insights on the dynamics of the model as displayed by the simulations and can shed light on the evolution of the network during the cycle. Summarising, the main contributions of this paper are the following:

- to provide a solution to the problem of the aggregation of HIAs by using methods inspired by the statistical mechanics and the complexity theory against the usual classical mechanics involved in RA based models;
- to propose a dynamic stochastic model for sub-populations of many HIAs interacting in an endogenous evolving network;
- to develop an analytical solution composed by a deterministic ordinary differential equation describing the dynamics of network degree, and a dynamic probability distribution for the fluctuations around this trend.

The dynamic stochastic aggregation is able to provide a complete and consistent analytical representation of the system, using numerical simulations only as a further verification of the results of the model.

The paper is structured as follows: section 2 provides some general indication about the use of the ME in the context of a credit network and defines the general characteristics of the framework; sections 3 and 4 present the hypotheses, respectively, for firms and for banks; section 5 explains the dynamics of the creation of links in the network and provides the results of the numerical simulations; section 6 develops and illustrates the endogenous dynamics of firms by means of the ME; section 7 concerns the analytical representation and final solution of the model, composed by a system of equations for the dynamics of the number of banks in each dimensional class and the degree distribution; section 8 offers some final considerations about possible extensions of the framework presented here.

2. Master equation, networks and the credit market

Previous studies on asymmetric information and financial fragility modelled the credit market as a network, whose links are given by the credit relationships between financial institutions and firms (Delli Gatti et al., 2006, 2009, 2010; Freixas et al., 2000; Furfine, 2003; Iori et al., 2006). These models describe different aspects of the generation of systemic financial distress and the transmission of shocks among the different sectors of the economy. In particular, a financial crisis may take the form of an avalanche of bankruptcies of firms and banks, which creates a domino effect impacting the level of output.

In this paper, we use as a work-horse the model by Delli Gatti et al. (2010) in which firms produce a homogeneous good sold at a stochastic price, demand for credit to banks in order to finance their wage bill and directly interact with banks within a local environment, under the assumption of asymmetric and limited information. Firms do not have complete information about their final demand: they will sell all their production but at uncertain price. This creates uncertainty also for the lending bank because a negative price shock for the borrowing firm can determine its bankruptcy and, as consequence, its insolvency. Firms have also limited information in the credit market: in each period they survey only a subset of all the banks.

As explained in the following sections, one of the consequences of asymmetric information is the existence a hierarchy of financial sources for firms: they prefer to finance their production with internal finance. If internal resources are smaller than production costs, they resort to banks to obtain credit. Thus, firms are classified into self-financing (: SF) and non self-financing (: NSF). This distinction is fundamental because only NSF firms give origin to a network by establishing credit links with banks. The structure of network can be represented as a bipartite graph, in the sense that the nodes can be sorted into two disjoint sets: firms and banks. Its structure is asymmetric since, by assumption, one firm can have only a lending bank while each bank can have multiple borrowers.

The main novelty with respect to the existing works on the credit network is the use of the ME to solve the model and to investigate the dynamics of the network in terms of its degree distribution. In particular, we employ two MEs, one nested into the other, to attain an integrated representation of the system. More precisely, a first ME yields the dynamics of the two types of firms, SF and NSF; another ME uses this first results to describe the evolution of the network in terms of its degree distribution. We show that this analysis is able to describe the dynamics of the network, its resilience and the emergence of systemic risk, in the same way as for the numerical solution in the cited papers. The analytical derivation and analysis of the degree distribution sheds light on the domino effect and the financial contagion that played such a relevant role in the recent crisis. In fact, the interaction among agents is modelled as an explicit functional form and it is at the root of the network evolution.

While the solution of the model and the interpretations of the results are provided by its analytical representation, the numerical simulations offer a benchmark for testing the theoretical predictions of the model as in Alfarano et al. (2008) and Chiarella and Di Guilmi (2011).

3. The firms sector

All firms adopt the same linear production technology, described by the following relationship

$$Q_{f,t} = (1/\gamma)N_{f,t}, \quad (1)$$

where $\gamma > 0$ is the inverse of labour productivity, assumed to be constant, and $N_{f,t}$ is the labour. The output is decided according to the rule

$$Q_{f,t} = \alpha A_{f,t}^\beta. \quad (2)$$

The variable $A_{f,t}$ indicates the internal financial resources, that we refer to as net worth. It is given by the sum of previous profits retained by the firm. The parameter $\alpha > 0$ and $\beta \in (0, 1)$ are constant.

As explained by Delli Gatti et al. (2010), the rationale for (2) is provided by a maximisation problem, as the one in Greenwald and Stiglitz (1993). In their model, firms maximise a profit function whose argument is the bankruptcy

cost weighted by the probability of bankruptcy. As the cost of bankruptcy is increasing with the scale of production and the probability is an inverse function of the equity, there is a direct proportionality between net worth and output. The functional form comes from the assumption of decreasing return to financial soundness: a firm which is already relatively safe from bankruptcy benefits progressively less of an increase in its net worth.

The demand for labour from the firm f is thus given by

$$N_{f,t} = \gamma Q_{f,t}. \quad (3)$$

It is easy to see that the demand for labour is an increasing and concave function of firms financial conditions, as, combining (2) and (3), it is possible to write: $N_{f,t} = \gamma \alpha A_{f,t}^\beta$. Assuming a constant wage w and a perfectly elastic supply of labour, the wage bill W is given by

$$W_{f,t} = w N_{f,t}. \quad (4)$$

Firms are classified in the two above mentioned groups depending on their capacity to finance the desired production with internal funding. In particular a firm is SF if $A_{f,t} \geq W_{f,t}$ and NSF in the the other case.

Due to the uncertainty on the final demand, the selling price of the output is modelled as a random process (Greenwald and Stiglitz, 1993). Following Delli Gatti et al. (2005), it is quantified as a stochastic realization of a uniform distribution

$$p_{f,t} \rightarrow U(u_{min}; u_{max}). \quad (5)$$

Consistently with the assumption of less than perfect information, firms rank their sources of financing, from internal to external finance. Therefore, the demand for credit for a single firm is residually determined as

$$D_{f,t} = \begin{cases} W_{f,t} - A_{f,t} & \text{if } W_t > A_t \\ 0 & \text{if } W_t \leq A_t \end{cases} \quad (6)$$

The profits $\pi_{f,t}$ are determined in the following way

$$\pi_{f,t} = p_{f,t} Q_{f,t} - W_{f,t} - r_{f,t} D_{f,t}. \quad (7)$$

The law of motion of the net worth is given by

$$A_{f,t+1} = A_{f,t} + \pi_{f,t}. \quad (8)$$

A firms fails when $A < 0$. There is a one-to-one replacement in order to keep the number of firms constant³.

³For simulation purposes, firms and banks are endowed with a random quantity of equity, as specified in the section about the simulations.

4. The banking sector

The structure of the network is asymmetric as each firm can have only one lending bank while a single bank can finance more firms. Banks have a degree of market power as they can set the level of interest rate.

The interest rate the bank b offers to the firm f is computed according to

$$r_{b,t}^f = r_{f,t} = a \left[A_{b,t}^{-a} + \left(\frac{B_{f,t}}{A_{f,t}} \right)^a \right] = a \left[A_{b,t}^{-a} + v_{f,t}^a \right], \quad (9)$$

where $a > 0$, $A_{b,t}$ is the bank's net worth and $v_{f,t}$ is the firm leverage ratio. The interest rate is therefore defined by two components:

- $a(A_{b,t})^{-a}$, which represents the fact that the better is the financial situation of a bank, the lower is the interest rate it can offer, and, consequently, the higher is the capacity of attracting customers;
- $a(v_{f,t})^a$, which quantifies the financial distress effect: the higher is the leverage the bigger the firm's bankruptcy probability is, and, thus, higher is the risk premium requested by the bank.

As for the first component, banks have less than perfect foresight about the capacity of the borrowers to pay back, due to the uncertainty on the final demand side. This implies incomplete information of the bank about the quality of its assets. As a consequence, investors who own bank capital or bonds will demand a higher premium the higher is ratio lending to equity of the bank. Therefore a higher net worth means lower premium paid by the bank to investors and allows the bank to charge a lower interest rate to customers (Stein, 1998; Gambacorta, 2008). The rationale of the second component is given by the fact that, as stressed by Bernanke and Gertler (1989), the loans accorded to relatively distressed firms require a higher premium of risk. Equation (9) plays a determinant role in modelling the financial contagion mechanism described by Stiglitz and Greenwald (2003, 145): the bankruptcy of one or more borrowers generates *bad debt* for the bank, erasing part of its net worth. According to rule in equation (9), the bank will apply higher interest rates on the other outstanding credit. The higher service on debt can lead to the insolvency of other customers in a negative feedback loop.

The bank pays an interest rate r^E , constant and uniform across banks and in time on the deposits $E_{b,t}$. For each bank, the amount of the deposits plus the equities must be a fraction equal to e of the deposits, with e set by the monetary authority. Deposits are residually quantified and assumed to be infinitely elastic. Accordingly, the profits of the bank b are quantified in the following way⁴

$$\pi_{b,t} = \sum_{f \in F_t^{B,b}} (1 + r_{b,f,t}) D_{b,f,t} - E_{b,t} r^E. \quad (10)$$

⁴By assumption, banks without active loans at a given time shut down for that period.

The law of motion of net worth follows the same rule as for firms, with a correction for the possible insolvencies (bad debt) BD , such that

$$A_{b,t+1} = A_{b,t} + \pi_{b,t} - BD_{b,t} \quad (11)$$

Analogously to firms, a bank is considered as bankrupted when $A_{b,t} < 0$, being replaced by a new one.

5. Creation of links and dynamics of the network

This section illustrates the dynamics of the network as a result of the behavioural assumptions for banks and firms. The first subsection specifies the hypotheses while the second provides the reader with some quantitative features of the model, presenting the outcomes of the computer simulations.

5.1. Formation of the links

In each period every NSF firm surveys a random pool of banks. The size of the pool is identical for all firms, whilst its composition can vary from firm to firm and in time. In particular every NSF firm randomly chooses a sample of $B_{f,t} = mB$ banks, with $m < 1$, among the whole population of banks and communicates to them the amount of its demand for credit. The banks in the sample reply by indicating the interest rate they are willing to apply, which is calculated according to (9). The firm sorts the banks according to the proposed interest rates and sends a signal to the bank which offers the lowest interest rate, demanding for credit.

The selected bank collects all the demands for credit it received and evaluates whether lending to a firm. The bank cannot provide unlimited credit since its individual risk is limited by a regulatory framework set by the monetary authority. We model this framework in its simplest form as a sort of the adequacy ratio of Basel II. We assume that it is given by $\bar{\theta} \leq A_{b,t}/L_{b,t}$, where $\bar{\theta} > 0$ is a constant parameter, quantifying the minimum equity ratio allowed, and L is the total lending of bank b . Thus, in order to provide a positive feedback to the signal sent by firms, the bank needs to satisfy its own necessary condition: all the accumulated demand signals have to be sustainable for the bank. Consequently, the limit for lending that the bank b can supply is

$$L_{b,t}^s = A_{b,t}/\bar{\theta}. \quad (12)$$

In order to select how many and which requests for credit will be satisfied, the bank adopts a prudential criterion. In particular it will satisfy the demands for credit of the firms which display the lowest level of leverage ratios v as long as $\theta \leq A_{b,t}/L_{b,t}$. The total demand for credit that the bank faces is

$$D_{b,t} = \sum_{f \in b_t} D_{f,b,t}. \quad (13)$$

A firm which is refused credit from the first bank in its list will demand for credit to second one, which of course requests a higher interest rate, and so on until it finds a bank willing to lend. If the list is exhausted without success, the firm is totally credit rationed and forced to reduce its production in order to be able to cover the costs with internal resources. The targeted production in such a case is $Q_{f,t} = A_t / (w * \gamma)$.

This network mechanism amplifies the financial distress of a productive unit. A firm with high leverage ratio is likely to be supplied only by the banks which offer the highest interest rate in its sample. In the worst scenario, credit cannot be obtained and the firm is forced to reduce its production and, therefore, it is expected to make a profit not large enough to improve its financial condition. The introduction of credit rationing generates a final result of the bargaining process which is substantially comparable to the one in Stiglitz and Weiss (1981). In both models, the expected return to the bank is, above a certain threshold, an inverse function of the interest rate charged. In Stiglitz and Weiss this is a consequence of the adverse selection process originated by the high interest rates. In this paper, the bank charges the highest interest rates to the most financially distressed firms, which, as a consequence, are unlikely to fulfil their debt commitments, creating a loss for the bank. We postpone the explicit modelling of this aspect to future extensions of the present project.

We have therefore two stochastic variables which influence the evolution of the network: the idiosyncratic shock in price u , which determines whether a firm is SF or NSF, and *which* banks are interrogated by each NSF firm, as the composition of the pool is random.

The network is composed of islands, or cliques, centred around a lending bank. An example is displayed by figure 1. The structure of the network changes in each period as an effect of: the change of the population of NSF firms, the different composition of the pools of “visible” banks for each NSF firm, the modification in the financial conditions of banks and firms which determine the interest rates and, accordingly, the formation of links. As a consequence, the composition and the number of cliques in each period changes, as a bank can be without borrowers in a given period.

There is a level of indirect interaction among firms. Two firms connected to the same bank affect each other through their performance: if they are able to repay the debt, they improve the financial soundness of the bank and, thus, reduce the requested interest rate in the following period; on the contrary, if a firm fails, it negatively affects the other firm through the same mechanism.

5.2. Simulations

This section provides a visualization of the dynamics of the model. We performed some simulations in order to help the reader to get a quantitative insight about the evolution of the network. The simulations provides a benchmark for the comparison with the analytical results.

The configuration of the parameter is: $\beta = 0.15$, $\alpha = 1$, $a = 0.03$, $\gamma = 1.2$, $w = 1$, $p = 2$, $m = 0.2$. Banks and firms are initially endowed with a random level of equity.

Figure 2 shows the time series of the aggregate output and the number of bankruptcies of banks. It is interesting to note that the major downturns in production are associated to avalanches of failures for banks and to drops in the percentage of financially sound firms. The lagged series of production shows a negative correlation (-0.18) with the bankruptcy ratio of banks. In other words, the model is able to generate financial crises which noticeably affect the real side of the economy. A further analysis reveals that the small cycles in production are a consequence of the debt dynamics. At the trough of the cycle firms increase production and take on new debt. This trend continues for a few periods until the debt burden becomes excessive for the weakest firms. Their failure worsens the financial conditions of the lending banks, which tighten the credit to other units leading to the bankruptcy of some of them and, possibly, of the bank themselves, according to the dynamics described in section 5. When all or most of the financially weakest units in the system have disappeared, a new upturn begins and the cycle starts again. The positive correlation of production with the lagged series of debt (around 0.23) reveals a mechanism that is reminiscent of Minsky's Financial Instability Hypothesis (Minsky, 1963).

The distribution of nodes per bank is clearly right skewed, as a few banks are connected with a large number of firms while most of the lenders have a very small number of customers (between 1 and 3). The tail of the degree distribution in this case is well fitted by a power law distribution function, as in figure 3, matching the evidence reported in De Masi et al. (2010). The interesting feature of this distribution is that it changes during the business cycle, being flatter during recessions and more concentrated during expansions. While empirical works found evidence of a right skewed distribution of banks size (Janicki and Prescott, 2006) and degree (Fujiwara et al., 2009), to the knowledge of the authors there is no study of the evolution of these distributions during the cycle. Nevertheless, the modification of the distribution can be interpreted in the light of Asea and Blomberg (1998), who demonstrate that banks tighten credit during recessions and expand it during upturns. In particular the concentration is higher at the peak of the cycle. This is a consequence of the higher production and, therefore, a larger total debt (meaning more profits for banks as the economy is in expansion) and more firms demanding debt (meaning more customers for sounder banks). On the contrary, during downturns the distribution is more even, with a smaller difference in the number of borrowers among banks. This feature also reveals that, when an avalanche of banks bankruptcies happens, it can also involve big banks.

The degree of the system is strongly correlated with some of the relevant macro-variables, and these correlations show a slow decay, as displayed by the table 1. In particular, the average degree is strongly correlated with the number of NSF firms and therefore with the fragility of the system (as confirmed by the positive correlation with the number of firms bankruptcies). As a consequence the correlation of the degree with the performance of the economy in terms of aggregate production is negative.

We perform a study of the sensitivity of the model to the parameter a , the reaction coefficient of banks to the equity for the computation of the interest

Macro-variables	lag=0	lag=1	lag=2	lag=3	lag=4
Aggregate output	-0.60	-0.40	-0.23	-0.14	-0.09
Self financing firms	-1.00	-0.43	-0.27	-0.17	-0.10
Bankruptcies of firms	0.26	0.14	0.09	0.05	0.03
Bankruptcies of banks	0.07	0.06	0.05	0.04	0.04

Table 1: Correlations between average degree and the most important macro-variables. Monte Carlo simulations with 1000 replications.

rate, and β , the exponent of the production function, measuring the sensitivity of firms to their equity. The results are displayed by figures 4 and 5 for what concerns the impact of these parameters on the percentage of self financing firms, rates of firms and banks bankruptcies and level of aggregate output. An increase in a causes a generally higher level of interest rates. As a consequence, firms fail more often and do not become too large. This is highlighted by the highest rate of bankruptcy for firm and a lower rate for banks, as the failure of small firms has a limited impact on their financial stability. The effect of average small firms size on the output is obviously negative. An increase in a brings about a higher concentration in the network. This is due to the fact that a higher a , widens the differences in the interest rates offered by the banks, bringing therefore a higher concentration of credit links.

An increase in β means that firms increase their level of production for each unit of equity. This implies a larger demand for credit (and thus a lower number of self financing firms) but also that the borrowing units are, on average, sounder. Therefore we have a lower number of failures for both firms and banks. As long as total output is concerned, the positive effect of the lower number of bankruptcies is offset by the lower equity of firms (and then production) caused by a soaring cost of production.

6. Stochastic evolution of firms

This section presents the study of the stochastic dynamics of the proportions of different firms along the lines of Di Guilmi (2008) and Di Guilmi et al. (2010). As specified in the introduction, this evolution is the result of the aggregation at the meso level by means of the master equation. We first identify the transition rates, then we set up the master equation and provide the solution. The solution of this first ME will be then used in the ME representing the evolution of the degree distribution for the NSF firms network.

6.1. The transition rates

Potentially each firm has a different balance sheet. This implies that each firm may have a different probability of changing state (from NSF to SF or vice versa). In order to make the model analytically tractable, we need to have just one probability for entries and one for exits from the NSF state. This can be obtained through the mean-field approximation which ultimately consists in

defining a representative unit for each of the states. We can calculate an average net worth for each the two clusters of firms (calling these average values A_0 for SF and A_1 for NSF firms) and, accordingly, determine the desired output (Q_0 and Q_1).

Let us indicate with ι_t the probability for a NSF firm to become SF and with ζ_t the probability of the opposite transition. The former depends on the capacity of the firm of having at time $t - 1$ a profit large enough to pay the salary bill at time t . This condition can be written as

$$A_{f,t-1} + u_{f,t-1}Q_{f,t-1} - r_{f,t-1}D_{f,t-1} - W_{f,t-1} \geq W_{f,t}. \quad (14)$$

As the price u_f is an exogenous stochastic quantity with a known distribution, it is possible to express condition (14) in terms of the probability function for u_f . Indeed, we know that if the price is above a certain threshold, the firm can obtain a profit sufficient to become SF. More precisely, rearranging equation (14) and indicating this threshold with \bar{u}_t , we can write

$$u_{f,t-1} \geq \frac{W_{f,t} + W_{f,t-1} + r_{f,t-1}D_{f,t-1} - A_{f,t-1}}{Q_{f,t-1}} = \bar{u}_{f,t}. \quad (15)$$

Using the mean-field approximation we can identify a single set of variables A_0 , Q_0 , W_0 and D_0 for all the SF firms and, accordingly, a unique \bar{u}_t . Therefore, ι_t can be quantified by using the probability function of u as

$$\iota_t = F(\bar{u}_t) = \frac{\bar{u}_t - u_{min}}{u_{max} - u_{min}}. \quad (16)$$

Analogously, ζ_t measures the average probability for a SF firm to become NSF and can be written as

$$A_{f,t-1} + u_{f,t-1}Q_{f,t-1} - W_{f,t-1} < W_{f,t}. \quad (17)$$

For this transition to occur, the price shock should be below a level that we indicate with \underline{u}_t . Thus we have that

$$u_{f,t-1} < \frac{W_{f,t} + W_{f,t-1} - A_{f,t-1}}{Q_{f,t-1}} = \underline{u}_{f,t}. \quad (18)$$

Finally, using the mean field values, the probability ζ_t can be written as

$$\zeta_t = 1 - F(\underline{u}_t) = 1 - \frac{\underline{u}_t - u_{min}}{u_{max} - u_{min}}. \quad (19)$$

The transition rates are the probability of observing a transition from a group to another in a unit of time. They are probabilities observed at the system level. As a consequence, their quantification requires to weight the micro-level probabilities ι_t and ζ_t by two other factors. The first one concerns the macroeconomic conditions. We need to consider whether the conditions of the system facilitate or hasten the transitions and condition the transition probabilities

accordingly. These environmental forces can be quantified by the so-called *externality functions*. Following Aoki and Yoshikawa (2006) and Alfarano et al. (2008), we assume the following functional forms⁵

$$\psi_0(t) = \psi_0 \left(\frac{N_1(t)}{N} \right) = \frac{b_0 + b[N - (N_1 + \vartheta)]}{N}, \quad (20)$$

$$\psi_1(t) = \psi_1 \left(\frac{N_1(t)}{N} \right) = \frac{b_1 + b(N_1 - \vartheta)}{N}. \quad (21)$$

The constant $\vartheta = |\theta|$ with $\theta = -1, 0, +1$ indicates *inflows*, when $\vartheta = 1$, or *outflows*, when $\vartheta = 0$, with respect to the NSF state. The symbols b , b_0 and b_1 indicate positive constants. We define the transition probabilities conditioned on the macroeconomic conditions as *rate functions*⁶. In particular, $\lambda(t)$ represents the rate function for entries into the NSF state while $\mu(t)$ represents the exits from it. Since the dynamics of this system can be assimilated to a birth and death process, from now on we will refer to the entries into the NSF state as *births* and to the exits as *deaths*. Accordingly, the birth and death rates are given by, respectively.

$$\begin{aligned} \lambda(t) &= \zeta_t \psi_1(t), \\ \mu(t) &= \iota_t \psi_0(t). \end{aligned} \quad (22)$$

Finally, the last factor needed to quantify the probabilities of transition involves the numerosity of the two populations. For example, the probability of observing a transition from NSF to SF is higher the larger is the number of NSF firms. Therefore, the birth transition rate is given by

$$\beta_t(N_1(t) - \vartheta) = \lambda_t[N - (N_1(t) - \vartheta)], \quad (23)$$

while death transition rate is

$$\delta_t(N_1(t) + \vartheta) = \mu_t(N_1(t) + \vartheta) \quad (24)$$

6.2. The NSF master equation

The master equation for the number of NSF firms can be expressed as a balance flow equation between in and out probability flows for a given state.

⁵The specification of the quantities involved in the analytical solution requires continuous time; therefore from this point we consider the time unit of reference as infinitesimal. The time references that appear in ρ_t and ξ_t are indexing parameters since these quantities are perturbed parameters in the macroscopic equation dynamics. In essence, they are not properly function of time in an autonomous sense, rather they are just indexed by time as if it was an iteration counter.

⁶With some abuse of terminology, (22) are here called rate functions since they concerns birth and death rates for a birth-death process. They are not to be confused with rate functions as usually defined in mathematics.

We take as a reference the number of NSF firms $N_1(t) = N_h$ and specify the master equation as

$$\frac{dP(N_h, t)}{dt} = \underbrace{[\beta(N_h - 1, t)P(N_h - 1, t) + \delta(N_h + 1, t)P(N_h + 1, t)]}_{\text{inflow probability}} - \underbrace{[(\beta(N_h, t) + \delta(N_h, t))P(N_h, t)]}_{\text{outflow probability}} \quad (25)$$

The meaning of (25) is that the change in the probability of having a number N_h of NSF firms at time t , within an infinitesimal time window, is the net balance between inflows and outflows probabilities. Inflow probabilities can be either due to a birth from $N_h - 1$ or a death from $N_h + 1$. Outflow probabilities concerns both a birth or a death from N_h .

6.3. ME solution and dynamics of the proportions of firms

A closed form solutions for MEs exist under very restrictive conditions. Nevertheless, an asymptotic solution can be always obtained with approximation methods⁷. Appendix A develops the asymptotic solution to identify the drift and the spread of the number of NSF firms about the most probable path. In order to isolate the two components in the solution, a convenient way is to preliminarily split the state variable N_1 according to the following hypothesis (Aoki, 2002)

$$N_1(t) := N\phi(t) + \sqrt{N}\epsilon(t). \quad (26)$$

The quantity $\phi(t) = \langle N_1(t)/N \rangle$ represents the drift of the proportion of NSF firms, while $\epsilon(t) \xrightarrow{i.i.i.d.} F_\epsilon(\mu_\epsilon(t), \sigma_\epsilon(t))$ quantifies the spread about the expected path. As shown in Appendix A, the average proportion of NSF firms evolves according to the following differential equation

$$\dot{\phi} = b(\zeta_t - \iota_t)\phi(1 - \phi) \quad : \quad b > 0. \quad (27)$$

Setting the initial condition $\phi(0) = \phi_0 \in (0, 1)$, the general solution of equation (27) reads as

$$\phi(t) = \frac{1}{1 + \left(\frac{1}{\phi_0} - 1\right) \exp[-\rho_t t]} \in (0, 1). \quad (28)$$

The dynamics is driven by the sign of $\rho_t = b(\zeta_t - \iota_t)$ which is a stochastically perturbed parameter, since it depends on ζ_t and ι_t . In case of constant ζ and ι equation (28) would provide a monotonic deterministic logistic trajectory. As shown in Appendix A, two equilibria are possible and they correspond to the

⁷See Aoki (1996, 2002); Aoki and Yoshikawa (2006); van Kampen (1992); Risken (1989); Gardiner (1985) and Di Guilmi et al. (2012a).

two extreme situations in which we have all NSF or all SF firms, depending on the initial condition. Namely, we have that

$$\begin{cases} 0 < \phi_0 < 0.5, \zeta_t < \nu_t \Rightarrow \phi(t) \xrightarrow{t \rightarrow +\infty} 0^+, \\ 0.5 < \phi_0 < 1, \zeta_t > \nu_t \Rightarrow \phi(t) \xrightarrow{t \rightarrow +\infty} 1^-. \end{cases} \quad (29)$$

The dependence on the initial condition and on the net rate ρ_t is illustrated in figure 6. The two equilibria are asymptotes as the drift approaches without reaching them due to the effect of the net rate ρ_t .

As long as the spread component is concerned, the asymptotic solution identifies the associated Fokker-Planck equation for the density of $\epsilon(t) \xrightarrow{i.i.i.d.} F_\epsilon(\mu_\epsilon(t), \sigma_\epsilon(t))$. As demonstrated in Appendix A, the general solution of the Fokker-Planck equation is in Gaussian form and expressed as

$$\Pi_t(\epsilon) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2(t)}} \exp \left[-\frac{(\epsilon - \mu_\epsilon(t))^2}{2\sigma_\epsilon^2(t)} \right]. \quad (30)$$

Therefore also the parameters of the spread distribution (30) are dependent on the transition rates.

The solution of the ME provides the endogenous dynamics not only of the proportions of the two types of firms, but also of the aggregate production. Since we identified a mean-field value of the output for each of the two types of firms, we can also analytically represent the dynamic of the aggregate output as

$$Q(t) = N[n_0(t)Q_0(t) + n_1(t)Q_1(t)]. \quad (31)$$

where the dynamics of n_1 , and $n_0 = 1 - n_1$, comes from equation (26) being $n_1 = N_1/N = \phi + \epsilon/\sqrt{N}$ and $N_0 = N - N_1$. Therefore, we have endogenously and analytically quantified the trend and the fluctuations of the aggregate output of the economy. Here we do not investigate this result further as it is already fully developed in Di Guilmi (2008) and Di Guilmi et al. (2010).

7. The stochastic evolution of the network

This section develops the analytical representation of the network through the ME and its solution. The solution of the model reproduces the main features of the network dynamics as presented in the previous section. The use of the ME for the network analysis in an economic model, introduced in Di Guilmi et al. (2012b) and here enhanced, represents a novelty⁸.

The ME is used to study the dynamics of the degree of the network, i. e. the number of links per node. In our model it corresponds to the number of borrowers per bank, as each firm can only have one bank as a lender. The reason for the choice of the degree as our state variable is twofold:

⁸Applications in other fields can be found in Ehrhardt et al. (2006); Erhardt et al. (2007); Boccaletti et al. (2006); Smith et al. (2009); de Aguiar et al. (2005); Gelenbe (2008) and Chen et al. (2003).

1. from a methodological point of view, this is the most informative variable as long as the structure of the network is concerned. In particular the evolution of its distribution provides useful details for the analysis, for instance, of the size of the banks, of the size of the cliques in the network and of its density⁹;
2. given the structure of this particular network (one link per firm and more links for each bank), the degree distribution is also the distribution of borrowing firms per banks. Thus, the average degree synthetically represents the level of concentration of the credit market, giving a quantitative dimension to the idea of “too interconnected to fail”. As shown in subsection 5.2, this variable is strongly correlated with the output and the number of self financing firms. The study of its evolution through time, therefore, reveals important features of the economic system as represented by the network. This ME provides a dynamic model for the concentration of the credit market and is suitable to describe the structure of a network emerging from a HIA model.

The stochastic representation is developed by studying the number of firms which are customer of banks with the same size. The probabilistic device is the probability of two firms to be connected to the same bank. In this treatment, the transitions are represented by the creation and destruction of such links between firms. The state variable of the new ME is the number of banks in a dimensional class. We indicate this quantity as C_l , where l is the number of customers, that is the degree. The symbol $N_{1,l} = K_l C_l$, $K_l = l$ is the degree level, identifies the number of firms borrowing from banks of size l .

The analysis is articulated in the following steps: first, the transition rates are quantified and the K-ME is defined accordingly; second, we derive the ODE and the probability distribution which compose the solution for the dynamics of the number of banks in each level of the degree; finally, the degree distribution is obtained as the vector of the number of banks for each level of the degree generated by the solution at a given time.

In Appendix B the mechanism of network formation is explained. By construction, the network degree distribution, that is the number of components with a given size (i.e. degree level), is equivalent to the size distribution for banks.

⁹The size of the clique is an informative indicator about the resilience of the network and the systemic risk. While traditional literature emphasizes the positive role of diversification in reducing the risk of financial contagion (Allen and Gale, 2001), others (Battiston et al., 2007; Gallegati et al., 2008) argue that a large number of borrowers increases the probability of being hit by a negative shock, offsetting the benefits of diversification. Hence, the high concentration (large cliques) in the credit market can increase instability, while a more fragmented market can better absorb the effect of bankruptcies.

7.1. The transition rates

The number of components (firms) in the network evolves over time according to the dynamics described by the NSF-ME. In order to set up the ME of the degree (K-ME), this time evolution must be considered. An analytical representation of this dependence can be provided by a nested model, in which the solution of the NSF-ME enters the K-ME. This subsection introduces the mechanics of the transition and derives the transition rates functional forms. The quantification of the transition rates involves the same three factors that we identified in subsection 6.1 for the NSF-ME. In order to consider these factors the transition should be specified according to

$$\begin{cases} \beta_t(C_l(t) - \vartheta J_l(t)) = \lambda_l(t) \times \mathcal{M}_{1,l}(t) \times \mathcal{N}_1(C_l(t) - \vartheta J_l(t)), \\ \delta_t(C_l(t) + \vartheta J_l(t)) = \mu_l(t) \times \mathcal{M}_{0,l}(t) \times \mathcal{N}_0(C_l(t) + \vartheta J_l(t)), \end{cases} \quad (32)$$

where

- $\lambda_l(t)$ and $\mu_l(t)$ are the rate functions;
- $\mathcal{M}_{1,l}(t)$ and $\mathcal{M}_{0,l}(t)$ has the same role played in the NSF-ME by the externality functions, accounting for the network size and components' size neighbouring effects. They can be defined as *Moran-like mechanisms* Moran (1962);
- $\mathcal{N}_1(C_l(t) - \vartheta J_l(t))$ and $\mathcal{N}_0(C_l(t) + \vartheta J_l(t))$ are the state specific *cardinality functions* that account for changes in the occupation numbers through time.

Appendix C demonstrates that the three components of the transition rates can be identified on the basis of the assumptions of the economic model and on its dynamics. Following this phenomenological approach the transition rates can be expressed by

$$\begin{cases} \beta_t(C_l(t) - \vartheta J_l(t)) = \zeta_l e^{c(t)} \left(\frac{C_l(t) - \vartheta J_l(t)}{\hat{B}} \right) [\hat{B} - (C_l(t) - \vartheta J_l(t))], \\ \delta_t(C_l(t) + \vartheta J_l(t)) = \iota_l e^{d(t)} \left(\frac{C_l(t) + \vartheta J_l(t)}{\hat{B}} \right) [C_l(t) + \vartheta J_l(t)]. \end{cases} \quad (33)$$

In equations (33):

- $\zeta_l e^{c(t)}$ and $\iota_l e^{d(t)}$ are the rate functions, indicated respectively with $\lambda(t)$ and $\mu(t)$ in (32). The symbols ζ_l and ι_l indicate the birth and death rates while $c(t)$ and $d(t)$ stand for the creation and destruction rates of links;
- the second factor is the functional form of the Moran-like mechanism with J_l indicating the variation in the occupation number C_l ;
- the third factor is the cardinality function.

7.2. The K master equation

The transition rates specified in (33) can be used to specify the phenomenological ME for the network degree. The single phenomenological $K_l - ME$ reads as

$$\frac{dP_t(C_h)}{dt} = \underbrace{[\beta_t(C_h - J_m)P(C_h - J_m) + \delta_t(C_h + J_m)P(C_h + J_m)]}_{\text{inflow probability}} + \underbrace{- [(\beta_t(C_h) + \delta_t(C_h))P(C_h + J_m)]}_{\text{outflow probability}} \quad (34)$$

where $J_l(t) = J_m$ is a realization of the jumps.

7.3. Master equation solution and dynamics of the degree

The solution method for (34) is the same used for (25) and detailed in Appendix A. Some adjustments are needed to take into account that we are dealing with a multistate ME (as the number of states is given by all the possible levels of the degree) and that the size of the transitions is variable. The solution algorithm is fully detailed in Appendix C.

As done for the NSF-ME in equation (26), the state variable is split into its trend and fluctuations component as for the following

$$C_l(t) = \hat{B}\phi_l(t) + \sqrt{\hat{B}}\epsilon_l(t), \quad (35)$$

where $\phi_l(t) = \langle C_l(t)/\hat{B} \rangle$ is the drift and $\epsilon_l(t) = [C_l(t) - \hat{B}\phi_l(t)]\hat{B}^{-1/2} \stackrel{i.i.d}{\rightarrow} F_{\epsilon_l}(\mu_{\epsilon_l}(t), \sigma_{\epsilon_l}^2(t))$ is the spread. Therefore ϕ_l indicates the percentage of banks of degree l in the network. The solution for the K-ME is represented by the following Cauchy problem

$$\dot{\phi}_l = \Delta_t(\phi_l) = \gamma_l^- \phi_l(1 - \phi_l) : \phi_l^0 = \phi_l(0), \quad (36)$$

where

$$\gamma_l^-(t) = \lambda_l(t) - \mu_l(t) = \zeta_l e^{c(t)} - \iota_l e^{d(t)}. \quad (37)$$

The equilibrium condition yields

$$\dot{\phi}_l = 0 \Rightarrow \phi_l^* \in \{0, 1\} \quad (38)$$

The point $\phi_l^* = 0$ implies that the l -th degree level is empty, while $\phi_l^* = 1$ means that all the components have the same size K_l . As for the NSF-ME, the solution of the macroscopic equation tends toward one of the two equilibria without reaching them due to the coefficient $\gamma_l^-(t)$. Hence, as a consequence, the pattern of the proportion of banks in a degree level would cyclically evolve between the two equilibria or oscillate around one of them in the long run.

The Cauchy problem (36) has a general solution which reads as

$$\phi_l(t) = \frac{1}{1 + \left(\frac{1}{\phi_l^0} - 1\right) e^{-t\gamma_l^-(t)}} \in (0, 1) \quad (39)$$

As one can note, it is the same functional form obtained as a solution for the NSF-ME.

The general solution of the Fokker-Planck equation is

$$\Pi_t(\epsilon_l) = \frac{1}{\sqrt{2\pi\sigma_{\epsilon_l}^2(t)}} \exp\left\{-\frac{\epsilon_l^2}{2\sigma_{\epsilon_l}^2(t)}\right\} \quad (40)$$

The volatility $\sigma_{\epsilon_l}^2(t)$ can be defined by using (A.35) to obtain

$$\begin{aligned} \sigma_{\epsilon_l}^2(t) &= \langle (\epsilon_l^*)^2 \rangle \{1 - \exp[2\Delta'_t(\phi_l)t]\} \text{ s.t.} \\ \langle (\epsilon_l^*)^2 \rangle &= -\frac{\Sigma_t(\phi_l)}{2\Delta'_t(\phi_l)} = -\frac{\gamma_l^+(t)\phi_l(t)(\phi_l(t))}{\gamma_l^-(t)(1-2\phi_l(t))} > 0 \end{aligned} \quad (41)$$

Equation (39) provides the dynamics of the drift for any value l of the degree, while (40) quantifies the distribution of fluctuations in the number of component with degree l around this drift. If we consider all the possible value of l at a given time, we are able to build a vector $\mathbf{C}(t) = \{C_l(t) : l \in \Lambda_K\}$ containing the occupation number (the number of banks) for each level of the degree. Thus this vector represents the degree distribution. The analytical solution therefore provides the degree distribution at each point in time.

7.4. Analysis of the results

The analytical model is also able to replicate the pattern generated by the underlying agent based model. Figure 7 compares the dynamics of the proportion of NSF produced by the numerical simulation of the model with full degree of heterogeneity and the one generated by the analytical solution. As one can see the gap between the two lines is always within the second decimal digit. The average correlation between the two filtered¹⁰ series over a Monte Carlo simulation with 1000 replications is 0.9577. Figure 8 contrasts the distributions of the degree generated by the two different solution. Also in this case the solution of the ME mimics to a good extent the behaviour of the underlying agent based model.

In order to provide some insights on the dynamics, it is possible to study the sensitivity of the macroscopic equation (36) to the initial condition and to the transition rates. Defining the functional $\phi_l(t) = \varphi(\gamma^-(t)|\phi_l^0)$, the derivative with respect to the initial condition is

$$\frac{d\varphi(\gamma_l^-(t)|\phi_l^0)}{d\phi_l^0} = \left(\frac{\phi_l(t)}{\phi_l^0}\right)^2 \exp(-t\gamma_l^-(t)) > 0 \quad (42)$$

Therefore, the higher the share of network components in the l -th level of degree at the beginning, i.e. ϕ_l^0 , the higher the share of components as time goes on. We need also to consider that

¹⁰The raw series need to be cleaned by the possible bias introduced by the stochastic fluctuations.

- if $\gamma_l^-(t) > 0$ then $d\varphi/d\phi_l^0 \rightarrow (\phi_l(t)/\phi_l^0)^2$ as $t \rightarrow \infty$;
- if $\gamma_l^-(t) = 0$ then $d\varphi/d\phi_l^0 = (\phi_l(t)/\phi_l^0)^2 \forall t$;
- if $\gamma_l^-(t) < 0$ then $d\varphi/d\phi_l^0 \rightarrow \infty$ as $t \rightarrow \infty$.

This is because the ratio $\phi_l(t)/\phi_l^0$ converges to a (large or small) constant through time, due to the logistic behaviour of the macroscopic equation. It is worth noticing that if $\gamma_l^-(t) = 0$ then $\phi_l(t) = \phi_l^0$ and (42) reduces to 1. Moreover, it can be shown that

$$\lim_{\phi_l^0 \rightarrow 1} \frac{d\varphi}{d\phi_l^0} \propto \exp(-t\gamma_l^-(t)) > 0 \Rightarrow \lim_{\substack{\phi_l^0 \rightarrow 1 \\ t \rightarrow \infty}} \frac{d\varphi}{d\phi_l^0} = \begin{cases} \text{const if } \gamma_l^-(t) \geq 0 \\ \infty \text{ if } \gamma_l^-(t) < 0 \end{cases} . \quad (43)$$

This can be extended to any initial condition value. Therefore, if the death rate is higher than the birth rate, the dynamics becomes very sensitive to the initial conditions. Indeed a negative net-rate, if persistent, can imply the collapse of the degree level hence no dynamics for that level can be found consistently. According to (39), if $\gamma_l^-(t) < 0$ persists the dynamics jumps to $\phi_l(t) = 1$, meaning that all the other levels of degree will become empty in the long run.

Let us consider now the derivative w.r.t. $\gamma_l^-(t)$, which is

$$\frac{\partial \varphi_1(\gamma_l^-(t)|\phi_l^0)}{\partial \gamma_l^-(t)} = \frac{t(1/\phi_l^0 - 1) \exp(-t\gamma_l^-(t))}{(\phi_l(t))^2} > 0 \quad (44)$$

Since $\phi_l^0 \in (0, 1)$, as a consequence we have that $1/\phi_l^0 > 1$. Therefore, the reaction of the dynamics to the net-rate is proportional to the change in the net-rate but the proportionality is not constant, as it depends on time and on $\phi_l(t)$. As for the initial condition, if $\phi_l^0 \rightarrow 1$ then it diminishes the effect of the net-rate because $1/\phi_l^0 - 1 \rightarrow 0$. Therefore, the more the dynamics is considered toward the highest levels of degree the less the net-rate will influence the dynamics.

Finally, it can be demonstrated that

$$\lim_{t \rightarrow \infty} \frac{\partial \varphi_1(\gamma_l^-(t)|\phi_l^0)}{\partial \gamma_l^-(t)} = 0^+ \quad \text{if } \gamma_l^-(t) \geq 0 \quad (45)$$

Thus, when the birth rate is higher than the death rate the effect of the net-rate on the dynamics lowers through time, in the opposite case the effect is large.

The solution highlights how systems with high degree of concentration are unlikely to change into more evenly distributed ones, given that for high level of the initial condition in a cluster, the effect of the net rate is small. The analytical solution confirms that concentration is a consequence of a phase of expansion in the economy. Indeed the impact of a reduction in the number of NSF firms (which accompanies an expansion) is stronger at lower levels of the degree that register a larger decrease in the number of customers. The distribution will then

be more right-skewed. On the contrary, during a recession the number of NSF rises and the increase in the number of components will be larger in the lower levels of the degree (small banks) bringing about a more even distribution.

The formulation of the general solution (41) of the Fokker-Planck equation reveals another consequence of the concentration. The variance of the number of banks in a dimensional class is positively related to the size of the dimensional class. It implies that, during expansions, it will be the largest ones to display a larger volatility in their size, with more pronounced consequences in the whole stability of the system. This outcome seems to be consistent with the simulations and with the narrative of the financial stability hypothesis, according to which “stability breeds instability”. The difference in the distribution of banks and therefore in the market structure signals that a highly concentrated market is somehow unstable and susceptible to shocks. In particular the simulations show that the transition to a more even distribution is driven by “avalanches” of banks’ bankruptcies.

Not surprisingly, it can be proved that $\partial\phi_l/\partial\bar{\theta} > 0$. As a consequence, a lower capital requirement for banks (higher $\bar{\theta}$) bolsters market concentration.

Given its analytical formulation, the solution of the master equation cannot provide a definitive answer to the relationship between concentration and economic performance. However, the analysis of the results gives evidence that a more concentrated distribution increases the volatility in the market structure and, as a consequence, makes the system more prone to sudden shocks.

8. Concluding remarks

This paper proposes an analytical solution for a network model in which heterogeneous firms and banks are linked by credit contracts. The analytical solution is attained by means of a system of two MEs, one nested into the other. A first ME provides the dynamics of the number of firms that resort to the credit market to finance their production and are therefore part of the network. The solution of this ME is plugged into the second one in order to study the evolution of the network’s structure and, in particular, of the degree distribution. The solution of the second ME takes into account all the possible patterns of evolution of the links due to the interaction among agents.

By means of a further development of recently proposed solution methods, we identify an ordinary differential equation for the trend of the network degree and a limit probability distribution for the fluctuations around this trend. In particular, the solution of the second master equation represents a considerable advancement in this field of research since this ME is a multi-state equation with a variable number of components. We show that, using a bottom-up approach, it is possible to obtain a complete description of the macroeconomic dynamics, dependent on the stochastic interaction of the units which compose the system. The deterministic component of the dynamics allows us to study the evolution of the degree distribution and to analyse the systemic fragility of the credit network. The solution algorithm also generates multiple equilibria with different degrees of systemic financial fragility which allow policy experiments.

These results come from the solution of the ME, which appears to be a highly valuable tool for macroeconomic modelling. Indeed it makes possible to deal with two issues which are not treatable with the standard tools of the economist: first, the analytical aggregation of HIAs, and, second, the study of dynamics of the network. The paper also presents the numerical solution of the model, obtained by means of numerical simulations, which is compared to the analytical result. The two solutions of the MEs are able to replicate to a relevant extent the outcomes of the simulations. The use of the ME proves to be a reliable tool to overcome the shortcomings of mainstream models (which are analytically solvable but theoretically constrained by the use of the representative agent) and of agent based models (which allow for heterogeneity and interaction but are limitedly insightful due to the use of numerical simulations).

This basic model presents a framework which is suitable of extensions in a number of directions. For a more realistic representation of the credit market, we can consider some rigidities and costs which prevent or limits the possibility for a firm to change bank, affecting the dynamics of the network structure. Another options would be the introduction of more refined mechanism of credit rationing as in Stiglitz and Weiss (1981). Also, the model could take into account a more elaborated credit contracts as, for example, the commercial papers. As regard the object of the model, it would be possible to model more sophisticated network topologies, for example allowing firms to obtain credit from more than one bank. Another interesting improvement is a more complete reconsideration of more sophisticated credit network models (Delli Gatti et al., 2006, 2010) within the approach adopted in this paper. The framework is also suitable to investigate possible policies for preventing a financial crisis. In particular, policy interventions can shape the network topology and structure or impact on links or nodes in order to make the system more resilient.

Appendix A: solution of the NSF-ME

This appendix illustrates the main steps for the solution of the ME for the number of NSF firms. The methodology develops Aoki and Yoshikawa (2006) along the lines of Di Guilmi (2008) and Chiarella and Di Guilmi (2011). The method is general will be applied, with minor adjustment, in Appendix C to solve the ME for the degree. The basic ME is (25). Indicating with ϑ the size of the jump, the phenomenological model (26) is described as

$$\left\{ \begin{array}{l} (i) \quad N_1(t) \pm \vartheta = N\phi(t) + \sqrt{N}\epsilon(t) \pm \vartheta, \text{ s.t.} \\ (ii) \quad \phi(t) = \langle \eta N_1(t) \rangle : \eta N = 1, \\ (iii) \quad \epsilon(t) = (N_1(t) - N\phi(t))\sqrt{\eta} \xrightarrow{i.i.d.} F_\epsilon(\mu_\epsilon(t), \sigma_\epsilon^2(t)). \end{array} \right. \quad (\text{A.1})$$

As specified in the main text, the constant $\vartheta = |\theta|$ with $\theta = -1, 0, +1$ distinguishes inflows ($\vartheta = 1$) and outflows ($\vartheta = 0$), with respect to the NSF state. It allows for the following intensive form representation

$$\left[\frac{N_1 \pm \vartheta}{N} = n_1 \pm \vartheta\eta \right] = \phi + \sqrt{\eta}(\epsilon \pm \vartheta\sqrt{\eta}) = \phi_{(\pm\vartheta)}, \quad (\text{A.2})$$

where

$$\phi_{(\pm\vartheta)} = \phi + \sqrt{\eta}\epsilon_{(\pm\vartheta)} \quad : \quad \epsilon_{(\pm\vartheta)} = \epsilon \pm \vartheta\sqrt{\eta}. \quad (\text{A.3})$$

As a consequence of the change of variable in (A.1:iii) the l.h.s. of (25) reads as

$$\frac{dP_t(N_h)}{dt} = \frac{\partial\Pi_t(\epsilon)}{\partial t} + \frac{d\epsilon}{dt} \frac{\partial\Pi_t(\epsilon)}{\partial\epsilon}, \quad (\text{A.4})$$

with $P_t(N_h) = \Pi_t(\epsilon)$. Hence, by computing $\dot{\epsilon}$, and considering N_h as fixed, (A.1:iii) gives

$$\frac{d\epsilon}{dt} = -\frac{1}{\sqrt{\eta}} \frac{d\phi}{dt}. \quad (\text{A.5})$$

The substitution of (A.5) into (A.4) yields

$$\frac{dP_t(N_h)}{dt} = \frac{\partial\Pi_t(\epsilon)}{\partial t} - \frac{1}{\sqrt{\eta}} \frac{d\phi}{dt} \frac{\partial\Pi_t(\epsilon)}{\partial\epsilon}. \quad (\text{A.6})$$

Time is rescaled according to

$$t = f(N)\tau \quad : \quad f(N) = N \Rightarrow [t] \neq [\tau]. \quad (\text{A.7})$$

It follows that¹¹

$$\eta \frac{dP_t(N_h)}{dt} = \eta \frac{\partial\Pi_t(\epsilon)}{\partial t} - \sqrt{\eta} \frac{d\phi}{dt} \frac{\partial\Pi_t(\epsilon)}{\partial\epsilon}. \quad (\text{A.8})$$

Also the r.h.s. should be reformulated according to the same change of variable used to rewrite its l.h.s. as (A.8). This means that transition rates in (22) and (23), assumed to be homogeneous functions w.r.t. to the system size N , should be re-expressed by using (A.3) as

$$\beta_t(N_h - \vartheta) = N\beta_t\left(\frac{N_h - \vartheta}{N}\right) = N\beta_t(\phi + \sqrt{\eta}(\epsilon - \vartheta\sqrt{\eta})) = N\beta_t(\phi_{(-\vartheta)}), \quad (\text{A.9})$$

$$\delta_t(N_h + \vartheta) = N\delta_t\left(\frac{N_h + \vartheta}{N}\right) = N\delta_t(\phi + \sqrt{\eta}(\epsilon + \vartheta\sqrt{\eta})) = N\delta_t(\phi_{(+\vartheta)}). \quad (\text{A.10})$$

Accordingly, also the density changes into

$$\begin{aligned} P_t(N_h \pm \vartheta) &= P_t(N\phi + \sqrt{N}\epsilon \pm \vartheta) = \\ P_t\left(N\phi + \sqrt{N}\left(\epsilon \pm \frac{\vartheta}{\sqrt{N}}\right)\right) &= \Pi_t(\epsilon_{\pm\vartheta}). \end{aligned} \quad (\text{A.11})$$

¹¹Of course t should be written as τ but, as usual in this literature, we consider again t for notation convenience: on the other hand (A.7) means that t and τ are just symbols for time, independently on its unit of measure.

Therefore, by substitution of (A.9), (A.10) and (A.11) into the r.h.s. of (25) as re-formulated in (A.8), we finally have the solvable ME

$$\eta \frac{\partial \Pi_t(\epsilon)}{\partial t} - \sqrt{\eta} \frac{d\phi}{dt} \frac{\partial \Pi_t(\epsilon)}{\partial \epsilon} = [\beta_t(\phi_{(-)})\Pi_t(\epsilon_{(-)}) + \delta_t(\phi_{(+)})\Pi_t(\epsilon_{(+)})] + \quad (A.12)$$

$$- [\Sigma_t(\phi_{(0)})\Pi_t(\epsilon_{(0)})],$$

with $\Sigma_t(\phi_{(0)}) = \beta_t(\phi_{(0)}) + \delta_t(\phi_{(0)})$.

Expanding the transition rates in a Taylor series about ϕ and for the density about ϵ , we obtain

$$\beta_t(\phi_{(-)}) = \sum_{p \geq 0} \frac{\eta^p/2}{p!} (\epsilon - \eta^{1/2})^p \beta_t^{(p)}(\phi) : \vartheta = 1 \quad \theta = -1, \quad (A.13)$$

$$\delta_t(\phi_{(+)}) = \sum_{p \geq 0} \frac{\eta^p/2}{p!} (\epsilon + \eta^{1/2})^p \delta_t^{(p)}(\phi) : \vartheta = 1 \quad \theta = +1, \quad (A.14)$$

$$\Sigma_t(\phi_{(0)}) = \sum_{p \geq 0} \frac{\eta^p/2}{p!} (\epsilon)^p \Sigma_t^{(p)}(\phi) \quad \vartheta = 0 : \theta = 0, \quad (A.15)$$

$$\Pi_t(\epsilon_{(\pm)}) = \sum_{p \geq 0} \frac{(\pm \eta^{1/2})^p}{p!} \Pi_t^{(p)}(\epsilon) \text{ if } \vartheta = 1 \text{ or } \Pi_t(s) \text{ if } \vartheta = 0. \quad (A.16)$$

These expansions can be plugged into (A.12) leading to long expressions in which most of the terms decay to zero very quickly since

$$\eta^{p/2} \rightarrow 0^+ \quad \forall p > 2 \quad N \rightarrow +\infty. \quad (A.17)$$

Therefore, considering $p = 0, 1, 2$, we attain an asymptotic second order approximation for

$$\eta \dot{Q}_t - \sqrt{\eta} \dot{\phi} \Pi_t' = -\sqrt{\eta} \Delta_t \Pi_t' - \eta \left[\Delta_t' \partial_\epsilon (\epsilon \Pi_t) - \frac{1}{2} \Sigma_t \Pi_t'' \right], \quad (A.18)$$

where $\Delta_t = \Delta_t(\phi) = \beta_t(\phi) - \delta_t(\phi)$, $\Delta_t' = \partial_\phi \Delta_t$, $\Pi_t' = \partial_\epsilon \Pi_t$ and $\ddot{\Pi}_t = \partial_t \Pi_t$ being $\Pi_t = \Pi_t(\epsilon)$. By using the polynomial identity principle we can compare terms with the same power of η and get

$$\dot{\phi} = \Delta_t(\phi) = \beta_t(\phi) - \delta_t(\phi) \quad (A.19)$$

$$\partial_t \Pi_t(\epsilon) = -\Delta_t'(\phi) \partial_\epsilon (\epsilon \Pi_t(\epsilon)) + \frac{1}{2} \Sigma_t(\phi) \partial_\epsilon^2 \Pi_t(\epsilon). \quad (A.20)$$

The Macroscopic Equation

Equation (A.19) can be defined as macroscopic equation: it is an ODE for the dynamics of the drifting trajectory of the intensive form expected value of the state variable, i.e. $\phi(t) = \langle \eta N_1(t) \rangle$. Hence, its solution provides the most probable path of the state variable. By substituting the transition rates (23) and (24) into (A.9) and (A.10), with $\vartheta = 0$, we can set up the following Cauchy problem

$$\dot{\phi} = \rho_t \phi(1 - \phi) \text{ s.t. } \phi(0) = \phi_0 : \rho_t = b(\zeta_t - \iota_t), b > 0. \quad (\text{A.21})$$

The solution of (A.21) is

$$\phi(t) = \frac{1}{1 + \left(\frac{1}{\phi_0} - 1\right) \exp[-\rho_t t]} \in (0, 1). \quad (\text{A.22})$$

The two steady states solution of equation (A.22) are 0 and 1. In particular it can be easily demonstrated that:

1. by definition $0 \leq \phi(0) \leq 1$ and $\phi(t) \rightarrow 1^-$, if almost all the firms are NSF, or $\phi(t) \rightarrow 0^+$ if almost all firms are SF. These two extreme scenarios are also the logistic equilibria;
2. as shown below, in order for (A.20) to make sense it must follow that $\Delta'_t(\phi) < 0$. As a consequence the behaviour of the macroscopic equation, as dependent on the initial conditions, can be summarised in the following way

$$\begin{cases} 0 < \phi_0 < 0.5, \zeta_t < \iota_t \Rightarrow \phi(t) \xrightarrow{t \rightarrow +\infty} 0^+, \\ 0.5 < \phi_0 < 1, \zeta_t > \iota_t \Rightarrow \phi(t) \xrightarrow{t \rightarrow +\infty} 1^-. \end{cases} \quad (\text{A.23})$$

$$\Delta'_t(\phi) = b(\zeta_t - \iota_t)(1 - 2\phi) < 0 \Rightarrow \begin{cases} \rho_t > 0 \wedge \phi_0 \in (0.5, 1) \\ \rho_t < 0 \wedge \phi_0 \in (0, 0.5) \end{cases} \quad (\text{A.24})$$

The Fokker-Planck Equation

Equation (A.20) is a Fokker-Planck equation, hence a second order partial differential equation of parabolic type. The current probability is defined as

$$S_t(\epsilon) = \Delta'_t(\phi)(\epsilon \Pi_t(\epsilon)) - \frac{1}{2} \Sigma_t(\phi) \Pi'_t(\epsilon). \quad (\text{A.25})$$

Accordingly, (A.20) reads as a continuity equation of the form

$$\partial_t \Pi_t(\epsilon) = -\partial_\epsilon S_t(\epsilon). \quad (\text{A.26})$$

Therefore the stationary condition is

$$S_t(\epsilon) = \text{const.} \Leftrightarrow \partial_s S_t(\epsilon) = 0 \Leftrightarrow \partial_t \Pi_t(\epsilon) = 0 \Leftrightarrow \lim_{t \rightarrow +\infty} \Pi_t(\epsilon) = \text{const.} \quad (\text{A.27})$$

By using (A.25)-(A.27) it follows that

$$\Delta'(\epsilon \Pi) = \frac{1}{2} \Sigma \Pi'. \quad (\text{A.28})$$

The integration of (A.28) gives

$$\Pi(\epsilon) = K \exp \left\{ \epsilon^2 \frac{\Delta'(\phi)}{\Sigma(\phi)} \right\} : K = \sqrt{-\frac{1}{\pi} \frac{\Delta'(\phi)}{\Sigma(\phi)}}. \quad (\text{A.29})$$

The coefficients of the Fokker-Planck equation (A.20) are independent on ϵ and can be written as

$$A = \Delta'_t(\phi) \quad \wedge \quad B = \Sigma_t(\phi). \quad (\text{A.30})$$

By substitution into the continuity equation (A.26) and computing derivatives we have that

$$\partial_t \Pi_t = -A \Pi_t + A \epsilon \Pi'_t + \frac{B}{2} \Pi''_t. \quad (\text{A.31})$$

Rather than solving directly (A.31), we plug it in the dynamic equation for moments.

$$\partial_t \langle \epsilon^k \rangle = \partial_t \int \epsilon^k \Pi_t(\epsilon) d\epsilon. \quad (\text{A.32})$$

Thus, since the integration variable is ϵ while the differentiation variable is t it then follows that the system for the dynamics of the first two moments is given by

$$\begin{cases} \partial_t \langle \epsilon \rangle = \int \epsilon \partial_t \Pi_t \epsilon = A \langle \epsilon \rangle, \\ \partial_t \langle \epsilon^2 \rangle = \int \epsilon^2 \partial_t \Pi_t \epsilon = 2A \langle \epsilon^2 \rangle - B. \end{cases} \quad (\text{A.33})$$

Therefore, once initial conditions are set for (A.33), by using (A.30) the Cauchy problem system provides

$$\begin{cases} \mu_\epsilon(t) = \langle \epsilon_0 \rangle \exp \{ \Delta'_t(\phi) t \}, \\ \langle \epsilon^2 \rangle = \langle (\epsilon^*)^2 \rangle + (\langle \epsilon_0^2 \rangle - \langle (\epsilon^*)^2 \rangle) \exp \{ 2\Delta'_t(\phi) t \} : \langle (\epsilon^*)^2 \rangle = -\frac{\Sigma_t(\phi)}{2\Delta'_t(\phi)} \end{cases} \quad (\text{A.34})$$

Under $\langle \epsilon_0 \rangle = 0$, it follows that the variance with $\mu_\epsilon(t) = 0$ is

$$\sigma_\epsilon(t) = \langle \epsilon^2 \rangle - \langle \epsilon \rangle^2 = \langle (\epsilon^*)^2 \rangle [1 - \exp(2\Delta'_t(\phi)t)]. \quad (\text{A.35})$$

As a consequence the general solution of the Fokker-Planck equation is

$$\Pi_t(\epsilon) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2(t)}} \exp \left[-\frac{(\epsilon - \mu_\epsilon(t))^2}{2\sigma_\epsilon^2(t)} \right]. \quad (\text{A.36})$$

The results of the solution algorithm for the NSF-ME are summarised here below

$$\left\{ \begin{array}{l}
N_1(t) = N\phi(t) + \sqrt{N}\epsilon(t) \text{ s.t.} \\
\phi(t) = \left[1 + \left(\frac{1}{\phi_0} - 1 \right) \exp(-\rho_t t) \right]^{-1} : \phi_0 \in [0, 1) \\
\epsilon(t) \xrightarrow{i.i.d} \mathcal{N}(\mu_\epsilon(t), \sigma_\epsilon^2(t)) \text{ where} \\
\left\{ \begin{array}{l}
\mu_\epsilon(t) = \langle \epsilon_0 \rangle \exp(-\Delta'_t(\phi)t) \quad \langle \epsilon_0 \rangle = 0 \\
\sigma_\epsilon^2(t) = \langle \epsilon_e^2 \rangle [1 - \exp(2\Delta'_t(\phi)t)] : \langle \epsilon_e^2 \rangle = -\frac{\Sigma_t(\phi)}{2\Delta'_t(\phi)} \\
\Pi_t(\epsilon) = \frac{1}{\sqrt{2\pi\sigma_\epsilon^2(t)}} \exp\left[-\frac{(\epsilon - \mu_\epsilon(t))^2}{2\sigma_\epsilon^2(t)}\right] \text{ and}
\end{array} \right. \\
\left\{ \begin{array}{l}
\Delta'_t(\phi) = \beta'_t(t) - \delta'_t(\phi) = \rho_t(1 - 2\phi(t)) \\
\Sigma_t(\phi) = \beta_t(t) + \delta_t(\phi) = \xi_t\phi(t)(1 - \phi(t)) \text{ with}
\end{array} \right. \\
\left\{ \begin{array}{l}
\rho_t = b(\zeta_t - \iota_t) \\
\xi_t = b(\zeta_t + \iota_t) \text{ being}
\end{array} \right. \\
\left\{ \begin{array}{l}
\beta_t(\phi) = \zeta_t b\phi(1 - \phi) \\
\delta_t(\phi) = \iota_t b\phi(1 - \phi)
\end{array} \right.
\end{array} \right. \quad (\text{A.37})$$

Appendix B: Projection algorithm and structure of the network

The system is composed by two populations of heterogeneous firms (\mathcal{F}_t) and banks (\mathcal{B}_t), of constant numerosity. The set of NSF firms is indicated by \mathcal{F}_t^B , while the one of SF is $\mathcal{F}_t - \mathcal{F}_t^B$. Each NSF firm subscribes a financing contract with one single bank, originating a network characterized by finance (lending) links. Accordingly, we can build an adjacency matrix \mathbf{L}_t for the bipartite network of firms and banks according to the following rule

$$\mathbf{L}_t = \left\{ l_{f,b}^t = \begin{cases} 1 & \Leftrightarrow (f \rightleftharpoons b) \\ 0 & \text{otherwise} \end{cases} : f \in \mathcal{F}_t^B, b \in \mathcal{B}_t^B \right\}. \quad (\text{A.38})$$

By means of a projection algorithm we can obtain the adjacency matrix for the network of NSF firms

$$\left\{ \begin{array}{l}
\mathbf{A}_t = \{a_{i,j}^t : \forall f_i, f_j \in \mathcal{F}_t^B\} \in \mathcal{M}_{N_{1,t}, N_{1,t}}(\{0, 1\}) \text{ s.t.} \\
a_{i,j}^t = \begin{cases} 1 & \Leftrightarrow l_{f_i,b}^t = l_{f_j,b}^t = 1 \\ 0 & \text{otherwise} \end{cases} : b \in \mathcal{B}_t^B
\end{array} \right. \quad (\text{A.39})$$

Hence, two firms are neighbours if and only if they are clients of the same bank. We can consider the set of NSF firms to be made of $N_{1,t}$ agents interacting through time on the credit market and defining a network $\mathcal{G}_t = \{\mathcal{F}_t^B, E_t\}$ where E_t is the set of edges linking firms randomly. The probability for a bank $b \in \mathcal{B}_t^B$ to enter the randomly drawn pool $\mathcal{B}_{f,t}$ of banks for the firm $f \in \mathcal{F}_t^B$ depends on the Basel II criterion. The firm-bank matching probability is assumed to be

$$\mathbb{P}(f \rightleftharpoons b) = \pi_1(\bar{\theta}) = 1 - \exp(-b\theta) : b_\theta > 0 \quad (\text{A.40})$$

Since the Basel II criterion is the same for every bank, this probability can be regarded as a mean-field matching probability. Moreover, since in the network two firms are connected if they share the same bank, then

$$\mathbb{P}(f_i \rightleftharpoons b, f_j \rightleftharpoons b) = \pi_1^2(\bar{\theta}), \quad (\text{A.41})$$

evaluates also the probability of a link between them.

Since $\mathbb{P}(l_{f_i,b}^t = 1) = \pi_t$ is the probability of the event $(f_i \rightleftharpoons b)$, and since $(f_i \rightleftharpoons b)$ and $(f_j \rightleftharpoons b)$ are independent, it follows that

$$\mathbb{P}(l_{f_i,b}^t = 1, l_{f_j,b}^t = 1) = \pi_t^2 = \mathbb{P}(a_{i,j}^t = 1). \quad (\text{A.42})$$

Appendix C: solution of the K-ME

The economic model is the data generating process of all the involved micro and macro observables, as such it also generates the credit network \mathcal{G}_t . This is a system of disconnected cliques or system of sub-systems. Each sub-system defines a network component $\mathcal{C}_{b,t}$ around a bank $b \in \mathcal{B}_t^B$ in such a way that $\bigcap_{b \in \mathcal{B}_t^B} \mathcal{C}_{b,t} = \emptyset$. A component $\mathcal{C}_{b,t}$ is therefore the group of $|\mathcal{C}_{b,t}| = K_l$ NSF clients drawn from \mathcal{F}_t^B for the financing bank $b \in \mathcal{B}_t^B$. Being $\mathcal{B}_{f,t}^B$ the pool of banks visible to a NSF firm, we define

$$\Lambda_K = \{K_l = l \in [1, L] \subset \mathbb{N}\}. \quad (\text{A.43})$$

It then follows that a component is defined as

$$\mathcal{C}_{b,t} = \left\{ f \in \mathcal{F}_t^B : \bigcap_{f \in \mathcal{F}_t^B} \mathcal{B}_{f,t}^B = b \in \mathcal{B}_t^B \right\} \subset \mathcal{F}_t^B \quad (\text{A.44})$$

and, more precisely,

$$\mathcal{C}_{b,t}^l \subset \mathcal{F}_t^B : |\mathcal{C}_{b,t}| = K_l \in \Lambda_K \quad (\text{A.45})$$

$\mathcal{G}_t = \bigcup_{b \in \mathcal{B}_t^B} \mathcal{C}_{b,t}$ and $|\mathcal{C}_{b,t}| = K_l$ evaluates the size of the group of clients/nodes for the b -th bank/component. Moreover, since $\mathcal{C}_{b,t}$ is a clique, K_l is also the degree of each firms in the component.

Mechanics of the transitions

In order to study the dynamics of the degree we need to identify all the events that can cause a modification in the number of customers of a bank and, thus, in its degree. Two main kinds of transitions are possible:

- **State Transition (ST)**: new (old) firms can entry (exit) into (out of) the network, $SF \rightleftharpoons NSF$: STs will define pure births/deaths with respect to the credit network;
- **Phase Transition (PS)**: NSF firms in the network can switch the financing bank, $\mathcal{C}_p \rightleftharpoons \mathcal{C}_q$, moving along the components' size distribution: PTs will define birth/death transitions within the credit network.

As for the state transitions, there are two possible types of events:

- a $ST : SF \rightarrow NSF$ is a pure birth into the network: $ST_{\downarrow+}$;
- a $ST : SF \leftarrow NSF$ is a pure death out of the network: $ST_{\uparrow-}$.

Let us define $K_L = \max \Lambda_K$ as the maximum allowed degree level and $k_l = K_l/K_L$ as its intensive value. The symbol $\mathcal{C}_{b,t}^l$ indicates the components in the l -th degree level on the distribution of banks size along a unitary but vanishing reference interval of time Δ . Hence, four kind of *phase transitions* can happen:

- a forward $PS : \mathcal{C}_{p,t-\Delta}^{l-1} \rightarrow \mathcal{C}_{q,t}^l$ is an inflow with a birth: $PS_{\downarrow+}^{\rightarrow}$;
- a backward $PS : \mathcal{C}_{p,t}^l \leftarrow \mathcal{C}_{q,t-\Delta}^{l+1}$ is an inflow with a death: $PS_{\downarrow-}^{\leftarrow}$;
- a forward $PS : \mathcal{C}_{p,t}^l \rightarrow \mathcal{C}_{q,t+\Delta}^{l+1}$ is an outflow with a birth: $PS_{\uparrow+}^{\rightarrow}$;
- a backward $PS : \mathcal{C}_{p,t+\Delta}^{l-1} \leftarrow \mathcal{C}_{q,t}^l$ is an outflow with a death: $PS_{\uparrow-}^{\leftarrow}$.

It is worth recalling that:

- N is the total of NSF+SF firms in the economy;
- $N_1(t)$ is the number of NSF firms;
- $\hat{N}_1(t)$ is the NSF-ME estimate of $N_1(t)$, as given in eq. (A.37);
- $N_1^*(t)$ is the number of NSF-financed firms, hence $N_0^*(t) = N_1(t) - N_1^*(t)$ are the NSF credit-rationed firms;
- $\hat{N}_1^*(t)$ is the K-ME estimate of $N_1^*(t)$;
- $\hat{n}_1^*(t) = N_1^*(t)/\hat{N}_1(t)$ is the estimate share of NSF-financed firms with the NSF-ME solution.

The agent based model is the data generating process that determines the sequence of credit networks $\mathcal{G} = \{\mathcal{G}_t : t \in \mathbb{T}\}$. Each component has its specific size $K_l \in \Lambda_K$ (see eq. (A.43)). This value also measures the degree of each firm within the component $\mathcal{C}_{b,t}^l$ (see eq. (A.45)). Therefore, for any given network $\mathcal{G}_t \in \mathcal{G}$ a distribution $\mathbf{C}(t) = \{C_l(t) : l \in \Lambda_K\}$ of $\hat{B}+1$ occupation numbers $C_l(t)$ is determined as

$$C_l(t) = \# \{|\mathcal{C}_{b,t}^l| = K_l \in \Lambda_K \forall b \in \mathcal{B}_t^B\} \in [0, \hat{B}] \subset \mathbb{N}. \quad (\text{A.46})$$

The occupation numbers $C_l(t)$ are bounded from above and below by two limiting cases: (i) by the maximum allowed value \hat{B} if $C_l(t) = \hat{B}$ which means that all the components have the same size K_l and all the other degree levels are empty; (ii) by the minimum allowed value $C_l(t) = 0$, that is in \mathcal{G}_t there are not components of size K_l at time t . The sequence of distributions $\{\mathbf{C}(t) : t \in \mathbb{T}\}$ determines the evolution of credit network degree distribution which is developed with a K-ME.

Due to the specific nature and structure of each \mathcal{G}_t , by combining (A.46) and (A.43) it follows that

$$N_{1,l}^*(t) = K_l C_l(t) \in [0, N_1^*(t)] \subseteq [0, L\hat{B}] \subset \mathbb{N}. \quad (\text{A.47})$$

This equation evaluates the number of NSF firms financed by banks of size K_l , hence it cannot exceed the value $L\hat{B}$ where $L = K_L = \max \Lambda_K$. Therefore, due to (A.47), $C_l(t)$ and $\hat{N}_{1,l}^*(t)$ are isomorphic. These aspects determine the following conservative constraint

$$\sum_{l \in \Lambda_K} C_l(t) = \hat{B} \Leftrightarrow \sum_{l \in \Lambda_K} \hat{N}_{1,l}^*(t) = N_1^*(t) \leq N_1(t) \quad \forall t \in \mathbb{T}. \quad (\text{A.48})$$

It implies that any firm $f \in \mathcal{F}_t$ in the economic system can assume three macro states: SF, NSF-financed and NSF-rationed. Consequently, at each date there are N firms among which $N_0(t)$ are SF and $N_1(t)$ are NSF; among these last ones $N_1^*(t)$ are NSF-financed and $N_0^*(t) = N_1(t) - N_1^*(t)$ are NSF-rationed. Since the degree distribution involves only the NSF-financed, the specification of transition rates for the K-ME is more complicated than for the NSF-ME because:

- there are six kinds of transitory events: two STs and four PSs;
- there are $L \geq 2$ possible states the NSF-financed firms can assume;
- the number $N_1^*(t)$ of constituents of each \mathcal{G}_t changes through time;
- the K-ME solution must satisfy the constraint (A.48);
- the intensity of transitions is heterogeneous, as it depends on the degree level along the distribution of the occupation numbers. Hence, jump rates can be constant through time but not across states.

As shown for the NSF-ME in equation (26), the state variable is split into its trend and fluctuations component as for the following

$$C_l(t) = \hat{B}\phi_l(t) + \sqrt{\hat{B}}\epsilon_l(t), \quad (\text{A.49})$$

where

$$\phi_l(t) = \left\langle C_l(t) / \hat{B} \right\rangle, \quad (\text{A.50})$$

and

$$\epsilon_l(t) = [C_l(t) - \hat{B}\phi_l(t)]\hat{B}^{-1/2} \stackrel{i.i.d.}{\rightarrow} F_{\epsilon_l}(\mu_{\epsilon_l}(t), \sigma_{\epsilon_l}^2(t)), \quad (\text{A.51})$$

being F_{ϵ_l} unknown. Dividing the constraint (A.48) by \hat{B} and using (A.49) we have

$$\sum_{l \in \Lambda_K} \phi_l(t) + \hat{B}^{-1/2} \sum_{l \in \Lambda_K} \epsilon_l(t) = 1. \quad (\text{A.52})$$

It then must follow that¹²

$$\sum_{l \in \Lambda_K} \epsilon_l(t) = 0 \Rightarrow \mu_{\epsilon_l}(t) = 0 \quad (\text{A.53})$$

The solution of the K-Me should provide analytical expressions for:

- drifts: $\mathbf{F} = \{\phi_l(t) : l \in \Lambda_K, t \in \mathbb{T}\}$;
- spreads: $\mathbf{E} = \{\epsilon_l(t) : l \in \Lambda_K, t \in \mathbb{T}\}$;
- volatilities:¹³ $\mathbf{s}(t) = \{\sigma_{\epsilon_l}^2(t) : l \in \Lambda_K\}$.

in order to obtain

$$\mathbf{N}^* = (\mathbf{1}_T \mathbf{K})' \mathbf{C} \text{ s.t. } \mathbf{C} = \hat{B} \mathbf{F} + \sqrt{\hat{B}} \mathbf{E}, \quad (\text{A.54})$$

being $\mathbf{1}_T$ a column unit vector of $T = |\mathbb{T}|$ components, $\mathbf{K} = (K_1, \dots, K_L)$ a row vector and \mathbf{N}^* a matrix ($T \times L$): each column vector is a time series $\left\{ N_{1,l}^*(t) \right\}_{t \in \mathbb{T}}$.

These results can be attained by means of the ME solution techniques in Appendix A for the NSF-ME. Indeed, having assumed δ -correlated and independent realizations for the spread, it is sufficient to solve one ME for the l -th occupation number. The functional forms for the other degree levels is the same with the caveat that the sequence of occupation numbers must satisfy the conservative constraint (A.48). The difference is in the transition rate functional form specifications.

Specification of the functionals

The Rate Functions

The rate functions provide the rates of births and deaths in a functional form concerning all the birth and death events in the transitory mechanics of components, that is from/to outside of the network ($ST_{\downarrow+}$ and $ST_{\uparrow-}$) and within the network ($PS_{\downarrow+}^{\rightarrow}$, $PS_{\downarrow-}^{\leftarrow}$, $PS_{\uparrow+}^{\rightarrow}$, $PS_{\uparrow-}^{\leftarrow}$).

When facing a pure birth ($ST_{\downarrow+}$) or an inflow ($PS_{\downarrow+}^{\rightarrow}$, $PS_{\downarrow-}^{\leftarrow}$) a *link creation rate* is needed to consider the positive effect of an increase in the Basel II parameter and the restrictive effect of an increase in the density of the NSF-financed firms. This rate can be specified as

$$c(t) = \alpha_c [1 - \hat{n}_1^*(t)] \pi_1^2(\bar{\theta}) : \alpha_c > 0 \quad (\text{A.55})$$

When facing a pure death ($ST_{\uparrow-}$) or an outflow ($PS_{\uparrow+}^{\rightarrow}$, $PS_{\uparrow-}^{\leftarrow}$) a *link destruction rate* is needed to consider the negative effect of an increase in the Basel II

¹²The natural estimator of $\mu_{\epsilon_l}(t) = \langle \epsilon_l(t) \rangle$ is $\bar{\epsilon}(t) = \hat{B}^{-1} \sum_{l \in \Lambda_K} \epsilon_l(t)$.

¹³This set up assumes that spreading volatilities are δ -correlated and *i.i.d.*

parameter and an extensive effect of an increase in the density of NSF-financed firms. Accordingly, we can write

$$d(t) = \alpha_d \hat{n}_1^*(t) [1 - \pi_1^2(\bar{\theta})] : \alpha_d > 0 \quad (\text{A.56})$$

Since $k_l = K_l/K_L$ where $K_L = L$ and $K_l = l \in \Lambda_K$, $K_L = L = \max \Lambda_K$, it follows that $k_l \in (\underline{k}, 1)$ being $\underline{k} = 1/L$. The *birth* $\zeta_l = \zeta(K_l)$ and *death* $\iota_l = \iota(K_l)$ rates are higher for lower levels of the degree, that is when approaching \underline{k} they are about $\zeta_1 = \bar{\zeta}$ and $\iota_1 = \bar{\iota}$, respectively. The maximum allowed values at $K_1 = 1 \Leftrightarrow k_1 = \underline{k}$, while $\zeta_L = \underline{\zeta}$ and $\iota_L = \underline{\iota}$ are the minimum allowed values at $K_L = L \Leftrightarrow k_L = 1$. Thus, ζ_l and ι_l display a negative slope as $k_l \rightarrow 1$. We can assume that

$$\frac{\partial \zeta_l}{\partial k_l} < 0 \quad \wedge \quad \frac{\partial \iota_l}{\partial k_l} < 0. \quad (\text{A.57})$$

All the possible decreasing behaviours can be summarized in a single equation model for both ζ_l and ι_l . Indeed, if we consider $x \in [\underline{x}, 1]$ as $k_l \in [\underline{k}, 1]$ and $y(x) \in [\underline{y}, \bar{y}]$ play the role of $\zeta_l \in [\underline{\zeta}, \bar{\zeta}]$ and $\iota_l \in [\underline{\iota}, \bar{\iota}]$, we can set up a Cauchy problem in the following form

$$\begin{cases} y'(x) = -ax^p(y(x))^q : y(\underline{x}) = \bar{y} \text{ s.t.} \\ y(x) \in [\underline{y}, \bar{y}], x \in [\underline{x}, 1], a > 0, y'(x) < 0, p, q \in \mathbb{R} \end{cases} \quad (\text{A.58})$$

It yields the following solution

$$y(x; p, q) = \left[-a \frac{1-q}{1+p} x^{1+p} + \frac{a \underline{x}^{1+p} (1-q) \bar{y}^q + (1+p) \bar{y}}{\bar{y}^q (1+p)} \right]^{\frac{1}{1-q}}. \quad (\text{A.59})$$

This functional forms can generate different behaviours that are dependent on the micro-level interaction:

- linear: $y(x; p \rightarrow 0, q \rightarrow 0) = C_{0,0} - ax$ where $C_{0,0} = a \underline{x} + \bar{y}$;
- concave: $y(x; p \rightarrow 1, q \rightarrow 0) = C_{1,0} - \frac{a}{2} x^2$ where $C_{1,0} = \frac{a \underline{x}^2 + 2 \bar{y}}{2}$;
- convex: $y(x; p \rightarrow 0, q \rightarrow 1) = C_{0,1} e^{-ax}$ where $C_{0,1} = \bar{y} e^{\underline{x}}$;
- with inflection: $y(x; p \rightarrow 1, q \rightarrow 1) = C_{1,1} e^{-\frac{a}{2} x^2}$ where $C_{1,1} = \bar{y} e^{\frac{a}{2} \underline{x}^2}$.

The parameters \underline{x} and \bar{y} can be estimated by means of Monte Carlo simulations of the agent based model. Only $a > 0$ needs to be calibrated while p and q are chosen to provide a better fit or by assuming specific behaviours. In what follows we consider convex decreasing shapes, in order to account for the power-law shape of the degree observed in real data, for ζ_l and for ι_l . Namely, we make use of the following

$$\zeta_l = \bar{\zeta} e^{-\alpha_1(k_l - \underline{k})}, \quad \iota_l = \bar{\iota} e^{-\alpha_0(k_l - \underline{k})} : \alpha_1, \alpha_0 > 0 \quad (\text{A.60})$$

All the terms needed to specify the rate functions are now given, therefore the birth rate function is defined as

$$\lambda_l(t) = \zeta_l e^{c(t)}. \quad (\text{A.61})$$

The death rate function is

$$\mu_l(t) = \iota_l e^{d(t)}. \quad (\text{A.62})$$

For further developments of the model and the solution of the K-ME, it is useful to define the following quantity

$$\gamma_l^-(t) = \lambda_l(t) - \mu_l(t) = \zeta_l e^{c(t)} - \iota_l e^{d(t)}, \quad (\text{A.63})$$

which evaluates the net-rate function at each date and at each position (i.e. degree level) on the support Λ_K . In the same way it is possible to define

$$\gamma_l^+(t) = \lambda_l(t) + \mu_l(t) = \zeta_l e^{c(t)} + \iota_l e^{d(t)} \quad (\text{A.64})$$

The Moran-like Mechanism

This term of the transition rates accounts for the neighbouring effects. As for the birth and death transition rates, their generic functional representations are, respectively

$$\mathcal{M}_{1,l}(t) = \psi_1(C_l(t), J_l(t)) \quad \wedge \quad \mathcal{M}_{0,l}(t) = \psi_0(C_l(t), J_l(t)), \quad (\text{A.65})$$

where ψ_1 and ψ_0 are the externality functions and J_l are the variations of the occupation number C_l .

Set $\nu(t)$ the expected number of occurrences and define $P(K_l, t) = p_l(t)$ to be Poisson for the probability to observe a component/bank of size K_l . It is defined as

$$p_l(t) = e^{-\nu(t)} \frac{\nu(t)^{K_l}}{K_l!} = Pr(K_l) \quad : \quad \nu(t) = \frac{\hat{N}_1(t)}{\hat{B}}. \quad (\text{A.66})$$

The number of components with size K_l are estimated by the following expression for jumps

$$J_l(t) = \hat{B} p_l(t) \quad (\text{A.67})$$

Both jumps and externality functions are time dependent, since the K-ME depends on the NSF-ME solution, but they also depend on K_l , that is the position along the distribution. Hence, the Moran-like mechanism changes through time as well as the rate functions. The Moran-like terms are then defined as the following externality functions

$$\begin{cases} \psi_1(C_l(t), J_l(t)) = \frac{C_l(t) - \vartheta J_l(t)}{\hat{B}} = \psi_{1,l}(t), \\ \psi_0(C_l(t), J_l(t)) = \frac{C_l(t) + \vartheta J_l(t)}{\hat{B}} = \psi_{0,l}(t). \end{cases} \quad (\text{A.68})$$

The first derivatives with respect to the number of components are

$$\begin{cases} \frac{\partial \psi_1(C_l(t), J_l(t))}{\partial K_l} = -\vartheta p_l(t) (\log \nu(t) - k_l), \\ \frac{\partial \psi_0(C_l(t), J_l(t))}{\partial K_l} = +\vartheta p_l(t) (\log \nu(t) - k_l). \end{cases} \quad (\text{A.69})$$

Therefore, since $\hat{N}_1(t) \gg \hat{B}$ and $k_l = K_l/L \in [1/L, 1]$, it follows that

$$\begin{cases} \frac{\partial \psi_1(C_l(t), J_l(t))}{\partial K_l} \leq 0 \text{ iff } \hat{N}_1(t) \geq e^{k_l} \hat{B}, \\ \frac{\partial \psi_0(C_l(t), J_l(t))}{\partial K_l} \leq 0 \text{ iff } \hat{N}_1(t) \leq e^{k_l} \hat{B}. \end{cases} \quad (\text{A.70})$$

By observing that $k_l = K_l/L \in [1/L, 1]$ it is clear that $e^{1/L} \hat{B} \leq e^{k_l} \hat{B} \leq e \hat{B} \approx 3\hat{B}$, therefore $\hat{B} < e^{k_l} \hat{B} < 3\hat{B}$ as the number L of degree level is large. Accordingly, it is possible to conclude that:

- for birth transition rates: the Moran-like mechanism has a decreasing effect $\partial \psi_{1,l}(t)/\partial K_l \leq 0$ if the estimated volume $\hat{N}_1(t)$ of NSF-firms is greater than a number between \hat{B} and $3\hat{B}$;
- for death transition rates: the Moran-like mechanism has an increasing effect $\partial \psi_{0,l}(t)/\partial K_l \geq 0$ if the estimated volume $\hat{N}_1(t)$ of NSF-firms is smaller than a number between \hat{B} and $3\hat{B}$.

The variable \hat{B} quantifies the number of banks active in the market. In case of outflows ($\vartheta = 0$) $\partial \psi_{1,l}(t)/\partial K_l = \partial \psi_{0,l}(t)/\partial K_l = 0$; in case of inflows ($\vartheta = 1$) the derivatives are $\partial \psi_{1,l}(t)/\partial K_l < 0$ and $\partial \psi_{0,l}(t)/\partial K_l > 0$.

For any value K_l of the degree, the occupation number is $C_l(t)$. Hence, for any component $\mathcal{C}_{p,t}^l$ with $K_l \in [2, L-1]$, the transition rates concern the following Phase Transitions among components

$$\mathcal{C}_{o,t-\Delta}^{l-1} \underbrace{\begin{array}{c} \text{inflow } \vartheta=1 \text{ with birth} \\ \text{=} \\ \text{outflow } \vartheta=0 \text{ with death} \end{array}}_{\text{}} \mathcal{C}_{p,t}^l \underbrace{\begin{array}{c} \text{outflow } \vartheta=0 \text{ with birth} \\ \text{=} \\ \text{inflow } \vartheta=1 \text{ with death} \end{array}}_{\text{}} \mathcal{C}_{q,t+\Delta}^{l+1}. \quad (\text{A.71})$$

The infinitesimal but unitary reference time is Δ .

The Cardinality Functions

The cardinality functions for birth and death transition rates can be defined as, respectively

$$\begin{cases} \mathcal{N}_1(C_l(t) - \vartheta J_l(t)) = \hat{B} - (C_l(t) - \vartheta J_l(t)), \\ \mathcal{N}_0(C_l(t) + \vartheta J_l(t)) = (C_l(t) + \vartheta J_l(t)). \end{cases} \quad (\text{A.72})$$

It follows that

- birth TRs with inflow, $PS_{\downarrow+}^-: \mathcal{N}_1(C_l(t) - \vartheta J_l(t)) = \hat{B} - (C_l(t) - J_l(t))$;
- birth TRs with outflow, $PS_{\uparrow+}^-: \mathcal{N}_1(C_l(t) - \vartheta J_l(t)) = \hat{B} - C_l(t)$;
- death TRs with inflow, $PS_{\downarrow-}^+: \mathcal{N}_0(C_l(t) + \vartheta J_l(t)) = C_l(t) + J_l(t)$;
- death TRs with outflow, $PS_{\uparrow-}^+: \mathcal{N}_0(C_l(t) + \vartheta J_l(t)) = C_l(t)$.

The Transition Rates

All the terms of transition rates are now specified and can be put together to get the phenomenological transition rates

$$\begin{cases} \beta_t(C_l(t) - \vartheta J_l(t)) = \zeta_l e^{c(t)} \left(\frac{C_l(t) - \vartheta J_l(t)}{\hat{B}} \right) \left[\hat{B} - (C_l(t) - \vartheta J_l(t)) \right], \\ \delta_t(C_l(t) + \vartheta J_l(t)) = \iota_l e^{d(t)} \left(\frac{C_l(t) + \vartheta J_l(t)}{\hat{B}} \right) [C_l(t) + \vartheta J_l(t)]. \end{cases} \quad (\text{A.73})$$

The transition rates must be homogeneous functions of a system size parameter, in this case \hat{B} : it is easy to check that w.r.t. cardinalities this requirement is satisfied. Indeed it can be shown that

$$\begin{cases} \hat{B} \beta_t \left(\frac{C_l(t) - \vartheta J_l(t)}{\hat{B}} \right) = \zeta_l e^{c(t)} \left(\frac{C_l(t) - \vartheta J_l(t)}{\hat{B}} \right) \left[\frac{\hat{B} - (C_l(t) - \vartheta J_l(t))}{\hat{B}} \right] \hat{B} \\ \hat{B} \delta_t \left(\frac{C_l(t) + \vartheta J_l(t)}{\hat{B}} \right) = \iota_l e^{d(t)} \left(\frac{C_l(t) + \vartheta J_l(t)}{\hat{B}} \right) \left[\frac{C_l(t) + \vartheta J_l(t)}{\hat{B}} \right] \hat{B} \end{cases} \quad (\text{A.74})$$

The K master equation

The transition rates specified in (A.73) can be used to specify the phenomenological master equation for the network degree. The K-ME can be solved by solving single components $K_l - MEs$ focusing the attention on single realizations $C_l(t) = C_h \in [0, \hat{B}]$ each. That is, the single phenomenological $K_l - ME$ reads as

$$\frac{dP_t(C_h)}{dt} = \underbrace{[\beta_t(C_h - J_m)P(C_h - J_m) + \delta_t(C_h + J_m)P(C_h + J_m)]}_{\text{inflow probability}} - \underbrace{[(\beta_t(C_h) + \delta_t(C_h))P(C_h + J_m)]}_{\text{outflow probability}} \quad (\text{A.75})$$

where $J_l(t) = J_m$ is a realization of the jumps, as defined in (A.67).

ME solution and dynamics of the degree

The solution method for (A.75) is the same used for (25) in Appendix A, to which the reader is referred, but with some additional devices. According to (A.50)-(A.53) the ansatz (A.49) gives¹⁴

$$\begin{cases} (i) & C_l(t) \pm \vartheta J_l(t) = \hat{B} \phi_l(t) + \sqrt{\hat{B}} \epsilon_l(t) \pm \vartheta J_l(t), \text{ s.t.} \\ (ii) & \phi_l(t) = \langle \hat{\beta} C_l(t) \rangle : \hat{\beta} \hat{B} = 1, \\ (iii) & \epsilon_l(t) = (C_l(t) - \hat{B} \phi_l(t)) \sqrt{\hat{\beta}} \xrightarrow{i.i.d.} F_{\epsilon_l}(0, \sigma_{\epsilon_l}^2(t)), \end{cases} \quad (\text{A.76})$$

¹⁴Here, $\hat{\beta} = 1/\hat{B}$ is not to be confused with the symbol β_t for the birth transition rates.

equivalently to (A.1). The intensive form representation reads as

$$\left[\frac{C_l(t) \pm \vartheta J_l(t)}{\hat{B}} = c_l(t) \pm \vartheta j_l(t) \right]_{j_l(t) = \hat{\beta} J_l(t)} = \phi_l(t) + \sqrt{\hat{\beta}} \left(\epsilon_l(t) \pm \vartheta \sqrt{\hat{\beta}} J_l(t) \right) = \phi_{l,(\pm \vartheta j_l(t))}(t), \quad (\text{A.77})$$

where

$$\phi_{l,(\pm \vartheta j_l(t))}(t) = \phi_l(t) + \sqrt{\hat{\beta}} \epsilon_{l,(\pm j_l(t))}(t) : \epsilon_{l,(\pm j_l(t))}(t) = \epsilon_l(t) \pm \vartheta \sqrt{\hat{\beta}} J_l(t). \quad (\text{A.78})$$

Equations (A.77) and (A.78) are equivalent to (A.2) and (A.3), the main difference is due to space-time varying jumps. By using the change of variable in (A.76-iii) and fixing $C_l(t) = C_h$ and $J_l(t) = J_m$, the l.h.s of (34) is equivalent to (A.4) and, by following the same steps up to (A.7), we obtain

$$\hat{\beta} \frac{dP_t(C_h)}{dt} = \hat{\beta} \frac{\partial \Pi_t(\epsilon_l)}{\partial \epsilon_l} - \frac{1}{\sqrt{\hat{\beta}}} \frac{d\phi_l}{dt} \frac{\partial \Pi_t(\epsilon_l)}{\partial \epsilon_l}. \quad (\text{A.79})$$

Since (A.74) shows that the transition rates are homogeneous functions of the system size parameter \hat{B} , then (A.9)-(A.11) are still suitable, in order to obtain

$$\beta_t(C_h - \vartheta J_m) = \hat{B} \beta_t(\phi_{l,(-\vartheta j_m)}), \quad (\text{A.80})$$

$$\delta_t(C_h + \vartheta J_m) = \hat{B} \delta_t(\phi_{l,(+\vartheta j_m)}), \quad (\text{A.81})$$

$$P_t(C_h \pm \vartheta J_m) = \Pi_t(\epsilon_{l,(\pm \vartheta j_m)}). \quad (\text{A.82})$$

Hence (A.79) gives the $K_l - ME$ which is going to be solved

$$\begin{aligned} & \hat{\beta} \frac{\partial \Pi_t(\epsilon_l)}{\partial \epsilon_l} - \sqrt{\hat{\beta}} \frac{d\phi_l}{dt} \frac{\partial \Pi_t(\epsilon_l)}{\partial \epsilon_l} = \\ & [\beta_t(\phi_{l,(-j_m)}) \Pi_t(\epsilon_{l,(-j_m)}) + \delta_t(\phi_{l,(+j_m)}) \Pi_t(\epsilon_{l,(+j_m)})] + \\ & - [\Sigma_t(\phi_{l,(0)}) \Pi_t(\epsilon_{l,(0)})], \end{aligned} \quad (\text{A.83})$$

where $\Sigma_t(\phi_{l,(0)}) = \beta_t(\phi_{l,(0)}) + \delta_t(\phi_{l,(0)})$, as found in (A.12).

It is now convenient to specify more suitably the transition rates and the density in (A.80)-(A.82) as follows

$$\beta_t(\phi_{l,(-\vartheta j_m)}) = \beta_t \left(\phi_l + \sqrt{\hat{\beta}} \left(\epsilon_l - \vartheta \sqrt{\hat{\beta}} J_m \right) \right), \quad (\text{A.84})$$

$$\delta_t(\phi_{l,(+\vartheta j_m)}) = \delta_t \left(\phi_l + \sqrt{\hat{\beta}} \left(\epsilon_l + \vartheta \sqrt{\hat{\beta}} J_m \right) \right), \quad (\text{A.85})$$

$$\Sigma_t(\phi_{l,(0)}) = \Sigma_t \left(\phi_l + \sqrt{\hat{\beta}} \epsilon_l \right), \quad (\text{A.86})$$

$$\Pi_t(\epsilon_{l,(\pm \vartheta j_m)}) = P_t \left(\hat{B} \phi_l + \sqrt{\hat{B}} \left(\epsilon_l \pm \vartheta \sqrt{\hat{\beta}} J_m \right) \right). \quad (\text{A.87})$$

The transition rates need to be approximated around ϕ_l while the density around ϵ_l . Hence, by using (A.13)-(A.16) and since, as in (A.17), it holds true that

$$\hat{\beta}^{p/2} \rightarrow 0^+ \quad \forall p > 2 \quad \forall \hat{B} \rightarrow +\infty \quad (\text{A.88})$$

The second order approximation is

$$\hat{\beta} \dot{\Pi}_t - \sqrt{\hat{\beta}} \dot{\phi}_l \Pi'_t = -\sqrt{\hat{\beta}} \Delta_t \Pi'_t - \sqrt{\hat{\beta}} \left[\Delta'_t \partial_{\epsilon_l} (\epsilon_l \Pi_t) - \frac{1}{2} \Sigma_t \Pi''_t \right], \quad (\text{A.89})$$

where $\Delta_t = \Delta_t(\phi_l) = \beta_t(\phi_l) - \delta_t(\phi_l)$, $\Delta'_t = \partial_{\phi_l} \Delta_t$, $\Pi_t = \Pi_t(\epsilon_l)$, $\Pi'_t = \partial_{\epsilon_l} \Pi_t$, $\dot{\Pi}_t = \partial_t \Pi_t$ and $\dot{\phi}_l = \partial_t \phi_l$; equivalently to (A.18). The application of the polynomial identity principle in equation (A.89) w.r.t. the parameter $\hat{\beta}$ yields the following system of coupled equation

$$\dot{\phi}_l = \Delta_t(\phi_l) = \beta_t(\phi_l) - \delta_t(\phi_l) \quad (\text{A.90})$$

$$\partial_t \Pi_t(\epsilon_l) = -\Delta'_t(\phi_l) \partial_{\epsilon_l} (\epsilon_l \Pi_t(\epsilon_l)) + \frac{1}{2} \Sigma_t(\phi_l) \partial_{\epsilon_l}^2 \Pi_t(\epsilon_l), \quad (\text{A.91})$$

In order to find the final solution of (A.83), the two equation above must be solved.

The Macroscopic Equation

The macroscopic equation (A.90) is an ODE for the drifting trajectory of $c_l(t) = \hat{\beta} C_l(t)$ on Λ_K . By expanding (A.84) and (A.85) as done for (A.13) and (A.14), it follows that (A.90) reads as the following Cauchy problem

$$\dot{\phi}_l = \Delta_t(\phi_l) = \gamma_l^- \phi_l (1 - \phi_l) : \phi_l^0 = \phi_l(0), \quad (\text{A.92})$$

where $\gamma_l^-(t)$ is defined in (A.63). The equilibrium condition yields

$$\dot{\phi}_l = 0 \Rightarrow \phi_l^* \in \{0, 1\} \quad (\text{A.93})$$

The point $\phi_l^* = 0$ implies that the l -th degree level is empty, while $\phi_l^* = 1$ means that all the components have the same size K_l .

The Cauchy problem (A.92) has the following general solution

$$\phi_l(t) = \frac{1}{1 + \left(\frac{1}{\phi_l^0} - 1 \right) e^{-t \gamma_l^-(t)}} \in (0, 1) \quad (\text{A.94})$$

The conditions for the macroscopic equation to satisfy the conservative constraint (A.52), under (A.53), reads as

$$\sum_{l \in \Lambda_K} \prod_{i \neq l} \left[1 + \left(\frac{1}{\phi_i^0} - 1 \right) e^{-\gamma_i^-(t)t} \right] = \sum_{l \in \Lambda_K} \left[1 + \left(\frac{1}{\phi_l^0} - 1 \right) e^{-t \gamma_l^-(t)} \right] \quad (\text{A.95})$$

The Fokker-Planck Equation

Equation (A.91) defines a Fokker-Planck equation for the distribution of spread

around the drifting trajectory (A.92). The Fokker-Planck equation has a stationary and a general solution. As long as the stationary solution is concerned, by using (A.25)-(A.28) we can write

$$\Pi(\epsilon_l) = H_l \exp \left\{ \epsilon_l^2 \frac{\Delta'(\phi_l)}{\Sigma(\phi_l)} \right\} : H_l = \sqrt{-\frac{1}{\pi} \frac{\Delta'(\phi_l)}{\Sigma(\phi_l)}}, \quad (\text{A.96})$$

for which

$$\frac{\Delta'(\phi_l)}{\Sigma(\phi_l)} = \frac{\gamma_l^-(t)}{\gamma_k^+(t)} \frac{1 - 2\phi_l(t)}{\phi_l(t)(1 - \phi_l(t))} < 0. \quad (\text{A.97})$$

The quantity $\gamma_l^+(t)$ has been defined in (A.64) and $\Sigma(\phi_l(t)) = \beta_t(\phi_l) + \delta(\phi_l)$.

As for the general solution, equation (A.53) shows that $\mu_{\epsilon_l} = 0$. It follows that

$$\Pi_t(\epsilon_l) = \frac{1}{\sqrt{2\pi\sigma_{\epsilon_l}^2(t)}} \exp \left\{ -\frac{\epsilon_l^2}{2\sigma_{\epsilon_l}^2(t)} \right\} \quad (\text{A.98})$$

The volatility $\sigma_{\epsilon_l}^2(t)$ can be defined by using (A.35) to obtain

$$\begin{aligned} \sigma_{\epsilon_l}^2(t) &= \langle (\epsilon_l^*)^2 \rangle \{1 - \exp[2\Delta'_t(\phi_l)t]\} \text{ s.t.} \\ \langle (\epsilon_l^*)^2 \rangle &= -\frac{\Sigma_t(\phi_l)}{2\Delta'_t(\phi_l)} = -\frac{\gamma_l^+(t)}{\gamma_l^-(t)} \frac{\phi_l(t)(\phi_l(t))}{1-2\phi_l(t)} > 0 \end{aligned} \quad (\text{A.99})$$

For reader's convenience we can summarise all the results obtained for the dynamics of the degree:

$$\left\{ \begin{array}{l} N_{1,i}^*(t) = K_i C_i(t) \\ C_i(t) = \hat{B} \phi_i(t) + \sqrt{\hat{B}} \epsilon_i(t) \text{ s.t.} \\ \phi_i(t) = \left\{ 1 + \left(\frac{1}{\phi_i^0} - 1 \right) \exp[-t\gamma_i^-(t)] \right\}^{-1} \\ \epsilon_i(t) \xrightarrow{i.i.d.} \mathcal{N}(\mu_{\epsilon_i}(t), \sigma_{\epsilon_i}^2(t)) \\ \left\{ \begin{array}{l} \mu_{\epsilon_i}(t) = 0 \\ \sigma_{\epsilon_i}^2(t) = \langle (\epsilon_l^*)^2 \rangle \{1 - \exp[2\Delta'_t(\phi_l)t]\} : \langle (\epsilon_l^*)^2 \rangle = -\frac{\Sigma_t(\phi_l)}{2\Delta'_t(\phi_l)} \\ \Pi_t(\epsilon_l) = \frac{1}{\sqrt{2\pi\sigma_{\epsilon_l}^2(t)}} \exp \left\{ -\frac{\epsilon_l^2}{2\sigma_{\epsilon_l}^2(t)} \right\}; \text{ with} \\ \Delta'_t(\phi_l) = \gamma_l^-(t)(1 - 2\phi_l(t)) \\ \Sigma_t(\phi_l) = \gamma_l^+(t)\phi_l(t)(1 - \phi_l(t)) \\ \gamma_l^\pm(t) = \lambda_l(t) \pm \mu_l(t) = \zeta_l e^{c(t)} \pm \nu_l e^{d(t)} \text{ with} \\ \zeta_l = \bar{\zeta} \exp[-\alpha_1(k_l - \underline{k})] \\ \nu_l = \bar{\nu} \exp[-\alpha_0(k_l - \underline{k})] \\ c(t) = \alpha_c [1 - \hat{n}_1^*(t)] \pi_1^2(\bar{\theta}) \\ d(t) = \alpha_d \hat{n}_1^*(t) [1 - \pi_1^2(\bar{\theta})] \text{ being} \\ \hat{n}_1^*(t) = \frac{N_1^*(t)}{\hat{N}_1(t)} : \hat{N}_1(t) \equiv \text{NSF-ME sol.}, N_1^*(t) \equiv \text{ABM NSF-financed} \\ \pi_1^2(\bar{\theta}) = 1 - \exp(-b_\theta \bar{\theta}) : \bar{\theta} \equiv \text{Basel II policy param.} \end{array} \right. \end{array} \right. \quad (\text{A.100})$$

References

- de Aguiar, M.A.M., Epstein, I.R., Bar-Yam, Y., 2005. Analytically solvable model of probabilistic network dynamics. *Phys. Rev. E* 72, 067102.
- Alfarano, S., Lux, T., Wagner, F., 2008. Time variation of higher moments in a financial market with heterogeneous agents: An analytical approach. *Journal of Economic Dynamics and Control* 32, 101 – 136.
- Allen, F., Gale, D., 2001. Financial contagion. *Journal of Political Economy* 108, 1–33.
- Aoki, M., 1996. *New approaches to macroeconomic modeling*. Cambridge University Press.
- Aoki, M., 2002. *Modeling aggregate behaviour and fluctuations in economics*. Cambridge University Press.
- Aoki, M., Yoshikawa, H., 2006. *Reconstructing Macroeconomics*. Cambridge University Press.
- Asea, P.K., Blomberg, B., 1998. Lending cycles. *Journal of Econometrics* 83, 89–128.
- Battiston, S., Delli Gatti, D., Gallegati, M., Greenwald, B., Stiglitz, J.E., 2007. Credit chains and bankruptcy propagation in production networks. *Journal of Economic Dynamics and Control* 31, 2061–2084.
- Bernanke, B., Gertler, M., 1989. Agency costs, net worth, and business fluctuations. *American Economic Review* 79, 14–31. Available at <http://ideas.repec.org/a/aea/aecrev/v79y1989i1p14-31.html>.
- Boccaletti, S., Latora, V., Moreno, Y., Chavez, M., Hwang, D., 2006. Complex networks: Structure and dynamics. *Physics Reports* 424, 175–308.
- Brock, W.A., Durlauf, S.N., 2001. Discrete choice with social interactions. *Review of Economic Studies* 68, 235–60.
- Chen, L., Wang, R., Aihara, K., 2003. Genetic networks with stochastic fluctuations. *Genome Informatics* , 356–357.
- Chiarella, C., Di Guilmi, C., 2011. The financial instability hypothesis: A stochastic microfoundation framework. *Journal of Economic Dynamics and Control* 35, 1151 – 1171.
- De Masi, G., Fujiwara, Y., Gallegati, M., Greenwald, B., Stiglitz, J.E., 2010. An analysis of the Japanese credit network. *Evolutionary and Institutional Economics Review* 7, 209–232.

- Delli Gatti, D., Di Guilmi, C., Gaffeo, E., Giulioni, G., Gallegati, M., Palestrini, A., 2005. A new approach to business fluctuations: heterogeneous interacting agents, scaling laws and financial fragility. *Journal of Economic Behavior and Organization* 56.
- Delli Gatti, D., Gallegati, M., Greenwald, B., Russo, A., Stiglitz, J.E., 2006. Business fluctuations in a credit-network economy. *Physica A: Statistical Mechanics and its Applications* 370, 68 – 74.
- Delli Gatti, D., Gallegati, M., Greenwald, B., Russo, A., Stiglitz, J.E., 2009. Business fluctuations and bankruptcy avalanches in an evolving network economy. *Journal of Economic Interaction and Coordination* 4, 195–212.
- Delli Gatti, D., Gallegati, M., Greenwald, B., Russo, A., Stiglitz, J.E., 2010. The financial accelerator in an evolving credit network. *Journal of Economic Dynamics and Control* 34, 1627–1650.
- Di Guilmi, C., 2008. *The generation of business fluctuations: financial fragility and mean-field interaction*. Peter Lang Publishing Group: Frankfurt/M.
- Di Guilmi, C., Gallegati, M., Landini, S., 2010. Financial fragility, mean-field interaction and macroeconomic dynamics: A stochastic model, in: Salvadori, N. (Ed.), *Institutional and social dynamics of growth and distribution*. Edward Elgar, UK, pp. 323–351.
- Di Guilmi, C., Gallegati, M., Landini, S., 2012a. *Interactive Macroeconomics*. [Http://finance.uts.edu.au/staff/corradodg/Interactive%20Macroeconomics.pdf](http://finance.uts.edu.au/staff/corradodg/Interactive%20Macroeconomics.pdf). Forthcoming.
- Di Guilmi, C., Gallegati, M., Landini, S., Stiglitz, J.E., 2012b. Towards an analytical solution for agent based models: an application to a credit network economy, in: Aoki, M., Binmore, K., Deakin, S., Gintis, H. (Eds.), *Complexity and Institutions: Markets, Norms and Corporations*. Palgrave: London, pp. 61–78. Forthcoming.
- Ehrhardt, G., Marsili, M., Vega-Redondo, F., 2006. Phenomenological models of socioeconomic network dynamics. *Phys. Rev. E* 74, 036106.
- Erhardt, G., Marsili, M., Vega-redondo, F., 2007. Emergence and resilience of social networks: a general theoretical framework. *Annales d’Economie et de Statistique* .
- Fisher, F.M., 1987. Aggregation problem, in: *The New Palgrave: A Dictionary of Economics*, First edition. pp. 53–55.
- Foley, D.K., 1994. A statistical equilibrium theory of markets. *Journal of Economic Theory* 62, 321–345.
- Freixas, X., Parigi, B.M., Rochet, J.C., 2000. Systemic risk, interbank relations, and liquidity provision by the central bank. *Journal of Money, Credit and Banking* 32, 611–38.

- Fujiwara, Y., Aoyama, H., Ikeda, Y., Iyetomi, H., Souma, W., 2009. Structure and temporal change of the credit network between banks and large firms in Japan. *Economics: The Open-Access, Open-Assessment E-Journal* 3.
- Furfine, C.H., 2003. Interbank exposures: Quantifying the risk of contagion. *Journal of Money, Credit and Banking* 35, 111–28.
- Gallegati, M., Greenwald, B., Richiardi, M.G., Stiglitz, J.E., 2008. The asymmetric effect of diffusion processes: Risk sharing and contagion. *Global Economy Journal* 8.
- Gambacorta, L., 2008. How do banks set interest rates? *European Economic Review* 52, 792–819.
- Gardiner, C.W., 1985. *Handbook of stochastic methods*. Springer-Verlag: New York.
- Gelenbe, R., 2008. Network of interacting synthetic molecules in steady state. *Proc. R. Soc.A* 464, 2219–2228.
- Gorman, W.M., 1953. Community preference fields. *Econometrica* , 63–80.
- Greenwald, B., Stiglitz, J.E., 1993. Financial markets imperfections and business cycles. *Quarterly journal of Economics* 108.
- Hildebrand, W., 2008. Aggregation (theory), in: *The New Palgrave: A Dictionary of Economics*, Second edition.
- Iori, G., Jafarey, S., Padilla, F.G., 2006. Systemic risk on the interbank market. *Journal of Economic Behavior & Organization* 61, 525–542.
- Janicki, H.P., Prescott, E.S., 2006. Changes in the size distribution of u.s. banks: 1960-2005. *Economic Quarterly* , 291–316.
- van Kampen, N.G., 1992. *Stochastic processes in physics and chemistry*. North-Holland, Amsterdam.
- Landini, S., Uberti, M., 2008. A statistical mechanic view of macro-dynamics in economics. *Computational Economics* 32, 121–146.
- Minsky, H.P., 1963. Can “it” happen again?, in: Carson, D. (Ed.), *Banking and Monetary Studies*. Richard D Irwin, Homewood. Reprinted in Minsky (1982).
- Minsky, H.P., 1982. *Inflation, recession and economic policy*. ME Sharpe: New York.
- Moran, P., 1962. *The Statistical Processes of Evolutionary Theory*. Clarendon Press: Oxford.
- Risken, H., 1989. *Fokker-Planck equation. Method of solutions and applications*. Berlin: Springer Verlag.

- Smith, D.M.D., Onnela, J.P., Jones, N.S., 2009. Master-equation analysis of accelerating networks. *Phys. Rev. E* 79, 056101.
- Stein, J.C., 1998. An adverse-selection model of bank asset and liability management with implications for the transmission of monetary policy. *RAND Journal of Economics* 29, 466–486.
- Stiglitz, J., Greenwald, B., 2003. *Towards a New Paradigm in Monetary Economics*. Cambridge University Press.
- Stiglitz, J.E., 2009. The current economic crisis and lessons for economic theory. *Eastern Economic Journal* 35, 281–296.
- Stiglitz, J.E., Gallegati, M., 2011. Heterogeneous Interacting Agent Models for Understanding Monetary Economies. *Eastern Economic Journal*, Vol. 37, Issue 1, pp. 6-12 .
- Stiglitz, J.E., Weiss, A., 1981. Credit rationing in markets with imperfect information. *American Economic Review* 71, 393–410. Available at <http://ideas.repec.org/a/aea/aecrev/v71y1981i3p393-410.html>.
- Weidlich, W., 2000. *Sociodynamics: A Systematic Approach to Mathematical Modelling in the Social Sciences*. Gordon and Breach: London.

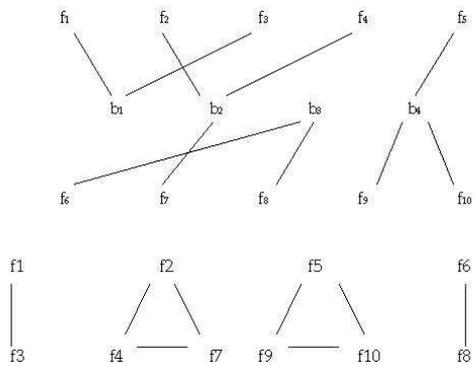


Figure 1: A bipartite network of banks (b_k) and firms (f_h) and its projection into a firms-firms network made of cliques..

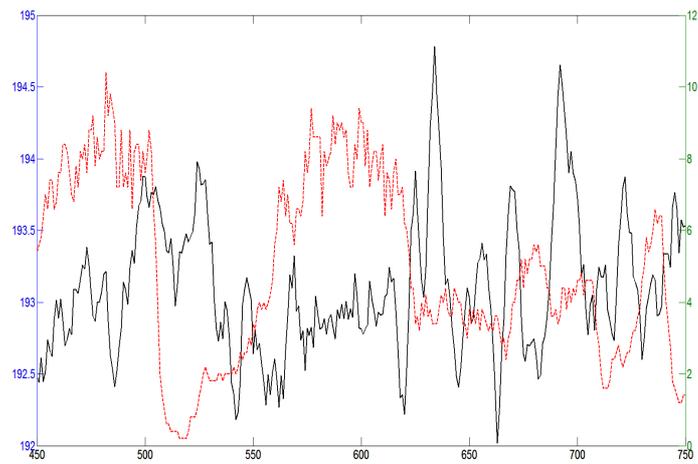


Figure 2: Aggregate production (solid line, left scale) and bankruptcies of banks (dashed line, right scale). Single simulation.

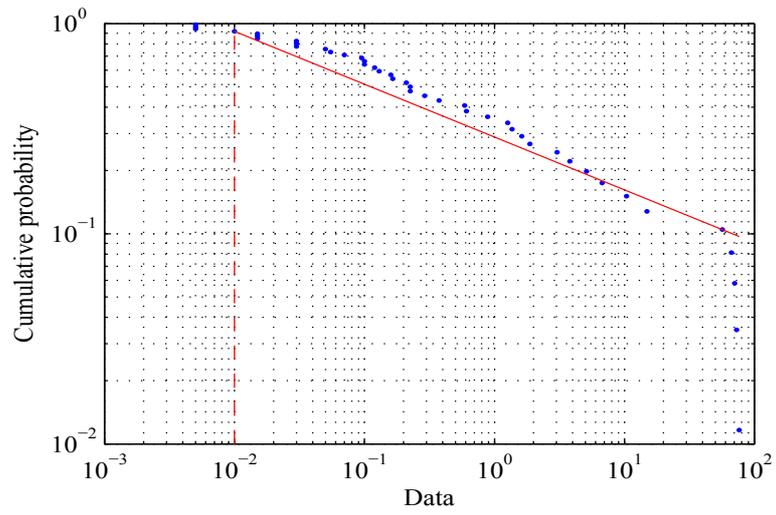


Figure 3: Degree distribution for simulations and power law fit.

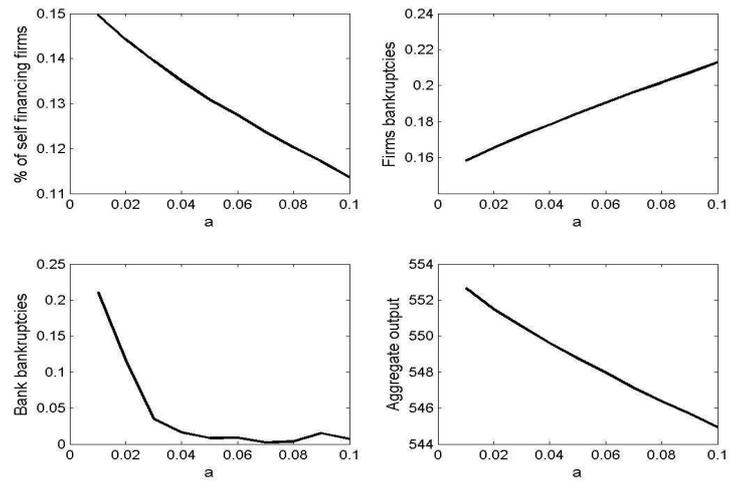


Figure 4: Average percentage of self financing firms, rates of firms and banks bankruptcies and level of aggregate output for different values of a .

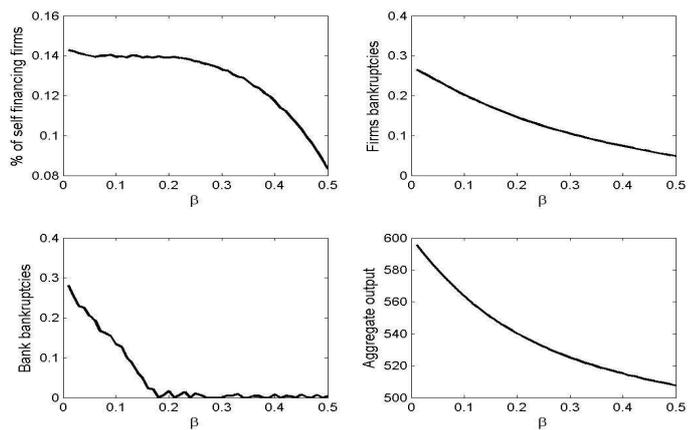


Figure 5: Average percentage of self financing firms, rates of firms and banks bankruptcies and level of aggregate output for different values of β .

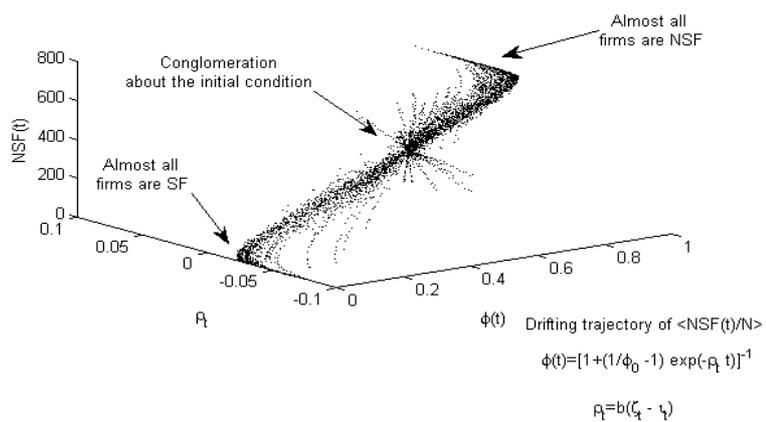


Figure 6: Dynamics of the number of NSF as a function of the net rate ρ .

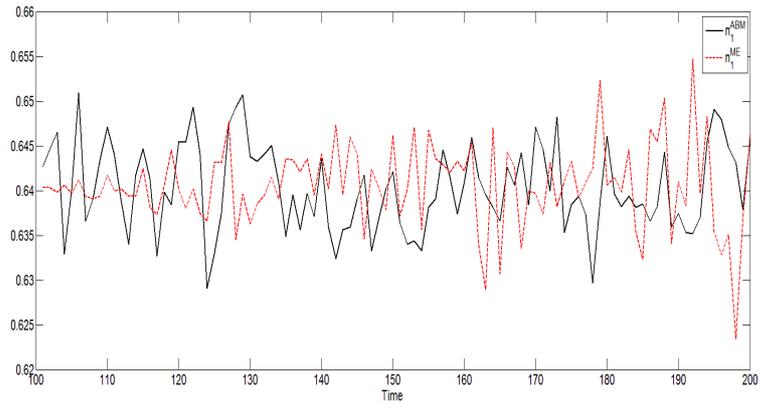


Figure 7: Share of non self financing firms for ABM and ME solution simulations.

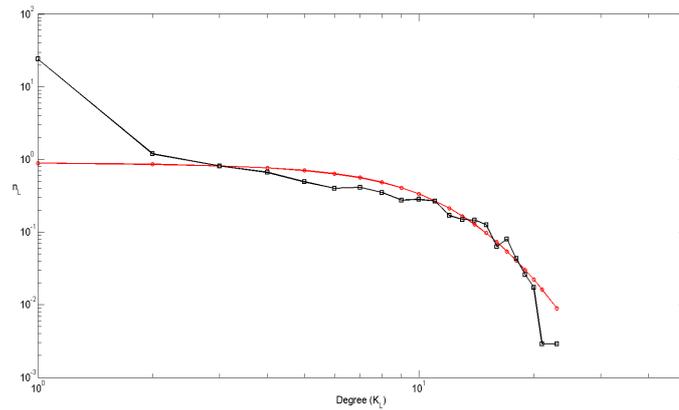


Figure 8: Degree distribution of ABM simulation (black line) and ME solution (red line).