Piketty meets Pasinetti: On public investment and intelligent machinery

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Abstract

Wealth inequality is currently rising in rich countries. Expectations that ever more intelligent machines might replace people's jobs amplify the concerns about an increasingly unequal wealth distribution. We examine how capital tax-financed public investment affects the distribution of wealth when the substitutability between capital and labor changes. We consider a setting with a labor-augmenting public capital stock and distinct wealth cohorts: dynastic savers and life-cycle savers. We prove that for every elasticity of substitution greater than a threshold, there exists a capital tax rate at which the dynastic savers disappear in the long term. For every elasticity below that threshold, there exists a capital tax rate at which the life-cycle savers disappear. Below these capital tax rates, both types of households co-exist in equilibrium. Finally, we elaborate on how these results depend on the role of public investment in production.

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1 Introduction

Wealth inequality is currently rising in rich countries. We examine how funding public investment and changes in the substitutability between capital and labor influence the wealth distribution. Recent technological advances that make machines more "intelligent" amplify concerns about a continuing increase in the wealth concentration in the top income percentiles (Piketty and Saez, 2014). These concerns are reminiscent of the Ricardian pessimism "on machinery" (Ricardo, 1821). Yet, the implications of the development of intelligent machines for wealth inequality are presently unclear. They comprise both a potential rise in the capital intensity of the economy as well as an enhanced substitutability between capital and labor. At the same time, it is widely recognized that public investment, notably in education and health, is underfunded, while there is no consensus about how such investments are to be financed. Piketty (2014) hence suggested that wealth inequality can be reduced by taxing capital and also makes the case for increasing for public investment (see also Stiglitz, 2016a). In this article, we examine the policy proposal to decrease wealth inequality by taxing capital and using the proceeds to finance public investment in the context of changes in the relative importance of capital as a production factor.

Our main contribution is to show that the success of this policy recommendation depends entirely on the elasticity of substitution between capital and labor. We show that wealth inequality decreases when a capital tax is introduced to finance public investment if substitution possibilities between capital and labor are high. This is arguably the case when capital consists of intelligent machines. In contrast, we prove that wealth inequality rises under this policy if substitution possibilities are low.

Stiglitz (2016b) argues that there is a new set of stylized facts regarding growth and distribution to be explained by macroeconomic modelling. These stylized facts include an increase in the wealth-income ratio, growing wealth disparity and a wealth distribution more skewed than the labor income distribution. Taken together, these stylized facts seem to replace the Kaldor facts that were integral in the development of the neoclassical growth model. These new facts cannot be entirely explained by conventional neoclassical models (Stiglitz, 2016b). Instead, one key component models need to account for are heterogeneous preferences, especially with respect to saving behavior (Stiglitz, 2016b). Indeed, much empirical evidence (Attanasio, 1994; Browning and Lusardi, 1996; Dynan et al., 2004; Saez and Zucman, 2016) suggests that individuals at the top of the wealth distribution display a saving behavior markedly different from the remainder of the population.

We thus analyze the proposal to combat increasing wealth inequality by taxing private capital and investing the proceeds in public capital in a two-class model. In our model, advances in automation are modeled as an increase in the relative importance of capital in production, either through an increase in capital intensity or in the substitutability between capital and labor. Households are distinguished through their income source, time preference rate and their saving behavior: "Workers" receive income from labor and capital and save for life-cycle purposes. "Capitalists", the top wealth owners, receive only capital income and have a dynastic saving motive. This type of model is originally associated with Pasinetti (1962) and has been taken up by Stiglitz (1967, 1969). More recently Baranzini (1991), Klenert et al. (2016), Mattauch et al. (2016), Michl (2009) and Stiglitz (2016b) analyzed the version presented below, in which workers also save, in a life-cycle fashion, thus accounting for the importance of retirement savings. In particular, Mattauch et al. (2016) proved that capital tax-financed public investment is Pareto-improving for low tax rates.

The present article characterizes all possible distributional outcomes that can result from introducing capital tax-financed public investment in an otherwise unregulated economy. We thus generalize and unify claims about the distributional outcome of two-class models with public capital made in Klenert et al. (2016), Mattauch et al. (2016) and Stiglitz (2016b). We prove that, depending on the value of the elasticity of substitution between capital and labor σ and on the level of the capital tax τ, three cases can occur. Either both classes co-exist ("Pasinetti-state"), or the capitalists ("Anti-Pasinetti-state") or the workers ("Anti-Anti-Pasinetti-state") disappear.

Specifically, the main result is that for any given elasticity of substitution between capital and labor σ there exists a capital tax rate τ_{lim} such that either workers or capitalists vanish. We prove that there exists a threshold elasticity σ_1 such that for any elasticity $\sigma \geq \sigma_1$, there exists a capital tax rate τ_{lim} at which the economy switches from a Pasinetti to an Anti-Pasinetti state. If $\sigma < \sigma_1$ there exists a capital tax rate τ_{lim} at which the economy switches from a Pasinetti to an Anti-Anti-Pasinetti state. In all cases the relationship between the elasticity of substitution and τ_{lim} is monotone: for the switch to the Anti-Pasinetti regime, the higher the elasticity, the lower the tax rate at which capitalists vanish. For the switch to the Anti-Anti-Pasinetti regime, the lower the elasticity, the lower the tax rate at which workers vanish.

Capital taxation as a means of reducing wealth inequality thus works if intelligent machines imply a high substitutability between capital and labor. The intuition behind our main result is that for high elasticities, capital taxes that finance labor-enhancing public investment influence the relative importance of capital and labor: workers earn more and can afford to save more. For low elasticities, labor-enhancing public investment implies that labor becomes cheap relative to capital, which suppresses wages and decreases workers' savings. Further, a higher capital intensity increases wealth inequality.

We also show that the results depend on the functional form in which public investment enters production. In the above, public investment is assumed to be labor-enhancing, such as investment in education would be. If instead it is akin to state capital, that is public and private capital are substitutes, we find that the Anti-Anti-Pasinetti regime can never occur.

Our article synthesizes two strands of literature. First, it has already been pointed out that a high income share of capital as a result of a higher capital-output ratio (Piketty, 2014) depends on the value of the elasticity of substitution between labor and capital σ (Rognlie, 2014). For example, Chirinko (2008) show that 26 out of 31 studies find an elasticity of substitution between capital and labor below 1. Rognlie (2014) thus concludes that more capital will rather mean a diminished return to capital. By contrast, Piketty and Saez (2014) and Piketty and Zucman (2015) argue, based on the data on changes in the capital-output ratio collected in Piketty and Zucman (2014), that the elasticity must be higher than 1. For standard parameter choices we find a threshold elasticity σ_1 between 0.8 and 0.9 below, indicating that both the Anti-Pasinetti and the Anti-Anti-Pasinetti case are conceivable. In particular, Piketty and Saez (2014) argue that "it makes sense to assume that σ tends to rise over the development process, as there are more diverse uses and forms for capital and more possibilities to substitute capital for labor (e.g., replacing delivery workers by drones or self-driving trucks)." Our contribution is the first to systematically analyze how the success of *capital tax-financed public investment* for combating inequality depends on σ. If one sees the development of intelligent machines as rising the elasticity close to or above 1, our contribution shows that capital-tax financed public investment emerges as the adequate recipe to keep wealth inequality in check and labor income sufficiently high.[1]

Second, previous work on two-class models usually focused on one or two, but never on all three steady-state outcomes of one, two or both classes existing. Samuelson and Modigliani (1966) were the first to describe the conditions under which an Anti-Pasinetti outcome can occur in a neoclassical framework – however, they did not relate their finding to capital tax-financed public investment and elasticities of substitution. Taylor (2014) and Zamparelli (2015), inspired by Piketty (2014), analyze the emergence of an Anti-Anti-Pasinetti case in a classical setting. Mattauch et al. (2016) and Klenert et al. (2016) focus mainly on the Pasinetti case and identify tax rates at which public investment constitutes a Pareto improvement. To our knowledge, our contribution is the first to identify all possible distributional outcomes of a two-class model with capital tax-financed public investment. The Anti-Anti-Pasinetti case, $\sigma < \sigma_1$, is particularly relevant for understanding dystopian visions about intelligent machines. In a world in which

[1] When the idea of intelligent machines is understood as capital-enhancing technological progress, it can only be captured as a transitory phenomenon, not a steady-state outcome, in view of Uzawa's Theorem. Our approach is differs, treating innovation in tasks that machines can execute as an enhancement of the *substitution possibility* between capital and labor.

labor is not productive any more, the only possibility for redistribution is to tax the income of the capitalists who own the majority of the machines. In previous work (Mattauch et al., 2016), we showed that Pareto improvements are only possible if the tax revenues are recycled through public investment, but that this financing option does not influence the wealth distribution. Here, we qualify this view, demonstrating that the way the investment enters production does indeed matter for wealth inequality.

The remainder of this article is structured as follows: In Section 2, we briefly lay out the model. In Section 3, we analyze the steady-state properties for a general production function. In Section 4, we derive the main results with a CES production function in which public capital is labor-enhancing. In Section 5, we consider the alternative specification of public capital being a substitute for private capital. Section 6 concludes with policy implications.

2 Model

We model a one-good economy in which the government can finance productivity-enhancing public investment by a capital tax. The population consists of two types of households, "capitalists" and "workers". The workers are modeled as a representative overlapping generations (OLG) agent that lives for two periods. It provides labor in the first period and saves for its own retirement in the second period, but leave no bequests to future generations. The capitalists are modeled as a representative infinitely-lived agent (ILA) that saves dynastically. Its source of income are interest payments on their capital holdings and potentially firms' profits. Both types of agents derive utility from consumption only. Factor markets clear and on the capital market, the supply consists of both agents' capital holdings. There are always decreasing returns to scale in private and public capital so that the economy converges to a steady state.

Capitalists The ILA owns a capital stock K_t^c and maximizes intertemporal utility given by

$$\sum_{t=0}^{\infty} \frac{1}{(1+\rho_c)^t} \ln(C_t^c), \tag{1}$$

with consumption C_t^c and time preference rate ρ_c. Its budget constraint is

$$K_{t+1}^c - K_t^c = (1-\tau) r_t K_t^c - C_t^c + \Pi_t, \tag{2}$$

where r_t is the interest rate and τ is the capital tax. Firms' profits Π_t may be zero, depending on the production structure. The initial capital stock is given as $K_1^c = K_0^c$. The ILA respects a transversality condition: $\lim_{t\to\infty} \left(K_t^c \prod_{s=1}^{t-1} \frac{1}{1+r_s} \right) \geq 0$.

Solving the maximization problem yields an Euler equation for this household:

$$\frac{C^c_{t+1}}{C^c_t} = \frac{1 + (1-\tau)r_{t+1}}{1+\rho_c}. \tag{3}$$

Workers The OLG agent lives for two periods, a 'young' (y) and an 'old' (o) stage. It maximizes its lifetime utility, with utility from consumption in the second period being discounted by the time preference rate ρ_w:

$$\ln(C^y_t) + \frac{1}{1+\rho_w}\ln(C^o_{t+1}). \tag{4}$$

In the first period, the agent rents its fixed labor L to the producing firm, which in turn pays a wage rate w_t. Labor income can either be consumed or saved for the old age:

$$w_t L = S_t + C^y_t. \tag{5}$$

In the second period the agent consumes its savings and the interest on them:

$$C^o_{t+1} = (1 + (1-\tau)r_{t+1})S_t. \tag{6}$$

Solving the optimization problem subject to the budget constraints leads to an Euler equation for this household:

$$\frac{C^o_{t+1}}{C^y_t} = \frac{1 + (1-\tau)\cdot r_{t+1}}{1+\rho_w}. \tag{7}$$

From Equations (5-7) an explicit expression for saving can be derived:

$$S_t = \frac{1}{2+\rho_w} w_t L. \tag{8}$$

This implies a constant savings rate of $1/(2+\rho_w)$, as is standard in discrete OLG models when the utility function is logarithmic.

Production Consider a production sector given by the production function $F(P_t, K_t, L)$ which fulfills the Inada conditions. Throughout we assume constant returns to scale in all three factors ($F(P_t, K_t, L) = F_P P + F_K K + F_L L$). This implies that constant returns in *accumulable* factors K and P are excluded, so that the economy always converges to a steady-state.

K_t denotes the sum of the individual capital stocks

$$K_t = K^c_t + S_{t-1}. \tag{9}$$

By Equation (8), this also implies

$$K^c_t = K_t + \frac{1}{2+\rho_w} w_{t-1} L. \tag{10}$$

Profit maximization yields the standard rates of return for capital and labor (with δ_K denoting depreciation of private capital):

$$r_t + \delta_K = \frac{\partial F(P_t, K_t, L)}{\partial K_t} \tag{11}$$

$$w_t = \frac{\partial F(P_t, K_t, L)}{\partial L} \tag{12}$$

Government The sole function of the government in this model is the provision of public capital. It finances its investments by the capital tax, thus influencing the interest rate. Hence the government's activity is summarized as the change in the stock of public capital (with δ_P denoting its depreciation):

$$P_{t+1} = (1 - \delta_P)P_t + \tau r_t K_t. \tag{13}$$

2.1 Rents from public capital

Given that the existence of productive public capital implies the presence of rents, there are a number of ways to close the model, which lead to different economic interpretations. Here we discuss three of them.

Rents appropriated and redistributed by the government First, the conceptually easiest case is that firms make profits equivalent to the returns on public capital:

$$\Pi_t = \frac{\partial F(P_t, K_t, L)}{\partial P} P_t \tag{14}$$

In Equation (2) we assumed that capitalists appropriate the profits, for example as shareholders of the firms. Alternatively, one may think of the government as appropriating the rent and redistributing the returns to the capitalists – and additionally to young and old workers.

Suppose young and old workers also receive a share of profits Π_1 and Π_2. One then obtains a modified savings behavior of the workers in Equations (5) and (6) above, since the profits enter as an additional source of income. In particular, Equation (8) is modified to

$$S_t = \frac{1}{2 + \rho_w}(w_t L + \Pi_1 - \frac{(1 + \rho)}{1 + (1 - \tau)r_{t+1}}\Pi_2). \tag{8a}$$

However, as this case precludes a characterization of the ratio of workers wealth to total wealth below, we focus on the case of the profits being appropriated only by the capitalists in the following sections.

Public investment as education Second, if one thinks of public capital as investment into education, it is natural to specify more structure on the production function. For this case, assume instead that workers' productivity is a constant-returns-to-scale sub-production function J

$$J(G_t, L) = LJ(G_t/L, 1), \tag{15}$$

a function of the labor supply and education expenditures. Total production is then given by $F(K, J)$ and is constant-returns to scale in K and J. With this modification it is natural to define the wage income of workers as

$$w_t = \frac{\partial F(K_t, J_t)}{\partial J}. \tag{12a}$$

We employ this version of the model in Section 4 below. Profits in Equation (2) are set to zero as a consequence.

Firms keep the rents Third, an opposite case is that firms themselves keep the rents from public capital and they earn it through a modified interest rate given by

$$r_t' + \delta_K = \frac{\partial F(P_t, K_t, L)}{\partial K_t} + \frac{\partial F(P_t, K_t, L)}{\partial P_t} \frac{P_t}{K_t}. \tag{16}$$

The idea behind the formulation of the model with a modified interest rate is that the share of each firm is proportional to their capital stock. As a benchmark case, we neglect the complications of entry, which would be beyond scope of the present analysis, and conceive of the rents as equally divided by a finite number of identical firms. Again, profits in Equation (2) are set to zero as a consequence.

3 Steady state for general production functions

The basic model is solved for the steady state for general production functions, assuming capitalists appropriate the profits.[2] Steady-state values of variables are denoted by a tilde.

First, consider the first version of the model discussed in the previous subsection, the case of governments redistributing profits to capitalists that includes Equation (14). It follows from the capitalist's Euler Equation (3) that the steady-state interest rate $\widetilde{r}$ is given by

$$\widetilde{r} = \frac{\rho_c}{(1-\tau)}. \tag{17}$$

This entails that in our model a form of the Pasinetti (1962) Theorem occurs: the steady-state interest rate is solely determined by the capitalists'

[2] For existence and stability see Mattauch et al. (2016).

time preference rate; by contrast, workers' saving propensity determines the distribution of capital between both classes.

The steady-state level of public capital is given by:

$$\widetilde{P} = \frac{1}{\delta_P}\tau\widetilde{r}\widetilde{K}. \tag{18}$$

The share of workers' wealth for a general production function can be determined by dividing Equation (8) by total capital:

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{1}{2+\rho_w}\frac{wL}{K}. \tag{19}$$

As we assumed constant returns to scale in all factors of production, this translates to:

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{1}{2+\rho_w}\Big(\frac{\widetilde{Y}}{\widetilde{K}} - F_P(\widetilde{P},\widetilde{K},L)\frac{\rho_c\tau}{\delta_p(1-\tau)} - \frac{\rho_c}{(1-\tau)} - \delta_K\Big). \tag{20}$$

with $F_P(.) = \partial F/\partial P(.)$ as usual.

Equations (10), (17) and (18) determine the allocation in the economy. However, as Equation (10) yields an expression for K^c in terms of K and P using (12), it suffices to consider (17) and (18) only.

Equations (17) and (18) thus define two equations in only two variables and can be solved for K and P, fixing the tax rate τ. By elementary real analysis ("Extreme Value Theorem"), the solution in $\widetilde{K}$ and $\widetilde{P}$ will always yield a maximum (and minimum) value for $\widetilde{S}/\widetilde{K}$ as a function of τ. This remains true as long as $\tau \in [0+\epsilon, 1-\epsilon]$ for $\epsilon > 0$ small, because of the continuity of the functions involved. The result establishes that a specific capital tax rate will be optimal from the point of view of redistributing as much wealth as possible.

We next discuss changes to the analysis just made for the other two formulations to close the model discussed in Subsection 2.1.

First, consider the case of public capital as education that solely benefits workers' income given by Equations ($12a$) and (15), but without Equation (14). In that case, Equation (20) simplifies to

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{1}{2+\rho_w}\Big(\frac{\widetilde{Y}}{\widetilde{K}} - \frac{\rho_c}{(1-\tau)} - \delta_K\Big). \tag{21}$$

Second, for the case of a modified interest rate, Equation (17) changes to

$$\widetilde{r'} = \frac{\rho_c}{(1-\tau)}. \tag{22}$$

implying Equation (18) changes to

$$\widetilde{P} = \frac{1}{\delta_P}\tau\widetilde{r'}\widetilde{K}. \tag{23}$$

Finally, the ratio of workers' to total wealth is also given by Equation (21).

The general result about the existence of maximum and minimum value for $\widetilde{S}/\widetilde{K}$ as a function of τ remains valid for both these cases.

However, the general solution in no case allows to determine whether the maximum and minimum values of the wealth ratio are such that both classes actually co-exist. Instead, in order to determine whether there are tax rates for which one class vanishes and what they depend on, a parametrization of the production function is needed, as in the next two sections.

4 Labor-enhancing public investment

This section investigates how the elasticity of substitution between capital and labor determines whether labor-enhancing public capital leads to a reduction in wealth inequality. In particular, we think of public investment as investment in education and thus consider the version of the general model in which workers appropriate the return to their modified productivity. Investment in public capital is financed through a tax on private capital. We think of the elasticity as the degree to which machines can replace people's jobs. We classify all possible long-term outcomes, that is whether one or both classes exist.

We proceed as follows: We first prove that for a given elasticity of substitution between capital and labor σ with $\sigma \geq 1$, there exists a capital tax rate τ_{lim} at which the economy switches from a Pasinetti to an Anti-Pasinetti state (Proposition 1). We then show that, if $\sigma < 1$, there exists a capital tax rate τ_{lim} at which the economy either switches from a Pasinetti to an Anti-Pasinetti or to an Anti-Anti-Pasinetti state (Proposition 2). Further, there exists a threshold value $\sigma_1 < 1$ such that for $\sigma \geq \sigma_1$ the switch is to an Anti-Pasinetti state, while for $\sigma \leq \sigma_1$, the switch is to an Anti-Anti-Pasinetti state (Corollary 3). For all cases the relationship between the elasticity of substitution and τ_{lim} is monotone. The results are displayed graphically in Figure 1.

As in this section public investment is assumed to be in education, we consider the version of the general model in which workers appropriate their return to their modified productivity, that is wages are given by Equation ($12a$). Further, we assume an explicit production function as given by the following equation, in which public capital P_t is labor-enhancing and which fulfills condition (15):

$$F(P_t, K_t, L) = \left(\alpha K^{\gamma} + (1-\alpha)(A(P_t)L^{(1-\beta)})^{\gamma}\right)^{\frac{1}{\gamma}} \tag{24}$$

with $A_t(P_t) = P_t^{\beta}$, and $0 < \alpha, \beta < 1$. So β denotes the efficiency factor of public capital P_t. Throughout, we assume $\alpha + \beta < 1$ to exclude the case of long-run or explosive growth. Also assume $\gamma < 1, \gamma \neq 0$. The elasticity of substitution between capital and labor σ is given by $\sigma = 1/(1-\gamma)$.

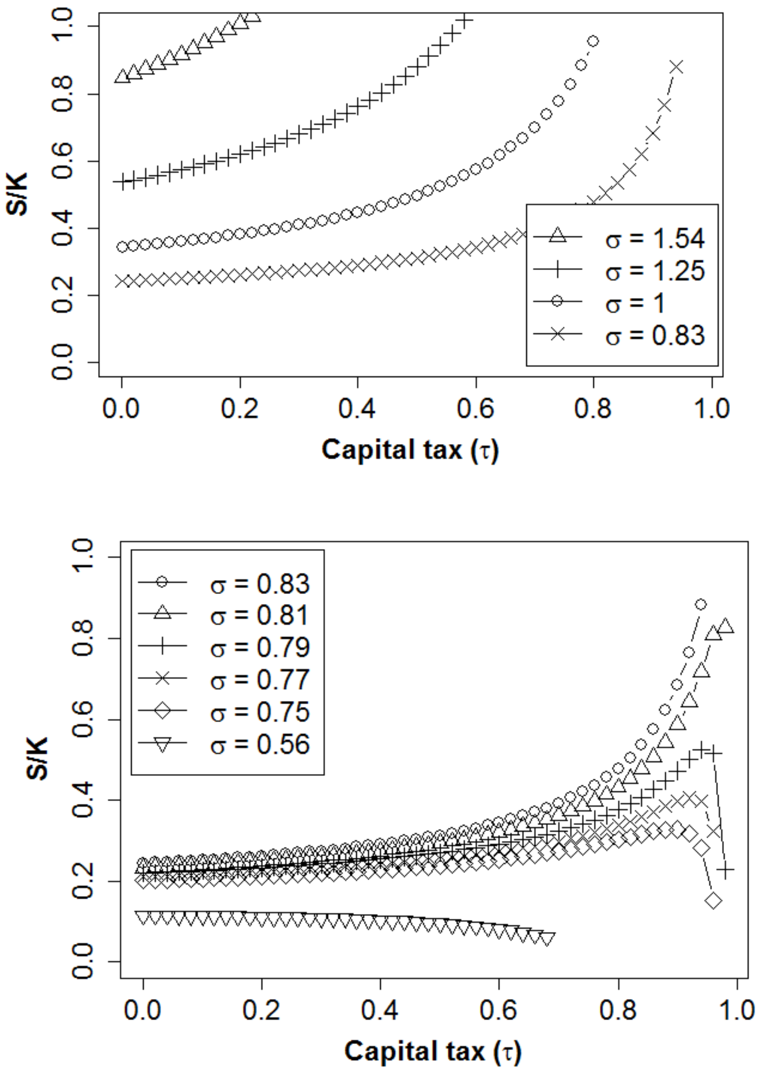

Figure 1: Wealth inequality as a function of the capital tax rate for various elasticities as an illustration of Proposition 1, 2 and Corollary 3. The upper panel shows selected cases of high elasticities (as in Proposition 1), the lower panel shows cases of low elasticities that illustrate the cases in Proposition 2 and Corollary 3. The plots are generated by using a calibration of the model given in Appendix D. Figure adapted from Mattauch et al. (2016).

In view of Equation ($12a$), note that here $J_t = A(P_t)L^{(1-\beta)}$.
We derive in Appendix A that

$$\frac{\widetilde{S}}{\widetilde{K}}(\tau;\gamma) = \frac{(1-\alpha)}{\alpha(2+\rho_w)}\Big(\frac{\rho_c}{1-\tau}+\delta_k\Big)\Big(\frac{1}{(1-\alpha)}\big((\frac{1}{\alpha}(\frac{\rho_c}{1-\tau}+\delta_k))^{\frac{\gamma}{1-\gamma}}-\alpha\big)\Big). \quad (25)$$

Note that one can show by straightforward computation that this expression is monotonically decreasing in α. This means that wealth inequality increases with higher capital intensity, as is to be expected. We hence focus on substitution elasticities. Note further that the expression is robust in the sense that if instead of wages as the improved marginal profit of labor, one would assume that firm's make profits that get appropriated by the capitalists, the expression would change by a factor of $(1-\beta)$.

We assume throughout this section that we are in an economically meaningful state in which both classes co-exist for a capital tax of zero (Pasinetti-case).

Assumption 1. *(a) For a capital tax of nearly zero both agents co-exist. This implies that* $0 < \widetilde{S}/\widetilde{K}(\epsilon,\gamma) < 1$ *for* $\epsilon > 0$ *small.*

(b) We further assume that $\delta_K > \alpha$.

Both of these assumption hold for the economically relevant range of the parameters used in our model by a very large margin, given the time horizon is 30 years. (For the content of Propostions 1 and 2 for Part (b) the weaker claim $\rho_c + \delta_K > \alpha$ would suffice.)

First, consider the case that $\gamma > 0$, that is, the substitution elasticity between capital and labor is greater than 1. Note that Assumption 1 (b) implies that $\rho_c + \delta_K > \alpha^{1/\gamma}$.

Proposition 1. *Let production be specified as above and assume* $\gamma > 0$.

(a) For every γ*, there exists a capital tax rate* $\tau_{\lim}$*, such that capitalists vanish (Anti-Pasinetti case).*

(b) This relationship is monotone: the higher the value of γ*, the lower the tax rate extinguishing capitalists.*

Proof. The idea of the proof is to show that $\widetilde{S}/\widetilde{K}(\tau;\gamma)$ is monotonically increasing in τ and γ for $\tau,\gamma \in (0,1)$, keeping the other quantity fixed.

To prove part (a), it can be shown using Equation (25) that

$$\lim_{\tau\to 1}\frac{\widetilde{S}}{\widetilde{K}}(\tau,\gamma) = \infty$$

This follows from the algebra of limits, noting

$$\lim_{\tau\to 1}\big(\frac{\rho_c}{1-\tau}+\delta_k\big) = \infty$$

and
$$\lim_{\tau\to 1}\left(\left(\frac{1}{\alpha}\left(\frac{\rho_c}{1-\tau}+\delta_k\right)\right)^{\frac{\gamma}{1-\gamma}}-\alpha\right)=\infty$$

$\widetilde{S}/\widetilde{K}(\tau;\gamma)$ is continuous in $\tau\in(0,1)$. So, by the intermediate value theorem, for all values of $\gamma>0$, there exists a capital tax τ_{lim} at which capitalists vanish, that is $\widetilde{S}/\widetilde{K}(\tau_{\text{lim}},\gamma)=1$.

Now consider monotonicity with respect to τ : The function $1/(1-\tau)$ is monotonically increasing with respect to τ on [0,1). As products and sums and positive exponentials of monotone functions are monotone, it follows from Equation (25) that $\widetilde{S}/\widetilde{K}(\tau;\gamma)$ is monotone in τ on $(0,1)$. So the value of τ_{lim} is unique for every γ.

To prove part (b), that is, the existence of different, monotonically decreasing tax rates τ_{lim} for increasing substitution elasticities, it remains to prove that $\widetilde{S}/\widetilde{K}(\tau;\gamma)$ is monotonically increasing in γ. This holds as $\gamma/(1-\gamma)$ is a monotonically increasing function for $\gamma<1$. Inserting this function as the exponential of a constant basis and the addition of multiplicative and additive constants, as in Equation (25), yields a monotonically increasing function. (Note that the relationship is monotonically increasing because of the condition $\delta_K>\alpha$, so that the basis of the exponent is greater than 1 for all $\tau\in(0,1)$.) This implies that for any fixed τ, the greater γ, the higher the value of $\widetilde{S}/\widetilde{K}$ and thus monotonically decreasing tax rates τ_{lim}.

□

Now consider the case $\gamma<0$, that is, the substitution elasticity between capital and labor is smaller than 1. Note that Assumption 1 now implies that $\rho_c+\delta_K<\alpha^{1/\gamma}$, since $\gamma<0$.

Proposition 2. *Let production be specified as above and assume $\gamma<0$. Assumption 1 is still given.*

(a) For every γ, there exists a capital tax rate, such that either capitalists vanish (Anti-Pasinetti case) or workers vanish (Anti-Anti-Pasinetti case).

(b) In both cases, the relationship is monotone: For the Anti-Pasinetti case, the higher the elasticity, the lower the tax rate that extinguishes capitalists. For the Anti-Anti-Pasinetti case, the lower the elasticity, the lower the tax rate that extinguishes workers.

Proof. The idea of the proof is to realize that for $\gamma<0$ with $|\gamma|$ small, the function $\widetilde{S}/\widetilde{K}(\tau;\gamma)$ has a unique maximum that may or may not be greater than 1 depending on parameter choices.

To prove part (a), first note that

$$\lim_{\tau\to 1}\frac{\widetilde{S}}{\widetilde{K}}(\tau;\gamma)=-\infty.$$

This again follows from the algebra of limits, noting

$$\lim_{\tau\to 1}\left(\frac{\rho_c}{1-\tau}+\delta_k\right)=\infty$$

and

$$\lim_{\tau\to 1}\left(\left(\frac{1}{\alpha}(\frac{\rho_c}{1-\tau}+\delta_k)\right)^{\frac{\gamma}{1-\gamma}}-\alpha\right)=-\alpha.$$

By straightforward computation, it can be shown that $\widetilde{S}/\widetilde{K}(\tau)$ has a unique maximum in $\tau\in(0,1)$ for

$$\alpha(\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}}>\delta_K. \tag{26}$$

Else it is monotonically decreasing in (0,1) (see Appendix B).

First consider the case that a unique maximum exists. If the value of this maximum is below 1 (or outside of the range (0,1)), the Anti-Anti-Pasinetti case occurs, by the intermediate value theorem, as $\widetilde{S}/\widetilde{K}(\tau)$ is continuous in $\tau\in(0,1)$. If instead the value of this maximum is above 1 and it is in the range $(0,1)$, the Anti-Pasinetti case occurs. However, if condition (26) is not fulfilled, the Anti-Anti-Pasinetti case occurs.

To prove part (b), note that the proof of monotonicity of $\widetilde{S}/\widetilde{K}(\tau;\gamma)$ in γ above (in the proof of Proposition 1) does not depend on γ being positive. $\gamma/(1-\gamma)$ is still a monotonically increasing function for $\gamma<0$, given Assumption 1. Part (b) of the proposition then follows from the monotonicity of $\widetilde{S}/\widetilde{K}(\tau;\gamma)$ in γ. □

The following corollary gives more detail on the conditions under which the Anti- and the Anti-Anti-Pasinetti case occur for $\gamma<0$.

Corollary 3. *Let production be specified as above and assume $\gamma<0$. Assumption 1 is still given.*

(a) There exists $\gamma_1<0$, such that: If $\gamma>\gamma_1$, for every γ, there exists a capital tax rate, such that capitalists vanish (Anti-Pasinetti case). If $\gamma<\gamma_1$, for every γ, there exists a tax rate such that workers vanish (Anti-Anti-Pasinetti case).

(b) In both cases, the relationship is monotone: For the Anti-Pasinetti case, the higher the elasticity, the lower the tax rate extinguishing capitalists. For the Anti-Anti-Pasinetti case, the lower the elasticity, the lower the tax that extinguishes workers.

Proof. It is established in Appendix B that $\widetilde{S}/\widetilde{K}(\tau)$ has a unique maximum for $\alpha(\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}}>\delta_K$. (If this condition is not fulfilled, which is the case for $|\gamma|$ large, the function has a minimum, but it occurs for $\tau>1$, so that it can be shown to be decreasing within $\tau \in (0,1)$. See Appendix B for details.) The value of this maximum is given by

$$\frac{\widetilde{S}}{\widetilde{K}}(\tau_z) = -\frac{\alpha\gamma}{(2+\rho_w)}\big(\alpha(1-\gamma)\big)^{\frac{1-\gamma}{\gamma}}. \tag{27}$$

Consider this value as a function of γ :

$$f(\gamma) = -\frac{\alpha\gamma}{(2+\rho_w)}\big(\alpha(1-\gamma)\big)^{\frac{1-\gamma}{\gamma}}. \tag{28}$$

The corollary is shown by proving the following properties:

1. $f(\gamma)$ has a unique minimum with respect to γ at $\gamma = \ln(\alpha)$. It is monotonically increasing with respect to γ for $\gamma > \ln(\alpha)$.

2. $\lim_{\gamma\to 0+} f(\gamma) = +\infty$.

Assumption 1 implies that $f(\ln(\alpha)) < 1$ because one can deduce that there exists a γ with $\alpha(\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}} > \delta_K$. such that $\tau_z = 0$.[3]

The corollary is then deduced from the two properties in the following way: by the intermediate value theorem, a value γ_1 exists, such that $f(\gamma_1) = 1$, since $f(\gamma)$ is continuous. This implies that for $\gamma < \gamma_1$, $f(\gamma_1)$ is lower than one and hence the Anti-Anti-Pasinetti case occurs. If $\gamma > \gamma_1$, then $f(\gamma_1)$ is greater than one and the economy is in an Anti-Pasinetti state.

We now complete the proof by showing the two properties. Regarding the first property, note that $f(\gamma)$ can be rewritten as

$$f(\gamma) = -\gamma\alpha^{1/\gamma}\Big(\frac{1}{(2+\rho_w)}\big(1-\gamma\big)^{\frac{1-\gamma}{\gamma}}\Big) \tag{29}$$

Let $g(\gamma) = -\gamma\alpha^{1/\gamma}$ and $h(\gamma) = 1/(2+\rho_w)\big((1-\gamma)\big)^{\frac{1-\gamma}{\gamma}}$. $h(\gamma)$ is monotonically increasing for all $\gamma > 0$, as is obtained from the fact that the function x^x is monotonically increasing. Further, it can be calculated that

$$\frac{\mathrm{d}g}{\mathrm{d}\gamma} = \alpha^{1/\gamma}\left(\frac{1}{\gamma}\ln(\alpha)\right). \tag{30}$$

This derivative equals zero for $\gamma = \ln(\alpha)$ and is positive for $\gamma > \ln(\alpha)$ and negative for $\gamma < \ln(\alpha)$. Since $f(\gamma)$ is the product of function g, which has a minimum at $\gamma = \ln(\alpha)$ and the monotonically increasing function h, it also has a minimum at $\gamma = \ln(\alpha)$. From this also follows that $f(\gamma)$ is monotonically increasing for $\gamma > \ln(\alpha)$.

Regarding the second property, factor $f(\gamma)$ into

[3] Condition $f(\ln(\alpha)) < 1$ is true if $-\frac{\alpha\ln(\alpha)}{(2+\rho_w)}\big(\alpha(1-\ln(\alpha))\big)^{\frac{1-\ln(\alpha)}{\ln(\alpha)}} < 1$, a condition that is satisfied by our standard parametrization (see Appendix D) by a large margin.

$$f(\gamma) = \big(-\gamma\alpha^{1/\gamma} \big) \cdot \frac{1}{(1-\gamma)(2+\rho_w)} \cdot \big(1-\gamma\big)^{\frac{1}{\gamma}}. \tag{31}$$

Taking limits with respect to $\gamma \to 0$ from below, the second factor of this product tends to $1/(2+\rho_w)$. Note the third factor is equivalent to $\exp(1/x \ln(1-x))$. Applying L'Hôpital's rule to its exponent yields that this factor tends to e^{-1}.

It remains to consider the first term, $-\gamma\alpha^{1/\gamma}$. Substituting $\gamma = -1/y$ and applying L'Hôpital's rule to $(1/\alpha)^y/y$ as $y \to +\infty$ shows that this term tends to $+\infty$. This establishes the behavior at $\gamma = 0$ from below and completes the proof of Part (a).

Finally, note that Part (b) would not follow if it were the case that $\ln(\alpha) > \gamma_{\text{crit}}$ with γ_{crit} given by $\alpha(\alpha(1-\gamma_{\text{crit}}))^{\frac{1-\gamma_{\text{crit}}}{\gamma_{\text{crit}}}} = \delta_K$. In fact, it would violate the monotonicity of $\widetilde{S}/\widetilde{K}(\tau)$ throughout. Appendix B shows why this cannot occur.

□

The previous results imply that for *fixed* τ, the workers' wealth share increases in γ. This is a consequence of the Pasinetti property of the model, as the interest rate remains fixed by the capitalists' time preference rate even if the elasticity between capital and labor is changed. By contrast, the capitalists' wealth share increases in α, the capital intensity (see discussion below Equation 25). Taken together, this means that changes in the capital structure of the economy would have ambiguous effects on wealth inequality. However, a detailed analysis is beyond scope here, as the focus of the article is on understanding in which situations the policy proposal of capital tax-financed public investment is effective, not on the impact of automation *per se*.

5 Public investment as a substitute for private capital

In this section we analyze the findings from Section 4 for their robustness. We consider an alternative way in which public investment might act on the economy: public capital as an imperfect substitute for private capital, as in the case of state-owned companies. We show in Subsection 5.1 that, for a nested CES production structure in which public and private capital are combined into a generic capital stock, the Anti-Anti-Pasinetti case cannot occur (Proposition 4). However, for an elasticity of substitution between the two capital stocks bigger than one, there is a capital tax rate τ_{lim} at which the economy switches from the Pasinetti to the Anti-Pasinetti state (Proposition 5). In Subsection 5.2, we elaborate on the special case of perfect substitutability between private and public capital.

5.1 Nested CES production function

In this subsection we analyze a production function of the nested CES type, instead of a single CES function with labor-enhancing public capital as in Section 4. In this case firms generate profits and we assume that they are appropriated by the capitalists (i.e. the first case described in Subsection 2.1).

Public and private capital G and K are combined into generic capital H_t by means of a CES function:

$$H_t(K_t, P_t) = (\epsilon K_t^\eta + (1-\epsilon)P_t^\eta)^{(1/\eta)}, \tag{32}$$

with $0 < \epsilon < 1$ being the share parameter of private capital and $s = 1/(1-\eta)$ being the elasticity of substitution between private and public capital with $-\infty < \eta \leq 1$. Generic capital and labor are then combined in a Cobb-Douglas function to produce the final good Y:

$$Y_t = F_t(H_t, L) = H_t^\alpha L^{(1-\alpha)}, \tag{33}$$

where $0 < \alpha < 1$ is the output elasticity of generic capital.

This specification of the production function has the property that for a tax rate of 0 the economy does not collapse: $F(0, K_t, L) > 0$.

The first-order conditions of the firm are:

$$w_t = \frac{\partial F_t(H_t, L)}{\partial L} = (1-\alpha)\frac{Y_t}{L},$$

and

$$r_t + \delta_K = \frac{\partial F_t(H_t(K_t, P_t), L)}{\partial K_t} = \alpha\epsilon Y_t H_t^{(-\eta)} K_t^{(\eta-1)}.$$

The equation above can be rearranged to obtain an explicit expression for Y:

$$Y_t = \frac{(r_t + \delta_K)}{\alpha\epsilon} H_t^{(\eta)} K_t^{(1-\eta)}. \tag{34}$$

As above, the S/K ratio is calculated by dividing Equation (8) by total private capital K:

$$\frac{S_t}{K_t} = \frac{1}{2+\rho_w}\frac{w_t L}{K_t}.$$

Eliminating w_t by means of Equation (12) yields:

$$\frac{S_t}{K_t} = \frac{(1-\alpha)}{2+\rho_w}\frac{Y}{K_t},$$

which can be further simplified by using the expression for Y derived in Equation (34):

$$\begin{aligned}\frac{S_t}{K_t} &= \frac{(1-\alpha)}{(2+\rho_w)}\frac{(r_t+\delta_K)}{\alpha\epsilon} H_t^{(\eta)} K_t^{(-\eta)} \\ &= \frac{(r_t+\delta_K)(1-\alpha)}{\alpha\epsilon(2+\rho_w)}\left(\epsilon + (1-\epsilon)(P_t/K_t)^\eta\right).\end{aligned}$$

Finally, by inserting the expression for the steady-state level of public capital derived in Equation (18), the steady-state capital ratio $\widetilde{S}/\widetilde{K}$ becomes:

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{(\widetilde{r}+\delta_K)(1-\alpha)}{\alpha\epsilon(2+\rho_w)}\left(\epsilon+(1-\epsilon)\left(\frac{\tau\widetilde{r}}{\delta_P}\right)^{\eta}\right). \tag{35}$$

From this expression, one can deduce the following results:

Proposition 4. *With a nested CES production structure as assumed in Equations* (32) *and* (33) *and for* $1 \geq \eta > 0$, *the Anti-Anti-Pasinetti state cannot occur.*

Proposition 5. *With a nested CES production structure as assumed in Equations* (32) *and* (33), *for every* $1 \geq \eta > 0$ *there exists a capital tax rate* τ_{lim} *from which on the the economy switches from a Pasinetti to an Anti-Pasinetti state.*

Two caveats to the significance of these propositions matter. First, the case $\eta < 0$, that is substitution elasticity $s < 1$, is not treated. The reason is that one can show that for small tax rates, capitalists vanish because the limit of S/K tends to infinity as the tax rate approaches 0. This is not a surprising finding: The assumption that private and public capital are highly complementary implies that, for low taxes, the value of private capital is strongly diminished and capitalists income is decreased. However, as this setting only considers good substitutability between capital and *labor,* this increases wages and explains how for low tax rates the Anti-Pasinetti case can reappear.

This leads to the second caveat. It concerns the assumption of an elasticity of 1 between capital and labor (as in Equation 33) above. In view of the results in Section 4, it is to be expected that the Anti-Anti-Pasinetti case does not occur for good substitutability between capital and labor (see also Mattauch et al. 2016). By contrast, one can expect that, if the two CES functions of Sections 5 and 4 would be combined, that the Anti-Anti-Pasinetti case would reappear for low elasticities of substitution between capital and labor.

In proving the propositions, we assume that Assumption 1 still holds, i.e. that $0 < \widetilde{S}/\widetilde{K}(0) < 1$, which is the case for the meaningful parameter range.

Proof of Propostion 4. This can be inferred directly from Equation (35): since $\widetilde{r}$, δ_K, δ_P, α, ϵ, ρ_w are greater than zero, and $0 \leq \tau \leq 1$, the expression for $\widetilde{S}/\widetilde{K}$ is always strictly positive. □

Proof of Proposition 5. The idea of the proof is to show that $\widetilde{S}/\widetilde{K}(\tau)$ is monotonically increasing in τ, starting from a value lower than one and converging to infinity for $\tau \to 1$. The proof proceeds in two steps:

1. we show that $\lim_{\tau\to 1} \widetilde{S}/\widetilde{K}(\tau) = \infty$.
2. we show that $\widetilde{S}/\widetilde{K}(\tau)$ is monotonically increasing in $0 \leq \tau < 1$.

Regarding the first step, we insert the explicit expression for $\widetilde{r} = \rho_c/(1-\tau)$ and expand the products in Equation (35). This yields the following expression:

$$\begin{aligned}\frac{\widetilde{S}}{\widetilde{K}} &= \frac{(1-\alpha)}{\alpha\epsilon(2+\rho_w)}\left[\left(\frac{\rho_c}{1-\tau}+\delta_K\right)\left(\epsilon+(1-\epsilon)\left(\frac{\tau\rho_c}{(1-\tau)\delta_P}\right)^{\eta}\right)\right]\\ &= \frac{(1-\alpha)}{\alpha\epsilon(2+\rho_w)}\left[a\epsilon+(1-\epsilon)\left(\left(\frac{\rho_c^{1+\eta}}{\delta_P^{\eta}}b\right)+\frac{\delta_K}{\delta_P^{\eta}}c\right)\right],\end{aligned}$$

with

$$\lambda(\tau) = \left(\frac{\rho_c}{1-\tau}+\delta_K\right),$$

$$\mu(\tau) = \frac{\tau^{\eta}}{(1-\tau)^{(1+\eta)}}$$

and

$$\nu(\tau) = \left(\frac{\tau\rho_c}{(1-\tau)}\right)^{\eta}.$$

It can be inferred from these equations directly that for $\tau \in (0,1)$

$$\lim_{\tau\to 1}\lambda(\tau) = \lim_{\tau\to 1}\mu(\tau) = \lim_{\tau\to 1}\nu(\tau) = \infty,$$

which implies that $\lim_{\tau\to 1} \widetilde{S}/\widetilde{K}(\tau) = \infty$.

Regarding the second step, it remains to show that $\widetilde{S}/\widetilde{K}(\tau)$ is monotonically increasing for all $\tau \in (0,1)$.

Since we only consider $\eta > 0$, that is, the case of elasticities between public and private capital greater than or equal to one, this is straightforward to show: $\widetilde{S}/\widetilde{K}(\tau)$ is the sum of the functions $1/(1-\tau)$, $\tau^{\eta}/(1-\tau)^{(1+\eta)}$ and $(\tau/(1-\tau))^{\eta}$, multiplied by positive constants. All these functions are monotonically increasing for $\eta > 0$. This implies that the function $\widetilde{S}/\widetilde{K}(\tau)$ is monotonically increasing.

Since we assume that $0 < \widetilde{S}/\widetilde{K}(0,\gamma) < 1$ and we showed that $\widetilde{S}/\widetilde{K}(\tau)$ is monotonically increasing in τ and $\lim_{\tau\to 1} \widetilde{S}/\widetilde{K}(\tau) = \infty$, it follows directly from the intermediate value theorem that for a given $0 < \eta < 1$, there exists a $\tau_{lim} \in (0,1)$, with $\widetilde{S}/\widetilde{K}(\tau_{lim}) = 1$. For this τ_{lim} the Pasinetti regime changes into an Anti-Pasinetti regime. □

5.2 The case of perfect substitutability between private and public capital

We next consider the special case of a perfect elasticity of substitution between public and private capital. While Proposition 5 covers this case, the value of the Anti-Pasinetti tax rate can be calculated explicitly here, which we prove next.

The calculation of the previous subsection (5.1) cover this case with the parameter values $\epsilon = 0.5$ and $\eta = 1$. It yields an S/K ratio of:

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{(\widetilde{r} + \delta_K)(1-\alpha)}{\alpha(2+\rho_w)} \left(1 + \left(\frac{\tau\widetilde{r}}{\delta_P}\right)\right). \tag{36}$$

Proposition 6. *For the case of a perfect elasticity of substitution between public and private capital, there exists a capital tax rate τ_{lim} at which the Pasinetti regime changes to the Anti-Pasinetti regime. This tax rate is given by that value of $\tau_{lim}^{1,2}$ which is in the economically meaningful range of* $(0, 1)$*:*

$$\tau_{lim}^{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{37}$$

with

$$a = \frac{1}{x} - \delta_k + \rho_c \frac{\delta_K}{\delta_P},$$

$$b = \rho_c \left(1 - \frac{\delta_K}{\delta_P} - \frac{\rho_c}{\delta_P}\right) + 2\left(\delta_k - \frac{1}{x}\right),$$

$$c = \frac{1}{x} - \delta_k - \rho_c$$

and $x = \frac{(1-\alpha)}{\alpha(2+\rho_w)}$.

Proof. Straightforward computation of the solution to a quadratic equation, see Appendix C. □

For the standard parametrization (see Appendix D) the economically meaningful tax rate at which the economy switches from a Pasinetti to an Anti-Pasinetti state is 54 %.

If we set $\delta_K = \delta_P = \delta$ in Equation (37), the expression for τ_{lim} can be simplified to:

$$\tau_{\mathrm{lim}}^{1,2} = \frac{\frac{\rho_c^2}{\delta} - 2(\delta - \frac{1}{x}) \pm \frac{\rho_c}{\delta}\sqrt{1 + 4\frac{\delta}{x}}}{2\frac{1}{x} - \delta + \rho_c}. \tag{38}$$

Equation (38) permits to study the dependency of the critical tax rate on parameters. For example, one finds that it increases monotonically in the pure time preference rate of the capitalists, while it decreases monotonically in the workers' time preference rate. Treating τ_{lim} as a function of each parameter, while keeping the other parameters fixed at their standard value, shows that the sensitivity to changes in the capitalists time preference rate is much stronger than the sensitivity to changes in the workers' time preference rate (details available upon request).

6 Conclusion and Outlook

Heterogeneous income sources for different cohorts and the development of ever more intelligent machines are two reasons why there is currently great concern that wealth inequality might continue to rise. This article is the first to consider these two reasons in a common framework. It studies the role of the substitution elasticity σ between capital and labor for the impact of public investment on the wealth distribution. A two-class model with dynastically saving top earners and life-cycle savers is introduced to prove the following results. There exists a threshold elasticity $\sigma_1 < 1$ such that: For any elasticity of substitution greater than σ_1, there exists a positive capital tax rate at which dynastic savers disappear. For an elasticity that is smaller than σ_1, there exists a positive capital tax rate at which life-cycle savers disappear. These relationships are monotone: for $\sigma > \sigma_1$, the higher the elasticity, the lower the tax rate at which capitalists cease to exist. For $\sigma < \sigma_1$, the lower the elasticity, the lower the tax rate at which workers cease to exist.

Our results have one major policy implication: we demonstrate that in a world with high substitution elasticities between capital and labor, capital tax-financed public investment is an effective strategy to combat wealth inequality. This is a major policy recommendation resulting from the discussion around Piketty (2014) (see also Stiglitz 2016a). However, it has so far not been assessed in the context of increasingly "intelligent" machines, as one can suppose that many who believe that artificial intelligence is a danger to job security would have in mind an increase in the substitution elasticity and capital intensity.

Conversely, our results can be seen as a note of caution against this policy recommendation if there is poor substitutability between capital and labor. While empirically there is disagreement about the value of the substitution elasticity (Chirinko, 2008; Piketty and Zucman, 2015), we find that for standard parameter values the threshold between these cases is around 0.85, indicating that both cases are conceivable. Clarification in future research is thus needed on the value of the elasticity and about whether developments in artificial intelligence raise it significantly (Rognlie, 2014).

Two extensions to our framework are conceivable: first, considering the case of the government as an inefficient provider of public capital, one could derive how the long-term equilibria depend on the conversion of tax revenue into public investment. Second, one could consider the case in which public investment is both labor- and capital-enhancing, but potentially to different degrees. This case would be reminiscent of factor-biased technological change. A large body of literature on this topic has elaborated on the role of innovation possibility curves, implying that there is a trade-off between labor- and capital-enhancing technological progress (Kennedy, 1964; Samuelson, 1965; Acemoglu, 2010). However, this literature has to date not been linked to analyses of wealth inequality by class models (Stiglitz, 2014).

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Appendices

A Derivation of $\widetilde{S}/\widetilde{K}$ in Section 4

In this section we derive an explicit formula for the capital share of the workers $\widetilde{S}/\widetilde{K}$ (Equation 25).

We start by dividing the expression for the OLG agent's saving (Equation 8) by total capital and then insert the firm's first-order conditions (Equations 11 and 12):

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{L\widetilde{w}}{(2+\rho_w)\widetilde{K}} = \frac{(1-\alpha)\widetilde{A}^{\gamma}L^{\gamma}}{(2+\rho_w)\widetilde{K}\widetilde{Y}^{-(1-\gamma)}} = \frac{(1-\alpha)}{\alpha(2+\rho_w)}\Big(\frac{\rho_c}{1-\tau}+\delta_k\Big)\Big(\frac{L\widetilde{A}}{\widetilde{K}}\Big)^{\gamma} \tag{A.1}$$

Here we used that

$$w_t = \frac{\partial F(K_t, J_t)}{\partial J}, \tag{A.2}$$

with $J_t = A(P_t)L^{(1-\beta)}$, see Equation ($12a$).

To find an explicit solution for expression (A.1), solve the model for $\widetilde{K}/(\widetilde{A}L)$. For this purpose, let $k = K/(AL)$, and let $y = Y/(AL)$. Then

$$y = \big(\alpha(k)^{\gamma} + (1-\alpha)1^{\gamma}\big)^{\frac{1}{\gamma}}. \tag{A.3}$$

From standard growth theory, we know that for any constant-returns-to-scale function
$r_t + \delta_k = \widetilde{Y}_K = \widetilde{y}'(k)$,
so that

$$\widetilde{y}'(k) = \frac{\rho_c}{1-\tau} + \delta_k. \tag{A.4}$$

To solve this, use that

$$\widetilde{y}'(\widetilde{k}) = \alpha\widetilde{k}^{\gamma-1}\big(\alpha\widetilde{k}^{\gamma} + (1-\alpha)\big)^{\frac{1-\gamma}{\gamma}}. \tag{A.5}$$

Substituting this into Equation (A.4) gives

$$\big(\frac{1}{\alpha}(\frac{\rho_c}{1-\tau} + \delta_k)\big)^{\frac{\gamma}{1-\gamma}} = \widetilde{k}^{-\gamma}(\alpha\widetilde{k}^{\gamma} + (1-\alpha)) = \alpha + (1-\alpha)\widetilde{k}^{-\gamma}. \tag{A.6}$$

This is an equation that can be solved for $\widetilde{k}$,[4] as it is equivalent to

$$\frac{\widetilde{K}}{\widetilde{A}L} = \widetilde{k} = \Big(\frac{1}{(1-\alpha)}\big((\frac{1}{\alpha}(\frac{\rho_c}{1-\tau} + \delta_k))^{\frac{\gamma}{1-\gamma}} - \alpha\big)\Big)^{\frac{-1}{\gamma}} \tag{A.7}$$

This expression can be substituted into Equation (A.1) to obtain an explicit solution for the capital ratio.

$$\frac{\widetilde{S}}{\widetilde{K}} = \frac{(1-\alpha)}{\alpha(2+\rho_w)}\big(\frac{\rho_c}{1-\tau} + \delta_k\big)\Big(\frac{1}{(1-\alpha)}\big((\frac{1}{\alpha}(\frac{\rho_c}{1-\tau} + \delta_k))^{\frac{\gamma}{1-\gamma}} - \alpha\big)\Big) \tag{A.8}$$

Inserting $\gamma = 0$ (that is, reducing to the Cobb-Douglas case), one recovers the case of Mattauch et al. (2016).

B Properties of $\widetilde{S}/\widetilde{K}$ in Section 4

Next we determine the sign and zero of the derivative of $\widetilde{S}/\widetilde{K}(\tau)$. For this purpose, let $x(\tau) = (\rho_c/(1-\tau) + \delta_K)$, and note that $x'(\tau) = \rho_c(1-\tau)^{-2}$.

Then:

$$\frac{\widetilde{S}}{\widetilde{K}}(\tau) = \frac{1}{\alpha(2+\rho_w)}\big((\frac{1}{\alpha})^{\frac{\gamma}{1-\gamma}}(x(\tau))^{\frac{1}{1-\gamma}} - \alpha x(\tau)\big) \tag{B.1}$$

[4]Evidently solutions to Equation (A.7) could be complex if the term inside the exponent is negative. This need not concern us: for the further economic analysis, only the term's appearance in Equation (A.8) eventually matters and it is shown to have an exponent equal to unity there.

Thus:

$$(\frac{\widetilde{S}}{\widetilde{K}})'(\tau) = \frac{1}{\alpha(2+\rho_w)(1-\gamma)}(\frac{x(\tau)}{\alpha})^{\frac{\gamma}{1-\gamma}}x'(\tau) - \frac{x'(\tau)}{(2+\rho_w)}$$
$$= (\frac{\rho_c}{(2+\rho_w)(1-\tau)^2})\Big(\frac{1}{\alpha(1-\gamma)}(\frac{1}{\alpha}(\frac{\rho_c}{(1-\tau)}+\delta_K))^{\frac{\gamma}{1-\gamma}} - 1\Big) \tag{B.2}$$

We now compute the zero of the derivative by setting the second term of the product to 0:

$$\frac{1}{\alpha(1-\gamma)}(\frac{1}{\alpha}(\frac{\rho_c}{(1-\tau)}+\delta_K))^{\frac{\gamma}{1-\gamma}} = 1 \tag{B.3}$$

This is equivalent to

$$(\frac{1}{\alpha}(\frac{\rho_c}{(1-\tau)}+\delta_K)) = (\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}} \tag{B.4}$$

and further equivalent to

$$\frac{\rho_c}{(1-\tau)} = \alpha(\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}} - \delta_K. \tag{B.5}$$

Therefore,

$$\tau_z = 1 - \frac{\rho_c}{\alpha(\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}} - \delta_K}. \tag{B.6}$$

Further, replacing the equalities by inequalities, one can determine the sign of the derivative. This is, in general, dependent on the value of all relevant parameters. However, for non-restrictive parameter conditions, its sign can be determined for the economically relevant cases as follows.

Consider the above four equations as inequalities: First, note that for values $\gamma < 0$ the direction of the inequality changes from Equation (B.3) to (B.4). Second, noting that $\tau \in (0,1)$, there is also a change in the direction of the inequality from Equation (B.5) to (B.6) if

$$\alpha(\alpha(1-\gamma))^{\frac{1-\gamma}{\gamma}} > \delta_K. \tag{B.7}$$

For $|\gamma|$ small, it can be verified that this inequality holds for $\gamma < 0$, but not for $\gamma > 0$, for a wide parameter range for α and δ_k around their standard values of 0.38 and 0.7, respectively. For part of this parameter range, it also holds for large values of $|\gamma|$. Taken together, this means that the derivative is positive for $\tau < \tau_z$ and negative for $\tau > \tau_z$. Thus τ_z is a local maximum. The only economically relevant case that differs is for $\gamma < 0$ and $|\gamma|$ large ($\gamma < -0.95$ for the standard parametrization): in this case τ_z is a local minimum. However, for this case, $\tau_z > 1$ and thus $\widetilde{S}/\widetilde{K}$ is decreasing on $\tau \in (0,1)$.

Further, it can be verified, by inserting τ_z into the function, that the value of the maximum is given by

$$\frac{\widetilde{S}}{\widetilde{K}}(\tau_z) = -\frac{\alpha\gamma_1}{(2+\rho_w)}\big(\alpha(1-\gamma_1)\big)^{\frac{1-\gamma_1}{\gamma_1}}. \tag{B.8}$$

We finally explain that given Assumption 1 it is always the case that $\ln(\alpha) < \gamma_{\text{crit}}$ as mentioned in the proof of Corollary 3. Recall that γ_{crit} is given by

$$\alpha(\alpha(1-\gamma_{\text{crit}}))^{\frac{1-\gamma_{\text{crit}}}{\gamma_{\text{crit}}}} = \delta_K. \tag{B.9}$$

Suppose for contradiction that $\ln(\alpha) > \gamma_{\text{crit}}$. Then by definition

$$\alpha(\alpha(1-\ln(\alpha)))^{\frac{1-\ln(\alpha)}{\ln(\alpha)}} > \delta_K. \tag{B.10}$$

Rearranging gives:

$$(1-\ln(\alpha))^{\frac{1}{\ln(\alpha)}} > \delta_K(1-\ln(\alpha)) \tag{B.11}$$

Noting that for $0 < \alpha < 1$, the right-hand side is bigger than δ_K and the left-hand side is smaller than α, one establishes a contradiction to Assumption 1 (b) in Section 4 of the main text.

C Proof of Proposition 6

To determine the capital tax rate τ_{lim} at which the regime changes from a Pasinetti to an Anti-Pasinetti state we set $S/K = 1$ in Equation (36):

$$1 = x\left(\frac{\rho_c}{(1-\tau_{\text{lim}})} + \delta_K\right)\left(1 + \left(\frac{\tau_{\text{lim}}}{\delta_P}\frac{\rho_c}{(1-\tau_{\text{lim}})}\right)\right). \tag{C.1}$$

Solving for τ yields the following quadratic equation:

$$(\tau_{\text{lim}})^2 \underbrace{\left[\frac{1}{x} - \delta_k + \rho_c\frac{\delta_K}{\delta_P}\right]}_{a} + \tau_{\text{lim}} \underbrace{\left[\rho_c\left(1 - \frac{\delta_K}{\delta_P} - \frac{\rho_c}{\delta_P}\right) + 2\left(\delta_k - \frac{1}{x}\right)\right]}_{b} + \underbrace{\left[\frac{1}{x} - \delta_k - \rho_c\right]}_{c} = 0 \tag{C.2}$$

D Standard calibration

The standard calibration of our model is summarized in Table 1.

We calibrated the time preference rates ρ such that for a capital tax of 21 %, which is the average capital tax rate in OECD countries between the years 1970 and 2000 (Carey and Rabesona, 2002), and an elasticity of substitution between capital and labor of one, the distribution of wealth is as in Wolff (2010): in the U.S. in 2007, 62 % of net worth are held by the top 5 % of the population and almost 38 % of net worth by the remaining 95 %.

The capital share of income α in the production function was chosen to be 0.38. This is in accordance with observations by the OECD, that in 26 OECD countries with reliable data available, the labor share of income was dropping from 66.1 % to 61.7 % from 1990 to 2009 (OECD, 2012). The productivity of public capital β, has been estimated to be between 0.08 and 0.19 (Bom and Ligthart, 2014), downwardly correcting higher estimates from earlier studies (Aschauer, 1989; Gramlich, 1994).

Labor L, the total working hours, is a fixed factor in our model. Its value scales all variables (for more details on this see Mattauch et al. 2016). We normalize labor $L = 100$ and measure the other variables in this unit to obtain values in a convenient range. Time is measured in steps of 30 years, as workers are assumed to live for two periods.

Parameter	Standard value	Corresponding annual value
ρ_c	0.56	1.5%
ρ_w	3.98	5.5%
δ_k	0.7	4%
δ_P	0.7	4%
α	0.38	–
β	0.2	–
L	100	–

Table 1: In the second and third column the standard values used in the simulation and the corresponding yearly values are given.

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