16

Samuelson and the Factor Bias of Technological Change: Toward a Unified Theory of Growth and Unemployment

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It is a great pleasure for me to be able to write this chapter in honor of Paul's ninetieth birthday. On such occasions, one's students traditionally write an essay inspired by one's work. Paul's long and prolific career—which continues almost unabated—makes this both an easy and a difficult task: easy, because on almost any subject one reflects upon, Paul has made seminal contributions; all of MIT's students—indeed, much of the economic profession for the past half century—has been simply elaborating on Paul's ideas. But by the same token, the task is difficult: there are so many of his ideas the elaboration of which remain on my research agenda, forty two years after leaving MIT, it is hard to make a choice.

Take, for instance, his development of the overlapping generations model, which has played such a central role in macroeconomics. Social security is one of the central issues facing American public policy, and his model remains the central model for analyzing theoretically the consequences of various proposals. Obviously, the results obtained in that

Joseph E. Stiglitz

model are markedly different from—and far more relevant than—those in an infinitely lived representative agent model.

Recently, I used the model in a quite different context,[1] to study the impact of capital market liberalization, one of the central issues under debate in the international arena. Again, the results are markedly different from those obtained in the perfect information, perfect capital markets, representative agent models, where liberalization allows a country facing a shock to smooth consumption: it helps stabilize the economy. The evidence, of course, was overwhelmingly that that was not the case, and using a variant of the overlapping generations model, one can understand why. Without capital market liberalization, a technology shock, say, to one generation is shared with succeeding generations, as savings increase, wages of successive generations increase, and interest rates fall (in response to the increased capital stock). But with capital market liberalization, the productivity shock may simply be translated into increased income in the period, and increased consumption of the lucky generation. By the same token, capital market liberalization exposes countries to external shocks from the global capital market. I had thought of using this occasion to elaborate on the life-cycle model in a rather different direction: a central feature of the standard life-cycle model and some of the subsequent elaborations (such as Diamond[2]) is the possibility of oversaving: if capital is the only store of value, then the demand for savings by households may be such that the equilibrium interest rate is *beyond* the golden rule, and the economy is dynamically inefficient. Introducing a life-cycle model with land, however, can have profound implications. Take the case, for instance, with no labor force growth and no technological change; being beyond the golden rule would imply a negative real interest rate, which would, in turn, mean an infinite value to land. Obviously, this cannot be an equilibrium. The problems of oversaving, on which so much intellectual energy was spent in the 1960s, simply cannot occur when there is land (and obviously, land does exist). Samuelson was the master of simple models that provided enormous insights, but the result shows the care that must be exercised in the use of such models: sometimes, small and realistic changes may change some of the central conclusions in important ways.

But I have chosen in this chapter to focus on another topic on which I remember so vividly Paul lecturing: endogenous technological change. Long—some two decades—before the subject of endogenous growth theory (which really focuses on growth theory where the rate of technological change is endogenous) became fashionable, Paul Samuelson, Hirofumi

Uzawa, and Ken Arrow and their students were actively engaged in analyzing growth models with endogenous technological progress, either as a result of learning by doing[3] or research.[4]

Of particular interest to Paul was the work of Kennedy[5] and Weizacker[6] (and others) on the *bias* of technological progress—whether it was labor or capital augmenting. Earlier, Kaldor[7] had set forth a set of stylized facts, one of which was the constancy of the capital output ratio. It was easy to show that that implied that technological change was labor augmenting. But what ensured that technological change was labor augmenting—if entrepreneurs had a choice between labor and capital augmentation? These authors had posited a trade-off between rates of capital and labor augmentation, and shown an equilibrium with pure labor augmentation.

Contemporaneously, economic historians, such as Salter[8] and Habakkuk,[9] had discussed economic growth arguing that it was a *shortage* of labor that motivated labor saving innovations, for example, in America. Of course, in standard neoclassical economics, there is no such thing as a shortage—demand equals supply. One might be tempted to say what they meant to say was "high" wages. But how do we know that wages are high? What does that even mean? Of course, with productivity increases, wages are high, but not relative to productivity.

Once we get out of the neoclassical paradigm, of course, markets may be characterized by "tightness" or "looseness." There can be unemployment. Firms may have a hard time finding employees. Moreover, if the unemployment rate is low, workers are more likely to leave, so firms face high turnover costs; what matters is not just the wage, but total labor costs.[10] Worse still, if the unemployment rate is low, workers may shirk—the penalty for getting caught is low.[11] Some economies are plagued by labor strife, again increasing the total cost of labor. One of the motivations for the model below was to try to capture (even if imperfectly) some aspects of this as affecting the *endogenous direction* of technological progress.

There is another motivation for this chapter. The early beginnings of growth theory derive from the basic model of Harrod and Domar, where there was a fixed capital-output ratio, a. With savings, s, a fixed fraction of output (income), Y,

$$I = sY = dK/dt \tag{16.1}$$

where K is the capital stock, so the rate of growth of capital is

$$d\ln K/dt = sY/K = s/a. \tag{16.2}$$

Moreover, as machines become more efficient, each machine requires less labor, so the number of jobs created goes up more slowly than the capital stock. If

$$L/Y = b \qquad (16.3)$$

is the labor requirement per unit output, then b/a is the labor required per unit capital, and job growth is

$$\mathrm{dln}\, L/\mathrm{d}t = s/a - \beta - \alpha \qquad (16.4)$$

where

$$-\mathrm{dln}\, b/\mathrm{d}t = \beta \qquad (16.5)$$

$$\mathrm{dln}\, a/\mathrm{d}t = \alpha \qquad (16.6)$$

s/a was sometimes referred to as the warranted rate of growth, what the system would support. Once technological change was incorporated, the warranted rate of growth is modified to $s/a - \beta - \alpha$.

By contrast, labor was assumed to grow at an exogenous rate, n. The problem was that, in general, n was not equal to $s/a - \beta - \alpha$.[12]

If (in the model without technological change), $s/a < n$, unemployment would grow continually; and if $s/a > n$, eventually the economy reached full employment—after which it would be profitable to invest only enough to keep full employment, an amount less than s/a.

The "dilemma" was resolved by Solow (1956), who proposed that the capital output ratio depended on the capital labor ratio, k: $a(k)$; and technological change was purely labor augmenting, so in equilibrium

$$s/a(k^\star) = n + \beta \qquad (16.7)$$

There is a capital labor ratio such that capital and *effective* labor (the demand for jobs and the supply of labor) grow precisely at the same rate.

The problem with Solow's "solution" is that it does away with the very concept of a job; alternatively, if there were ever a job shortage, simply by lowering the wage, more jobs would be created until the economy reached full employment. In developing countries, this means there is never a capital shortage; if there is unemployment, it must simply be

because wages are too high. By the same token, there is never "technological unemployment." Technology may reduce the demand for labor *at a particular wage*, but whatever technology does, wage adjustments can undo. In practice, of course, at least in the short run, there is not such flexibility.[13]

In this short note, we take seriously the notion of jobs (perhaps more seriously than the concept should be taken). Given today's technology and capital stock, wage adjustments will not lead to full employment. There is a maximum employment which they can support.

In the model here, it is the combination of changes in capital stock and technology which drive changes in employment. Wages make a difference, through their effects on technology (and possibly capital accumulation). In short, we construct a model where, over time, technological change leads to either increases or decreases in the capital output ratio, so that *eventually*

$$s/a^\star = n + \beta \qquad (16.8)$$

That is,

$$a = a^\star = s/(n+\beta). \qquad (16.9)$$

It is technological change that ensures that jobs grow at the same rate as the labor force.

The problem with standard versions of the fixed coefficients model (where, at any moment of time, a and b are fixed) is that the distribution of income is very fragile: if N is the supply of labor and L is the demand,

$$L = (L/Y)\,(Y/K)\,K = (b/a)\,K \qquad (16.10)$$

$$\text{If } (b/a)\,K < N, \quad \text{then } w = 0$$

$$\text{If } (b/a)\,K > N, \quad \text{then } r = 0$$

where $w =$ (real) wage, $r =$ (real) return on capital. If $(b/a)K = N$, the distribution of income is indeterminate.

Here, however, we present an alternative version, based on agency theory (Shapiro and Stiglitz, 1984). If workers are paid too low a wage, they prefer to shirk; there is the lowest wage which firms can pay at any unemployment rate to induce them not to shirk. That wage depends on the payment an unemployed worker receives. We write this as

$$w = f(v)w_{\min} \qquad (16.11)$$

Joseph E. Stiglitz

where v is the employment ratio,

$$v = L/N = (b/a)(K/N) \tag{16.12}$$

So

$$d\ln v/dt = s/a - n - \beta - \alpha \tag{16.13}$$

Finally, firms have a choice of innovations. Total cost of production per unit output is

$$c = ar + bw \tag{16.14}$$

The firm has a given technology today $\{a_0, b_0\}$. It can, however, decide on the nature of the technology by which it can produce next period (Figure 16.1).

Technology defines next year's feasibility locus. Taking for a moment r and w as given, the firm can reduce (next year's) cost by balancing out changes in a and b:

$$dc/dt = \alpha m - \beta(1 - m) \tag{16.15}$$

where m is the share of capital in costs, or the share of capital in income:
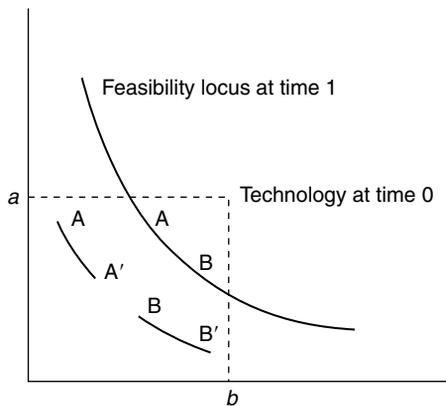
$$m = ar = rK/Y \tag{16.16}$$

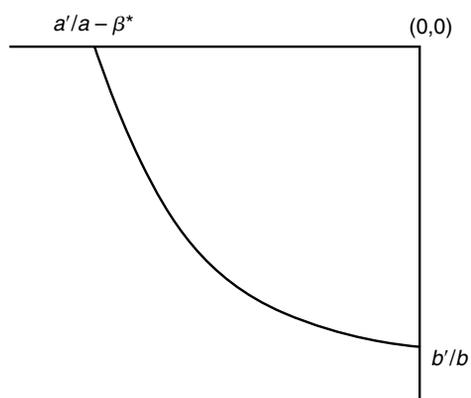**Figure 16.1** Technology feasibility locus.

Figure 16.2 Labor vs. capital augmentation.

Assume that there is a trade-off between labor and capital augmenting progress, so that

$$\beta = \beta(\alpha). \quad \beta' > 0 \quad \beta'' < 0 \tag{16.17}$$

depicted in Figure 16.2.

Then cost reductions are maximized when

$$\beta'(\alpha) = m/(1 - m) \tag{16.18}$$

16.1 Steady State Equilibrium

We can now fully describe the steady state equilibrium. In the long run, we have argued that a must converge to $a^\star$, which means that

$$\alpha^\star = 0, \tag{16.19}$$

which in turn means that

$$m^\star/(1 - m^\star) = \beta'(0) \equiv \beta^\star \tag{16.20}$$

or

$$r^\star = m^\star/a^\star. \tag{16.21}$$

Joseph E. Stiglitz

$a^\star$, in turn, solves

$$s/a^\star = n + \beta(\alpha^\star) \tag{16.22}$$

We can similarly solve for the wage (conditional on productivity):

$$w^\star b_0 = (1 - m^\star) \tag{16.23}$$

Finally, we can use this to solve for the equilibrium unemployment rate:

$$w^\star/w_{\min} = (w^\star b_0)/(b_0 w_{\min}) = (1 - m^\star)/(b_0 w_{\min}) = f(v^\star) \tag{16.24}$$

That is, once we set the unemployment compensation ($w_{\min}$) relative to labor market productivity, then we know what the unemployment rate is.

16.2 Heuristic Dynamics

In this model, there is a simple adjustment process. If the capital output ratio is too high, too few jobs will be created (given the savings rate) and unemployment grows. Growing unemployment means that wages will become depressed—in the story told here, firms can pay a lower wage without workers' shirking, but there are other stories (such as bargaining models) which yield much the same outcome. Lower wages mean, of course, that the return to capital is increased. As wages get depressed, and labor becomes easier to hire, and the return (cost) of capital increases, firms seek ways of economizing on capital, and pay less attention to economizing on labor. The new technologies that are developed are capital saving and labor using. The capital output ratio falls, and the labor output ratio increases. Given the savings rate, more jobs are created, and the unemployment rate starts to fall.

16.3 Formal Dynamics

The convergence to equilibrium, however, may neither be direct nor fast. Figure 16.3 depicts the phase diagram, in $\{a, v\}$ space.

The locus of points for which $da/dt = 0$ is a vertical line, given by

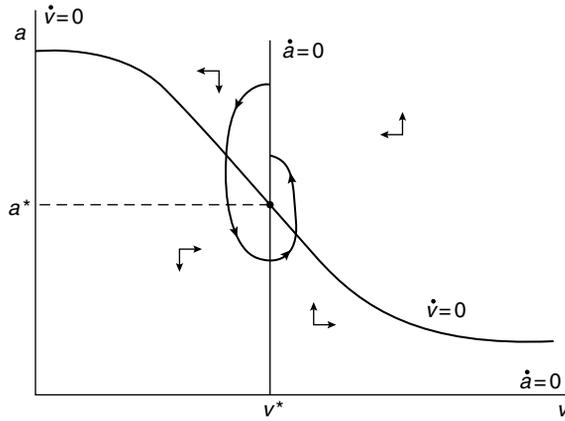$$\alpha = 0, \text{ that is,} \tag{16.25}$$

Figure 16.3

from Equations (16.24), (16.23) and (16.20), given $b_0 w_{min}$ (which we will take as set), there is a unique value of v^* for which $\alpha = 0$, that is, a is constant; and if

$$v > v^*, \quad \mathrm{d}a/\mathrm{d}t > 0$$

$$v < v^*, \quad \mathrm{d}a/\mathrm{d}t < 0$$

when employment is high, wages are high; firms economize on labor, and the capital output ratio increases, and conversely when employment is low.

The locus of points for which $\mathrm{d}v/\mathrm{d}t = 0$ is the negatively sloped curve defined by

$$a = s/(n + \beta + \alpha) \qquad (16.26)$$

and it is easy to show that below the curve, v is increasing, and above it, it is decreasing. We show a sample path converging through oscillations into the equilibrium.

In the appendix we provide sufficient conditions for this type of stability of the equilibrium.

16.4 Micro-economics

As noted earlier, at any point of time, the representative firm has a given technology, defined by $\{a, b\}$. Its choice is about its technology next period.

Joseph E. Stiglitz

Figure 16.1 showed the original point, and its opportunity set, the locus of points which (given the *technology of change*) it can achieve next period. This is, of course, the framework that Atkinson and I set forth in our 1969 paper.

The firm chooses the point on the feasibility locus that will minimize next period's cost, at expectations concerning next period's factor prices. Thus, what is relevant in the cost minimization described earlier are not current factor prices, but next period's factor prices; but in the continuous time formulation used here, there is no real difference. On the other hand, a firm today should be aware that its choices today affect its choice set tomorrow, that is, in Figure 16.1, if it chooses point A, its choice set tomorrow is the locus AA′, while if it chooses point B, its choice set tomorrow is the locus BB′. And of course, each of those subsequent choices are affected by wages prevailing then and in the future. Hence, in reality, what should matter for a firm is not just tomorrow's wage, but the entire wage profile. The full solution of this complicated dynamic programming problem is beyond the scope of this brief note. The steady state *equilibrium* which emerges is identical to that described here, though the dynamics are somewhat more complicated.

16.5 A Generalization

Earlier, we set forth the notion that what mattered in the choice of technology was not just factor prices, but the total cost of labor, including turnover costs, how hard it was to hire workers, etc. These variables too will, in general, be related to the unemployment rate, so that the point along the feasibility set chosen by the firm will depend not only on relative shares, which depend on v, but also on v directly:

$$\alpha = \Phi(m(v), v). \tag{16.27}$$

Equilibrium still requires that

$$\alpha = 0, \beta = \beta^\star = \beta(0) \tag{16.28}$$

and so $a = a^\star$. Indeed, once we set the policy variable ($b_0 w_{\min}$), the analysis is changed little, except now, using Equation (16.24), $m^\star$ is defined by

$$0 = \Phi(m, v) = \Phi(m^\star, f^{-1}(1 - m^\star)/(b_0 w_{\min})) \tag{16.29}$$

16.6 A Kaldorian Variant

Kaldor provided an alternative approach to reconciling the "warranted" and "natural" rate of growth, the disparity between $s/a - \beta - \alpha$, the rate at which jobs are created, and n, the rate at which the labor force grows. He suggested that the average savings rate depends on the distribution of income, and by changing the distribution of income, s can be brought into line. Thus, he posited (in simplified form) that none of the wages are saved, but a fraction s_p of profits, so

$$\mathrm{d}K/\mathrm{d}t = s_p r K \qquad (16.30)$$

Hence, we replace the differential equation for v, (16.13) with

$$\mathrm{d}\ln v/\mathrm{d}t = s_p r - n - \beta - \alpha \qquad (16.13')$$

$$= s_p(1/a - w(v)b/a) - n - \beta - \alpha$$

and the equilibrium Equation (16.8) defining a^* with

$$s_p r^* = n + \beta(0) \qquad (16.8')$$

defining the equilibrium interest rate. Before, given a, we used (16.21) to solve for r. Now, we use (16.21) to solve for a, given r:

$$a^* = m^*/r^* \qquad (16.21')$$

The dynamics are also modified only slightly. As (16.13′) makes clear, it is still the case that above the locus $\mathrm{d}v/\mathrm{d}t = 0$ (i.e. for higher values of a), the rate of growth of capital is lower (as before), so v (the employment rate) is falling; below the curve, it is rising. Hence, the qualitative dynamics remains unchanged.[14]

16.7 A Related Model

Some years ago, George Akerlof and I formulated a related model of the business cycle (another area in which Samuelson's contributions were seminal.)[15]

Real wages were portulated to depend positively on capital per worker, As here, an increase in capital accumulation led to increases in wages which reduced funds available for savings, which slowed growth and led to lower wages.[16] In that model, we again obtained oscillatory dynamic behavior.

16.8 Why it Matters: a Distinction with a Difference

At one level of analysis, the difference between this model and the standard Solow model is small. In the standard model, firms choose the *current* technology among a set of available technologies so that the capital output ratio adjusts and eventually the warranted and natural rate are equated. Here, firms choose *future technologies*, and again, eventually the warranted and natural rates are equated. In both models, at the microeconomic level, firms are choosing technologies in response to maximizing profits (minimizing costs), given factor prices.

There are, of course, important differences in dynamics: in the Solow model, convergence is monotonic. Here, the dynamics are far more complicated. Convergence may be oscillatory.

But there are some more profound differences, some of which relate to economic policy, to which I want to call attention. The first relates to the determination of the distribution of income and the choice of technique. In the Solow model, wages adjust *so that there is always full employment*. The choice of technique is, in effect, dictated by factor supplies. Though firms *choose* the technology to employ, factor prices always adjust so that the technology they choose is such that factors are fully employed. Thus, the distribution of income really plays no role—and in Solow's exposition, one could tell the entire dynamic story without reference to it, or without reference to firms "choosing" a technology. If there is unemployment, it is only because wages are too high and lowering wages would eliminate the unemployment (but increase growth only slightly and temporarily).

In the model here, the choice of (future) technology is central. Wages are not determined by marginal productivities, but by firms, as the lowest wage they can pay to avoid shirking on the part of workers. If minimum wages pushed wages above this level, they would result in increased unemployment; but for most workers, the minimum wage is set below that level so that lowering the minimum wage has little effect on wages actually paid, and hence on unemployment or growth. (An increase in unemployment compensation in this model does, however, increase the unemployment rate, by forcing firms to pay higher wages to avoid shirking.)

High wages do have an effect on unemployment, through the impact on the evolution of technology. This has two implications. First, it takes considerable time before any action to lower wages (even if it were successful) has any effect. The short-run effect on unemployment is nil.[17] Second, there are other ways by which the government could affect the evolution of the system and the creation of jobs. There are two ways by which this can be done in the medium run. First, policies which increase the national savings rate would be just as or more effective in increasing employment in the medium term. Second, marginal wage subsidies reduce the cost of labor, and it is the high cost of labor (at the margin) which induces firms to shift the direction of technological developments toward excessive labor savings and capital using technologies.

16.9 Concluding Remarks

For almost half a century, the Solow growth model, in which technological change was exogenous, has dominated discussions of growth theory. But almost half a century ago, Samuelson helped lay the foundations of an alternative approach to explaining the "stylized" facts of economic growth, based on endogenous technological change. What was needed, however, to close the model was a plausible theory of wage determination, which subsequent work in the economics of information (efficiency wage theory) has helped provide. By unifying these two disparate strands of literature, we have provided here a general theory of growth and employment which makes sense of discussions of technological unemployment or job shortages—concepts which have no meaning in Solow's formulation. We have suggested that the policy implications of this theory are markedly different from those arising from Solow's model.

It will be a long time before the fruit of the seeds which Paul sowed so many years ago are fully mature.

Notes

1. Stiglitz (2004)
2. Diamond (1965)
3. Arrow (1962a)
4. Here again, Arrow's (1962b) contribution was seminal. This is not the occasion to go into the large literature, except to mention Karl Shell's volume of essays (1967), Nordhaus' thesis (1969), and my own work with Tony Atkinson (1969).

   5. Kennedy (1964)

   6. Weizacker (1966)

   7. Kaldor (1961)

   8. Salter (1960)

   9. Habbakuk (1962)

10. Stiglitz (1974)

11. Shapiro and Stiglitz (1984)

12. Harrod and Domar's original analysis did not include technological change. This is a slight generalization of their analysis.

13. Standard models have formalized this in the notion of putty-clay models.

14. The stability conditions are of course changed. See the appendix for details.

15. The accelerator-multiplier model has gone out of fashion, partly because the assumption of fixed coefficients on which it relied has become unfashionable, partly because it was not based on rational expectations (which has become fashionable). But one can obtain much the same results from a model in which investment increases not because sales have increased, but because profits have increased. Stiglitz and Greenwald (1993) have explained both why capital (equity) market imperfections exist and how they can lead to such a *financial accelerator*.

16. That model differed in the wage determination function (we used a real-Phillips curve) and, as in the Solow model, wages determined current choice of technique, as opposed to here, where it affects the evolution of future technology. In some cases, we showed that the economy could be characterized by a limit cycle.

17. Early students of growth theory recognized that this would be true even within the neoclassical model; they focused on putty-clay models in which after investments have been made, the ability to change its characteristics (the labor required to work it) is very limited. Dynamics of growth in putty-clay models are markedly different from those of standard neoclassical models. See Cass and Stiglitz (1969). Unfortunately, the models were not easy to work with, and the distinction seems to have been lost in discussions of growth in recent decades.

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Appendix: Stability Conditions

In order to analyze stability, we simplify by writing

$$\mathrm{dln}\, a/\mathrm{d}t = \alpha(v); \quad \alpha' > 0$$

$$-\mathrm{dln}\, b/\mathrm{d}t = \beta(\alpha); \quad \beta' > 0$$

Joseph E. Stiglitz

The locus of points for which $dv/dt = 0$ is the negatively sloped curve defined by

$$a = s/(n + \beta(\alpha(v)) + \alpha(v)) \quad da/dv = -a^2 (\beta'\alpha' + \alpha')/s \qquad (16.27)$$

Below the $dv/dt = 0$ curve, v is increasing and above the curve, v is decreasing.

To evaluate the stability conditions of the pair of differential equations:

$$v' = v (s/a - n - \beta(\alpha(v)) - \alpha(v))$$

$$a' = a \, \alpha(v)$$

we look at the Jacobian evaluated at $\{v^*, a^*\}$ as follows:

$$J(v^*, a^*) = \left\{ \begin{array}{cc} -v^*(\beta'(0)\alpha'(v^*) + \alpha'(v^*)) & -v^*s/a^{*2} \\[2ex] a^*\alpha'(v^*) & 0 \end{array} \right\}$$

The conditions for the steady state to be a stable spiral (converging to equilibrium through oscillations) is:

$$-v^*(\beta'(0)\alpha'(v^*) + \alpha'(v^*)) < 0$$

which is always satisfied; and

$$[v^*(\beta'(0)\alpha'(v^*) + \alpha'(v^*))]^2 - 4(v^*s/a^{*2})a^*\alpha'(v^*) < 0$$

which can be simplified to

$$v^*\alpha'(1 + \beta')^2 < 4(n + \beta(0)).$$

Provided the limit as α goes to zero of $d\ln \alpha/d\ln v$ is finite, then the limit of the LHS of the above condition is always satisfied.[1]

16A.1 Kaldorian Variant

For the Kaldorian variant, the dv/dt equation is now:

$$v' = v (s_p r - n - \beta - \alpha) \text{ from Eq. (13') in the text}$$

$$= v (s_p(1/a - w(v)b/a) - n - \beta(\alpha(v)) - \alpha(v)) \text{ from the fact} \\ \text{that } Y = rK + wL$$

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[1] If $\lim d\ln \alpha/d\ln v$ is infinite, then the stability condition will be satisfied only if the derivatives of the technology functions with respect to employment are sufficiently small, that is in the limit, as α goes to zero

$$(1+\beta')^2 \alpha' < 4(n + \beta(0))/v^*$$

The Jacobian under the Kaldorian variant becomes:

$$J(v^*, a^*) = \begin{Bmatrix} -v^*(s_p(b_0/a^*)w' + \beta'(0)\alpha'(v^*) + \alpha'(v^*)) & -v^*s_p(1 - w(v^*)b_0)/a^{*2} \\ \\ a^*\alpha'(v^*) & 0 \end{Bmatrix}$$

The conditions for local stability with oscillations in this case are:

$$[v^*(s_p(b_0/a^*)w' + \beta'(0)\alpha'(v^*) + \alpha'(v^*))]^2 - 4(v^*s_p(1 - w(v^*)b_0)/a^{*2})a^*\alpha'(v^*) < 0$$

$$\text{and} - v^*(s_p(b_0/a^*)w' + \beta'(0)\alpha'(v^*) + \alpha'(v^*)) < 0$$

Again, the latter condition is always satisfied, but now if the limit as α goes to zero of dln α/dln v is finite, the former condition is never satisfied; but if the limit is finite, the former condition requires that real wages not be too sensitive to employment. To see this, we rewrite the former condition as

$$[v^*(s_p(b_0/a^*)w' + \beta'(0)\alpha'(v^*) + \alpha'(v^*))]^2 < \{(v^*s_p r^*\alpha' = 4v^*(n + \beta(0))\alpha'\}$$

$$\text{LHS} = [v^*s_p(b_0/a^*)w' + v^*\alpha'(1 + \beta')]^2$$

$$= [v^*s_p b_0/a^* w']^2 + [2(v^*s_p(b_0/a^*)w'(v^*\alpha(1 + \beta')] + [v^*\alpha(1 + \beta')]^2$$

It is apparent, first, that if dln α/dln v is finite, the condition for stable oscillations is never satisfied (in marked contrast to the standard case), because the LHS is strictly positive, the RHS is zero. If lim $\alpha'(v)$ is strictly positive, the condition can be satisfied only if w' is not too large. If the condition is not satisfied, the equilibrium is locally stable and the approach is not oscillatory.