Essays on Asset Pricing in Open Economies

Andreas Stathopoulos

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Chapter 1 proposes a two-country general equilibrium model with external habits and home-biased preferences that addresses a number of international finance puzzles. Specifically, the model reconciles the high degree of international risk sharing implied by relatively smooth exchange rates with the modest cross-country consumption growth correlations seen in the data, resolving the Brandt, Cochrane and Santa-Clara (2006) puzzle. Furthermore, the model matches the empirically observed low correlation between exchange rate changes and international consumption growth rate differentials. For both effects, the fundamental mechanism is time variation in consumption growth volatility, which is endogenously generated through international trade. Asset prices depend on a weighted average of the two countries' time-varying risk aversion, with the weights determined by the wealth and degree of home bias of each country. Simulation results indicate that the model is successful in matching key empirical asset pricing, exchange rate and international trade moments.

Chapter 2 examines international portfolio choice in a two-country general equilibrium setting which features time-varying risk aversion generated by external habit formation. I show that, under complete markets, home bias in the consumption preferences leads to significant portfolio home bias due to differential hedging demands. Domestic (foreign) agents shift their portfolio towards domestic (foreign) assets, so as to better hedge against adverse changes in their conditional risk aversion. Furthermore, the model generates realistic asset pricing moments, thus reconciling international portfolio choice with asset pricing.
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To my parents
Chapter 1

Asset Prices and Risk Sharing in Open Economies

1.1 Introduction

This Chapter presents a model which highlights, in a tractable way, the links between asset prices, exchange rates and international risk sharing generated by international trade in goods and assets. The model proposes a solution to a number of international finance puzzles that are related to the connection between the aforementioned three economic concepts, with the primary focus being on the international risk sharing puzzle. Furthermore, the model clearly illustrates how international trade affects equity prices and risk-free rates vis-à-vis the closed economy benchmark and explicitly connects real exchange rates and equity prices.

The international risk sharing puzzle, illustrated in detail in Brandt, Cochrane and Santa-Clara (2006), is the apparent disconnect between relatively modest empirical cross-country consumption growth rate correlations and the extremely high degree of international risk sharing implied by the relatively low real exchange rate volatility observed in the data.\(^1\) If financial markets are complete, the key relationship that generates the links between asset and currency prices and risk sharing is the no-arbitrage relationship

\[
\frac{M_{t+1}^*}{M_{t+1}} = \frac{E_{t+1}}{E_t}
\]  

\(^1\)As will be discussed later, real exchange rate volatility is low only compared to asset return volatility; it is quite high compared to macroeconomic (income or consumption) volatility.
where $M_{t+1}$ and $M_{t+1}^*$ are the home and foreign stochastic discount factor (SDF), respectively, and $E_t$ is the real exchange rate (domestic price of foreign currency, in real terms, i.e. an increase in $E_t$ denotes a real depreciation of the domestic currency). This relationship is not without assumptions: it holds only when financial markets are frictionless, in the sense that investors in each country can freely invest in assets denominated in any of the two currencies.\footnote{Complete markets are not necessary for (1.1) to hold. In the presence of market incompleteness, (1.1) holds with $M_{t+1}$ being the (unique) projection of all (i.e. domestic and foreign) investors' intertemporal marginal rate of substitution (IMRS), expressed in domestic currency units, on $X_{t+1}$, and $M_{t+1}^*$, the (unique) projection of all investors' IMRS, expressed in foreign currency units, on $X_{t+1}^*$, where $X_{t+1}$ ($X_{t+1}^*$) is the space spanned by the domestic (foreign) currency returns of all assets, domestic and foreign. See Backus, Foresi and Telmer (2001) and Brandt, Cochrane and Santa-Clara (2006).} Perfect risk sharing between the two countries means $M_{t+1}^* = M_{t+1}$, which implies a constant real exchange rate.\footnote{In the international macroeconomics and finance literature, optimal risk sharing is sometimes defined as being equivalent to the achievement of a Pareto optimal allocation. In that case, (1) is the optimal risk sharing condition; for special cases of this condition, see, for example, Cole and Obstfeld (1991), Backus and Smith (1993), Lewis (1996) and Obstfeld and Rogoff (2000). This paper, following Brandt et al. (2006), uses the term "perfect risk sharing" to refer to the more stringent condition $M_{t+1}^* = M_{t+1}$; the reason is that, as explained in detail in Brandt et al. (2006) and in this paper, the international risk sharing puzzle regards SDF correlations, not Pareto optimality.}

Taking logs and then unconditional variances on each side of (1.1), we get

\[ var(m_{t+1}^*) + var(m_{t+1}) - 2\sigma(m_{t+1})\sigma(m_{t+1}^*)\rho(m_{t+1}, m_{t+1}^*) = var(\Delta e_{t+1}) \]

with small letters denoting the log of their capital letter counterpart. Using the logic of Hansen and Jagannathan (1991) volatility bounds, the high Sharpe ratios we observe in asset markets imply very high pricing kernel volatility. Unless the correlation between the two pricing kernels is extremely high, high kernel volatility cannot be reconciled with the empirically observed modest levels of real exchange rate volatility. Setting, for example, $\sigma(m_{t+1}) = \sigma(m_{t+1}^*) = 50\%$ and considering $\sigma(\Delta e_{t+1}) = 10\%$ (in line with empirical real exchange rate volatility for major currency pairs), we can easily see that prices imply that $\rho(m_{t+1}, m_{t+1}^*) = 0.98$. Under CRRA preferences, $m_{t+1} = \log \beta - \gamma \Delta c_{t+1}$ and $m_{t+1}^* = \log \beta - \gamma \Delta c_{t+1}^*$, so $\rho(m_{t+1}, m_{t+1}^*) = \rho(\Delta c_{t+1}, \Delta c_{t+1}^*)$. Then, to square prices with quantities, we need $\rho(\Delta c_{t+1}, \Delta c_{t+1}^*) = 0.98$; unfortunately, the observed cross-country consumption growth correlations are typically much lower, with correlations of 0.9 and above not being even remotely plausible. The only way we can have e.g. $\rho(m_{t+1}, m_{t+1}^*) = 0.3$ (in line with empirical cross-country consumption growth correlations) is if $\sigma(\Delta c_{t+1}) = 65\%$; as mentioned above, this is highly counterfactual. This, in a nutshell, is the puzzle: prices tell
us that risk is nearly perfectly shared among countries, but quantities tell us otherwise. Consequently, any model that aims to explain the relationship between asset returns and exchange rates should address international risk sharing, reconciling high unconditional pricing kernel correlations with relatively modest unconditional consumption growth rate correlations.

A related puzzle to be addressed is the exchange rate disconnect puzzle, illustrated by Backus and Smith (1993). Starting with (1.1), it is easy to see that for CRRA preferences we get

\[ \text{corr}(\Delta c_{t+1} - \Delta c^*_t, \Delta e_{t+1}) = 1 \] (1.2)

irrespective of the value of \( \gamma \). Backus and Smith (1993) derive this result in a more general setting. However, in the data, consumption growth rate differentials appear to be decoupled from real exchange rate changes; using data from 8 OECD countries, Backus and Smith (1993) show that the average correlation between per capita consumption growth rate differentials and real exchange rate changes is -0.056, with a range of [-0.63, 0.21], a far cry from the theoretical value of 1.

Another issue in international macroeconomics is the "remarkable", in the words of Obstfeld and Rogoff (2000), volatility of real exchange rates. As mentioned earlier, from the perspective of asset pricing, real exchange rate volatility is relatively small: asset return volatility is around 15% - 20% per year, while pricing kernel volatility is even higher, of the order of 50%. However, from the perspective of international macroeconomics, the volatility of real exchange rate changes should not be far from consumption or income growth volatility, around 1% - 3% per year; it is, instead, almost an order of magnitude higher.

This Chapter proposes a two-country endowment model that incorporates external habits and consumption home bias in preferences. In the model, the global economy is comprised of two countries, each represented by a stand-in agent endowed with a stream of a differentiated perishable good. Each of the two agents has Menzly, Santos and Veronesi (2004) external habit preferences, with the habit defined on a home-biased, CES aggregated consumption basket of the two goods. This model leads to an economically intuitive solution...
to both the international risk sharing puzzle and the exchange rate disconnect puzzle.

Regarding the international risk sharing puzzle, the model implies that countries indeed share risk to a very large degree through trade in goods and assets. Specifically, trade generates endogenous time variation in consumption growth volatility: the conditionally relatively less risk averse country assumes more of the global endowment risk. In other words, the conditionally more risk averse country has low consumption risk, while the less risk averse country has high consumption risk. On the other hand, the conditionally less risk averse country has low conditional sensitivity to consumption growth risk, while the more risk averse country is very sensitive to consumption risk. To understand how this resolves the puzzle, consider the market price of consumption risk. Since, for each country, the conditional market price of consumption risk is an increasing function of both conditional risk aversion and conditional consumption growth volatility, the two effects (volatility and sensitivity) push the relative market price of risk of the two countries to different directions. However, since the magnitude of the two effects is almost equal, those two effects, combined, balance each other, leading to a very high cross-country correlation of market prices of risk and, thus, pricing kernels. However, cross-country consumption growth correlation is modest, since there is no sensitivity effect to counter the volatility effect.

Regarding the exchange rate disconnect puzzle, habits decouple marginal utility growth from consumption growth. This effect generates very low correlation between consumption growth rate differentials and real exchange rate changes, despite the fact that the correlation between pricing kernel differentials and real exchange rate changes is, by construction, perfect.

The model also sheds light on the issue of asset pricing in open economies. Specifically, the model generates an economically intuitive solution for the price of the two countries' total wealth portfolios: the price-dividend ratio of each total wealth portfolio is determined by a weighted average of the two countries' time-varying relative risk aversion coefficients, with the weights depending on the initial wealth and the degree of home bias of the two countries. When the economy is closed, the solution collapses to the Menzly, Santos and Veronesi (2004) pricing results, so the model clearly illustrates how international trade affects asset prices and returns. The connection of asset prices with exchange rates is also straightforward: the real exchange rate is a function of both the endowment ratio and the price-dividend ratios of the two total wealth portfolios. Thus, real exchange rate volatility
is generated by two economic mechanisms: time variation in relative endowments and time variation in price-dividend ratios. The latter, asset pricing-related, mechanism amplifies the effects of the former, endowment-related, mechanism, so real exchange rate changes are much more volatile than endowment growth rates. The failure of most standard international macroeconomic models to generate substantial real exchange rate volatility can, thus, be traced to their inability to generate time-varying asset price-dividend ratios.

This Chapter is part of the recent literature that focuses on the connections between asset prices and exchange rates. The model in this Chapter builds on Pavlova and Rigobon (2007). They use a Lucas (1982) two-country, two-good model to examine the effects of the terms of trade on asset prices and exchange rates when preferences are characterized by demand shocks. Inter alia, they use financial data to extract latent factors implied by their model and show that those factors can be used to predict macroeconomic variables and ameliorate puzzles arising in the international real business cycle literature. Despite its success in addressing macroeconomic questions, the ability of their model to match asset prices and returns is limited by its inability to generate time-varying asset price-dividend ratios. Other international asset pricing models that focus on terms of trade effects are Cole and Obstfeld (1991), Zapatero (1995) and Serrat (2001).

Recent papers extend standard asset pricing models to examine international finance issues. Regarding habits, Verdelhan (2008a) uses a two-country, one-good model in which each country has an exogenously specified i.i.d. consumption growth process and Campbell and Cochrane (1999) external habit preferences. The model is able to explain the forward premium puzzle, but generates real exchange rates that are both highly volatile, implying poor international risk sharing, and excessively linked to consumption growth. Verdelhan (2008b), a companion paper, assumes i.i.d. endowment growth and allows for international trade characterized by proportional and quadratic trade costs, thus endogenizing consumption; the ability to share risk by international trade lowers real exchange rate volatility to realistic levels, but the real exchange rate remains very closely linked to consumption growth, so the Backus and Smith (1993) puzzle cannot be resolved. Bekaert (1996) ex-

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5To be precise, price-dividend ratios are non-stochastic. Specifically, when the time horizon is finite, the price-dividend ratio of each country’s total wealth portfolio is a deterministic function of time. When the time horizon is infinite, the price-dividend ratio is constant.

6The author suggests that this result may be caused by the one-good assumption. It should be noted that Verdelhan (2008b) does not examine international consumption growth correlations, so the paper does not explore the international risk sharing puzzle.

Colacito and Croce (2008a) utilize the Bansal and Yaron (2004) long-run risks framework in order to address the international risk sharing puzzle. They consider a two-country, two-good closed economy endowment model in which each country has Epstein and Zin (1989) preferences and an exogenously specified consumption growth process featuring a slow moving, predictable component. They show that the puzzle can be resolved if the two predictable components, the domestic and the foreign one, are highly correlated. In Colacito and Croce (2008b), they extend their model to open economies, allowing for international trade and endogenizing consumption, in order to revisit the Cole and Obstfeld (1991) results; they show that international portfolio diversification may produce significant welfare gains in the presence of long-run risk. Bansal and Shaliastovich (2007) also use a long-run risks model in order to, inter alia, address the forward premium puzzle.


One of the key assumptions of this Chapter is external habit formation. Habits, internal or external, have been used in much of the recent asset pricing literature. The present work postulates Menzly et al. (2004) external habits, which share the motivation of Campbell and Cochrane (1999) habits, but model the inverse surplus consumption ratio. Buraschi and Jiltsov (2007) use the same mean-reverting process for the inverse surplus

\footnote{As in Colacito and Croce (2007), matching the empirical volatility level of real exchange rate changes requires that long-run endowment shocks be highly internationally correlated.}

consumption ratio in order to study the term structure of interest rates.\footnote{Santos and Veronesi (2006) use a similar formulation to examine the cross section of stock returns. Bekaert, Engstrom and Grenadier (2005) also model the inverse surplus consumption ratio: in their model, it is a mean-reverting process driven by two shocks, a consumption growth shock and an exogenous shock in risk appetite (or "mood").}

The rest of the Chapter is organized as follows. Section 2 describes the model. Section 3 presents the equilibrium macroeconomic prices and quantities and explains how the international risk sharing puzzle is resolved. Section 4 explores the asset pricing implications of the model. Section 5 reports the simulation results. Section 6 concludes. The Appendix contains the proofs and all supplementary material not included in the main body of this Chapter.

\section{The model}

\subsection{The structure of the economy}

The world economy is comprised of two countries, Domestic and Foreign, each of which is populated by a single risk-averse representative agent who receives an endowment stream of a single differentiated perishable good: the domestic agent is endowed with the domestic good, while the foreign agent is endowed with the foreign good. Economic activity takes place in the time interval $[0, \infty)$. Uncertainty in this economy is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$, where $\mathbf{F} = \{\mathcal{F}_t\}_{t \in [0, \infty)}$ is the filtration generated by the two-dimensional Brownian motion $\mathbf{B} = [B^X, B^Y]'$, augmented by the null sets. All stochastic processes introduced in the remainder of the Chapter are assumed to be progressively measurable with respect to $\mathbf{F}$ and to satisfy all the necessary regularity conditions for them to be well-defined. All (in)equalities that involve random variables hold $P$-almost surely.

The endowment sequence of the domestic good is denoted by $\{X_t\}$ and that of the foreign good by $\{Y_t\}$. Both processes are assumed to be of the form:

\begin{equation}
\log X_t = \mu_t^X dt + \sigma^X dB^X_t
\end{equation}

and

\begin{equation}
\log Y_t = \mu_t^Y dt + \sigma^Y dB^Y_t
\end{equation}

Note that both drifts are left unspecified. On the other hand, I specify that endowment growth is homoskedastic for both countries. This is a key point: any conditional hetero-
eroskedasticity arising in this model is endogenously generated.\footnote{This specification is adopted for simplicity; the extension to arbitrary diffusion processes $\sigma^X_t$ and $\sigma^Y_t$ is trivial.} The two endowment shocks $dB^X_t$ and $dB^Y_t$ are correlated with instantaneous correlation $\rho^{XY}$. Thus, the instantaneous covariance matrix of $dB_t$ is

\[
\Sigma = \begin{bmatrix}
1 & \rho^{XY} \\
\rho^{XY} & 1
\end{bmatrix}
\]

Both goods are frictionlessly traded internationally, so the price of each good (in units of the numeraire good) is the same in both countries (law of one price). Denote by $Q$ and $Q^*$ the price of the domestic good and the foreign good, respectively, in terms of the numeraire. Since this is a non-monetary economy, only relative prices are determined; without loss of generality, the domestic good is considered the numeraire good, so $Q_t \equiv 1, \forall t \in [0, \infty)$. Then, $Q^* = \frac{Q^*}{Q}$ denotes the terms of trade (the ratio of the price of exports over the price of imports) for the foreign country, which are the inverse terms of trade for the domestic country; in the remainder of the Chapter, $Q^*$ will be called terms of trade without further specification.\footnote{This definition of the terms of trade (price of exports over the price of imports) is the one used in international trade. In international macroeconomics, sometimes the inverse definition is applied - see, for example, Chapter 11 in Cooley (1995).}

Finally, financial markets are complete and there are no frictions in the international trade of financial assets, so the no arbitrage condition (1.1) holds $\forall t \in [0, \infty)$.

### 1.2.2 Preferences

The domestic representative agent maximizes expected discounted utility

\[
E_0 \left[ \int_0^\infty e^{-\rho t} u(X_t, Y_t) dt \right]
\]

where $\rho > 0$ is her subjective discount rate, and her instantaneous utility function is

\[
u(X_t, Y_t) = \log(X_t^{a} Y_t^{1-a} - H_t) = \log(C_t - H_t)
\]

where $X_t$ and $Y_t$ is the quantity of the domestic and foreign good, respectively, she consumes at time $t$, $C \equiv X^a Y^{1-a}$ is the domestic consumption basket and $H_t$ is the time $t$ habit level associated with that consumption basket.

Two main assumptions about the domestic agent's preferences are adopted here. The first assumption is that the domestic consumption basket is a Cobb-Douglas aggregate
of the two goods. Then, the elasticity of substitution between the two goods is unity, so the goods are imperfect substitutes. A second implication is that the domestic agent may exhibit home bias, in the sense that her preferences over the two goods may not be necessarily symmetric. Parameter $a \in [0,1]$ denotes the degree of relative preference for the domestic good. When $a > 0.5$, the agent is home biased: one unit of the domestic good provides her with more utility than one unit of the foreign good. When $a = 1$, the agent is completely home biased: she only gets utility from the domestic good, so no international trade occurs in equilibrium. When $a = 0.5$, the agent has symmetric preferences towards the two goods, so no home bias exists.

The second main assumption regarding preferences is the existence of an external habit. It should be noted that the habit is over the consumption basket and not over individual goods' consumption. This specification is in line with the standard asset pricing literature: although asset pricing models usually assume a single good, empirically this good is taken to be aggregate consumption, which consists of many goods. I further assume that the external habit is of the Menzly et al. (2004) form. Specifically, it is assumed that the inverse surplus consumption ratio $G = \left(\frac{C-H}{C}\right)^{-1}$ solves the stochastic differential equation

$$dG_t = k (G - G_t) dt - \delta (G_t - l) \left( \frac{dC_t}{C_t} - E_t \left( \frac{dC_t}{C_t} \right) \right)$$ (1.6)

The inverse surplus consumption ratio $G$ is a mean-reverting process, reverting to its long-run mean of $\bar{G}$ at speed $k > 0$ and is driven by consumption growth shocks. The parameter $\delta > 0$ scales the impact of the consumption growth shock and the parameter $l \geq 1$ is the lower bound of the inverse surplus consumption ratio. Obviously, $\bar{G} > l$. The local curvature of the utility function is $-\frac{u_{cc}(C,H)}{u_{c}(C,H)} C = G$; for that reason, and in a slight abuse of terminology, in the rest of the Chapter I will refer to $G$ as domestic risk aversion.

The preferences of the foreign stand-in agent are similar. Her instantaneous utility function is

$$u^*(x^*_t, y^*_t) = \log \left( \frac{x^*_t}{y^*_t} \right) = \log \left( \frac{C^*_t}{H^*_t} \right)$$ (1.7)

where $x^*_t$ and $y^*_t$ is the agent’s time $t$ consumption of the domestic and foreign good.

\[\text{The Menzly et al. (2004) model shares many of the properties of the Campbell and Cochrane (1999) model, which assumes a specification for the process of the surplus consumption ratio $S_t = \frac{C_t - H_t}{C_t}$. In the Campbell and Cochrane (1999) model, the support of $S$ is $(0, 5]$. In the Menzly et al. (2004) model the support of $G$ is $[l, \infty)$, so $S$ is bounded in $(0, l]$; the support of $S$ is the same for $l = \frac{1}{5}$. However, the two models are not isomorphic: for example, see Hansen (2008) for a discussion of their differing implications for long-run returns.}\]
respectively, \( C^* = (X^*)^{a^*} (Y^*)^{1-a^*} \) is the foreign consumption basket and \( H_t^* \) is the foreign habit level at time \( t \). Note that home consumption bias for the foreign agent implies \( a^* < 0.5 \).

The results discussed in the remainder of the Chapter refer to non-boundary parameter values \( a \in (0,1) \) and \( a^* \in (0,1) \), unless otherwise noted. Furthermore, the empirically relevant case is \( a^* < 0.5 < a \), with both countries exhibiting home bias. However, the weaker condition \( a^* < a \) suffices for the qualitative characterization of the results in this Chapter.\(^{13}\) Thus, when discussing the results, I will focus on the case \( 0 < a^* < a < 1 \). The difference in the preferences for the domestic good \( a - a^* \) will be called the degree of home bias.

The foreign agent also has external habits, with her inverse surplus consumption ratio process satisfying:

\[
dG_t^* = k (\bar{G} - G_t^*) \, dt - \delta (G_t^* - l) \left( \frac{dG_t^*}{C_t^*} - E_t \left( \frac{dG_t^*}{C_t^*} \right) \right) \tag{1.8}
\]

For simplicity, it is assumed that the preference parameters \( k, \delta, \bar{G} \) and \( l \) are the same in both countries.\(^{14}\)

### 1.2.3 Prices and exchange rates

Given that the domestic consumption basket is \( C = X^a Y^{1-a} \), the associated time \( t \) price index is:

\[
P_t = \left( \frac{Q_t}{a} \right)^a \left( \frac{Q_t^*}{1-a} \right)^{1-a} \tag{1.9}
\]

\( P_t \) is the time \( t \) price of one unit of domestic consumption in units of the numeraire good; it is defined as the minimum expenditure required to buy a unit of the domestic consumption basket \( C \) and is derived by minimizing the relevant expenditure function.

Similarly, the foreign price index is:

\[
P_t^* = \left( \frac{Q_t}{a^*} \right)^{a^*} \left( \frac{Q_t^*}{1-a^*} \right)^{1-a^*}
\]

\(^{13}\)Under that weaker condition, each country cares more about its good than the other country does, but does not necessarily care more about its own good than about the other country's good; the latter requires the stronger condition \( a^* < 0.5 < a \). Condition \( a^* < a \) can be called relative home bias, while condition \( a^* < 0.5 < a \) can be called absolute home bias. In the remainder of the Chapter, the term home bias will be used to refer to relative home bias.

\(^{14}\)This assumption is made for convenience and can be easily relaxed without any qualitative difference in the results.
which is the price, in terms of the numeraire good, of one unit of the foreign consumption basket.

Therefore, the time $t$ real exchange rate, which expresses the price of a unit of the foreign consumption basket in units of the domestic consumption basket, is:

$$E_t = \frac{P_t^*}{P_t} = \frac{a^a(1-a)^{1-a}}{(a^*)^{a^*}(1-a^*)^{1-a^*}} \left(Q_t^*\right)^{a-a^*}$$

using the fact that $Q_t = 1$, $\forall t \in [0, \infty)$. Trivially, when the preferences of the two countries are identical ($a = a^*$), the two consumption baskets are also identical ($C = C^*$). Then, since the absence of trade frictions implies the law of one price, the price of the two baskets is the same, and the real exchange rate is constant at 1 (Purchasing Power Parity). This is the case of perfect risk sharing: the absence of market frictions allows agents with identical preferences to fully share risk. In a frictionless world, what generates real exchange rate volatility is the difference in the two countries’ preferences, and thus the fact that the two consumption baskets are not identical: $C \neq C^*$. Then, volatility in the terms of trade $Q^*$ generates variation in the relative price of the two consumption baskets. In fact, in this model, due to the assumption of unit elasticity of substitution between the two goods, real exchange rate change volatility is proportional to the degree of home bias $a - a^*$.

### 1.3 Equilibrium prices and quantities

#### 1.3.1 The planner’s problem

Under the assumption of market completeness, the competitive equilibrium (CE) allocation is equivalent to a central planner’s allocation, with the planner taking the laws of motion for $G$ and $G^*$ as given.\(^{15}\) For the CE solution to be identical to the planner’s problem solution, the welfare weights must be determined endogenously.\(^{16}\) We will see in a later section that the appropriate welfare weights can be easily calculated in this model, so we can first solve the planner’s problem and then calculate the welfare weights that equate the planner’s problem equilibrium with the CE.

The social planner maximizes a weighted average of the two countries’ expected

\(^{15}\)For the planner’s solution to coincide with the CE solution, the planner has to take into account the externality arising from external habit formation. Thus, the CE solution will not be uncontrained Pareto optimal, but constrained Pareto optimal, with the constraint being the assumed external habit processes.

\(^{16}\)Specifically, welfare weights are related to the intertemporal budget constraints of the two countries.
utility, with welfare weights being \( \lambda \) and \( \lambda^* = 1 - \lambda \), for the domestic and foreign country, respectively:

\[
\max_{\{X_t, Y_t, X_t^*, Y_t^*\}} \mathbb{E}_0 \left[ \int_0^\infty e^{-\rho t} \left( \lambda \log(C_t - H_t) + \lambda^* \log(C_t^* - H_t^*) \right) dt \right]
\]

subject to the resource constraints \( X_t + X_t^* = \tilde{X}_t \) and \( Y_t + Y_t^* = \tilde{Y}_t \).

### 1.3.2 Consumption

Solving the planner's problem (see Appendix, section 1.7.1), we get the equilibrium consumption allocation. For the home agent:

\[
X_t = \omega_t \tilde{X}_t, \quad Y_t = \omega_t \tilde{Y}_t
\]

and for the foreign agent:

\[
X_t^* = (1 - \omega_t) \tilde{X}_t, \quad Y_t^* = (1 - \omega_t^*) \tilde{Y}_t
\]

where I introduce the share functions \( \omega_t \) and \( \omega_t^* \)

\[
\omega_t = \omega \left( \frac{G_t^*}{G_t} \right) \equiv \frac{a\lambda}{a\lambda + a^*\lambda^* \left( \frac{G_t}{G_t} \right)}
\]

\[
\omega_t^* = \omega^* \left( \frac{G_t^*}{G_t} \right) \equiv \frac{(1 - a)\lambda}{(1 - a)\lambda + (1 - a^*)\lambda^* \left( \frac{G_t}{G_t} \right)}
\]

with \( \omega_t \) (\( \omega_t^* \)) being the proportion of domestic (foreign) endowment consumed by the domestic agent. In the case of complete home bias \((a = 1, a^* = 0)\), it can easily be shown that \( \omega_t = 1 \) and \( \omega_t^* = 0, \forall t \in [0, \infty) \): each country consumes its endowment, so no trade occurs and both economies are closed in equilibrium. Both share functions are decreasing in the risk aversion ratio \( \frac{G_t}{G_t^*} \). It should be noted that under home bias \((a > a^*)\), \( \omega_t^* \) is more sensitive than \( \omega_t \) to the risk aversion ratio \( \frac{G_t}{G_t^*} \), so \( \frac{\omega_t^*}{\omega_t} \) is increasing in \( \frac{G_t}{G_t^*} \).

Therefore, domestic consumption is

\[
C_t = \omega_t^a (\omega_t^*)^{1-a} \tilde{X}_t^a \tilde{Y}_t^{1-a}
\]

and foreign consumption is

\[
C_t^* = (1 - \omega_t)^a (1 - \omega_t^*)^{1-a} \tilde{X}_t^{a^*} \tilde{Y}_t^{1-a^*}
\]
Consumption depends on three state variables: the two endowment levels $X_t$ and $Y_t$, and the ratio of risk aversions $\frac{G^*_t}{G_t}$. The effects of endowment levels on consumption are straightforward: since both countries consume both goods, both domestic and foreign consumption are increasing in both endowments. However, the degree that each country’s consumption is affected by each endowment’s fluctuations depends on relative preferences: unsurprisingly, under home-biased preferences ($a > a^*$), each country’s consumption is more sensitive to its own endowment than to the other country’s endowment.

More interesting are the effects of the ratio $\frac{G^*_t}{G_t}$, through the share functions: domestic consumption ($X_t, Y_t$ and so $C_t$) is decreasing in $\frac{G^*_t}{G_t}$, while foreign consumption is increasing in $\frac{G^*_t}{G_t}$. When the foreign agent becomes relatively more risk averse than the domestic agent, consumption is shifted from the domestic country to the foreign country, and vice versa. This is an international risk sharing effect: each period, consumption flows to the country that needs it the most, i.e. the country which is closer to its habit and is more averse to further consumption reduction.

Since consumption and habit level are jointly determined, understanding the evolution of domestic and foreign consumption over time requires explicitly solving for the two equilibrium consumption processes as functions of the two exogenous shocks $dB_t^X$ and $dB_t^Y$. The following proposition, the proof of which can be found in the Appendix, presents the result.\textsuperscript{17}

**Proposition 1** *The equilibrium consumption process for the domestic representative agent* is

$$\frac{dC_t}{C_t} - E_t \left( \frac{dC_t}{C_t} \right) = \sigma_t^C dB_t = \sigma_t^{CX} dB_t^X + \sigma_t^{CY} dB_t^Y$$

with

\begin{align*}
\sigma_t^{CX} &= \frac{1}{D_t} \left( a + (ak_t^* + a^*k_t) \delta \left( \frac{G_t^* - l}{G_t^*} \right) \right) \sigma^X \quad \text{and} \\
\sigma_t^{CY} &= \frac{1}{D_t} \left( (1 - a) + ((1 - a)k_t^* + (1 - a^*)k_t) \delta \left( \frac{G_t^* - l}{G_t^*} \right) \right) \sigma^Y
\end{align*}

(1.16) (1.17)

and the equilibrium consumption process for the foreign representative agent is

$$\frac{dC_t^*}{C_t^*} - E_t \left( \frac{dC_t^*}{C_t^*} \right) = \sigma_t^{C^*} dB_t = \sigma_t^{C^*X} dB_t^X + \sigma_t^{C^*Y} dB_t^Y$$

\textsuperscript{17}Proposition 1 focuses on the diffusion terms of the two consumption processes; the two drift terms depend on endowment drifts, which have been left unspecified. The simulation section of the paper considers specific endowment growth drift specifications and their results for the mean of consumption growth rates.
with

$$\sigma_t^{\pi X} = \frac{1}{D_t} \left( a + (a k_t^* + a^* k_t) \delta \left( \frac{G_t - l}{G_t} \right) \right) \sigma^X$$ and
$$\sigma_t^{\pi Y} = \frac{1}{D_t} \left( (1 - a^*) + ((1 - a) k_t^* + (1 - a^*) k_t) \delta \left( \frac{G_t - l}{G_t} \right) \right) \sigma^Y$$ (1.18)

where $k_t$, $k_t^*$ and $D_t$ are functions of $G_t$ and $G_t^*$ defined in the Appendix (equations (1.55), (1.56) and (1.57), respectively). For $0 < a^* < a < 1$, it holds that $0 < k_t < 1$ and $0 < k_t^* < 1$ and $D_t > 0$, $\forall t \in [0, \infty)$.

The key result is that both consumption growth processes have time-varying volatility, even though both endowment growth processes are homoskedastic. Note that $\sigma_t^{\pi X}$ and $\sigma_t^{\pi Y}$ are (roughly) proportional to $\frac{G_t - l}{G_t}$; thus, domestic conditional consumption growth volatility is roughly scaled by $\frac{G_t - l}{G_t}$. Conversely, foreign conditional consumption growth volatility is scaled by $\frac{G_t - l}{G_t}$. Thus, each country’s conditional consumption growth volatility is increasing in the other country’s conditional risk aversion. This, again, is the result of risk sharing: the conditionally less risk averse country insures the more risk averse country by assuming more of the global endowment risk. This way, international trade in goods and assets allows countries to allocate endowment risk efficiently.

To examine the impact of habit preferences, consider the log utility economy, to which the model economy reduces in the absence of external habit formation. In that case, consumption growth is homoskedastic:

$$\sigma_t^{\pi X} = a \sigma^X, \sigma_t^{\pi Y} = (1 - a) \sigma^Y$$

and

$$\sigma_t^{\pi X} = a^* \sigma^X, \sigma_t^{\pi Y} = (1 - a^*) \sigma^Y$$

Habit preferences lead countries to share risk through the reallocation of consumption growth risk; standard log (and, in general, CRRA) preferences do not.

Since this is a complete markets setting, the aforementioned risk reallocation occurs through transactions in Arrow-Debreu securities. International risk sharing, in this context, means that, at each period, the conditionally more risk averse country holds the Arrow-Debreu consumption claims that ensure low consumption growth volatility, i.e. that hedge against big swings in its consumption growth. To express the asset allocation decisions of the two countries in terms of more realistic assets (such as stocks and bonds), we would need
to fully specify the assets that can be traded in the financial markets; the only requirement would be that the assets specified are sufficient for market completeness.\(^{18}\)

### 1.3.3 International risk sharing

Consider the discounted marginal utility of domestic and foreign consumption, \(\Lambda_t = e^{-\rho C_t^*}G_t\) and \(\Lambda_t^* = e^{-\rho C_t^*}G_t^*\), respectively. The log pricing kernel (stochastic discount factor) of the domestic country is

\[
d\log \Lambda_t = -\rho dt + d\log G_t - d\log C_t
\]

and the log pricing kernel of the foreign country is

\[
d\log \Lambda_t^* = -\rho dt + d\log G_t^* - d\log C_t^*
\]

Note that \(d\log \Lambda_t\) and \(d\log \Lambda_t^*\) are the continuous-time equivalents of \(m_{t+1}\) and \(m_{t+1}^*\), seen in the introduction.

It is shown in the Appendix (section 1.7.1) that

\[
d\log \Lambda_t^* - d\log \Lambda_t = d\log E_t
\]

which is the continuous-time equivalent of (1.1) in logs.

As mentioned in the introduction, the international risk sharing puzzle is the coexistence of extremely high international pricing kernel correlation and relatively low international consumption growth rate correlation. In this model, the key for the explanation of the international risk sharing puzzle is the endogenously generated time-varying conditional consumption growth volatility discussed in the previous section. To understand how this endogeneity explains the puzzle, recall that: \(^{19}\)

\[
d\log G_t = drift - \delta \frac{G_t - l}{G_t} \left( \sigma_t C_t dB_t \right)
\]

so the domestic log pricing kernel is:

\[
d\log \Lambda_t = drift - \left( 1 + \delta \frac{G_t - l}{G_t} \right) \left( \sigma_t C_t dB_t \right)
\]

\(^{18}\)Current work in progress examines portfolio choice in a two-good, two-country economy characterized by external habit formation and consumption home bias in preferences. It is shown that, under home bias, equilibrium portfolios are home biased; in fact, they are superbiased, in the Bennett and Young (1999) sense.

\(^{19}\)This follows from an application of Itô’s lemma to (2.1).
and, thus, the market price of domestic consumption risk is:

\[ \eta_t^C = \left( 1 + \frac{G_t - 1}{G_t} \right) \sigma_t^C \]

Similarly, the foreign log pricing kernel is:

\[ d \log \Lambda_t^* = \text{drift} - \left( 1 + \frac{G_t^* - 1}{G_t^*} \right) \left( \sigma_t^{C^*} dB_t \right) \]

so the market price of foreign consumption growth risk is:

\[ \eta_t^{C^*} = \left( 1 + \frac{G_t^* - 1}{G_t^*} \right) \sigma_t^{C^*} \]

Note that the diffusion of the pricing kernel of each country - in other words, the market price of consumption risk - can be decomposed into two components: consumption growth volatility (the quantity of risk that the agent undertakes) and sensitivity to consumption growth shocks (which depends on conditional risk aversion).

To fix ideas, assume that the domestic country is conditionally more risk averse than the foreign country: \( G_t > G_t^* \). First, consider the case of closed economies \( (\alpha = 1, \alpha^* = 0) \). In that case, as we have seen, each country consumes its endowment, so, under the assumption of homoskedastic endowment growth, conditional consumption growth volatility is constant for both countries (constant \( \sigma_t^C \) and \( \sigma_t^{C^*} \)). Since each country’s sensitivity to consumption growth shocks is increasing in its conditional risk aversion \( 1 + \delta_t^C \) and \( 1 + \delta_t^{C^*} \) are increasing in \( G_t \) and \( G_t^* \), respectively), the condition \( G_t > G_t^* \) implies, ceteris paribus, that the domestic pricing kernel is conditionally more volatile than its foreign counterpart. This is the sensitivity effect. In that case, high correlation between \( d \log G_t \) and \( d \log G_t^* \) - and thus high correlation between the two log pricing kernels \( d \log \Lambda_t \) and \( d \log \Lambda_t^* \) - requires high correlation between the two countries’ consumption growth processes. In other words, the market prices of consumption risk of the two countries are highly correlated only if their endowment growth processes are highly correlated.

However, this is not true for open economies: as we saw in the previous section, the condition \( G_t > G_t^* \) also implies, ceteris paribus, that domestic conditional consumption growth volatility \( \sigma_t^C \) is lower than foreign conditional consumption growth volatility \( \sigma_t^{C^*} \). This is the consumption volatility effect and it has the opposite direction of the sensitivity effect, decreasing the relative conditional volatility of the domestic pricing kernel and increasing the relative volatility of the foreign kernel. Simply stated, for the domestic
country, relatively high sensitivity to consumption growth risk is multiplied by relatively low consumption growth volatility. Exactly the opposite happens for the foreign country: relatively low sensitivity is multiplied by relatively high consumption growth volatility. The two components, sensitivity and volatility, have opposing effects on the relative market price of risk of the two countries. Apart from having opposite signs, the two components have similar magnitudes: this is because \( \sigma^C_t \) is roughly scaled by \( \delta \frac{G^*_t - l}{G_t} \) (which is roughly foreign sensitivity), whereas \( \sigma^*_t \) is roughly scaled by \( \delta \frac{G_t - l}{G_t} \) (roughly domestic sensitivity). The end result is that sensitivity and volatility balance each other out almost completely, bringing the two countries' market prices of consumption risk very close to each other and, thus, generating very high correlation between the two pricing kernels. The following corollary illustrates the above argument more formally.

**Corollary 2** Let \( \phi_t \) be the 2x1 vector such that

\[
\phi_t dB_t = \frac{1}{D_t} \left( (ak^*_t + a^* k_t) \sigma^X dB^X_t + ((1 - a) k^*_t + (1 - a^*) k_t) \sigma^Y dB^Y_t \right)
\]

The domestic consumption growth rate process is:

\[
d\log C_t = drift + \frac{1}{D_t} \left( a \sigma^X dB^X_t + (1 - a) \sigma^Y dB^Y_t \right) + \delta \left( \frac{G^*_t - l}{G_t} \right) \phi_t dB_t
\]

and the foreign consumption growth rate process is:

\[
d\log C^*_t = drift + \frac{1}{D^*_t} \left( a^* \sigma^X dB^X_t + (1 - a^*) \sigma^Y dB^Y_t \right) + \delta \left( \frac{G^*_t - l}{G^*_t} \right) \phi^*_t dB_t
\]

Furthermore, the domestic log pricing kernel is:

\[
d\log A_t = drift - \left( 1 + \delta \frac{G^*_t - l}{G_t} \right) \frac{1}{D_t} \left( a \sigma^X dB^X_t + (1 - a) \sigma^Y dB^Y_t \right) - \delta \left( \frac{G^*_t - l}{G_t} \right) \phi_t dB_t - \delta^2 \left( \frac{G^*_t - l}{G_t} \right) \left( \frac{G^*_t - l}{G^*_t} \right) \phi^*_t dB_t
\]

and the foreign log pricing kernel is:

\[
d\log A^*_t = drift - \left( 1 + \delta \frac{G^*_t - l}{G^*_t} \right) \frac{1}{D^*_t} \left( a^* \sigma^X dB^X_t + (1 - a^*) \sigma^Y dB^Y_t \right) - \delta \left( \frac{G^*_t - l}{G^*_t} \right) \phi^*_t dB_t - \delta^2 \left( \frac{G^*_t - l}{G^*_t} \right) \left( \frac{G^*_t - l}{G^*_t} \right) \phi^*_t dB_t
\]

First, consider the two pricing kernels \( d\log A_t \) and \( d\log A^*_t \). Since the sensitivity parameter \( \delta \) is large\(^{20}\), the dominant term for both kernels is the last one:

\[
-\delta^2 \left( \frac{G_t - l}{G_t} \right) \left( \frac{G^*_t - l}{G^*_t} \right) \phi_t dB_t
\]

\(^{20}\)It is close to 80 in the calibration of Menzly et al. (2004).
which is identical for both processes. This term represents the "canceling out" of the sensitivity and volatility effects described above. The fact that the dominant term is identical leads, of course, to unconditional correlation between the two kernels that is extremely close to 1. On the other hand, the two consumption growth processes do not include sensitivity terms, so there is nothing to counterbalance the volatility effect. Mathematically, the dominant term is \( \delta \left( \frac{G_t - 1}{G_t} \right) \phi_t dB_t \) for the domestic consumption growth rate and \( \delta \left( \frac{G_t - 1}{G_t} \right) \phi_t dB_t \) for the foreign consumption growth rate. It can be easily seen that those two expressions have values that are close to each other when \( G_t \) and \( G_t^* \) are not too far apart. Thus, the correlation between those two terms is decreasing in the volatility of \( \frac{G_t}{G_t^*} \): the more \( G_t^* \) and \( G_t \) diverge, the more the dominant terms of the two consumption growth rate processes diverge.

Finally, we can intuitively see how the assumption of external habit formation can help resolve the Backus and Smith (1993) puzzle. From (1.20), (1.21) and (1.22), we get:

\[
d\log E_t = (d\log C_t - d\log C_t^*) + (d\log G_t^* - d\log G_t)
\]

Real exchange rate changes are driven by both the consumption growth rate differential and an additional, habit-induced differential term, which breaks the perfect relationship between real exchange rates and consumption growth rates. In turn, this habit-related term depends, inter alia, on \( G_t \) and \( G_t^* \), which are driven by past consumption realizations. In other words, it is not only present consumption that matters for real exchange rate changes; past consumption also matters.

### 1.3.4 International trade and the real exchange rate

We have so far described the equilibrium quantities of the model economy. The planner's problem can be easily decentralized to generate solutions for the terms of trade and, thus, the real exchange rate. The terms of trade are:

\[
Q_t^* = \frac{1 - a \frac{\omega_t}{\omega_t^*} \frac{X_t}{Y_t}}{a \frac{\omega_t}{\omega_t^*} \frac{X_t}{Y_t}}
\]  

(1.23)

The relative price of the foreign good \( Q_t^* \) depends on two ratios: the endowment ratio \( X_t / Y_t \) and the risk aversion ratio \( G_t^* / G_t \), the latter through the share ratio \( \frac{\omega_t}{\omega_t^*} \). The dependence of the terms of trade on the endowment ratio is not surprising, as it is well established in standard two-country models: the endowment ratio reflects the relative scarcity of the two
goods; high $\tilde{X}_t$ relative to $\tilde{Y}_t$ means that the foreign good is relatively scarcer and thus commands a high relative price $Q_t^*$. What is new in this model is the dependence on the risk aversion ratio $\frac{G_t^*}{G_t}$. Recall that under home bias ($a > a^*$), $\omega_t^*$ is increasing in $\frac{G_t^*}{G_t}$, so the terms of trade are increasing in the ratio of risk aversions. This is because, as seen earlier, high values of $\frac{G_t^*}{G_t}$ correspond to elevated consumption demand in the foreign country and reduced consumption demand in the domestic country. If consumption in both countries is home biased, then most of the high foreign consumption demand is expressed as high demand for the foreign good; correspondingly, there is low demand for the domestic good. The end result is a high relative price $Q_t^*$ for the foreign good.

It is also important to note that the assumption of external habit preferences, by adding dependence on the risk aversion ratio $\frac{G_t^*}{G_t}$, significantly increases the volatility of the terms of trade: $Q_t^*$ is now driven by the two endowment shocks through two mechanisms: there is a direct effect of the shocks though the endowment ratio $\frac{\tilde{X}_t}{\tilde{Y}_t}$ and an indirect (and larger) effect through the risk aversion ratio $\frac{G_t^*}{G_t}$. Those effects reinforce each other: a relative positive domestic endowment shock tends to increase both $\frac{\tilde{X}_t}{\tilde{Y}_t}$ and $\frac{G_t^*}{G_t}$, thus greatly enhancing terms of trade volatility vis-a-vis the benchmark of standard preferences.

Since the real exchange rate is proportional (in logs) to the terms of trade, the two variables share the same characteristics. Specifically, the real exchange rate is:

$$E_t = \left( \frac{a}{a^*} \right)^{a^*} \left( \frac{1-a}{1-a^*} \right)^{1-a^*} \left( \frac{\omega_t^*}{\omega_t} \right)^{a-a^*} \left( \frac{X_t}{Y_t} \right)^{a-a^*}$$

Unsurprisingly, $E_t$ is increasing in the endowment ratio: a positive endowment shock in a country depreciates its currency in real terms. Furthermore, under home bias, an increase in the risk aversion ratio $\frac{G_t^*}{G_t}$ leads to a real depreciation of the domestic currency; this is because an increase in $Q_t^*$ increases the foreign price level $P_t^*$ much more than it increases the domestic price level $P_t$. What is true for the volatility of the terms of trade is also true for the volatility of the real exchange rate: the addition of external habit preferences greatly enhances real exchange rate volatility.

Finally, the domestic net exports ratio, i.e. the ratio of the value of net exports over the value of the endowment, is:

$$NX_t = \frac{\tilde{X}_t - C_t P_t}{\tilde{X}_t} = \frac{X_t^* - Y_t^* Q_t^*}{\tilde{X}_t} = 1 - \frac{\omega_t}{a}$$
It is important to note that the net exports ratio only depends on the risk aversion ratio \( \frac{G_t}{G^*} \); only relative risk aversion matters for the external sector. Furthermore, the domestic net exports ratio is increasing in \( \frac{G_t}{G^*} \): high values of the risk aversion ratio mean that, as we have seen before, goods flow from the domestic to the foreign country, since the latter needs consumption more.

### 1.3.5 Wealth and welfare weights

To close the model, we need to calculate the endogenous welfare weights \( \lambda \) and \( \lambda^* \). The following proposition, proven in the Appendix, illustrates the connections between wealth and welfare weights.

**Proposition 3** Domestic wealth \( W_t \) in units of the domestic good is:

\[
W_t = \frac{\rho G_t + kG}{a \lambda G_t + a^* \lambda^* G_t} \frac{\lambda \hat{X}_t}{\rho (p + k)}
\]

(1.26)

and foreign wealth \( W_t^* \) in units of the foreign good is:

\[
W_t^* = \frac{\rho G_t^* + kG}{(1 - a) \lambda G_t + (1 - a^*) \lambda^* G_t} \frac{\lambda \hat{X}_t^*}{\rho (p + k)}
\]

(1.27)

where \( \lambda = \frac{a^* (\rho G_0 + kG)}{(1 - a)(\rho G_0 + kG) + a^*(\rho G_0^* + kG)} \) and \( \lambda^* = 1 - \lambda = \frac{(1 - a)(\rho G_0 + kG)}{(1 - a)(\rho G_0 + kG) + a^*(\rho G_0^* + kG)} \). Initial domestic wealth, as a proportion of global wealth, is:

\[
\frac{W_0}{W_0 + W_0^* Q_0^*} = \frac{a^*}{1 - a + a^*}
\]

(1.28)

Each country's wealth, in units of its own good, is increasing in its endowment and decreasing in the other country's risk aversion. However, the sign of dependence on its own risk aversion is not clear, as it depends on the parameter values and the value of the other country's risk aversion. Nevertheless, it is easy to characterize the wealth ratio of the two countries; it is:

\[
\frac{W_t}{W_t^* Q_t^*} = \frac{\lambda}{\lambda^*} \frac{\rho G_t + kG}{\rho G_t^* + kG}
\]

The wealth ratio is increasing in domestic risk aversion \( G_t \) and decreasing in foreign risk aversion \( G_t^* \).

The domestic share of global initial wealth \( \frac{W_0}{W_0 + W_0^* Q_0^*} \) is increasing in both \( a \) and \( a^* \). This makes sense: the stronger the preference for the domestic good from either the
domestic \((a)\) or the foreign \((a^*)\) country, the wealthier the domestic country is. In the limit, as the domestic country becomes completely home biased \((a \to 1)\) but the foreign country is not \((a^* \in (0, 1))\), then the domestic country has all the wealth \((\frac{w_0}{w_0 + w_0^* Q_0^*} \to 1)\); conversely, when the foreign country is completely home biased \((a^* \to 0)\) and the domestic country is not \((a \in (0, 1))\), then the foreign country has all the wealth \((\frac{w_0}{w_0 + w_0^* Q_0^*} \to 0)\).

This is intuitive: for example, when the domestic country is completely home biased and the foreign country is not, the foreign country wants to import from the domestic country, but it has nothing that the domestic country wants; the terms of trade \(Q^*\) approach zero, so the foreign country has a valueless endowment. It can be shown that when both countries are completely home biased \((a = 1\) and \(a^* = 0)\), the initial wealth ratio is indeterminate: since no country has preferences over both goods, there is no way to determine their relative price \(Q^*\) and, ultimately, the relative wealth of the two countries.

The domestic welfare weight \(\lambda\) is increasing in both \(a\) and \(a^*\). Furthermore, \(\lambda\) is decreasing in \(G_0\) and increasing in \(G_0^*\): the more initially risk averse country has a lower welfare weight, ceteris paribus. The limit behavior is illuminating:

\[
\lim_{G_0 \to \infty} \lambda = 0 \quad \text{and} \quad \lim_{G_0^* \to \infty} \lambda = 1
\]

Furthermore, \(\lambda\) has the same behavior as \(\frac{w_0}{w_0 + w_0^* Q_0^*}\) for boundary values of the two parameters: it approaches 1 (0) when the domestic (foreign) country approaches complete home bias and it is indeterminate when both countries are completely home biased.

For \(G_0 = G_0^*\):

\[
\lambda = \frac{a^*}{1 + a^* - a} = \frac{w_0}{w_0 + w_0^* Q_0^*}
\]

Thus, if initial risk aversion is equal for both countries, \(\lambda\) is equal to the proportion of initial wealth that the domestic country owns, so \(\lambda\) has a very natural interpretation. However, this is not true for \(G_0 \neq G_0^*\).

### 1.4 Asset prices

So far I have assumed complete markets, without explicitly specifying the securities in which the agents can invest. Under market completeness, all assets can be priced by no arbitrage, using the prices of Arrow-Debreu securities. In this section, I consider four assets: two total wealth portfolios, the domestic and the foreign one, and two locally riskless assets, the
domestic bond and the foreign bond. The domestic (foreign) total wealth portfolio is the asset that pays as dividend, each period, the endowment of the domestic (foreign) country. The net supply of each of those two portfolios is normalized to one. The domestic bond is a locally riskless asset in terms of the domestic good, in the sense that its return in terms of the domestic good is the same across states of the world; similarly for the foreign bond. Both bonds are in zero net supply. The price of the four assets is, respectively, $V_t$, $V_t^*$, $D_t$ and $D_t^*$; all prices are denoted in units of the local good, so $V_t$ and $D_t$ are expressed in units of the domestic good and $V_t^*$ and $D_t^*$ are expressed in units of the foreign good.

1.4.1 Risk-free rates

The price of the domestic bond satisfies the stochastic differential equation $dD_t = r_D^t D_t dt$, where $r_D^t$ is the continuously compounded domestic risk-free rate, i.e. the real rate of return demanded from a riskless investment in the domestic good. Similarly, the price of the foreign bond solves $dD_t^* = r^f D_t^* dt$. Note that neither of those bonds is riskless in terms of any of the two consumption baskets. Thus, there are consumption risk premia associated with both of those bonds; those premia are, in effect, compensation for terms of trade risk.\footnote{We can also define consumption bonds, with the domestic (foreign) consumption bond being locally riskless in terms of the domestic (foreign) consumption basket. See the Appendix (section 1.7.2) for a discussion.}

Proposition 4, the proof of which can be found in the Appendix, reports the domestic and foreign risk-free rates.

**Proposition 4** Let $e_1 \equiv [1 0]'$, $e_2 \equiv [0 1]'$. Also, denote $\sigma^C_t = \delta \left( \frac{G_{t-1}}{G_t} \right) \sigma^C_t$ and $\sigma^{G*}_t = \delta \left( \frac{G_{t-1}}{G_t} \right) \sigma^{G*}_t$.

The domestic risk-free rate is:

$$r_D^t = \rho + \mu^X + k \left[ \omega_t \left( \frac{G_t - \bar{G}}{G_t} \right) + \left( 1 - \omega_t \right) \left( \frac{G_t^* - \bar{G}}{G_t^*} \right) \right]$$

$$- \left[ \omega_t \sigma^C_t + \left( 1 - \omega_t \right) \sigma^{G*}_t \right] \Sigma e_1 \sigma^X - \frac{1}{2} \left( \sigma^X \right)^2$$

and the foreign risk-free rate is:

$$r_{D*}^f = \rho + \mu^Y + k \left[ \omega_t^* \left( \frac{G_t - \bar{G}}{G_t} \right) + \left( 1 - \omega_t^* \right) \left( \frac{G_t^* - \bar{G}}{G_t^*} \right) \right]$$

$$- \left[ \omega_t^* \sigma^C_t + \left( 1 - \omega_t^* \right) \sigma^{G*}_t \right] \Sigma e_2 \sigma^Y - \frac{1}{2} \left( \sigma^Y \right)^2$$

I focus on the domestic risk-free rate $r_D^t$; for $r_{D*}^f$, the analysis is identical. Unsurprisingly, the first term is the subjective discount rate $\rho$: the higher the agents discount
the future, the higher the interest rate has to be. The next two terms are marginal utility-smoothing terms: \( \mu_t^X \) is the familiar endowment-smoothing term, while the second term results from the agents' desire to smooth their conditional risk aversion. Specifically, when \( G_t \) and \( G^*_t \) are above their unconditional mean \( \bar{G} \), marginal utility is high, so the agents' willingness to save is low and equilibrium interest rates are high. Importantly, both risk aversions matter for both risk-free rates: the risk aversion-smoothing term depends on a weighted average of the two percentage deviations from unconditional risk aversion, with the weighted average largely depending on the home bias parameters \( a \) and \( a^* \). The last two terms are related to precautionary savings, so they enter with a negative sign: the more conditionally volatile domestic or foreign consumption growth (or, less importantly, domestic endowment growth) are, the more precautionary savings will the domestic agent desire, decreasing the equilibrium riskless rate.

### 1.4.2 Total wealth portfolio prices

After considering the two bonds, we now turn to the two total wealth portfolios. Proposition 5, proven in the Appendix, presents the equilibrium price of the two portfolios.

**Proposition 5** The price-dividend ratio of the domestic total wealth portfolio is:

\[
\frac{V_t}{X_t} = \frac{1}{\rho} \left( \frac{k}{\rho + k} \frac{(a\lambda + a^*\lambda^*)\bar{G}}{\alpha_a G_t} + \frac{\rho}{\rho + k} \right)
\]

and the price-dividend ratio of the foreign total wealth portfolio is:

\[
\frac{V_t^*}{Y_t} = \frac{1}{\rho} \left( \frac{k}{\rho + k} \frac{((1 - a)\lambda + (1 - a^*)\lambda^*)\bar{G}}{\alpha_{a^*} G^*_t} + \frac{\rho}{\rho + k} \right)
\]

To realize the effects of international trade on asset prices, first consider the closed economy case \( (a = 1 \) and \( a^* = 0) \). Then, it can easily be shown that the price-dividend ratio of the domestic total wealth portfolio is

\[
V_t = \frac{1}{\rho} \left( \frac{k}{\rho + k} \frac{\bar{G}}{G_t} + \frac{\rho}{\rho + k} \right) \bar{X}_t
\]

which, unsurprisingly, is the solution obtained in Menzly et al. (2004). In the power utility benchmark, which here obtains by setting \( G_t = \bar{G} \), the price-dividend ratio is \( \frac{1}{\rho} \), so changes in the endowment have a linear impact on prices. With habit preferences, however, the price-dividend ratio is no longer constant, but varies with \( G_t \). In this case, a positive
(for example) shock to $\tilde{X}_t$ increases $V_t$ both directly and indirectly, the latter through its negative effect on $G_t$. Thus, habits considerably magnify the effects of endowment shocks on asset prices by adding a second, multiplicative effect of endowment shocks on asset prices.

In an open economy, this dual effect of endowment shocks on asset prices is the same, with the difference being that what matters is not just the local endowment shock, but also the foreign one. The two endowment shocks affect prices in a more complicated way than in the closed economy case. Specifically, the domestic endowment shock affects both $G_t$ and $G^*_t$, since both processes are driven by consumption growth shocks, and both consumption growth shocks are, in turn, affected by both endowment shocks (consider the solution for the two countries' consumption growth rate in the previous section). Under home biased preferences, the two shocks are not equally important, of course: the domestic endowment shock affects $G_t$ more than $G^*_t$. Similarly, the foreign endowment shock affects both $G_t$ and $G^*_t$, but primarily the latter. Furthermore, only the domestic shock has a direct effect on $V_t$, through $\tilde{X}_t$, and, conversely, only the foreign shock directly affects $V^*_t$.

The expressions for the two portfolios' price-dividend ratios, (1.31) and (1.32), are economically intuitive. Since each country's total wealth portfolio represents claims to its endowment good, the price of the two total wealth portfolios will depend on both domestic and foreign demand for the endowment goods. As shown in the previous section, under certain conditions, the domestic welfare weight $\lambda$ is equal to the proportion of global wealth that the domestic country initially owns. Thus, the domestic price-dividend ratio will depend on both countries' time-varying risk aversions, weighted by each country's wealth ($\lambda, \lambda^*$) and desire for the domestic good ($a, a^*$). As a corollary, the foreign country's time-varying risk aversion will have a big impact on the domestic country's price-dividend ratio if either the foreign country is wealthy compared to the domestic country (i.e. $\lambda^*$ is high relative to $\lambda$), or if it has a strong preference for the domestic good (i.e. $a^*$ is high). This means, for example, that US risk aversion has a large effect on other countries' price-dividend ratios (and hence asset prices and returns), since the US is large compared to almost all other economies. Conversely, foreign countries' risk aversions have a relatively small effect on US asset prices, since the US economy is both large and relatively closed. Furthermore, it is the asset prices of small countries and countries with a significant volume
of exports (large $a^*$) that will be heavily affected by foreign risk preferences $G_t$.\(^{22}\)

### 1.4.3 Total wealth portfolio excess returns

After analyzing prices, we need to examine excess returns $\frac{dV_t}{V_t} + \tilde{x}_tdt - r_t^d dt$ and $\frac{dV_t^*}{V_t^*} + \tilde{x}_tdt - r_t^{d^*} dt$ for the domestic and foreign total wealth portfolio, respectively. The domestic total wealth portfolio pays domestic good dividends $\{\tilde{x}_t\}_{t=0}^{\infty}$, discounted by the domestic good marginal utility $\Xi_t$,\(^{23}\) which satisfies

$$\frac{d\Xi_t}{\Xi_t} = -r_t^d dt - \eta_t dB_t$$

where $\eta_t$ is the market price of domestic good risk. Similarly for the foreign total wealth portfolio and the foreign good marginal utility $\Xi_t^*$. An explicit solution for the excess returns of the two portfolios is provided in the following proposition, the proof of which can be found in the Appendix.\(^{24}\)

**Proposition 6** Let $e_1 \equiv [1 \ 0]'$, $e_2 \equiv [0 \ 1]'$. Also, denote $\sigma_t^C = \delta \left( \frac{G_t - 1}{G_t} \right) \sigma_t^C$ and $\sigma_t^{C^*} = \delta \left( \frac{G_t - 1}{G_t^*} \right) \sigma_t^{C^*}$. The excess return, in terms of the domestic good, of the domestic total wealth portfolio is

$$dB_t = \frac{\tilde{x}_t}{\Xi_t} dt + \sigma_t^C dB_t$$

where $\eta_t$ is the market price of domestic good risk, given by

$$\eta_t = \sigma^X e_1 + \left( \omega_t \sigma_t^C + (1 - \omega_t) \sigma_t^{C^*} \right)$$

(1.33)

and $\sigma_t^{R}$ is the diffusion process of the domestic total wealth portfolio excess return, given by

$$\sigma_t^{R} = \sigma^X e_1 + \frac{(a\lambda + a^*\lambda^*)k\tilde{G}}{(a\lambda + a^*\lambda^*)k\tilde{G} + \rho (a\lambda G_t + a^*\lambda^* G_t)} \left( \omega_t \sigma_t^C + (1 - \omega_t) \sigma_t^{C^*} \right)$$

(1.34)

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\(^{22}\)Note that country wealth refers to aggregate wealth, not per capita wealth. In this model, each country's population has been normalized to 1, so per capita and aggregate figures coincide. However, it should be stressed that, if the model is to be mapped to real data, all quantities mentioned are aggregate quantities. It can be shown that the results for aggregate variables are identical under any assumption regarding the two countries' population measures, as long as both measures are constant over time. The following section discusses the mapping of the model to empirical data.

\(^{23}\)In fact, the two agents, domestic and foreign, have different, but proportional, marginal utility of the domestic good. $\Xi_t$ is defined such that $\Xi_t = \lambda MU_t^X = \lambda^* MU_t^{X^*}$; thus, $\Xi_t$ is proportional to both the domestic $\lambda MU_t^X$ and the foreign $\lambda^* MU_t^{X^*}$ marginal utility of the domestic good. For more details, see the Appendix.

\(^{24}\)There is a tight connection between the marginal utility of domestic and foreign consumption $\Lambda_t$ and $\Lambda_t^*$, respectively, and the marginal utility of the domestic and foreign good $\Xi_t$ and $\Xi_t^*$, respectively. Details are provided in the Appendix (section A.2).
The excess return, in terms of the foreign good, of the foreign total wealth portfolio is

\[ dR_t^* = \left( \eta_t^* \boldsymbol{\Sigma} \sigma_t^{R*} \right) dt + \sigma_t^{R*} d\mathbf{B}_t \]

where \( \eta_t^* \) is the market price of foreign good risk, given by

\[ \eta_t^* = \sigma^Y e_2 + \left( \omega_t^* \sigma_t^{G*} + (1 - \omega_t^*) \sigma_t^{G*} \right) \]

(1.35)

and \( \sigma_t^{R*} \) is the diffusion process of the foreign total wealth portfolio excess return, given by

\[ \sigma_t^{R*} = \sigma^Y e_2 + \frac{((1 - a) \lambda + (1 - a^*) \lambda^*) kG}{((1 - a) \lambda + (1 - a^*) \lambda^*) kG + \rho ((1 - a) \lambda G_t + (1 - a^*) \lambda^* G^*_t)} \]

For each portfolio, the expected excess return in terms of the local good is determined by the covariation of the portfolio return with the relevant marginal utility growth: since the domestic (foreign) portfolio pays domestic (foreign) good dividends, its risk premium is compensation for domestic (foreign) good risk. As in the closed economy benchmark, holding the domestic total wealth portfolio is risky in terms of the domestic good, as it tends to generate low payoffs exactly when domestic good marginal utility is high, and vice versa. As it can be easily seen from the functional forms of \( \sigma_t^R \) and \( \eta_t \), it will have a positive risk premium. The same applies to the foreign total wealth portfolio: it pays a lot in terms of the foreign asset exactly when the foreign asset is not very valuable in marginal utility terms. After establishing that the two portfolios have positive risk premia, we can now examine the magnitude of those premia. Since the sensitivity parameter \( \delta \) is very high, the habit-induced second term of \( \eta_t \) (and \( \eta_t^* \)) contributes to a big increase of market price of risk over the power utility benchmark. In other words, the habit-induced multiplicative mechanism that generates high risk premia in models of closed economies retains its potency in open economies.

Furthermore, the market price of risk is time-varying: both \( \eta_t \) and \( \eta_t^* \) are increasing in both domestic and foreign conditional consumption growth volatilities \( \sigma_t^G \) and \( \sigma_t^{G*} \). In addition to that, returns are conditionally heteroskedastic, as both \( \sigma_t^R \) and \( \sigma_t^{R*} \) are time-varying. As a result, risk premia of the two total wealth portfolios (\( \eta_t \Sigma \sigma_t^R \) for the domestic one and \( \eta_t^* \Sigma \sigma_t^{R*} \) for the foreign one) are also time-varying, as in the closed economy case.
1.4.4 Asset prices and exchange rates

As seen in (1.24), the addition of habits generates additional variability in the real exchange rate through the mechanism of time-varying risk aversions $G$ and $G^*$. Since time-varying risk aversion also generates time variation in price-dividend ratios, we can clearly see the relationship between asset prices and the real exchange rate by rewriting (1.24) as follows:

$$E_t = \frac{a^a(1-a)^{1-a}}{(a^*)^{a^*}(1-a^*)^{1-a^*}} \left( \frac{(1-a)\lambda + (1-a^*)\lambda^*}{a\lambda + a^*\lambda^*} \right)^{a-a^*} \left( \frac{\dot{X}_t}{X_t} - \frac{1}{\rho+k} \right)^{a-a^*} \left( \frac{\dot{Y}_t}{Y_t} - \frac{1}{\rho+k} \right)^{a-a^*}$$

In our model, real exchange rate volatility is caused by two economic mechanisms: time variation in relative endowments (endowment mechanism) and time variation in price-dividend ratios (asset pricing mechanism). Since the variability of price-dividend ratios is typically much higher than the variability of macroeconomic variables, the most important mechanism for real exchange rate volatility is the asset pricing mechanism. In the absence of habits, $\frac{\dot{X}_t}{X_t}$ and $\frac{\dot{Y}_t}{Y_t}$ are constant; the asset pricing mechanism shuts down, leaving only the endowment mechanism to generate real exchange rate volatility. This is the reason standard international macroeconomic models, which do not generate time-varying price-dividend ratios, severely undershoot the empirical level of real exchange rate volatility.

1.5 Simulation

1.5.1 Definitions and data

In order to calibrate the model, I discretize it at the quarterly frequency; the United States is the domestic country and the United Kingdom is the foreign country. Since this Chapter considers an endowment model which includes neither investment nor government spending, real endowment is mapped to the sum of consumption of non-durables and services (NDS) and total net exports, in real per capita terms. This is consistent with Verdelhan (2008b) and Colacito and Croce (2008b). It should be noted that all imports and exports

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25 As previously mentioned, the population of each country is normalized to 1, but this normalization does not affect the results of the model; the analysis is identical under any assumption regarding the two countries' population measures. However, the model cannot accommodate population growth; population measures have to be constant. To correct for population growth in the empirical data, we can either adjust aggregate data for population growth, or use per capita data. The only difference between the two approaches regards the scale of the model, i.e. the level of quantities; growth rates, scaled quantities (such as price-dividend ratios and net exports ratios) and asset returns are unaffected. In this calibration, I follow the second approach and use per capita data.
(regardless of country of origin and destination, respectively) are taken into account for the calculation of each country’s endowment. On the other hand, exports and imports in the model are mapped to bilateral US-UK trade flows. The adopted mapping of model variables to real-world variables is meant to accommodate the two-country nature of the model under consideration. Specifically, regarding endowments, consumption data already include imported goods and services from all the other countries of the world, so, in order to derive the correct measure of home production, it is imperative that total imports are subtracted from consumption (and total exports added back). On the other hand, if model trade flows were mapped to total trade flows, trade between the US and the UK would be greatly exaggerated: the calibrated model would be pushed to generate unrealistic trade patterns between the two countries.

The sample period is 1975:Q1 to 2007:Q2, for a total of 130 quarterly observations. The data on seasonally-adjusted consumption, total imports and total exports, nominal and real, are from the US Bureau of Economic Analysis (BEA) and the UK Office for National Statistics (ONS). Implicit price deflators are constructed as the ratio of nominal to real quantities. Population data, used to calculate per capita figures where necessary, are constructed as the ratio of nominal GDP over nominal GDP per capita. Non-seasonally adjusted bilateral US-UK trade data (in USD) are from the BEA; they are seasonally adjusted with the US Census Bureau’s X12 seasonal adjustment program and, where necessary, are converted to GBP using the quarterly average exchange rate from the IMF International Financial Statistics (IFS). The total wealth portfolio for each country is proxied by the corresponding country’s Datastream equity index; real returns are constructed using the Total Return Index, while the series for the price-dividend ratio is constructed using the Total Return Index and the Price Index series. The nominal US risk-free rate is proxied by the 3-month Treasury bill rate (from CRSP) and the UK risk-free rate is proxied by the UK government 3-month bill yield (from ONS). The data for the nominal end-of-quarter USD/GBP exchange rate are from MSCI. CPI data are from the IMF IFS. Nominal endowment and nominal NDS consumption are deflated by the corresponding country’s CPI.

26 Under the adopted mapping, the part of the US (UK) endowment that is, in reality, exported to countries other than the UK (US) is implicitly assumed to be consumed by the US (UK) representative agent.

27 Specifically, the Total Return Index \( P_t \), the Price Index \( P_t \) and the dividend \( D_t \) are connected by the following relationship: \[ \frac{P_{t+1}}{P_t} = \frac{P_{t+1} D_{t+1}}{P_t}. \] Then, it can easily be seen that \[ \frac{P_{t+1}}{P_t} = \frac{P_{t+1}}{P_t} \left( \frac{P_{t+1} - P_{t+1}}{P_t} \right)^{-1}. \] Bansal, Dittmar and Lundblad (2005) use an equivalent procedure to calculate portfolio dividends.
The terms of trade $Q^*$ are the ratio of the implicit price deflator for total UK exports over the implicit price deflator for total US exports.

1.5.2 The endowment processes

In the previous sections, the drift processes of the endowments were left unspecified. To calibrate the model, I have to assume specific functional forms for the two endowment drifts. I choose the functional form in a way that imposes on the drifts some structure based on empirical evidence. Specifically, I would like to allow for the possibility that the real exchange rate between the two countries is stationary.²⁸ From (1.24), it follows that real exchange rate stationarity requires that the ratio $z$ of the two endowment processes be stationary. This is intuitive: a non-stationary ratio of the two endowments would imply that the output of one economy would almost surely approach zero as a proportion of the other country’s endowment as $t \to \infty$, as illustrated by Cochrane, Longstaff and Santa-Clara (2007).

For this purpose, I assume that the log of the ratio of the two endowment processes $z_t \equiv \frac{\tilde{Y}_t}{\tilde{X}_t}$ is an Ornstein-Uhlenbeck (first order autoregressive) process:

$$d \log z_t = \theta (\log \tilde{z} - \log z_t) dt + \sigma_z dB_t^z$$

(1.37)

where $\sigma^2 = (\sigma^X)^2 + (\sigma^Y)^2 - 2\sigma^X \sigma^Y \rho^{XY}$ and $dB_t^z \equiv \frac{\sigma^Y dB_t^Y - \sigma^X dB_t^X}{\sigma_z}$. The endowment ratio $z_t$ is positive (almost surely) and mean reverts to its long-run level $\tilde{z}$. The speed of mean reversion (in logs) is $\theta$ and the volatility of the (log) ratio is $\sigma_z$.

The process for $\tilde{X}_t$ is assumed to be:

$$d \log \tilde{X}_t = [\mu - \psi \theta (\log \tilde{z} - \log z_t)] dt + \sigma_X dB_t^X$$

(1.38)

Parameter $\mu$ is the unconditional domestic endowment growth rate. Parameter $\psi$ measures the degree of adjustment done by $\tilde{X}_t$. Specifically, since $\log z_t$ is stationary, $\log \tilde{X}_t$ and $\log \tilde{Y}_t$ are cointegrated, so their paths are not independent of each other: either $\log \tilde{X}_t$ adjusts to $\log \tilde{Y}_t$ (in other words, $\log \tilde{X}_t$ error-corrects), or $\log \tilde{Y}_t$ error-corrects or both do. For $\psi = 0$, $\log \tilde{X}_t$ does not adjust at all to the movements of $\log \tilde{Y}_t$, so stationarity is preserved solely by the adjustment of $\log \tilde{Y}_t$. On the other hand, for $\psi = 1$, all adjustment is done by $\log \tilde{X}_t$. For $\psi \in (0,1)$, both processes error-correct.

²⁸For a discussion of the empirical evidence regarding the stationarity of real exchange rates, see Sarno and Taylor (2002).
The process for the foreign country’s endowment $\tilde{Y}_t = z_t X_t$ is given by an application of Itô’s lemma using (1.38) and (1.37):

$$d \log \tilde{Y}_t = [\mu + (1 - \psi)(\log \tilde{z} - \log z_t)] dt + \sigma^Y dB^Y_t$$  \hspace{1cm} (1.39)

The long-run mean of the foreign endowment growth rate is also $\mu$, since, to achieve cointegration, the two countries must grow at the same rate, on average. The proportion of global error-correction performed by the foreign economy is $1 - \psi$.

It should be noted that the non-cointegration case is a special case in this setup, achieved when $\theta = 0$, in which case both log endowment processes $\log \tilde{X}$ and $\log \tilde{Y}$ are geometric Brownian motions and the log endowment ratio $\log z$ is a scaled Brownian motion.

### 1.5.3 Parameter calibration

The model includes 14 parameters in total, 7 endowment-related and the rest preference-related. The 7 endowment-related parameters ($\mu$, $\theta$, $\psi$, $\tilde{z}$, $\sigma^X$, $\sigma^Y$ and $\rho^{XY}$) are calibrated using US and UK endowment data (as defined in the previous section). Specifically, discretizing (1.37) and (1.38), I get 7 moment conditions, which are used to infer the 7 endowment parameters using exactly identified GMM estimation; details are provided in the Appendix (section 1.7.3). The parameter estimates, along with their standard errors, appear in Table 1.1. The point estimate of $\theta$, the mean-reversion speed of $\log z$ is 0.05 (0.19 annualized) and it appears that cointegration is achieved because the UK endowment adjusts to the US one, rather than vice versa: the point estimate for $\psi$ is less than 0.02 and not statistically significant, which implies that more than 98% (if not all) of the endowment adjustment is done by the UK. Clearly, the US economy has an impact on the UK one, but not vice versa. Regarding $\tilde{z}$, it should be noted that its value only affects the scale of the model variables; it can be normalized to 1 without any effect to the moments. Lastly, it appears that the US economy is less volatile than the UK one: the point estimate of US quarterly endowment growth rate standard deviation $\sigma^X$ is 0.74% (1.49% annualized), while the corresponding UK volatility $\sigma^Y$ is more than double that, standing at 1.95% (3.91% annualized). The correlation of the two endowment growth rates is around 0.16 and is marginally statistically significant, with its t-statistic being 1.98.

Regarding the 5 habit-related preferences parameters, the values for $\rho$, $\delta$, $\bar{G}$ and $l$ are the ones used in Menzly et al. (2004), while the value of $k$, the speed of mean reversion
of $G$, is adjusted downwards (from 0.16 in Menzly et al. (2004) to 0.12) to get a better fit with the return data.\textsuperscript{29} Regarding the home bias parameters $a$ and $a^*$, they are calibrated to match the share of the domestic good expenditure and foreign good expenditure in each country's consumption expenditure. In the model, the share of foreign good expenditure in the domestic consumption expenditure is

$$\frac{Y_t Q_t^*}{C_t P_t} = 1 - a$$

while the share of domestic good expenditure in the foreign consumption expenditure is

$$\frac{X_t^*}{C_t^* P_t^*} = a^*$$

Since, on average, imports from the UK represent 1.0% of US consumption expenditure and US imports correspond to 8.2% of UK consumption, the calibrated values are $a = 0.990$ and $a^* = 0.082$. For those values of the home bias parameters, the US welfare weight is $\lambda = 0.89$, with the UK weight being $\lambda^* = 0.11$; thus, the US has 8.2 times the weight of the UK.

The values for the calibrated parameters are presented in Table 1.2.

1.5.4 Simulation results

I simulate 10,000 sample paths of the model economy, each consisting of 170 quarterly observations. The system is initialized at the steady state $(z_1 = \bar{z}, G_1 = G_1^* = \bar{G})$ and I adopt the normalization $\tilde{X}_1 = 1$. Of the 170 observations, the first 40 (10 years) are discarded to reduce the dependence on initial conditions, so, at the end, each sample path consists of 130 observations, as many as available in the dataset. All moments of interest are calculated for each of the 10,000 simulated paths. For each of the moments of interest, Tables 1.3, 1.4 and 1.5 present the sample average across the 10,000 simulations, as well as the 2.5% and 97.5% percentiles across simulations.

1.5.4.1 Endowment, consumption and risk sharing

Table 1.3 presents moments pertaining to endowment and consumption growth rates and risk sharing. To begin with, the simulated model adequately captures the relative size of the

\textsuperscript{29}In Menzly et al. (2004) the 5 habit-related preference parameters are calibrated to match the following 5 moment conditions for US data: $E(P/D)$, $E(R^*)$, $\text{var}(R^*)$, $E(r^f)$ and $\text{var}(r^f)$. Since the Menzly et al. calibration is designed to match asset pricing moments, the downward adjustment of $k$ in this paper, aimed at a better fit of the asset pricing moments, is consistent with the spirit of the original calibration.
two economies: in the model, the US economy is, on average, 8.23 times the size of the UK economy, very close to the empirical value of 8.29. As outlined above, the mean, standard deviation and correlation of endowment growth rates are calibrated moments. However, the model also adequately matches the autocorrelation of the two countries’ endowment growth rates, which hints that the postulated functional form for the endowment processes is not implausible.

Since consumption is endogenous in the model, the most important moments of Table 1.3 are the ones related to consumption growth rates. Given the endogeneity of consumption, the performance of the model is unambiguously good. Simulated consumption growth rate means and, more importantly, standard deviations are very close to their empirical counterparts. Although consumption growth rate autocorrelation is not perfectly matched, the disparities between simulated and empirical data are not big.

The final two moments showcase the ability of the model to tackle the international risk sharing puzzle. As expected, the model generates very high correlation between the two pricing kernels; not only is the correlation high on average (0.88), but it is also high across simulations, with the 95% confidence interval being [0.72, 0.97]. On the other hand, consumption growth correlation is kept at moderate levels (it is 0.50 on average, and the confidence interval is [0.20, 0.66]), with the empirical value (0.33) being somewhat lower than the simulated one, but comfortably within the 95% simulation confidence interval. Figure 1.1 presents the empirical probability density functions of the domestic and foreign surplus consumption ratio \( S = \frac{1}{c} \) and \( S^* = \frac{1}{c^*} \), respectively, (Panel (a)) and their ratio \( \frac{S}{S^*} = \frac{c^*}{c} \) (Panel (b)): the latter is almost symmetrically distributed around 1, with almost all the probability mass falling in the interval [0.5, 1.5]. In fact, great disparities between the two countries’ risk aversion ratios are rare: a significant amount of mass is accumulated tightly around 1.

1.5.4.2 International trade and the real exchange rate

Table 1.4 presents moments related to the real exchange and international trade. First of all, while the model generates (log) real exchange rate changes that are much more volatile than fundamentals (endowment and consumption growth rates), but much less volatile than the pricing kernel, it overshoots empirical real exchange rate change volatility by about 50%. Hence, under the chosen calibration values, the model generates a degree of international
risk sharing that is lower than what the data imply. As will be discussed in the next section, this is a not serious shortcoming: even a slight decrease of the domestic home bias parameter can considerably increase international risk sharing and substantially reduce exchange rate change volatility.

On the other hand, the disparity between the simulated and the empirical values for (log) terms of trade change volatility and the correlation between (log) terms of trade movements and (log) real exchange rate movements highlight one of the limitations of the model. Specifically, terms of trade changes are much more volatile in the model than in the data and, while the correlation between terms of trade and real exchange rate changes is perfect in the model, it is far from perfect in the data. Taken together, those two facts imply that the terms of trade are not the only driver of real exchange rates; in reality, a part of the endowment is not tradable, so the existence of non-tradables in both countries affects the properties of the real exchange rate. Nonetheless, the poor performance of the model with respect those two moments may be exaggerated by the fact that the empirical data used to calculate the terms of trade do not match the model definition. Specifically, the terms of trade are calculated as the price of total UK exports over the price of total US exports, when the true definition, according to the model, would be the price of UK exports to the US over the price of US exports to the UK. Since there are no data that would enable the construction of the latter variable, the adopted measure of the terms of trade is imperfect.

Regarding international trade flows, simulated openness ratios (with openness defined as the ratio of the sum of imports and exports over endowment) almost exactly match their empirical values. Furthermore, the net exports ratio for both countries is pro-cyclical in both simulated and empirical data.\(^{30}\) The last moment of Table 1.4 shows the ability of the model to resolve the Backus and Smith (1993) exchange rate disconnect puzzle: the simulation-generated correlation of consumption growth rate differentials and real exchange rate changes is not only very far from 1 (at 0.14), but also in line with the data.

\(^{30}\)To be consistent with the international macroeconomics literature, this moment is calculated using the cyclical component of the time-series for both the net exports ratio and the log endowment. The cyclical component for the relevant theoretical and empirical time-series is calculated by applying the Hodrick-Prescott (1997) filter, with sensitivity parameter 1600, on the original series.
1.5.4.3 Asset prices and returns

Table 1.5 evaluates the ability of the model to match asset returns and the relationships between asset prices and the real exchange rate. Regarding the two countries' equity price-dividend ratios, the model produces plausible values for their unconditional mean and generates a cross-country price-dividend ratio correlation that is close to the empirical value. Furthermore, the model matches equity excess returns and risk-free rates in both first and second moments, with the exception that simulated excess return volatility is counterfactually high, leading to low model Sharpe ratios. The model also matches empirical excess equity return and risk-free rate cross-country correlations, with a small upward bias: the simulated values are 0.83 and 0.65, with the empirical values being 0.69 and 0.47, respectively.

The rest of the moments focus on the relation between equity prices and the real exchange rate. The model comes close to replicating the zero correlation between real US equity excess returns and US/UK log real exchange rate changes, and it also captures the negative sign - but not the magnitude - of the correlation between UK excess returns and the log real exchange rate changes. Regarding the relationship between the level of the two (log) price-dividend ratios and the level of the (log) real exchange rate, it is clear that, under the model null, those moments are very close to being uninformative: for both countries, the 95% confidence interval is very high.

In short, although the model exhibits considerable ability in capturing most of the asset pricing-related moments, it generates more cross-country correlation between risk-free rates and equity excess returns and more cross-asset correlation between equities and exchange rates than what is found in the data. This should not be very surprising: it is unlikely that two shocks are able to capture all the economic uncertainty in the US and the UK. Another important point is that although the simulated moments correspond to the total wealth portfolio for each country, the actual moments are based on Datastream equity market indices. Since consumption is not equivalent to dividends (and, thus, the total wealth portfolio is not equal to the market portfolio), there is not a perfect mapping between the model and the data in that respect.
1.5.4.4 Sensitivity with respect to the home bias parameters

The values chosen for the two home bias parameters $a$ and $a^*$, although clearly motivated by the data, may be extreme, especially regarding the domestic home bias parameter $a$. To examine the sensitivity of the model results to the home bias parameters, I fix $a^* = 8.2(1-a)$, so as to capture the relative openness (and size) of the two economies, and perform the same simulation exercise as before with $a = 0.95 + 0.005j$, $j = \{0, ..., 10\}$. The results are presented in Figure 1.2; the horizontal axis measures the value of $a$ and the vertical axis the value of the moment of interest.

First, note that a slight perturbation of the home bias parameter $a$ from 0.99 to 0.98 is sufficient to considerably improve the risk sharing properties of the model: the cross-country pricing kernel correlation increases to 0.96, with the trade-off being that the consumption growth rate correlation increases to 0.59; that is higher than its empirical value, but still much lower than 1. The correlation of consumption growth rate differentials with real exchange rate changes, which is the object of the Backus-Smith puzzle, does not change almost at all. The sharp increase in cross-country risk sharing is reflected in the volatility of the real exchange rate, which drops sharply; notice that, for $a = 0.98$, it is slightly lower than the empirical value. The small decrease of domestic home bias also pushes the international correlations of asset prices and returns higher: recall that for the benchmark case of $a = 0.99$, the model cross-country correlation was higher than the empirical correlation for price-dividend ratios, but lower for the risk-free rates and excess equity returns. Furthermore, increased risk sharing appears to weaken the links between asset prices and exchange rates, both in levels and growth rates (returns), bringing the simulated data moments closer to their empirical counterparts. It is clear that a small reduction of the domestic home bias parameter from 0.99 to 0.98 leads to a non-trivial improvement of the model fit with the data.

In contrast to the sometimes sharp changes in unconditional moments when $a$ is reduced from 0.99 to 0.98, further reductions to the value of $a$ do not change much any of the moments of interest. In general, it appears that whether an economy is open or closed has a first order effect on risk sharing and asset price correlations, whereas the degree of openness appears to be a second-order issue. It is, thus, strongly hinted that an

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31 Of course, note that $a^*$ changes much more: specifically, it increases from about 0.08 to about 0.16 in order for relative openness to remain constant.
economy that trades minimally with the outside world cannot be approximated by a closed economy. The United States, in particular, is relatively closed; however, the fact that it trades, however little, with the rest of the world has a substantial effect on its ability to share risk. That result casts significant doubt to the ability of closed economy asset pricing models to describe economic behavior in economies that exhibit even a minimal amount of international trade.

1.5.4.5 Robustness with respect to the endowment specification

The assumption that the two countries' (log) endowments are cointegrated is necessary for the (log) real exchange rate to be stationary, so it has a clear economic motivation. However, as discussed in the Appendix (section 1.7.4), an econometric study of the properties of the two endowment processes does not lead to a conclusive answer on whether they are cointegrated. For that reason, in this section I examine the robustness of the simulation results when the cointegration assumption is relaxed. Specifically, assume that endowments satisfy

\[
d\log X_t = \mu_X dt + \sigma_X dB_t^X
\]  

(1.40)

and

\[
d\log Y_t = \mu_Y dt + \sigma_Y dB_t^Y
\]  

(1.41)

In that case, \( \log z_t \) is a unit root process:

\[
d\log z_t = (\mu_Y - \mu_X) dt + \sigma_z dB_t^z
\]  

(1.42)

and, consequently, the (log) real exchange rate is non-stationary.

The 5 endowment parameters \((\mu^X, \mu^Y, \sigma^X, \sigma^Y \text{ and } \rho^{XY})\) are calibrated by exactly identified GMM estimation as in the stationary case; the estimated parameter values are given in Table 1.6. The preference parameters are calibrated to have the same values as in the previous section. In results not reported here, after repeating the simulation exercise, the simulated moments are almost exactly identical to the moments calculated in the stationary case, indicating that the model results are robust to endowment specifications that allow for a non-stationary real exchange rate.\(^{32}\)

\(^{32}\)The results are not reported in the interests of space; they are available from the author.
1.6 Conclusion

This Chapter shows that a two-good, two-country endowment model that incorporates consumption home bias in preferences and external habit formation is able to match several key risk sharing, international trade and asset pricing moments and resolve significant international finance puzzles, including the Brandt, Cochrane and Santa-Clara (2006) international risk sharing puzzle and the Backus and Smith (1993) exchange rate disconnect puzzle. Furthermore, the model shows that, in open economies, foreign preferences and economic conditions can have a significant effect on domestic asset prices and returns. The increasing volume of international transactions in the last few years implies that the asset pricing effects generated by international trade tend to increase with time and cannot be ignored anymore, even for large economies like the US. Hence, one the future goals of asset pricing should be to enrich our understanding of the links between asset prices, exchange rates and international risk sharing that characterize open economies.

The model proposed in this Chapter is quite successful, and, interestingly, that success is accomplished in a stylized environment that exhibits both complete financial markets and frictionless international trade in goods and assets. Naturally, the model is unable to completely describe the economic dynamics of the US and the UK economy. An obvious reason for those shortcomings is that some of the features of the model are unrealistic. Richer models, incorporating frictions in the international trade in goods and assets or incomplete financial markets, may provide a more accurate description of economic reality; this task is left to future research.
1.7 Appendix

1.7.1 Solution to the planner's problem

The first order conditions of the planner's problem are:

\[
\lambda e^{-\rho t} \pi(\omega, t) a \frac{G_t}{X_t} = \Theta_t
\]

\[
\lambda e^{-\rho t} \pi(\omega, t)(1 - a) \frac{G_t}{Y_t} = \Theta_t^*
\]

\[
\lambda^* e^{-\rho t} \pi(\omega, t) a^* \frac{G_t^*}{X_t^*} = \Theta_t
\]

\[
\lambda^* e^{-\rho t} \pi(\omega, t)(1 - a^*) \frac{G_t^*}{Y_t^*} = \Theta_t^*
\]

where \(\Theta_t\) and \(\Theta_t^*\) are the Lagrange multipliers associated with the market clearing condition for the domestic and foreign good, respectively, and \(\pi(\omega, t)\) is the \(P\) measure probability that state \(\omega\) occurs at time \(t\).

We now adopt the following notation: \(MU_t^X = e^{-\rho t} a \frac{G_t}{X_t}\) is the domestic agent discounted marginal utility of the domestic good and \(MU_t^Y = e^{-\rho t}(1 - a) \frac{G_t}{Y_t}\) is the domestic agent discounted marginal utility of the foreign good. Similarly for the foreign agent, domestic good marginal utility is \(MU_t^{X^*} = e^{-\rho t} a^* \frac{G_t^*}{X_t^*}\) and foreign good marginal utility is \(MU_t^{Y^*} = e^{-\rho t}(1 - a^*) \frac{G_t^*}{Y_t^*}\). Note that \(MU_t^{X^*} = \frac{\lambda}{\lambda^*} MU_t^X\) and \(MU_t^{Y^*} = \frac{\lambda}{\lambda^*} MU_t^Y\), i.e. the foreign agent discounted marginal utility for each good is proportional to the respective domestic agent marginal utility. This results from the absence of frictions in international trade for goods: the two agents are able to equalize their marginal utility growth for each of the two goods. Furthermore, the ratio of marginal utilities for the two goods is the same for both countries: \(\frac{MU_t^Y}{MU_t^X} = \frac{MU_t^{Y^*}}{MU_t^{X^*}} = \frac{Q_t^*}{Q_t} = Q_t^*\), so the law of one price holds. Denote \(\Xi_t = \lambda MU_t^X = \lambda^* MU_t^{X^*}\) and \(\Xi_t^* = \lambda MU_t^Y = \lambda^* MU_t^{Y^*}\), so that each agent's marginal utility is proportional to \(\Xi_t\) and \(\Xi_t^*\), for the respective good. It follows that \(\Xi_t^* = \Xi_t Q_t^*\).

Further, \(\xi^*_0\) and \(\xi^*_t\) are the state-price deflator processes for the domestic and foreign good, respectively.

Using the FOCs, along with the two market clearing conditions, we get the sharing rules (1.12) and (1.13). Furthermore:

\[
\Xi_t = e^{-\rho t} (a \lambda G_t + a^* \lambda^* G_t^*) \frac{1}{X_t}
\]

(1.43)
\[ \Xi_t^* = e^{-pt} \left( (1 - a) \lambda G_t + (1 - a^*) \lambda^* G_t^* \right) \frac{1}{Y_t} \]  

(1.44)

Then, the terms of trade \( Q_t^* \) are

\[ Q_t^* = \frac{\Xi_t^*}{\Xi_t} = \frac{(1 - a) \lambda + (1 - a^*) \lambda^*}{a \lambda + a^* \lambda^*} \left( \frac{G_t}{G_t^*} \right) \frac{X_t}{Y_t} \]

The domestic agent marginal utility of consumption is \( \Lambda_t = e^{-pt} \frac{G_t}{G_t^*} \) and the foreign agent marginal utility of consumption is \( \Lambda_t^* = e^{-pt} \frac{G_t^*}{G_t} \), with \( \frac{\Lambda_t}{\lambda_0} \) and \( \frac{\Lambda_t^*}{\lambda_0^*} \) being the state-price deflator processes for domestic and foreign consumption, respectively. It can easily be shown that

\[ \Lambda_t = \frac{1}{\lambda} \left( \frac{\Xi_t}{a} \right)^a \left( \frac{\Xi_t^*}{1 - a} \right)^{1-a} \]  

(1.45)

\[ \Lambda_t^* = \frac{1}{\lambda^*} \left( \frac{\Xi_t}{a^*} \right)^{a^*} \left( \frac{\Xi_t^*}{1 - a^*} \right)^{1-a^*} \]  

(1.46)

so, using (1.10), we have

\[ E_t = \frac{\lambda^* \Lambda_t^*}{\lambda \Lambda_t} \]

which generates the condition

\[ d \log \Lambda_t^* - d \log \Lambda_t = d \log E_t \]

1.7.2 Pricing kernels

There are four goods in the world economy (the domestic and the foreign good and the domestic and the foreign consumption basket, each of which is a composite good), so we define 4 marginal utility processes: \( \Xi_t \) (marginal utility of the domestic good), \( \Xi_t^* \) (marginal utility of the foreign good), \( \Lambda_t \) (marginal utility of the domestic consumption basket) and \( \Lambda_t^* \) (marginal utility of the foreign consumption basket). Thus, we can define 4 distinct market price of risk processes, each for one of the 4 aforementioned goods. Formally, the market price of domestic good risk is the bivariate process \( \eta_t \) such that

\[ \frac{d \Xi_t}{\Xi_t} = -r_t^f dt - \eta_t^f dB_t \]  

(1.47)

where \( r_t^f \) is the continuously compounded domestic good risk-free rate, i.e. the instantaneous return, in terms of the domestic good, of a locally riskless asset. This asset, called domestic bond, has price (in units of the domestic good) \( D_t \) and price process \( dD_t = r_t^f D_t dt \).
Similarly, the market price of foreign good risk is the bivariate process $\eta^*_t$ such that 

$$\frac{d\Xi^*_t}{\Xi^*_t} = -r^*_t dt - \eta^*_t dB_t$$

(1.48)

where $r^*_t$ is the foreign good risk-free rate. The locally riskless (in terms of the foreign good) asset is called the foreign bond; its price (in units of the foreign good) is $D^*_t$ and its price process is $dD^*_t = r^*_t D^*_t dt$.

We can now define the equivalent terms for the two consumption baskets, $C$ and $C^*$. The market price of domestic (foreign) consumption risk is the bivariate process $\eta^C_t$ ($\eta^{C*}_t$) such that 

$$\frac{d\Lambda^C_t}{\Lambda^C_t} = -r^C_t dt - \eta^C_t dB_t$$

and 

$$\frac{d\Lambda^{C*}_t}{\Lambda^{C*}_t} = -r^{C*}_t dt - \eta^{C*}_t dB_t$$

where, $r^C_t$ and $r^{C*}_t$ are, respectively, the domestic and the foreign consumption risk-free rate. In other words, $r^C_t$ ($r^{C*}_t$) is the return of the domestic (foreign) consumption bond, an asset that is locally riskless in terms of the domestic (foreign) consumption basket.

It should be noted that, in the case of complete home bias ($a = 1$ and $a^* = 0$), $C = X$ and $C^* = Y$, i.e. the domestic (foreign) consumption basket coincides with the domestic (foreign) good. In that case, $\Xi = \Lambda$ and $\Xi^* = \Lambda^*$, so they have equal market prices ($\eta^C_t = \eta^C_t$ and $\eta^{C*}_t = \eta^{C*}_t$) and risk-free rates ($r^C_t = r^C_t$ and $r^{C*}_t = r^{C*}_t$). This is the case of the standard one-good asset pricing paradigm, in which the consumption good and the endowment good coincide. If home bias is not complete, then the endowment good and the consumption good are not identical. In that case, we will call risk-free rate $r^f_t$ the return of the asset that is riskless in terms of the domestic good (not domestic consumption); similarly for the foreign risk-free rate $r^{f*}_t$.

The aforementioned terms are connected. An application of Itô’s lemma to (1.45) and (1.46) results in:

$$\eta^C_t = a\eta^C_t + (1 - a)\eta^C_t$$

$$\eta^{C*}_t = a^*\eta^{C*}_t + (1 - a^*)\eta^{C*}_t$$

so the market price of consumption risk (domestic and foreign) is a weighted average (with home biased weights) of the two goods’ market price of risk. This makes sense: each country’s consumption basket is nothing but a home biased basket of the two goods.
Furthermore, using (1.45) and (1.46), we get:

\[ r^C_t = ar_t^f + (1 - a)r_t^f + \frac{1}{2}a(1 - a)(\eta^*_t - \eta_t)' \Sigma (\eta^*_t - \eta_t) \]

\[ r^C t^* = a^* r_t^f + (1 - a^*)r_t^f + \frac{1}{2}a^*(1 - a^*)(\eta^*_t - \eta_t)' \Sigma (\eta^*_t - \eta_t) \]

The two consumption risk-free rates are home biased weighted averages of the two risk free rates, adjusted by a Jensen inequality term.

### 1.7.3 Endowment parameter calibration

To calibrate the 7 endowment-related parameters (\( \mu, \theta, \psi, \bar{z}, \sigma^X, \sigma^Y \) and \( \rho^{XY} \)), I consider the discretized versions of (1.37) and (1.38):

\[
\begin{bmatrix}
\Delta \log \bar{Y}_{t+1} \\
\Delta \log \bar{X}_{t+1}
\end{bmatrix} = \begin{bmatrix}
\mu + (1 - \psi)\theta \log \bar{z} \\
\mu - \psi \theta \log \bar{z}
\end{bmatrix} + \begin{bmatrix}
- (1 - \psi) \theta \\
\psi \theta
\end{bmatrix} \log z_t + \begin{bmatrix}
u^Y_{t+1} \\
u^X_{t+1}
\end{bmatrix} \tag{1.49}
\]

with

\[
\begin{bmatrix}
u^Y_{t+1} \\
u^X_{t+1}
\end{bmatrix} \sim N \left( \begin{bmatrix}0 \\
0 \end{bmatrix}, \begin{bmatrix}(\sigma^Y)^2 & \rho^{XY} \sigma^X \sigma^Y \\
\rho^{XY} \sigma^X \sigma^Y & (\sigma^X)^2 \end{bmatrix} \right) \tag{1.50}
\]

Then, the following 7 moments:

\[
E\left( u^Y_{t+1} \right) = 0 \tag{1.51}
\]

\[
E\left( u^Y_{t+1} \log z_t \right) = 0
\]

\[
E\left( u^X_{t+1} \right) = 0
\]

\[
E\left( u^X_{t+1} \log z_t \right) = 0
\]

\[
E\left( \left( u^Y_{t+1} \right)^2 - (\sigma^Y)^2 \right) = 0
\]

\[
E\left( \left( u^X_{t+1} \right)^2 - (\sigma^X)^2 \right) = 0
\]

\[
E\left( u^X_{t+1} u^Y_{t+1} \right) - \rho^{XY} \sigma^X \sigma^Y = 0
\]

constitute a system of 7 equations and 7 parameters which is estimated by (exactly identified) GMM. The first four moments are the OLS moments for the two processes, while the last 3 moments identify the covariance matrix of the two endowment shocks. The spectral density matrix is Newey-West with 5 lags. The estimation results are presented in Table 1.6.
1.7.4 Empirical examination of endowment cointegration

Consider the following specification of the two endowment processes:

\[
d\log \hat{X}_t = (\mu - \psi (\delta_0 + \delta_1 \log z_t)) \, dt + \sigma^X dB_t^X
\]

and

\[
d\log \hat{Y}_t = (\mu + (1 - \psi) (\delta_0 + \delta_1 \log z_t)) \, dt + \sigma^Y dB_t^Y
\]

Then, the process for \( \log z_t \) is

\[
d\log z_t = (\delta_0 + \delta_1 \log z_t) \, dt + \sigma^z dB_t^z
\]

It can easily be shown that the above specification yields the stationary case described by (1.38), (1.39) and (1.37) when \( \delta_1 < 0 \) and the non-stationary case of (1.40), (1.41) and (1.42) when \( \delta_1 = 0 \) (with constants appropriately renamed).

Discretizing, we can write:

\[
\begin{bmatrix}
\Delta \log \hat{Y}_{t+1} \\
\Delta \log \hat{X}_{t+1}
\end{bmatrix} = 
\begin{bmatrix}
\mu + (1 - \psi) \delta_0 \\
\mu - \psi \delta_0
\end{bmatrix} + 
\begin{bmatrix}
-1 \\
\delta_0 + \delta_1
\end{bmatrix} \log z_t + 
\begin{bmatrix}
u_{Y_{t+1}}^Y \\
u_{X_{t+1}}^X
\end{bmatrix}
\]

with

\[
\begin{bmatrix}
u_{Y_{t+1}}^Y \\
u_{X_{t+1}}^X
\end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix}
0 \\
0
\end{bmatrix}, 
\begin{bmatrix}
(\sigma^Y)^2 & \rho^{XY} \sigma^X \sigma^Y \\
\rho^{XY} \sigma^X \sigma^Y & (\sigma^X)^2
\end{bmatrix} \right)
\]

This a system of balanced regressions, in the sense that the regressor and the dependent variable are of the same order of integration, only if \( \log z \) is stationary.\(^{33}\) In that case, the system above is an error-correction system. To determine whether \( \log z \) is a stationary variable, I use the Augmented Dickey-Fuller (ADF) unit root test, when \( k = 0, 1, \ldots, 4 \) lagged differences are included in the regression. In results not reported here, the null of \( \log z \) being a unit root process cannot be rejected for any conventional levels of significance, for all \( k \).

An alternative procedure is to examine whether \( \log \hat{Y} \) and \( \log \hat{X} \) are cointegrated. Note that we can write:

\[
\begin{bmatrix}
\Delta \log \hat{Y}_{t+1} \\
\Delta \log \hat{X}_{t+1}
\end{bmatrix} = 
\begin{bmatrix}
\mu \\
\mu
\end{bmatrix} + 
\begin{bmatrix}
1 - \psi \\
-\psi
\end{bmatrix} \left( \begin{bmatrix}
\delta_0 + \delta_1 \\
1 - 1
\end{bmatrix} \begin{bmatrix}
\log \hat{Y}_t \\
\log \hat{X}_t
\end{bmatrix} \right) + 
\begin{bmatrix}
u_{Y_{t+1}}^Y \\
u_{X_{t+1}}^X
\end{bmatrix}
\]

\(^{33}\)In results not reported here, the null of unit root cannot be rejected by the Augmented Dickey-Fuller (ADF) test for neither \( \log \hat{X} \) nor \( \log \hat{Y} \), for any conventional level of significance. Furthermore, the ADF test comfortably rejects the unit root null for both \( \Delta \log \hat{X} \) and \( \Delta \log \hat{Y} \). Therefore, both \( \log \hat{X} \) and \( \log \hat{Y} \) are unit root processes and there is scope for examining whether there exists a cointegrating vector.
This is a Vector Error Correction Model (VECM), with the cointegration vector imposed to be $\begin{bmatrix} 1 & -1 \end{bmatrix}$. Without imposing this restriction, the system can be written:

$$\begin{bmatrix} \Delta \log \tilde{Y}_{t+1} \\ \Delta \log \tilde{X}_{t+1} \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \end{bmatrix} + \begin{bmatrix} 1 - \psi \\ -\psi \end{bmatrix} \begin{bmatrix} \delta_0 + \delta_1 \begin{bmatrix} 1 & \gamma \end{bmatrix} \begin{bmatrix} \log \tilde{Y}_t \\ \log \tilde{X}_t \end{bmatrix} \end{bmatrix} + \begin{bmatrix} u_{t+1}^Y \\ u_{t+1}^X \end{bmatrix}$$

for $\gamma \in \mathbb{R}$.

I consider the following more general VECM:

$$\begin{bmatrix} \Delta \log \tilde{Y}_{t+1} \\ \Delta \log \tilde{X}_{t+1} \end{bmatrix} = A + \Gamma_0 \begin{bmatrix} 1 & \gamma_0 \\ \log \tilde{Y}_t \\ \log \tilde{X}_t \\ 1 \end{bmatrix} + \sum_{j=1}^{k-1} \Gamma_j \begin{bmatrix} \Delta \log \tilde{Y}_{t-j+1} \\ \Delta \log \tilde{X}_{t-j+1} \end{bmatrix} + \begin{bmatrix} u_{t+1}^Y \\ u_{t+1}^X \end{bmatrix}$$

(1.52)

For $k = 1, \ldots, 4$, I estimate the number of cointegrating relationships using the Maximum Eigenvalue ($\lambda$-max) and the Trace test statistics, as suggested by Johansen (1988, 1991). For each $r$, the null hypothesis of both the Maximum Eigenvalue and the Trace tests is that there exist exactly $r$ cointegrating relationships; the alternative hypothesis of the former test is that there are $r + 1$ cointegrating relationships, whereas the alternative of the latter test is that there are 2 cointegrating relations. The results are presented in Table 1.7. For all lags, the two test statistics indicate that the null of $r = 0$ cannot be rejected in favor of either $r = 1$ or $r = 2$ for any conventional significance level. However, the two tests also indicate that the null of $r = 1$ cannot be rejected in favor of $r = 2$.

Our empirical results appear to point against the existence of cointegration between the two (log) endowment processes. However, given the properties of the sample under consideration (slightly more than 30 years of quarterly observations) and the economic arguments in favor of real exchange rate stationarity, a definite answer regarding the existence and the properties of a cointegrating vector for the two endowment processes is elusive in this sample.

1.7.5 Proofs

Proof of Proposition 1

Let

$$\frac{dC_t}{C_t} - E_t \left( \frac{dC_t}{C_t} \right) = \sigma_t^X dB_t^X + \sigma_t^Y dB_t^Y$$

(1.53)
and

\[ \frac{dC_t^*}{C_t^*} - E_t \left( \frac{dC_t^*}{C_t^*} \right) = \sigma_t^X dB_t^X + \sigma_t^Y dB_t^Y \]  \hspace{1cm} (1.54)

Using (1.6) and (1.8) and applying Itô's lemma, the process of the ratio \( \frac{G_t^*}{C_t} \) is:

\[ \frac{d \left( \frac{G_t^*}{C_t} \right)}{\left( \frac{G_t^*}{C_t} \right)} = \text{drift} + s_t^X dB_t^X + s_t^Y dB_t^Y \]

where

\[ s_t^X = \delta \left( \frac{G_t - l}{G_t} \right) \sigma_t^X - \delta \left( \frac{G_t^* - l}{G_t^*} \right) \sigma_t^X \]
\[ s_t^Y = \delta \left( \frac{G_t - l}{G_t} \right) \sigma_t^Y - \delta \left( \frac{G_t^* - l}{G_t^*} \right) \sigma_t^Y \]

Applying Itô's lemma to (1.14) and (1.15), we get, respectively:

\[ d\log C_t = \text{drift} + a\sigma^X dB_t^X + (1 - a)\sigma^Y dB_t^Y - k_t (s_t^X dB_t^X + s_t^Y dB_t^Y) \]
\[ d\log C_t^* = \text{drift} + a^*\sigma^X dB_t^X + (1 - a^*)\sigma^Y dB_t^Y + k_t^* (s_t^X dB_t^X + s_t^Y dB_t^Y) \]

where we have used the following definitions for \( k_t \) and \( k_t^* \):

\[ k_t \equiv \frac{\lambda^*}{\lambda} \frac{a(1 - a)\lambda + a^*(1 - a^*)\lambda^* \left( \frac{G_t^*}{C_t^*} \right)}{(a\lambda + a^*\lambda^* \left( \frac{G_t^*}{C_t^*} \right)) \left( (1 - a)\lambda + (1 - a^*)\lambda^* \left( \frac{G_t^*}{C_t^*} \right) \right)} \]  \hspace{1cm} (1.55)
\[ k_t^* \equiv \frac{\lambda}{\lambda^*} \frac{a(1 - a)\lambda + a^*(1 - a^*)\lambda^* \left( \frac{G_t}{C_t} \right)}{(a\lambda + a^*\lambda^* \left( \frac{G_t}{C_t} \right)) \left( (1 - a)\lambda + (1 - a^*)\lambda^* \left( \frac{G_t}{C_t} \right) \right)} \]  \hspace{1cm} (1.56)

Regarding \( k_t \), \( \lim \limits_{\frac{G_t}{C_t} \to 0} k_t = 0 \) and \( \lim \limits_{\frac{G_t}{C_t} \to \infty} k_t = 1 \). Further, it can be shown that, for the empirically relevant case \( 0 < a^* < a < 1 \), \( k_t \) is globally increasing in \( \frac{G_t}{C_t} \), so it is bounded in \( (0, 1) \).

Similarly, \( \lim \limits_{\frac{G_t}{C_t} \to 0} k_t^* = 1 \) and \( \lim \limits_{\frac{G_t}{C_t} \to \infty} k_t^* = 0 \). For \( 0 < a^* < a < 1 \), \( k_t^* \) is globally decreasing in \( \frac{G_t}{C_t} \), so it is also bounded in \( (0, 1) \).

On the other hand, applying Itô's lemma to (1.53) and (1.54), we get

\[ d\log C_t = \text{drift} + \sigma_t^X dB_t^X + \sigma_t^Y dB_t^Y \]
\[ d\log C_t^* = \text{drift} + \sigma_t^* X dB_t^X + \sigma_t^{* Y} dB_t^Y \]
Matching diffusions, we get the following system of equations:

\[
\begin{align*}
\sigma_t^{CX} &= -k_t \left[ \delta \left( \frac{G_t}{G_t} \right) \sigma_t^{CX} - \delta \left( \frac{G_t^*}{G_t} \right) \sigma_t^{*X} \right] + a\sigma_X \\
\sigma_t^{CY} &= -k_t \left[ \delta \left( \frac{G_t}{G_t} \right) \sigma_t^{CY} - \delta \left( \frac{G_t^*}{G_t} \right) \sigma_t^{*Y} \right] + (1-a)\sigma_Y \\
\sigma_t^{*X} &= k_t^* \left[ \delta \left( \frac{G_t}{G_t} \right) \sigma_t^{CX} - \delta \left( \frac{G_t^*}{G_t} \right) \sigma_t^{*X} \right] + a^*\sigma_X \\
\sigma_t^{*Y} &= k_t^* \left[ \delta \left( \frac{G_t}{G_t} \right) \sigma_t^{CY} - \delta \left( \frac{G_t^*}{G_t} \right) \sigma_t^{*Y} \right] + (1-a^*)\sigma_Y \\
\end{align*}
\]

the solution of which is

\[
\begin{bmatrix}
\sigma_t^{CX} \\
\sigma_t^{*X}
\end{bmatrix} = \frac{1}{D_t^{C}} \begin{bmatrix}
a + (ak_t^* + a^*k_t) \delta \left( \frac{G_t-1}{G_t} \right) \\
a^* + (ak_t^* + a^*k_t) \delta \left( \frac{G_t^*-1}{G_t^*} \right)
\end{bmatrix} \sigma_X
\]

and

\[
\begin{bmatrix}
\sigma_t^{CY} \\
\sigma_t^{*Y}
\end{bmatrix} = \frac{1}{D_t^{C}} \begin{bmatrix}
(1-a) + ((1-a)k_t^* + (1-a^*)k_t) \delta \left( \frac{G_t-1}{G_t} \right) \\
(1-a^*) + ((1-a)k_t^* + (1-a^*)k_t) \delta \left( \frac{G_t^*-1}{G_t^*} \right)
\end{bmatrix} \sigma_Y
\]

where

\[
D_t^{C} = 1 + k_t \delta \left( \frac{G_t-1}{G_t} \right) + k_t^* \delta \left( \frac{G_t^*-1}{G_t^*} \right) \tag{1.57}
\]

**Proof of Proposition 3**

As mentioned in the main text, time \(t\) domestic country wealth is the sum of its appropriately discounted future consumption flows. To calculate domestic wealth in units of the domestic good, we convert all future good flows in units of the domestic good and discount using the domestic state-price deflator. Thus

\[
W_t = E_t \left[ \int_t^\infty \frac{\Xi_s}{\Xi_t} (X_s + Q_s^* Y_s) \, ds \right]
\]

Similarly, to calculate foreign wealth in units of the foreign good we use:

\[
W_t^* = E_t \left[ \int_t^\infty \frac{\Xi_s}{\Xi_t} \left( \frac{X_s^*}{Q_s^*} + Y_s^* \right) \, ds \right]
\]

Using the solutions for \(\Xi_t, \Xi_t^*, X_t, Y_t, X_t^*, Y_t^*\) and \(Q_t^*\), we can calculate the wealth of the two countries, given in (1.26) and (1.27).

On the other hand the time \(t\) value of the domestic country endowment, in units of the domestic good, is \(E_t \left[ \int_t^\infty \frac{\Xi_s}{\Xi_t} \bar{X}_s \, ds \right]\). After some algebra:

\[
E_t \left[ \int_t^\infty \frac{\Xi_s}{\Xi_t} \bar{X}_s \, ds \right] = \frac{(a\lambda + a^*\lambda^*)k\bar{G} + \rho (a\lambda G_t + a^*\lambda^* G_t^*)}{\rho (\rho + k) (a\lambda G_t + a^*\lambda^* G_t^*)} \bar{X}_t \tag{1.58}
\]
To calculate $\lambda$, we use the fact that, at $t = 0$, the wealth of each country equals the value (in units of the numeraire) of each country’s endowment. Therefore, for the domestic country, it holds that

$$W_0 = E_0 \left[ \int_0^\infty \frac{\Xi_t}{\Xi_0} \tilde{X}_t dt \right]$$

Setting $t = 0$, we can get closed-form expressions for $W_0$ and $E_0 \left[ \int_0^\infty \frac{\Xi_t}{\Xi_0} \tilde{X}_t dt \right]$ from (1.26) and (1.58), respectively. Equating the two expressions, as above, and using the normalization $\lambda + \lambda^* = 1$, we derive the expression for $\lambda$ given in Proposition 3.

Finally, to derive (1.28), consider that, after substituting the expressions for $W_t$, $W_t^*$ and $Q_t^*$, we get

$$W_t = \frac{(\rho G_t + k\tilde{G}) \lambda}{(\rho G_t + k\tilde{G}) \lambda + (\rho G_t^* + k\tilde{G}) \lambda^*}$$

Setting $t = 0$ and substituting the expressions for $\lambda$ and $\lambda^*$ in Proposition 3, we get (1.28).

**Proof of Proposition 4**

From (1.47) and (1.48) we get

$$r_t^f = -\frac{1}{dt} E_t \left( \frac{d\Xi_t}{\Xi_t} \right)$$

$$r_t^{f*} = -\frac{1}{dt} E_t \left( \frac{d\Xi_t^*}{\Xi_t^*} \right)$$

so, applying Itô’s lemma to (1.43) and (1.44) to derive the SDEs that $\Xi_t$ and $\Xi_t^*$ solve and then taking conditional expectations, we arrive at (1.29) and (1.30).

**Proof of Proposition 5**

The price of the domestic total wealth portfolio is (in units of the domestic good):

$$V_t = E_t \left[ \int_t^\infty \frac{\Xi_s}{\Xi_t} \tilde{X}_s ds \right]$$

and, similarly, the price of the foreign total wealth portfolio is (in units of the foreign good):

$$V_t^* = E_t \left[ \int_t^\infty \frac{\Xi_s^*}{\Xi_t} \tilde{Y}_s ds \right]$$

Using the expressions for $\Xi_t$, $\Xi_t^*$, $\tilde{X}_t$ and $\tilde{Y}_s$, we get, after some algebra, (1.31) and (1.32).

**Proof of Proposition 6**

We can define the diffusion processes of the domestic and the foreign total wealth portfolio excess return, $\sigma_t^f$ and $\sigma_t^{f*}$, respectively, as the bivariate processes such that

$$dR_t^f = \frac{dV_t}{V_t} + \frac{\tilde{X}_t}{V_t} dt - r_t^f dt = drift + \sigma_t^f dB_t$$
\[
\frac{dR^*}{R^*} = \frac{dV^*_t}{V^*_t} + \rho_1^* dt - \rho_2^* dt = \text{drift} + \sigma^{R_1^*}_t dB_t
\]

From (1.31), we can use Itô's lemma to obtain the SDE satisfied by \( V_t \); then, it is easy to see that the diffusion process of the domestic total wealth portfolio is given by (1.34).

The domestic good market price of risk \( \eta_t \) is defined in (1.47) and, applying Itô's lemma to (1.43), we get that:
\[
\frac{d\Xi_t}{\Xi_t} = \text{drift} - \left( \sigma^X \eta_1 + \left( \omega_t \sigma^G_t + (1 - \omega_t) \sigma^{G_1}_t \right) \right) dB_t
\]
so, equating diffusions, we get (1.33).

Similarly, applying Itô's lemma to get the SDE for \( V^*_t \), we get that the diffusion process of the foreign total wealth portfolio is given by (1.36). An application of Itô's lemma to (1.44) gives
\[
\frac{d\Xi^*_t}{\Xi^*_t} = \text{drift} - \left( \sigma^Y \eta_2 + \left( \omega^*_t \sigma'^G_t + (1 - \omega^*_t) \sigma'^{G}_t \right) \right) dB_t
\]
so the foreign good market price of risk, defined in (1.48), is given by (1.35).

Therefore, the excess return of the domestic total wealth portfolio is
\[
\frac{dR^*_t}{R^*_t} = \mu^R_t dt + \sigma_t^R dB_t
\]
where \( \mu^R_t \) is the domestic total wealth portfolio conditional risk premium, calculated as
\[
\mu^R_t = -\frac{1}{dt} E_t \left( \frac{dR^*_t}{R^*_t} \right) = \eta_t \Sigma \sigma_t^R
\]
Similarly, the excess return of the foreign total wealth portfolio is
\[
\frac{dR^{*_t}}{R^{*_t}} = \mu^{R^*_t} dt + \sigma_t^{R^*_t} dB_t
\]
where \( \mu^{R^*_t} \) is the foreign total wealth portfolio conditional risk premium, given by
\[
\mu^{R^*_t} = -\frac{1}{dt} E_t \left( \frac{dR^{*_t}}{R^{*_t}} \right) = \eta_t^{R^*_t} \Sigma \sigma_t^{R^*_t}
\]
Parameter & Estimate \\ 
\hline
\mu & 0.0037 \\ & (0.0006) \\
\theta & 0.0520 \\ & (0.0204) \\
\psi & 0.0195 \\ & (0.1436) \\
\log z & -1.0348 \\ & (0.0243) \\
\sigma^X & 0.0074 \\ & (0.0007) \\
\sigma^Y & 0.0195 \\ & (0.0018) \\
\rho^{XY} & 0.1553 \\ & (0.0786) \\
\hline

Table 1.1: Endowment calibration

The endowment parameters in (1.37) and (1.38) are estimated by exactly identified GMM, with moment conditions given in (1.51). The spectral density matrix is Newey-West with 5 lags. Standard errors in parentheses. Note: the parameters are not annualized.
## Endowment parameter

<table>
<thead>
<tr>
<th>Endowment parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady-state of endowment ratio $z$</td>
<td>$\bar{z}$</td>
<td>1</td>
</tr>
<tr>
<td>Speed of $z$ mean reversion</td>
<td>$\theta$</td>
<td>0.192</td>
</tr>
<tr>
<td>Endowment growth rate</td>
<td>$\mu$</td>
<td>0.015</td>
</tr>
<tr>
<td>Domestic contribution to endowment adjustment</td>
<td>$\psi$</td>
<td>0.02</td>
</tr>
<tr>
<td>Domestic endowment growth volatility</td>
<td>$\sigma^X$</td>
<td>0.015</td>
</tr>
<tr>
<td>Foreign endowment growth volatility</td>
<td>$\sigma^Y$</td>
<td>0.039</td>
</tr>
<tr>
<td>Endowment growth correlation</td>
<td>$\rho^{XY}$</td>
<td>0.155</td>
</tr>
</tbody>
</table>

## Preference parameter

<table>
<thead>
<tr>
<th>Preference parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domestic preference for the domestic good</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>Foreign preference for the domestic good</td>
<td>$\alpha^*$</td>
</tr>
<tr>
<td>Subjective rate of time preference</td>
<td>$\rho$</td>
</tr>
<tr>
<td>Speed of $G$ mean reversion</td>
<td>$k$</td>
</tr>
<tr>
<td>$G$ sensitivity to consumption growth shocks</td>
<td>$\delta$</td>
</tr>
<tr>
<td>Lower bound of $G$</td>
<td>$l$</td>
</tr>
<tr>
<td>Steady-state value of $G$</td>
<td>$\bar{G}$</td>
</tr>
</tbody>
</table>

**Table 1.2: Calibration parameters**

Calibration parameters. All parameters are annualized. The endowment parameters in Table 1.2 are the annualized counterparts of the parameters in Table 1.1, with the exception of the steady-state endowment ratio $\bar{z}$, which is normalized to 1.
Table 1.3: Simulation results: endowment and consumption

<table>
<thead>
<tr>
<th>Moment</th>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>US</td>
<td>UK</td>
</tr>
<tr>
<td>Relative endowment value mean</td>
<td>8.23</td>
<td>[6.97, 9.47]</td>
</tr>
<tr>
<td>Endowment growth mean</td>
<td>1.50%</td>
<td>[1.00%, 2.02%]</td>
</tr>
<tr>
<td>Endowment growth st. dev.</td>
<td>1.50%</td>
<td>[1.31%, 1.69%]</td>
</tr>
<tr>
<td>Endowment growth corr.</td>
<td>0.15</td>
<td>[-0.02, 0.32]</td>
</tr>
<tr>
<td>Endowment growth autocorr.</td>
<td>-0.01</td>
<td>[-0.18, 0.16]</td>
</tr>
<tr>
<td>Consumption growth mean</td>
<td>1.50%</td>
<td>[1.00%, 2.01%]</td>
</tr>
<tr>
<td>Consumption growth st. dev.</td>
<td>1.46%</td>
<td>[1.25%, 1.68%]</td>
</tr>
<tr>
<td>Consumption growth autocorr.</td>
<td>-0.01</td>
<td>[-0.18, 0.16]</td>
</tr>
<tr>
<td>Consumption growth corr.</td>
<td>0.50</td>
<td>[0.20, 0.66]</td>
</tr>
<tr>
<td>Log pricing kernel corr.</td>
<td>0.88</td>
<td>[0.72, 0.97]</td>
</tr>
</tbody>
</table>

A comparison of endowment and consumption simulated and empirical moments. To calculate the former, I simulate 10,000 sample paths of the model economy, with each path consisting of 170 quarterly observations. The system is initialized at $z_t = z, G_1 = G = G_1$ and $X_1 = 1$. Of the 170 observations, the first 40 (10 years) are discarded to reduce the dependence on initial conditions. Thus, each sample path consists of 130 observations, as many as available in the dataset. For each of the moments of interest, Table 1.3 presents the sample average across the 10,000 simulations, as well as the 2.5% and 97.5% percentiles across simulations (in brackets). The empirical moments are calculated by using data from 1975:Q1 to 2007:Q2, for a total of 130 quarterly observations.
<table>
<thead>
<tr>
<th>Moment</th>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log real exchange rate change st. dev.</td>
<td>US 16.27%</td>
<td>US 10.26%</td>
</tr>
<tr>
<td></td>
<td>[12.27%, 23.19%]</td>
<td></td>
</tr>
<tr>
<td>Log terms of trade change st. dev.</td>
<td>US 17.92%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[13.51%, 25.54%]</td>
<td>3.75%</td>
</tr>
<tr>
<td>$corr(\log Q^*, \log E)$</td>
<td>1.00</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>[1.00, 1.00]</td>
<td></td>
</tr>
<tr>
<td>Openness mean</td>
<td>0.02 [0.02, 0.02]</td>
<td>0.16 [0.15, 0.18]</td>
</tr>
<tr>
<td>Corr. of NX with endowment</td>
<td>0.24 [-0.16, 0.60]</td>
<td>0.63 [0.24, 0.88]</td>
</tr>
<tr>
<td>$corr(\Delta e_{t+1}, \Delta c_{t+1}^* - \Delta c_{t+1})$</td>
<td>0.14 [-0.49, 0.52]</td>
<td>0.06</td>
</tr>
</tbody>
</table>

Table 1.4: Simulation results: international trade and the real exchange rate

A comparison of international trade and real exchange rate simulated and empirical moments. To calculate the former, I simulate 10,000 sample paths of the model economy, with each path consisting of 170 quarterly observations. The system is initialized at $z_1 = \bar{z}$, $G_1 = G^\top = \bar{G}$ and $\bar{X}_1 = 1$. Of the 170 observations, the first 40 (10 years) are discarded to reduce the dependence on initial conditions. Thus, each sample path consists of 130 observations, as many as available in the dataset. For each of the moments of interest, Table 1.4 presents the sample average across the 10,000 simulations, as well as the 2.5% and 97.5% percentiles across simulations (in brackets). The empirical moments are calculated by using data from 1975:Q1 to 2007:Q2, for a total of 130 quarterly observations.
<table>
<thead>
<tr>
<th>Moment</th>
<th>Model</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>US</td>
<td>UK</td>
</tr>
<tr>
<td>Log pricing kernel st. dev.</td>
<td>34.50%</td>
<td>37.09%</td>
</tr>
<tr>
<td></td>
<td>[16.81%, 69.53%]</td>
<td>[19.46%, 73.10%]</td>
</tr>
<tr>
<td>Equity P/D mean</td>
<td>32.10</td>
<td>31.90</td>
</tr>
<tr>
<td></td>
<td>[27.09, 34.88]</td>
<td>[26.38, 35.14]</td>
</tr>
<tr>
<td>Equity P/D corr.</td>
<td>0.79</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td>[0.45, 0.97]</td>
<td></td>
</tr>
<tr>
<td>Equity excess return mean</td>
<td>6.74%</td>
<td>8.46%</td>
</tr>
<tr>
<td></td>
<td>[2.88%, 10.01%]</td>
<td>[3.66%, 12.70%]</td>
</tr>
<tr>
<td>Equity excess return st. dev.</td>
<td>22.03%</td>
<td>25.15%</td>
</tr>
<tr>
<td></td>
<td>[13.95%, 30.90%]</td>
<td>[17.38%, 33.78%]</td>
</tr>
<tr>
<td>Equity excess return corr.</td>
<td>0.83</td>
<td>0.69</td>
</tr>
<tr>
<td></td>
<td>[0.69, 0.92]</td>
<td></td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>0.32</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td>[0.11, 0.54]</td>
<td>[0.13, 0.58]</td>
</tr>
<tr>
<td>Risk-free rate mean</td>
<td>0.46%</td>
<td>0.14%</td>
</tr>
<tr>
<td></td>
<td>[-1.25%, 3.54%]</td>
<td>[-2.66%, 4.09%]</td>
</tr>
<tr>
<td>Risk-free rate st. dev.</td>
<td>1.67%</td>
<td>2.01%</td>
</tr>
<tr>
<td></td>
<td>[0.80%, 3.03%]</td>
<td>[1.07%, 3.26%]</td>
</tr>
<tr>
<td>Risk-free rate corr.</td>
<td>0.65</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td>[0.18, 0.93]</td>
<td></td>
</tr>
<tr>
<td>$corr(\Delta \log E, R^e)$</td>
<td>0.10</td>
<td>-0.40</td>
</tr>
<tr>
<td></td>
<td>[-0.29, 0.45]</td>
<td>[-0.72, 0.02]</td>
</tr>
<tr>
<td>$corr(\log E, \log(P/D))$</td>
<td>0.13</td>
<td>-0.39</td>
</tr>
<tr>
<td></td>
<td>[-0.58, 0.74]</td>
<td>[-0.88, 0.42]</td>
</tr>
</tbody>
</table>

Table 1.5: Simulation results: asset prices and returns

A comparison of asset price and return simulated and empirical moments. To calculate the former, I simulate 10,000 sample paths of the model economy, with each path consisting of 170 quarterly observations. The system is initialized at $z_1 = \bar{z}$, $G_1 = G_1^* = \bar{G}$ and $\tilde{X}_1 = 1$. Of the 170 observations, the first 40 (10 years) are discarded to reduce the dependence on initial conditions. Thus, each sample path consists of 130 observations, as many as available in the dataset. For each of the moments of interest, Table 1.5 presents the sample average across the 10,000 simulations, as well as the 2.5% and 97.5% percentiles across simulations (in brackets). The empirical moments are calculated by using data from 1975:Q1 to 2007:Q2, for a total of 130 quarterly observations.
Table 1.6: Calibration parameters: no endowment cointegration

The parameters are estimated by exactly identified GMM, using as moment conditions the first, third, fifth, sixth and seventh conditions in (1.51). The spectral density matrix is Newey-West with 5 lags. Standard errors in parentheses. Note: the parameters are not annualized.
Table 1.7: Johansen cointegration tests

Results of Johansen cointegration tests for log $\tilde{X}$ and log $\tilde{Y}$; the econometric specification is given by (1.52). For the null hypothesis of $r = 0$ and $r = 1$ cointegrated vectors, the Maximum Eigenvalue ($\lambda$-max) and Trace test statistics are presented in the second and fourth column, respectively; the $p$-values for each test are presented in the third and fifth columns.
Figure 1.1: Empirical probability density functions of surplus consumption ratios

Empirical probability density functions (PDFs). Panel (a) presents the empirical PDF of the domestic and foreign surplus consumption ratio $S = \frac{1}{\sigma}$ and $S^* = \frac{1}{\sigma^*}$ (for the United States and United Kingdom, respectively). Panel (b) presents the empirical PDF of the ratio $\frac{S}{S^*} = \frac{\sigma}{\sigma^*}$. The empirical estimate is calculated by using a normal kernel.
Figure 1.2: Simulated moments for different values of $a$

Simulated moments. I examine the sensitivity of model results to the home bias parameters by fixing $a^* = 8.2(1-a)$, so as to capture the relative openness of the two economies, and vary the domestic home bias parameter $a$: $a = 0.95 + 0.005j$, $j = \{0, ..., 10\}$. For each of the values of $a$, I simulate 10,000 sample paths of the model economy and, for each of the moments of interest, I calculate the sample average across the 10,000 simulations. The horizontal axis measures the value of $a$ and the vertical axis the value of the moment of interest. Panel (a) presents the correlation between the domestic and the foreign consumption growth rate, the correlation between the domestic and the foreign pricing kernel and the Backus and Smith (1993) puzzle correlation $corr(\Delta c_{t+1}, \Delta c_{t+1}^* - \Delta c_{t+1})$. Panel (b) presents the standard deviation of: the domestic consumption growth rate, the domestic pricing kernel and log real exchange rate changes. Panel (c) presents the correlation between domestic and foreign risk-free rates and between the excess returns and price-dividend ratios of the two countries' total wealth portfolios. Panel (d) presents, for each of the two countries, the correlation: (i) between the excess return of the total wealth portfolio and the change in the log real exchange rate, and (ii) between the log price-dividend ratio of the total wealth portfolio and the log real exchange rate.
Chapter 2

Portfolio Choice in Open Economies with External Habit Formation

2.1 Introduction

One of the most extensively documented stylized facts in international finance is portfolio home bias, the overwhelming tendency of investors to hold portfolios heavily skewed towards assets of their home country. It has long been recognized that portfolio home bias is puzzling, as it suggests that investors forego significant international diversification benefits.\(^1\) Indeed, two of the most prevalent international finance models, the international version of the Capital Asset Pricing Model (CAPM) and the Lucas (1982) two-country general equilibrium model, imply that the desire for international risk sharing should lead countries to hold identical, completely diversified equity portfolios. Extensions of those models that incorporate frictions and non-tradeabilities have been largely unsuccessful in explaining the home bias puzzle and, in some cases, have exacerbated the puzzle, suggesting that optimal portfolios should have a foreign bias.

This Chapter proposes a two-country general equilibrium model in which optimal risk sharing implies a very high degree of home bias; countries share risk because of, not despite, portfolio home bias. The key mechanism is novel: home bias results from the

\(^1\)There is a long literature on the importance and benefits of international diversification; see, for example, Grubel (1968), Levy and Sarnat (1970), Lessard (1973), Solnik (1974a) and Errunza (1983).
desired of investors to hedge against adverse shocks in risk aversion. Agents' preferences are characterized by external habit formation, which generates countercyclical risk aversion and, thus, increases the marginal utility of consumption in recessions. Domestic equity is a better hedge against domestic risk aversion shocks than foreign equity; conversely, foreign equity is a better hedge against foreign risk aversion shocks. Consequently, domestic agents hedge their exposure to risk aversion fluctuations by tilting their portfolio towards domestic equity; foreign investors bias their portfolio towards foreign equity. Furthermore, the model proposed in this Chapter is able to reconcile international portfolio choice with realistic asset price dynamics. Given the overwhelming importance of asset return dynamics for portfolio choice, this aspect of the model satisfies a long-standing need of the general equilibrium literature in international finance, which has focused on standard preferences that are unable to match key asset pricing moments.\(^2\)

The model considered in the Chapter is a two-country general equilibrium model, in the tradition of Lucas (1982). The global economy is comprised of two countries, Domestic and Foreign, each populated by a representative agent. There are two goods, the domestic and the foreign one, and each of the agents is endowed with a Lucas tree that pays continuous dividends in the country's home good. Each agent has preferences over a home-biased Cobb-Douglas consumption basket comprised of both goods, so each of the agents derives more utility by consuming her home good than by consuming the other country's good. As expected, in equilibrium the home bias in preferences generates consumption home bias. However, it is not sufficient to generate portfolio home bias; indeed, if preferences are of the standard CRRA form, there can be a range of portfolios that can support the competitive equilibrium allocation.\(^3\) To generate portfolio home bias, a second assumption about preferences is crucial: external habit formation. What external habit formation does is to cause the heterogeneity in consumption, generated by the heterogeneity in preferences, to give rise to differential hedging demands. In a nutshell, agents do not bias their portfolio towards the home asset because they want to consume the dividends of that asset; rather, they want to hold their home country's equity because home equity is the best hedge against adverse movements in their marginal utility. In fact, in this model, consumption

\(^2\)Lewis (1999) notes that "[A] major problem with reconciling investor home bias with international consumption movements is that the volatility of the implicit intertemporal marginal rate of substitution is not high enough to explain stock price movements."

\(^3\)Zapatero (1995) also illustrates that point; he proposes a two-good, two country model with log preferences and shows that, in equilibrium, portfolios are indeterminate.
home bias and portfolio home bias are negatively, not positively related: more home biased preferences generate more consumption home bias, but less portfolio home bias. The reason is that higher consumption home bias limits the hedging benefits provided by home equity. Thus, this Chapter shows that it is possible to reconcile portfolio choice with consumption choice. As expected, the link is through consumption home bias; what may not be expected is that the mechanism is more complicated than simple intuition may suggest: home bias in preferences generates deviations from purchasing power parity (PPP), so that countries face different investment opportunity sets in real terms, and external habit formation causes the PPP deviations to generate different hedging demands.

The literature on international portfolio choice and, more specifically, the portfolio home bias puzzle is extensive. As mentioned above, the canonical models are the international CAPM and the Lucas (1982) general equilibrium model. The key assumption of both models is that country representative agents are identical, so their portfolio holdings should reflect that symmetry. In slightly more detail, both models imply that real exchange rates are constant and PPP holds; the PPP assumption implies that all agents face the investment opportunity set in real terms, so, if they are identical, country portfolios should also be identical. Specifically, the international CAPM is a partial equilibrium model which, taking asset returns as given, leads to the result that the capitalization-weighted world market portfolio is \textit{ex-ante} mean-variance efficient. On the other hand, Lucas (1982), from a general equilibrium standpoint, examines both a barter and a monetary version of a complete markets, two-good, two-country endowment economy with no frictions in trade and finds that the two countries perfectly pool their risks and, as a result, their consumption growth is perfectly correlated and is financed by holding identical internationally diversified portfolios.\footnote{Lucas recognizes that the portfolio results of his paper are not empirically validated, mentioning that his result is "[G]rossly at variance with what we know about the spatial distribution of portfolios: Americans hold a disproportionately high fraction of claims to American earnings in their portfolios, Japanese a high fraction of Japanese assets, and so on."}
The reasoning is straightforward: identical agents share consumption risk by holding the portfolios that allow them to have perfectly correlated consumption growth.

Naturally, the corollary of that reasoning is that differences among country portfolios can only be generated by relaxing the assumption that countries are identical, so a large and diverse international finance literature has focused in exploring what kind of differences between countries can bias investors towards holding their home assets. The literature has
proposed classes of models that relax different assumptions of the standard models. The first class of models retain the assumption of frictionless financial markets and no arbitrage opportunities and introduce frictions in the international trade for goods, in the form of finite or infinite deadweight costs (in the latter case, the literature calls the goods internationally non-tradeable). Those frictions allow for different equilibrium consumption across countries and, thus, a non-constant real exchange rate (i.e. PPP deviations). The motivation for those models arises from the observation that portfolio home bias may be linked to consumption home bias, the tendency of consumption growth to be highly correlated with output growth in a given country. The reasoning is that differences in equilibrium consumption expose countries to different consumption risks, so countries have to hold different portfolios in order to hedge those risks. Under this reasoning, home bias in portfolios can be justified if domestic assets hedge domestic consumption risk better than foreign assets.

In that vein, Adler and Dumas (1983) modify the international CAPM by proposing a partial equilibrium model which emphasizes the role of inflation risk. Specifically, they consider a setup in which agents consume different baskets of goods. In the presence of PPP deviations, the optimal portfolio for each agent contains an idiosyncratic hedging component against country inflation and the standard international CAPM does not hold. Other models emphasizing the importance of hedging demand for domestic inflation are Solnik (1974b), Sercu (1980) and Stulz (1981b). However, Adler and Dumas (1983), using data from 1971-1979, show that hedging portfolios are almost entirely comprised of home bonds, since risk averse agents prefer to assume some small inflation risk rather than hold more substantial exchange rate risk. Similarly, Cooper and Kaplanis (1994) show that the data strongly reject the claim that portfolio home bias can be explained by hedging for inflation risk.

In a general equilibrium setting, Stockman and Dellas (1989) consider a complete markets endowment economy in which each agent has separable preferences over traded and non-traded goods; preferences over traded good are identical across agents, while preferences over non-traded goods can differ. They show that the optimal domestic portfolio is comprised by the entirety of claims on the domestic non-traded goods and a diversified

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5See, for example, the argument in Obstfeld and Rogoff (2000).
6Indeed, they show that the pricing model that emerges is a multi-beta model, in which the riskiness of each asset does not only depend on its covariance with the world market portfolio, but also on its covariance with each of the individual countries' inflation.
portfolio of the claims on traded goods. Thus, consumption and portfolio home bias are both increasing in the share of non-tradeables in domestic consumption. Baxter, Jermann and King (1998) consider a generalized version of the Stockman and Dellas (1989) model by relaxing the separability assumption between traded and non-traded goods and show that, in a world with identical preferences for all agents, each country holds a diversified portfolio of the claims on tradeable goods, as in the Stockman and Dellas (1989) model. However, regarding the claims on non-tradeable goods, each agent holds appropriate - depending on the degree of substitutability between traded and non-traded goods - quantities of two other funds: i) a diversified portfolio of the claims on non-tradeable goods and ii) the claim on the country's non-traded good. It follows that, under separable preferences, portfolio home bias is not likely to be justified by the presence of non-tradeables. On the other hand, Tesar (1993) and Pesenti and van Wincoop (2002) consider partial equilibrium portfolio choice in economies in which preferences are non-separable and claims on non-tradeable goods are themselves non-tradeable. Tesar (1993) argues that, under those assumptions, home bias in the shares of claims on tradeable goods is possible, but Pesenti and van Wincoop (2002) show that the degree of home bias generated in such an economy would be insufficient to account for the bias observed in the data.\footnote{In the same vein, Serrat (2001) proposes a model with heterogeneous preferences and non-tradeables, but Kollman (2006) shows that Serrat's model is characterized by portfolio indeterminacy. The only aspect of portfolio choice that can be pinned down is that, regarding claims to tradeable goods, each agent holds an internationally diversified portfolio.}

Uppal (1993) departs from the assumption of outright non-tradeability and considers finite trade costs. Specifically, he proposes a complete markets general equilibrium one-good, two-country model with identical preferences across countries. Although financial markets are frictionless, there is a proportional deadweight cost when real capital is transferred internationally. In this economy, investors more (less) risk verse than a log utility investor bias their portfolios against (towards) home equity, so for realistic levels of risk aversion the model generates equity foreign bias. The reason is that risk averse agents want to hold the less risky asset and, due to the negative correlation between exchange rate changes and foreign equity returns in the model, the less risky asset is foreign equity. Coeurdacier (2009) confirms that the home bias puzzle cannot be explained by trade costs: he proposes a two-good, two-country static model with identical preferences across countries and finds that, for reasonable trade costs, optimal portfolios exhibit a foreign, rather than a home,
bias.

None of the aforementioned models takes into account labor income. Baxter and Jermann (1997) argue that domestic human capital returns are highly correlated with domestic equity returns and, thus, to hedge against their labor income risk, investors should hold foreign biased equity portfolios; therefore, the international diversification puzzle is worse than asset markets suggest. However, Heathcote and Perri (2008) show that, in a two-good, two-country stochastic growth model, domestic human capital growth is negatively correlated with domestic equity returns. The reason is that positive productivity shocks, which generate positive human capital returns, depreciate the domestic good and, thus, lower the price of domestic equity. Thus, in their model, hedging against labor income shocks entails holding a home-biased portfolio.

To sum, frictions in the goods market have been unable to generate sufficient portfolio home bias and, in some cases, have exacerbated the puzzle. More forcefully, van Wincoop and Warnock (2006) show that, under standard CRRA preferences, home bias in assets cannot be linked to home bias in goods without generating unrealistic results about asset returns and exchange rates. Partly in response to that failure, other strands of the international finance literature have proposed alternative sources of home bias. This literature has focused on models that relax either the assumption of frictionless asset markets or the assumption of full information or full rationality.

The models that examine asset market frictions associated with cross-country investing typically focus on taxes, trading costs and investment restrictions and argue that frictions may be high enough to wipe out almost all gains from international diversification. Black (1974) considers the effects of taxation of net foreign investment and finds that such taxes would lead to significant portfolio home bias. Stulz (1981a) focuses on taxes on gross foreign investment and finds that domestic investors' risky holdings should be tilted towards assets with high expected returns, but Kang and Stulz (1997) find that this result is not supported by Japanese data. Cooper and Kaplanis (1994) propose a model that incorporates deadweight costs of foreign investment and show that, unless risk aversion is much lower than conventionally thought, the level of deadweight costs that is needed to justify the empirical levels of portfolio home bias is higher than observable costs, such as withholding taxes.

\footnote{Julliard (2002) argues that Baxter and Jermann (1997) overstate the advantage of foreign equity over domestic equity for hedging domestic labor income risk, so considering human capital does not necessarily worsen the home bias puzzle.}
taxes. Furthermore, Tesar and Werner (1995) show that foreign equity investments have much higher turnover rates than domestic equity investments, so international transaction costs are unlikely to explain the lack of sufficient international portfolio diversification.

The second class of models emphasize informational or behavioral explanations: domestic and foreign investors differ in either their information or their beliefs about the distribution of asset returns and those differences are such that tilt their portfolios towards their home assets. French and Poterba (1991) show that the observed levels of portfolio home bias can be reconciled with mean-variance optimizing behavior only if agents have very optimistic beliefs about the expected returns of their home assets. To explain their results, they argue that either agents are indeed overoptimistic about their home assets or they misperceive foreign assets as excessively risky due to uncertainty. Kang and Stulz (1997) show that the holdings of foreigners who invest in Japan are skewed towards large firms and provide some evidence that this large firm bias can be partly explained by the fact that large firms tend to be better known internationally. Brennan and Cao (1997) also provide empirical support to the notion of informational asymmetries between domestic and foreign investors. Van Nieuwerburgh and Veldkamp (2008) show that, if information acquisition is not limitless, domestic agents rationally specialize their information acquisition to domestic securities and, as a result, are less uncertain about domestic assets and tilt their portfolios towards them.

Apart from being a part of the extensive international portfolio literature, this Chapter also belongs to the recent literature that embeds complex preferences in general equilibrium open economy models, thus bridging the asset pricing literature with the international finance literature. The model in this Chapter extends the model in Chapter 1, which, in a complete markets framework, shows how external habit formation can reconcile very high risk sharing across countries, in the sense of very high pricing kernel cross-country correlations, with relatively low consumption growth correlations, thus solving the Brandt, Cochrane and Santa-Clara (2006) international risk sharing puzzle. Both Chapters of this dissertation build on Pavlova and Rigobon (2007, 2008). They propose a two-country, two-good model in which preferences are characterized by demand shocks; Pavlova and Rigobon (2007) assumes complete financial markets, whereas Pavlova and Rigobon (2008) allows for market incompleteness. Their model is successful in addressing macroeconomic questions, but their assumed preferences are unable to generate realistic asset pricing moments.
Verdelhan (2008a) proposes a two-country, one-good model with Campbell and Cochrane (1999) external habit preferences, while Bekaert (1996) presents a two-country monetary model which features durability and habit persistence. Moore and Roche (2006) propose a monetary model with Campbell and Cochrane (1999) preferences and "deep" habits (Ravn, Schmitt-Grohe and Uribe (2006)). Aydemir (2008) uses a two-country, one-good external habit model to study international asset prices. Colacito and Croce (2008a, 2008b) and Bansal and Shaliastovich (2007) study international risk sharing, the forward premium puzzle and the welfare gains from international portfolio diversification in a two-country Bansal and Yaron (2004) long-run risks framework. Farhi and Gabaix (2008) examine the asset pricing implications of a two-country rare disasters model. Lustig and Verdelhan (2006) use the Yogo (2006) model to explain currency risk premia in a consumption asset pricing setting. None of those models examine equilibrium international portfolios, with the exception of Shore and White (2006), which address the portfolio home bias puzzle with a model that incorporates external habit formation. In their model, portfolio home bias results from the attempt of unconstrained investors to mimic, in a "catching up with the Joneses" spirit, the portfolio behavior of small entrepreneurs, who are forced to hold domestic equity for agency reasons.

One of the key assumptions of the model proposed in this Chapter is external habit formation, a preference specification with a long history in economics and finance. The present work assumes Menzly, Santos and Veronesi (2004) external habits; this specification shares the motivation of the Campbell and Cochrane (1999) external habit formation model, but what is assumed is not a process for the surplus consumption ratio, as Campbell and Cochrane do, but for the inverse surplus consumption ratio.\(^9\) The assumption of home bias in preferences is a common modelling device in international finance; see, for example, Zapatero (1995).

The model in this Chapter allows for financial market incompleteness. Specifically, I examine the case no international trade in assets (portfolio autarky). Portfolio autarky is also considered by Cole and Obstfeld (1991). Other open economy models focusing on financial market incompleteness include Baxter and Crucini (1995), Kollman (1996) and Kehoe and Perri (2002), all of which propose models in the international real business cycle

\(^9\)See Buraschi and Jiltsov (2007), Santos and Veronesi (2006) and Bekaert, Engstrom and Grenadier (2005) for specifications that model the inverse surplus consumption ratio.
(RBC) paradigm. In the first two papers, market incompleteness is due to the restriction that countries can only trade real, non-contingent bonds; in Kehoe and Perri (2002) markets are incomplete due to the ability of countries to elect to default on their liabilities.

The rest of the Chapter is organized as follows. Section 2 presents the model and Section 3 describes the equilibrium prices and quantities in a general setting that allows for market incompleteness. The next two Sections focus on special cases: Section 4 examines the case of portfolio autarky, while Section 5 the case of complete markets. Section 6 concludes. The Appendix contains the proofs and all supplementary material not included in the main body of this Chapter.

2.2 The economic setting

2.2.1 Endowments

The world economy is comprised of two countries, Domestic and Foreign, each of which is populated by a single risk-averse representative agent who receives an endowment stream of a single differentiated perishable good: the domestic agent is initially endowed with the claim on the domestic good endowment, while the foreign agent is endowed with the claim on the foreign good endowment. Uncertainty in this economy is represented by a filtered probability space \((\Omega, F, \mathbb{F}, P)\), where \(F = \{\mathcal{F}_t\}\) is the filtration generated by the standard 2-dimensional Brownian motion \(B_t, t \in [0, \infty)\), augmented by the null sets. All the stochastic processes introduced in the remainder of the Chapter are assumed to be progressively measurable with respect to \(F\) and to satisfy all the necessary regularity conditions for them to be well-defined. All (in)equalities that involve random variables hold \(P\)-almost surely.

The endowment stream of the domestic good is denoted by \(\{X_t\}\) and that of the foreign good by \(\{Y_t\}\). Both processes are assumed to be Itô processes driven by the 2-dimensional Brownian shock \(dB_t\), satisfying:

\[
d\log X_t = \mu_t^X dt + \sigma_t^X dB_t
\]

and

\[
d\log Y_t = \mu_t^Y dt + \sigma_t^Y dB_t
\]

Both goods are frictionlessly traded internationally, so the price of each good, in units of the numeraire good, is the same in both countries; in other words, the law of one
price holds. \( Q \) and \( Q^* \) denote, respectively, the price of the domestic good and the foreign good in terms of the numeraire. Without loss of generality, I set the domestic good as the numeraire good, so \( Q_t \equiv 1, \forall t \in [0, \infty) \). Then, \( Q^* = \frac{Q}{Q} \) denotes the terms of trade (the ratio of the price of exports over the price of imports) for the foreign country and the inverse terms of trade for the domestic country; in the remainder of this Chapter, \( Q^* \) will be called terms of trade without further specification.

### 2.2.2 Assets

Both agents can internationally trade \( k + 1 \) assets, \( k \) of which are non-redundant risky assets, with \( 0 \leq k \leq 2 \). Let the \( k \) risky asset returns, in units of the numeraire good, be given by the \( k \)-dimensional process

\[
dR_t = \mu_t dt + \sigma_t dB_t
\]

where \( \mu_t \) is the \( k \times 1 \) vector of expected returns and \( \sigma_t \) is the \( k \times 2 \) asset return diffusion matrix.

The last asset is locally riskless in terms of the numeraire good and is called *domestic bond*. The price of that asset in units of the numeraire is denoted by \( D_t \) and solves

\[
dD_t = r_f^t D_t dt
\]

where \( r_f^t \) is the continuously compounded numeraire riskless rate.

Then, we can define excess returns as

\[
dR_t^e = \mu_t^e dt + \sigma_t dB_t
\]

where \( \mu_t^e = \mu_t - r_f^t 1_k \), with \( 1_k \) representing a \( k \times 1 \) vector of ones.

If \( k < 2 \), there are not enough risky assets to span the risk in the economy, so financial markets are incomplete. For \( k = 2 \), financial markets are complete if asset returns are not perfectly conditionally correlated; in that case the diffusion matrix \( \sigma_t \) is a non-singular square matrix.

### 2.2.3 Preferences

The domestic representative agent has expected discounted utility

\[
E_0 \left[ \int_0^\infty e^{-\rho t} u(X_t, Y_t) dt \right]
\]
where $\rho > 0$ is her subjective discount rate, and her instantaneous utility function is

$$u(X_t, Y_t) = \log(X_t^{a}Y_t^{1-a} - H_t) = \log(C_t - H_t)$$

where $X_t$ and $Y_t$ is the quantity of the domestic and foreign good, respectively, she consumes at time $t$, $C \equiv X^aY^{1-a}$ is the domestic consumption basket and $H$ is the habit level associated with that consumption basket. I further assume that the external habit is of the Menzly, Santos and Veronesi (2004) form. Specifically, it is assumed that the inverse surplus consumption ratio $G = \left(\frac{C-H}{C}\right)^{-1}$ solves the stochastic differential equation

$$dG_t = k\left(G - G_t\right)dt - \delta\left(G_t - l\right)\left(\frac{dC_t}{C_t} - E_t\left(\frac{dC_t}{C_t}\right)\right)$$

(2.1)

The inverse surplus consumption ratio is a mean-reverting process, reverting to its long-run mean of $\bar{G}$ at speed $k$ and is driven by consumption growth shocks. The parameter $\delta > 0$ scales the impact of a consumption growth shock and the parameter $l \geq 1$ is the lower bound of the inverse surplus ratio $G_t$. Obviously, $\bar{G} > l$. The local curvature of the utility function is $-\frac{\partial^2 u}{\partial C^2}$ for that reason, and in a slight abuse of terminology, in the rest of this Chapter I will refer to $G$ as domestic risk aversion.

The preferences of the foreign representative agent are similar. Her instantaneous utility function is

$$u^*(X_t^*, Y_t^*) = \log\left((X_t^*)^a (Y_t^*)^{1-a} - H_t^*\right) = \log(C_t^* - H_t^*)$$

where $X_t^*$ and $Y_t^*$ is the agent's time $t$ consumption of the domestic and foreign good, respectively, $C^* \equiv (X^*)^a (Y^*)^{1-a}$ is the foreign consumption basket and $H^*$ is the foreign habit level. Note that home consumption bias for the foreign agent implies $a^* < 0.5$. The foreign agent's inverse surplus consumption ratio $G^*$ satisfies

$$dG^*_t = k\left(G^* - G_t^*\right)dt - \delta\left(G_t^* - l\right)\left(\frac{dC_t^*}{C_t} - E_t\left(\frac{dC_t^*}{C_t}\right)\right)$$

(2.2)

2.2.4 Prices and exchange rates

The time $t$ price of the domestic consumption basket $C = X^aY^{1-a}$, in units of the numeraire good, is

$$P_t = \left(\frac{Q_t}{a}\right)^a \left(\frac{Q_t^*}{1-a}\right)^{1-a}$$

(2.3)

and is defined as the minimum expenditure required to buy a unit of the domestic consumption basket $C$. 
Similarly, the foreign price index is:

\[ P_t^* = \left( \frac{Q_t}{a^*} \right)^{a^*} \left( \frac{Q_t^*}{1-a^*} \right)^{1-a^*} \]

Therefore, the time \( t \) real exchange rate, which expresses the price of a unit of the foreign consumption basket in units of the domestic consumption basket, is:

\[ E_t = \frac{P_t^*}{P_t} = \frac{a^*(1-a)^{1-a}}{(a^*)^{a^*} (1-a^*)^{1-a^*}} (Q_t^*)^{a-a^*} \]  

using the fact that \( Q_t = 1, \forall t \in [0,\infty) \). When the preferences of the two countries are identical \((a = a^*)\), the two consumption baskets are also identical \((C = C^*)\) and have the same price \((P = P^*)\), so the real exchange rate is constant at 1 (absolute PPP). If the two countries’ preferences differ \((a \neq a^*)\), the two consumption baskets are not identical, so the two countries are exposed to different price shocks. Indeed, the higher the difference in preferences (i.e. the higher \( a - a^* \)), the more volatile the real exchange rate.

### 2.2.5 The agents’ problem

As mentioned above, at \( t = 0 \), the domestic (foreign) agent is endowed with a claim on the domestic (foreign) good endowment sequence \( \{X_t\} \) \( \{Y_t\} \) and nothing else. Since this claim may or may not be one of the \( k \) internationally tradable financial assets, denote by \( N_t \) the real flow, in terms of the numeraire, paid at period \( t \) by the non-tradeable part of the domestic claim. Then:

\[ N_t = \begin{cases} 
0, & \text{if the domestic claim is tradeable} \\
\tilde{X}_t, & \text{if the domestic claim is not tradeable}
\end{cases} \]

Similarly, \( N_t^* \) is defined as the real flow, in units of the numeraire, paid each period \( t \) by the non-tradeable part of the foreign claim:

\[ N_t^* = \begin{cases} 
0, & \text{if the foreign claim is tradeable} \\
\tilde{Y}_t Q_t^*, & \text{if the foreign claim is not tradeable}
\end{cases} \]

Let the \( k \times 1 \) vector \( \pi_t \) describe the investment decision of the domestic agent: the \( j^{th} \) element of \( \pi_t \) describes the amount, in units of the numeraire good, that the domestic agent invests in the \( j^{th} \) risky asset at period \( t \). Thus, the domestic agent chooses consumption shares \( X_t \) and \( Y_t \) and investment amounts \( \pi_t \) so as to maximize her expected discounted utility

\[
\max_{\{X_t, Y_t, \pi_t\}} E_0 \left[ \int_0^\infty e^{-\rho t} \log (C_t - H_t) \ dt \right]
\]
subject to the intertemporal flow budget constraint:

\[ dA_t = \pi_t' (\mu_t' dt + \sigma_t dB_t) + A_t r_t^f dt + (N_t - C_t P_t) dt \]

where \( A_t \) is the domestic agent's period \( t \) asset wealth. Consequently, the domestic agent's investment in the locally riskless asset is \( A_t - \pi_t' 1_k \). If \( A_t \neq 0 \), we can also define the domestic portfolio weight vector \( x_t \); it denotes the share of each risky asset in the domestic portfolio and is given by \( x_t = \frac{1}{A_t} \pi_t \). The portfolio weight of the riskless asset is, thus, \( 1 - x_t' 1_k \).

Similarly, the foreign agent chooses investment amounts \( \pi_t^* \) and consumption shares \( X_t^* \) and \( Y_t^* \) so as to maximize her expected discounted utility

\[ \max \left\{ x_t^*, Y_t^*, \pi_t^* \right\} \mathbb{E}_0 \left[ \int_0^\infty e^{-\rho t} \log(C_t^* - H_t^*) dt \right] \]

subject to

\[ dA_t^* = \pi_t^{*f} (\mu_t^* dt + \sigma_t dB_t) + A_t^{*f} r_t^f dt + (N_t^* - C_t^* P_t^*) dt \]

where \( A_t^* \) is foreign asset wealth. If \( A_t^* \neq 0 \), the foreign portfolio weight vector \( x_t^* \) is given by \( x_t^* = \frac{1}{A_t^*} \pi_t^* \).

### 2.3 Equilibrium

Following Pavlova and Rigobon (2008), each country's two-good partial optimization problem can be expressed as an one-good problem, where the good in question is total domestic expenditure \( C_t \), expressed in units of the numeraire. To express the problem in terms of \( C_t \), rather than \( X_t \) and \( Y_t \), I first solve the static consumer problem:

\[ \max_{\{X_t, Y_t\}} u(X_t, Y_t) = \log(X_t^{aY_t^{1-a}} - H_t) \]

s.t.

\[ X_t + Y_t Q_t^* \leq C_t \]

Solving, we get:

\[ X_t = a C_t, \quad Y_t = (1-a) \frac{C_t}{Q_t^*} \]

so the indirect utility function \( U(C_t; H_t, Q_t^*) = u(X_t, Y_t) \mid X_t, Y_t \text{ optimal} \) is

\[ U(C_t; H_t, Q_t^*) = \log \left( \frac{a^a(1-a)^{1-a}}{(Q_t^*)^{1-a}} C_t - H_t \right) \]
Then, the domestic agent’s problem can be written as the following one-good problem:

\[
\max_{\{c_t, \pi_t\}} E_0 \left[ \int_0^\infty e^{-\rho t} \log \left( \frac{a^d(1-a)^{1-a}c_t - h_t}{(q^d_t)^{1-a}} \right) dt \right]
\]

s.t.

\[
dA_t = \pi_t (\mu_t^e dt + \sigma_t dB_t) + A_t r_t^f dt + (N_t - C_t) dt
\]

(2.6)

The dynamic budget constraint (2.6) can be written in static form as follows:

\[
E_0 \left[ \int_0^\infty \xi_t^\nu (c_t - N_t) dt \right] \leq A_0
\]

(2.7)

where \( \xi_t^\nu \) is the domestic agent’s numeraire state-price density. To understand this relationship, consider that, since \( N_t \) represents the flows from the non-tradeable part of the domestic endowment, \( C_t - N_t \) is the part of domestic consumption that has to be financed by domestic asset wealth. Then, condition (2.7) is nothing but the familiar restriction that the present value of the part of domestic consumption that is financed by asset wealth cannot exceed the initial domestic asset wealth. Under incomplete financial markets, there is an infinite number of state-price densities \( \xi_t^\nu \) which are consistent with no arbitrage and which satisfy (2.7). However, He and Pearson (1991) show that all those state-price densities \( \xi_t^\nu \) have to satisfy the SDE

\[
\frac{d\xi_t^\nu}{\xi_t^\nu} = -r_t^f dt - (\eta_t + \tilde{\nu}_t) dB_t
\]

where \( \eta_t = \sigma_t^i (\sigma_t \sigma_t^i)^{-1} \mu_t^i \) is the market price of risk process and \( \tilde{\nu}_t \) is the domestic agent’s price of non-marketed risk and satisfies \( \sigma_t \tilde{\nu}_t = 0, \forall t \in [0, \infty) \).\(^{10}\) In order words, we can decompose the domestic agent’s price of risk \( \eta_t + \tilde{\nu}_t \) to a market price of risk, which is in the asset span\(^{11}\), and an idiosyncratic price of risk \( \tilde{\nu}_t \), which is orthogonal to the asset span.\(^{12}\) Under market incompleteness, there are infinite non-marketed prices of risk \( \tilde{\nu}_t \) that

\(^{10}\)It also satisfies \( \int_0^\infty |\tilde{\nu}_t|^2 dt < \infty \).

\(^{11}\)To see this, note that expected excess returns are given by \( \mu_t^i = \sigma_t \eta_t \), so it holds that \( \eta_t = \sigma_t^i (\sigma_t \sigma_t^i)^{-1} \mu_t^i \). In this model, any shock \( dU_t \) will be of the form \( dU_t = \sigma_t^u dB_t \) for some diffusion \( \sigma_t^u \). The market component of the \( dB \) shock is \( \sigma_t^i (\sigma_t \sigma_t^i)^{-1} \sigma_t^u dB_t \) and its non-marketed component is \( \tilde{\nu}_t dB_t \). Thus, the marketed component of the \( dU_t \) risk is \( dU_t^M \equiv \sigma_t^u \left( \sigma_t^i (\sigma_t \sigma_t^i)^{-1} \sigma_t^u \right) dB_t \) and its non-marketed component is \( dU_t^N \equiv \sigma_t^u \left[ (I - \sigma_t^i (\sigma_t \sigma_t^i)^{-1} \sigma_t^i) \right] dB_t \). The total compensation required by the domestic agent for holding the \( dU_t \) risk is \( -E_t \left[ \frac{d\xi_t^\nu}{\xi_t^\nu} dU_t^M \right] = \sigma_t^u \eta_t \). The compensation for holding the market part is \( -E_t \left[ \frac{d\xi_t^\nu}{\xi_t^\nu} dU_t^M \right] = \sigma_t^u \eta_t \), using \( \eta_t = \sigma_t^i (\sigma_t \sigma_t^i)^{-1} \mu_t^i \) and \( \sigma_t \eta_t = 0 \). The compensation for holding the non-marketed component is \( -E_t \left[ \frac{d\xi_t^\nu}{\xi_t^\nu} dU_t^N \right] = \sigma_t^u \tilde{\nu}_t \).
are consistent with no arbitrage. However, He and Pearson (1991) also show that, for each agent, there is only one of those prices \( \bar{v}_t \) that minimizes her maximum domestic expected utility. In the remainder of this Chapter, I will denote the aforementioned domestic agent's unique price of non-marketed risk by \( \nu_t \) and the corresponding state-price density by \( \Xi_t \).

Thus, we have:

\[
\frac{d\Xi_t}{\Xi_t} = -r_t^f dt - (\eta_t + \nu_t)' dB_t
\]  

(2.8)

For the foreign agent, the analysis is the same. She solves:

\[
\max_{\{C^*, \pi_t^*\}} E_0 \left[ \int_0^\infty e^{-\rho t} \log \left( \frac{(a^*)^{a^*} (1-a^*)^{1-a^*}}{(Q^*)^{1-a^*}} \right) \frac{C_t^* - H_t^*}{C^*_t} \right] dt
\]

s.t.

\[
dA_t^* = \pi_t^* \left( \mu_t^* dt + \sigma_t dB_t \right) + A_t^* r_t^f dt + (N_t^* - C_t^*) dt
\]

(2.9)

which can be written in static form as

\[
E_0 \left[ \int_0^\infty \frac{C_t^* - N_t^*}{\Xi^*_t} dt \right] \leq A_t^*
\]

(2.10)

with the foreign price of risk process \( \Xi_t^\nu \) satisfying

\[
\frac{d\Xi_t^\nu}{\Xi_t^\nu} = -r_t^f dt - (\eta_t + \bar{\nu}_t)' dB_t
\]

where \( \sigma_t \bar{\nu}_t = 0 \). The foreign agent's unique price of non-marketed risk will be denoted by \( \nu_t^* \) and the foreign agent's state-price density by \( \Xi_t^* \). In short:

\[
\frac{d\Xi_t^*}{\Xi_t^*} = -r_t^f dt - (\eta_t + \nu_t^*)' dB_t
\]

(2.11)

### 2.3.1 The planner's problem formulation

We can directly solve for the competitive equilibrium by solving each agent's optimization problem and imposing market clearing; the details are provided in the Appendix (section 2.7.1). However, it can easily be shown that the competitive equilibrium solution is equivalent to the solution of a non-standard planner's problem, as introduced in Cuoco and He (1994). Specifically, we can solve a fictional planner's problem in which the welfare weights are not constant, as in the complete markets case, but stochastic, so as to accommodate potential market incompleteness. Specifically, if markets are incomplete, the competitive equilibrium allocation is not Pareto optimal, since agents are unable to equalize their marginal utility growth for each good. This inefficiency is expressed in the fictitious planner's
problem in the form of stochastic welfare weights: the welfare weights are additional state
variables, providing information as to the distribution of wealth at each time.

The fictional planner's problem has the form:

\[
\max_{\{X_t, Y_t\}} E_0 \left[ \int_0^\infty e^{-\rho t} \left( \log(C_t - H_t) dt + \lambda_t \log(C_t - H_t) \right) dt \right]
\]

subject to the resource constraints

\[X_t + X_t^* = \tilde{X}_t \text{ and } Y_t + Y_t^* = \tilde{Y}_t\]

The domestic welfare weight is normalized to 1, whereas the foreign weight is \(\lambda_t\).

It is shown in the Appendix (section 2.7.2) that the competitive equilibrium corre­
sponds to a planner's solution with

\[\lambda_t = \frac{\zeta}{\zeta^* \Xi_t} \]

(2.12)

where \(\zeta\) and \(\zeta^*\) are constants. The expression above clearly illustrates that \(\lambda\) is the wedge
between the two numeraire state-price densities \(\Xi\) and \(\Xi^*\).

In more detail, consider the fictitious representative agent and note that, at period \(t\)

\[U(\tilde{X}_t, \tilde{Y}_t; \lambda_t) = \max_{X_t + X_t^* = \tilde{X}_t, Y_t + Y_t^* = \tilde{Y}_t} \{u(X_t, Y_t) + \lambda_t u(X_t^*, Y_t^*)\}\]

so

\[
\frac{\partial U(\tilde{X}_t, \tilde{Y}_t)}{\partial \tilde{X}_t} = \frac{\partial u(X_t, Y_t)}{\partial X_t} = \lambda_t \frac{\partial u(X_t^*, Y_t^*)}{\partial X_t^*}
\]

or

\[\lambda_t = \frac{\partial u(X_t, Y_t)}{\partial X_t} \frac{\partial X_t}{\partial X_t^*} \]

Similarly, for the foreign good it holds that:

\[
\frac{\partial U(\tilde{X}_t, \tilde{Y}_t)}{\partial \tilde{Y}_t} = \frac{\partial u(X_t, Y_t)}{\partial Y_t} = \lambda_t \frac{\partial u(X_t^*, Y_t^*)}{\partial Y_t^*}
\]

or

\[\lambda_t = \frac{\partial u(X_t, Y_t)}{\partial Y_t} \frac{\partial Y_t}{\partial Y_t^*} \]

In other words, the stochastic welfare weight \(\lambda\) represents the wedge that financial
market incompleteness introduces to the two agents' marginal utility growth. In complete
markets, \(\lambda\) is constant: perfect risk sharing stipulates that marginal utility growth is equal­
ized across agents. When markets are incomplete, though, there are not enough assets to
efficiently share risk and marginal utility growth cannot be equalized across agents.
Furthermore, the foreign welfare weight $\lambda$ is connected to the wealth distribution between the two countries. Specifically, define as domestic aggregate wealth $W$ the present value, in units of the numeraire good, that future consumption expenditure has for the domestic agent (i.e. the value calculated by applying the domestic state-price density $\Xi$):

$$W_t = E_t \left[ \int_t^\infty \frac{\Xi_s}{\Xi_t} C_s ds \right]$$

Foreign wealth $W^*$ is defined similarly:$^{13}$

$$W^*_t = E_t \left[ \int_t^\infty \frac{\Xi^*_s}{\Xi^*_t} C^*_s ds \right]$$

The following proposition, proven in the Appendix, clarifies the relationship between wealth and welfare weights.

**Proposition 7** Domestic wealth, in units of the numeraire good, is

$$W_t = \frac{\bar{X}_t}{\alpha G_t + \beta\lambda_t G_t^*} \frac{\rho G_t + k\bar{G}}{\rho (\rho + k)}$$  \hspace{1cm} (2.13)

and foreign wealth, in units of the numeraire good, is

$$W^*_t = \frac{\lambda_t \bar{X}_t}{\alpha G_t + \beta\lambda_t G_t^*} \frac{\rho G_t^* + k\bar{G}}{\rho (\rho + k)}$$  \hspace{1cm} (2.14)

Therefore,

$$\lambda_t = \frac{\rho G_t + k\bar{G}}{\rho G_t^* + k\bar{G}} \frac{W^*_t}{W_t}$$

We can easily see that the welfare weight $\lambda$ is partly representing the ratio of the two countries’ wealth; under this interpretation, the higher $\lambda$ is, the more wealthy the foreign country is. However, it would be more accurate to discuss $\lambda$ not in terms of wealth, but in terms of welfare. In that sense, $\lambda$ represents the conditional welfare gain or loss that the foreign agent experiences. Under incomplete markets, the two agents cannot completely share risk; the two agents have to take different risks. A high $\lambda$ characterizes a state of the world in which the foreign agent has low marginal utility - and, thus, high utility - because the idiosyncratic risks she has taken have had a positive outcome. This may or may not mean that the agent is wealthier; all it means is that the agent has taken idiosyncratic risks

$^{13}$Note that, if markets are incomplete and the densities $\Xi$ and $\Xi^*$ differ, then, under the definition used here, the two countries may reach different valuation conclusions for each country’s wealth. In that sense, wealth, as defined here, is a local, rather than a global, measure.
that ex post paid off. This is exactly the meaning of a higher planner’s welfare weight: a high \( \lambda \) means that the foreign agent is better off in welfare terms.

Using (2.8), (2.11) and (2.12), a characterization of the stochastic process \( \lambda \) can be easily obtained; it is presented in the following lemma.

**Lemma 8** The foreign welfare weight process \( \lambda \) satisfies

\[
\frac{d\lambda_t}{\lambda_t} = (\nu_t^* - \nu_t)' \nu_t^* dt + (\nu_t^* - \nu_t)' dB_t
\]

In the complete markets case, all economic risk is marketed, since there are enough assets to span all risk, so \( \sigma_t \) is non-singular and \( \nu_t = \nu_t^* = 0, \forall t \in [0, \infty) \); in that case, \( \lambda_t = \lambda_0 = \bar{\lambda} \). However, under incomplete markets, non-marketed risk is priced differently by the economic agents, so differences in the price of non-marketed risk \( (\nu_t^* - \nu_t) \) affect equilibrium prices and quantities. In that case, to fully characterize the behavior of \( \lambda \) we need to solve for the idiosyncratic prices of non-marketed risk \( \nu \) and \( \nu^* \). For that, we need, in turn, to solve for the asset return diffusion matrix \( \sigma \), so that we can identify the asset span and, thus, the marketed risk space. Thus, as expected, the behavior of the fictional welfare weight \( \lambda \) depends on the structure of international asset markets - in other words, on which assets are internationally tradeable.

2.3.2 **Quantities and prices**

In this section, we characterize all quantities and prices of interest in terms of the five state variables \( \tilde{X}, \tilde{Y}, G, G^* \) and \( \lambda \); of course, the latter state variable is non-trivial only when financial markets are incomplete. Under market incompleteness, the conditional cross-country distribution of wealth, as encapsulated in the value of \( \lambda_t \), affects equilibrium prices and quantities; thus, under incomplete markets, the divergence of equilibrium prices and quantities from their complete markets counterparts depends on the divergence of the conditional value of \( \lambda \) from its constant, complete markets, value.

2.3.2.1 **Macroeconomic quantities and prices**

The domestic equilibrium consumption allocation is

\[
X_t = \omega_t \tilde{X}_t, \quad Y_t = \omega_t \tilde{Y}_t
\]  

(2.15)
and the foreign consumption allocation is

\[ X_t^* = (1 - \omega_t) \bar{X}_t, \quad Y_t^* = (1 - \omega_t^*) \bar{Y}_t \]  \hspace{1cm} (2.16)

where I introduce the share functions \( \omega_t \) and \( \omega_t^* \)

\[ \omega_t = \omega \left( \lambda_t, \frac{G_t^*}{G_t} \right) \equiv \frac{a}{a + a^* \lambda_t \left( \frac{G_t^*}{G_t} \right)} \]

\[ \omega_t^* = \omega^* \left( \lambda_t, \frac{G_t^*}{G_t} \right) \equiv \frac{(1 - a)}{(1 - a) + (1 - a^*) \lambda_t \left( \frac{G_t^*}{G_t} \right)} \]

with \( \omega_t \) (\( \omega_t^* \)) being the proportion of domestic (foreign) endowment consumed domestically. Naturally, in the case of complete home bias \( (a = 1, a^* = 0) \), \( \omega_t = 1 \) and \( \omega_t^* = 0 \), \( \forall t \in [0, \infty) \), since each country consumes its endowment. Both share functions are decreasing in the welfare weight \( \lambda_t \); the higher the foreign welfare weight \( \lambda_t \), the more of the world endowment is consumed by the foreign country. This effect makes economic sense: since \( \lambda_t \) is a rough proxy for the relative wealth of the foreign country, it stands to reason that higher \( \lambda_t \), implying a relatively wealthier foreign country, leads to an elevated foreign consumption share. More importantly, both share functions are decreasing in the ratio \( \frac{G_t^*}{G_t} \); in other words, an increase in a given country's risk aversion increases its consumption share for both goods. This is due to risk sharing; when a country is in a bad state of the world and its risk aversion is high, it needs to consume more, as its marginal utility of consumption is higher.

Trivially, domestic consumption expenditure is given by

\[ C_t = \frac{1}{a + a^* \lambda_t \left( \frac{G_t^*}{G_t} \right)} \bar{X}_t \]  \hspace{1cm} (2.17)

and foreign consumption expenditure by

\[ C_t^* = \frac{\lambda_t \left( \frac{G_t^*}{G_t} \right)}{a + a^* \lambda_t \left( \frac{G_t^*}{G_t} \right)} \bar{X}_t \]  \hspace{1cm} (2.18)

Domestic (foreign) consumption expenditure is decreasing (increasing) in both \( \lambda_t \) and \( \frac{G_t^*}{G_t} \). Furthermore, domestic consumption is

\[ C_t = \omega_t^a (\omega_t^*)^1-a \bar{X}_t^a \bar{Y}_t^{1-a} \]  \hspace{1cm} (2.19)
and foreign consumption is

\[ C_t^* = (1 - \omega_t)^a^* (1 - \omega_t^*)^{1-a^*} \tilde{X}_t \tilde{Y}_t^{1-a^*} \]  

(2.20)

The terms of trade are given by

\[ Q_t^* = \frac{(1-a) + (1-a^*)\lambda_t \left( \frac{G_t^*}{G_t} \right) \tilde{X}_t}{a + a^* \lambda_t \left( \frac{G_t^*}{G_t} \right) \tilde{Y}_t} \]  

(2.21)

Under home bias \((a > a^*)\), the price of the foreign good, in units of the numeraire, is increasing in the welfare weight \(\lambda_t\). The reason for this is that an increase of foreign country wealth when the foreign country is home biased implies a relative increase of the demand for the foreign good, pressuring its relative price upward. Furthermore, the terms of trade are increasing in \(\frac{G_t^*}{G_t}\); high foreign risk aversion implies high foreign consumption demand and, thus, due to home bias, a high relative price for the foreign good.

Finally, the domestic trade balance (domestic net exports) is given by

\[ TB_t = \tilde{X}_t - C_t P_t = X_t^* - Y_t Q_t^* = \left( 1 - \frac{\omega_t}{a} \right) \tilde{X}_t \]

Since \(\omega_t\) is decreasing in \(\lambda_t\), \(TB_t\) is increasing in \(\lambda_t\). This also makes economic sense: when the foreign country is wealthier, it consumes more of the global goods endowment. The only way for the foreign country to increase its consumption is to increase its net imports - which translates to higher net exports for the domestic country.

### 2.3.2.2 The pricing of risk

The price of numeraire risk for each of the two agents is given by their numeraire state-price densities \(\Xi\) and \(\Xi^*\); they are their discounted marginal utilities for the domestic good. The numeraire state-price density of the domestic agent, \(\Xi\), is

\[ \Xi_t = e^{-\rho t} \frac{1}{\tilde{X}_t} \frac{aG_t + a^* \lambda_t G_t^*}{\zeta} \]  

(2.22)

while the numeraire state-price density of the foreign agent, \(\Xi^*\), is

\[ \Xi_t^* = e^{-\rho t} \frac{1}{\tilde{X}_t} \frac{1}{\tilde{X}_t} \frac{aG_t + a^* \lambda_t G_t^*}{\zeta} \]  

(2.23)

Both marginal utilities are increasing in conditional risk aversions \(G\) and \(G^*\); the marginal utility of consuming the numeraire good is higher when either the domestic or the foreign
risk aversion are high. Furthermore, both $\Xi$ and $\Xi^*$ are decreasing in $X$; for both agents, the marginal utility of the numeraire good is proportional to its scarcity. The result is that both agents require high compensation (in the form of expected return) from assets that have low payoffs when either risk aversion is high or when the state of nature is unfavorable (low endowment realization).

Where the behaviors of $\Xi$ and $\Xi^*$ diverge is with respect to the welfare weight $\lambda$: the domestic state-price density $\Xi$ is increasing in $\lambda$, whereas the foreign density $\Xi^*$ is decreasing in $\lambda$. This result has a natural economic interpretation: high values of $\lambda$ correspond to states of nature in which the foreign agent is relatively wealthy and the domestic agent relatively poor, so the marginal value of the numeraire good is high for the "poor" domestic agent and low for the "rich" foreign agent. This difference in the behavior of the two agents' state-price densities introduces differences in the pricing of numeraire good flows. Specifically, each agent is willing to pay a high price for flows negatively correlated with her wealth. In other words, numeraire flows that are positively correlated with foreign relative wealth $\lambda$ are valued more highly by the domestic agent than the foreign agent; the opposite is true for flows that are negatively correlated with $\lambda$.

2.3.3 Consumption volatility and risk sharing

Having described the levels of the variables of interest, in this section we focus on cross-country risk sharing, which entails characterizing the conditional variance of consumption growth rates. The following proposition, proven in the Appendix, sheds lights on equilibrium consumption risk.

**Proposition 9** The equilibrium consumption process for the domestic representative agent is

$$\frac{dC_t^D}{C_t^D} - E_t \left( \frac{dC_t^D}{C_t^D} \right) = \sigma_t^D dB_t,$$

with

$$\sigma_t^D = \frac{1}{D_t^D} \left( a \sigma_t^X + (1-a) \sigma_t^Y \right)$$

$$+ \frac{1}{D_t^D} \left\{ \left( (a^* k_t + a k_t^*) \sigma_t^X + ((1-a^*) k_t + (1-a) k_t^*) \sigma_t^Y \right) \delta \left( \frac{G_t^* - 1}{G_t^*} \right) \right\}$$

$$- \frac{k_t}{D_t^D} (\nu^*_t - \nu_t)$$

and the equilibrium consumption process for the foreign representative agent is

$$\frac{dC_t^F}{C_t^F} - E_t \left( \frac{dC_t^F}{C_t^F} \right) =$$
\[
\sigma_t^{C^*} dB_t, \text{ with }
\]
\[
\sigma_t^{C^*} = \frac{1}{D_t^c} \left( a^* \sigma_t^X + (1 - a^*) \sigma_t^Y \right) \\
+ \frac{1}{D_t^c} \left\{ \left[ (a^* k_t + a k_t^*) \sigma_t^X + ((1 - a^*) k_t + (1 - a) k_t^*) \sigma_t^Y \right] \frac{G_t - 1}{G_t} \right\} \\
+ \frac{k_t^*}{D_t^c} (\nu_t^* - \nu_t)
\]

where \( k_t, k_t^* \) and \( D_t^c \) are functions of \( G_t, G_t^* \) and \( \lambda_t \) defined in the Appendix (equations (2.37), (2.39) and (2.41), respectively).

Denoting the consumption growth diffusion process in the complete markets case by \( \tilde{\sigma}_t^C \) and \( \tilde{\sigma}_t^{C^*} \), for domestic and foreign consumption growth, respectively, the incomplete markets solution can be written as:

\[
\sigma_t^C = \tilde{\sigma}_t^C - \frac{k_t}{D_t^c} (\nu_t^* - \nu_t)
\]

and

\[
\sigma_t^{C^*} = \tilde{\sigma}_t^{C^*} + \frac{k_t^*}{D_t^c} (\nu_t^* - \nu_t)
\]

Domestic (foreign) consumption growth volatility is decreasing (increasing) in the price difference of non-marketed risk \( \nu_t^* - \nu_t \); for each of the agents, higher compensation to bear non-marketed risk is associated with higher consumption growth risk relative to the complete markets benchmark.

### 2.4 The portfolio autarky case

In this section, I assume that no financial assets can be traded internationally; the only trade between countries is trade in goods. Trivially, this implies that the each country's trade balance is always identically zero: the value of each country's imports need to equal the value of its exports. Furthermore, since with no asset trade the current account is equal to the trade balance, both countries' current accounts are always zero. Thus:

\[
TB_t = TB_t^* = CA_t = CA_t^* = 0, \forall t \in [0, \infty)
\]

In this setup, there is no portfolio decision. The domestic agent chooses \( \{X_t, Y_t\} \) to maximize her expected discounted utility under a set of zero trade balance constraints.
Thus, for the risky assets, \( \pi_t = 0 = \pi^*_t \). Since no bonds are traded, we also have \( A_t - \pi^*_t 1_k = 0 \), so \( A_t = 0 \) and, similarly, \( A^*_t = 0 \). Therefore, the domestic agent’s problem is:

\[
\max_{(X, Y)} E_0 \left[ \int_0^\infty e^{-\rho t} \log(X_t^{\alpha} Y_t^{1-\alpha} - H_t) dt \right]
\]

such that domestic trade balance is zero at each time \( t \):

\[
X_t + Y_t Q_t^* = \bar{X}_t
\]

Similarly, the foreign agent maximizes her expected discounted utility under the zero trade balance constraint \( X^*_t + Y^*_t Q_t^* = \bar{Y}_t Q_t^* \).

It can easily be determined that the competitive equilibrium allocation is

\[
X_t = a \bar{X}_t \quad \text{and} \quad Y_t = a^* \bar{Y}_t
\]

for the domestic agent, and

\[
X^*_t = (1-a) \bar{X}_t \quad \text{and} \quad Y^*_t = (1-a^*) \bar{Y}_t
\]

for the foreign agent.

Since the equilibrium allocation differs from the complete markets allocation, market incompleteness implies a welfare loss for the agents, caused by their inability to use financial markets in order to efficiently share risk. This also implies that the two agents’ marginal utility is not equalized. Indeed, the domestic agent’s state-price density is

\[
\Xi_t = e^{-\rho t} \frac{1}{\zeta} \frac{G_t}{X_t}
\]

where the foreign state-price density is

\[
\Xi^*_t = e^{-\rho t} \frac{1}{\zeta^*} \frac{a^* G^*_t}{1-a \bar{X}_t}
\]

Thus, the wedge \( \lambda \) between the two densities is

\[
\lambda_t = \frac{1-a}{a^*} \frac{G_t}{G^*_t}
\]

The fictional welfare weight \( \lambda \) adjusts in a way that represents the impact of imperfect risk sharing to consumption and, thus, welfare. Without asset markets, the shock in domestic risk aversion \( G \), given by \( \frac{dC^*_t}{C_t} - E_t \left( \frac{dC^*_t}{C_t} \right) \), satisfies

\[
\frac{dC^*_t}{C_t} - E_t \left( \frac{dC^*_t}{C_t} \right) = a \left( \frac{d\bar{X}_t}{\bar{X}_t} - E_t \left( \frac{d\bar{X}_t}{\bar{X}_t} \right) \right) + (1-a) \left( \frac{d\bar{Y}_t}{\bar{Y}_t} - E_t \left( \frac{d\bar{Y}_t}{\bar{Y}_t} \right) \right)
\]
so it is a home-biased weighted average of the domestic and the foreign endowment shock. To clarify ideas, assume, for example, that domestic endowment shocks have been mostly negative in the recent past, whereas foreign endowment shocks have been mostly positive. Then, due to home bias, domestic consumption shocks have been mostly negative, while foreign consumption shocks have been mostly positive. Since negative consumption shocks imply positive shocks in risk aversion, \( G_t > G_t^* \): the country which has experienced mostly negative consumption shocks is more risk averse than the country that has experienced positive shocks. This is not surprising: since countries cannot efficiently share risk, home bias implies that each country has to shoulder most of the burden of unfavorable realizations of its home good endowment process.

The differential welfare of the two countries, given those shocks, is expressed by the fictional welfare weight \( \lambda \). In the case we consider, \( \lambda \) will be above its steady-state value, illustrating the fact that the foreign country is "wealthier", in the sense that it consumes more than it would consume in the complete markets case. Indeed, under complete markets, the domestic country would have insured itself against adverse relative shocks in a way that keeps \( \lambda \) constant. Without financial assets, on the other hand, each country has to bear most of the burden of bad realizations of its endowment. The welfare weight \( \lambda \) measures exactly this burden; in complete markets, allocations are a function of the risk aversion ratio \( \frac{G^*}{G} \), since Pareto optimality dictates that consumption should shift to the conditionally more risk averse country. Without assets, however, the link between consumption and the risk aversion ratio is broken, since countries cannot efficiently allocate risk - in that regime, consumption only depends on endowment realizations. Thus, the welfare weight \( \lambda \) completely eliminates all the dependence of the consumption allocation on the ratio \( \frac{G^*}{G} \).

To examine the properties of equilibrium prices and quantities in the portfolio autarky setup, I simulate 20,000 sample paths of the model economy, each consisting of 200 quarterly observations (50 years), for different degrees of preference home bias.\(^{14}\) I specify that both endowments follow a Geometric Brownian motion, so that \( \mu_t^X = \mu^X, \mu_t^Y = \mu^X, \sigma_t^X = [\sigma^X 0]' \) and \( \sigma_t^Y = [0 \sigma^Y]' \); this implies that endowment growth is uncorrelated across countries.

The system is initialized at the steady state \( (G_1 = G_1^* = \bar{G}) \) and I adopt the

---

\(^{14}\)To reduce the dependence on initial conditions, for each of the 20,000 paths I generate a time series of 240 quarterly observations (60 years) and eliminate the first 40 observations (10 years).
normalization $\tilde{X}_1 = \tilde{Y}_1 = 1$. The calibration is symmetric; the annualized parameter values used are as follows: $\rho = 0.04$, $k = 0.12$, $\delta = 79.39$, $G = 34$, $l = 20$, $\mu = 0.015$, $\sigma^X = \sigma^Y = 0.015$ and $\rho^{XY} = 0.15$. I consider domestic home bias parameter values of $a = 0.5, 0.7, 0.9, 0.95, 0.99$ and 1. For each of the values of $a$, $a^* = 1 - a$, so the degree of home bias is identical in the two countries.

Table 2.1 presents the properties of several key moments in this economy. Due to symmetry, consumption growth and pricing kernel volatility are virtually identical in the two countries for all levels of home bias. As expected, the less divergent are the two countries' preferences, the higher the correlation of their consumption growth and marginal utility is. Due to home bias, the agents take different amounts of each risk, so they price them differently. The domestic agent is more exposed to domestic endowment shocks, so she prices domestic endowment risk higher than her foreign counterpart: for all home bias parameterizations, the first element of the $\nu^* - \nu$ vector, $(\nu^* - \nu)_1$, which denotes the cross-country difference in the price of domestic endowment risk, is (weakly) negative. The opposite is true for foreign risk: $(\nu^* - \nu)_2$ is always weakly positive. The two agents price the risks identically only when $a = a^* = 0.5$, in which case they have the same preferences and, thus, perfectly pool all risks.

2.5 The complete markets case

In this section, I assume that both endowment claims are internationally tradeable: the agents can trade the domestic total wealth portfolio and the foreign total wealth portfolio: the domestic (foreign) total wealth portfolio is the asset that pays as dividend, each period, the endowment of the domestic (foreign) good. The net supply of each of those two assets is normalized to one. The period $t$ price of the domestic (foreign) claim, in units of the domestic (foreign) good, is denoted by $V_t$ ($V^*_t$); thus, the numeraire price of the domestic asset is $V_t$ and that of the foreign asset is $V^*_t Q^*_t$. All returns are expressed in units of the numeraire good.

Recall that both agents can also trade the domestic bond, which is locally riskless.

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15 The mean and standard deviation of endowment growth are set equal to the corresponding moments of quarterly US endowment growth from 1975:Q1 to 2007:Q2, with endowment defined as the sum of consumption of non-durables and services and net exports. The habit parameters are those used in Menzly, Santos and Veronesi (2004), with the exception of $k$, which is set slightly lower. The calibration here follows the calibration in Chapter 1.
in terms of the numeraire. Thus, both agents have access to two risky assets (the two total wealth portfolios) and one riskless asset (the domestic bond).\footnote{We could also introduce a foreign bond, an asset that is locally riskless in terms of the foreign good. Due to terms of trade risk, this is a risky asset in terms of the numeraire. Since there are only two sources of risk in the world economy, if the two total wealth portfolios have linearly independent returns, then the two assets induce market completeness and the foreign bond is a redundant security; its payoffs can be replicated by a portfolio consisting of the domestic total wealth portfolio and the foreign total wealth portfolio. As it will be shown shortly, this is true for $a \neq a^*$, so we can ignore the foreign bond.}

Since both endowment claims are marketed, the analysis presented in the previous section applies with $N_t = N_t^* = 0$.

Since there are two risky assets ($k = 2$) and two sources of risk in the world economy, financial markets are potentially complete; they are incomplete only if the conditional returns of the two assets are perfectly correlated. The following proposition illustrates that, unless the two countries have the same preferences ($a = a^*$), trading in the two endowment claims is sufficient to dynamically induce market completeness. In that case, the welfare weight is constant: $\lambda_t = \lambda, \forall t \in [0, \infty)$.

**Proposition 10** If $a \neq a^*$, the asset return diffusion matrix $\sigma_t$ is non-singular. Then, $\nu_t = \nu_t^* = 0$ and financial markets are complete. If $a = a^*$, the two total wealth portfolios have perfectly correlated returns in terms of the numeraire good, so $\sigma_t$ is singular.

In the rest of the section we will focus in the complete markets case $a \neq a^*$.

### 2.5.1 Equilibrium portfolios

Equilibrium portfolios are given by the following proposition, proven in the Appendix.

**Proposition 11** For $a \neq a^*$, the domestic country equilibrium portfolio weights are given by $x_t = [x_t^V, x_t^V^*]'$ where

$$x_t^V = (1 - a^*) \frac{(a + a^*\lambda)kG + apG_t + a^* p\lambda G_t^*}{(a - a^*) (kG + \rho G_t)}$$

and

$$x_t^{V*} = 1 - x_t^V$$

while the foreign country equilibrium portfolio weights are given by $x_t^* = [x_t^{*V}, x_t^{*V^*}]'$ where

$$x_t^{*V} = - (1 - a) \frac{(a + a^*\lambda)kG + apG_t + a^* p\lambda G_t^*}{(a - a^*) \lambda (kG + \rho G_t^*)}$$

$$x_t^{*V^*} = 1 - x_t^{*V}$$
and

\[ x_t^{*V} = 1 - x_t^{*V} \]  

(2.28)

The domestic agent holds a fixed portfolio comprised of \( \theta_t^V = \frac{1-a^V}{a-a^*} \equiv \theta^V \) shares of the domestic total wealth portfolio and \( \theta_t^{V*} = \frac{-a^*}{a-a^*} \equiv \theta^{V*} \) shares of the foreign total wealth portfolio. Consequently, the foreign agent holds a fixed portfolio comprised of \( \theta_t^{V*} = -\frac{1-a^*}{a-a^*} \equiv \theta^{V*} \) shares of the domestic total wealth portfolio and \( \theta_t^{V} = \frac{a}{a-a^*} \equiv \theta^{V} \) shares of the foreign total wealth portfolio.

The first important point is that, in equilibrium, no country holds domestic bonds. Furthermore, each country's equilibrium portfolio is comprised by a constant number of shares: both agents adopt a buy-and-hold strategy and never reallocate. In fact, under home bias \( (a > a^*) \), each country holds a long-short portfolio: each country holds a leveraged position in its own asset, financed by short-selling the other country's asset; for the domestic portfolio, \( \theta^V > 1 \) and \( \theta^{V*} < 0 \), and for the foreign portfolio, \( \theta^{V*} < 0 \) and \( \theta^{V*} > 1 \). Equilibrium portfolios are not just home biased, they are superbiased.\(^{17}\) However, although the amount of shares in each country's portfolio is constant, the portfolio weights \( x_t \) and \( x_t^* \) are time-varying: an increase in domestic risk aversion \( G \) leads both countries to reallocate their portfolio towards the foreign asset, while an increase in \( G^* \) induces reallocation towards the domestic asset for both countries; this is the result of valuation effects, explained in more detail below.

To understand those results and to clarify the elements introduced by external habit formation, it is useful to consider the benchmark of an economy without habits, keeping everything else the same. Under standard log preferences over a home biased Cobb-Douglas consumption basket, each country consumes a fixed proportion of the endowment of each good. In that economy, the only risk present is endowment risk: fluctuations of the two endowments \( \bar{X} \) and \( \bar{Y} \). Equivalently, we can consider the relative endowment \( z = \frac{\bar{Y}}{\bar{X}} \) and write the two endowments as \( \bar{X} \) and \( z \bar{X} \). Using this formulation, we can see that, in this setup, risk sharing refers to the sharing of relative endowment risk; fluctuations in \( \bar{X} \) in effect represent undiversified global risk. This is evident when we consider the consumption expenditure for each of the countries: domestic expenditure is \( C_t = \bar{X}_t \), while foreign expenditure is \( C_t^* = (\frac{1-a^*}{a}) \bar{X}_t \), so equilibrium expenditure is perfectly correlated across

\(^{17}\)The possibility that Cobb-Douglas heterogeneous preferences across countries could give rise to super-biased portfolios is examined in Bennett and Young (1999).
countries and is exposed only to global risk \( \bar{X} \) and not to relative endowment risk \( z \). It is important to note that the diversification of relative endowment risk \( z \) is achieved through fluctuations of the terms of trade. For example, when a positive shock increases the domestic endowment, a proportional reduction in the relative price of the domestic good leaves the consumption expenditure of both countries completely unaffected. Thus, international trade in goods leads to the perfect sharing of relative endowment risk. Naturally, this eliminates the need for risk sharing through financial markets: international trade in goods is sufficient for Pareto optimality. In fact, financial markets are not complete in this setup, since the terms of trade fluctuations also induce perfect correlation in (numeraire) asset real flows. Specifically, at period \( t \), the domestic asset pays numeraire flows of \( \bar{X}_t \), while the foreign asset pays numeraire flows of \( \bar{Y}_t Q_t^* = \frac{1-a}{a} \bar{X}_t \), so one of the two assets is redundant and the competitive equilibrium allocation can be achieved by any buy-and-hold portfolio \( [\theta^V, \theta^{V^*}] \) that satisfies \( \theta^V + \frac{1-a}{a} \theta^{V^*} = 1 \). However, this market incompleteness does not matter for welfare purposes, as asset trading is not necessary. Note that the aforementioned set of portfolios includes both the habit equilibrium portfolio \( \left[ \frac{1-a}{a-a^*}, -\frac{a^*}{a-a^*} \right] \) and the autarky portfolio \([1,0]\). Furthermore, since countries can achieve Pareto optimality by trading in the spot market for goods, they can gain no risk sharing benefits through current account fluctuations. Indeed, both the trade balance and the current account are identically zero at all periods for both countries.

In the habit economy, countries trade assets in equilibrium. Specifically, their behavior is very straightforward: each country holds a static portfolio which, each period, pays exactly the amount of resources that the country needs in order to afford its Pareto optimal consumption allocation. Then, each country achieves that allocation by trading in the spot market for goods. For example, each period the domestic country portfolio pays dividends of \( \theta^V \bar{X}_t \) units of the domestic good and \( \theta^{V^*} \bar{Y}_t \) units of the foreign good; those dividends are worth \( \theta^V \bar{X}_t + \theta^{V^*} \bar{Y}_t Q_t^* \) numeraire units. What the domestic country needs to consume - \( X_t \) units of the domestic good and \( Y_t \) units of the foreign good - is worth \( X_t + Y_t Q_t^* \) numeraire units, so the domestic agent has exactly the resources she needs to buy that consumption bundle: \( \theta^V \bar{X}_t + \theta^{V^*} \bar{Y}_t Q_t^* = X_t + Y_t Q_t^* \). All the domestic agent has to do to achieve the desirable allocation is to exchange \( \theta^V \bar{X}_t - X_t > 0 \) units of the domestic good for \( Y_t - \theta^{V^*} \bar{Y}_t > 0 \) units of the foreign good in the time \( t \) spot goods market. It is notable that, with the exception of period \( t = 0 \), the countries do not need to trade assets
in order to share risk; the only trading that takes place is in the spot market for goods. This intratemporal - rather than intertemporal - risk sharing behavior is further illustrated by the fact that countries do not hold any bonds at all.

Naturally, this behavior leads to current account balance at each period, as the countries do not need to smooth consumption by moving wealth across time. Formally, recall that the domestic current account \( CA \) is comprised of the sum of the domestic trade balance and the net flows (dividends and interest payments) from the domestic net foreign portfolio. Thus:

\[
CA_t = TB_t + \left( \theta_t^{V^*} \hat{Y}_t Q^*_t - \theta_t^{V^*} \hat{X}_t \right) = TB_t + \left( \frac{\omega_t}{\alpha} - 1 \right) \hat{X}_t = 0
\]

In the complete markets case, net foreign portfolio flows completely counterbalance net trade flows; although the domestic trade balance is non-zero, the current account is always zero. Trivially, the foreign current account is also identically zero: \( CA^*_t = 0, \forall t \in [0, \infty) \).

The long-short nature of equilibrium portfolios is motivated by the need to hedge against risk aversion fluctuations. In contrast to the simple log economy analyzed above, the marginal utility of the agents in the habit economy also depends on conditional risk aversion \( G_t \) and \( G^*_t \). Risk sharing is achieved by consuming time-varying shares of the global endowments: recall that domestic expenditure \( c_t \) is decreasing in \( G_t \). This is due to risk sharing: relatively high domestic risk aversion pushes domestic marginal utility upwards, so marginal utility growth can be equalized across countries only if the domestic country consumes more. Since domestic consumption is decreasing in \( G_t \), the domestic agent needs to hold a portfolio that is a good hedge for fluctuations in \( G_t \), so that it can finance her consumption. The numeraire cashflows of the domestic asset are \( \hat{X}_t \), while those of the foreign asset are \( \hat{Y}_t Q^*_t = \frac{(1-a)G_t+(1-a^*)\lambda G^*_t}{aG_t+a^*\lambda G^*_t} \hat{X}_t \), positively correlated with \( G^*_t \). In other words, for the domestic agent, the domestic asset is a better hedge than the foreign asset. To be more precise, the reason the domestic asset is a hedge for the domestic asset and risky for the foreign agent is due to the behavior of the terms of trade (and thus the real exchange rate). Under home bias, the terms of trade \( Q_t^* \) are increasing in \( G_t^* \), so the foreign good become cheaper exactly when the domestic agent is very risk averse and needs high consumption; for the domestic agent, holding the foreign asset is very risky. Exactly the opposite is true for the foreign agent. This heterogeneity between the two agents, generated by their home biased preferences, leads to the formation of long-short equilibrium portfolios.
To see the effects of investor heterogeneity in portfolio choice, we can decompose each agent’s equilibrium portfolio in a mean-variance efficient portfolio and a hedging portfolio; the former is common for both agents, but the latter differs, due to differing preferences. The decomposition is given in the following lemma, the proof of which can be found in the Appendix.

**Lemma 12** The equilibrium risky portfolio for the domestic agent can be written as

\[
x_t = (\text{var}_t(dR^*_t))^{-1} E_t(dR^*_t) + \left( \frac{\rho G_t}{\rho G_t + kG} \right) (\text{var}_t(dR^*_t))^{-1} \text{cov}_t\left( dR^*_t, \frac{dG_t}{G_t} \right)
\]

and the optimal risky portfolio for the foreign agent can be written as

\[
x^*_t = (\text{var}_t(dR^*_t))^{-1} E_t(dR^*_t) + \left( \frac{\rho G^*_t}{\rho G^*_t + kG^*} \right) (\text{var}_t(dR^*_t))^{-1} \text{cov}_t\left( dR^*_t, \frac{dG^*_t}{G^*_t} \right)
\]

The conditional mean-variance efficient portfolio is \((\text{var}_t(dR^*_t))^{-1} E_t(dR^*_t)\) and is common for both agents, while hedging portfolios are agent-specific. Specifically, the domestic hedging portfolio is the projection of domestic risk aversion shocks to the asset span and, thus, is maximally conditionally correlated with shocks in domestic risk aversion. It follows that the domestic hedging portfolio provides the maximum possible insurance against increases in \(G\). The analysis is similar for the foreign agent; her hedging portfolio is the portfolio that is maximally correlated with shocks in \(G^*\) and, thus, provides the best possible insurance against increases in \(G^*\).

It is important to note that the each agent’s hedging demand for each individual asset does not depend only on that asset’s hedging properties, but also on the hedging properties of the other asset and the covariance of the assets’ returns. The signs of hedging demands are given by the following corollary.

**Corollary 13** The mean-variance optimal holding of the asset \(i\) is

\[
\text{corr}_t\left( dR^*_t, \frac{dR^*_i}{G^*_t} \right) - \text{corr}_t\left( dR^*_t, \frac{dG^*_t}{G^*_t} \right) \text{corr}_t\left( dR^*_t, dR^*_i \right), \quad i \neq j.
\]

Therefore, the mean-variance optimal portfolio, common to both agents, depends heavily on the Sharpe ratios of the two assets. Regarding the hedging portfolio, an agent
can maintain a positive hedging demand for an asset that has poor hedging properties if the alternative is to hold even worse hedging instruments; on the other hand, the agent may short a good hedge if she can use that short position to finance a large long position in an even better hedge.

2.5.2 Home bias

The static nature of countries' portfolios does not mean that equity home bias is constant; home bias will be time-varying, due to valuation effects. To see this more clearly, define domestic home bias as:

\[
HB_t = 1 - \frac{\frac{\text{Value of domestic holdings of foreign assets}}{\text{Value of total domestic holdings}}}{\frac{\text{Total value of foreign assets}}{\text{Total value of world assets}}}
\]

When the domestic agent invests in foreign assets in proportion to their market capitalization, the home bias measure takes the value of 0; this is the case of no home bias. When the domestic agent invests solely in domestic assets, then the above measure takes the value of 1; the domestic agent is completely home biased. Here, we have:

\[
HB_t = 1 - \frac{\frac{\theta^\nu V_t^* Q_t^*}{\theta^\nu V_t + \theta^\nu V_t^* Q_t^*}}{V_t + V_t^* Q_t^*} = 1 - \frac{1 - x_t^V}{V_t^* Q_t^*} = 1 - \alpha^* - \alpha^* \left( \frac{V_t^* Q_t^*}{V_t} \right)
\]

Under home biased preferences (\(\alpha > \alpha^*\)), the degree of domestic home bias is time-varying solely because the relative valuation term \(\frac{V_t^* Q_t^*}{V_t^* V_t^* Q_t^*}\) is time-varying. The end result is procyclical home bias: domestic home bias is more pronounced when the domestic economy is in a relative expansion. More specifically, domestic home bias is increasing in two variables: the proportion of wealth that the domestic investor invests in domestic assets \(x_t^V\) and the relative value of the foreign assets \(\frac{V_t^* Q_t^*}{V_t + V_t^* Q_t^*}\). It can easily be shown that, under home bias in preferences \((\alpha > \alpha^*)\), both those variables are procyclical. Thus, in a domestic relative expansion, the domestic agent invests a higher proportion of her wealth in the domestic asset, exactly when the domestic asset constitutes a smaller proportion of global asset wealth. Each of those effects alone would generate procyclical home bias; taken together, the two effects reinforce each other, enhancing the procyclical character of asset home bias.

To understand how this result arises, assume that the domestic country is in a relative expansion \((G_t < G_t^*)\). First, simply consider the prices of the two assets expressed in units of the local good: although both \(V_t\) and \(V_t^*\) are decreasing in both risk aversion
measures $G_t$ and $G_t^*$, each country's risk aversion matters more for its local asset, due to the home bias in preferences. That would imply that a relative expansion in the domestic country increases domestic asset values more than it does foreign asset values. However, that analysis does not take into account changes in the terms of trade $Q_t^*$: the relative expansion in the domestic country depreciates the domestic good, increasing $Q_t^*$. The end result is that $V_t^*Q_t^*$, the price of the foreign asset in units of the numeraire, increases more than $V_t$. Since the domestic agent is long in the domestic asset and short in the foreign asset, those valuation changes have the result that a bigger proportion of her wealth is invested in the domestic asset; $x_t^V$ increases, pressuring home bias upward. At the same time, the foreign assets comprise a higher percentage of global wealth, pushing domestic home bias even higher.

2.5.3 Simulation results

Tables 2.2 and 2.3 present the properties of key moments in this economy; the calibration and simulation methodology is identical to that followed in the previous section. As evident in Table 2.2, the absence of complete financial markets affects consumption growth and marginal utility growth: compared to the portfolio autarky case, both variables are less volatile and more internationally correlated, since agents are able to use assets in order to intertemporally smooth their utility and consumption and share risks efficiently. As expected, the more home biased preferences are, the more volatile and less internationally correlated consumption growth is, which reflects a lower degree of risk sharing, as evidenced by the lower correlation between the two pricing kernels and the higher volatility of the real exchange rate. Importantly, the asset pricing moments (risk-free rate and excess return means and standard deviations) have empirically plausible magnitudes and are relatively unaffected by the degree of home bias in the preferences; the latter result is due to the fact that asset prices and returns depend on a weighted average of risk aversions $G$ and $G^*$ and, unconditionally, the two risk aversions have identical behavior, due to the symmetric calibration.\(^\text{18}\) Thus, the model is able to generate portfolio home bias, along with realistic asset pricing implications. The lower part of Table 2.2 presents the properties of the excess returns of the two assets and is useful in understanding equilibrium portfolios, presented in

\(^{18}\)Since the real exchange rate is not defined in the case of complete home bias ($a = 1$ and $a^* = 0$), I do not report the foreign asset returns for this parameterization.
Table 2.3, apart from reporting both agents' portfolios, decomposes the number of shares each agent holds in each asset into two components: the number of shares in the mean-variance portfolio and the number of shares in the hedging portfolio. As explained in the previous section, each country holds a home-biased long-short portfolio. Furthermore, the degree of portfolio home bias is decreasing in the degree of consumption home bias. The inverse relationship between consumption home bias and portfolio home bias may be surprising at first glance, since, in this model, home bias in preferences (and, thus, consumption) is necessary for the existence of portfolio home bias. However, the reason countries are superbiased towards their local asset is not because they want to consume its dividend, but because they want to sell a part its dividend in the goods market in order to finance their consumption of the non-local good. Consumption home bias is enough to generate portfolio home bias, but not superbias. Superbias is generated because the agents want to hedge against consumption expenditure fluctuations. Hedging against those fluctuations is very important for an agent who consumes much of the non-local good, but not very important for a very home-biased agent who mostly consumes her home good and is, thus, minimally exposed to terms of trade fluctuations. In the case of absolute home bias \((a = 1 \text{ and } a^* = 0)\), we know that consumption home bias gives rise to an absolutely biased domestic portfolio, but there is no superbias, as there is no foreign good consumption and, therefore, no exposure to terms of trade risk.

Note that the mean-variance portfolio, common for both agents, is long on the domestic asset and short on the foreign asset; since both returns are expressed in terms of the domestic good (which is the numeraire good), the foreign asset return embeds real exchange rate fluctuations that reduce its mean-variance attractiveness. It turns out that \(SR^V\) is marginally higher than \(SR^{V*}\), so the holding of the domestic (foreign) asset is positive (negative). What generates the extreme size of the absolute value of the holdings is the high correlation between the two assets; \(\text{corr}(dR_t^{V}, dR_t^{V*})\) is so close to 1 that small differences in the two Sharpe ratios are significantly amplified. In economic terms, the two assets are so correlated that the agent needs to hold an very leveraged portfolio in order to be able to extract the maximum advantage of small differences in Sharpe ratios.

On the other hand, both agents' hedging portfolios are long on the foreign asset and short on the domestic asset; the reason is that, although both assets are poor hedges for
either $G$ or $G^*$, the domestic asset is riskier than the foreign asset for both agents:

$$\text{corr} \left( \frac{dR_{t}^{e,V}}{G_{t}}, \frac{dG_{t}}{G_{t}} \right) < \text{corr} \left( \frac{dR_{t}^{e,V^*}}{G_{t}}, \frac{dG_{t}}{G_{t}} \right) < 0$$

and

$$\text{corr} \left( \frac{dR_{t}^{e,V}}{G_{t}^*}, \frac{dG_{t}^*}{G_{t}^*} \right) < \text{corr} \left( \frac{dR_{t}^{e,V^*}}{G_{t}^*}, \frac{dG_{t}^*}{G_{t}^*} \right) < 0$$

However, the domestic hedging portfolio is less biased than the foreign portfolio; the reason is that both asset returns are more correlated with changes in $G$ than to changes in $G^*$. The foreign bias in the foreign hedging portfolio is enough to reverse the domestic bias in the mean-variance portfolio and, thus, the foreign portfolio ends up being superbiased towards the foreign asset. On the other hand, the domestic bias in the domestic hedging portfolio is not enough to reverse the domestic bias of the mean-variance portfolio, so the domestic portfolio is superbiased towards the domestic asset.

### 2.6 Conclusion

This Chapter shows that external habit formation, coupled with home bias in preferences, can generate significant portfolio home bias, along with realistic asset pricing moments. The mechanism that generates portfolio home bias is novel: countries bias their equity portfolio towards the home asset due to their desire to hedge against adverse movements in their conditional risk aversion. Surprisingly, portfolio home bias is not increasing in consumption home bias; the reason is that portfolio dividends are not directly consumed by the countries, but used as a means to finance their consumption expenditure after transactions in the market for goods.

However, it is unlikely that time-varying risk aversion, generated by external habit formation, can fully explain the complex dynamics of portfolio choice in open economies. The extant international finance literature has pointed to other factors that may affect international portfolio choice in equilibrium: frictions in the market for goods, asset market frictions and restrictions and informational differences across countries, as well as behavioral biases, may well cause first-order effects in the behavior of investors. More complex models that build on the success of the external habit formation model and extend it by adopting more complex - and more realistic - assumptions would likely enhance our understanding of the portfolio home bias phenomenon.
2.7 Appendix

2.7.1 Competitive equilibrium

The first order condition in the domestic agent’s problem is

$\Xi_t = \frac{1}{\mu} e^{-\rho t} \frac{a^a(1-a)^{1-a}}{(Q^*_t)^{1-a}} \frac{\tilde{C}_t - H_t}{a^a(1-a)^{1-a}}$

where $\zeta$ is the Lagrange multiplier associated with the domestic budget constraint holding with equality. As a first step, we want to express everything in terms of expenditures $C_t$ and $C^*_t$. Using

$C_t = X_t + Y_tQ^*_t = C_tP_t = C_t \frac{(Q^*_t)^{1-a}}{a^a(1-a)^{1-a}}$

the FOC can be rewritten as follows:

$\Xi_t = \frac{1}{\zeta} e^{-\rho t} \frac{C_t}{\tilde{C}_t - H_t} = \frac{1}{\zeta} e^{-\rho t} \frac{C_t}{\tilde{C}_t - H_t} = \frac{1}{\zeta} e^{-\rho t} \frac{G_t}{\tilde{C}_t}$

or

$C_t = \frac{1}{\zeta} e^{-\rho t} \frac{G_t}{\Xi_t}$

(2.29)

Working in the same way as above, we can also write

$C^*_t = \frac{1}{\zeta^*} e^{-\rho t} \frac{G^*_t}{\Xi^*_t}$

(2.30)

where $\zeta^*$ is the Lagrange multiplier associated with the foreign budget constraint holding with equality.

Define

$\lambda_t \equiv \frac{\zeta}{\zeta^*} \frac{\Xi_t}{\Xi^*_t}$

(2.31)

For now, this is just a change in notation, but we will later show that the competitive equilibrium solution is such that $\lambda_t$, as defined above, coincides with the foreign welfare weight in the fictional planner’s problem.

We now impose market clearing:

$\tilde{C}_t = \tilde{X}_t + \tilde{Y}_tQ^*_t$

(2.32)

where $\tilde{C}_t \equiv C_t + C^*_t$ is aggregate global consumption expenditure. From (2.29), (2.30), (2.32) and definition (2.31) we get, after some algebra:

$C_t = \frac{G_t}{G_t + \lambda_t G^*_t} \tilde{C}_t$ and $C^*_t = \frac{\lambda_t G^*_t}{G_t + \lambda_t G^*_t} \tilde{C}_t$
as well as

$$\Xi_t = \frac{e^{-\rho t}}{\zeta C_t} (G_t + \lambda_t G_t^*)$$

So far our solutions are in terms of expenditures. Now, we need to solve for the allocations of each good. To do this, first we need to calculate the relative price of two goods. To do this, we use (2.5) and the corresponding result for the foreign agent

$$X_t^* = a^* C_t^*, \ Y_t^* = (1 - a^*) \frac{C_t^*}{Q_t^*}$$

to write the domestic good resource constraint $X_t + X_t^* = \bar{X}_t$ as

$$aC_t + a^* C_t^* = \bar{X}_t$$

and using (2.29), (2.30) and (2.32) we solve for $Q_t^*$:

$$Q_t^* = \frac{(1 - a) G_t + (1 - a^*) \lambda_t G_t^* \bar{X}_t}{a G_t + a^* \lambda_t G_t^* Y_t}$$

Then, from (2.32) we get:

$$\bar{C}_t = \bar{X}_t + Y_t Q_t^* = \frac{G_t + \lambda_t G_t^*}{a G_t + a^* \lambda_t G_t^*} \bar{X}_t$$

so

$$C_t = \frac{G_t}{G_t + \lambda_t G_t^*} \bar{C}_t = \frac{G_t}{a G_t + a^* \lambda_t G_t^*} \bar{X}_t$$

(2.33)

and

$$C_t^* = \frac{\lambda_t G_t^*}{G_t + \lambda_t G_t^*} \bar{C}_t = \frac{\lambda_t G_t^*}{a G_t + a^* \lambda_t G_t^*} \bar{X}_t$$

(2.34)

Thus,

$$X_t = aC_t = \frac{a G_t}{a G_t + a^* \lambda_t G_t^*} \bar{X}_t \text{ and } Y_t = (1 - a) \frac{C_t}{Q_t^*} = \frac{(1 - a) G_t}{(1 - a) G_t + (1 - a^*) \lambda_t G_t^* Y_t}$$

From market clearing, we get:

$$X_t^* = \frac{a^* \lambda_t G_t^*}{a G_t + a^* \lambda_t G_t^*} \bar{X}_t \text{ and } Y_t^* = \frac{(1 - a^*) \lambda_t G_t^*}{(1 - a) G_t + (1 - a^*) \lambda_t G_t^* Y_t}$$

Finally,

$$\Xi_t = \frac{e^{-\rho t}}{\zeta C_t} (G_t + \lambda_t G_t^*) = \frac{1}{\zeta} e^{-\rho t} \frac{a G_t + a^* \lambda_t G_t^*}{\bar{X}_t}$$
2.7.2 The planner's problem solution

Consider the fictional planner’s problem; the first order conditions are

\[ e^{-\rho t} \pi(\omega, t) a \frac{G_t}{X_t} = \Theta_t \]

\[ e^{-\rho t} \pi(\omega, t) (1 - a) \frac{G_t}{Y_t} = \Theta^*_t \]

\[ \lambda_t e^{-\rho t} \pi(\omega, t) a^* \frac{G^*_t}{X^*_t} = \Theta_t \]

\[ \lambda_t e^{-\rho t} \pi(\omega, t) (1 - a^*) \frac{G^*_t}{Y^*_t} = \Theta^*_t \]

where \( \Theta_t \) and \( \Theta^*_t \) are the Lagrange multipliers associated with the market clearing condition for the domestic and the foreign good, respectively, and \( \pi(\omega, t) \) is the \( P \) measure probability that state \( \omega \) occurs at time \( t \). Using the resource constraints, it is easy to see that the system of equations yields (2.15) and (2.16), the same sharing rule as the competitive equilibrium solution. Thus, the variable \( \lambda \), as defined in (2.31), corresponds to the fictional welfare weight of the planner’s problem.

The terms of trade are equal to the ratio of foreign good marginal utility to domestic good marginal utility for both agents, i.e.

\[ Q^*_t = \frac{\partial u(X_t, Y_t)}{\partial Y_t} \frac{\partial u(X^*_t, Y^*_t)}{\partial Y^*_t} = \frac{\partial u(X^*_t, Y^*_t)}{\partial X^*_t} \frac{\partial u(X^*_t, Y^*_t)}{\partial X^*_t} \]

For the domestic agent

\[ \frac{\partial u(X_t, Y_t)}{\partial X_t} = a \frac{G_t}{X_t}, \quad \frac{\partial u(X_t, Y_t)}{\partial Y_t} = (1 - a) \frac{G_t}{Y_t} \] (2.35)

and for the foreign agent

\[ \frac{\partial u(X^*_t, Y^*_t)}{\partial X^*_t} = a^* \frac{G^*_t}{X^*_t}, \quad \frac{\partial u(X^*_t, Y^*_t)}{\partial Y^*_t} = (1 - a^*) \frac{G^*_t}{Y^*_t} \] (2.36)

so, using (2.15) and (2.16) we get (2.21).

It is trivially easy to show that all other prices and quantities that the planner’s problem solution yields are identical to their competitive equilibrium counterparts.
2.7.3 Proofs

Proof of Proposition 7

To calculate $W_t$ and $W^*_t$, we first need to strengthen the conditions on $\nu_t$ and $\nu^*_t$. As mentioned in the main body of this Chapter, $\nu$ satisfies $\int_0^\infty |\tilde{\sigma}_t|^2 dt < \infty$; the same is true for $\nu^*$. In the remainder of the Chapter, I will strengthen the conditions on $\nu_t$ and $\nu^*_t$ by assuming that $\nu_t$ and $\nu^*_t$ satisfy the appropriate regularity conditions so that the resulting stochastic integrals are true martingales. Economically, this extra assumption implies the absence of bubbles in equilibrium asset prices. Under this assumption, we can consider the definition

$$W_t = \mathbb{E}_t \left[ \int_t^\infty \tilde{\sigma}_s ds \right]$$

and using (2.33) and (2.22), we get (2.13). Similarly, using the definition of $W^*$ and plugging in the expressions in (2.34) and (2.23), we get (2.14).

Proof of Proposition 9

We have:

$$C_t = \frac{a^a(1-a)^{1-a}}{(a + a^*\lambda_t \left( \frac{G^*_t}{G_t} \right))^a \left( (1-a) + (1-a^*)\lambda_t \left( \frac{G^*_t}{G_t} \right) \right)^{1-a}} \tilde{X}_t \tilde{Y}^{1-a}$$

Then, by Itô's lemma:

$$d \log C_t = \text{drift} + (a\sigma^X_t + (1-a)\sigma^Y_t)' dB_t$$

$$-k_t \left( (\nu_t - \nu^*_t) + \delta \left( \frac{G_t - l}{G_t} \right) \sigma^C_t - \delta \left( \frac{G^*_t - l}{G^*_t} \right) \sigma^{C*}_t \right)' dB_t$$

where

$$k_t = \frac{a(1-a) + a^*(1-a^*)\lambda_t \left( \frac{G^*_t}{G_t} \right)}{(a + a^*\lambda_t \left( \frac{G^*_t}{G_t} \right))^a \left( (1-a) + (1-a^*)\lambda_t \left( \frac{G^*_t}{G_t} \right) \right)^{1-a}} \lambda_t \left( \frac{G^*_t}{G_t} \right)$$

(2.37)

Matching coefficients:

$$\sigma^C_t = -k_t \left( \delta \left( \frac{G_t - l}{G_t} \right) \sigma^C_t - \delta \left( \frac{G^*_t - l}{G^*_t} \right) \sigma^{C*}_t + (\nu_t - \nu^*_t) + a\sigma^X_t + (1-a)\sigma^Y_t \right)$$

(2.38)

19 Since this extra assumption is not innocuous, it has to be justified. In this paper, I consider two asset market specifications. Under the first specification, in which both endowment claims are securitizable, financial markets are complete in equilibrium, so $\nu_t = \nu^*_t = 0, \forall t \in [0, \infty]$. In that case, the regularity conditions are trivially satisfied. Under the second specification, in which there is no international trade in assets, financial markets are incomplete. Considering the equilibrium consumption allocation in that case, we can easily solve for all the variables of interest and note that for $\lambda_t = \frac{1-a}{a^* \frac{G^*_t}{G_t}}$, which is the equilibrium value of the fictional welfare weight under this particular form of market incompleteness, the solutions coincide with those in the main body of the Chapter.
Similarly:

\[ C_t^* = \frac{(a^*)^{a^*} (1 - a^*)^{1-a^*} \lambda_t \left( \frac{G_t^*}{G_t} \right)}{(a + a^* \lambda_t \left( \frac{G_t^*}{G_t} \right))^{a^*} \left( (1 - a) + (1 - a^*) \lambda_t \left( \frac{G_t^*}{G_t} \right) \right)^{1-a^*} X_t^* Y_t^{1-a} } \]

so, by Itô's lemma:

\[ d \log C_t^* = drift + (a^* \sigma_t^X + (1 - a^*) \sigma_t^Y)' dB_t \]

\[ k_t^* \left( \nu_t^* - \nu_t \right) + \delta \left( \frac{G_t - l}{G_t} \right) \sigma_t^C - \delta \left( \frac{G_t^* - l}{G_t^*} \right) \sigma_t^C^* \right)' dB_t \]

where

\[ k_t^* = \frac{a(1-a) + a^*(1-a^*) \lambda_t \left( \frac{G_t^*}{G_t} \right)}{(a + a^* \lambda_t \left( \frac{G_t^*}{G_t} \right)) \left( (1 - a) + (1 - a^*) \lambda_t \left( \frac{G_t^*}{G_t} \right) \right)} \] (2.39)

Matching coefficients:

\[ \sigma_t^C^* = k_t^* \left( \delta \left( \frac{G_t - l}{G_t} \right) \sigma_t^C - \delta \left( \frac{G_t^* - l}{G_t^*} \right) \sigma_t^C + \nu_t^* - \nu_t \right) + a^* \sigma_t^X + (1 - a^*) \sigma_t^Y \] (2.40)

Thus, we have a system of equations ((2.38) and (2.40)), the solution of which is:

\[ \sigma_t^C = \frac{1}{D_t^C} \left( a \sigma_t^X + (1 - a) \sigma_t^Y \right) \]

\[ + \frac{1}{D_t^C} \left\{ \left( a^* k_t + a k_t^* \right) \sigma_t^X + ((1 - a^*) k_t + (1 - a) k_t^*) \sigma_t^Y \right\} \delta \left( \frac{G_t - l}{G_t} \right) \]

\[ - \frac{k_t}{D_t^C} \left( \nu_t^* - \nu_t \right) \]

and

\[ \sigma_t^C^* = \frac{1}{D_t^C} \left( a^* \sigma_t^X + (1 - a^*) \sigma_t^Y \right) \]

\[ + \frac{1}{D_t^C} \left\{ \left( a^* k_t + a k_t^* \right) \sigma_t^X + ((1 - a^*) k_t + (1 - a) k_t^*) \sigma_t^Y \right\} \delta \left( \frac{G_t^* - l}{G_t^*} \right) \]

\[ + \frac{k_t^*}{D_t^C} \left( \nu_t^* - \nu_t \right) \]

where

\[ D_t^C = 1 + k_t \delta \left( \frac{G_t - l}{G_t} \right) + k_t^* \delta \left( \frac{G_t^* - l}{G_t^*} \right) \] (2.41)

**Proof of Proposition 10**

We adopt an "assume and verify" strategy. We first assume that \( \sigma_t \) is non-singular and, after solving for asset prices and returns, verify that the resulting asset return diffusion matrix \( \sigma_t \) is, indeed, non-singular. Assuming that \( \sigma_t^{-1} \) exists, we have:

\[ \sigma_t \nu_t = 0 \Rightarrow \nu_t = 0 \]
and, similarly, $\nu^*_t = 0$, so

$$\frac{d\lambda_t}{\lambda_t} = 0 \Rightarrow \lambda_t = \lambda_0 = \tilde{\lambda}, \forall t \in [0, \infty)$$

The price of the domestic asset is

$$V_t = E_t \left[ \int_{\tilde{t}}^{\infty} \tilde{X}_s ds \right] = E_t \left[ \int_{\tilde{t}}^{\infty} \tilde{X}^*_s ds \right] = \frac{1}{\rho} \left( \frac{k}{\rho + k} \frac{(a + a^* \tilde{\lambda}) \tilde{G} + \rho G^*_t + k \tilde{G}}{aG_t + a^* \lambda G^*_t} \right) \tilde{X}_t$$

Similarly, the price of the foreign total wealth portfolio, in units of the numeraire, is:

$$V^*_t = E_t \left[ \int_{\tilde{t}}^{\infty} \tilde{Y}_s Q^*_s ds \right] = E_t \left[ \int_{\tilde{t}}^{\infty} \tilde{Y}^*_s Q^*_s ds \right] = \frac{1}{\rho} \left( (1 - a) \rho G_t + k \tilde{G} + (1 - a^*) \tilde{\lambda} \rho G^*_t + k \tilde{G} \right) \tilde{X}_t$$

Define $\sigma^V_t$ and $\sigma^{V^*Q^*}_t$ as the processes that satisfy

$$\frac{dV^*_t}{V^*_t} = \text{drift} + \sigma^V_t dB_t \quad \text{and} \quad \frac{d(V^*_t Q^*_t)}{V^*_t Q^*_t} = \text{drift} + \sigma^{V^*Q^*}_t dB_t$$

An application of Itô’s lemma shows that

$$\sigma^V_t = \frac{(a + a^* \tilde{\lambda}) k \tilde{G}}{(a + a^* \tilde{\lambda}) k \tilde{G} + a \rho G_t + a^* \lambda \rho G^*_t} \left[ (1 - \omega_t) \delta \left( \frac{G_t - l}{G_t} \right) \sigma^C_t + (1 - \omega_t) \delta \left( \frac{G^*_t - l}{G^*_t} \right) \sigma^{C^*}_t \right] + \sigma^X_t$$

and

$$\sigma^{V^*Q^*}_t = \left[ \frac{G_t}{a G_t + a^* \lambda G^*_t} \left( (a - a^*) \rho \tilde{\lambda} G_t + a \tilde{\lambda} (1 - a) \tilde{\lambda} \tilde{G} + (1 - a^*) \tilde{\lambda} \rho G^*_t \right) \left( (1 - a) + (1 - a^*) \tilde{\lambda} \tilde{G} \right) \right. \delta \left( \frac{G_t - l}{G_t} \right) \sigma^C_t$$

$$- \left[ \frac{G^*_t}{a G_t + a^* \lambda G^*_t} \left( (a - a^*) \rho \tilde{\lambda} G_t - a^* \tilde{\lambda} (1 - a) \tilde{\lambda} \tilde{G} + (1 - a^*) \tilde{\lambda} \rho G^*_t \right) \left( (1 - a) + (1 - a^*) \tilde{\lambda} \tilde{G} \right) \right] \delta \left( \frac{G^*_t - l}{G^*_t} \right) \sigma^{C^*}_t$$

$$+ \sigma^X_t$$

By definition

$$\sigma_t = \begin{bmatrix} \sigma^V_t \\ \sigma^{V^*Q^*}_t \end{bmatrix}$$

It can easily be shown that $|\sigma_t| \neq 0$ (almost surely) if $a \neq a^*$. When $a = a^*$, it is easy to see that

$$V_t = \left( \frac{1}{a} \right) V^*_t$$
The price of the foreign asset, in units of the numeraire, is a multiple of the price of the domestic asset; thus, their returns are perfectly positively correlated and the asset return matrix $\sigma_t$ is non-singular. Thus, the only restriction that market completeness imposes on the model specification is that the two agents need to have different consumption baskets.

**Proof of Proposition 11**

Assuming that $a \neq a^*$, we can calculate equilibrium portfolios. Under the assumptions of the current Section, domestic and foreign asset wealth, $A$ and $A^*$, respectively, are given by $A_t = W_t$ and $A^*_t = W^*_t$. To calculate the domestic portfolio, we apply Itô’s lemma to (2.13) and get

$$\frac{dA_t}{A_t} = \text{drift} + \left( \eta_t - \left( \frac{\rho G_t}{\rho G_t + kG} \right) \delta \left( \frac{G_t - l}{G_t} \right) \sigma_t^C \right)' dB_t$$

Considering the intertemporal budget constraint of the domestic agent:

$$dA_t = \pi_t' \left( \mu_t \text{dt} + \sigma_t dB_t \right) + (A_t - \pi_t' 1_k) r^f_t dt - C_t dt$$

and matching diffusions, we get:

$$\pi_t' \sigma_t = A_t \left( \eta_t - \left( \frac{\rho G_t}{\rho G_t + kG} \right) \delta \left( \frac{G_t - l}{G_t} \right) \sigma_t^C \right)'$$

or, dividing by $A_t$ and solving for the portfolio weights $x_t$:

$$x_t = (\sigma_t')^{-1} \eta_t - (\sigma_t')^{-1} \left( \frac{\rho G_t}{\rho G_t + kG} \right) \delta \left( \frac{G_t - l}{G_t} \right) \sigma_t^C$$

(2.42)

After a substantial amount of algebra, we derive (2.25), (2.26), (2.27) and (2.28).

Similarly, it can easily be shown that the foreign equilibrium portfolio weights $x^*_t$ are:

$$x^*_t = (\sigma_t'^*)^{-1} \eta_t - (\sigma_t'^*)^{-1} \left( \frac{\rho G_t}{\rho G_t^* + kG^*} \right) \delta \left( \frac{G_t - l}{G_t^*} \right) \sigma_t^C$$

(2.43)

**Proof of Lemma 12**

Recall that $\eta_t = \sigma_t' (\sigma_t \sigma_t')^{-1} \mu_t$, so (2.42) can be written as

$$x_t = (\sigma_t')^{-1} \sigma_t' (\sigma_t \sigma_t')^{-1} \mu_t - (\sigma_t')^{-1} \left( \frac{\rho G_t}{\rho G_t + kG} \right) \delta \left( \frac{G_t - l}{G_t} \right) \sigma_t^C$$

Since $\sigma_t$ is invertible, simple algebra yields

$$x_t = (\sigma_t \sigma_t'^{-1})^{-1} \mu_t + \left( \frac{\rho G_t}{\rho G_t + kG} \right) (\sigma_t \sigma_t'^{-1})^{-1} \sigma_t \left( -\delta \left( \frac{G_t - l}{G_t} \right) \sigma_t^C \right)$$

which is equivalent to the expression for $x_t$ in Lemma 12. The expression for $x^*_t$ is derived in a similar fashion.
<table>
<thead>
<tr>
<th>Moment</th>
<th>$a = 0.5$</th>
<th>$a = 0.7$</th>
<th>$a = 0.9$</th>
<th>$a = 0.95$</th>
<th>$a = 0.99$</th>
<th>$a = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dom. end. growth st. dev.</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
</tr>
<tr>
<td>Fgn. end. growth st. dev.</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
</tr>
<tr>
<td>Endowment growth corr.</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
</tr>
<tr>
<td>Dom. cons. growth st. dev.</td>
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<td>1.14%</td>
<td>1.36%</td>
<td>1.43%</td>
<td>1.48%</td>
<td>1.50%</td>
</tr>
<tr>
<td>Fgn. cons. growth st. dev.</td>
<td>1.06%</td>
<td>1.14%</td>
<td>1.36%</td>
<td>1.42%</td>
<td>1.48%</td>
<td>1.50%</td>
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<td>Consumption growth corr.</td>
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<td>0.723</td>
<td>0.218</td>
<td>0.103</td>
<td>0.019</td>
<td>-0.001</td>
</tr>
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<td>Dom. pr. kernel st. dev.</td>
<td>28.73%</td>
<td>30.20%</td>
<td>33.78%</td>
<td>34.85%</td>
<td>35.75%</td>
<td>35.98%</td>
</tr>
<tr>
<td>Fgn. pr. kernel st.dev.</td>
<td>28.73%</td>
<td>30.21%</td>
<td>33.87%</td>
<td>34.97%</td>
<td>35.88%</td>
<td>36.11%</td>
</tr>
<tr>
<td>Pricing kernel corr.</td>
<td>1.000</td>
<td>0.605</td>
<td>0.129</td>
<td>0.056</td>
<td>0.010</td>
<td>-0.000</td>
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<td>$\Delta \log Q^*$ st. dev.</td>
<td>2.12%</td>
<td>2.12%</td>
<td>2.12%</td>
<td>2.12%</td>
<td>2.12%</td>
<td>-</td>
</tr>
<tr>
<td>$\Delta \log E$ st. dev.</td>
<td>0.00%</td>
<td>0.85%</td>
<td>1.70%</td>
<td>1.91%</td>
<td>2.08%</td>
<td>-</td>
</tr>
<tr>
<td>$(\nu^* - \nu)_1$</td>
<td>0.000</td>
<td>-0.054</td>
<td>-0.095</td>
<td>-0.103</td>
<td>-0.109</td>
<td>-0.111</td>
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<tr>
<td>$(\nu^* - \nu)_2$</td>
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<td>0.055</td>
<td>0.096</td>
<td>0.104</td>
<td>0.110</td>
<td>0.111</td>
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</tbody>
</table>

Table 2.1: Moments: portfolio autarky case

Simulated moments assuming portfolio autarky. I simulate 20,000 sample paths of the model economy, with each path consisting of 240 quarterly observations (60 years). The system is initialized at $G_1 = G_1^* = \bar{G}$ and $X_1 = Y_1 = 1$. Of the 240 observations, the first 40 (10 years) are discarded to reduce the dependence on initial conditions. Thus, each sample path consists of 200 observations (50 years). For each of the moments of interest, Table 2.1 presents the sample average across the 20,000 simulations. The calibration is symmetric; for each of the values of $a$, $a^* = 1 - a$, so the degree of home bias is identical in the two countries.
<table>
<thead>
<tr>
<th>Moment</th>
<th>$a = 0.5$</th>
<th>$a = 0.7$</th>
<th>$a = 0.9$</th>
<th>$a = 0.95$</th>
<th>$a = 0.99$</th>
<th>$a = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dom. end. growth st. dev.</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
</tr>
<tr>
<td>Fgn. end. growth st. dev.</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
<td>1.50%</td>
</tr>
<tr>
<td>End. growth corr.</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
</tr>
<tr>
<td>Dom. cons. growth st. dev.</td>
<td>1.06%</td>
<td>1.07%</td>
<td>1.12%</td>
<td>1.16%</td>
<td>1.31%</td>
<td>1.50%</td>
</tr>
<tr>
<td>Fgn. cons. growth st. dev.</td>
<td>1.06%</td>
<td>1.07%</td>
<td>1.12%</td>
<td>1.16%</td>
<td>1.31%</td>
<td>1.50%</td>
</tr>
<tr>
<td>Cons. growth corr.</td>
<td>1.000</td>
<td>0.954</td>
<td>0.809</td>
<td>0.709</td>
<td>0.398</td>
<td>-0.001</td>
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<tr>
<td>Dom. pr. kernel st. dev.</td>
<td>28.73%</td>
<td>28.72%</td>
<td>28.71%</td>
<td>28.78%</td>
<td>30.58%</td>
<td>35.98%</td>
</tr>
<tr>
<td>Fgn. pr. kernel st. dev.</td>
<td>28.73%</td>
<td>28.72%</td>
<td>28.71%</td>
<td>28.77%</td>
<td>30.58%</td>
<td>36.11%</td>
</tr>
<tr>
<td>Pricing kernel corr.</td>
<td>1.000</td>
<td>0.999</td>
<td>0.986</td>
<td>0.950</td>
<td>0.611</td>
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<td>$\Delta \log Q^*$ st. dev.</td>
<td>2.12%</td>
<td>2.49%</td>
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<td>9.33%</td>
<td>26.87%</td>
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<tr>
<td>$\Delta \log E$ st. dev.</td>
<td>0.00%</td>
<td>1.00%</td>
<td>4.33%</td>
<td>8.40%</td>
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<td>-</td>
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<tr>
<td>$r_f$ mean</td>
<td>1.65%</td>
<td>1.65%</td>
<td>1.65%</td>
<td>1.64%</td>
<td>1.28%</td>
<td>0.33%</td>
</tr>
<tr>
<td>$r_f$ std</td>
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<td>1.77%</td>
<td>1.76%</td>
<td>1.77%</td>
<td>1.83%</td>
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<tr>
<td>Dom. excess ret. mean</td>
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<td>4.83%</td>
<td>4.84%</td>
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<td>18.65%</td>
<td>18.69%</td>
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<td>19.94%</td>
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<tr>
<td>Dom. Sharpe ratio</td>
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<td>0.271</td>
<td>0.271</td>
<td>0.272</td>
<td>0.285</td>
<td>0.318</td>
</tr>
<tr>
<td>Fgn. excess ret. mean</td>
<td>4.83%</td>
<td>4.83%</td>
<td>4.86%</td>
<td>4.96%</td>
<td>6.28%</td>
<td>-</td>
</tr>
<tr>
<td>Fgn. excess ret. st. dev.</td>
<td>18.64%</td>
<td>18.65%</td>
<td>18.79%</td>
<td>19.18%</td>
<td>23.59%</td>
<td>-</td>
</tr>
<tr>
<td>Fgn. Sharpe ratio</td>
<td>0.271</td>
<td>0.271</td>
<td>0.271</td>
<td>0.271</td>
<td>0.280</td>
<td>-</td>
</tr>
<tr>
<td>Excess return corr.</td>
<td>1.000</td>
<td>1.000</td>
<td>0.999</td>
<td>0.995</td>
<td>0.949</td>
<td>-</td>
</tr>
<tr>
<td>$corr(R_m^V, d \log E)$</td>
<td>0.000</td>
<td>0.067</td>
<td>0.122</td>
<td>0.190</td>
<td>0.422</td>
<td>-</td>
</tr>
<tr>
<td>$corr(R_m^{\hat{V}_m}, d \log E)$</td>
<td>0.000</td>
<td>0.071</td>
<td>0.160</td>
<td>0.272</td>
<td>0.640</td>
<td>-</td>
</tr>
<tr>
<td>$corr(R_m^V, \frac{d \hat{Q}}{\hat{Q}})$</td>
<td>-0.979</td>
<td>-0.980</td>
<td>-0.980</td>
<td>-0.980</td>
<td>-0.977</td>
<td>-0.958</td>
</tr>
<tr>
<td>$corr(R_m^{\hat{V}_m}, \frac{d \hat{Q}}{\hat{Q}})$</td>
<td>-0.979</td>
<td>-0.980</td>
<td>-0.978</td>
<td>-0.973</td>
<td>-0.940</td>
<td>-</td>
</tr>
<tr>
<td>$corr(R_m^V, \frac{d \hat{Q}}{\hat{Q}})$</td>
<td>-0.979</td>
<td>-0.977</td>
<td>-0.958</td>
<td>-0.915</td>
<td>-0.571</td>
<td>0.001</td>
</tr>
<tr>
<td>$corr(R_m^{\hat{V}_m}, \frac{d \hat{Q}}{\hat{Q}})$</td>
<td>-0.979</td>
<td>-0.977</td>
<td>-0.950</td>
<td>-0.883</td>
<td>-0.361</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2.2: Quantities and prices: complete markets case

Simulated moments assuming complete markets. I simulate 20,000 sample paths of the model economy, with each path consisting of 240 quarterly observations (60 years). The system is initialized at $G_1 = G_1^* = \tilde{G}$ and $\tilde{X}_1 = \tilde{Y}_1 = 1$. Of the 240 observations, the first 40 (10 years) are discarded to reduce the dependence on initial conditions. Thus, each sample path consists of 200 observations (50 years). For each of the moments of interest, Table 2.2 presents the sample average across the 20,000 simulations. The calibration is symmetric; for each of the values of $a$, $a^* = 1 - a$, so the degree of home bias is identical in the two countries.
### Table 2.3: Portfolios: complete markets case

<table>
<thead>
<tr>
<th>Number of shares</th>
<th>$a = 0.7$</th>
<th>$a = 0.9$</th>
<th>$a = 0.95$</th>
<th>$a = 0.99$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Domestic portfolio</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta^V$</td>
<td>1.750</td>
<td>1.125</td>
<td>1.056</td>
<td>1.010</td>
</tr>
<tr>
<td>$\theta^{V*}$</td>
<td>-0.750</td>
<td>-0.125</td>
<td>-0.056</td>
<td>-0.010</td>
</tr>
<tr>
<td>$\theta^V$ mean-variance</td>
<td>4.665</td>
<td>1.646</td>
<td>1.433</td>
<td>1.294</td>
</tr>
<tr>
<td>$\theta^{V*}$ mean-variance</td>
<td>-3.378</td>
<td>-0.360</td>
<td>-0.148</td>
<td>-0.017</td>
</tr>
<tr>
<td>$\theta^V$ hedging</td>
<td>-2.915</td>
<td>-0.521</td>
<td>-0.378</td>
<td>-0.283</td>
</tr>
<tr>
<td>$\theta^{V*}$ hedging</td>
<td>2.628</td>
<td>0.235</td>
<td>0.093</td>
<td>0.007</td>
</tr>
<tr>
<td><strong>Foreign portfolio</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta^{*V}$</td>
<td>-0.750</td>
<td>-0.125</td>
<td>-0.0556</td>
<td>-0.010</td>
</tr>
<tr>
<td>$\theta^{<em>V</em>}$</td>
<td>1.750</td>
<td>1.125</td>
<td>1.0556</td>
<td>1.010</td>
</tr>
<tr>
<td>$\theta^{*V}$ mean-variance</td>
<td>4.665</td>
<td>1.646</td>
<td>1.434</td>
<td>1.304</td>
</tr>
<tr>
<td>$\theta^{<em>V</em>}$ mean-variance</td>
<td>-3.378</td>
<td>-0.360</td>
<td>-0.148</td>
<td>-0.016</td>
</tr>
<tr>
<td>$\theta^{*V}$ hedging</td>
<td>-5.415</td>
<td>-1.771</td>
<td>-1.490</td>
<td>-1.314</td>
</tr>
<tr>
<td>$\theta^{<em>V</em>}$ hedging</td>
<td>5.128</td>
<td>1.485</td>
<td>1.203</td>
<td>1.026</td>
</tr>
</tbody>
</table>

Average number of shares in the domestic and foreign portfolio. Simulated data assuming complete markets. I simulate 20,000 sample paths of the model economy, with each path consisting of 240 quarterly observations (60 years). The system is initialized at $G_1 = G_1^* = \bar{G}$ and $\bar{X}_1 = \bar{Y}_1 = 1$. Of the 240 observations, the first 40 (10 years) are discarded to reduce the dependence on initial conditions. Thus, each sample path consists of 200 observations (50 years). Each number is the sample average of shares across time, across the 20,000 simulations. The calibration is symmetric; for each of the values of $a$, $a^* = 1 - a$, so the degree of home bias is identical in the two countries.
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