Essays on Volatility Derivatives and Portfolio Optimization

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ABSTRACT

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Ashish Jain

This thesis is a collection of four papers: 1) Discrete and continuously sampled volatility and variance swaps, 2) Pricing and hedging of volatility derivatives, 3) VIX index and VIX futures, and 4) Asset allocation and generalized buy and hold trading strategies.

The first three papers answer various questions relating to the volatility derivatives. Volatility derivatives are securities whose payoff depends on the realized variance of an underlying asset or an index. These include variance swaps, volatility swaps and variance options. All of these derivatives are trading in over-the-counter market. With the popularity of these products and increasing demand of these OTC products, the Chicago Board of Options Exchange (CBOE) changed the definition of VIX index and launched VIX futures on VIX index. The new definition of VIX index approximates the one month variance swap rate. In second chapter we investigate the effect of discrete sampling and asset price jumps on fair variance swap strikes. We calculate the fair discrete volatility strike and the fair discrete variance strike in different models of the underlying evolution of the asset price: the Black-Scholes model, the Heston stochastic volatility model, the Merton jump-diffusion model and the Bates and Scott stochastic volatility model with jumps. We determine fair discrete and continuous variance strikes analytically and fair discrete and continuous volatility strikes using simulation and variance reduction techniques and numerical integration techniques in all models. Numerical results are provided to show that the well known convexity correction formula doesn't work well to
approximate volatility strikes in the jump-diffusion models. We find that, for realistic contract specifications and realistic risk-neutral asset price processes, the effect of discrete sampling in minimal while the effect of jumps can be significant.

In the third chapter we present pricing and hedging of variance swaps and other volatility derivatives, e.g., volatility swaps and variance options, in the Heston stochastic volatility model using partial differential equation techniques. We formulate an optimization problem to determine the number of options required to best hedge a variance swap. We propose a method to dynamically hedge volatility derivatives using variance swaps and a finite number of European call and put options.

In the fourth chapter we study the pricing of VIX futures in the Heston stochastic volatility (SV) model and the Bates and Scott stochastic volatility with jumps (SVJ) model. We provide formulas to price VIX futures under the SV and SVJ models. We discuss the properties of these models in fitting VIX futures prices using market VIX futures data and SPX options data. We empirically investigate profit and loss of strategies which invest in variance swaps and VIX futures empirically using historical data of the SPX index level, VIX index level and VIX futures data. We compare the empirical results with theoretical predictions from the SV and SVJ model.

In fifth chapter we present the generalized buy-and-hold (GBH) portfolio strategies which are defined to be the class of strategies where the terminal wealth is a function of only the terminal security prices. We solve for the optimal GBH strategy when security prices follow a multi-dimensional diffusion process and when markets are incomplete. Using recently developed duality techniques, we compare the optimal GBH portfolio to the
optimal dynamic trading strategy. While the optimal dynamic strategy often significantly outperforms the GBH strategy, this is not true in general. In particular, when no-borrowing or no-short sales constraints are imposed on dynamic trading strategies, it is possible for the optimal GBH strategy to significantly outperform the optimal dynamic trading strategy. For the class of security price dynamics under consideration, we also obtain a closed-form solution for the terminal wealth and expected utility of the classic constant proportion trading strategy and conclude that this strategy is inferior to the optimal GBH strategy.
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I further want to express my gratitude to my parents, who always supported me. I dedicate this dissertation to my parents.
To My Parents
Chapter 1

Introduction

This thesis develops pricing and hedging formulas for volatility derivatives, e.g., variance swaps, volatility swaps, variance options, VIX futures, in different asset pricing models which include stochastic volatility and jumps. The last chapter of this thesis develops new class of trading strategies, the generalized buy-and-hold (GBH) portfolio strategies in an incomplete market setting and compares the utility of these strategies with the optimal dynamic trading strategy using duality techniques.

In this chapter we give a brief motivation for the following chapters. Section 1.1 gives a brief overview of the chapters on volatility derivatives and section 1.2 gives an overview of last chapter on asset allocation and generalized buy and hold trading strategies.

1.1 Volatility Derivatives

Volatility and variance swaps are forward contracts in which one counterparty agrees to pay the other a notional amount times the difference between a fixed level and a realized level of variance and volatility, respectively. The fixed level is called the variance strike for variance swaps and the volatility strike for volatility swaps. This is typically set initially so that the net present value of the payoff is zero. The realized variance is
determined by the average variance of the asset over the life of the swap. Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a partition of the time interval $[0, T]$ into $n$ equal segments of length $\Delta t$, i.e., $t_i = iT/n$ for each $i = 0, 1, \ldots, n$. Most traded contracts define the realized variance to be

$$V_d(0, n, T) = \frac{AF}{n-1} \sum_{i=0}^{n-1} \left( \ln \left( \frac{S_{i+1}}{S_i} \right) \right)^2$$

for a swap covering $n$ return observations. Here $S_i$ is the price of the asset at the $i^{th}$ observation time $t_i$ and $AF$ is the annualization factor, e.g., $252 (= n/T)$ if the maturity of the swap, $T$, is one year with daily sampling. This definition of realized variance differs from the usual sample variance because the sample average is not subtracted from each observation. Since the sample average is approximately zero the realized variance is close to the sample variance.

The analysis in most papers in the literature is based on an idealized contract where realized variance and volatility are defined with continuous sampling, e.g., a continuously sampled realized variance, $V_c(0, T)$, defined by:

$$V_c(0, T) = \lim_{n \to \infty} V_d(0, n, T)$$

In second chapter we analyze the differences between actual contracts based on discrete sampling and idealized contracts based on continuous sampling. We calculate the fair discrete volatility strike and the fair discrete variance strike in different models of the underlying evolution of the asset price: the Black-Scholes model, the Heston stochastic volatility model (SV), the Merton jump-diffusion model (J) and a combined Bates (1996) and Scott (1997) stochastic volatility jump model (SVJ). We determine fair discrete and
continuous variance strikes analytically and fair discrete and continuous volatility strikes using simulation and variance reduction techniques and numerical integration techniques in all models. Brockhaus and Long (2000) provide a convexity correction formula for calculating the fair volatility strike using a Taylor’s expansion of the square root function. We provide a theoretical condition required for the convexity correction formula to provide a good approximation to fair volatility strikes. The theoretical condition implies that the convexity correction formula is a good approximation if the realized variance on each sample path is less than twice the expected value of the realized variance. We quantify this condition in terms of the excess probability and compute this for all four models. We show that the convexity correction formula doesn’t work well to approximate volatility strikes in the SV, J and SVJ models. We prove that the expected discrete realized variance converges linearly with the number of sampling dates to the expected continuous realized variance in all models. Numerical results show that the expected discrete realized volatility converges linearly with the number of sampling dates to the expected continuous realized volatility in all models.

In third chapter we propose a methodology for hedging volatility swaps and variance options using variance swaps. The no arbitrage relationship can be exploited in the pricing and hedging of volatility derivatives. We compute fair volatility strikes and price variance options by deriving a partial differential equation that must be satisfied by volatility derivatives in the Heston stochastic volatility model. We compute the risk management parameters (greeks) of volatility derivatives by solving a series of partial differential equations. We formulate an optimization problem to determine the number
of options required to best hedge a variance swap. We propose a method to dynamically hedge volatility derivatives using variance swaps and a finite number of European call and put options.

In fourth chapter we study the pricing of VIX futures in the Heston stochastic volatility (SV) model and the Bates and Scott stochastic volatility with jumps (SVJ) model. VIX futures are exchange traded contracts on a one month volatility index level (VIX) derived from a basket of S&P 500 (SPX) index options. We study how sensitive the VIX formula is to the interval between discrete set of strikes and a finite range of strikes of SPX options used in the computation. We provide formulas to price VIX futures under the SV and SVJ models. We discuss the properties of these models in fitting VIX futures prices using market VIX futures data and SPX options data. We empirically investigate profit and loss of strategies which invest in variance swaps and VIX futures empirically using historical data of the SPX index level, VIX index level and VIX futures data. We compare the empirical results with theoretical predictions from the SV and SVJ model.

In an attempt to make the chapters as self contained as possible, some material is repeated in some chapters.

1.2 Asset Allocation and Generalized Buy-and-Hold Strategies

In the last chapter of this thesis we introduce a particular class of strategies, the \textit{generalized buy-and-hold} (GBH) strategies. We define the GBH strategies to be the class of strategies where the terminal wealth is a function of \textit{only} the terminal security prices. In contrast, the terminal wealth of a standard static buy-and-holy strategy is always an
affiliate function of terminal security prices. However, it should be possible to approximate the payoff of a GBH strategy using a static portfolio consisting of positions in the cash account, the underlying securities and some judiciously chosen European-style options on these securities. When the expected utility of the optimal GBH portfolio is comparable to the expected utility of the optimal dynamic strategy, then many investors should benefit by instead adopting the more static-like optimal GBH portfolio.

Indeed, when investors face position constraints such as no short-sales or no borrowing constraints, the GBH portfolio can have a significantly higher expected utility than the optimal dynamic strategy that trades only in the underlying securities.

We solve for the optimal GBH strategy when security prices follow a multi-dimensional diffusion process and when markets are incomplete. Using recently developed duality techniques, we compare the optimal GBH portfolio to the optimal dynamic trading strategy. While the optimal dynamic strategy often significantly outperforms the GBH strategy, this is not true in general. In particular, when no-borrowing or no-short sales constraints are imposed on dynamic trading strategies, it is possible for the optimal GBH strategy to significantly outperform the optimal dynamic trading strategy.

The main contributions of fourth chapter are: First, we extend the applicability of the dual methods developed in Haugh, Kogan and Wang (2006) to evaluate a new class of strategies, i.e. the GBH strategies. Second, we also derive a closed form solution for the optimal wealth and expected utility of that wealth when security dynamics are

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1 Haugh and Lo (2001) show how a static position with just a few well-chosen vanilla European options can be used to approximate the payoff of a GBH strategy when there is just one risky security.

2 In particular, those investors for whom dynamic trading is impractical either due to large trading costs or trading constraints.

3 Hereafter, referred to as HKW.
predictable and a constant proportion portfolio strategy is employed. This strategy is often considered by researchers who wish to estimate the value of predictability in security prices to investors. Indeed, we use this closed form solution to improve on HKW's analysis of the constant proportion portfolio trading strategy.
Chapter 2

Effect of Jumps and Discrete Sampling on Volatility and Variance Swaps

2.1 Introduction

Volatility and variance swaps are forward contracts in which one counterparty agrees to pay the other a notional amount times the difference between a fixed level and a realized level of variance and volatility, respectively. The fixed level is called the variance strike for variance swaps and the volatility strike for volatility swaps. This is typically set initially so that the net present value of the payoff is zero. The realized variance is determined by the average variance of the asset over the life of the swap.

The variance swap payoff is defined as

\[(V_d(0, n, T) - K_{\text{var}}(n)) \times N\]

where \(V_d(0, n, T)\) is the realized stock variance (as defined below) over the life of the contract, \([0, T]\), \(n\) is the number of sampling dates, \(K_{\text{var}}(n)\) is the variance strike, and
$N$ is the notional amount of the swap in dollars. The holder of a variance swap at expiration receives $N$ dollars for every unit by which the stock’s realized variance $V_d(0, n, T)$ exceeds the variance strike $K_{var}(n)$. The variance strike is quoted as volatility squared, e.g., $(20\%)^2$.

The volatility swap payoff is defined as

$$(\sqrt{V_d(0, n, T)} - K_{vol}(n)) \times N$$

where $\sqrt{V_d(0, n, T)}$ is the realized stock volatility (quoted in annual terms as defined below) over the life of the contract, $n$ is the number of sampling dates, $K_{vol}(n)$ is the volatility strike, and $N$ is the notional amount of the swap in dollars. The volatility strike $K_{vol}(n)$ is typically quoted as volatility, e.g., 20%. The procedure for calculating realized volatility and variance is specified in the contract and includes details about the source and observation frequency of the price of the underlying asset, the annualization factor to be used in moving to an annualized volatility and the method of calculating the variance.

Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a partition of the time interval $[0, T]$ into $n$ equal segments of length $\Delta t$, i.e., $t_i = iT/n$ for each $i = 0, 1, \ldots, n$. Most traded contracts define the realized variance to be

$$V_d(0, n, T) = \frac{AF}{m} \sum_{i=0}^{n-1} \left( \ln \left( \frac{S_{i+1}}{S_i} \right) \right)^2$$

for a swap covering $n$ return observations. In most traded contracts $m$ is equal to
$n - 1$. Here $S_i$ is the price of the asset at the $i^{th}$ observation time $t_i$ and $AF$ is the annualization factor, e.g., 252 ($= n/T$) if the maturity of the swap, $T$, is one year with daily sampling. This definition of realized variance differs from the usual sample variance because the sample average is not subtracted from each observation. Since the sample average is approximately zero the realized variance is close to the sample variance.


The analysis in most of these papers is based on an idealized contract where realized variance and volatility are defined with continuous sampling, e.g., a continuously sampled realized variance, \( V_c(0,T) \), defined by:

\[
V_c(0,T) = \lim_{n \to \infty} V_d(0,n,T)
\] (2.1.2)

In this chapter we analyze the differences between actual contracts based on discrete sampling and idealized contracts based on continuous sampling. Another objective of this chapter is to analyze the effect of ignoring jumps in the underlying on fair variance swap strikes.

In financial models we typically specify the dynamics of the stock price and variance using stochastic differential equations (SDE) and discrete and continuous realized variance depend on the modeling assumptions. The Black-Scholes model proposed in the early 1970’s assumes that a stock price follows a lognormal distribution and the volatility term is constant. This constant volatility assumption is not typically satisfied by options trading in the market and subsequently many different models have been proposed. Merton (1973) extended the constant volatility assumption in Black-Scholes model to a
term structure of volatility, i.e., $\sigma = \sigma(t)$. Derman and Kani (1994) and Derman, Kani and Zou (1996) extended this to local volatility models where volatility is a function of two parameters, time and the current level of the underlying, i.e., $\sigma = \sigma(t, S(t))$. Several models have been developed where volatility is modeled as a stochastic process often including mean reversion. Hull and White (1987) proposed a lognormal model for the variance process with independence between the driving Brownian motions of the stock price and variance processes. Heston (1993) proposed a mean reverting model for variance that allows for correlation between volatility and the asset level. Stein and Stein (1991) and Schobel and Zhu (1999) proposed a stochastic volatility model in which volatility of underlying asset follows Ornstein-Uhlenbeck process. Bates (1996) and Scott (1997) proposed a stochastic volatility with jumps model by adding log-normal jumps in stock price process in the Heston stochastic volatility model.

Continuous realized variance depends on the model assumed for the underlying asset price. Depending on the model, discrete realized variance and continuous realized variance can be different. The fair strike of a variance swap (with discrete or continuous sampling) is defined to be the strike which makes the net present value of the swap equal to zero. We call it the fair variance strike. The fair discrete volatility strike and fair discrete variance strikes are defined similarly. In this chapter we analyze discrete variance swaps and continuous variance swaps and the effect of the number of sampling dates on fair variance strikes and fair volatility strikes. Various authors have proposed to replicate a variance swap using a static portfolio of out-of-money call and put options.
This ignores the effect of jumps in the underlying. The fair variance swap strike will differ from the static replicating portfolio of options if the underlying has jumps. In this chapter we investigate the following questions:

- What is the effect of ignoring jumps in the underlying on fair variance swap strikes?

- What is the relationship between fair variance strikes and fair volatility strikes?

- How do fair variance strikes and fair volatility strikes vary in different models?

- What is the convergence rate of expected discrete realized variance to expected continuous realized variance with the number of sampling dates? Are fair discrete variance strikes and fair discrete volatility strikes with daily, weekly or monthly sampling significantly different than fair continuous variance strikes and fair continuous volatility strikes, respectively?

- How well does the convexity correction formula approximate fair volatility strikes?

In this chapter, we analyze all these issues under four different models of underlying evolution of asset price: the Black Scholes model, the Heston stochastic volatility model, the Merton jump-diffusion model and stochastic volatility model with jumps.

The rest of the chapter is organized as follows. We begin briefly by introducing volatility derivatives in section 2.2 and provide the formulas available to price these derivatives. In section 2.3 we analyze variance and volatility swaps in the Black-Scholes model and determine the convergence rate of the discrete variance strike to the continuous
variance strike. In sections 2.4, 2.5 and 2.6, we present analysis in the stochastic volatility model, the jump-diffusion model and stochastic volatility model with jumps respectively. In section 2.7 we present numerical results and concluding remarks are given in section 2.8.

2.2 Volatility Derivatives

2.2.1 Variance swaps

In this section, we provide definitions of discretely sampled realized variance and continuously sampled realized variance and review how to replicate variance swaps when the stock price process is continuous. We assume the risk neutral dynamics of the underlying asset $S_t$ are given by:

$$\frac{dS_t}{S_t} = r dt + \sigma_t dW_t^Q$$  \hspace{1cm} (2.2.1)

where $r$ is the risk free rate, $W$ is a standard Brownian motion under the risk neutral measure $Q$. We assume throughout in this chapter that there exists a unique risk neutral measure $Q$. The parameter $\sigma_t$ represents the level of volatility. The standard Black-Scholes model assumes that this parameter is constant, while in stochastic volatility models $\sigma_t$ is specified by another diffusion process. In this chapter we assume that it is given by the Heston stochastic volatility model. We will specify its dynamics later.

A variance swap is a forward contract on the realized variance of underlying security. The floating leg of variance swap is the realized variance and is calculated using the
second moment of log returns of the underlying asset:

\[ R_i = \ln \left( \frac{S_t}{S_{t-1}} \right), \quad i = 1, 2, ..., n \]

where \( 0 = t_0 < t_1 < ... < t_n = T \) is a partition of the time interval \([0, T]\) into \( n \) equal segments of length \( \Delta t \), i.e., \( t_i = iT/n \) for each \( i = 0, 1, ..., n \). The discrete realized variance, \( V_d(0, n, T) \), from equation (3.2.1) can be written as:

\[
V_d(0, n, T) = \frac{1}{(n-1)\Delta t} \sum_{i=1}^{n} R_i^2 = \sum_{i=0}^{n-1} \frac{(\ln(S_{i+1}) - \ln(S_i))^2}{(n-1)\Delta t} \tag{2.2.2}
\]

The variable leg of the variance swap, or the discretely sampled realized variance, in the limit approaches the continuously sampled realized variance, \( V_c(0, T) \), that is:

\[
V_c(0, T) \equiv \lim_{n \to \infty} V_d(0, n, T) = \lim_{n \to \infty} \frac{n}{(n-1)T} \sum_{i=1}^{n} R_i^2 \tag{2.2.3}
\]

Jacod and Protter (1998) provide necessary and sufficient conditions for the rate of convergence of the Euler scheme approximation of the solution to a stochastic differential equation to be \( 1/\sqrt{n} \). The discrete realized variance, \( V_d(0, n, T) \), is the Euler scheme approximation of the stochastic differential equation (5.2.1) followed by underlying asset \( S_t \) when sampling size is \( n \). Thus, the rate of convergence of discrete realized variance, \( V_d(0, n, T) \), to continuous realized variance, \( V_c(0, T) \), is \( 1/\sqrt{n} \).

In the case of the Black-Scholes model and the Heston stochastic volatility model\(^1\), continuous realized variance is given by:

\(^1\)Equation (2.2.4) holds for asset price models following the dynamics in (5.2.1). When jumps are introduced the definition of \( V_c(0, T) \) will be different.
Continuous realized variance can be replicated by a static position in a log contract (Demeterfi et al. 1999) and a dynamic trading strategy in the underlying asset. Applying Itô’s lemma to equation (5.2.1) we get

\[
\ln \left( \frac{S_T}{S_0} \right) = \int_0^T \left( r - \frac{1}{2} \sigma_t^2 \right) dt + \int_0^T \sigma_t dW_t^Q
\]  

Subtracting equation (2.2.5) from equation (5.2.1) and rearranging we get,

\[
\frac{2}{T} \left( \int_0^T \frac{dS_t}{S_t} - \ln \frac{S_T}{S_0} \right) = \frac{1}{T} \int_0^T \sigma_t^2 dt = V_c(0, T)
\]  

Equation (3.5.1) shows that continuous realized variance can be replicated by a short position in the log contract and payoffs from a dynamic trading strategy which holds \(1/S_t\) shares of the underlying stock at each instant of time \(t\). In particular, equation (3.5.1) holds in the Black-Scholes model and the Heston stochastic volatility model.

Next, we give definitions of realized variance and accumulated variance with discrete sampling and continuous sampling. Continuous realized variance between time \(t\) and \(T\) is given by

\[
V_c(t, T) = \frac{1}{T} \int_t^T \sigma^2_s ds
\]
Continuous accumulated variance from the start of the contract (time 0) until time \( t \) is defined by

\[
I_c(t, T) = \frac{1}{T} \int_0^t \sigma_s^2 ds
\]

(2.2.8)

Thus, from equations (2.2.7) and (2.2.8) we can write

\[
V_c(0, T) = I_c(t, T) + V_c(t, T)
\]

We define \( P_c(t, T, K, I) \), as the expected present value at time \( t \) of the payoff of a continuous variance swap with variance strike, \( K \), i.e.,

\[
P_c(t, T, K, I) = E_t^Q \left( e^{-r(T-t)} \left( I + V_c(t, T) - K \right) \right)
\]

(2.2.9)

where the superscript \( Q \) denotes the risk neutral measure and the subscript \( t \) denotes expectation at time \( t \). Throughout this chapter expectation is always in the risk neutral measure so we will drop the superscript. The fair continuous variance strike, \( K^{\text{var}} \), is defined to be the strike such that the net present value of the swap at time \( t = 0 \) is zero, i.e.,

\[
P_c(0, T, K^{\text{var}}, I) = E_0 \left( e^{-rT} \left( V_c(0, T) - K^{\text{var}} \right) \right) = 0
\]

(2.2.10)

Solving (3.2.6) for \( K^{\text{var}} \) gives

\[
K^{\text{var}} = E_0[V_c(0, T)] = E_0 \left[ \frac{1}{T} \int_0^T \sigma_s^2 ds \right]
\]

(2.2.11)
The discrete realized variance between times \( t_i = iT/n \) and \( T \) when there are \( n \) sampling dates between the start of contract at \( t = 0 \) and its maturity at \( t = T \) is given by

\[
V_d(i, n, T) = \frac{\sum_{j=i}^{n-1} (\ln(\frac{S_{j+1}}{S_j}))^2}{(n-1)\Delta t}
\] (2.2.12)

The discrete accumulated variance from the start of the contract, \( t = 0 \), until time \( t_i \) is defined by

\[
I_d(i, n, T) = \frac{\sum_{j=0}^{i-1} (\ln(\frac{S_{j+1}}{S_j}))^2}{(n-1)\Delta t}
\] (2.2.13)

From equations (2.2.12) and (2.2.13) we can write

\[
V_d(0, n, T) = I_d(i, n, T) + V_d(i, n, T)
\]

We define \( P_d(i, n, T, K, I_d(i, n, T)) \), as the expected present value at time \( t_i = iT/n \) of the payoff of a discrete variance swap with strike \( K \). It is given by

\[
P_d(i, n, T, K, I) = E_t \left( e^{-r(T-t_i)} \left( I + V_d(i, n, T) - K \right) \right)
\] (2.2.14)

The fair discrete variance strike, \( K_{\text{var}}(n) \), is defined to be the strike such that the expected net present value of the swap at time \( t = 0 \) is zero. i.e.,

\[
P_d(0, n, T, K_{\text{var}}(n), I) = E_0 \left( e^{-rT} \left( V_d(0, n, T) - K_{\text{var}}(n) \right) \right) = 0
\] (2.2.15)
At time $t = 0$, $P_d(0, n, T, K, I)$ can be written as

$$P_d(0, n, T, K, I) = P_c(0, T, K, I) + E_0 \left( e^{-rT} \left( V_d(0, n, T) - V_c(0, T) \right) \right)$$

$$= P_c(0, T, K, I) + e^{-rT} \left( K_{\text{vol}}^* (n) - K_{\text{vol}}^* \right)$$

(2.2.16)

We will use these definitions to show the linear convergence rate of $P_d(0, n, T, K, I)$ to $P_c(0, T, K, I)$.

### 2.2.2 Volatility swaps

The floating leg of a volatility swap on an asset $S$ is the realized volatility of that asset’s price. This volatility is commonly calculated using the square root of the realized variance defined in equation (2.2.2). The fair strike $K_{\text{vol}}^*$ of a continuous volatility swap is set at the initiation of the contract so that the contract net present value is equal to zero, i.e.,

$$E_0 \left[ e^{-rT} \left( \sqrt{V_c(0, T)} - K_{\text{vol}}^* \right) \right] = 0$$

(2.2.17)

Solving (3.2.8) for the fair continuous volatility strike, $K_{\text{vol}}^*$, we get

$$K_{\text{vol}}^* = E[\sqrt{V_c(0, T)}] = E_0 \left[ \sqrt{\frac{1}{T} \int_0^T \sigma^2 s ds} \right]$$

Similarly, the fair discrete volatility strike is given by:

$$K_{\text{vol}}^*(n) = E_0 \left[ \sqrt{V_d(0, n, T)} \right]$$

(2.2.18)
2.2.3 Convexity correction formula

In this section we present the convexity correction formula (Brockhaus and Long 2000) to approximate fair volatility strikes. Then we present an argument to show that it may not provide an accurate approximation.

Jensen’s inequality shows that the fair volatility strike is bounded above by the square root of the fair variance strike\(^2\).

\[ K_{vol}^* = E_0[\sqrt{V_c(0,T)}] \leq \sqrt{E_0[V_c(0,T)]} = \sqrt{K_{var}^*} \quad (2.2.19) \]

A similar result holds in the discrete case:

\[ K_{vol}^*(n) = E_0[\sqrt{V_d(0,n,T)}] \leq \sqrt{E_0[V_d(0,n,T)]} = \sqrt{K_{var}^*(n)} \quad (2.2.20) \]

Brockhaus and Long (2000) provide a convexity correction formula for calculating the fair volatility strike using a Taylor’s expansion of the square root function. A second order Taylor’s expansion of \( f(x) = \sqrt{x} \) around \( x_0 \) gives

\[ \sqrt{x} = \sqrt{x_0} + \frac{(x - x_0)}{2 \sqrt{x_0}} - \frac{(x - x_0)^2}{8x_0^{\frac{3}{2}}} + f^{(3)}(\varepsilon) \frac{(x - x_0)^3}{3!} \quad (2.2.21) \]

where \( f^{(3)} \) is the 3\(^{rd} \) derivative of function \( f(x) \) for some \( \varepsilon \) in \( (x_0, x) \). The first three terms on the right hand side provide a good approximation of \( \sqrt{x} \) for all values of \( x \) in the

\(^2\)For the concave square root function Jensen’s inequality is:
\[ E(\sqrt{x}) \leq \sqrt{E(x)} \]
neighborhood of \( x_0 \) for which Taylor's series converges. For Taylor's series to converge, \( x - x_0 \) should lie in the radius of convergence, which for the square root function is

\[
|x - x_0| \leq x_0
\] (2.2.22)

When this condition holds, the last term in equation (2.2.21) is bounded and the first three terms provide a good estimate to compute the value of function at a point, in this case \( \sqrt{x} \). Now, substitute \( x = V_c(0, T) \) and \( x_0 = E[V_c(0, T)] \) in equation (2.2.21) to get:

\[
\sqrt{V_c(0, T)} \approx \sqrt{E[V_c(0, T)]} + \frac{(V_c(0, T) - E[V_c(0, T)])}{2\sqrt{E[V_c(0, T)]}} - \frac{(V_c(0, T) - E[V_c(0, T)])^2}{8E[V_c(0, T)]^{3/2}}
\] (2.2.23)

The terms on the right hand side in equation (2.2.23) provide a good estimate of the square root of the realized variance \( \sqrt{V_c(0, T)} \) on a single stock price path if the realized variance \( V_c(0, T) \) satisfies the condition:

\[
|V_c(0, T) - E(V_c(0, T))| \leq E(V_c(0, T))
\] (2.2.24)

which can also be rewritten as

\[
0 \leq V_c(0, T) \leq 2E(V_c(0, T))
\] (2.2.25)

If condition (2.2.25) holds on all stock price paths under the risk neutral measure then the right hand side of equation (2.2.23) provides a good estimate of square root
of realized variance $\sqrt{V_c(0,T)}$ on all stock price paths. Hence, we can take expectation under risk neutral measure on both sides of equation (2.2.23) to get:

$$K_{\text{vol}}^* \approx \sqrt{K_{\text{var}}} - \frac{\text{Var}(V_c(0,T))}{8E[V_c(0,T)]^\frac{3}{2}} \quad (2.2.26)$$

Formula (2.2.26) is called the convexity correction formula and the 2nd order term in equation (2.2.26) is the convexity correction term. It can be used to approximate the fair volatility strike. As explained above this will be a good approximation if condition (2.2.25) holds on all sample paths. We can rewrite this condition in terms of the excess probability

$$p = P(VC(0,T) > 2E(V_c(0,T))) \quad (2.2.27)$$

Thus, condition (2.2.25) to use the convexity correction formula translates to the excess probability being equal to zero, i.e., $p = 0$. Equation (2.2.26) also holds for fair discrete volatility strikes if condition (2.2.25) is satisfied by the discrete realized variance.

When the excess probability (2.2.27) is not equal to zero then the higher order terms in the Taylor’s expansion are not negligible compared to the first three terms in the expansion. If we include the 3rd and 4th order expansion terms in equation (2.2.26) we get,

$$E[\sqrt{V_c(0,T)}] \approx \sqrt{E[V_c(0,T)]} + \frac{(V_c(0,T) - E[V_c(0,T)])}{2\sqrt{E[V_c(0,T)]}} - \frac{(V_c(0,T) - E[V_c(0,T)])^2}{8E[V_c(0,T)]^\frac{3}{2}}$$

$$+ \frac{(V_c(0,T) - E[V_c(0,T)])^3}{16E[V_c(0,T)]^\frac{5}{2}} - \frac{5(V_c(0,T) - E[V_c(0,T)])^4}{128E[V_c(0,T)]^\frac{7}{2}} \quad (2.2.28)$$
We refer the last two terms in Taylor's expansion as the $3^{rd}$ and $4^{th}$ order terms. When $p$ is not equal to zero then the higher moments of $V_c(0, T) - E(V_c(0, T))$ are not negligible.

In the Black-Scholes model, the excess probability is equal to zero in the continuous case and the higher moments of continuous realized variance are zero since volatility is constant. Hence, the convexity correction formula holds with equality in (2.2.26) in the Black-Scholes model and can be used to compute the fair continuous volatility strike.

In the discrete case, i.e., for a finite number of sampling dates $n$, the excess probability is not equal to zero and the higher moments of the discrete realized variance in the Black-Scholes model are not zero. The magnitude of the $3^{rd}$ and $4^{th}$ order terms are comparable to first two terms and the excess probability $p$ is not zero with discrete sampling in the Black-Scholes model. Hence, the convexity correction formula (2.2.26) will not provide a good approximation of the fair volatility strike in the Black-Scholes model when the number of sampling dates $n$ is small.

In the Heston stochastic volatility model, the excess probability $p$ is not equal to zero, the $3^{rd}$ and $4^{th}$ order terms in equation (2.2.28) are not small and hence the convexity correction formula will not provide a good estimate of the fair volatility strike. This is true in the Merton jump-diffusion model as well. Section 2.7 provides numerical results illustrating the computation of volatility strikes from the convexity correction formula, the $3^{rd}$ and $4^{th}$ order terms in equation (2.2.28) and the excess probability in all three.
models.

2.3 Black-Scholes Model

In this section we present an analysis of the convergence of expected discrete realized variance to expected continuous realized variance with number of sampling dates in the Black-Scholes model. This result gives the relationship between fair discrete variance strikes and fair continuous variance strikes. The Black-Scholes model assumes the underlying asset follows the process in (5.2.1) with $\sigma_t$ set to the constant value $\sigma$.

In the case of continuous sampling, the fair continuous variance strike using (3.2.2) and (3.2.6) is given by

$$K_{var}^* = E_0[V_c(0,T)] = E_0\left[\frac{1}{T} \int_0^T \sigma_s^2 ds \right] = \sigma^2$$

(2.3.1)

and

$$K_{vol}^* = E_0[\sqrt{V_c(0,T)}] = E_0\left[\sqrt{\frac{1}{T} \int_0^T \sigma_s^2 ds} \right] = \sigma$$

(2.3.2)

since in the Black-Scholes model volatility $\sigma$ is constant and so the fair continuous volatility strike is square root of the fair continuous variance strike. But in the discrete case this result does not hold.

2.3.1 Black-Scholes Model: Discrete Variance Strike

In this section we compute the fair discrete variance strike in the Black-Scholes model and compute the variance of the discrete realized variance (2.2.2).
Proposition 1  In the Black-Scholes model

\[
E_0 \left( V_d(0,n,T) \right) = E_0 \left( V_c(0,T) \right) + \frac{\sigma^2 + (r - \frac{1}{2}\sigma^2)^2 T}{n-1}
\]

\[
= \sigma^2 + \frac{\sigma^2 + (r - \frac{1}{2}\sigma^2)^2 T}{n-1}
\]

(2.3.3)

and the expectation of discrete realized variance converges to the continuous realized variance linearly with the number of sampling dates \((n = T/\Delta t)\). As a consequence

\[
K_{\text{var}}^\star(n) = K_{\text{var}}^\star + \frac{\sigma^2 + (r - \frac{1}{2}\sigma^2)^2 T}{n-1}
\]

(2.3.4)

and the fair discrete variance strike converges to the fair continuous variance strike linearly with the number of sampling dates \((n = T/\Delta t)\).

Proof:  In the case of discrete sampling we derive the variance strike as follows.

Applying Itô’s lemma to \(\ln S_t\) we get,

\[
d(\ln S_t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t
\]

(2.3.5)

Integrating equation (2.3.5) from \(t_i\) to \(t_{i+1}\) we get,

\[
\ln \left( \frac{S_{t_{i+1}}}{S_{t_i}} \right) = (r - \frac{1}{2}\sigma^2)\Delta t + \sigma \sqrt{\Delta t} Z_{i+1}
\]

(2.3.6)

where \(Z_{i+1} \sim N(0,1)\). Squaring both sides of equation (2.3.6) and summing from time 0 to time \(n - 1\) we get,
\[ \sum_{i=0}^{n-1} \left( \ln \left( \frac{S_{i+1}}{S_i} \right) \right)^2 = \sum_{i=0}^{n-1} \left( r - \frac{1}{2} \sigma^2 \right)^2 \Delta t + \sigma^2 \Delta t Z_{i+1}^2 + 2 \sigma \left( r - \frac{1}{2} \sigma^2 \right) \Delta t Z_{i+1} \]

(2.3.7)

Dividing equation (2.3.7) on both sides by \((n - 1)\Delta t\) and taking expectation under the risk neutral measure and using equation (2.3.2) we get

\[ E_0[V_d(0,n,T)] = E_0 \left[ \left( r - \frac{1}{2} \sigma^2 \right)^2 \Delta t \frac{n}{n-1} + \sigma^2 \frac{n}{n-1} + 2 \sigma \left( r - \frac{1}{2} \sigma^2 \right) \Delta t \frac{n}{n-1} \right] \]

\[ = \left( r - \frac{1}{2} \sigma^2 \right)^2 \Delta t \frac{n}{n-1} + \sigma^2 \frac{n}{n-1} - E_0[Z_{i+1}^2] \]

\[ = \left( r - \frac{1}{2} \sigma^2 \right)^2 \frac{T}{n(1 - \frac{1}{n})} + \sigma^2 \frac{1}{(1 - \frac{1}{n})} \]

(2.3.8)

since \(\Delta t = T/n\), \(E_0[Z_{i+1}^2] = 0\) and \(E_0[Z_{i+1}^2] = 1\). Rearranging gives (2.3.3) and (2.4.7) is immediate from the definitions of \(K_{var}^*(n)\) and \(K_{var}^*\). □

Hence, from equation (2.2.16) the initial value of a discrete variance swap, \(P_d(0,n,T,K,I)\), converges linearly to the initial value of a continuous variance swap, \(P_c(0,T,K,I)\), with the number of sampling dates. This \(1/n\) convergence rate is similar to many weak convergence results since fair discrete strikes are expectations of a smooth function of the sample path of the underlying asset price (see, e.g., (Kloeden and Platen 1999)). In contrast, Jacod and Protter (1998) provide necessary and sufficient condition for the rate of convergence of discrete realized variance to continuous realized variance to be \(1/\sqrt{n}\). This slower rate occurs because the convergence is in a pathwise or strong sense.
Next we compute the variance of the discrete realized variance $V_d(0,n,T)$ and its convergence rate with number of sampling dates.

**Proposition 2** In the Black-Scholes model

\[
\text{Var}[V_d(0,n,T)] = \frac{2\sigma^4 n}{(n-1)^2} + \frac{4\sigma^2(r - \frac{1}{2}\sigma^2)^2 T}{(n-1)^2}
\]  

(2.3.9)

and the variance of the discrete realized variance converges to 0 as the sampling interval $(\Delta t = T/n)$ goes to zero.

**Proof:** The variance of the realized variance is given by

\[
\text{Var}[V_d(0,n,T)] = \text{Var}\left[\sum_{i=0}^{n-1}\frac{\sigma^2 Z_{i+1}^2}{n-1}\right] + \text{Var}\left[\sum_{i=0}^{n-1}2\sigma(r - \frac{1}{2}\sigma^2)\Delta t^\frac{1}{2} \frac{Z_{i+1}}{n-1}\right]
\]

\[
+ 4\sigma^3(r - \frac{1}{2}\sigma^2)\frac{\Delta t^\frac{1}{2}}{(n-1)^2}\text{Cov}\left[\sum_{i=0}^{n-1}Z_{i+1}^2 \sum_{i=0}^{n-1}Z_{i+1}\right]
\]

\[
= 2\sigma^4 \frac{n}{(n-1)^2} + 4\sigma^2(r - \frac{1}{2}\sigma^2)^2 \frac{T}{(n-1)^2}
\]

(2.3.10)

Thus, in the Black-Scholes model the variance of the discrete realized variance converges to zero (2.3.9). This also holds for higher moments of discrete realized variance.

Hence, in the Black-Scholes model the fair continuous volatility strike (2.2.26) is equal to the square root of the fair continuous variance strike, $K_{\text{var}}^*$. 


2.3.2 Black-Scholes Model: Discrete Volatility Strike

In this section we compute the fair discrete volatility strike in the Black-Scholes model.

The square root function can be expressed (Schurger 2002) as:

\[ \sqrt{x} = \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1 - e^{-sz}}{s^{3/2}} ds \]  
\[(2.3.11)\]

Taking expectations on both sides of (2.3.11) and interchanging the expectation and integral using Fubini’s theorem we get,

\[ E(\sqrt{x}) = \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} \frac{1 - E(e^{-sz})}{s^{3/2}} ds \]  
\[(2.3.12)\]

Using this formula we can compute the discrete volatility strike in the Black-Scholes model.

**Proposition 3** In the Black-Scholes model, the Laplace transform of the discrete realized variance, \( E(\exp(-sV_d(0,n,T))) \), is given by

\[ E(\exp(-sV_d(0,n,T))) = \frac{\exp \left( -sT(\frac{1}{n-1} + \frac{1}{n^2}) \right)}{\left( 1 + \frac{2s^2\sigma^2}{n-1} \right)^{\frac{n}{2}}} \]  
\[(2.3.13)\]

**Proof:** Using the definition of discrete realized variance in (2.2.2), the Laplace transform of the realized variance can be expressed as

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Using equation (2.3.6) we get

\[
E\left(\exp(-sV_d(0, n, T))\right) = E\left(\exp\left(-s\sum_{i=0}^{n-1} \left(\frac{\ln\left(\frac{S_{i+1}}{S_i}\right)}{(n-1)\Delta t}\right)\right)\right)
\]

\[
= \prod_{i=0}^{n-1} E\left(\exp\left(-s\left(\frac{\ln\left(\frac{S_{i+1}}{S_i}\right)}{(n-1)\Delta t}\right)^2\right)\right)\tag{2.3.14}
\]

Using this we can compute the expectation in equation (2.3.14),

\[
\ln\left(\frac{S_{i+1}}{S_i}\right) = N((r - \frac{1}{2}\sigma^2)\Delta t, \sigma^2\Delta t) \tag{2.3.15}
\]

which proves (2.3.13). □

### 2.4 Heston Stochastic Volatility Model

In this section, we present an analysis of the convergence of discrete variance strikes to continuous variance strikes with number of sampling dates in the Heston stochastic volatility (SV) model. The Heston (1993) model is given by:

\[
dS_t = rS_t dt + \sqrt{\nu_t}S_t(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)
\]

\[
d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t}dW_t^1
\]
Equation (4.2.5) gives the dynamics of the stock price: \( S_t \) denotes the stock price at time \( t \), \( \mu \) is the risk neutral drift, and \( \sqrt{\nu_t} \) is the volatility. Equation (4.2.6) gives the evolution of the variance which follows a square root process: \( \theta \) is the long run mean variance, \( \kappa \) represents the speed of mean reversion, and \( \sigma_\nu \) is a parameter which determines the volatility of the variance process. The processes \( W^1_t \) and \( W^2_t \) are independent standard Brownian motions under risk neutral measure \( Q \), and \( \rho \) represents the instantaneous correlation between the return process and the volatility process. First we derive the continuous variance strike.

### 2.4.1 SV Model: Continuous Variance Strike

**Proposition 4** In the Heston stochastic volatility model, the fair continuous variance strike \( K_{var}^* = E[V_c(0, T)] \) is given by:

\[
E \left( \frac{1}{T} \int_0^T v_s ds \right) = \theta + \frac{\nu_0 - \theta}{\kappa T} (1 - e^{-\kappa T})
\]  

(2.4.3)

**Proof:** The Laplace transform of \( \int_0^T v_s ds \) is given by (Cairns 2000)

\[
E_0[e^{-s \int_0^T v_s dt} \mid v(0) = v_0] = \exp[A(T, s) - B(T, s)v_0]
\]  

(2.4.4)

where

\[
A(T, s) = \frac{2\kappa\theta}{\sigma^2} \log \left( \frac{2\gamma(s)e^{(\gamma(s) + \kappa)T}}{(\gamma(s) + \kappa)(e^{(\gamma(s))T} - 1) + 2\gamma(s)} \right) \\
B(T, s) = \frac{2s(e^{\gamma(s))T} - 1}{T(\gamma(s) + \kappa)(e^{(\gamma(s))T} - 1) + 2\gamma(s)} \\
\gamma(s) = \sqrt{\kappa^2 + 2\frac{\sigma^2 s}{T}}
\]
From the Laplace transform of $\int_0^T v_s ds$ we can derive the first moment:

$$
E \left[ \int_t^T v_s ds \right] = -\frac{d}{dv} \left( E^Q [e^{-\nu \int_t^T v_s ds} \mid v(t) = v_t] \right) \bigg|_{(v=0)} = \theta(T-t) + \frac{v_t - \theta}{\kappa} (1 - e^{-\kappa(T-t)})
$$

which proves (3.2.7). ⊓⊔

The fair continuous variance strike in the Heston stochastic volatility model is independent of the volatility of variance $\sigma_v$. Similarly, the variance of the continuous realized variance, $\text{Var}(V_c(0,T))$, can be derived by calculating the second moment of the Laplace transform.

$$
\text{Var} \left( \frac{1}{T} \int_t^T v_s ds \right) = \frac{\sigma_v^2 e^{-2\kappa(T-t)}}{2\kappa^3T^2} \left( 2(e^{2\kappa(T-t)} - 2e^{\kappa(T-t)}\kappa(T-t) - 1)(v_t - \theta) + (4e^{\kappa(T-t)} - 3e^{2\kappa(T-t)} + 2e^{2\kappa(T-t)}\kappa(T-t) - 1)\theta \right) (2.4.5)
$$

The variance of the continuous realized variance (2.4.5) depends on the volatility of variance. Since the variance of the continuous realized variance is not equal to zero, there will be a convexity correction (2.2.26) in the volatility strike and the fair volatility strike will not be equal to the square root of the fair variance strike. However, in the Heston stochastic volatility model, the realized variance on a sample path doesn’t satisfy condition (2.2.25), and the convexity correction formula (2.2.26) doesn’t provide a good estimate of the fair volatility strike. Numerical results are given in section 2.7.

We compute the fair continuous volatility strike in the stochastic volatility model by using the formula (2.3.11) and the Laplace transform of the realized variance from
equation (3.2.7). Broadie and Jain (2006b) present an alternative partial differential equation approach to compute the same quantities, as well as to price variance options. Next, we compute the fair discrete variance strike in the Heston stochastic volatility model and show that the expected discrete realized variance converges linearly to the expected continuous realized variance with the number of sampling dates.

2.4.2 SV Model: Discrete Variance Strike

**Proposition 5** In the Heston stochastic volatility model,

\[
E_0 \left( V_{d}(0,n,T) \right) = E_0 \left( V_{c}(0,T) \right) + g(r, \rho, \sigma_v, \kappa, \theta, n) \quad (2.4.6)
\]

The function \( g(\cdot) \) is given explicitly in appendix A. It converges to zero linearly with the number of sampling dates:

\[
g(r, \rho, \sigma_v, \kappa, \theta, n) = O \left( \frac{1}{n} \right)
\]

and the expectation of discrete realized variance converges to the expected continuous realized variance linearly with the sampling size \( n = T/\Delta t \). Hence,

\[
K_{\text{var}}^*(n) = K_{\text{var}}^* + g(r, \rho, \sigma_v, \kappa, \theta, n) \quad (2.4.7)
\]

and the discrete variance strike converges to the continuous variance strike linearly with the number of sampling dates \( \Delta t = T/n \).
A proof is given in appendix A. Hence, from equation (2.2.16) the initial value of a discrete variance swap, $P_d(0,n,T,K,I)$, converges linearly to the initial value of a continuous variance swap, $P_c(0,T,K,I)$, with the number of sampling dates.

2.5 Merton Jump-Diffusion Model

In this section, we present an analysis of the convergence of discrete variance strikes to continuous variance strikes with number of sampling dates in the Merton jump-diffusion (J) model. The risk neutral dynamics of the jump-diffusion model are given by:

$$\frac{dS_t}{S_t} = (r - \lambda m)dt + \sigma dW^Q_t + dJ_t$$  \hspace{1cm} (2.5.1)

where $J_t = \sum_{i=1}^{N_t} (Y_j - 1)$ and $N_t$ is a Poisson process with rate $\lambda$ and $Y_j$ is the relative jump size in the stock price. When jump occurs at time $\tau_j$, then $S(\tau_j^+) = S(\tau_j^-)Y_j$, where the distribution of $Y_j$ is LN[$a$, $b^2$] and $m$ is the mean proportional size of jump $E(Y_j - 1) = m$. The parameters $a$ and $m$ are related to each other by the equation: $e^{a + \frac{1}{2} b^2} = m + 1$ and only one of them needs to be specified.

In the case of continuous sampling, realized variance consists of two components. The first is the accumulated variance of the underlying stock until the maturity of swap contributed from the diffusive Brownian motion and second is the contribution from jumps in the underlying stock. If there are $N(T)$ number of jumps in the stock in $[0,T]$ then the contribution to the realized variance from jumps is given by
Thus, the continuous realized variance in the Merton jump-diffusion model can be expressed as

\[
\frac{1}{T} \left( \sum_{i=1}^{N(T)} (\ln(Y_i))^2 \right)
\]

Thus, the continuous realized variance in the Merton jump-diffusion model can be expressed as

\[
V_c(0,T) = \frac{1}{T} \int_0^T \sigma^2 dt + \frac{1}{T} \left( \sum_{i=1}^{N(T)} (\ln(Y_i))^2 \right) = \sigma^2 + \frac{1}{T} \left( \sum_{i=1}^{N(T)} (\ln(Y_i))^2 \right) (2.5.2)
\]

The fair continuous variance strike is obtained by taking the expectation of the continuous realized variance:

\[
K_{var}^* = E_0[V_c(0,T)] = \sigma^2 + \lambda(a^2 + b^2) (2.5.3)
\]

In the jump-diffusion model, the fair continuous variance strike depends on the continuous volatility parameter \(\sigma\) and the volatility of the stock from the jumps during the life of contract. Depending on the relative size of the jump parameters, realized variance can be significantly different than in other models.

2.5.1 Jump-Diffusion Model: Continuous Volatility Strike

In this section we derive the fair continuous volatility strike in the Merton jump-diffusion model. The continuous realized variance in the Merton jump-diffusion model is given by equation (2.5.2). We can compute the fair continuous volatility strike by using formula (2.3.12).
Proposition 6  In the Merton jump-diffusion model, the Laplace transform of the continuous realized variance $E(\exp(-sV_c(0,T)))$ is given by:

$$E(\exp(-sV_c(0,T))) = \exp \left( -\sigma^2 + \lambda T \left( \frac{\exp\left(\frac{-sn^2}{T+2ab^2}\right)}{\sqrt{1 + \frac{2ab^2}{T}}} - 1 \right) \right)$$  \hspace{1cm} (2.5.4)

Proof: The Laplace transform of the continuous realized variance can be expressed as

$$E(\exp(-sV_c(0,T))) = E(\exp(-s(\sigma^2 + \frac{1}{T} \sum_{i=1}^{N(T)} (\ln(Y_i))^2))) = \exp(-s\sigma^2) E\left(\exp\left(\frac{1}{T} \sum_{i=1}^{N(T)} (\ln(Y_i))^2 | N(T) = n \right)\right)$$ \hspace{1cm} (2.5.5)

where second equality follows by taking an expectation conditional on Poisson random variable, $N(T) = n$. Since $\ln(Y_i) \sim N(a, b^2)$ follows the normal distribution, the inner expectation can be computed as:

$$E(\exp(-sV_c(0,T))) = \exp(-s\sigma^2) \frac{\exp\left(\frac{-sn^2}{T+2ab^2}\right)}{\left(1 + \frac{2ab^2}{T}\right)^{\frac{n}{2}}}$$ \hspace{1cm} (2.5.6)

We can compute the outer expectation as follows:

$$E(\exp(-sV_c(0,T))) = \exp(-s\sigma^2) \sum_{n=0}^{\infty} \frac{\exp(-\lambda T)(\lambda T)^n}{n!} \frac{\exp\left(\frac{-sn^2}{T+2ab^2}\right)}{\left(1 + \frac{2ab^2}{T}\right)^{\frac{n}{2}}}$$ \hspace{1cm} (2.5.7)
Simplifying the infinite sum gives

\[ E(\exp(-sV_c(0,T))) = \exp(-s\sigma^2)\exp(-\lambda T)\exp\left(\frac{\lambda T\exp(-\frac{s\sigma^2}{T+2b^2})}{\sqrt{1+2b^2}}\right) \]

\[ = \exp(-s\sigma^2 + \lambda T\left(\frac{\exp(-\frac{s\sigma^2}{T+2b^2})}{\sqrt{1+2b^2}} - 1\right)) \]

\[ = \exp\left(-s\sigma^2 + \lambda T\left(\frac{\exp(-\frac{s\sigma^2}{T+2b^2})}{\sqrt{1+2b^2}} - 1\right)\right) \quad \Box \]

2.5.2 Merton Jump-Diffusion Model: Discrete Variance Strike

Proposition 7 In the Merton jump-diffusion model

\[ E_0\left(V_d(0,n,T)\right) = E_0\left(V_c(0,T)\right) + f(r,a,b,\sigma,\lambda,m,T,n) \]

\[ = \sigma^2 + (a^2 + b^2)\lambda + f(r,a,b,\sigma,\lambda,m,T,n) \quad (2.5.8) \]

where the function \( f(r,a,b,\sigma,\lambda,m,T,n) \) converges to zero linearly with number of sampling dates \( n \) and

\[ f(r,a,b,\sigma,\lambda,m,T,n) = \frac{\sigma^2 + (a^2 + b^2)\lambda + (r - \lambda m - \frac{1}{2}\sigma^2)^2T + a^2\lambda^2T + 2(r - \lambda m - \frac{1}{2}\sigma^2)\lambda T}{n - 1} \]

The expectation of discrete realized variance converges to the expected continuous realized variance linearly with the sampling size \( (n = T/\Delta t) \). Consequently

\[ K_{\text{var}}^*(n) = K_{\text{var}}^* + f(r,a,b,\sigma,\lambda,m,T,n) \quad (2.5.9) \]

and the fair discrete variance strike converges to the fair continuous variance strike linearly with number of sampling dates \( (n = T/\Delta t) \).
Proof: Applying Itô’s lemma in the jump-diffusion model (2.5.1) and integrating from \( t_i \) to \( t_{i+1} \) gives

\[
\ln \left( \frac{S_{i+1}}{S_i} \right) = (r - \lambda m - \frac{1}{2} \sigma^2) \Delta t + \sigma \sqrt{\Delta t} Z_{i+1} + \ln \left( \prod_{j=1}^{n_j} Y_j \right)
\]  

(2.5.10)

where \( n_j \) is number of jumps in the stock price during time \( t_{i+1} - t_i \). Squaring equation (2.5.10) on both sides and summing from time 0 to time \( n - 1 \) we get

\[
\sum_{i=0}^{n-1} \left( \ln \left( \frac{S_{i+1}}{S_i} \right) \right)^2 = \sum_{i=0}^{n-1} \left[ \left( r - \lambda m - \frac{1}{2} \sigma^2 \right)^2 \Delta t^2 + \sigma^2 \Delta t Z_{i+1}^2 + 2(\sigma(r - \lambda m - \frac{1}{2} \sigma^2) \Delta t Z_{i+1} Z_{i+1} + \left( \sum_{j=1}^{n_j} \ln Y_j \right)^2 + 2\left( r - \lambda m - \frac{1}{2} \sigma^2 \right) \Delta t \left( \sum_{j=1}^{n_j} \ln Y_j \right) + 2\sigma \Delta t Z_{i+1} \left( \sum_{j=1}^{n_j} \ln Y_j \right) \right]
\]

(2.5.11)

The fair discrete variance strike can be calculated by dividing equation (2.5.11) on both sides by \((n - 1)\Delta t\) and taking expectation under the risk neutral measure.

\[
K_{\text{var}}^{*}(n) = E \left[ V_d(0, n, T) \right] = E \left[ \left( r - \lambda m - \frac{1}{2} \sigma^2 \right)^2 \Delta t \frac{n}{n-1} + \sigma^2 Z_{i+1}^2 \frac{n}{n-1}
\right.
\]

\[
+ 2\sigma(\sigma - \lambda m - \frac{1}{2} \sigma^2) \Delta t Z_{i+1} \frac{n}{n-1} + (\Sigma_j^{n_j} \ln Y_j)^2 \frac{n}{(n-1)\Delta t} \\
\left. + 2(\sigma - \lambda m - \frac{1}{2} \sigma^2)(\Sigma_j^{n_j} \ln Y_j) \frac{n}{n-1} \\
+ 2\sigma Z_{i+1}(\Sigma_j^{n_j} \ln Y_j) \frac{n}{(n-1)\Delta t} \right]
\]

(2.5.12)

Using properties of the normal and Poisson distributions.
\[
E(Z_i) = 0 \quad E[Z_{i+1}^2] = 1 \quad E\left[\sum_{j=1}^{n_j} \ln Y_j\right] = a\lambda \Delta t
\]

\[
E\left[\sum_{j=1}^{n_j} \ln Y_j\right]^2 = b^2 E[n_j] + a^2 (E(n_j)^2) = (a^2 + b^2)(\lambda \Delta t) + (\lambda \Delta t)^2 a^2
\]

Substituting these in equation (2.5.12) we get

\[
K_{var}^* (n) = (r - \lambda m - \frac{1}{2} \sigma^2)^\frac{T}{n-1} + \sigma^2 \frac{n}{n-1} + \frac{(a^2 + b^2)\lambda n}{n-1} + \frac{a^2 \lambda^2 T}{n-1} + 2(r - \lambda m - \frac{1}{2} \sigma^2) \frac{a \lambda T}{n-1}
\]

(2.5.13)

The previous expression gives the fair discrete variance strike. Rearranging terms gives (2.5.9). □

Hence, from equation (2.2.16) the initial value of a discrete variance swap, \(P_d(0, n, T, K, I)\), converges linearly to the initial value of a continuous variance swap, \(P_c(0, T, K, I)\), with the number of sampling dates.

2.5.3 Merton Jump-Diffusion Model: Discrete Volatility Strike

In this section we compute the fair discrete volatility strike in the Merton jump-diffusion model. We can compute the fair discrete volatility strike by using formula (2.3.12).

**Proposition 8** In the Merton jump-diffusion model, the Laplace transform of the discrete realized variance \(E(\exp(-sV_d(0, n, T)))\) is given by:
\[
E\left( \exp(-sV_d(0,n,T)) \right) = \left( \sum_{n_i=0}^{\infty} \frac{\exp(-\lambda \Delta t)(\lambda \Delta t)^{n_i}}{n_i!} \left( \frac{\exp\left(-s(\sigma^2 \Delta t + b^2 n_i)\right)}{\sqrt{1 + \frac{2s(\Delta t + b^2 n_i)}{(n-1)\Delta t}}} \right) \right)^n
\]

(2.5.14)

**Proof:** The Laplace transform of the discrete realized variance can be expressed as

\[
E\left( \exp(-sV_d(0,n,T)) \right) = E\left( \exp\left(-s\sum_{i=0}^{n-1} \left( \ln\left( \frac{S_{i+1}}{S_i} \right) \right)^2 \right) \right) = E\left( E\left( \exp\left(-s\sum_{i=0}^{n-1} \ln\left( \frac{S_{i+1}}{S_i} \right)^2 \right) \bigg| N(0), N(t_1), ..., N(T) \right) \right)
\]

\[
= E\left( \prod_{i=0}^{n-1} E\left( \exp\left(-s\ln\left( \frac{S_{i+1}}{S_i} \right)^2 \right) \bigg| N(0), N(t_1), ..., N(T) \right) \right)
\]

(2.5.15)

The third equality follows since a Poisson process has stationary and independent increments where

\[
\ln\left( \frac{S_{i+1}}{S_i} \right) = (r - \lambda m - \frac{1}{2} \sigma^2) \Delta t + \sigma \sqrt{\Delta t} Z_{i+1} + \sum_{j=1}^{n_i} \left( \ln(Y_j) \right)
\]

(2.5.16)

and \( n_i \) is number of jumps in the stock price during the time \( t_{i+1} - t_i \). The random variables \( n_i \) are independent and identically distributed with Poisson rate \( \lambda \Delta t \) for each \( i = 0, 1, ..., n-1 \). Since \( \ln(Y_j) \sim N(a, b^2) \) the distribution of log return given \( n_i \) jumps is

\[
\ln\left( \frac{S_{i+1}}{S_i} \right) \sim N((r - \lambda m - \frac{1}{2} \sigma^2) \Delta t + an_i, \sigma^2 \Delta t + b^2 n_i)
\]

(2.5.17)
The inner expectation in equation (2.5.15) can be solved using property (2.5.17):

\[
E\left(\exp(-sV_d(0, n, T))\right) = E\left(\prod_{i=0}^{n-1} \left(\frac{\exp(-s((r-\lambda m - \frac{1}{2}\sigma^2)\Delta t + \kappa n_i)^2)}{\sqrt{1 + \frac{2s(\sigma^2\Delta t + \kappa^2 n_i)}{(n-1)\Delta t}}}\right)\right)
\]

\[
= \left(\sum_{n_i=0}^{\infty} \frac{\exp(-\lambda \Delta t (\lambda \Delta t)^{n_i})}{n_i!} \left(\frac{\exp\left(-s((r-\lambda m - \frac{1}{2}\sigma^2)\Delta t + \kappa n_i)^2\right)}{\sqrt{1 + \frac{2s(\sigma^2\Delta t + \kappa^2 n_i)}{(n-1)\Delta t}}}\right)\right)^n
\]

(2.5.18)

The second equality follows since \(n_i\) are independent. \(\square\)

The expectation in (2.5.18) can be computed numerically since the sum converges very fast. We use the Laplace transform and formula (2.3.12) to compute the fair discrete volatility strike in the Merton jump-diffusion model.

### 2.6 Stochastic Volatility Model with Jumps

In this section, we present an analysis of the convergence of discrete variance strikes to continuous variance strikes with number of sampling dates in the stochastic volatility (SVJ) model with jumps. The Bates (1996) and Scott (1997) stochastic volatility with jumps (SVJ) is an extension of SV (4.2.5, 4.2.6) model to include jumps in the stock price process. The risk-neutral dynamics are:

\[
\frac{dS_t}{S_t} = (r - \lambda m)dt + \sqrt{\nu_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2) + dJ_t
\]

\[
d\nu_t = \kappa(\theta - \nu_t)dt + \sigma_\nu \sqrt{\nu_t}dW_t^1
\]

(2.6.1)
The specifications of different parameters are same as in the Heston stochastic volatility model specified in equations (4.2.5) and (4.2.6) and the Merton jump-diffusion (J) model specified by equation (2.5.1). The jump process, $N_t$ and the Brownian motion are independent.

From equation (2.5.2) the continuous realized variance in SVJ model can be expressed as

$$V_c(0, T) = \frac{1}{T} \int_0^T \nu_t dt + \frac{1}{T} \left( \sum_{i=1}^{N(T)} (\ln(Y_i))^2 \right)$$  \hspace{1cm} (2.6.2)

The fair continuous variance strike in SVJ model is obtained by taking the expectation of the continuous realized variance and using equations (3.2.7) and (2.5.3) we get:

$$K_{var}^* = \mathbb{E}[V_c(0, T)] = \theta + \frac{\nu_0 - \theta}{\kappa T} (1 - e^{-\kappa T}) + \lambda (a^2 + b^2)$$ \hspace{1cm} (2.6.3)

### 2.6.1 SVJ Model: Continuous Volatility Strike

In this section we derive the fair continuous volatility strike in SVJ model. The continuous realized variance in SVJ model is given by equation (2.6.2).

**Proposition 9** In SVJ model, the fair continuous volatility strike is given by the following equation:

$$K_{vol}^* = \mathbb{E}[\sqrt{V_c(0, T)}] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - E(e^{-sV_c(0, T)})}{s^{3/2}} ds$$ \hspace{1cm} (2.6.4)
where Laplace transform of the continuous realized variance \( E(\exp(-sV_c(0,T))) \) is given by:

\[
E(\exp(-sV_c(0,T))) = \exp \left( A(T, s) - B(T, s)v_0 + \lambda T \left( \frac{\exp\left(\frac{-sg^2}{\sqrt{T+2s\sigma^2}}\right)}{\sqrt{1 + \frac{2s\sigma^2}{T}}} - 1 \right) \right) \tag{2.6.5}
\]

\( A(T, s) \) and \( B(T, s) \) are given by equation (2.4.4).

**Proof:** Equation (2.6.4) follows from (2.3.11) and equation (2.6.5) follows from propositions 5 and 8. \( \square \)

### 2.6.2 SVJ Model: Discrete Variance Strike

**Proposition 10** In SVJ model,

\[
E_0 \left( V_d(0, n, T) \right) = E_0 \left( V_c(0, T) \right) + h(r, \rho, \sigma_v, \kappa, \theta, \lambda, m, b, n) \tag{2.6.6}
\]

The function \( h(\cdot) \) is given explicitly in appendix A. It converges to zero linearly with the number of sampling dates:

\[
h(r, \rho, \sigma_v, \kappa, \theta, \lambda, m, b, n) = O\left(\frac{1}{n}\right)
\]

and the expectation of discrete realized variance converges to the expected continuous realized variance linearly with the sampling size (\( n = T/\Delta t \)). Hence,

\[
K_{\text{var}}^*(n) = K_{\text{var}}^* + h(r, \rho, \sigma_v, \kappa, \theta, \lambda, m, b, n) \tag{2.6.7}
\]
and the discrete variance strike converges to the continuous variance strike linearly with the number of sampling dates ($\Delta t = T/n$).

A proof is given in appendix A. Hence, from equation (2.2.16) the initial value of a discrete variance swap, $P_d(0, n, T, K, I)$, converges linearly to the initial value of a continuous variance swap, $P_c(0, T, K, I)$, with the number of sampling dates.

2.7 Numerical Results

In this section we present numerical results for the computation of fair variance strikes and fair volatility strikes. We price variance swaps and volatility swaps of one year maturity with monthly, weekly and daily sampling i.e., with $n = 12, 52, 252$ respectively. For each sampling size $n$ we compute variance strikes and volatility strikes using analytical formulas and simulation. Using Monte Carlo simulation, we calculate the realized variance, the realized volatility, the convexity correction term and the $3^{rd}$ and $4^{th}$ order correction terms in equation (2.2.28). We use the model parameters similar to those estimated in Duffie, Pan and Singleton (2000). These were found by minimizing the mean squared errors for market option prices for S&P500 on November 2, 1993. We adjust the parameters slightly so that the fair continuous variance strike, $(13.261\%)^2$, is same in these models. We used the stochastic volatility jump model parameters and equation (3.2.7) to calculate the volatility in the Merton jump-diffusion (J) model. Table 5.1 gives these parameters.
Table 2.1: Model parameters used in numerical experiments

<table>
<thead>
<tr>
<th>Parameters</th>
<th>BS model</th>
<th>SV model</th>
<th>J model</th>
<th>SVJ model</th>
</tr>
</thead>
<tbody>
<tr>
<td>risk free rate $r$</td>
<td>3.19%</td>
<td>3.19%</td>
<td>3.19%</td>
<td>3.19%</td>
</tr>
<tr>
<td>initial volatility $\sqrt{V_0}$</td>
<td>13.261%</td>
<td>10.101%</td>
<td>11.394%</td>
<td>9.4%</td>
</tr>
<tr>
<td>correlation $\rho$</td>
<td>n/a</td>
<td>-0.70</td>
<td>n/a</td>
<td>-0.79</td>
</tr>
<tr>
<td>long run mean variance $\theta$</td>
<td>n/a</td>
<td>0.019</td>
<td>n/a</td>
<td>0.014</td>
</tr>
<tr>
<td>speed of mean reversion $\kappa$</td>
<td>n/a</td>
<td>6.21</td>
<td>n/a</td>
<td>3.99</td>
</tr>
<tr>
<td>volatility of variance $\sigma_v$</td>
<td>n/a</td>
<td>0.31</td>
<td>n/a</td>
<td>0.27</td>
</tr>
<tr>
<td>jump arrival rate $\lambda$</td>
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<td>n/a</td>
<td>0.11</td>
<td>0.11</td>
</tr>
<tr>
<td>mean proportional size of jump $m$</td>
<td>n/a</td>
<td>n/a</td>
<td>-0.12</td>
<td>-0.12</td>
</tr>
<tr>
<td>jump size volatility $b$</td>
<td>n/a</td>
<td>n/a</td>
<td>0.15</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Table 2.2: Fair variance strikes and fair volatility strikes versus the number of sampling dates in the Merton jump-diffusion model

<table>
<thead>
<tr>
<th>$n$</th>
<th>Simulation</th>
<th>Analytical</th>
<th>Simulation</th>
<th>Num. Int.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K_{var}(n)$ (%)</td>
<td>$K_{var}^*(n)$ (%)</td>
<td>$\rho_n$</td>
<td>$K_{vol}^*(n)$ (%)</td>
</tr>
<tr>
<td>Monthly</td>
<td>13.867 0.784</td>
<td>13.868 0.947</td>
<td>12.806 0.004</td>
<td>12.798</td>
</tr>
<tr>
<td>Weekly</td>
<td>13.407 0.735</td>
<td>13.394 0.966</td>
<td>12.565 0.003</td>
<td>12.559</td>
</tr>
<tr>
<td>Daily</td>
<td>13.305 0.734</td>
<td>13.288 0.973</td>
<td>12.504 0.003</td>
<td>12.498</td>
</tr>
<tr>
<td>Cont.</td>
<td>13.261</td>
<td></td>
<td></td>
<td>12.481</td>
</tr>
</tbody>
</table>

The first column shows the sampling size in computing the realized variance of one year maturity swap in the jump-diffusion model. The second column shows the fair variance strike for the respective number of sampling dates computed using simulation. The third column shows the standard error in the estimate of the fair variance strike computed using simulation. The fourth column shows the fair variance strike values computed using analytical formula (2.5.9). The fifth column shows the correlation coefficient $\rho_n$. The sixth column shows fair volatility strikes obtained using simulation with a control variate. The seventh column shows the standard error in the estimate of the fair volatility strike and last column shows the fair volatility strike computed using numerical integration. The last row shows the fair continuous variance strike and the fair continuous volatility strike.

2.7.1 Merton Jump-Diffusion Model

In the Merton jump-diffusion model we compute fair discrete variance strikes using the formula (2.5.9). We compute the fair continuous variance strike using the formula (2.5.3).

We compute fair discrete volatility strikes using the integration formula (2.3.12) and Laplace transform of the discrete realized variance (2.5.14). Equation (2.3.12) can be

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represented in the following way:

\[
E\left(\sqrt{\sigma_K^2(n)}\right) = \frac{1}{2\sqrt{\pi}} \int_0^b \frac{1 - E(e^{-soK(n)})}{s^{3/2}} \, ds + \frac{1}{2\sqrt{\pi}} \int_b^\infty \frac{1 - E(e^{-soK(n)})}{s^{3/2}} \, ds
\]

\[ (2.7.1) \]

We can bound the second integral as follows:

\[
\frac{1}{2\sqrt{\pi}} \int_b^\infty \frac{1 - E(e^{-soK(n)})}{s^{3/2}} \, ds \leq \frac{1}{2\sqrt{\pi}} \int_b^\infty \frac{1}{s^{3/2}} \, ds = \frac{1}{\sqrt{\pi}b}
\]

\[ (2.7.2) \]

There are two types of errors in computing the fair discrete variance strike numerically using the integration formula. The first one is the discretization error in evaluating the first integral in equation (2.7.1) and second is the tail sum error in the second integral in equation (2.7.1). We compute discrete volatility strikes so that both errors are less than \(10^{-8}\). Thus, we choose the parameter \(b = 10^{16}\) and evaluate first integral between 0 and \(b = 10^{16}\) such that the discretization error in the first integral is less than \(10^{-8}\).

Similarly, we compute the fair continuous volatility strike using the integration formula and the Laplace transform of the continuous realized variance (2.5.4). We also compute the fair variance strike and the fair volatility strike using simulation.

For computing the fair variance strike and the fair volatility strike using simulation we need to simulate the jump diffusion model at fixed dates. We used the following equation to simulate the stock prices at every time \(t_i = iT/n\) in the partition \(0 = t_0 < \)
\( t_1 < \ldots < t_n = T \) of \([0, T]\).

\[
S_{i+1} = S_i \left[ e^{(r-\lambda m - \frac{1}{2} \sigma^2) \Delta t + \sigma \sqrt{\Delta t} \sigma_{i+1}} \right]^{N(t_{i+1})} \prod_{j=N(t_i)+1}^{N(t_{i+1})} Y_j
\]  

(2.7.3)

where \( N(t_i) \) refers to total jumps in time \([0, t_i]\). We simulated \( N = 1,000,000 \) paths of the underlying asset in the Merton jump-diffusion model.

![Figure 2.1: Convergence of fair strikes with sampling size in jump-diffusion model. This figure plots on log-log scale difference in the fair discrete strike and the fair continuous strike versus the number of sampling dates.](image)

Table 2.2 shows the results in the Merton jump-diffusion model. We used equation (2.5.13) to compute the fair variance strike. Since we know the exact value of the fair discrete variance strike we used the control variate method to obtain more reliable estimates of fair volatility strikes from simulation. We used the following equation to compute the fair volatility strike:
\[ K_{\text{vol}}^*(n) = E(\sqrt{V_d(0,n,T)}) - b_n \left( E((V_d(0,n,T)) - K_{\text{var}}^*(n) \right) \quad (2.7.4) \]

where \( E(\sqrt{V_d(0,n,T)}) \) is the simulation estimate of the fair volatility strike with number of sampling dates \( n \), \( E((V_d(0,n,T)) \) is the simulation estimate of the fair variance strike with same sampling size computed using same simulation paths, \( K_{\text{var}}^*(n) \) is the fair discrete variance strike computed using equation (2.5.13) and \( b_n \) is the optimal coefficient which minimizes the variance and is given by:

\[ b_n = \frac{\text{Cov}((V_d(0,n,T), \sqrt{(V_d(0,n,T)}))}{\text{Var}((V_d(0,n,T)))} \]

The ratio of the variance of the optimally controlled estimator \( K_{\text{vol}}^*(n) \) to that of the uncontrolled estimator is \( 1 - \rho_n^2 \) where \( \rho_n \) is the correlation coefficient between \( (V_d(0,n,T) \) and \( \sqrt{(V_d(0,n,T)} \). We also report the correlation coefficient \( \rho_n \) for all sampling sizes \( n \) in Table 2.2.

From the analytical results we plot \( \log(K_{\text{vol}}^*(n) - K_{\text{var}}^*) \) versus \( \log(n) \). Figure 2.1 shows the convergence plot of the fair variance strike and the fair volatility strike with the number of sampling dates. These results show that in the Merton jump-diffusion model the fair discrete variance strike converges linearly to the fair continuous variance strike with the number of sampling dates consistent with Proposition 8.

For computing the convergence rate of the discrete volatility strikes to continuous volatility strikes we do the following. From the numerical integration results we plot
Table 2.3: Approximation of the fair volatility strike using the convexity correction formula in jump-diffusion model

<table>
<thead>
<tr>
<th></th>
<th>Conv. corr.(cc)</th>
<th>3rd order</th>
<th>4th order</th>
<th>Excess Prob.(p)</th>
<th>$K_{vol}^*(n)$(%) using cc</th>
<th>$K_{vol}^*(n)$(%) True value</th>
<th>Diff (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monthly</td>
<td>2.881</td>
<td>13.801</td>
<td>111.630</td>
<td>0.053</td>
<td>10.985</td>
<td>12.798</td>
<td>1.813</td>
</tr>
<tr>
<td>Weekly</td>
<td>2.799</td>
<td>13.422</td>
<td>109.350</td>
<td>0.053</td>
<td>10.608</td>
<td>12.559</td>
<td>1.951</td>
</tr>
<tr>
<td>Daily</td>
<td>2.861</td>
<td>13.973</td>
<td>122.290</td>
<td>0.053</td>
<td>10.444</td>
<td>12.498</td>
<td>2.054</td>
</tr>
</tbody>
</table>

The first column shows the sampling size in computing the realized variance of 1 year maturity volatility swap. The second column shows the convexity correction term (2.2.26) with the different number of sampling dates. The third and fourth columns show the 3rd and 4th order term in equation (2.2.28). The fifth column shows the excess probability (2.2.27). The sixth column column shows the fair volatility strike computed using the convexity correction formula. The seventh column shows the fair volatility strike computed using numerical integration and the last column shows the absolute difference between the fair volatility strike computed using the numerical integration in Table 2.2 and the convexity correction formula.

$$\log(K_{vol}^*(n) - K_{vol}^*)$$ versus $\log(n)$ and compute the coefficient $\beta$ using regression:

$$\log(K_{vol}^*(n) - K_{vol}^*) = \log(\gamma_1) - \beta \log(n)$$  \hspace{1cm} (2.7.5)

For the volatility strike we get,

$$\log(K_{vol}^*(n) - K_{vol}^*) = -1.014 \log(n) - 4.554 \hspace{1cm} R^2 = 0.999$$  \hspace{1cm} (2.7.6)

These results show that the fair discrete volatility strike converges to the fair continuous volatility strike linearly with number of sampling dates.

Table 2.3 shows the results of computing the fair volatility strike by the convexity correction formula. The excess probability $p$ from (2.2.27) is not equal to zero and the 3rd and 4th order terms in equation (2.2.28) are comparable in magnitude with convexity correction term. We can see from the last column the differences between the fair...
volatility strike computed using numerical integration and computed using convexity correction formula is quite significant. Hence, the convexity correction formula doesn't work well to compute the fair volatility strike in the Merton jump-diffusion model.

2.7.2 Heston Stochastic Volatility Model

In the Heston stochastic volatility model we compute fair discrete variance strikes from equation (2.4.7) the fair continuous variance strike from equation (3.2.7). We also compute fair discrete variance strikes and fair discrete volatility strikes using Monte Carlo simulation. We compute the fair continuous volatility strike by numerical integration using (2.3.12) and the Laplace transform of the continuous realized variance (2.4.4).

For the stochastic volatility model, we used the Euler discretization with modified drift to simulate the paths of the stock price and the variance process on a discrete time grid with continuous approximation to the drift part. Let \(0 = t_0 < t_1 < \ldots < t_n = T\) be a partition of \([0, T]\) into \(n\) equal segments of length \(\Delta t = T/n\), i.e. \(t_i = iT/n\) for each \(i = 0, 1, \ldots, n\). The discretization of the stock price process is:

\[
S_{t_i} = S_{t_{i-1}} + rS_{t_{i-1}} \Delta t + \sqrt{v_{t_{i-1}}} S_{t_{i-1}} (\rho \Delta W^1_t + \sqrt{1 - \rho^2} \Delta W^2_t)
\]

where \(\Delta W^j_t = W^j_{t_i} - W^j_{t_{i-1}}, j = 1, 2\). The discretization of the variance process is:

\[
v_{t_i} = \theta (1 - e^{-\kappa \Delta t}) + v_{t_{i-1}} e^{-\kappa \Delta t} + \sqrt{v_{t_{i-1}}} \sigma_v \Delta W^1_t
\]

Here, we used the exact solution of the drift part of the variance process. We used the
parameters in Table 5.1 for simulation. We simulated $N = 1,000,000$ paths of stock prices to compute the fair strikes and the convexity approximation terms for different number of sampling dates. In our simulation we set the variance process to zero if variance goes negative.

Table 2.4: Fair variance strikes and fair volatility strikes with different numbers of sampling dates in the SV model

<table>
<thead>
<tr>
<th>$n$</th>
<th>Simulation</th>
<th>Analytical</th>
<th>Simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K_{\text{var}}^*(n)(%)$</td>
<td>$SE(K_{\text{var}}^*(n))$</td>
<td>$K_{\text{var}}^*(n)(%)$</td>
</tr>
<tr>
<td>Monthly</td>
<td>13.789</td>
<td>0.122</td>
<td>13.782</td>
</tr>
<tr>
<td>Weekly</td>
<td>13.373</td>
<td>0.078</td>
<td>13.365</td>
</tr>
<tr>
<td>Daily</td>
<td>13.287</td>
<td>0.061</td>
<td>13.282</td>
</tr>
<tr>
<td>Cont.</td>
<td>13.261</td>
<td></td>
<td>13.261</td>
</tr>
</tbody>
</table>

The first column shows the number of sampling dates in computing the realized variance of one year maturity swap in stochastic volatility model. The second column shows the fair variance strike for respective number of sampling dates computed using simulation. The third column shows the standard error in the simulation estimate of the fair variance strike $K_{\text{var}}^*(n)$. The fourth column shows the fair variance strike computed using the formula in proposition 5. The fourth and fifth columns show fair volatility strikes and their standard error. The last row shows the fair variance strike and the fair volatility strike in the case of a continuously sampled swap.

Table 2.4 shows fair discrete and continuous variance and volatility strikes in the SV model. In the Heston stochastic volatility model, the fair discrete variance strike in the case of monthly sampling ($n = 12$) is $(13.782\%)^2$ versus the fair continuous variance strike $(13.261\%)^2$. This corresponds to a relative percentage difference of $8.01\%$. The fair discrete volatility strike $13.40\%$ in the monthly sampling ($n = 12$) differs from the fair continuous volatility strike $13.09\%$. This corresponds to a relative percentage difference of $2.31\%$. The third column in Table 2.4 shows the standard error in the estimate of the fair variance strike, $\sqrt{\text{Var}(V_d(0,n,T))/N}$, where $N$ is the number of simulation.
Figure 2.2: Convergence of fair strikes with number of sampling dates in stochastic volatility model. This figure plots on log-log scale the difference in the fair discrete strike and the fair continuous strike versus the number of sampling dates.

paths. The number of simulation paths $N$ is same for all sampling sizes $n$. The numerical results show that the variance of the realized variance, $\text{Var}(V_d(0,n,T))$, doesn’t converge to zero in the Heston stochastic volatility model. For $n = 252$ we compute $\sqrt{\text{Var}(V_d(0,n,T))} = 0.0062$ using equation (2.4.5) and it is consistent with numerical results from the simulation. Unlike the Black-Scholes model, in the Heston stochastic volatility model the fair continuous volatility strike (13.09%) is not equal to the square root of the fair continuous variance strike (13.26%).

From the analytical results we plot $\log(K_{\text{var}}^*(n) - K_{\text{var}}^*)$ versus $\log(n)$. We plot the same for the volatility strikes using the simulation results. Figure 2.2 shows the convergence plot of the fair variance strikes and the fair volatility strikes with the number of sampling dates. These results show that in the Heston stochastic volatility model the
fair discrete variance strike converges linearly to the fair continuous variance strike with the number of sampling dates, consistent with Proposition 5.

We determine the convergence rate of the fair discrete volatility strike to the fair continuous volatility strike numerically using the same procedure from section 2.7.1. The regression equation of the volatility strike is

\[ \log(K_{vol}^*(n) - K_{vol}^*) = -1.098 \log(n) - 4.562 \quad R^2 = 0.999 \quad (2.7.7) \]

These results show that in the Heston stochastic volatility model the fair discrete volatility strike converges linearly to the fair continuous volatility strike with sampling size.

2.7.3 Stochastic Volatility Model with Jumps

In stochastic volatility model with jumps (SVJ) we compute fair discrete variance strikes from equation (2.6.7) the fair continuous variance strike from equation (2.6.3). We also compute fair discrete variance strikes and fair discrete volatility strikes using Monte Carlo simulation. We compute the fair continuous volatility strike by numerical integration using (2.6.4).

For simulating stochastic volatility model with jumps, we used the Euler discretization with modified drift to simulate the paths of the variance process on a discrete time grid with continuous approximation to the drift part as described in section 2.7.2 and
Table 2.5: Fair variance strikes and fair volatility strikes with different numbers of sampling dates in the SVJ model

<table>
<thead>
<tr>
<th>$n$</th>
<th>Simulation</th>
<th>Analytical</th>
<th>Simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K_{var}(n)$($%$)</td>
<td>$SE(K_{var}(n))$</td>
<td>$K_{var}(n)$($%$)</td>
</tr>
<tr>
<td>Monthly</td>
<td>13.854</td>
<td>0.796</td>
<td>13.891</td>
</tr>
<tr>
<td>Weekly</td>
<td>13.398</td>
<td>0.749</td>
<td>13.397</td>
</tr>
<tr>
<td>Daily</td>
<td>13.311</td>
<td>0.757</td>
<td>13.290</td>
</tr>
<tr>
<td>Cont.</td>
<td>13.261</td>
<td></td>
<td>13.222</td>
</tr>
</tbody>
</table>

The first column shows the number of sampling dates in computing the realized variance of one year maturity swap in stochastic volatility model with jumps. The second column shows the fair variance strike for respective number of sampling dates computed using simulation. The third column shows the standard error in the simulation estimate of the fair variance strike $K_{var}(n)$. The fourth column shows the fair variance strike computed using the formula in proposition 5. The fifth and sixth columns show fair volatility strikes and their standard error. The last row shows the fair variance strike and the fair volatility strike in the case of a continuously sampled swap.

Figure 2.3: Convergence of fair strikes with sampling size in SVJ model. This figure plots on log-log scale difference in the fair discrete strike and the fair continuous strike versus the number of sampling dates.

the stock price process with jumps in it. We used the parameters in Table 5.1 for simulation. We simulated $N = 100,000$ paths of stock prices to compute the fair strikes and the convexity approximation terms for different number of sampling dates. In our
simulation we set the variance process to zero if variance goes negative.

Table 2.6: Approximation of the fair volatility strike using the convexity correction formula in SVJ model

<table>
<thead>
<tr>
<th>n</th>
<th>Conv corr.(cc)</th>
<th>3rd order</th>
<th>4th order</th>
<th>Excess Prob.(p)</th>
<th>$K_{var}^*(n)(%)$ using cc</th>
<th>$K_{var}^*(n)(%)$ correct value</th>
<th>Diff (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monthly</td>
<td>2.980</td>
<td>12.759</td>
<td>99.433</td>
<td>0.061</td>
<td>10.874</td>
<td>12.756</td>
<td>1.882</td>
</tr>
<tr>
<td>Weekly</td>
<td>2.915</td>
<td>12.488</td>
<td>97.879</td>
<td>0.062</td>
<td>10.483</td>
<td>12.313</td>
<td>1.830</td>
</tr>
<tr>
<td>Daily</td>
<td>3.039</td>
<td>13.454</td>
<td>122.090</td>
<td>0.072</td>
<td>10.272</td>
<td>12.131</td>
<td>1.859</td>
</tr>
</tbody>
</table>

The first column shows the number of sampling dates in computing the realized variance of one year maturity volatility swap. The second column shows the convexity correction term (2.2.26) with the different number of sampling dates. The third and fourth columns show the 3rd and 4th order term in equation (2.2.28). The fifth column shows the excess probability (2.2.27). The sixth column column shows the fair volatility strike computed using convexity correction formula. The seventh column shows the fair volatility strike computed using simulation and the last column shows the difference between the fair volatility strike computed using the simulation in Table 2.5 and the convexity correction formula (2.2.26).

Table 2.5 shows fair discrete and continuous variance and volatility strikes in the SVJ model. In SVJ model, the fair discrete variance strike in the case of monthly sampling ($n = 12$) is $(13.891\%)^2$ versus the fair continuous variance strike $(13.26\%)^2$. This corresponds to a relative percentage difference of 9.68%. The fair discrete volatility strike 12.756% in the monthly sampling ($n = 12$) differs from the fair continuous volatility strike 12.222%. This corresponds to a relative percentage difference of 4.43%. Even though fair variance strikes are similar in SV, J and SVJ models, the fair volatility strikes in SVJ model are less compared to the SV and J model. This implied there is more convexity value in the SVJ model.

From the analytical results we plot $\log(K_{var}^*(n) - K_{var}^* )$ versus $\log(n)$. We plot the same for the volatility strikes using the simulation results. Figure 2.3 shows the conver-
gence plot of the fair variance strikes and the fair volatility strikes with the number of sampling dates. These results show that in SVJ model the fair discrete variance strike converges linearly to the fair continuous variance strike with the number of sampling dates, consistent with Proposition 10.

We determine the convergence rate of the fair discrete volatility strike to the fair continuous volatility strike numerically using the same procedure from section 2.7.1. The regression equation of the volatility strike is

\[
\log(K_{vol}^*(n) - K_{vol}^*) = -1.046 \log(n) - 4.428 \\
R^2 = 0.99 
\] (2.7.8)

These results show that in SVJ model the fair discrete volatility strike converges linearly to the fair continuous volatility strike with sampling size.

Table 2.6 shows the results of computing the fair volatility strike by the convexity correction formula. The excess probability \( p \) from (2.2.27) is not equal to zero and the 3\(^{rd} \) and 4\(^{th} \) order terms in equation (2.2.28) are comparable in magnitude with convexity correction term. We can see from the last column the differences between the fair volatility strike computed using numerical integration and computed using convexity correction formula is quite significant. Hence, the convexity correction formula doesn’t work well to compute the fair volatility strike in SVJ model.

Table 2.7 shows fair variance strikes and fair volatility strikes in different models with
Table 2.7: Comparison of fair variance strikes and fair volatility strikes in different models

<table>
<thead>
<tr>
<th>Sampling Size (n)</th>
<th>$K_{var}^*$ (%)</th>
<th>$K_{vol}^*$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monthly</td>
<td>BS 13.86</td>
<td>SV 13.78</td>
</tr>
<tr>
<td>Daily</td>
<td>BS 13.27</td>
<td>SV 13.28</td>
</tr>
<tr>
<td>Continuous</td>
<td>BS 13.26</td>
<td>SV 13.26</td>
</tr>
</tbody>
</table>

The first column shows the number of sampling dates. Then next four columns show fair variance strikes in the Black-Scholes (BS), the Heston stochastic volatility (SV), the Merton jump-diffusion model (J) and stochastic volatility model with jumps (SVJ) respectively. The last three columns show fair volatility strikes in respective models.

different sampling sizes. This table illustrates answers to the questions about variance and volatility swaps we have investigated in this chapter. The fair volatility strike is less than the square root of the fair variance strike due to the concave payoff of volatility swap in realized variance. This is true for all models and all sampling sizes except for continuous sampling in the Black-Scholes model in which case they are equal. Even though fair continuous variance strikes are identical in all models, fair continuous volatility strikes are significantly different. Fair discrete strikes under monthly and weekly sampling are considerably different than under continuous sampling. Formulas and results derived for idealized contracts with continuous sampling should be applied with caution to instruments which use discrete sampling.

Next we compute the variance strikes and volatility strikes with an alternative definition of the realized variance in equation (3.2.1) and with varying maturities of the variance swap. All the results so far were computed for a maturity of one year and with $m = n - 1$ in the realized variance definition specified in equation (3.2.1). We have seen
that with this definition \((m = n - 1)\) discrete variance strikes are different than the continuous variance strikes. Table 2.8 shows the variance strikes computed using two definition of the realized variance and with different maturities in the SVJ model. As can be seen from the results using \(m = n\) in the definition of the realized variance removes the effect of discrete sampling in the fair variance strike. There is not a significant difference in the fair variance strike with different sampling sizes when \(m = n\) in the definition of the realized variance. Also the effect of discrete sampling is more pronounced in the shorter maturity swaps. In the market, the typical maturity of swaps varies from one month to 30 years with one month being most popular and 30 year is quite less. In most of the contracts sampling is done daily or weekly and sometimes monthly. As can be seen from these results there is about 29 basis points difference between discrete sampling (daily) and continuous sampling one month fair variance strike. The effect of discreteness decreases as maturity increases.

Table 2.8: Comparison of fair variance strikes with an alternative definition of realized variance and with maturities in the SVJ model

<table>
<thead>
<tr>
<th>Sampling Size (n)</th>
<th>1 month (n - 1)</th>
<th>6 months (n - 1)</th>
<th>1 year (n - 1)</th>
<th>30 years (n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous</td>
<td>12.29</td>
<td>12.97</td>
<td>13.26</td>
<td></td>
</tr>
</tbody>
</table>
The difference in the fair volatility strike and the square root of the fair variance strike is called the convexity value. Table 2.9 shows fair volatility strikes computed using the convexity correction formula and true fair volatility strikes in all three models. The convexity correction formula only works well in the Black-Scholes model and performs poorly in the Heston stochastic volatility model and even worse in the models with jumps (J, SVJ). It can be seen from the results that the convexity value depends on the model and the sampling size.

Table 2.9: Comparison of fair volatility strikes and approximations using the convexity correction formula in different models

<table>
<thead>
<tr>
<th>Sampling Size</th>
<th>BS $K_{vol}^*(%)$</th>
<th>SV $K_{vol}^*(%)$</th>
<th>J $K_{vol}^*(%)$</th>
<th>SVJ $K_{vol}^*(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>cc true</td>
<td>cc true</td>
<td>cc true</td>
<td>cc true</td>
</tr>
<tr>
<td>Monthly</td>
<td>13.54 13.54</td>
<td>13.31 13.40</td>
<td>10.99 12.80</td>
<td>10.87 12.76</td>
</tr>
<tr>
<td>Daily</td>
<td>13.28 13.28</td>
<td>13.09 13.10</td>
<td>10.44 12.50</td>
<td>10.27 12.13</td>
</tr>
</tbody>
</table>

The first column shows the number of sampling dates. Then next two columns show fair volatility strikes in the Black-Scholes (BS) model computed using the convexity correction formula and true fair volatility strikes. The fourth and the fifth columns provide results in the Heston stochastic volatility (SV) model and the sixth and seventh columns give results for the Merton jump-diffusion (J) model and last two columns give results for the stochastic volatility model with jumps (SVJ).

Next we analyze the effect of ignoring jumps in the computation of the fair variance strike. Broadie and Jain (2007) provides the VIX index formula in the SVJ model. The VIX index provides the one month realized variance as computed from portfolio of out-of-money S&P 500 (SPX) put and call options of one month maturity. In the underlying has no jumps the VIX value and the fair variance strike are same or in other words the static portfolio of out-of-money call and put options replicates the one month continuous realized variance. But when there are jumps in the underlying the VIX or
the portfolio of options doesn’t capture the realized variance completely. Hence, in the case of jumps the market $\tilde{VIX}$ value and the fair variance strike value differs. We want to analyze the magnitude of this difference which also specifies the effect of ignoring jumps in computing the fair variance strike from the portfolio of options. Broadie and Jain (2007) shows that the $\tilde{VIX}$ index in the SVJ model is given by

$$\tilde{VIX} = \sqrt{\theta + \frac{1 - e^{-\kappa T}}{\kappa T} (\nu_0 - \theta) + 2\lambda (m - a)}$$

(2.7.9)

where $\tau = 30/365$. The fair variance strike in the SVJ model is given by (2.6.3). Hence the effect of ignoring jumps in the fair variance strike can be computed as:

$$\sqrt{K_{var}^*} - \tilde{VIX} = \sqrt{\theta + \frac{1 - e^{-\kappa T}}{\kappa T} (\nu_t - \theta) + \lambda (a^2 + b^2))} - \sqrt{\theta + \frac{1 - e^{-\kappa T}}{\kappa T} (\nu_t - \theta) + 2\lambda (m - a)}$$

(2.7.10)

From the above equation, we can expand the individual terms to understand the effect of jumps on the fair variance swap strike. On expanding the right hand side terms in equation (2.7.10) we get

$$K_{var}^* - \tilde{VIX}^2 = \lambda b^2 (-a - \frac{1}{4} b^2) + \frac{1}{3} (a + \frac{1}{2} b^2)^3 + O((a + \frac{1}{2} b^2)^3)$$

(2.7.11)

Also depending upon the direction of jump the VIX index can under-approximate or over-approximate the fair variance strike.

$$\tilde{VIX} \leq \sqrt{K_{var}^*} \quad \text{Negative Jumps}$$

$$\tilde{VIX} \geq \sqrt{K_{var}^*} \quad \text{Positive Jumps}$$

(2.7.12)
Table 2.10: Effect of jumps in the fair variance strike

<table>
<thead>
<tr>
<th>( \lambda = 0.4 )</th>
<th>( \sqrt{K^*_\text{var}} )</th>
<th>VIX</th>
<th>Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b/m )</td>
<td>0.2</td>
<td>0.2</td>
<td>-0.2</td>
</tr>
<tr>
<td>0</td>
<td>18.81</td>
<td>10.24</td>
<td>16.46</td>
</tr>
<tr>
<td>0.2</td>
<td>24.50</td>
<td>17.52</td>
<td>20.90</td>
</tr>
<tr>
<td>0.4</td>
<td>36.94</td>
<td>30.61</td>
<td>30.94</td>
</tr>
<tr>
<td>0.5</td>
<td>44.28</td>
<td>37.86</td>
<td>37.03</td>
</tr>
</tbody>
</table>

This Table shows the fair variance strike and the VIX value and their difference with different jump parameters. This Table also shows the effect of ignoring jumps in the computation of the fair variance strike. The first column shows the jump size volatility \( b \) and the second row shows the mean proportional size of jump, \( m \).

Table 2.10 shows the effect of ignoring jumps in the fair variance strike as computed from equation (2.7.10). We use the SVJ model parameters in Table 5.1 and varying the jump mean size and volatility to compute these differences. The first three columns show the fair variance strikes with different mean jump size \( m \) and jump size volatility \( b \). The next three columns show the strike computed from static portfolio of options or the VIX value and the last three columns show the difference between the fair variance strike and VIX index value. As can be seen with negative jump size the strike from portfolio of options or the VIX index value under approximates the fair variance strike and vice versa. Also we saw in results before that discrete fair variance strike is more than the continuous fair variance strike. Hence when the underlying has negative jumps (which in general is true in equity markets) then the effect of discrete sampling and jumps add up and the fair discrete variance strike when the underlying has jump can be significantly different from the continuous fair variance strike value. The parameters reported here are typical range of parameters in the SVJ model as reported in the (Gatheral 2006). These results show that the effect of jumps is quite significant.
2.8 Conclusion

In this chapter we study the pricing of variance swaps and volatility swaps in four financial models. We derive analytical formulas for fair discrete variance strikes in the Black-Scholes model, the Heston stochastic volatility model, the Merton jump-diffusion model and the Bates and Scott stochastic volatility model with jumps. We investigate the effect of discrete sampling and jumps in the underlying on fair variance strikes. We found that the effect of discrete sampling is less significant as compared to the effect of jumps in the underlying. The discrete sampling effect depends on the maturity of swap and size of sampling. The effect of jumps in the underlying on fair variance strikes depends on direction and magnitude of jumps. Hence one month discrete variance strike when the underlying has negative jumps can be significantly different from the continuous fair variance strike. We also present an argument to show that the convexity correction formula to approximate fair volatility strikes doesn't provide good estimates in jump-diffusion models. We present numerical approaches to compute fair volatility strikes. In particular we compute fair discrete volatility strikes from numerical integration and Monte Carlo simulation techniques. We show numerically that in all models fair discrete variance and volatility strikes converge linearly to fair continuous volatility strikes as the sampling size increases.
Chapter 3

Pricing and Hedging Volatility Derivatives

3.1 Introduction

Volatility derivatives are securities whose payoff depends on the realized variance of an underlying asset or an index. The realized variance is the average variance of the underlying asset price over the life of the volatility derivative. A variance swap has a payoff which is a linear function of the realized variance. A volatility swap has a payoff which is a concave function of the realized variance and a variance call option payoff is a convex function of the realized variance. We provide definitions of different volatility derivatives in section 3.2.

In this work we propose a methodology for hedging volatility swaps and variance options using variance swaps. Since the price of both variance swaps and volatility swaps depend on the realized variance of the underlying asset, there must be a relationship between their prices to avoid arbitrage. Since variance swaps can be priced and hedged using actively traded European call and put options, by exploiting the no arbitrage relationship between volatility derivatives and variance swaps we can price and hedge volatility derivatives.
The volatility of asset prices is an indispensable input in both pricing options and in risk management. Through the introduction of volatility derivatives, volatility is now, in effect, a tradable market instrument. Previously traders would use a delta-hedged option position as a means to trade volatility. However, this does not provide a pure volatility exposure since the return also depends on the underlying stock price. Variance and volatility swaps provide pure exposure to volatility and have become quite popular in the market. Three different groups of traders have emerged: directional traders, spread traders, and volatility hedgers. Directional traders bet on the future level of volatility, while spread traders trade the spread between realized and implied volatility. In contrast, a volatility hedger typically covers short volatility positions. Variance and volatility swaps capture the volatility of the underlying asset over a specified time period and are effective hedging instruments for volatility exposure. Based on the demand from volatility traders, the market in volatility and variance swaps has developed rapidly over the last few years and is expected to grow more in the future. Hence, the pricing and hedging of these derivatives have become an important research problem in academia and industry.

Demeterfi et al. (1999) examined the properties of variance and volatility swaps. They showed that variance swaps can be replicated by a static position in European call and put options of all strikes and a dynamic trading strategy in the underlying asset. Brockhaus and Long (2000) provided an analytical approximation for the pricing of volatility swaps. Javaheri et al. (2002) discussed the valuation of volatility swaps in the GARCH(1,1) stochastic volatility model. They used a partial differential equation...
approach to determine the first two moments of the realized variance and then used a convexity approximation formula to price the volatility swaps. Little and Pant (2001) developed a finite difference method for the valuation of variance swaps in the case of discrete sampling in an extended Black-Scholes framework. Detemple and Osakwe (2000) priced European and American options on the terminal value of volatility when volatility follows a diffusion process. Carr et al. (2005) priced options on realized variance by directly modeling the quadratic variation of underlying process using a Lévy process. Carr and Lee (2005) priced arbitrary payoffs of realized variance provided a zero correlation assumption between stock price process and variance process. In chapter two we show that the convexity correction formula doesn't provide a good estimate of fair volatility strikes in the Heston stochastic volatility and the Merton jump-diffusion models.

In this chapter we price variance and volatility swaps when the variance process is a continuous diffusion given by the Heston stochastic volatility model. We compute fair volatility strikes and price variance options by deriving a partial differential equation that must be satisfied by volatility derivatives. We compute the risk management parameters (greeks) of volatility derivatives by solving a series of partial differential equations. Independently, Sepp (2006) priced options on realized variance in the Heston stochastic volatility model by solving a partial differential equation. Then we present a numerical analysis to determine the number of options required to hedge a variance swap. We propose a method to dynamically hedge volatility derivatives using variance swaps and a finite number of European call and put options.

The rest of the chapter is organized as follows. We begin briefly by introducing volatility
derivatives in section 3.2. In section 3.3 we present the pricing of volatility swaps and the variance options using a partial differential equation approach in the Heston stochastic volatility model. In section 3.4 we present the computation of greeks of volatility derivatives in the Heston stochastic volatility model. In section 3.5 we present an optimization approach to hedge variance swaps using a finite number of options. We also present a dynamic hedging approach to hedge volatility swaps using variance swaps. Concluding remarks are given in section 4.6.

### 3.2 Volatility Derivatives

Volatility and variance swaps are forward contracts in which one counterparty agrees to pay the other a notional amount, $N$, times the difference between a fixed level and a realized level of volatility and variance, respectively. The fixed level is called the variance strike for variance swaps and the volatility strike for volatility swaps. The realized variance is determined by the average variance of the asset over the life of the swap.

The variance swap payoff is defined as

$$ (V_d(0, n, T) - K) \times N $$

where $V_d(0, n, T)$ is the realized stock variance (defined below) over the life of the contract, $[0, T]$, where $n$ is the number of sampling dates, $K$ is the variance strike, and $N$ is the notional amount of the swap in dollars. The holder of a variance swap at expiration receives $N$ dollars for every unit by which the stock's realized variance $V_d(0, n, T)$ exceeds the variance strike $K$. The variance strike is quoted in units of volatility squared,
For example, suppose an investor takes a long position in a variance swap with strike \((20\%)^2 = 0.04\) and a notional of one million dollars. If over the life of the contract realized variance is \((25\%)^2 = 0.0625\) then investor would make a profit of \((0.0625 - 0.04) \times 1000000 = $22500\).

The volatility swap payoff is defined as

\[
(\sqrt{V_d(0,n,T)} - K) \times N
\]

where \(\sqrt{V_d(0,n,T)}\) is the realized stock volatility (quoted in annual terms as defined below) over the life of the contract where \(n\) is the number of sampling dates, \(K\) is the volatility strike, and \(N\) is the notional amount of the swap in dollars. The volatility strike \(K\) is typically quoted in units of percent, e.g., 20\%. An investor who is long a volatility swap with strike 20\% and a notional of one million dollars would make a profit of \((.25 - 0.2) \times 1000000 = $50000\) in the previous example.

The procedure for calculating realized volatility and variance is specified in the derivative contract and includes details about the source and observation frequency of the price of the underlying asset, the annualization factor to be used in moving to an annualized volatility and the method of calculating the variance. Let \(0 = t_0 < t_1 < \ldots < t_n = T\) be a partition of the time interval \([0, T]\) into \(n\) equal segments of length \(\Delta t\), i.e., \(t_i = iT/n\) for each \(i = 0, 1, \ldots, n\). Most traded contracts define realized variance to be

\[
V_d(0,n,T) = \frac{AF}{n - 1} \sum_{i=0}^{n-1} \left( \ln \left( \frac{S_{i+1}}{S_i} \right) \right)^2
\]

for a swap covering \(n\) return observations. Here \(S_i\) is the price of the asset at the \(i^{th}\) observation time \(t_i\) and \(AF\) is the annualization factor, e.g., 252 (= \(n/T\)) if the maturity
of the swap, $T$, is one year with daily sampling. This definition of realized variance dif-
fers from the usual sample variance because the sample average is not subtracted from
each observation. Since the sample average is approximately zero, the realized variance
is close to the sample variance.

We call $V_d(0,n,T)$, the discretely sampled realized variance, and $V_c(0,T)$, the continu-
ously sampled realized variance. The variable leg of variance swap, or discrete realized
variance, in the limit approaches the continuously sampled realized variance, that is,

$$V_c(0,T) \equiv \lim_{n \to \infty} V_d(0,n,T)$$  \hspace{1cm} (3.2.2)

In this chapter we price volatility derivatives assuming sampling is done continuously.

In chapter two we compute fair variance strikes and fair volatility strikes when realized
variance is computed discretely.

A European variance call option gives the holder the right to receive a payoff $V_c(0,T)$ in
exchange for paying the strike $K$ at the maturity of variance call option, i.e., its payoff
is

$$C_T = \max(V_c(0,T) - K, 0) \times N$$  \hspace{1cm} (3.2.3)

Similarly the payoff of the variance put option is:

$$P_T = \max(K - V_c(0,T), 0) \times N$$  \hspace{1cm} (3.2.4)

where $N$ is the notional amount in dollars. Unlike European equity options, the payoff of
variance options depends on realized variance $V_c(0,T)$, which is not a traded instrument
in the market.
In the Heston stochastic volatility model, continuous realized variance is given by

\[ V_c(0, T) = \frac{1}{T} \int_0^T v_s ds \]  

(3.2.5)

The fair variance strike, \( K_{\text{var}}^\star \), is defined as the value which makes the contract's net present value equal to zero, i.e., it is the solution of

\[ E_0^Q \left[ e^{-rT} (V_c(0, T) - K_{\text{var}}^\star ) \right] = 0 \]  

(3.2.6)

where the superscript \( Q \) indicates the risk neutral measure and the subscript 0 denotes expectation at time \( t = 0 \). In the Heston stochastic volatility model, the fair variance strike is given by

\[ K_{\text{var}}^\star = E[V_c(0, T)] = E \left( \frac{1}{T} \int_0^T v_s ds \right) = \theta + \frac{\nu_0 - \theta}{\kappa T} (1 - e^{-\kappa T}) \]  

(3.2.7)

where the last equality follows from Proposition 4 in chapter 2. The fair volatility strike is defined as the value which makes the contract net present value equal to zero, i.e., it solves the equation

\[ E_0 \left[ e^{-rT} (\sqrt{V_c(0, T)} - K_{\text{vol}}^\star ) \right] = 0 \]  

(3.2.8)

Hence, the fair volatility strike can be expressed as

\[ K_{\text{vol}}^\star = E \left[ \sqrt{\frac{1}{T} \int_0^T v_s dt} \right] = E[\sqrt{V_c(0, T)}] \]  

(3.2.9)

Using Jensen’s inequality\(^1\) we can obtain an upper bound on the fair volatility strike:

\[ K_{\text{vol}}^\star = E_0[\sqrt{V_c(0, T)}] \leq \sqrt{E_0[V_c(0, T)]} = \sqrt{K_{\text{var}}^\star} \]  

(3.2.10)

\(^1\)For the concave square root function Jensen’s inequality is:

\[ E(\sqrt{x}) \leq \sqrt{E(x)} \]
Hence, the fair volatility strike is bounded above by the square root of the fair variance strike. The difference in the square root of the fair variance strike and the fair volatility strike is called the convexity correction. Some authors have tried to obtain an approximation of this convexity correction using Taylor’s expansion, but we show that in chapter 2 that it is not accurate in the Heston stochastic volatility model. We compute the fair volatility strike by deriving a partial differential equation which exploits a no arbitrage relationship between a variance and a volatility swap.

### 3.3 Pricing Volatility Derivatives

In this section we derive a partial differential equation to price volatility derivatives, compute the fair volatility strike and price variance call and put options.

#### 3.3.1 Pricing Volatility Swaps

We define the price process of a security $X^T_t$ which represents the variable leg of a variance swap:

$$X^T_t = E^Q_t \left[ \frac{1}{T} \int_0^T \nu_s ds \right]$$

This security price $X^T_t$ depends on the variance, $\nu_s$, of the underlying asset from time $t = 0$ until maturity $T$. This security has a payoff at maturity, $T$, which is same as the variable leg of a continuous variance swap. At time 0 it represents the fair variance strike

$$K_{\text{var}}^* = X^T_0$$  \hspace{1cm} (3.3.1)
From equation (3.2.7) we know the value of this security at time 0 and we can derive the stochastic differential equation satisfied by the security $X_t^T$:

$$dX_t^T = \frac{1 - e^{-k(T-t)}}{kT} \sigma_v v_t dW_t^1$$

(3.3.2)

This price process has zero drift since it is a forward price process. The process $X_t^T$ is driven by the same Brownian motion $W_t^1$ as the variance process in the Heston stochastic volatility model. The volatility of the price process, $X_t^T$, goes to 0 as $t$ approaches $T$.

Next, we define the price process of a security $Y_t^T$ which represents the variable leg of the volatility swap:

$$Y_t^T = E_t^Q \left[ \sqrt{\frac{1}{T} \int_0^T v_s ds} \right]$$

This security has a payoff at time $T$ which depends on the variance process from time $t = 0$ until maturity. At time $T$ it represents the payoff of the variable leg of the volatility swap. At time 0 it gives the fair volatility strike

$$K_{vol} = Y_0^T$$

These securities are similar to the interest rate derivatives. The price of a zero coupon bond trading in the market depends on the interest rate process from time 0 until the maturity of bond. An interest rate is not a tradable market instrument so for hedging any interest rate product we use some other interest rate derivatives which are traded in the market. Similarly, the security $Y_t^T$ depends on the variance process, $v_s$, which is not a traded instrument in the market. Since the security $X_t^T$ also depends on the variance process, there must be a relationship between the price processes of $Y_t^T$ and $X_t^T$ to avoid...
arbitrage in the market. Using that relationship we can hedge the volatility derivatives using variance swaps.

Next, we define a state variable \( I_t \) to measure the accumulated variance so far:

\[
I_t = \int_0^t v_s ds
\]

This state variable is a known quantity at time \( t \) and satisfies the differential equation:

\[
dI_t = v_t dt
\]

The forward price process \( Y_t^T \) can be expressed as

\[
Y_t^T = E_t \left[ \frac{1}{T} \left( I_t + \int_t^T v_s ds \right) \right] = F(t, v_t, I_t)
\]

It is a function of time \( t \), a stochastic variance \( v_t \) and a deterministic quantity \( I_t \). Applying Itô’s lemma to \( F(\cdot) \) we get

\[
dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial v} dv + \frac{\partial F}{\partial I} dI + \frac{1}{2} \frac{\partial^2 F}{\partial v^2} dv^2
\]

which can be simplified using equation (2.3.3) to

\[
dF = \left[ \frac{\partial F}{\partial t} + \frac{\partial F}{\partial v} (\theta - v_t) + \frac{\partial F}{\partial I} v_t + \frac{1}{2} \frac{\partial^2 F}{\partial v^2} v_t \sigma_v^2 \right] dt + \frac{\partial F}{\partial v} \sigma_v \sqrt{v_t} dW_t
\]  

(3.3.3)

Since \( F \) is a forward price process, its drift under the risk neutral measure must be zero. Hence,

\[
\frac{\partial F}{\partial t} + \frac{\partial F}{\partial v} (\theta - v_t) + \frac{\partial F}{\partial I} v_t + \frac{1}{2} \frac{\partial^2 F}{\partial v^2} v_t \sigma_v^2 = 0
\]

(3.3.4)

Thus, the forward price process satisfies the partial differential equation (3.3.4) in the Heston stochastic volatility model. We solve the partial differential equation (3.3.4) in
the region: \( 0 \leq t \leq T, I_{\text{min}} \leq I \leq I_{\text{max}}, \nu_{\text{min}} \leq \nu \leq \nu_{\text{max}} \) with the boundary condition

\[
Y^T_T = F(T, \nu_T, I_T) = \sqrt{\frac{I_T}{T}}
\]

(3.3.5)

At other boundaries (\( I \) and \( V \)) we set the second order variation of the price process to zero. In particular, we use the boundary conditions:

\[
\frac{\partial^2 F}{\partial I^2}
\bigg|_{(I=I_{\text{max}}, I_{\text{min}})} = 0 \quad \frac{\partial^2 F}{\partial \nu^2}
\bigg|_{(\nu=\nu_{\text{max}}, \nu_{\text{min}})} = 0
\]

(3.3.6)

Thus by solving the equation (3.3.4) with boundary conditions (3.3.5) and (3.3.6) we can compute the fair volatility strike. By solving this partial differential equation we get the price at all times until maturity. The variance swap forward price process \( X^T_t \) satisfies the same differential equation (3.3.4). The boundary condition in the case of a variance swap will be different at maturity and is given by

\[
X^T_T = G(T, \nu_T, I_T) = \frac{I_T}{T}
\]

(3.3.7)

The analytical formula for the variance strike given by equation (3.2.7) solves the partial differential equation (3.3.4) with boundary conditions (3.3.6) and (3.3.7).

Table 3.1: Black-Scholes and stochastic volatility model parameters used in pricing and hedging

<table>
<thead>
<tr>
<th>Parameters</th>
<th>BS</th>
<th>SV</th>
</tr>
</thead>
<tbody>
<tr>
<td>correlation ( \rho )</td>
<td>n/a</td>
<td>-0.7</td>
</tr>
<tr>
<td>long run mean variance ( \theta )</td>
<td>n/a</td>
<td>0.019</td>
</tr>
<tr>
<td>speed of mean reversion ( \kappa )</td>
<td>n/a</td>
<td>6.21</td>
</tr>
<tr>
<td>volatility of variance ( \sigma_v )</td>
<td>n/a</td>
<td>0.31</td>
</tr>
<tr>
<td>initial volatility ( \sqrt{\nu_0} )</td>
<td>13.261%</td>
<td>10.10%</td>
</tr>
<tr>
<td>risk free rate ( r )</td>
<td>3.19%</td>
<td>3.19%</td>
</tr>
<tr>
<td>real world growth rate ( \mu )</td>
<td>7.0%</td>
<td>7.0%</td>
</tr>
</tbody>
</table>
Next we present numerical results to illustrate the computation of fair variance and fair volatility strikes. We use model parameters similar to those estimated in Duffie et al. (2000), which were found by minimizing the mean squared errors for market option prices for S&P500 on November 2, 1993. We adjust the parameters slightly so that the fair continuous variance strike is same in the two models. We assume a risk free rate of 3.19%. Table 3.1 gives these parameters. Table 3.2 shows the fair variance strike and fair volatility strike of a one year maturity swap computed by solving the partial differential equation (3.3.4) with appropriate boundary conditions. We solve the PDE (3.3.4) on a three dimensional grid with 400 points each in the $V$ and $I$ directions and 2000 intervals in the $t$ direction. We also compute the fair variance and fair volatility strikes using Monte Carlo simulation and numerical integration approach as given in chapter 2. The theoretical value of the fair variance strike is computed using equation (3.2.7). We have reported the square root of fair variance strike, $\sqrt{\bar{K}_{var}}$, in the results. The fair variance strike for the parameters in Table 3.1 is $(13.26\%)^2 = 0.017585$. The results from the PDE approach in this section match the values obtained by other methods.

Figure 3.1 illustrates the dependence of fair variance and fair volatility strikes on initial variance. One advantage of the PDE method over simulation is that we get fair variance and fair volatility strikes for all values of initial variance and accumulated variance. Also, this approach gives prices at all times until maturity. The left graph in Figure 3.1 presents the fair variance strike (plotted as the square root of fair variance strike, $\sqrt{\bar{K}_{var}}$) and the fair volatility strike versus initial volatility $\sqrt{\nu_0}$. It can be seen from
Table 3.2: Comparison of fair variance and fair volatility strikes using different numerical methods

<table>
<thead>
<tr>
<th>Method</th>
<th>$K^*_\text{var} (%)^2$</th>
<th>$K^*_\text{vol} (%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulation price</td>
<td>13.259 (0.057)</td>
<td>13.094 (0.002)</td>
</tr>
<tr>
<td>PDE</td>
<td>13.261</td>
<td>13.096</td>
</tr>
<tr>
<td>Analytical</td>
<td>13.261</td>
<td></td>
</tr>
<tr>
<td>Numerical integration</td>
<td></td>
<td>13.096</td>
</tr>
</tbody>
</table>

The first column shows the fair variance strike computed using PDE method, simulation and analytical value in the Heston stochastic volatility model. The second column shows the respective values of the fair volatility strike.

equation (3.2.7) that the fair variance strike is a linear function of the initial variance.

The fair volatility strike is a not a linear function of the initial variance since its payoff is not a linear function of realized variance. Also, as known from equation (3.2.10), the fair volatility strike is less than the fair variance strike.

![Figure 3.1](image)

Figure 3.1: The left plot shows the square root of the fair variance strike and fair volatility strike versus initial volatility. The right plot shows the convexity value (3.2.10) versus initial volatility.

We define *convexity value* to be the difference in the square root of fair variance strike and the fair volatility strike. The right graph in Figure 3.1 plots the convexity value with
initial volatility. This illustrates that the convexity value is a decreasing function of initial volatility, $\sqrt{v_0}$.

### 3.3.2 Pricing Variance Options

The price of a variance call option is given by:

$$C_t = E_t^Q[e^{-r(T-t)} \max(V_c(0,T) - K_0, 0)] \times N$$

(3.3.8)

We derive a partial differential equation to price a variance call option using a similar no arbitrage argument as presented in the previous section. Since the prices of a variance call option and a variance swap depend on the variance process from initial time until maturity, we can hedge the variance call option using variance swaps and thus compute the price of variance call option.

We form a portfolio of one variance call option and $\alpha$ units of variance swaps. At time 0 the portfolio value is

$$\Pi_0 = \alpha(X^T_0 - K_{var}^*) + C_0$$

(3.3.9)

This portfolio value is the same as the variance call option value since there is no cost to buy one unit of a variance swap at the inception of the contract. The variance call price process, $C_t^T$, can be represented as

$$C_t^T = G(t, v_t, I_t)$$

Applying Itô's lemma to $G(\cdot)$ we get

$$dG = \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial v} dv + \frac{\partial G}{\partial I} dI + \frac{1}{2} \frac{\partial^2 G}{\partial v^2} dv^2$$

(3.3.10)
which can be simplified using equation (??) to

\[ dG = \left[ \frac{\partial G}{\partial t} + \frac{\partial G}{\partial \nu} \kappa(\theta - \nu_t) + \frac{\partial G}{\partial \nu} \nu_t + \frac{1}{2} \frac{\partial^2 G}{\partial \nu^2} \nu_t \sigma^2 \right] dt + \frac{\partial G}{\partial \nu} \sigma \nu_t dW_t \] (3.3.11)

From equation (3.3.9), the change in portfolio value in a small time \( dt \) is

\[ d\Pi_t = adF + dG \] (3.3.12)

Substituting equations (3.3.4), (3.3.3) and (3.3.11) in (3.3.12) and simplifying we obtain

\[ d\Pi_t = \alpha \left( \frac{\partial F}{\partial \nu} \sigma \nu_t dW_t \right) + \left[ \frac{\partial G}{\partial t} + \frac{\partial G}{\partial \nu} \kappa(\theta - \nu_t) + \frac{\partial G}{\partial \nu} \nu_t + \frac{1}{2} \frac{\partial^2 G}{\partial \nu^2} \nu_t \sigma^2 \right] dt + \frac{\partial G}{\partial \nu} \sigma \nu_t dW_t \] (3.3.13)

If we choose \( \alpha = \frac{-\partial G}{\partial \nu} / \frac{\partial F}{\partial \nu} \) then the stochastic component in the portfolio vanishes and equation (3.3.13) simplifies to

\[ d\Pi_t = \left[ \frac{\partial G}{\partial t} + \frac{\partial G}{\partial \nu} \kappa(\theta - \nu_t) + \frac{\partial G}{\partial \nu} \nu_t + \frac{1}{2} \frac{\partial^2 G}{\partial \nu^2} \nu_t \sigma^2 \right] dt \] (3.3.14)

Since the portfolio \( \Pi_t \) is riskless, it should earn the risk free rate of return, and so

\[ \left[ \frac{\partial G}{\partial t} + \frac{\partial G}{\partial \nu} \kappa(\theta - \nu_t) + \frac{\partial G}{\partial \nu} \nu_t + \frac{1}{2} \frac{\partial^2 G}{\partial \nu^2} \nu_t \sigma^2 \right] dt = rGdt \] (3.3.15)

which can be rewritten as

\[ \frac{\partial G}{\partial t} + \frac{\partial G}{\partial \nu} \kappa(\theta - \nu_t) + \frac{\partial G}{\partial \nu} \nu_t + \frac{1}{2} \frac{\partial^2 G}{\partial \nu^2} \nu_t \sigma^2 - rG = 0 \] (3.3.16)

We solve the partial differential equation (3.3.16) in the region: \( 0 \leq t \leq T, I_{\min} \leq \nu \leq \nu_{\max} \) with the boundary conditions (3.3.6).

We compute the price of variance call and variance put options of maturity one year for different strikes, \( K \), equal to at-the-money strike, \( K_{\text{at}}^* \), and in the money and out of

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the money strikes. The at the money strike is \( K = (13.261\%)^2 = 0.017585 \) from Table 3.2. We use the Heston stochastic volatility parameters in Table 3.1. The other strikes are given in Table 3.3. We solve the partial differential equation on a three dimensional grid with 400 points each in the \( V \) and \( I \) directions and 2000 intervals in the \( t \) direction.

We assume a notional \( N = \$1000 \) in our calculations. Option prices are given in Table 3.3. When the call and put options are both at-the-money, their prices are the same due to the put-call parity relationship:

\[
C_t - P_t = X_t^T - Ke^{-r(T-t)}
\]  

(3.3.17)

Table 3.3: Prices of variance call and put options in the Heston stochastic volatility model

<table>
<thead>
<tr>
<th>Strike K (%)^2</th>
<th>Call ($)</th>
<th>Put ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.272</td>
<td>7.127</td>
<td>0.314</td>
</tr>
<tr>
<td>11.095</td>
<td>5.575</td>
<td>0.465</td>
</tr>
<tr>
<td>12.581</td>
<td>3.101</td>
<td>1.398</td>
</tr>
<tr>
<td>13.261</td>
<td>2.220</td>
<td>2.220</td>
</tr>
<tr>
<td>14.527</td>
<td>1.068</td>
<td>4.475</td>
</tr>
<tr>
<td>15.691</td>
<td>0.481</td>
<td>7.295</td>
</tr>
</tbody>
</table>

These prices are for maturity one year and the Heston stochastic volatility parameters given in Table 3.1.

3.4 Risk Management Parameters of Volatility Derivatives

In this section we compute greeks of variance and volatility swaps using partial differential equations and discuss properties of the greeks. These greeks are required for hedging the volatility derivatives. We need delta of volatility derivatives to dynamically hedge
the volatility swaps with variance swaps as explained in section 3.5.2. Other greeks are useful to understand sensitivity of different parameters of the Heston stochastic volatility model in pricing of volatility derivatives.

### 3.4.1 Delta of Volatility Derivatives

We define the delta of variance and volatility swaps as the first order variation in the fair strike with respect to the variance $v_t$. Thus, the delta of the variance swap is

$$
\delta_t \equiv \frac{\partial X^T}{\partial v_t}
$$

(3.4.1)

and similarly the delta for a volatility swap. We can compute the delta of a variance swap analytically using equation (3.2.7) to get

$$
\delta_t = \frac{\partial X^T}{\partial v_t} = \frac{1 - e^{-\kappa(T-t)}}{\kappa}
$$

(3.4.2)

![Graphs showing sensitivity and difference in delta of variance and volatility strikes](image)

**Figure 3.2:** The left plot shows the sensitivity of fair variance strike (3.4.3) and fair volatility strike with initial variance versus initial volatility. The right plot shows the difference in the delta of fair variance strike and fair volatility strike.

The delta of variance swap is constant and positive since the payoff of the variance swap is a linearly increasing function of realized variance. The delta of the variance swap approaches zero as time to maturity decreases since at maturity the payoff of the var-
The variance swap is independent of the initial variance. We compute the delta of the volatility swap numerically using first order finite differences. The left plot in Figure 3.2 shows the delta of variance and volatility swaps versus initial volatility.

To make variance and volatility swap deltas comparable, note that:

\[
\frac{\partial \sqrt{X_0^T}}{\partial v_0} = \frac{1}{2\sqrt{X_0^T}} \frac{\partial X_0^T}{\partial v_0} = \frac{1}{2\sqrt{K_{var}^*}} \frac{\partial X_0^T}{\partial v_0}
\]

(3.4.3)

and in Figure 3.2 we plot \( \frac{\partial \sqrt{X_0^T}}{\partial v_0} \) as the variance swap delta and \( \frac{\partial \sqrt{X_0^T}}{\partial \sigma} \) as the volatility swap delta. We computed volatility swap delta numerically using first order finite difference and we used (3.4.2) and (3.4.3) to compute \( \frac{\partial \sqrt{X_0^T}}{\partial v_0} \). For all greeks in the following subsections we plot slight variations of fair variance strike sensitivities to make sensitivities of fair variance and fair volatility strikes comparable.

It can be seen from the Figure 3.2 that the delta of fair variance strike, \( \frac{\partial \sqrt{X_0^T}}{\partial v_0} \), is 60.2% = 0.602. We compute the approximate change in the fair variance strike if initial volatility changes from 10.1% to 11% as follows. The change in initial volatility of 10.1% to 11% implies change in initial variance from 0.010201 to 0.0121 or \( \Delta v_0 = 0.001899 \). The change in the fair variance strike is \( \partial X_0^T \approx 2 \sqrt{K_{var}^*} \cdot \Delta v_0 \cdot 0.602 \approx 0.0003 \). This implies the fair variance strike changes from \( (13.261\%)^2 = 0.017585 \) to 0.017885 = \( (13.375\%)^2 \).

As can be seen from the Figure 3.2, the delta of the fair volatility strike is a positive and decreasing function of variance and volatility. Since the volatility swap payoff is a concave function of realized variance, its delta decreases with initial variance and volatility. The right plot in Figure 3.2 shows the difference in the deltas of variance and volatility swaps.
versus initial volatility.

Next we define the sensitivities of strikes with respect to the parameters of the model.

### 3.4.2 Volatility Derivatives: \( \hat{\kappa} \)

We define \( \hat{\kappa} \) as first order variation in fair strikes with respect to the mean reversion speed, \( \kappa \). For variance swaps it is defined as:

\[
\hat{\kappa} \equiv \frac{\partial X^T_I}{\partial \kappa}
\]

(3.4.4)

Using equation (3.2.7) we get

\[
\hat{\kappa} = \frac{\partial X^T_I}{\partial \kappa} = (v_t - \theta) \left( \frac{(T - t)e^{-\kappa(T - t)}}{kT} - \frac{1 - e^{-\kappa(T - t)}}{\kappa^2 T} \right)
\]

(3.4.5)

Observe that \( \hat{\kappa} \) approaches zero as time to maturity decreases since at maturity the realized variance is fixed so all the sensitivities must approach zero. We compute the \( \hat{\kappa} \) of the volatility swap by differentiating the partial differential equation (3.3.4) with respect to the parameter \( \kappa \):

\[
\frac{\partial \hat{\kappa}}{\partial t} + \frac{\partial \hat{\kappa}}{\partial v}(\theta - v)\kappa + \frac{\partial F}{\partial v}(-\kappa) + \frac{1}{2} \frac{\partial^2 \kappa}{\partial v^2} v \sigma^2 = 0
\]

(3.4.6)

We solve this partial differential equation in the same domain \( 0 \leq t \leq T, I_{\text{min}} \leq I \leq I_{\text{max}}, V_{\text{min}} \leq V \leq V_{\text{max}} \) with the boundary conditions:

\[
\hat{\kappa}_{|t=T} = 0
\]

(3.4.7)

\[
\frac{\partial^2 \hat{\kappa}}{\partial I^2} \bigg|_{(I=I_{\text{max}}, I_{\text{min}})} = 0 \quad \frac{\partial^2 \hat{\kappa}}{\partial v^2} \bigg|_{(V=V_{\text{max}}, V_{\text{min}})} = 0
\]

(3.4.8)
The left plot in Figure 3.3 shows the sensitivity of the fair strikes with mean reversion speed $\kappa$ versus initial volatility. It represents the change in the fair strike as mean reversion speed, $\kappa$, changes for a given level of initial volatility. Again, we have plotted the following quantity for variance strike sensitivity.

$$\frac{\partial \sqrt{X_0^T}}{\partial \kappa} = \frac{1}{2\sqrt{X_0^T}} \frac{\partial X_0^T}{\partial \kappa} = \frac{1}{2\sqrt{K_{\text{var}^*}}} \frac{\partial X_0^T}{\partial \kappa}$$  \hspace{1cm} (3.4.9)

We plot $\kappa = \frac{\partial \sqrt{X_0^T}}{\partial \kappa}$ for the volatility swap which we compute by solving the PDE (3.4.6).

The fair variance strike sensitivity, $\frac{\partial \sqrt{X_0^T}}{\partial \kappa}$, to mean reversion speed $\kappa$ is approximately 0.08% = 0.0008 at an initial volatility of $\sqrt{v_0} = 10.10\%$. We compute the approximate change in the fair variance strike if the mean reversion speed, $\kappa$, changes from its initial level 6.21 to 7.21 at an initial volatility, 10.1%, as follows. The change in the fair variance strike is $\partial X_0^T \approx 2 \sqrt{K_{\text{var}^*}} \partial \kappa \approx 0.0008 \approx 0.0002$. This implies the fair variance strike changes from $(13.261\%)^2 = 0.017585$ to $0.017785 = (13.345\%)^2$. From the graphs it can be seen that sensitivity changes sign from positive to negative as initial variance, $v_0$, or volatility $\sqrt{v_0}$ increases in both cases. It changes sign at the long run mean variance.
\( \theta \). When initial variance is lower than the long run mean variance \( \theta \), increasing the mean reverting speed will result in increase in the variance level, the realized variance will be higher and hence, higher fair variance strike, \( X_0^T = K_{\text{var}} \), which implies positive \( \kappa \). Similar arguments apply when initial variance is greater than the long run mean variance. The right plot in Figure 3.3 shows the difference in the sensitivity of the fair variance strike and the fair volatility strike to the mean reversion speed \( \kappa \) versus initial volatility.

### 3.4.3 Volatility Derivatives: \( \dot{\theta} \)

We define \( \dot{\theta} \) as the first order variation in the fair strike with respect to the long run mean variance, \( \theta \). For variance swaps it is defined as:

\[
\dot{\theta} = \frac{\partial X_0^T}{\partial \theta}
\]

Using equation (3.2.7) we compute the \( \dot{\theta} \) of the variance swap and get

\[
\dot{\theta} = \frac{\partial X_0^T}{\partial \theta} = \frac{T - t}{T} \frac{1 - e^{-\kappa(T-t)}}{\kappa T}
\]

(3.4.11)

The fair variance strike sensitivity to the long run mean variance is constant and positive since the realized variance increases as the long run mean variance increases. We compute the \( \dot{\theta} \) of the volatility swap by differentiating the partial differential equation (3.3.4) with respect to the parameter \( \theta \):

\[
\frac{\partial \dot{\theta}}{\partial t} + \frac{\partial \dot{\theta}}{\partial \nu}(\theta - \nu)\kappa + \frac{\partial F}{\partial \nu} \kappa + \frac{1}{2} \frac{\partial^2 \dot{\theta}}{\partial \nu^2} \nu \sigma^2 = 0
\]

(3.4.12)

We solve this partial differential equation in the same domain \( 0 \leq t \leq T, I_{\text{min}} \leq I \leq I_{\text{max}}, V_{\text{min}} \leq V \leq V_{\text{max}} \) with the boundary conditions.

\[
\dot{\theta}|_{(t=T)} = 0
\]

(3.4.13)
\[
\frac{\partial^2 \hat{\theta}}{\partial \theta^2} (t = t_{\text{max}}, J_{\text{min}}) = 0 \quad \frac{\partial^2 \hat{\theta}}{\partial \psi^2} (V = V_{\text{max}}, V_{\text{min}}) = 0 \quad (3.4.14)
\]

Figure 3.4: The left plot shows the sensitivity \( \hat{\theta} \) of fair variance strike (3.4.15) and fair volatility strike (3.4.12) with long run mean variance \( \theta \) (3.4.10) versus initial volatility. The right plot shows the difference between the two sensitivities.

The left plot in Figure 3.4 shows the sensitivity of the strikes with long run mean variance versus initial volatility. It represents change in the fair strike as long run mean variance, \( \theta \), changes for a given level of initial volatility. Again, we have plotted the following quantity for variance strike sensitivity.

\[
\frac{\partial \sqrt{X_T^2}}{\partial \theta} = \frac{1}{2 \sqrt{X_T^2}} \frac{\partial X_T^2}{\partial \theta} = \frac{1}{2 \sqrt{K_{\text{var}}}} \frac{\partial X_T^2}{\partial \theta} \quad (3.4.15)
\]

For the fair volatility strike sensitivity we have plotted \( \hat{\theta} = \frac{\partial \psi}{\partial \theta} \) which we compute by solving the PDE (3.4.12). It can be seen from the Figure 3.4 that the fair variance strike sensitivity, \( \frac{\partial \sqrt{X_T^2}}{\partial \theta} \), is approximately 316\% = 3.16 at an initial volatility of 10.1\%. We compute the approximate change in the fair variance strike if long run mean variance, \( \theta \), changes from its value, 0.019 to 0.021 at an initial volatility 10.1\% as follows. The change in the fair variance strike is \( \partial X_T^2 \approx 2 \cdot \sqrt{K_{\text{var}}} \cdot \partial \theta \cdot 3.16 \approx 0.0017 \). This implies the fair variance strike changes from \((13.261\%)^2 = 0.017585\) to \(0.01926 = (13.879\%)^2\).
The variance swap strike sensitivity to theta is constant at all variance levels. For volatility swaps, \( \theta \) is positive, implying higher the long run variance level the higher the fair volatility strike. The \( \theta \) of the volatility swap is a decreasing function of initial variance. The right plot in Figure 3.4 shows the difference in the sensitivity of the fair variance strike and the fair volatility strike to the long run mean variance \( \theta \) versus initial volatility.

### 3.4.4 Volatility Derivatives: \( \hat{\sigma} \)

We define \( \hat{\sigma} \) as the first order variation in fair strikes with respect to the volatility of variance parameter, \( \sigma_v \). For variance swaps it is defined as:

\[
\hat{\sigma}_v = \frac{\partial X^F_t}{\partial \sigma_v} \tag{3.4.16}
\]

Using equation (3.2.7) we find that the fair variance strike is independent of the volatility of variance. We compute the \( \hat{\sigma}_v \) of the volatility swap by differentiating the partial differential equation (3.3.4) with respect to the parameter \( \sigma_v \):

\[
\frac{\partial \hat{\sigma}_v}{\partial t} + \frac{\partial \hat{\sigma}_v}{\partial \nu} (\theta - \nu) \kappa + \frac{1}{2} \frac{\partial^2 \sigma_v}{\partial \nu^2} \nu^2 \sigma_v^2 + \frac{\partial^2 F}{\partial \nu^2} \nu \sigma_v = 0 \tag{3.4.17}
\]

We solve this partial differential equation in the domain \( 0 < t < T, I_{\text{min}} < I < I_{\text{max}}, V_{\text{min}} < V < V_{\text{max}} \) with the boundary conditions:

\[
\hat{\sigma}_v |_{(t=T)} = 0 \tag{3.4.18}
\]

\[
\left. \frac{\partial^2 \hat{\sigma}_v}{\partial T^2} \right|_{(t=I_{\text{max}}, I_{\text{min}})} = 0 \quad \left. \frac{\partial^2 \hat{\sigma}_v}{\partial \nu^2} \right|_{(\nu=V_{\text{max}}, \nu_{\text{min}})} = 0 \tag{3.4.19}
\]
Figure 3.5: The left plot shows the sensitivity $\sigma_v$ of fair variance strike and fair volatility strike with volatility of variance $\sigma_v$ (3.4.16) versus initial volatility. The right plot shows the difference between the two sensitivities.

Figure 3.5 shows the sensitivity of fair strikes to the volatility of variance parameter versus initial volatility. As known from the theoretical results (see equation 3.2.7) the fair variance strike is independent of volatility of variance. The fair volatility strike has a negative dependence on the volatility of variance. This implies if we increase the volatility of variance parameter, $\sigma_v$, in the Heston stochastic volatility model the fair volatility strike will decrease. Since the fair volatility strike is a concave function of the realized variance, the fair volatility strike decreases with the increase in the volatility of variance parameter $\sigma_v$. For convex payoff functions, e.g, variance call and put options, the sensitivity with respect to the volatility of variance is positive.

Thus all the greeks can be computed by either solving the pricing partial differential equation (3.3.4) and using finite difference approximations (for delta) or by solving the other partial differential equations and appropriate boundary conditions.
3.5 Hedging Volatility Derivatives

In this section we present an approach to hedge volatility derivatives using variance swaps. Other authors (Demeterfi et al. 1999) have shown that variance swaps can be replicated using an infinite number of European call and put options. We formulate an optimization problem to find the best portfolio of European call and put options to closely replicate a variance swap for a given finite number of options. We also analyze how replication error decreases as we increase the number of European call and put options in the replicating portfolio. Then we present an approach to dynamically hedge volatility swaps using variance swaps and a finite number of European call and put options.

3.5.1 Replicating Variance Swaps

In this section we formulate an optimization problem for replicating a variance swap using a static portfolio consisting of a finite number of European call and put options. Applying Itô's lemma to the stock price diffusion (??, ??) we can express the realized variance as

\[ V_c(0,T) = \frac{1}{T} \int_0^T v_t dt = \frac{2}{T} \left( \int_0^T \frac{dS_t}{S_t} - \ln \frac{S_T}{S_0} \right) \]  

(3.5.1)

This result holds in both the Heston stochastic volatility model and the Black-Scholes model. From (3.5.1) the realized variance can be replicated by shorting a log contract and dynamically holding \( 1/S_t \) shares of stock until the maturity of the contract. Next we review how to replicate a European style payoff, in particular a log contract payoff (Neuberger 1994) statically using call and put options. Let \( f \) be a twice continuously...
differentiable function which represents the payoff of a European style path independent derivative security. It can be expressed as (Breeden and Litzenberger 1978)

\[
f(S_T) = f(x) + f'(x)(S_T - x) + \int_x^\infty f''(K)(S_T - K)^+ dK + \int_0^x f''(K)(K - S_T)^+ dK
\]  
(3.5.2)

Thus, the payoff function \( f \) can be replicated (Carr, Ellis and Gupta 1998) by holding positions in a zero coupon bond with face value \( f(x) \), a forward contract with strike \( x \), and call and put options of all strikes using equation (3.5.2). The time zero value of the claim can be expressed in terms of the European call \( C_0(K) \) and put \( P_0(K) \) prices of maturity \( T \):

\[
V_0 = E_0^Q[e^{-rT}f(S_T)]
\]
\[
= e^{-rT}f(x) + f'(x)[C_0(x) - P_0(x)] + \int_x^\infty f''(K)C_0(K)dK + \int_0^x f''(K)P_0(K)dK
\]  
(3.5.3)

Now, let \( f(S_T) = \ln(S_T/S_0) \) and \( x = S_0 \), and substitute in equation (3.5.2) to get:

\[
\ln \left( \frac{S_T}{S_0} \right) = \frac{S_T - S_0}{S_0} - \int_{S_0}^\infty \frac{1}{K^2} (S_T - K)^+ dK - \int_0^{S_0} \frac{1}{K^2} (K - S_T)^+ dK
\]  
(3.5.4)

Substituting this in equation (3.5.1), we get

\[
V_c(0, T) = \frac{2}{T} \int_0^T \frac{dS_t}{S_t} \left[ \frac{S_T - S_0}{S_0} - \int_{S_0}^\infty \frac{1}{K^2} C_T(K)dK + \int_0^{S_0} \frac{1}{K^2} P_T(K)dK \right]
\]  
(3.5.5)

Thus, the variable leg of the variance swap can be replicated (Demeterfi et al. 1999) by a portfolio having a short position in a forward contract struck at \( S_0 \), a long position
in $1/K^2$ put options of strike $K$, for all strikes from 0 to $S_0$, a long position in $1/K^2$ call options for all strikes from $S_0$ to $\infty$ and payoffs from a dynamic trading strategy which instantaneously holds $1/S_t$ shares of stock worth $\$1$ in the portfolio. In particular, equation (3.5.5) shows that continuously realized variance can be replicated in both the Black-Scholes and Heston stochastic volatility models.

Thus, to replicate the variance swap we need a short position in a log contract, and this can be replicated using call and put options of all strikes (3.5.4). In practice we can form a portfolio of only a finite number of options with a limited set of strikes. We analyze how well we can replicate the log contract (and variance swaps) with a finite number of options.

Suppose we want to replicate the log contract with $n_p$ put options and $n_c$ call options of various strikes and common maturity $T$. We define the portfolio of a log contract and a forward contract as portfolio $B$. It's payoff at maturity $T$ when the stock price is $S_T$ is given by

$$V_B(S_T) = \frac{S_T - S_0}{S_0} - \ln \left( \frac{S_T}{S_0} \right)$$

(3.5.6)

Let $w^c_i$ represent the number of call options having strike $K^c_i$ and $w^p_i$ the number of put options having strike $K^p_i$ in portfolio $A$. The payoff of portfolio $A$ at maturity $T$ when the stock price is $S_T$ is given by

$$V_A(S_T) = \sum_{i=1}^{n_p} w^p_i (K^p_i - S_T)^+ + \sum_{i=1}^{n_c} w^c_i (S_T - K^c_i)^+$$

(3.5.7)
where \((K_f^p - S_T)^+\) is the payoff of the put option and \((S_T - K_f^c)^+\) is the payoff of the call option. If we include options of all strikes in portfolio \(A\), then portfolio \(A\) exactly replicates portfolio \(B\) from equation (3.5.4). The quantities of options in portfolio \(A\) are unknown and we compute these values using optimization so that the payoff of portfolios \(A\) and \(B\) match as closely as possible for fixed number \((n_c \text{ and } n_p)\) of call and put options.

To compute the number of options in portfolio \(A\) which replicates portfolio \(B\) we solve the optimization problem:

\[
\begin{align*}
(P1) \quad & \min_{w^c, w^p} \quad \sum_{j=1}^{n} \left( V_A(S_T^j) - V_B(S_T^j) \right)^2 \\
& \text{s.t.} \quad \sum_{i=1}^{n_p} w_i^p P_0(S_0, K_f^p) + \sum_{i=1}^{n_c} w_i^c C_0(S_T, K_f^c) = P_B(S_0)
\end{align*}
\]

In the problem \((P1)\), the decision variables are vectors \(w^p, w^c\) of sizes \(n_p\) and \(n_c\), respectively, which represent the quantities of call and put options in the portfolio. The value \(V_B(S_T^j)\) is the payoff of the portfolio of log contract and forward contract when the terminal stock price is \(S_T^j\). The value \(V_A(S_T^j)\) is the payoff of the portfolio of call and put options when the terminal stock price is \(S_T^j\). The value \(P_B(S_0)\) represents the initial value of the portfolio of the log contract and forward contract. The value \(P_0(S_0, K_f^p)\) represents the initial value of the put option with strike \(K_f^p\) and \(C_0(S_0, K_f^c)\) represents the initial value of the call option with strike \(K_f^c\). We choose the stochastic volatility model parameters and Black-Scholes parameters in Table 3.1, and set the maturity value to be one year. We choose the strikes of call and put options to be equally distributed in a three

\footnote{We have chosen portfolio \(A\) to hold both call and put options. We can also choose this portfolio to consist of call options or put options only as we can replace the put options by call options and stock using put-call parity.}
standard deviation range defines as follows. For $S_0 = 100$ we choose the put strikes to be equally distributed between $S_0 e^{(rT - \frac{1}{2} \sigma^2 T - 3\sigma \sqrt{T})} = 68$ and $100$. Here, we have chosen $\sigma = \sqrt{\frac{1}{2} \sigma^2 T} = 13.26\%$. Thus, for $n_p$ put options in the optimization problem, $(P1)$, the put strikes are $K_i^P = 68 + (i - 1)(100 - 68)/(n_p - 1), i = 1, ..., n_p$. Similarly we choose call strikes to be equally distributed between $100$ and $S_0 e^{(rT - \frac{1}{2} \sigma^2 T + 3\sigma \sqrt{T})} = 152$. The call strikes are $K_i^C = 100 + (i - 1)(152 - 100)/(n_c - 1), i = 1, ..., n_c$. The objective function in the problem $(P1)$ minimizes the sum of squared differences in two portfolio payoffs at maturity $T$ over $n$ scenarios. In a similar manner, we choose the $n$ scenarios within a four standard deviation range. In particular, we take $n = 200$ scenarios of stock prices, $S^j_T = 60 + (j - 1)(173 - 60)/(n - 1), j = 1, ..., n$. The constraint enforces the initial values of both portfolios to be equal to each other. Thus the portfolio optimization problem $(P1)$ minimizes the sum of squared differences in two portfolio’s payoffs given the constraint that initial value of the two portfolios must be equal.

To compare the performance of the replicating portfolio of call and put options we compute three types of error.

$$e_1 : \quad \frac{E[V_A(S_T) - V_B(S_T)]}{P_B(S_0)}$$

$$e_2 : \quad \frac{\sqrt{E(V_A(S_T) - V_B(S_T))^2}}{P_B(S_0)}$$

$$e_\infty : \quad \frac{\max |V_A(S_T) - V_B(S_T)|}{P_B(S_0)}$$

(3.5.8)  

(3.5.9)  

(3.5.10)

where $P_B(S_0)$ represents the value of portfolio $B$, given in (3.5.6), at $t = 0$ when the
stock price is $S_0$. The expectation is under the real world probability measure. The error measures $e_1$, $e_2$ and $e_\infty$ are $L_1$-norm, $L_2$-norm and $L_\infty$-norm, respectively, normalized by the initial value of portfolio $B$ respectively. In the optimization problem (PI), the objective function is unweighted while the error measures $e_1$ and $e_2$ are weighted. We used the unweighted objective function to ensure the payoffs of portfolio’s $A$ and $B$ are close in the worst case scenario as well. We solve (PI) by forming the Lagrangian and solving the resulting system of linear equations. We compute these error measures in the Black-Scholes real world probability measure and the Heston stochastic volatility real world probability measure. The value of the drift in the real world probability measure from Table 3.1 is $\mu = 7\%$. For computing these error measures we simulated 10,000 terminal stock prices under the Black-Scholes and stochastic volatility real world probability measures.

Table 3.4: Error in static replication of log contract with a finite number of options

<table>
<thead>
<tr>
<th>Number of Options</th>
<th>Black-Scholes Error $e_1$</th>
<th>Black-Scholes Error $e_2$</th>
<th>Black-Scholes Error $e_\infty$</th>
<th>Stochastic volatility Error $e_1$</th>
<th>Stochastic volatility Error $e_2$</th>
<th>Stochastic volatility Error $e_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.371</td>
<td>0.469</td>
<td>1.541</td>
<td>0.384</td>
<td>0.483</td>
<td>4.919</td>
</tr>
<tr>
<td>8</td>
<td>0.088</td>
<td>0.112</td>
<td>0.275</td>
<td>0.091</td>
<td>0.126</td>
<td>4.158</td>
</tr>
<tr>
<td>16</td>
<td>0.018</td>
<td>0.024</td>
<td>0.106</td>
<td>0.019</td>
<td>0.026</td>
<td>3.340</td>
</tr>
<tr>
<td>32</td>
<td>0.004</td>
<td>0.006</td>
<td>0.070</td>
<td>0.005</td>
<td>0.006</td>
<td>2.923</td>
</tr>
</tbody>
</table>

The first column shows the number of options used in replicating a log contract of maturity one year. The second column shows the error measure defined in (3.5.8) and third column shows the error measure defined in (3.5.9). The fourth column shows the error measure defined in (3.5.10). The fifth, sixth and seventh column shows the respective error measures in the stochastic volatility model. These errors are computed for an interval of one year.

Table 3.4 shows the log contract replication error versus the number of options in the Black-Scholes and stochastic volatility models. The replication of the log contract
is static which means no rebalancing is required. The results show all error measures decrease as we increase the number of options in the replicating portfolio. With 16 options (8 puts and 8 calls in the option portfolio) the mean absolute replication error for the Black-Scholes model is about 1.8% of the initial value of the portfolio and 1.9% for the stochastic volatility model. Figure 3.6 shows the replication errors and number of options on log scale in the Black-Scholes and stochastic volatility models. These results illustrate that the error measures $e_1$ and $e_2$ converge quadratically with number of options.

Next we analyze the dynamic replication of a variance swap using a finite number of options. There are two types of errors in replicating a variance swap with a finite number of options. The first type of error comes from replicating a log contract by a finite number of options. The second type of error comes from the discrete rebalancing of $1/S_t$ shares of stock worth $\$1$ in the portfolio. If we do continuous rebalancing in computing

---

**Figure 3.6**: Replication error in log contract with number of options. These figures show error measures defined in (3.5.8) and (3.5.9) in replicating portfolio to replicate a log contract with finite number of options in the Black-Scholes model and the stochastic volatility model. These figures are plotted on a log-log scale. These errors are computed for an interval of one year.
Table 3.5: Error in dynamic replication of variance swap with a finite number of options

<table>
<thead>
<tr>
<th>Number of Options</th>
<th>Black-Scholes</th>
<th>Stochastic volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Error $e_1$</td>
<td>Error $e_2$</td>
</tr>
<tr>
<td>4</td>
<td>0.381</td>
<td>0.483</td>
</tr>
<tr>
<td>8</td>
<td>0.091</td>
<td>0.115</td>
</tr>
<tr>
<td>16</td>
<td>0.019</td>
<td>0.024</td>
</tr>
<tr>
<td>32</td>
<td>0.004</td>
<td>0.006</td>
</tr>
</tbody>
</table>

The first column shows number of options used in replicating a continuous variance swap of maturity one year. The second column shows the error measure defined in (3.5.8) (divided by the variance strike in this case to normalize error measures) in the Black-Scholes model and third column shows the error measure defined in (3.5.9). The fourth column shows the error measure defined in (3.5.10). The fifth, sixth and seventh column shows the respective error measures in the stochastic volatility model. These error measures are computed for an interval of one year for a continuous variance swap where rebalancing is done continuously.

Figure 3.7: Replication error in a variance swap with number of options. These figures show error measures defined in (3.5.8) and (3.5.9) in replicating portfolio to replicate a variance swap with a finite number of options in the Black-Scholes model and the stochastic volatility model. The error measures are computed for an interval of one year for a continuous variance swap where rebalancing is done continuously. These figures are plotted on a log-log scale.

payoffs from the dynamic trading strategy then the only replication error is from a finite number of options. These results are given in Table 3.5 and Figure 3.7. In these results we are replicating a continuous variance swap (3.5.5) of maturity one year with a finite number of options and continuous rebalancing to get payoffs from the dynamic trading strategy. We use the same static portfolio of options as determined from optimization (P1) to replicate a variance swap. We compute the error measures $e_1$, $e_2$ and $e_{\infty}$ as
defined in equation (3.5.8), (3.5.9) and (3.5.10), respectively, in the Black-Scholes and stochastic volatility models. In the variance swap replication case, the portfolio $A$ payoff in the error measures is the realized variance payoff at maturity and the portfolio $B$ payoff is composed of the payoffs from the options portfolio, short forward contract (3.5.5) and from a dynamic trading strategy with continuous rebalancing. To normalize the results, we set $P_B(S_0)$ to $K_{var}$ (3.2.7) in the three error measures. To compute the error measures we simulated 10000 stock price paths under real world measure Black-Scholes and stochastic volatility models. The results show all three error measures decrease as we increase the number of options in the replicating portfolio. Figure 3.7 shows replication errors and number of options on log scale with different number of options in the Black-Scholes model and stochastic volatility model. These results show error measures $e_1$ and $e_2$ converge quadratically to zero as the number of option increases. The results in Table 3.4 of replicating a log contract using a finite number of options and in Table 3.5 to replicate a continuous variance swap using same portfolio of options and a continuous dynamic trading strategy are very similar. This implies that payoffs from a dynamic trading strategy doesn't affect replication of a variance swap significantly if rebalancing is done continuously.

Next we analyze the effect of a discrete rebalancing interval in computing payoffs from the dynamic trading strategy in replicating a variance swap. Here, we assume that the sampling interval, $n$, in computing the realized variance in a variance swap is same as the rebalancing interval in the dynamic trading strategy. Thus we analyze errors in
Table 3.6: Error in dynamic replication of a discrete variance swap with a finite number of options

<table>
<thead>
<tr>
<th>Number of Options</th>
<th>Black-Scholes</th>
<th>Stochastic volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Error $e_1$</td>
<td>Error $e_2$</td>
</tr>
<tr>
<td>4</td>
<td>0.381</td>
<td>0.485</td>
</tr>
<tr>
<td>8</td>
<td>0.090</td>
<td>0.114</td>
</tr>
<tr>
<td>16</td>
<td>0.019</td>
<td>0.024</td>
</tr>
<tr>
<td>32</td>
<td>0.004</td>
<td>0.005</td>
</tr>
</tbody>
</table>

The first column shows number of options used in replicating a discrete variance swap of maturity one year with daily sampling. The rebalancing interval in computing payoffs from the dynamic trading in underlying stock is daily as well. The second column shows the error measure defined in (3.5.8) (divided by the variance strike in this case to normalize error measures) in the Black-Scholes model and third column shows the error measure defined in (3.5.9). The fourth column shows the error measure defined in (3.5.10). The fifth, sixth and seventh column shows the respective error measures in the stochastic volatility model.

replicating a discrete variance swap with a dynamic trading strategy and a finite number of call and put options. Table 3.6 shows the results when sampling in the variance swap is done daily and dynamic trading in the stock is done once per day. The results in Table 3.5 and Table 3.6 show that the error measures are very similar in hedging a discrete variance swap and a continuous variance swap with options.

Next, we compare the performance of replicating portfolio from optimization (PI) with the replicating portfolio proposed in Demeterfi et al. (1999) (portfolio $D$ hereafter).

Demeterfi et al. (1999) show that the portfolio $B$ (3.5.6) can be replicated using a finite number of out-of-money put and call options. We briefly present the formula given in appendix A of Demeterfi et al. (1999).

Assuming call options of strikes $K^c_1 = S_0 < K^c_2 < K^c_3 < \ldots < K^c_{n_c}$ and put options with strikes $K^p_1 = S_0 > K^p_2 > K^p_3 > \ldots > K^p_{n_p}$ are available. Let $n^c_i$ represent the number
of call options having strike $K^c_i$ and $\bar{w}_i^p$ the number of put options having strike $K^p_i$ in portfolio $D$. The payoff of portfolio $D$ at maturity $T$ when the stock price is $S_T$ is given by

$$V_D(S_T) = \sum_{i=1}^{n_p} \bar{w}_i^p (K^p_i - S_T)^+ + \sum_{i=1}^{n_c} \bar{w}_i^c (S_T - K^c_i)^+$$

(3.5.11)

where $(K^p_i - S_T)^+$ is the payoff of the put option and $(S_T - K^c_i)^+$ is the payoff of the call option. This portfolio is similar to the portfolio $A$ except that number of call and put options in two portfolios are different. The number of call options $\bar{w}_i^c$ and put options $\bar{w}_i^p$ which replicates portfolio $B$ as given in Demeterfi et al. (1999) are:

$$\bar{w}_i^c = \frac{V_B(K^c_{i+1}) - V_B(K^c_i)}{K^c_{i+1} - K^c_i} - \sum_{j=1}^{i-1} \bar{w}_j^c$$

$$\bar{w}_i^p = \frac{V_B(K^p_{i+1}) - V_B(K^p_i)}{K^p_{i+1} - K^c_i} - \sum_{j=1}^{i-1} \bar{w}_j^p$$

(3.5.12)

The number of maximum call strike, $K^c_n$ and minimum put strike, $K^p_n$ are also given by the above equations but these numbers require $K^c_{n+1}$ and $K^p_{n+1}$ respectively. These numbers are chosen so that they lie outside the strikes range $(K^c_n, K^c_{n+1})$. In our numerical results we choose these to be $(-/+)$ four standard deviation range of terminal stock price respectively.

Table 3.7 shows the number of put and call options in replicating portfolio $B$ with portfolio $A$ and portfolio $D$ with four puts and four calls in options portfolio. The strikes in portfolio $A$ are chosen in the same way as described before and number of call and put options in portfolio $A$ are obtained by solving the optimization problem $P1$ as explained above. The number of puts and calls in portfolio $D$ are obtained from equation (3.5.12).
Table 3.7: Number of call and put options in replicating a variance swap with four call and four put options

<table>
<thead>
<tr>
<th>Option Type</th>
<th>Strike Value (K)</th>
<th>Number of options portfolio (D) ($10^{-4}$)</th>
<th>Number of options portfolio (A) ($10^{-4}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Put</td>
<td>100.00</td>
<td>5.60</td>
<td>3.45</td>
</tr>
<tr>
<td></td>
<td>89.58</td>
<td>13.07</td>
<td>15.88</td>
</tr>
<tr>
<td></td>
<td>79.17</td>
<td>16.77</td>
<td>15.87</td>
</tr>
<tr>
<td></td>
<td>68.75</td>
<td>30.16</td>
<td>20.93</td>
</tr>
<tr>
<td>Call</td>
<td>100.00</td>
<td>7.83</td>
<td>5.80</td>
</tr>
<tr>
<td></td>
<td>117.45</td>
<td>12.79</td>
<td>16.27</td>
</tr>
<tr>
<td></td>
<td>134.89</td>
<td>9.67</td>
<td>7.97</td>
</tr>
<tr>
<td></td>
<td>152.34</td>
<td>12.39</td>
<td>8.64</td>
</tr>
</tbody>
</table>

This table shows the number of put and call options in replicating a variance swap with four puts and four calls using optimization approach (3.5.7) and options portfolio $D$ (3.5.12). The first and second column show the option type and strike value respectively. The third column shows the number of options in portfolio $V_D$ and fourth column shows the number of options in portfolio $V_A$.

Figure 3.8: This figure shows the payoff of portfolio $B$ (3.5.6, 'true payoff'), payoff of portfolio $A$ (3.5.7, 'options portfolio $A$ payoff') and payoff of portfolio $D$ (3.5.12, 'options portfolio $D$ payoff') versus terminal stock price when there are four puts and four calls in options portfolio. The right plot shows the difference in portfolio payoffs (portfolio $A$ and portfolio $D$) from true payoff.

Figure 3.8 shows the payoff of these portfolios with terminal stock price. The left plot in figure 3.8 shows the payoff of portfolio $B$ (true payoff), payoff of options portfolio $D$ ('Options portfolio $D$ payoff') and payoff of options portfolio $A$ ('Options portfolio $A$ payoff'). The right plot in figure 3.8 shows the difference of options portfolio $A$ and options portfolio $D$ payoff from the true payoff. The right plot shows that the options portfolio $A$ payoff matches with true payoff better than the options portfolio $D$ payoff.

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The options portfolio $D$ always over approximates the payoff of portfolio $B$ and hence the initial cost of these two portfolios will be different. The constraint in the optimization problem $(P1)$ ensures that the initial cost of portfolio $A$ and $B$ are same. Table 3.8 shows these error measures in replicating a discrete variance swap with two different options portfolio ($A$ and $D$). As can be seen from the results, the portfolio obtained from the optimization approach is better than the portfolio $D$.

Table 3.8: Comparison of errors in dynamic replication of a discrete variance swap with a finite number of option portfolios $A$ and $D$

<table>
<thead>
<tr>
<th>Number of Options</th>
<th>Portfolio ($D$)</th>
<th>Portfolio ($A$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Error $e_1$</td>
<td>Error $e_2$</td>
</tr>
<tr>
<td>4</td>
<td>1.732</td>
<td>1.965</td>
</tr>
<tr>
<td>8</td>
<td>0.366</td>
<td>0.393</td>
</tr>
<tr>
<td>16</td>
<td>0.069</td>
<td>0.075</td>
</tr>
<tr>
<td>32</td>
<td>0.016</td>
<td>0.022</td>
</tr>
</tbody>
</table>

The first column shows number of options used in replicating a discrete variance swap of maturity one year with daily sampling. The rebalancing interval in computing payoffs from the dynamic trading in underlying stock is daily as well. The second column shows the error measure defined in (3.5.8) (divided by the variance strike in this case to normalize error measures) and third column shows the error measure defined in (3.5.9) in replicating variance swap using options portfolio $D$ (3.5.12). The fourth column shows the error measure defined in (3.5.10). The fifth, sixth and seventh column shows the respective error measures in replicating variance swap using options portfolio $A$ (3.5.7). These error measure are computed in the SV model.

Next, we compare the performance of replicating portfolio to replicate a variance swap from two approaches using historical options data. We have S&P500 (SPX) index option quotes from the Chicago Mercantile Exchange (CME) from 1985-2005. For a given day in each month there are options trading with 30 days to maturity. Using these options we compute the replicating portfolio of options to replicate a log contract using portfolio $A$ (3.5.7) and portfolio $D$ (3.5.12). For our numerical results we choose eight (four calls and
four puts) options and sixteen (eight calls and eight puts) options trading in the market.

The number of these options to replicate a log contract is determined from optimization approach (3.5.7) and options portfolio $D$ (3.5.11) in Demeterfi et al. (1999). Then we compute the realized variance over next 30 days from daily SPX index values and payoffs from dynamic trading strategy from holding $1/S_t$ shares of SPX index worth $\$1$ in the portfolio. The options payoff at expiry is determined for both option portfolios $A$ and $D$. Together these payoffs gives the hedging error from replicating a variance swap using options portfolio $A$ and options portfolio $D$. We compute these hedging errors for all months from 1985-2005. Again these results show that the options portfolio $A$ to hedge a variance swap is better than the options portfolio $D$.

Table 3.9: Historical performance of options portfolio $A$ and options portfolio $D$ in replicating a discrete variance swap.

<table>
<thead>
<tr>
<th>Number of Options</th>
<th>Portfolio ($D$)</th>
<th>Portfolio ($A$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Error $e_1$</td>
<td>Error $e_2$</td>
</tr>
<tr>
<td>8</td>
<td>0.188</td>
<td>0.301</td>
</tr>
<tr>
<td>16</td>
<td>0.069</td>
<td>0.172</td>
</tr>
</tbody>
</table>

The first column shows number of options used in replicating a discrete variance swap of maturity 30 days with daily sampling. The rebalancing interval in computing payoffs from the dynamic trading in underlying stock is daily as well. The second column shows the error measure defined in (3.5.8) (divided by the variance strike in this case to normalize error measures) and third column shows the error measure defined in (3.5.9) in replicating variance swap using options portfolio $D$ (3.5.12). The fifth and sixth shows the respective error measures in replicating variance swap using options portfolio $A$ (3.5.7).

3.5.2 Hedging Volatility Derivatives in the SV Model

In this section we present an approach to dynamically hedge volatility swaps using variance swaps in a stochastic volatility model. Suppose we take a long position in one unit of volatility swap at $t = 0$ of maturity $T$ with fair volatility strike, $K_{vol}^*$. The volatility...
swap is initially costless. At time \( t \), the value of volatility swap contract is

\[
P_t = E_t(e^{-r(T-t)}(\sqrt{V_c(0,T)} - K_{\text{vol}}^*)) = e^{-r(T-t)}(Y_t^T - K_{\text{vol}}^*) \tag{3.5.13}
\]

We assume the notional amount of swap to be $1. To hedge a long position in volatility swap at time \( t \), we construct a portfolio having one unit of volatility swap and \( \beta_t \) units of variance swaps. Thus the portfolio value at time \( t \) equals

\[
\Pi_t = E_t\left[e^{-r(T-t)}\left(\beta_t(V_c(0,T) - K_{\text{var}}^*)\right) + (\sqrt{V_c(0,T)} - K_{\text{vol}}^*)\right] = e^{-r(T-t)}\left(\beta_t(X_t^T - K_{\text{var}}^*) + (Y_t^T - K_{\text{vol}}^*)\right) \tag{3.5.14}
\]

The change in this portfolio in a small amount of time \( dt \) is given by

\[
d\Pi_t = r\Pi_t dt + e^{-r(T-t)} \left( \beta_t dX_t^T + dY_t^T \right)
\]

which can be written using equation (3.3.3) as

\[
d\Pi_t = r\Pi_t dt + e^{-r(T-t)} \left[ \beta_t \left( \frac{\partial X_t^T}{\partial t} \sigma_v \sqrt{v_t} dW_t^1 + \frac{\partial X_t^T}{\partial v} \kappa (\theta - v_t) dW_t^1 + \frac{\partial X_t^T}{\partial I} v_t + \frac{1}{2} \frac{\partial^2 X_t^T}{\partial v^2} v_t^2 \sigma_v^2 \right) \right. \right.
\]
\[
+ \left. \frac{\partial X_t^T}{\partial v} \sigma_v \sqrt{v_t} dW_t^1 \right] + \left[ \frac{\partial Y_t^T}{\partial t} \sigma_v \sqrt{v_t} dW_t^1 + \frac{\partial Y_t^T}{\partial v} \kappa (\theta - v_t) dW_t^1 + \frac{\partial Y_t^T}{\partial I} v_t + \frac{1}{2} \frac{\partial^2 Y_t^T}{\partial v^2} v_t^2 \sigma_v^2 \right] dt
\]
\[
+ \frac{\partial Y_t^T}{\partial v} \sigma_v \sqrt{v_t} dW_t^1
\]

Since the processes \( X_t^T \) and \( Y_t^T \) satisfy the pricing partial differential equation (3.3.4), the \( dt \) terms in the previous equation vanish. Hence the change in the portfolio value can be rewritten as:

\[
d\Pi_t = r\Pi_t dt + e^{-r(T-t)} \left[ \beta_t \frac{\partial X_t^T}{\partial v} \sigma_v \sqrt{v_t} dW_t^1 + \frac{\partial Y_t^T}{\partial v} \sigma_v \sqrt{v_t} dW_t^1 \right] \tag{3.5.15}
\]

We define \( \beta_t \) as the volatility swap hedge ratio:

\[
\beta_t = \frac{\frac{\partial Y_t^T}{\partial v}}{\frac{\partial X_t^T}{\partial v}} \tag{3.5.16}
\]
If we choose $\beta_t$ as in equation (3.5.16), the stochastic component of portfolio vanishes and the portfolio value is hedged. Thus for hedging a volatility swap we can take a short position in $\beta_t$ units of the variance swap and the portfolio value is hedged dynamically.

Next, we present numerical results for the volatility swap hedging performance. We compute the profit and loss of two different hedging strategies and compare with no hedging. The two hedge portfolios are: a portfolio containing one unit of volatility swap and $\beta_t$ (3.5.16) units of a variance swap and a portfolio containing one unit of volatility swap and a portfolio of European call and put options which replicates $\beta_t$ units of variance swap. We compute the portfolio of call and put options which replicates a variance swap as described in section 3.5.1.

No hedging: We price the variance and volatility swap of maturity one year using the partial differential equation described in section 3.3.1. We also compute the deltas at time zero of the variance and volatility swaps using (3.4.1). Together these give the hedge ratio at time zero. (3.5.16). We generate 4800 scenarios of the stock price and variance level at $t = 1/252$ years. The variance and volatility swaps are initially costless. The profit and loss of a long position in unhedged volatility swap at $t = 1/252$ years is equal to the price of volatility swap contract at time $t$:

$$P_t = E_t(e^{-r(T-t)}(\sqrt{V_t(0,T)} - K_{vol}^*)) = e^{-r(T-t)}(Y^T_t - K_{vol}^*)$$  

(3.5.17)

Hedging with variance swap: We form a portfolio containing one unit of a volatil-
ity swap and $\beta_0$ units of variance swaps. The value of this portfolio is zero at $t = 0$.

$$\Pi_0 = \beta_0(X_0^T - K_{var}^*) + (Y_0^T - K_{vol}^*)$$

where $\beta_0 = -\frac{\partial X_0^T}{\partial v}/\frac{\partial X_0^T}{\partial v}$ is the hedge ratio. The profit and loss of this hedged portfolio at $t = 1/252$ years is equal to the value of this portfolio at time $t$ is:

$$\Pi_t = E_t \left[ e^{-r(T-t)} \left( \beta_0(V_c(0, T) - K_{var}^*) + (\sqrt{V_c(0, T)} - K_{vol}^*) \right) \right]$$

$$= e^{-r(T-t)} \left( \beta_0(X_t^T - K_{var}^*) + (Y_t^T - K_{vol}^*) \right)$$

(3.5.18)

Table 3.10: Hedging volatility swap using variance swaps and a finite number of options

<table>
<thead>
<tr>
<th></th>
<th>Error $e_1$ Avg. (SE) (%)</th>
<th>Error $e_2$ Avg. (SE) (%)</th>
<th>Error $e_{\infty}$ Avg. (SE) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volswap Unhedge</td>
<td>5.293 (0.066)</td>
<td>6.646 (1.044)</td>
<td>21.811 (0.745)</td>
</tr>
<tr>
<td>Varswap Hedge</td>
<td>0.029 (0.001)</td>
<td>0.042 (0.001)</td>
<td>0.289 (0.021)</td>
</tr>
<tr>
<td>Options Hedge (8)</td>
<td>0.146 (0.002)</td>
<td>0.184 (0.031)</td>
<td>0.617 (0.018)</td>
</tr>
<tr>
<td>Options Hedge (32)</td>
<td>0.057 (0.001)</td>
<td>0.076 (0.014)</td>
<td>0.342 (0.018)</td>
</tr>
</tbody>
</table>

The second, third and fourth column show the error measures $e_1$ (3.5.8), $e_2$ (3.5.9) and $e_{\infty}$ (3.5.10) of profit and loss in different hedging strategies for hedging volatility swap over an interval of 1/252 years. The first row shows the mean and standard error of profit and loss of unhedged volatility swap. The second row shows the mean and standard error of profit and loss of a hedged volatility swap using variance swaps. The third row shows the mean and standard error of profit and loss of hedged volatility swap with eight options. The fourth row shows the mean and standard error of profit and loss of hedged volatility swap with 32 options.

**Hedging with options:** We form a portfolio containing one unit of volatility swap and $\beta_t$ units of a portfolio of call and put options which replicates a variance swap. We replicate a variance swap using portfolio of call and put options as described in section 3.5.1. In these results we are replicating a continuous variance swap with a portfolio of put and call options and a forward contract and payoff from dynamic trading strategy which holds $1/S_t$ shares of stock worth $\$1$ in the portfolio (3.5.5). We have assumed rebalancing is done continuously in computing payoffs from the dynamic trading strategy.
In this hedging exercise we present results using eight (four calls and four puts) options and 32 (16 calls and 16 puts) options. We compute the profit and loss of this portfolio at time $t = 1/252$ years in both cases: hedging with eight options and hedging with 32 options.

Table 3.10 shows the hedging performance of volatility swaps with variance swaps and a finite number of options. We compute error measures $e_1$ (3.5.8), $e_2$ (3.5.9) and $e_{oo}$ (3.5.10) of profit and loss using 4800 scenarios of stock price and variance level at $t = 1/252$ years. In these results as well the error measures are normalized by the variance strike, $K_{var}^*$, defined in (3.2.7). We do batching to compute the standard error estimates in error measures. In particular, we used 12 batches to compute the standard error estimates in the error measures. These hedging errors are for hedging over an interval of 1/252 year compared to the Table 3.5 where the hedging interval is one year. From Table 3.10 we can see that the absolute value of the volatility swap profit and loss is about 5.29% of the variance strike, $K_{var}^*$, over a single day. Hedging a volatility swap with a variance swap reduces this to 0.03% which is quite significant. Hence, a volatility swap can be effectively hedged dynamically using variance swaps. The results also show that hedging with eight options reduces the absolute value of the profit and loss to 0.15% and with 32 options to 0.059%. Thus the error in hedging volatility swaps with options decreases as we increase the number of options. Hence, dynamic hedging of volatility swaps with variance swaps and options is quite effective.
3.6 Conclusion

In this chapter we presented a partial differential equation approach to price volatility derivatives. We derived a partial differential equation using a no arbitrage argument to price volatility swaps and variance options in the Heston stochastic volatility model. The pricing of volatility derivatives (volatility swaps and variance options) is difficult as the underlying variable, realized variance, is not a market traded instrument. We exploited a no arbitrage relationship between variance swaps and other volatility derivatives to price and hedge these volatility derivatives. We also computed greeks of these volatility derivatives in the Heston stochastic volatility model using partial differential equations. Then we presented approaches to hedge these products. A variance swap can be replicated with a static position in a log contract and gains from a dynamic trading strategy in the underlying stock. The log contract can be replicated, though not perfectly, using a finite number of put and call options. The replication error in log contract decreases linearly with the number of options in the portfolio. Hence, a variance swap can be effectively replicated using a finite portfolio of call and put options and a forward contract. We presented an approach to hedge volatility derivatives using variance swaps, and showed the hedge to be very effective.
Chapter 4

VIX Index and VIX Futures

4.1 Introduction

The Chicago Board of Options Exchange (CBOE) introduced the volatility index VIX in 1993. It is a key measure of volatility implied by S&P500 (SPX) index option prices. When it was introduced in 1993, it was based on at-the-money volatilities of S&P 100 (OEX) index options. A drawback of this approach was that it was impossible to replicate this index with a static position in options. On September 22, 2003, the CBOE made two changes to the VIX definition. The new VIX is based on the S&P 500 (SPX) index option prices and is based on a broader range of strike prices. Each option price is weighted inversely proportional to the square root of its strike. We discuss this formula in section 4.2. The CBOE back-calculated the index value according to this new methodology until 1990. The CBOE continues to calculate the original-formula index with the ticker VXO.

Carr and Wu (2006) show that the square of the VIX index is an approximation of the one month variance swap rate up to discretization error under the assumption that the SPX index does not jump. When there are jumps in the SPX index we show in
Appendix A that the square of the VIX index is different from the one month variance swap rate. In that case the square of VIX index is an approximation of the value of negative of the payoff of a log contract of maturity one month up to discretization error. The new VIX captures the volatility across all options trading on a given day and thus reflects the option skew, as opposed to the previous formula which was based on very few option prices. The CBOE launched a new exchange, the Chicago Futures Exchange (CFE), in 2004. On March 26, 2004 VIX futures became the first product listed on the CFE and the first in a new family of exchange traded volatility products. In 2006 the CBOE offered options on the VIX as well.

In this work we study how sensitive the VIX formula is to the interval between strikes and the range of strikes used in the computation. We present the pricing of VIX futures in the Heston (1993) stochastic volatility (SV) model and the Bates (1996) and Scott (1997) stochastic volatility with jumps (SVJ) model. We propose semi-analytical approaches to price VIX futures in the SV and SVJ models and discuss the properties of futures prices predicted by these models. Then we empirically test the VIX futures prices predicted by these models from the parameters obtained by fitting these models to SPX market option prices. We analyze the historical profit and loss from investing in variance swaps and in VIX futures and discuss the reasons for difference in profit and loss from investing in these two products.

The rest of the chapter is organized as follows. We review the methodology to compute
the VIX index level and its sensitivity to the interval between strikes and the range of strikes used in the computation in section 4.2. In section 4.3 we present formulas to price VIX futures under the SV and SVJ models. In section 4.4 we present the performance of these models in fitting VIX futures prices using market VIX futures data and SPX options data. In section 4.5 we analyze the historical profit and loss from investing in variance swaps and VIX futures. Concluding remarks are given in section 4.6.

4.2 VIX Replication from SPX Options

In this section we first describe the methodology used to compute the VIX index level and then analyze the sensitivity of the VIX formula to the interval between strikes and the range of strikes used in the computation.

The new methodology to compute VIX index is described in detail in CBOE (2003). The VIX is computed from nearest and second nearest maturity with at least 8 days left to expiration and then weights them to give a measure of 30 day expected volatility. The formula to compute VIX is as follows:

\[
\sigma^2_j = \frac{2}{T_j} \sum_i \frac{\Delta K_i}{K_i^2} e^{rT_j} Q_j(K_i) - \frac{1}{T_j} \left[ \frac{F_j}{K_0} - 1 \right]^2
\]

(4.2.1)

where

- \( j \): Index denoting nearest and second nearest maturity \((j = 1, 2)\)
- \( T_j \): Time to expiration of maturity \( j \)
- \( F_j \): Forward index level derived from index option prices
- \( K_i \): Strike price of \( i^{th} \) out-of-the-money option; a call if \( K_i > F_j \) and a put if \( K_i < F_j \)
\( \Delta K_i \): Interval between strikes prices; \( \frac{K_{i+1} - K_{i-1}}{2} \)

\( K_0 \): First strike below the forward index level, \( F_j \)

\( r \): Risk-free interest rate to expiration

\( Q_j(K_i) \): The midpoint of bid-ask spread of each option with strike \( K_i \)

From equation (4.2.1), \( \sigma_j^2 \) is computed for two different maturities and then linear interpolation is used to compute a value corresponding to a 30 day maturity.

\[
\text{VIX}_t = 100 \sqrt{\frac{365}{30} \left[ T_1 \sigma_1^2 \frac{N_{T_2} - 30}{N_{T_2} - N_{T_1}} + T_2 \sigma_2^2 \frac{30 - N_{T_1}}{N_{T_2} - N_{T_1}} \right]}
\]  

(4.2.2)

where \( N_{T_1} \) and \( N_{T_2} \) denote the number of days to expiration for the two maturities.

Equation (4.2.1) is a discretized version of the realized variance equation from Demeterfi et al. (1999). Demeterfi et al. (1999) showed that when the underlying has no jumps the continuous realized variance is given by

\[
V_c(0,T) = \frac{2}{T} \left( \int_0^T \frac{dS_t}{S_t} - \ln \frac{S_T}{S_0} \right)
\]

\[
= \frac{2}{T} \left[ \int_0^T \frac{dS_t}{S_t} - \frac{S_T - S_0}{S_0} + \int_0^\infty \frac{1}{K^2} C_T(K) dK + \int_0^{S_0} \frac{1}{K^2} P_T(K) dK \right]
\]  

(4.2.3)

Carr and Wu (2006) show that the square of the VIX index is an approximation of the one month variance swap rate up to discretization error under the assumption that the SPX index doesn’t jump. In continuous diffusion models (e.g., in the SV model) the one month variance swap rate is equal to the expected realized variance given by
(4.2.3) under risk neutral probability measure. When the price of the underlying can jump this relation no longer holds. However, in this case the square of the VIX index level approximates the value of negative of the payoff of a log contract of maturity one month up to discretization error. The latter result is a more general result and holds in the SVJ model as well.

There are two sources of error in the VIX formula. The first is due to using discrete strikes instead of continuous strip of options. The second is due to the finite range of strikes of SPX options in the market. For a given date, S&P index options are available in intervals of $\Delta K = \$5$ and there is a range, $(K_{\text{min}}, K_{\text{max}})$ between which strikes are available. We call the VIX index computed in the limit when strike interval goes to zero $(\Delta K \to 0)$ and range of strikes is infinite $(K_{\text{min}} \to 0, K_{\text{max}} \to \infty)$ as theoretical VIX, denoted $\tilde{\text{VIX}}$, i.e.,

$$\tilde{\text{VIX}} \equiv \lim_{\substack{\Delta K \to 0 \\ K_{\text{min}} \to 0 \\ K_{\text{max}} \to \infty}} \text{VIX}$$ (4.2.4)

Throughout this chapter theoretical VIX means $\tilde{\text{VIX}}$ from equation (4.2.4) and market VIX means the VIX index level as computed from the VIX formula (4.2.1). In this section we study the difference between the market VIX index level and theoretical VIX index level due to the discrete strikes and due to a finite range of strikes of SPX options available in the market.

To study these effects we compute the VIX index level using equation (4.2.2) for a given range of strikes. We have volatility data of S&P500 options for strikes ranging
from $K_{\min}$ to $K_{\max}$ where the values $K_{\min}$ are $K_{\max}$ are the minimum and maximum strike of S&P500 options available on a particular day. We fit the SV and SVJ model to current market option prices to obtain the parameters of the models.

The risk neutral dynamics of the underlying asset $S_t$ under the Heston stochastic volatility model (SV) is given by:

$$dS_t = rS_t dt + \sqrt{\nu_t S_t} (\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2)$$  \hspace{1cm} (4.2.5)$$

$$d\nu_t = \kappa (\theta - \nu_t) dt + \sigma_\nu \sqrt{\nu_t} dW_t^1$$  \hspace{1cm} (4.2.6)$$

Equation (4.2.5) gives the dynamics of the stock price: $S_t$ denotes the stock price at time $t$, $r$ is the riskless interest rate, and $\sqrt{\nu_t}$ is the volatility. Equation (4.2.6) specifies the evolution of the variance as a square root process: $\theta$ is the long run mean variance, $\kappa$ represents the speed of mean reversion, and $\sigma_\nu$ is a parameter which determines the volatility of the variance process. The processes $W_t^1$ and $W_t^2$ are two independent standard Brownian motion under the risk neutral measure $Q$, and $\rho$ represents the instantaneous correlation between the return and volatility processes.

The Bates (1996) and Scott (1997) stochastic volatility with jumps (SVJ) is an extension of the SV model to include jumps in the stock price process. The risk-neutral dynamics are:
\[
\frac{dS_t}{S_t} = (r - \lambda m)dt + \sqrt{\nu_t} (\rho dW^1_t + \sqrt{1 - \rho^2} dW^2_t) + dJ_t
\]
\[
d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dW^1_t
\]  
(4.2.7)

where \( J_t = \sum_{i=1}^{N_t} (Y_j - 1) \) and \( N_t \) is a Poisson process with rate \( \lambda \) and \( Y_j \) is the relative jump size in the stock price. When jump occurs at time \( \tau_j \), then \( S(\tau_j^+) = S(\tau_j^-)Y_j \), where the distribution of \( Y_j \) is LN\([a, b^2]\) and \( m \) is the mean proportional size of jump \( E(Y_j - 1) = m \). The parameters \( a \) and \( m \) are related to each other by the equation:
\[ e^{a + \frac{1}{2}b^2} = m + 1 \]  
and only one of them needs to be specified. All other specifications are same as in the Heston stochastic volatility model specified in equations (4.2.5) and (4.2.6). The jump process, \( N_t \), and the Brownian motions are independent.

In order to estimate model parameters, we minimize the squared differences between the Black-Scholes implied volatility from market option prices and the Black-Scholes implied volatility from the model option prices. From the least squares minimization using market option prices on March 23, 2005, we get the parameters given in Table 4.3. The risk free rate, \( r \), on the same day was 2.84% and SPX index level was 1172.53.

In this chapter we use VIX futures and VIX index data from the CFE website. We have SPX options quotes from 1985 to 2005. In order to check the accuracy of our options data we compute the historical VIX index level using equation (4.2.2) and options prices from our data and compare with historical VIX levels available from the CBOE site. These results are given in Appendix B. On March 23, 2005, the two nearest maturities
<table>
<thead>
<tr>
<th>Strike Interval $\Delta K$</th>
<th>VIX level</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>13.828</td>
</tr>
<tr>
<td>1</td>
<td>13.829</td>
</tr>
<tr>
<td>2</td>
<td>13.831</td>
</tr>
<tr>
<td>4</td>
<td>13.837</td>
</tr>
<tr>
<td>5</td>
<td>13.842</td>
</tr>
<tr>
<td>10</td>
<td>13.882</td>
</tr>
<tr>
<td>15</td>
<td>13.950</td>
</tr>
<tr>
<td>20</td>
<td>14.045</td>
</tr>
</tbody>
</table>

Figure 4.1: The left table shows the effect of discrete strikes in computing the VIX index level. The first column shows the strike interval in computing the VIX level and second column shows the theoretical VIX level given by equation (4.2.2). These results are obtained using option prices from the SVJ model with parameters from column four in Table 4.3. The right figure shows the plot of the VIX level versus strike interval.

of SPX options are $T_1 = 23$ and $T_2 = 58$ days, respectively. In our options data set, the minimum and maximum strikes for these two maturities are (1030, 1310) and (950, 1340) respectively. Using equation (4.2.1) and strike intervals ranging from continuous $\Delta K_i \approx 0$ to a finite interval $\Delta K_i = 20$ we compute the variance level for both maturities, $T_1$ and $T_2$, and then use linear interpolation to estimate VIX level using equation (4.2.2).

Figure 4.1 shows the effect of discrete strikes in computing the VIX index level. The left table shows the value of VIX level with different strike intervals and right plot shows the VIX index level with different strike intervals with in given range of strikes. The results show that a discrete range of strikes in $K_{min}, K_{max}$ over approximates the VIX index level compared to using a continuous strip of strikes. Since strikes on S&P 500 options are available in a range of $\$5$ the over approximation of realized variance due to discrete strikes is quite small (13.842 versus 13.828 in this example).

Next we study the difference in market VIX level and theoretical VIX (4.2.4) due
Table: Strike Range $x$ vs VIX level

<table>
<thead>
<tr>
<th>Strike Range $x$</th>
<th>VIX level</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.04</td>
<td>11.656</td>
</tr>
<tr>
<td>1.08</td>
<td>12.883</td>
</tr>
<tr>
<td>1.12</td>
<td>13.327</td>
</tr>
<tr>
<td>1.16</td>
<td>13.570</td>
</tr>
<tr>
<td>1.2</td>
<td>13.722</td>
</tr>
<tr>
<td>1.4</td>
<td>13.983</td>
</tr>
<tr>
<td>1.6</td>
<td>14.017</td>
</tr>
<tr>
<td>1.8</td>
<td>14.021</td>
</tr>
</tbody>
</table>

Figure 4.2: The left table shows the effect of a finite range of strikes in computing the VIX index level. The first column shows the factor $x$ which defines strike range as $K_{\text{min}} = SPX/x, K_{\text{max}} = x(SPX)$ in computing the VIX level. These results are obtained using option prices from the SVJ model with parameters from column four in Table 4.3. The right figure shows the plot of the VIX level versus strike range.

to a finite range of strikes. The theoretical VIX ($\overline{\text{VIX}}$) is obtained from an infinite number of options which requires all strikes between 0 and $\infty$. The market VIX index level is computed from strikes available in a market, which is a finite range from $K_{\text{min}}$ to $K_{\text{max}}$. Hence, the contribution from strikes ranging from $(0, K_{\text{min}}) \cup (K_{\text{max}}, \infty)$ is not included in computing the market VIX index level. To estimate this effect we compute the VIX index level using equation (B.1.3) for a range of strikes with $K_{\text{min}}^i = SPX/x_i, K_{\text{max}}^i = x_i(SPX)$ where $x_i = 1 + i(0.04), i = 1, \ldots, 20$. Beyond this range there is not a significant contribution to the VIX index level. We compute these results for strike interval $\Delta K_i$ equals zero. Figure 4.2 shows the VIX level with different range of strikes of SPX options. The left table shows the value of VIX level with different range of strikes of SPX options and right plot shows the VIX index level with different strike intervals with different range of strikes of SPX options. As can be seen from the results that the VIX index level computed from a finite range of strikes under approximates the theoretical VIX ($\overline{\text{VIX}}$) computed from an infinite range of strikes. Typically
in the market the range of strikes varies from day to day. Mostly this range lies between 1.2 to 1.3 but sometimes it can be lower. Since the market VIX index level is computed from a finite range of strikes available on that particular day it under approximates the theoretical VIX.

Thus, there are two sources of deviations of the market VIX index level from the theoretical VIX value. The first is due to the discrete strikes and second is due to the finite range of strikes. Both effects work in opposite directions. Since the strike range of SPX options is $5 for short maturity options which are used to compute market VIX index, the first effect is quite small. Hence, the second effect causes the main discrepancy between VIX level and the theoretical VIX value. The difference between the market VIX level and the theoretical VIX value on a particular day depends on the strike range of SPX options available on that day. Typically for SPX options the range of strikes available in market is large enough so that this difference is about 20-30 basis points or 3-5% of the VIX index value.

In Appendix A we show that in the continuous diffusion models (e.g., in the SV model) the square of the theoretical VIX ($\bar{V}X$) value is equal to the one month continuous variance swap rate and when there are jumps in the underlying (SPX) index the square of the theoretical VIX is different from the one month continuous variance swap rate. In the SVJ model the square of the theoretical VIX is equal to the value of negative of the payoff of a log contract of maturity one month. The market VIX
index value is independent of jumps in underlying (SPX) index. Since it is given by SPX market option prices the market VIX index value and the theoretical VIX value is same in the SV and SVJ models. When there are no jumps in the SPX index the square of the market VIX index approximates the one month continuous variance swap rate upto the error between the market VIX and the theoretical VIX (4.2.4). Since market VIX is less than the theoretical VIX, the square of the market VIX index level is less than the one month continuous variance swap rate by the same amount when there are no jumps. If there are jumps in the SPX index the square of the market VIX index deviates from the theoretical VIX value by the same amount as in the case of no jumps but in this case square of the theoretical VIX is also different from the one month continuous variance swap rate. We characterize the theoretical VIX value in the SV and SVJ models in section 4.3. Using parameters of the SVJ model in column four from Table 4.3, the theoretical VIX under approximates the one month continuous variance swap rate by 15 basis points in this particular case. Broadie and Jain (2006a) showed an analysis of this difference and this difference can be quite high depending on the conditions in the market. Hence, in the case of negative jumps the square of the market VIX level under approximates the one month continuous variance swap rate by a slightly larger value than in the case of no jumps.

4.3 Pricing VIX Futures

In this section we briefly overview the VIX futures contracts listed on the CFE and then provide different methods to price VIX futures in the SV and SVJ models. We compare
these prices with bounds on the VIX futures given in Carr and Wu (2006).

The CBOE volatility index VIX futures track the level of the VBI index which is ten times the value of VIX. The VIX futures have a ticker symbol VX. VIX futures contracts expire at the open on Wednesday of each month that is two days prior to monthly option expiration. The VIX futures are listed on the February quarterly cycle (February, May, August and November) with two near term contract months and two additional quarterly cycle expirations being listed at any given time. Hence, at any given time there are 4 different VIX futures contracts trading on the CFE with different maturities. VIX futures are similar to forward starting volatility swaps. The price at time $t$ of VIX futures maturing at time $T$ is given by:

$$F_{t,T} = E_t^Q \left[ \text{VBI}_T \right] = E_t^Q \left[ \text{VIX}_T \right] \times 10 \quad (4.3.1)$$

Now we state the various properties of the variance process in the Heston stochastic volatility (SV) model. The transition probability density of the variance process, $\nu_T$, at time $T$ conditional on the variance, $\nu_t$ at time $t$ is:

$$f^Q(\nu_T|\nu_t) = f_1 \left( \frac{\nu_T}{b} \right) / b \quad T > t \quad (4.3.2)$$

where $f_1(\frac{\nu_T}{b})$ is the noncentral chi-squared probability density function with degrees of freedom $d$ and non-centrality parameter $\lambda$ evaluated at $\frac{\nu_T}{b}$. It is given by the following
equations:

\[ f_1 \left( \frac{v_T}{b} \right) = \frac{1}{2} \exp \left( \frac{v_T}{b + \lambda} \right) \left( \frac{v_T/b}{\lambda} \right)^{(d/4-1/2)} I_{d/2-1} \left( \sqrt{\lambda v_T/b} \right) \]

\[ d = \frac{4\theta \kappa}{\sigma_v^2} \]

\[ \lambda = \frac{v_t e^{-\kappa(T-t)}}{b} \]

\[ b = \frac{\sigma_v^2(1 - e^{-\kappa(T-t)})}{4\kappa} \]  

(4.3.3)

where \( I_v(x) \) is the modified Bessel function of the first kind. The expectation and variance of \( v_T \) conditional on \( v_t \) is given by

\[ E(v_T|v_t) = \theta(1 - e^{-k(T-t)}) + e^{-k(T-t)} v_t = A_2 + B_2 v_t \]

\[ \text{Var}(v_T|v_t) = \frac{A_2 \sigma_v^2}{2\kappa \theta} (A_2 + 2B_2 v_t) \]  

(4.3.4)

The SVJ model is given by equation (4.2.7). Since jumps in the SVJ model are independent of the variance process, the variance process will have the same properties as in the SV model specified by equations (4.3.3) and (4.3.4).

Now we characterize the VIX index level in the SV and SVJ model. Throughout in this section we mean the theoretical VIX (\( \overline{\text{VIX}} \)) which is the limit of market VIX as shown in equation (4.2.4). The square of the VIX index level, \( \overline{\text{VIX}}_t^2 \) at time \( t \) is equal to the value of one month log contract as shown in Appendix A.

\[ \overline{\text{VIX}}_t^2 = \frac{-2}{\tau} \mathbb{E}_t^Q \left[ \ln \left( \frac{S_{t+\tau}}{F_t} \right) \right] \]  

(4.3.5)

where \( \tau = 30/365 \) (i.e., 30 calendar days) and \( F_t = S_t e^{(\tau-\delta)\tau} \) is the forward value of the SPX index at time \( t \) and \( \delta \) is the dividend yield of the SPX index. This result holds both
in the case of jumps and without jumps. In continuous diffusion models (e.g., the SV model), the square of the VIX index level is also equal to the expectation of continuous realized variance under the risk neutral measure $Q$ or the one month continuous variance swap rate.

\[
\overline{\text{VIX}}_t^2 = \frac{-2}{\tau} E^Q_t \left[ \ln \left( \frac{S_{t+\tau}}{F_t} \right) \right] = E^Q_t \left[ \frac{2}{\tau} \left( \int_t^{t+\tau} \frac{dS_u}{S_u} - \ln \frac{S_{t+\tau}}{S_t} \right) \right] = E^Q_t \left[ \frac{1}{\tau} \int_t^{t+\tau} v_s d\tau \right]
\]

(4.3.6)

In the following propositions we are omitting the factor of 10 in equation (4.3.1) for the sake of brevity.

**Proposition 11** In the SVJ model, the VIX index level, $\overline{\text{VIX}}_t$ at time $t$ can be represented as:

\[
\overline{\text{VIX}}_t = \sqrt{-\frac{2}{\tau} E^Q_t \left[ \ln \left( \frac{S_{t+\tau}}{F_t} \right) \right]} = \sqrt{E^Q_t \left[ \frac{2}{\tau} \left( \int_t^{t+\tau} \frac{dS_u}{S_u} - \ln \frac{S_{t+\tau}}{S_t} \right) \right]}
\]

(4.3.7)

\[
= \sqrt{C_1 + C_2 v_t + 2\lambda(m-a)}
\]

where

\[
C_2 = \frac{1 - e^{-\kappa \tau}}{\kappa \tau}
\]

\[
C_1 = \theta(1 - C_2)
\]

\[
\tau = 30/365
\]
Proof: Applying Itô’s lemma to $\ln(S_t)$ in equation (4.2.7) and integrating from $t$ to $t + \tau$ gives

$$\ln\left(\frac{S_{t+\tau}}{S_t}\right) = \int_t^{t+\tau} (r - \lambda m - \frac{1}{2}v_s)ds + \int_t^{t+\tau} \sqrt{\nu_s}(\rho dW_s^1 + \sqrt{1-\rho^2}dW_s^2) + \ln\left(\prod_{j=1}^{n_j} Y_j\right)$$

(4.3.8)

where $n_j$ is number of jumps in the stock price during time $t + \tau - t$. Hence,

$$E_t^Q \left[\frac{2}{\tau} \left( \int_t^{t+\tau} \frac{dS_u}{S_u} - \ln \frac{S_{t+\tau}}{S_t} \right) \right] = E_t^Q \left[\frac{2}{\tau} \left( \lambda m + \int_t^{t+\tau} \frac{1}{2}v_s ds + \ln\left(\prod_{j=1}^{n_j} Y_j\right)\right) \right]$$

$$= \sqrt{\theta + \frac{1 - e^{-\kappa \tau}}{\kappa \tau}} (v_t - \theta) + 2\lambda(m - a)$$

$$= \sqrt{C_1 + C_2 v_t + 2\lambda(m - a)}$$

(4.3.9)

It follows from Proposition 10 in Broadie and Jain (2006a). □

Using Proposition 11 the VIX index level, $\text{VIX}_t$ at time $t$ in the SV model, can be represented as:

$$\text{VIX}_t = \sqrt{C_1 + C_2 v_t}$$

(4.3.10)

In the SV model the square of the VIX index level is same as the one month continuous variance swap rate and in the SVJ model the one month variance swap rate from Broadie and Jain (2006a) is
\[
E_t^Q \left[ \frac{1}{\tau} \int_t^{t+\tau} v_s^2 ds \right] = C_1 + C_2 v_t + \lambda(a^2 + b^2) 
\]  
(4.3.11)

Hence in the SVJ model the difference between the square root of one month continuous variance swap rate and theoretical VIX value is \( \sqrt{(C_1 + C_2 v_t + \lambda(a^2 + b^2))} - \sqrt{C_1 + C_2 v_t + 2\lambda(m-a)} \).

**Proposition 12** In SVJ model, the price at time \( t \) of VIX futures contract expiring at time \( T \) is given by:

\[
F_{t,T} = \int_0^\infty \sqrt{C_1 + C_2 v_T + 2\lambda(m-a)} f^Q(v_T|v_t) dv_T 
\]  
(4.3.12)

where \( f^Q(v_T|v_t) \) is the transition probability density of variance process, \( v_T \) at time \( T \) conditional on variance, \( v_t \) at time \( t \) given by equation (4.3.2).

Using Proposition 12 the price at time \( t \) of VIX futures contract expiring at time \( T \) in the SV model is given by:

\[
F_{t,T} = \int_0^\infty \sqrt{C_1 + C_2 v_T} f^Q(v_T|v_t) dv_T 
\]  
(4.3.13)

Since the density of variance process is known in closed form we can obtain the price of VIX futures by numerical integration. We can also compute the price of VIX futures using the inverse transform formulas presented in previous chapters. We modify those formulas to price the VIX futures.
**Proposition 13** In the SVJ model, the price at time $t$ of VIX futures contract expiring at time $T$ can be computed by the following transform inversion formula:

$$F_{t,T} = E_t^Q \left[ \sqrt{\text{VIX}_T} \right] = E_t^Q \left[ \sqrt{\frac{\text{VIX}_T}{\text{VIX}_T}} \right] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - E_t^Q(e^{-s\text{VIX}_T^2})}{s^{3/2}} ds \quad (4.3.14)$$

where

$$E_t^Q(e^{-s\text{VIX}_T^2}) = E_t^Q(e^{-s(C_1 + C_2\text{VIX}_T + 2\lambda(m - a))}) = \exp[-sC_1 + 2\lambda(m - a) + \psi_1(T - t) - \psi_2(T - t)]$$

where

$$\psi_1(T - t) = \frac{2s\theta}{\sigma^2} \log \left( \frac{2\kappa e^\kappa(T-t)}{(sC_1^2 + 2\kappa e^\kappa(T-t) - 1) + 2\kappa e^\kappa(T-t)} \right)$$

$$\psi_2(T - t) = \frac{2s\theta C_2}{(sC_1^2 + 2\kappa e^\kappa(T-t) - 1) + 2\kappa e^\kappa(T-t)}$$

The pricing in the SV model follows directly from Proposition 13. We can also obtain an approximation to the price of VIX futures using the convexity correction formula.

**Proposition 14** In the SVJ model, the price at time $t$ of VIX futures contract expiring at time $T$ can be approximated by the convexity correction formula:

$$F_{t,T} = E_t^Q \left[ \sqrt{\text{VIX}_T} \right] = E_t^Q \left[ \frac{\text{VIX}_T^2}{\text{VIX}_T} \right] \approx \sqrt{E_t^Q\left(\text{VIX}_T^2\right)} - \frac{E_t^Q\left(\text{VIX}_T^2 - E_t^Q(\text{VIX}_T^2)\right)^2}{8E_t^Q(\text{VIX}_T^2)^{3/2}} \quad (4.3.15)$$

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where the following can be derived using equation (4.3.4):

\[
E_t^Q(\overline{\text{VIX}_T^2}) = E_t^Q(C_1 + 2\lambda(m - a) + C_2v_T)
\]

\[
= C_1 + 2\lambda(m - a) + C_2A_2 + v_tC_2B_2
\]

\[
E_t^Q\left[\overline{\text{VIX}_T^2} - E_t^Q(\overline{\text{VIX}_T^2})\right]^2 = \text{Var}(\overline{\text{VIX}_T^2}) = \text{Var}(C_1 + 2\lambda(m - a) + C_2v_T|v_t)
\]

\[
= \frac{C_2^2A_2^2\sigma_v^2}{2\kappa^2} (A_2 + 2B_2v_t)
\]

(4.3.16)

It follows from the Broadie and Jain (2006a) that the convexity approximation formula is a good approximation if the following technical condition is satisfied:

\[
0 \leq \overline{\text{VIX}_T^2} \leq 2E_t^Q(\overline{\text{VIX}_T^2})
\]

(4.3.17)

which can be quantified using the excess probability defined in Broadie and Jain (2006a).

**Proposition 15** In the SVJ model, the excess probability can be represented using the noncentral chi-squared cumulative distribution function in the following form:

\[
p = 1 - F_1\left(\frac{(C_1 + 2\lambda(m - a))/C_2 + 2(A_2 + B_2v_t)}{b}\right)
\]

(4.3.18)

where \(F(\cdot)\) is noncentral chi-squared cumulative density function with degrees of freedom \(d\) and non-centrality parameter \(\lambda\) as specified in equation (4.3.3).

**Proof:** The excess probability is given by

\[
p = P(\overline{\text{VIX}_T^2} \geq 2E_t^Q(\overline{\text{VIX}_T^2}))
\]

\[
= P(C_1 + 2\lambda(m - a) + C_2v_T \geq 2(C_1 + 2\lambda(m - a) + C_2A_2 + v_tC_2B_2))
\]

\[
= P(v_T \geq (C_1 + 2\lambda(m - a))/C_2 + 2(A_2 + v_tC_2B_2))
\]

\[
= P\left(v_T/b \geq \frac{(C_1 + 2\lambda(m - a))/C_2 + 2(A_2 + B_2v_t)}{b}\right)
\]

\[
= 1 - F_1\left(\frac{(C_1 + 2\lambda(m - a))/C_2 + 2(A_2 + B_2v_t)}{b}\right)
\]

(4.3.19)
The last equality follows since \( \nu_T/b \) has a noncentral chi-squared distribution with degrees of freedom \( d \) and non-centrality parameter \( \lambda \) as specified in equation (4.3.3). □

Carr and Wu (2006) have provided upper and lower bounds on the VIX futures using the Jensen’s inequality. The VIX futures value using equations (4.3.1) and (4.3.6) can be represented as follows:

\[
F_{t,T} = E_t^Q \left[ \text{VIX}_T \right] = E_t^Q \sqrt{E_t^Q \left[ \frac{1}{\tau} \int_T^{T+\tau} \nu_s \, ds \right]} \tag{4.3.20}
\]

The second equality is true for the continuous diffusion models (e.g., the SV model) and hence the following bounds don’t hold in the case of jumps in the SPX index. Applying Jensen’s inequality we get the following bounds:

\[
E_t^Q \left[ \sqrt{\frac{1}{\tau} \int_T^{T+\tau} \nu_s \, ds} \right] \leq F_{t,T} \leq \sqrt{E_t^Q \left[ \frac{1}{\tau} \int_T^{T+\tau} \nu_s \, ds \right]} \tag{4.3.21}
\]

For the upper bound we apply Jensen’s on the outer expectation and square root function and for the lower bound we apply Jensen’s on the inner expectation and the square root function. Thus upper bound on the VIX futures value is the square root of forward starting one month variance swap rate and lower bound on the VIX futures value is the forward starting one month volatility swap rate. Carr and Wu (2006) have shown that under a zero correlation assumption between stock and variance process the lower bound can be approximated by the implied volatility of a forward starting at-the-money call option. We compute the upper bound using equation (4.3.21) in our numerical example but we don’t compute the lower bound as the zero correlation assumption is quite restrictive.
Table 4.1: Model (SV and SVJ) parameters used in pricing VIX Futures

<table>
<thead>
<tr>
<th>Parameters</th>
<th>SV model</th>
<th>SVJ model</th>
</tr>
</thead>
<tbody>
<tr>
<td>risk free rate $r$</td>
<td>3.19%</td>
<td>3.19%</td>
</tr>
<tr>
<td>initial volatility $\sqrt{V_0}$</td>
<td>10.1%</td>
<td>9.4%</td>
</tr>
<tr>
<td>correlation $\rho$</td>
<td>-0.7</td>
<td>-0.79</td>
</tr>
<tr>
<td>long run mean variance $\theta$</td>
<td>0.019</td>
<td>0.014</td>
</tr>
<tr>
<td>speed of mean reversion $\kappa$</td>
<td>6.21</td>
<td>3.99</td>
</tr>
<tr>
<td>volatility of variance $\sigma_v$</td>
<td>0.31</td>
<td>0.27</td>
</tr>
<tr>
<td>jump arrival rate $\lambda$</td>
<td>n/a</td>
<td>0.11</td>
</tr>
<tr>
<td>mean proportional size of jump $m$</td>
<td>n/a</td>
<td>-0.12</td>
</tr>
<tr>
<td>jump size volatility $b$</td>
<td>n/a</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Table 4.2: Pricing of VIX Futures in the SV and SVJ models using different methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Value (SV)</th>
<th>Value (SVJ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>132.8036</td>
<td>130.4416</td>
</tr>
<tr>
<td>Inversion</td>
<td>132.8036</td>
<td>130.4416</td>
</tr>
<tr>
<td>UB</td>
<td>136.6663</td>
<td></td>
</tr>
<tr>
<td>CC</td>
<td>132.4424</td>
<td>129.8217</td>
</tr>
<tr>
<td>Prob(excess)</td>
<td>0.0448</td>
<td>0.0494</td>
</tr>
</tbody>
</table>

We use the parameters in Table 4.1 and maturity of futures is 0.5 years.

Figure 4.3: This figure shows the VIX futures price with maturity for two different values of initial variance, $v_t$. The left plot shows prices when initial variance, $v_t$ is less than $\theta$. For this case we use all the parameters in Table 4.1. The right plot shows futures prices when initial variance, $v_t$ is more than $\theta$. In this case we use all the parameters in Table 4.1 except $v_t$ which is equal to 0.025 and hence $v_t > \theta$.

Next we present numerical results of pricing VIX futures in the SV and SVJ models using the parameters in Table 4.1.
Table 4.2 shows the prices of VIX futures obtained using different methods. The prices obtained using numerical integration with density and Laplace inversion are accurate prices. The UB is within 3-4% of the true price. The excess probability for the VIX futures pricing is about 0.045 in the SV model and slightly higher in the SVJ model. The convexity correction formula price is slightly different than the true price in both models.

Next we discuss the term structure of VIX futures prices. As the maturity increases, the VIX futures prices on a given day can be increasing or decreasing depending on market conditions. In the SV and SVJ models when initial variance, $v_t$, is less than long run mean variance, $\theta$, i.e., $v_t < \theta$, then with increasing maturity the futures prices are always increasing. Similarly when $v_t > \theta$, the futures prices predicted by the SV and SVJ models are decreasing with maturity. Figure 4.3 shows sample term structures of VIX futures prices in both cases. This shows that both the SV and SVJ model are capable of predicting futures prices.

We also analyze the excess probability with different maturity values. The excess probability will depend on the parameters of the SV and SVJ models and maturity of VIX futures. Figure 4.4 plots the $1 - p$, the excess probability with maturity for two different cases: when initial variance is less than the long run mean variance and second when initial variance is more than the long run mean variance. It can be seen from the curve that the convexity correction formula is more inaccurate in computing longer
An image of a page from a document containing graphs and text. The text is related to financial modeling, specifically comparing SV and SVJ models with different initial variance values.

### Figure 4.4
This figure shows the $(1 - p(\text{excess probability} \ 4.3.18))$ with maturity for two different values of initial variance, $v_t$. The left plot shows probability when initial variance, $v_t$, is less than $\theta$. For this case we use all the parameters in Table 4.1. The right plot shows the probability when initial variance, $v_t$, is more than $\theta$. In this case we use all the parameters in Table 4.1 except $v_t$ which is equal to 0.025 and hence $v_t > \theta$.

### Table 4.3: Model parameters obtained using empirical fitting

<table>
<thead>
<tr>
<th>Parameters</th>
<th>SV (Price)</th>
<th>SV (Vol)</th>
<th>SVJ (Price)</th>
<th>SVJ (Vol)</th>
</tr>
</thead>
<tbody>
<tr>
<td>initial volatility $\sqrt{V_0}$</td>
<td>12.34%</td>
<td>12.87%</td>
<td>11.48%</td>
<td>11.59%</td>
</tr>
<tr>
<td>speed of mean reversion $\kappa$</td>
<td>2.73</td>
<td>2.70</td>
<td>3.03</td>
<td>3.75</td>
</tr>
<tr>
<td>long run mean volatility $\sqrt{\theta}$</td>
<td>19.52%</td>
<td>18.52%</td>
<td>15.89%</td>
<td>15.26%</td>
</tr>
<tr>
<td>volatility of variance $\sigma_v$</td>
<td>67.56%</td>
<td>76.16%</td>
<td>44.26%</td>
<td>44.57%</td>
</tr>
<tr>
<td>correlation $\rho$</td>
<td>-0.50</td>
<td>-0.48</td>
<td>-0.52</td>
<td>-0.55</td>
</tr>
<tr>
<td>jump arrival rate $\lambda$</td>
<td></td>
<td></td>
<td>0.15</td>
<td>0.14</td>
</tr>
<tr>
<td>mean proportional size of jump $m$</td>
<td>-0.08</td>
<td>-0.06</td>
<td></td>
<td></td>
</tr>
<tr>
<td>jump size volatility $b$</td>
<td></td>
<td></td>
<td>16.15%</td>
<td>18.21%</td>
</tr>
<tr>
<td>RMSE (price)</td>
<td>0.18</td>
<td>0.34</td>
<td>0.11</td>
<td>0.13</td>
</tr>
<tr>
<td>RMSE (vol)</td>
<td>0.43%</td>
<td>0.32%</td>
<td>0.22%</td>
<td>0.19%</td>
</tr>
</tbody>
</table>

This table shows the model parameters obtained by minimizing mean squared errors between model prices and market prices. The first column shows the respective parameters of the SV and SVJ models. The second column shows the SV model parameters obtained by minimizing mean squared errors between the SV model option prices and market option prices. A total of 72 options observed on the same day are used. The third column shows the SV model parameters obtained by minimizing mean squared errors between the SV model implied volatilities and market option implied volatilities. The last two columns show the respective parameters in the SVJ model. The second last row shows the square root mean squared error between model option prices and market options prices. The last row shows the square root of mean squared error between model implied volatilities and market option implied volatilities.
Figure 4.5: This figure shows the smile implied by model and market implied smile. The upper left plot shows the smile implied by the SV model parameters (obtained by minimizing mean squared error between prices) and market implied smile for three different maturities $T_1 = 23$ days, $T_2 = 58$ days and $T_3 = 86$ days of options available on March 23, 2005. The upper right plot shows the smile implied by the SV model parameters (obtained by minimizing mean squared error between volatilities) and market implied smile. The bottom plot shows the same for the SVJ model.

Table 4.4: Futures prices from different model parameters obtained using empirical fitting

<table>
<thead>
<tr>
<th>Maturity (days)</th>
<th>Market Value</th>
<th>SV (Price)</th>
<th>SV (Vol)</th>
<th>SVJ (Price)</th>
<th>SVJ (Vol)</th>
</tr>
</thead>
<tbody>
<tr>
<td>VIX</td>
<td>14.06</td>
<td>13.27</td>
<td>13.56</td>
<td>14.02</td>
<td>14.15</td>
</tr>
<tr>
<td>56 (K05)</td>
<td>146.4</td>
<td>138.8</td>
<td>131.5</td>
<td>146.8</td>
<td>147.8</td>
</tr>
<tr>
<td>84 (M05)</td>
<td>150.4</td>
<td>143.8</td>
<td>134.5</td>
<td>149.9</td>
<td>150.6</td>
</tr>
<tr>
<td>147 (Q05)</td>
<td>156.1</td>
<td>152.2</td>
<td>140.2</td>
<td>155.1</td>
<td>154.8</td>
</tr>
<tr>
<td>238 (X05)</td>
<td>160.7</td>
<td>159.1</td>
<td>145.2</td>
<td>159.1</td>
<td>157.8</td>
</tr>
<tr>
<td>RMSE</td>
<td>5.45</td>
<td>15.56</td>
<td>0.99</td>
<td>1.79</td>
<td></td>
</tr>
</tbody>
</table>

This table shows the futures prices from different model parameters obtained by minimizing mean squared errors between model prices and market prices. The first column shows VIX index value and four different VIX futures maturities in days trading on March 23, 2005. The second column shows the market value of VIX and VIX futures on the particular day. The last four columns show the VIX futures values using different model parameters obtained using market option prices we reported in last four columns in Table 4.3. The last row shows the square root of mean squared error between model implied futures prices and market futures prices.
4.4 Empirical Testing of Futures Prices

In this section we empirically test the futures prices implied by different models we discussed in this chapter. We study the performance of the SV and SVJ models in fitting the market future prices. For any given day there are four VIX futures contract of different expiries trading in the market. For example, on March 23, 2005 four futures contracts K05, M05, Q05 and X05 maturing in May, June, August and November respectively were trading. For the same day we have SPX options prices of three different maturities and all strikes trading in the market. The three different maturities available are $T_1 = 23$ days, $T_2 = 58$ days and $T_3 = 86$ days.

We fit the SV and SVJ models to the market option prices to obtain the parameters of these models. Table 4.3 shows parameters obtained by fitting different models and objective function values used in minimization. The second column (SV(Price)) in Table 4.3 shows the SV model parameters obtained by minimizing mean squared errors between the SV model option prices and market option prices weighted by inverse of strike squares. This form of the objective function puts weight on each option as in VIX index computation (4.2.1). This optimization objective function puts more weight on lower strikes and less on higher strikes and hence this objective fits options with lower values of strikes better than higher strikes. The third column (SV(Vol)) in Table 4.3 shows the SV model parameters obtained by minimizing mean squared errors between the SV model implied volatilities and market option implied volatilities. This objective function puts equal weight on all strikes. The last two columns show the respective parameters
in the SVJ model obtained by price fitting and implied volatility fitting. A total of 72 options observed on the same day are used in optimization to compute the parameters. The second last row shows the square root of mean squared error between model option prices and market options prices. The last row shows the square root of mean squared error between model implied volatilities and market option implied volatilities. As can be seen from the results of the SV and SVJ model fit parameters, the SVJ model is better in fitting market option prices.

Figure 4.5 shows the market implied smile and implied volatility smile using model parameters in Table 4.3. The upper left plot shows the smile implied by the SV model parameters (obtained by minimizing mean squared error between prices, second column in Table 4.3) and market implied smile for three different maturities $T_1 = 23$ days, $T_2 = 58$ days and $T_3 = 86$ days of options available on March 23, 2005. The upper right plot shows the smile implied by the SV model parameters (obtained by minimizing mean squared error between volatilities, third column in Table 4.3) and market implied smile. The bottom plot shows the same for the SVJ model. As can be seen from these smile plots, the SVJ model provides a better fit to the market implied smile.

Next using the parameters estimated in Table 4.3 we compute the market futures prices on March 23, 2005. Table 4.4 shows the prices of VIX futures as implied by different model parameters. There are four futures contracts of different maturities trading on the same day. The maturities of futures contracts are listed in first column in Table 4.4.
and second column shows the market values of these contracts. The third and fourth column in Table 4.4 show the futures prices obtained from the SV model parameters. The SV model parameters obtained by minimizing mean squared differences between prices do a better job in fitting market futures prices compared to the parameters obtained by minimizing mean squared differences between implied volatilities. The last two columns in Table 4.4 show the futures prices obtained from the SVJ model parameters. In this case as well parameters obtained by minimizing mean squared differences between prices do a better job in fitting market futures prices compared to the parameters obtained by minimizing mean squared differences between implied volatilities. The RMSE errors in fitting VIX futures from the SVJ model parameters are much smaller than in the SV model which is expected as the SVJ model is better fit to market option prices.

4.5 Realized Volatility, Implied Volatility and VIX Index

In this section we are investigating the following questions:

- What are the historical profit and loss by investing in variance swaps?
- What are the historical profit and loss by investing in VIX futures?
- How would investing in the VIX futures compare with investing in variance swaps and what is the reason for any difference?

To answer these questions we compute the one month realized volatility from S&P 500 index daily prices and one month at-the-money implied volatility from S&P 500 index option prices and one month realized volatility as implied by VIX index level. As
we discussed in section 4.2, when the underlying (SPX) diffusion is continuous, e.g., in the SV model, the square of the theoretical VIX ($\text{VIX}$) value is equal to the one month variance swap rate which is equal to the expected one month realized variance under the risk neutral measure $Q$ or one month continuous variance swap rate. Since market VIX is computed from a finite range of strikes and discrete strike intervals it under approximates the square root of one month expected realized variance under the risk neutral measure or the square root of one month continuous variance swap rate, i.e.,

$$VIX_t^2 \leq E_t^Q \left[ \frac{1}{\tau} \int_t^{t+\tau} v_s ds \right] = E_t^Q \left[ V_c(t, t+\tau) \right] = \frac{-2}{\tau} E_t^Q \left[ \ln \left( \frac{S_{t+\tau}}{F_t} \right) \right] = \sqrt{\text{VIX}_t^2}$$

(4.5.1)

When the underlying has jumps as in the SVJ model, the market VIX index level under approximates the theoretical VIX index level which is equal to the square root of negative value of one month log contract and the theoretical VIX value under approximates the one month continuous variance swap rate, i.e.,

$$VIX_t^2 \leq \frac{-2}{\tau} E_t^Q \left[ \ln \left( \frac{S_{t+\tau}}{F_t} \right) \right] = \sqrt{\text{VIX}_t^2} \leq E_t^Q \left[ V_c(t, t+\tau) \right]$$

(4.5.2)

Let $t = t_0 < t_1 < \ldots < t_n = t+\tau$ be a partition of the time interval $[t, t+\tau]$ into $n$ equal segments of length $\Delta t$, i.e., $t_i = t+i\tau/n$ for each $i = 0, 1, \ldots, n$. Then continuous realized variance is the limit of the discrete realized variance. In the SVJ model, the one month realized variance consists of two components. The first is the accumulated variance of the underlying stock during one month contributed from the diffusive Brownian motion and second is the contribution from jumps in the underlying stock. In the SVJ model
one month continuous realized variance is given by the following:

\[
V_c(t, t + \tau) = \frac{1}{\tau} \int_t^{t+\tau} v_s ds + \frac{1}{\tau} \left( \sum_{i=1}^{N(\tau)} (\ln(Y_i))^2 \right) = \lim_{n \to \infty} \frac{n}{n-1}\sum_{i=0}^{n-1} \left( \ln\left( \frac{S_{i+1}}{S_i} \right) \right)^2
\]

(4.5.3)

where \( \tau = 30/365 \) or 30 calendar days. In the SV model the one month realized variance is given by only the first term in on right side of first equality in equation (4.5.3).

Broadie and Jain (2006a) show that the expected discrete realized variance is larger than the continuous realized variance i.e.,

\[
E_t^Q \left[ V_c(t, t + \tau) \right] \leq E_t^Q \left[ \frac{n}{(n-1)\tau} \sum_{i=0}^{n-1} \left( \ln\left( \frac{S_{i+1}}{S_i} \right) \right)^2 \right]
\]

(4.5.4)

We compute the one month realized volatility using daily close of the S&P500 index level. We then compare this with the market VIX index level which under approximates the square root of the one month continuous variance swap rate both in the SV (4.5.1) and SVJ (4.5.2) models. The amount of under approximation is larger in the case of jumps. We also compute the at-the-money implied volatility from S&P 500 index option prices. All of these three quantities provide different measures of market volatility. The expectation of volatilities provided by the VIX and implied volatility are under the pricing measure or the risk neutral measure.

Table 4.5 shows the descriptive statistics of VIX index level from 1990-2006 and the square root of one month daily realized variance computed from daily close of the S&P500 index. It also shows the descriptive statistics of one month at-the-money implied
Table 4.5: Descriptive statistics of historical VIX index level and one month realized volatility

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>14.43</td>
<td>19.08</td>
<td>17.38</td>
</tr>
<tr>
<td>Stdev</td>
<td>6.80</td>
<td>6.42</td>
<td>5.87</td>
</tr>
<tr>
<td>Skewness</td>
<td>1.51</td>
<td>0.98</td>
<td>0.95</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>5.80</td>
<td>3.80</td>
<td>3.92</td>
</tr>
<tr>
<td>Difference</td>
<td>4.65</td>
<td>2.95</td>
<td></td>
</tr>
</tbody>
</table>

This table shows the descriptive statistics of one month realized volatility, the VIX index level and at-the-money implied volatility from S&P 500 index option prices for three different historical periods. The first row shows three different historical periods. The first column shows the descriptive statistic computed from the VIX index and one month realized volatility and implied volatility. The last row and columns two, five and eight show the difference between the volatilities from VIX and daily close of the S&P 500 index (RVOL) for three different historical periods. The last row and columns three, six and nine show the difference between the implied volatilities from S&P 500 index option prices and one month realized volatility from daily close of the S&P 500 index (RVOL) for three different historical periods.

Volatility of S&P 500 index option prices. The implied volatility values are from January 1990 to August 2005. We don’t have data for options expiring after August 2005. As can be seen from the results the average VIX index level is higher than average realized volatility (RVOL). Table shows the statistics for the combined period from 1990 to 2006 and also for periods from 1990-2003 and 2004-2006. We break the sample since the first exchange traded volatility products were launched in 2004. As discussed before the square of the market VIX index gives an under approximation of the expectation of one month realized variance under the risk neutral measure $Q$ or one month variance swap rate. Hence the mean of historical one month realized variance under the risk neutral measure $Q$ is higher than the average of the one month historical realized variance. From these results we also see that the mean of at-the-money one month implied volatility is higher than the average realized volatility. This again confirms the observation that
volatility expectations are higher under the pricing measure and lower under the real-world measure. The volatility expectations are higher from VIX compared to at-the-money implied volatility. This is because the VIX index captures the volatilities from entire option skew while the implied volatility is the volatility of a particular strike.

Figure 4.6: This figure shows the VIX index level historical series and one month realized volatility from 1990-2006.

Figure 4.6 shows the plot of VIX index level and square root of one month daily realized variance computed from daily close of the S&P500 index. The square root of one month daily realized variance is always below the VIX index level. Also, because of the following inequalities:
Table 4.6: Mean and standard deviation of yearly VIX index level and one month realized volatility and at-the-money implied volatility.

<table>
<thead>
<tr>
<th>Year</th>
<th>RVOL Mean</th>
<th>RVOL Stdev</th>
<th>VIX Mean</th>
<th>VIX Stdev</th>
<th>Imp Vol. Mean</th>
<th>Imp Vol. Stdev</th>
<th>Difference VIX-RVOL</th>
<th>Difference Imp Vol.-RVOL</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>15.57</td>
<td>4.53</td>
<td>23.06</td>
<td>4.74</td>
<td>19.59</td>
<td>4.11</td>
<td>7.49</td>
<td>4.01</td>
</tr>
<tr>
<td>1991</td>
<td>13.71</td>
<td>2.94</td>
<td>18.37</td>
<td>3.69</td>
<td>16.18</td>
<td>2.85</td>
<td>4.67</td>
<td>2.47</td>
</tr>
<tr>
<td>1992</td>
<td>9.51</td>
<td>1.94</td>
<td>15.44</td>
<td>2.10</td>
<td>13.38</td>
<td>1.66</td>
<td>5.93</td>
<td>3.87</td>
</tr>
<tr>
<td>1993</td>
<td>8.49</td>
<td>2.13</td>
<td>12.70</td>
<td>1.32</td>
<td>10.86</td>
<td>1.07</td>
<td>4.21</td>
<td>2.37</td>
</tr>
<tr>
<td>1994</td>
<td>9.66</td>
<td>2.18</td>
<td>13.92</td>
<td>2.08</td>
<td>11.61</td>
<td>1.88</td>
<td>4.26</td>
<td>1.95</td>
</tr>
<tr>
<td>1995</td>
<td>8.16</td>
<td>1.83</td>
<td>12.42</td>
<td>0.99</td>
<td>10.44</td>
<td>0.83</td>
<td>4.26</td>
<td>2.28</td>
</tr>
<tr>
<td>1996</td>
<td>11.59</td>
<td>2.70</td>
<td>16.40</td>
<td>1.93</td>
<td>14.30</td>
<td>1.70</td>
<td>4.80</td>
<td>2.71</td>
</tr>
<tr>
<td>1997</td>
<td>17.77</td>
<td>5.83</td>
<td>22.30</td>
<td>4.12</td>
<td>20.10</td>
<td>3.13</td>
<td>4.53</td>
<td>2.33</td>
</tr>
<tr>
<td>1998</td>
<td>18.88</td>
<td>8.44</td>
<td>25.62</td>
<td>6.86</td>
<td>21.93</td>
<td>6.03</td>
<td>6.75</td>
<td>3.05</td>
</tr>
<tr>
<td>1999</td>
<td>18.25</td>
<td>2.89</td>
<td>24.37</td>
<td>2.89</td>
<td>21.50</td>
<td>2.67</td>
<td>6.11</td>
<td>3.25</td>
</tr>
<tr>
<td>2000</td>
<td>21.81</td>
<td>6.21</td>
<td>23.25</td>
<td>3.36</td>
<td>21.06</td>
<td>2.98</td>
<td>1.44</td>
<td>-0.75</td>
</tr>
<tr>
<td>2002</td>
<td>25.14</td>
<td>8.60</td>
<td>27.20</td>
<td>6.93</td>
<td>24.81</td>
<td>6.71</td>
<td>2.06</td>
<td>-0.32</td>
</tr>
<tr>
<td>2003</td>
<td>15.92</td>
<td>5.03</td>
<td>22.13</td>
<td>5.27</td>
<td>20.47</td>
<td>5.37</td>
<td>6.21</td>
<td>4.55</td>
</tr>
<tr>
<td>2004</td>
<td>11.08</td>
<td>1.80</td>
<td>15.56</td>
<td>1.90</td>
<td>13.78</td>
<td>1.60</td>
<td>4.47</td>
<td>2.70</td>
</tr>
<tr>
<td>2005</td>
<td>10.18</td>
<td>2.21</td>
<td>12.81</td>
<td>1.47</td>
<td>11.73</td>
<td>1.18</td>
<td>2.63</td>
<td>1.55</td>
</tr>
<tr>
<td>2006</td>
<td>9.65</td>
<td>2.73</td>
<td>12.89</td>
<td>2.24</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This table shows the mean and standard deviation of yearly VIX index and one month realized volatility and implied volatility from S&P 500 index prices.

\[
E_t^P \left[ V_c(t, t + \tau) \right] = E_t^P \left[ \lim_{n \to \infty} \frac{n}{(n - 1)\tau} \sum_{i=0}^{n-1} \left( \ln \left( \frac{S_{i+1}}{S_i} \right) \right)^2 \right]
\]

\[
\leq_1 E_t^P \left[ \frac{n}{(n - 1)\tau} \sum_{i=0}^{n-1} \left( \ln \left( \frac{S_{i+1}}{S_i} \right) \right)^2 \right]
\]

\[
\leq_2 \text{VIX}_t^2 \leq_3 \sqrt{\text{VIX}_t^2} \leq_4 E_t^Q \left[ V_c(t, t + \tau) \right]
\]

\[
\leq_5 E_t^Q \left[ \frac{n}{(n - 1)\tau} \sum_{i=0}^{n-1} \left( \ln \left( \frac{S_{i+1}}{S_i} \right) \right)^2 \right]
\]

(4.5.5)

The difference between one month continuous realized volatility under the risk neutral measure or the square root of the one month variance swap rate and the one month continuous realized volatility should be larger than the difference shown in the Table.
4.5. The first and fifth inequalities state that the discrete realized variance is more than the continuous realized variance (4.5.4). The second inequality comes from empirical results in Table 4.6 that the square of the market VIX is more than one month realized variance. The third inequality is due to the finite range of strikes used in computing VIX. The fourth inequality states that the square of the VIX index level is smaller than the one month variance swap rate in the case of negative jumps and vice versa.

Table 4.6 shows the mean of VIX index level and mean of square root of one month daily realized variance yearly from 1990-2006. It appears that the VIX index level is mean reverting. We have shown in section 4.3 that if variance follows a mean reverting process as in the SV and SVJ models, then the square of VIX is an affine function of instantaneous variance and hence it is also mean reverting.

Next we investigate the empirical and model profit and loss from investing in variance swaps and VIX futures. An investor who has a short position in the discrete variance swap will have a payoff equal to the difference between the expectation of one month realized variance under the risk neutral pricing measure $Q$ or the one month variance swap rate and the realized one month variance:

\[
P&L_1 = E^Q \left[ \frac{n}{(n-1)\tau} \sum_{i=0}^{n-1} \left( \ln \left( \frac{S_{i+1}}{S_i} \right) \right)^2 \right] - \left[ \frac{n}{(n-1)\tau} \sum_{i=0}^{n-1} \left( \ln \left( \frac{S_{i+1}}{S_i} \right) \right)^2 \right] \\
\geq VIX^2 - \left[ \frac{n}{(n-1)\tau} \sum_{i=0}^{n-1} \left( \ln \left( \frac{S_{i+1}}{S_i} \right) \right)^2 \right] \tag{4.5.6}
\]
From equation (4.5.5) the square of the VIX index value is an under approximation of one month discrete realized variance under the risk neutral probability measure $Q$ or one month variance swap rate. Hence in Table 4.6 the column RVOL gives average one month discrete realized volatility and the column VIX can be used as a proxy for the square root of one month variance swap rate. The second-to-last column in Table 4.6 shows the yearly difference between VIX index level and one month realized volatility. It can be seen from the Table 4.6 that the difference values varies from year to year but is always positive. This difference is due to the $P$ and $Q$ measures and these results show that volatility has a negative risk premium historically.

Table 4.7: Empirical and theoretical monthly returns from investing in variance swaps

<table>
<thead>
<tr>
<th>Observations</th>
<th>Average (%)</th>
<th>Stdev (%)</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Empirical (4273)</td>
<td>39.6</td>
<td>36.6</td>
<td>-2.4</td>
<td>12.3</td>
</tr>
<tr>
<td>SVJ model</td>
<td>46.1</td>
<td>41.5</td>
<td>-5.5</td>
<td>107.2</td>
</tr>
</tbody>
</table>

This Table shows the empirical and theoretical monthly returns from a short position in variance swaps. The empirical returns are computed using historical SPX index and SPX options data from 1985-2005. The SVJ model returns are computed using simulation.

Figure 4.7: These figures show the historical and theoretical (from SVJ model) monthly returns from a short position in one month variance swaps.
Table 4.7 shows the historical and theoretical returns from a short position in one month variance swaps. The empirical returns are computed from SPX index and SPX options data. Using the historical daily SPX index data we compute the one month realized variance and from the SPX options data we compute the one month variance swap rate. We compute the monthly return as profit and loss divided by the variance swap rate. We compute these monthly returns historically from 1985-2005. To compute the model monthly returns we compute the one month variance swap rate in the SVJ model and compute the one month realized variance on 100,000 simulations paths of the SVJ model. As can be seen from the results that from a short position in variance swaps monthly returns are highly positive empirically and theoretically which implied that volatility has negative risk premium.

Next, we analyze the historical and theoretical monthly returns from investing in the VIX futures. An investor who takes a short position at time $t$ in the VIX future maturing at time $t + \tau$ will have a payoff equal to the difference between the VIX futures value at time $t$, $F_{t,t+\tau}$ and the VIX index level at time $t + \tau$, $\text{VIX}_{t+\tau}$. Hence the payoff can be represented as:

$$P&L = F_{t,t+\tau} - \text{VIX}_{t+\tau}$$  (4.5.7)

Since VIX futures started trading in 2004, we have limited observations of VIX futures contracts available to compute the historical monthly returns. Table 4.8 shows the monthly returns from investing in VIX futures of one month maturity from all historical futures contracts available and from the SVJ model. Results shows the mean, standard
deviation, 95% confidence interval, maximum and minimum value of monthly returns from a short position in VIX futures contract.

Table 4.8: Monthly returns from investing in VIX futures

<table>
<thead>
<tr>
<th>maturity (days)</th>
<th>Observations</th>
<th>Average</th>
<th>Stdev</th>
<th>CI (95 %) Upper</th>
<th>CI (95 %) Lower</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 empirical (28)</td>
<td>9.84</td>
<td>12.27</td>
<td>14.38</td>
<td>5.30</td>
<td>-0.35</td>
<td>2.25</td>
<td></td>
</tr>
<tr>
<td>30 SVJ model</td>
<td>5.21</td>
<td>8.81</td>
<td>5.25</td>
<td>5.18</td>
<td>-0.43</td>
<td>2.97</td>
<td></td>
</tr>
</tbody>
</table>

This table shows the empirical and theoretical monthly returns from investing in VIX futures. The empirical returns are computed from all VIX futures contract of one month maturity. There are 28 such contracts available. The model returns are computed using SVJ model.

Figure 4.8: These figures show the historical and theoretical (from SVJ model) monthly returns from a short position in one month VIX futures contracts.

4.6 Conclusion

In this chapter we discuss the replication of VIX index level and pricing of VIX futures. We show that there is a difference in the market VIX index level and the theoretical VIX index level. This difference is due to two reasons. First because of discrete strikes of SPX options available and second due to a finite range of strikes of SPX options. In general, mostly the second effect dominates and hence, VIX index is an under approximation of the theoretical VIX value. Also, from historical VIX index series we observe that the VIX level is mean reverting and hence, using the SV and SVJ models we can model the
VIX index and VIX futures prices. In the SV and SVJ models the square of the VIX index is an affine function of the instantaneous variance. When there are no jumps then the square of the theoretical VIX is equal to the one month continuous variance swap rate and in the case of jumps it under approximates the one month continuous variance swap rate. Hence the square of the market VIX under approximates the one month continuous variance swap rate by a slightly larger amount in the case of negative jumps (the SVJ model) as compared to no jumps (the SV model) case. Then we present two different methods to price the VIX futures in the SV and SVJ models. Both methods, numerical integration using density and transform inversion are quite fast and very accurate in computing the price. We also show that the convexity correction formula is not accurate to compute the VIX futures price and the excess probability depends on the time to maturity and the SV and SVJ model parameters. Then we test the pricing of futures using market options prices. We fit the market option prices to the SV and SVJ model and then using those parameters we compute the VIX futures prices and compared with market VIX futures prices. We found that the SVJ model is better in fitting market VIX futures prices and market options prices compared to the SV model. Then we present the descriptive statistics of the VIX index level and one month realized variance computed using daily S&P 500 index prices and at-the-money implied volatility from S&P 500 index option prices. These results show that the one month volatility from VIX index is higher than the mean at-the-money one month implied volatility from S&P 500 option prices which is higher than the one month realized volatility of S&P 500 index prices. We analyze the historical profit and loss from investing in a short position in variance
swaps. This historical monthly returns are highly positive implying a negative volatility risk premium. The historical monthly returns from investing in a short position in VIX futures is also positive but returns are lower compared to the variance swaps.
Chapter 5

Asset Allocation and Generalized Buy-and-Hold Strategies

5.1 Introduction

Ever since the pioneering work of Merton (1969, 1971), Samuelson (1969) and Hakansson (1970), considerable progress has been made in solving dynamic portfolio optimization problems. These problems are ubiquitous: individual agents, pension and mutual funds, insurance companies, endowments and other entities all face the fundamental problem of dynamically allocating their resources across different securities in order to achieve a particular goal. These problems are often very complex owing to their dynamic nature and high dimensionality, the complexity of real-world constraints, and parameter uncertainty. Using optimal control techniques, these researchers and others\footnote{See, for example, Kim and Omberg (1996), Liu (1998), Merton (1990) and the many references cited therein.} solve for the optimal dynamic trading strategy under various price dynamics in frictionless markets.

Optimal control techniques dominated until martingale techniques were introduced by Cox and Huang (1989) and Karatzas, Lehocky and Shreve (1987). Under complete market assumptions, they showed how the portfolio choice problem could be decomposed into two subproblems. The first subproblem solved for the optimal terminal wealth, a
problem which could be formulated as a static optimization problem given the complete
markets assumption. The second subproblem then solved for the trading strategy that
replicated the optimal terminal wealth. This new approach succeeded in expanding the
class of dynamic problems that could be solved.

Dual methods were then used by a number of authors (Xu 1990, Shreve and Xu 1992a
1991b) to extend the martingale approach to problems where markets are incomplete
and agents face portfolio constraints. Duality methods have since been very popular\(^2\)
for tackling other classes of portfolio optimization problems. These include, for example,
problems with transaction costs and models where trading impacts security prices.

Applying some of these dual methods, Haugh, Kogan and Wang\(^3\) (2006) showed how
suboptimal dynamic portfolio strategies could be evaluated by using them to compute
lower and upper bounds on the expected utility of the true optimal dynamic trading
strategy. In general, the better the suboptimal solution, the narrower the gap between
the lower and upper bounds, and the more information you therefore have regarding
how far the sub-optimal strategy is from optimality. These techniques apply directly to
multidimensional diffusion processes with incomplete markets and portfolio constraints
such as no-short selling or no borrowing constraints.

The first goal of this chapter is to evaluate in further detail the dual-based approach
of HKW. Assuming the same price dynamics as HKW, we use a simple application
of Ito's Lemma to derive a closed form solution for the optimal wealth and expected
\(^2\)See Rogers (2003) for a survey of some of the more recent advances.
\(^3\)Hereafter referred to as HKW.
utility of terminal wealth when security price dynamics are predictable and a static\textsuperscript{4}, i.e., constant proportion, trading strategy is employed. This strategy is often considered by researchers who wish to estimate the value of predictability in security prices to investors. Though this closed form solution is particularly simple to derive, we have not seen it presented elsewhere and it has a number of applications which enable us to further analyze the dual-based portfolio evaluation approach proposed by HKW. First, we can use it to compute the precise upper bound as originally proposed by HKW. In order to compute this upper bound, the static strategy's value function and its derivatives with respect to any state variables are required. Because they were unknown to HKW, they constructed an alternative upper bound. Though this alternative upper bound appears to perform well when the sub-optimal strategy is close to optimal, it is less satisfying from a theoretical perspective. The closed-form solution for the static value function that we derive in this chapter enables us to compute the more theoretically satisfying upper bound and compare it to the alternative bound used by HKW.

Second, we can use the closed-form solution for the static value function to optimize the upper bound over all static strategies when markets are incomplete. We show that in general, there are infinitely many strategies that minimize the upper bound. While none of these coincide with the static strategy, \( \theta_{\text{static}} \), that maximizes the lower bound, i.e., the expected utility, \( \theta_{\text{static}} \) does generate an upper bound that is almost indistinguishable from the optimal upper bound.

Third, our analytic solution to the static strategy value function may also be used

\textsuperscript{4}One of the principal results of the early literature (Merton 1969 and Samuelson 1969) is that a static trading strategy is optimal when the optimizing agent has constant relative risk aversion and security returns are independent and identically distributed.
to analytically compute the *myopic* strategy as a function of time and state variable(s). This obviates the need to compute the myopic strategy numerically and results in greater computational efficiency when using myopic strategies to generate upper bounds as in HKW.

Finally, since the analytic solution for the static strategy's terminal wealth depends only on the terminal security price we can conclude that the optimal GBH strategy (defined below) is always superior to the optimal static strategy.

The second goal of this chapter is to use the dual-based portfolio evaluation technique to evaluate the optimal _generalized buy-and-hold_ (GBH) strategy. We define the GBH strategies to be the class of strategies where the terminal wealth is a function of only the terminal security prices. In contrast, the terminal wealth of a static buy-and-hold strategy is always an affine function of terminal security prices. When markets are incomplete, we analyze the GBH strategy in a dynamic framework and use the dual evaluation approach to determine when the optimal GBH strategy is close to optimal.

Haugh and Lo (2001) is most relevant to this work. They considered buy-and-hold portfolios where at time 0 the optimizing agent could take positions in a stock, European options of various strikes on that stock and a cash account earning the risk-free rate. When the objective of the agent is to maximize the expected utility of terminal wealth, $W_T$, they showed that the optimal buy-and-hold portfolio was often comparable (in terms of expected utility) to the optimal dynamic strategy where the agent was free

---
5 More generally, any study of myopic strategies under the same class of price dynamics will benefit from the closed-form solution to the static strategy value function.
6 Other researchers have also considered the problem of adding options to the portfolio optimization problem. See, for example, Evnine and Henriksson (1987) and Carr and Madan (2001).
to trade continuously in the time interval $[0, T]$. They considered the case of a single stock and assumed that the dynamic strategy was not subject to portfolio constraints or other market frictions. In the case of geometric brownian motion (GBM) dynamics, we know from Merton (1969, 1971) that the terminal wealth, $W^*_T$, resulting from the optimal dynamic strategy is a function of only the terminal stock price, $P_T$, so that $W^*_T = W^*_T(P_T)$. In that case it is clear that the buy-and-hold portfolio with European options can approach the optimal dynamic solution if options with all possible strikes are available.

When security dynamics incorporate predictability, however, and are therefore no longer governed by GBM dynamics, the expected utility of the optimal buy-and-hold strategy, $E[u(W^{bh}_T)]$, will in general be bounded\(^7\) away from the expected utility of the optimal dynamic trading strategy, $E[u(W^*_T)]$. This will be true even when options with all possible strikes are available for the buy-and-hold portfolio. This is clear because $W^{bh}_T$ must, by construction, be a function of the terminal value, $P_T$ say, of the underlying security, whereas $W^*_T$ will in general be path dependent. In that case, Haugh and Lo computed the optimal GBH portfolio and compared it to the optimal dynamic trading strategy. Because a buy-and-hold portfolio with just a few well chosen options were sufficient to approximate the optimal GBH portfolio, they could therefore determine when a static buy-and-hold portfolio with just a few options could be used instead of adopting a dynamic trading strategy.

\(\text{In this chapter we use the techniques of HKW in order to extend Haugh and Lo}\)

\(^7\text{In related work, Kohn and Papazoglu (2004) identify those diffusion processes where the optimal dynamic trading strategy results in a terminal wealth that is a function of only the terminal security price. They do this in a complete markets setting.}\)
(2001) to the multi-dimensional diffusion setting with portfolio constraints. Rather than explicitly considering static portfolios with (generalized) European options, we consider instead the GBH strategies\(^8\) where terminal wealth is restricted to be a function of only the terminal security prices. Dual methods are used to accurately estimate\(^9\) the value function of the optimal dynamic trading strategy which we can then compare to the optimal GBH value function.

We consider how well the optimal GBH strategy performs when compared to the optimal dynamic strategy under various portfolio constraints. We find that when markets are incomplete and there are no portfolio constraints, the dynamic unconstrained portfolio often significantly outperforms the optimal GBH strategy. Once portfolio constraints are imposed, however, the GBH portfolio can often have a much higher expected utility than the optimal dynamic portfolio. In order to draw this conclusion it will be necessary to assume the existence of some non-constrained agents in the market-place who can “sell” the GBH terminal wealth to constrained investors. While it is true that under this assumption there is nothing to stop these agents selling more general path-dependent portfolios to the constrained agents, we believe the simplicity of the GBH portfolios are more realistic and merit further study. We emphasize that once portfolio constraints are imposed, we can no longer view the GBH terminal wealth as the outcome of a dynamic strategy. Instead it is necessary to view it as a random variable that may be purchased from an unconstrained agent who can price it uniquely in the market-place.

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\(^8\)Note that the terminal wealth of any GBH portfolio in a multi-dimensional setting could be approximated, for example, by a static buy-and-hold portfolio that includes European options with payoffs of the form \(\prod_i (P_i^T - K_i)^+\) where \(P_i^T\) is the terminal price of the \(i^{th}\) risky asset and \(K_i\) is a fixed strike.

\(^9\)In fact, we use the myopic strategy to estimate the optimal value function. We can do this by using the dual evaluation technique to show that the myopic strategy is approximately optimal for the price processes and parameters that we consider later in the chapter.
The remainder of this chapter is organized as follows. Section 5.2 formulates the portfolio optimization problem and describes the classes of portfolio policies that we will consider in this chapter. Section 5.3 briefly reviews the portfolio duality theory and the dual approach for bounding the optimal expected utility for a given dynamic portfolio optimization problem. Section 5.4 contains numerical results and in Section 5.5 we solve for the static strategies that minimize the upper bound on the optimal expected utility. Finally we conclude in Section 5.6.

5.2 Problem Formulation and Trading Strategies

We now formulate\(^{10}\) the dynamic portfolio optimization problem and specify the three trading strategies that we will analyze: (i) the constant proportion or static\(^{11}\) trading strategy (ii) the myopic trading strategy and (iii) the GBH strategy. But first we formulate the general dynamic portfolio optimization problem.

5.2.1 The Portfolio Optimization Problem

The Investment Opportunity Set and Security Price Dynamics

We assume there are N risky assets and a single risk-free asset available in the economy. The time t vector of risky asset prices is denoted by \(P_t = (P_t^{(1)}, \ldots, P_t^{(N)})\) and the instantaneously risk-free rate of return is denoted by \(r_t\). Security price dynamics are

\(^{10}\)We follow HKW in our problem formulation.

\(^{11}\)The constant proportion trading strategy is commonly called the static strategy as the portfolio weights in the risky assets do not vary with time or changes in state variables. Nonetheless, it is a dynamic strategy.
driven by the \(M\)-dimensional vector of state variables \(X_t\) so that

\[
\begin{align*}
\tau_t &= r(X_t) \\
dP_t &= P_t[\mu_P(X_t)dt + \Sigma_P dB_t] \\
dX_t &= \mu_X(X_t)dt + \Sigma_X dB_t
\end{align*}
\tag{5.2.1}
\]

where \(X_0 = 0, B_t = (B_{1t}, ..., B_{Nt})\) is a vector of \(N\) independent Brownian motions, \(\mu_P\) and \(\mu_X\) are \(N\) and \(M\) dimensional drift vectors, and \(\Sigma_P\) and \(\Sigma_X\) are constant diffusion matrices of dimensions \(N\) by \(N\) and \(M\) by \(N\), respectively. We assume that the diffusion matrix, \(\Sigma_P\), of the asset return process is lower-triangular and non-degenerate so that \(x^T \Sigma_P \Sigma_P^T x \geq \epsilon \| x \|^2\) for all \(x\) and some \(\epsilon > 0\). Then we can define a process, \(\eta_t\), as

\[
\eta_t = \Sigma_P^{-1}(\mu_P t - \tau_t).
\]

In a market without portfolio constraints, \(\eta_t\) corresponds to the market price of risk process (Duffie 1996). We make the standard assumption that the process \(\eta_t\) is square integrable so that

\[
E_0 \left[ \int_0^T \| \eta_t \|^2 dt \right] < \infty.
\]

Note that return predictability in the price processes in (5.2.1) is induced only through the drift vector, \(\mu_P(X_t)\), and not the volatility, \(\Sigma_P\). In this case it is well known that in the absence of trading constraints, European option prices can be uniquely determined despite the fact that the market is incomplete. It is for precisely the same reason that any random variable, \(X_T\), can be attained as the terminal wealth of some dynamic trading strategy if \(X_T\) is a function of only the terminal security prices. It is this fact that will
allow us to use the straightforward martingale technique to identify the optimal GBH strategy.

Later in the chapter we will compare the performance of the optimal GBH strategy to the optimal dynamic trading strategy when there are dynamic trading constraints. This would appear to give rise to an inconsistency, however, since we can only determine the optimal GBP strategy when there are no trading constraints. This is resolved by making the assumption that it is only the agent in question who faces dynamic trading constraints. In particular, we implicitly assume\(^{12}\) that the market-place contains agents who do not face any such trading constraints. As a result, the presence of these non-constrained agents imply that any GBH terminal wealth can be priced uniquely. It is these unique prices that our agent uses when determining his optimal GBH wealth.

**Portfolio Constraints**

A portfolio consists of positions in the \( N \) risky assets and the risk-free cash account. We denote the proportional holdings of the risky assets in the total portfolio value by \( \theta_t = (\theta_{1t}, \ldots, \theta_{Nt}) \). The proportion in the risk-free asset is then given by \((1 - \theta^T_t 1)\) where \( 1 \) is the unit vector of length \( N \). To rule out arbitrage, we require the portfolio strategy to satisfy a square integrability condition, namely that \( \int_0^T \| \theta_t \|^2 \, dt < \infty \) almost surely.

\(^{12}\)A similar argument is often used to justify complete-market models and the Black-Scholes model, in particular, for pricing options. For example, a common criticism of these models states that if markets are complete then we shouldn't need derivative securities in the first place. The response to this is that the market is complete only for a small subset of agents whose presence allows us to uniquely price derivatives. For the majority of investors, the presence of trading frictions and constraints implies that derivative securities do add to the investment opportunity set.
The value of the portfolio, $W_t$, then has the following dynamics

$$
\frac{dW_t}{W_t} = \left[(1 - \theta_t^T \mathbf{1})r_t + \theta_t^T \mu_p(X_t)\right] dt + \theta_t^T \Sigma_p dB_t.
$$

(5.2.2)

We assume that the proportional holdings in the portfolio are restricted to lie in a closed convex set, $K$, that contains the zero vector. In particular, we assume that

$$
\theta_t \in K.
$$

(5.2.3)

If short sales are not allowed, for example, then the constraint set takes the form

$$
K = \{\theta : \theta \geq 0\}.
$$

(5.2.4)

If, in addition, borrowing is not allowed then the constraint set takes the form

$$
K = \{\theta : \theta \geq 0, 1^T \theta \leq 1\}
$$

(5.2.5)

where $1^T = (1, ..., 1)$.

**Investor Preferences**

We assume that the portfolio policy is chosen to maximize the expected utility, $E_0[U(W_T)]$, of wealth at the terminal date $T$. The function $U(W)$ is assumed to be strictly monotone with positive slope, concave and smooth. It is assumed to satisfy the Inada conditions at zero and infinity so that $\lim_{W \to 0} U'(W) = \infty$ and $\lim_{W \to \infty} U'(W) = 0$. In our numerical results, we assume the investor's preferences to be of the constant relative risk aversion (CRRA) type so that

$$
U(W) = W^{1-\gamma}/(1 - \gamma).
$$

(5.2.6)
The investor's dynamic portfolio optimization problem is to solve for the value function, $V_0$, at $t = 0$ where

$$V_0 \equiv \sup_{\theta_t} E_0[U(W_T)]$$  \hspace{1cm} (5.2.7)

subject to constraints (5.2.1), (5.2.2) and (5.2.3).

We now discuss in turn the static, myopic and GBH strategies, all three of which are suboptimal solutions to (5.2.7). Our principal goals in this chapter are twofold. First, we use Proposition 16 below to further analyze the dual approach to portfolio optimization. Second, we study the performance of the optimal GBH strategy and compare it to the classic constant proportion or static strategy, as well as the optimal dynamic strategy that solves (5.2.7). Since it is not possible to obtain the optimal strategy in closed form or even numerically for high-dimensional problems, we will use the myopic strategy instead as a proxy. For the class of price dynamics under consideration in this chapter, HKW used the dual evaluation approach to demonstrate that the myopic strategy is indeed often very close\textsuperscript{13} to optimal. We use the same duality techniques in this chapter to determine how far these sub-optimal strategies are from optimality.

5.2.2 The Static Trading Strategy

The static strategy ignores the predictability of stock returns and it is defined using the unconditional average returns, $\mu_0$, instead of time varying conditional expected returns.

\textsuperscript{13}In particular, the hedging demand is insignificant and investment decisions at any point in time are driven primarily by the instantaneous Sharpe ratios.
on the stocks. It may be found as the solution\textsuperscript{14} to
\[
\theta^{\text{static}} = \max_{\theta \in \mathcal{C}} \left( \mu_0^T - r \right) \theta - \frac{1}{2} \gamma \theta^T \Sigma \theta
\]  
(5.2.8)

In particular, under the optimal static trading strategy the agent re-balances his portfolio at each time $t$ so that he always maintains a constant (vector) proportion, $\theta^{\text{static}}$, of his time wealth invested in the risky assets. It is well known\textsuperscript{15} that this static strategy is an optimal policy in a dynamic model with a constant investment opportunity set and cone constraints on portfolio positions. We have the following simple proposition showing that, under the price dynamics in (5.2.1), the terminal wealth resulting from any static trading strategy is a function of only the terminal security prices.

**Proposition 16** Suppose price dynamics satisfy (5.2.1) and a static trading strategy is followed so that at each time $t \in [0, T]$ a proportion, $\theta$, of time $t$ wealth is invested in the risky assets with $1 - \theta^T \mathbf{1}$ invested in the risk-free asset. Then the terminal wealth, $W_T$, resulting from this strategy only depends on the terminal prices of the risk assets, $P_T$. In particular, we have
\[
W_T = W_0 \exp \left( (1 - \theta^T \mathbf{1}) \gamma T + \frac{1}{2} \theta^T (\text{diag}(\Sigma \Sigma^T) - \Sigma \Sigma^T) \gamma T + \theta^T \left( \ln \frac{P_T}{P_0} \right) \right)
\]  
(5.2.9)

**Proof:** See Appendix A.

While straightforward to derive and perhaps not particularly surprising, we have not seen the statement of Proposition (16) elsewhere. Moreover, it has a number of\textsuperscript{14}Both (5.2.8) and (5.2.11) are standard and can easily be obtained by formulating the agent’s problem as a standard control problem. The corresponding HJB equations then lead immediately to (5.2.8) and (5.2.11).
\textsuperscript{15}See Merton (1969,1971) or more recently, Section 6.6 of Karatzas and Shreve (1998).
applications. First, the static strategy is typically used as a base case when researchers study the value of predictability in security prices. Since predictability is often induced via the drift term as in (5.2.1), the expression in (5.2.9) applies. This means that the expected utility of the static strategy can often be determined in closed form when the distribution of $P_t$ is also known. For example, if $\log(P_t)$ is a (vector) Gaussian process, then $W_T$ is log-normally distributed and

$$V_t^{\text{static}} = E_t[W_T^{1-\gamma} / (1 - \gamma)]$$  \hspace{1cm} (5.2.10)

can be computed analytically. This is obviously much more efficient than computing $V_t^{\text{static}}$ numerically by simulating the underlying stochastic differential equations for $X_t$, $P_t$ and $W_t$.

This latter simulation approach was used by HKW when using the static strategy to compute lower and upper bounds on the expected utility, $V_0$, of the true optimal dynamic trading strategy. Moreover, because the analytic expression for $V_t^{\text{static}}$ in (5.2.9) was unavailable, HKW were unable to compute the more theoretically satisfying upper bound on $V_0$ that their algorithm prescribed. Using (5.2.9), it is straightforward to compute that precise upper bound and we present the corresponding results in Section 5.4 and Appendix A. This is the second application of Proposition 16.

Third, the ability to compute $V_t^{\text{static}}$ analytically also implies that the optimal static strategy can be solved by directly maximizing $V_t^{\text{static}}$ over $\theta$ instead of solving (5.2.8). This is also useful when solving for the optimal myopic strategy which we describe in Section 5.2.3. In particular, solving for the optimal myopic strategy requires solving for a particular optimal static strategy at each time $t \in [0, T]$. An analytic expression for
$V_t^{\text{static}}$ would therefore reduce the computational burden of simulating and solving for the myopic trading strategy.

Fourth, we can use Proposition 16 to determine the static strategy or strategies that minimizes the upper bound. We will see that the optimal static strategy, $\theta^{\text{static}}$, does not coincide with any of the static strategies that minimize the upper bound. While this result is interesting in its own right, we observed nonetheless that $\theta^{\text{static}}$ generates upper bounds that are almost indistinguishable from those generated by the static strategies that minimize the upper bound.

Finally, because the terminal wealth of the optimal static strategy depends only on the terminal security prices, we also have the following corollary.

**Corollary 1** Assuming the price dynamics in (5.2.1), an agent with CRRA utility will always prefer the optimal GBH strategy to any static strategy.

### 5.2.3 The Myopic Trading Strategy

The *myopic* strategy is defined in the same way as the static policy except now the instantaneous moments of asset returns are fixed at their current values, as opposed to their long-run average values. In particular, at each time $t$ the agent invests a (vector) proportion, $\theta_t^{\text{myopic}}$, of his time $t$ wealth in the risky assets where $\theta_t^{\text{myopic}}$ solves

$$\theta_t^{\text{myopic}} = \arg \max_{\theta \in \mathcal{K}} \left( \mu^T_p(X_t) - r \right) \theta - \frac{1}{2} \gamma \theta^T \Sigma_p \Sigma_p^T \theta. \quad (5.2.11)$$

The approximate policy in (5.2.11) ignores the hedging component of the optimal trading strategy. In particular, at each time $t$ the agent observes the instantaneous
moments of asset returns, $\mu^P_t(X_t)$ and $\Sigma_P$, and, assuming that these moments are fixed from time $t$ onwards, he solves for the optimal static trading strategy.

Because we do not have a closed-form expression for the terminal wealth resulting from the myopic strategy, we estimate its expected utility by simulating the stochastic differential equations for $X_t$, $P_t$ and $W_t$, solving (5.2.11) at each discretized point on each simulated path. Moreover, we can use Proposition 16 to solve this problem analytically.

5.2.4 The Generalized Buy and Hold (GBH) Trading Strategy

As stated earlier, a GBH strategy is any strategy resulting in a terminal wealth that is a function of only the terminal security prices. When the optimizing agent does not face any trading constraints, the optimal GBH strategy may be implemented through a dynamic trading strategy. Using the results of Section 5.3, we can therefore use this dynamic strategy to compute both lower and upper bounds on the optimal value function, thereby indicating how far the optimal GBH strategy is from optimality.

When the agent does face trading constraints, the optimal GBH strategy is in general no longer attainable from a dynamic trading strategy. In this case we rely on our implicit assumption that there are other unconstrained agents in the marketplace who can replicate and therefore uniquely price any GBH strategy. The constrained agent is then assumed to 'purchase' his optimal GBH terminal wealth from one of these unconstrained agents. We then compare the expected utility of the agent's optimal GBH terminal wealth to the optimal wealth that could be attained from a constrained dynamic trading strategy. We will see in Section 5.4 that the GBH strategy often significantly outperforms the optimal constrained dynamic trading strategy.
One possible criticism of this analysis is to ask why the constrained agent should restrict himself to purchasing a GBH terminal wealth from an unconstrained agent. Instead, acting as though he was unconstrained, the agent could compute his optimal terminal wealth and then ‘purchase’ this wealth from one of the unconstrained agents who can actually replicate it. While this criticism has some merit, we believe that the GBH strategies are simple to understand and, as a straightforward generalization of the well known buy-and-hold strategy, deserve attention in their own right.

Moreover, we believe that many investors care more about the final level of security prices, rather than the path of security prices, when they are evaluating their investment performance. They are aware that they generally do not possess market-timing skills but, at the same time, they do not wish to ‘miss the boat’ on a sustained bull market, for example. Clearly, access to buy-and-hold strategies would be of particular interest to such investors.

We now outline the steps required for computing and evaluating the optimal GBH strategy. Further details are provided in Appendix B where we specialize to the price dynamics assumed in Section 5.4.

1. **Solving the SDE**: We first solve the stochastic differential equation (SDE) for the price processes, $P_t$, and state variable, $X_t$, under both the real world probability measure, $P$, and any risk neutral probability measure, $Q$. Recall that since markets are incomplete, a unique risk-neutral measure does not exist.

2. **Compute the Conditional State Price Density**: We compute the state price
density, $\pi^b_t$, conditional on the terminal security prices. In particular, we solve for

$$\pi^b_T(\omega) = E^P_0 \left( \pi^b_T(\omega) \mid P^D_T(\omega) = b, i = 1, \ldots, N \right)$$  \hspace{1cm} (5.2.12)

where $\omega$ is a sample outcome. It is worth mentioning that while there are infinitely many state price-density processes, $\pi_t$, we can use any such process on the right-hand-side of (C.2.10) and obtain the same conditional state price density, $\pi^b_T$.

3. **Compute Optimal GBH Wealth**: We then use static martingale approach to solve for the optimal GBH strategy. In particular we solve

$$V^gbh = \sup_{W_T} E^P_0 \left[ \frac{W_T^{1-\gamma}}{1 - \gamma} \right]$$

subject to $\ E^P_0 [\pi^b_T W_T] = W_0. \hspace{1cm} (5.2.13)$

Note that because of our use of the conditional state price density, $\pi^b_T$, in (C.2.12), we did not need to explicitly impose the constraint that $W_T$ be a function of only the terminal security prices. This constraint will be automatically satisfied.

4. **Determine the Value Function at all Intermediate Times**: Compute the GBH value function, $V^gbh_t$, for all $t \in [0,T]$. This is the same problem we solved in step 4.

5. **Determine the Replicating Trading Strategy**: Once the optimal GBH wealth, $W^gbh_T$, has been determined, we compute the replicating strategy, $\theta^gbh_t$, that attains $W^gbh_T$.

\[16\text{This is consistent with our earlier observation that all European options prices can be uniquely determined despite the market incompleteness.}\]
6. Compute Lower and Upper Bounds on the Global Optimal Value function: Using the GBH trading strategy, $\theta_t^{gbh}$, and the GBH value function, $V_t^{gbh}$, we can compute an upper bound on the optimal value function, $V_t$, for the problem in (5.2.7). This step is done using the duality-based algorithm described in Section 5.3. Note that $V_t^{gbh}$ constitutes a lower bound on the optimal value function. If the lower and upper bounds are close to one another, then we can conclude that the optimal GBH strategy is indeed close to the true optimal solution. When the GBH investor faces trading constraints we will compare $V_t^{gbh}$ to the optimal value function $V_t$ that results from dynamic trading with constraints.

5.3 Review of Duality Theory and Construction of Upper Bounds

In this section we briefly review the duality approach of HKW for analyzing the quality of a suboptimal strategy. This is done by using the suboptimal strategy to construct a lower and upper bound on the true value function. If the difference between the two bounds is large, i.e. the duality gap is wide, then it suggests that the suboptimal policy is not close to the optimal solution. If the duality gap is narrow, then (i) we know that the suboptimal strategy is close to optimal and (ii) we know approximately the optimal value function. In this chapter we will use the myopic policy to construct an upper bound on the optimal dynamic trading strategy. As we shall see\textsuperscript{17} in Section 5.4, the upper bound will be close to the corresponding lower bound. We will therefore have an accurate approximation to the expected utility of the optimal dynamic trading strategy.

\textsuperscript{17}These results were reported in Haugh, Kogan and Wang (2006).
with which we can compare the optimal GBH strategy.

Starting with the portfolio optimization problem of Section 5.2.1, we can define a fictitious problem \( P(\nu) \), based on a different financial market and without the portfolio constraints. First we define the support function of \( K \), \( \delta(\cdot) : \mathbb{R}^N \rightarrow \mathbb{N} \cup \infty \), by setting

\[
\delta(\nu) = \sup_{x \in K}(-\nu^T x). \tag{5.3.1}
\]

The effective domain of the support function is given by

\[
\tilde{K} = \{ \nu \in K : \delta(\nu) < \infty \}.
\]

Because the constraint set \( K \) is convex and contains zero, the support function is continuous and bounded from below on its effective domain \( \tilde{K} \). We then define the set \( D \) of \( \mathcal{F}_t \)-adapted \( \mathbb{R}^N \) valued processes to be

\[
D = \left\{ \nu_t, 0 \leq t \leq T : \nu_t \in \tilde{K}, \mathbb{E}_0 \left[ \int_0^T \delta(\nu_t) dt \right] + \mathbb{E}_0 \left[ \int_0^T \|\nu_t\|^2 dt \right] < \infty \right\}. \tag{5.3.2}
\]

For each process \( \nu \) in \( D \), we define a fictitious market \( M(\nu) \). In this market, one can trade the \( N \) stocks and the risk-free cash account. The diffusion matrix of stock returns in \( M(\nu) \) is the same as in the original market. However, the risk-free rate and the vector of expected stock returns are different. In particular, the riskfree rate process and the market price of risk in the fictitious market are defined respectively by

\begin{align*}
\eta^{(\nu)}_t &= \eta_t + \delta(\nu_t) \tag{5.3.3a} \\
\eta^{(\nu)}_t &= \eta_t + \Sigma_p^{-1} \nu_t \tag{5.3.3b}
\end{align*}

where \( \delta(\nu) \) is the support function defined in (5.3.1). We assume that \( \eta^{(\nu)}_t \) is square-integrable. Following Cox and Huang (1989), the state-price density process \( \pi^{(\nu)}_t \) in the
fictitious market is given by

\[ \pi_t^{(\nu)} = \exp \left( - \int_0^t r_s^{(\nu)} ds - \frac{1}{2} \int_0^t \eta_s^{(\nu)\top} \eta_s^{(\nu)} ds - \int_0^t \eta_s^{(\nu)\top} dB_s \right) \]  

(5.3.4)

and the vector of expected returns is given by

\[ \mu_t^{(\nu)} = r_t^{(\nu)} + \Sigma_P \eta_t^{(\nu)}. \]

The dynamic portfolio choice problem in the fictitious market without position con­straints can be equivalently formulated in a static form\(^{18}\):

\[ V^{(\nu)} = \sup_{\{W_t\}} E_0 \left[ U(W_T) \right] \text{ subject to } E_0 \left[ \pi_T^{(\nu)\top} W_T \right] \leq W_0. \]  

(\(\mathcal{P}^{(\nu)}\))

Due to its static nature, the problem (\(\mathcal{P}^{(\nu)}\)) is easy to solve. For example, when the utility function is of the CRRA type with relative risk aversion \( \gamma \) so that \( U(W) = W^{1-\gamma}/(1-\gamma) \), the corresponding value function in the fictitious market is given explicitly by

\[ V_0^{(\nu)} = \frac{W_0^{1-\gamma}}{1-\gamma} E_0 \left[ \pi_T^{(\nu)\top} W_T \right]^{\frac{1}{1-\gamma}}. \]  

(5.3.5)

It is easy to see that for any admissible choice of \( \nu \in D \), the value function in (5.3.5) gives an upper bound for the optimal value function of the original problem. In the fictitious market, the wealth dynamics of the portfolio are given by

\[ dW_t^{(\nu)} = W_t^{(\nu)} \left[ \left( r_t^{(\nu)} + \theta_t^{(\nu)\top} \Sigma_P \eta_t^{(\nu)} \right) dt + \theta_t^{(\nu)\top} \Sigma_P dB_t \right] \]  

(5.3.6)

so that

\[ \frac{dW_t^{(\nu)}}{W_t^{(\nu)}} - \frac{dW_t}{W_t} = \left[ r_t^{(\nu)} - r_t + \theta_t^{(\nu)\top} \Sigma_P \left( \eta_t^{(\nu)} - \eta_t \right) \right] dt + \left( \delta(v_t) + \theta_t^{(\nu)\top} \nu_t \right) dt. \]

---

The last expression is non-negative according to (5.3.1) since $\theta_t \in K$. Therefore, $W_t^{(\nu)} \geq W_t \forall t \in [0, T]$ and so

$$V_0^{(\nu)} \geq V_0.$$  \hfill (5.3.7)

Under fairly general assumptions, it can be shown that there exists a process, $\nu^*$, such that (5.3.7) holds with equality. While one can pick any fictitious market from the admissible set $D$ to compute an upper bound, HKW showed how a given suboptimal strategy, $\tilde{\theta}_t$, may be used to select a particular $\tilde{\nu}_t \in D$. If the suboptimal strategy is in fact optimal, then the lower bound associated with the suboptimal strategy will equal the associated upper bound, thereby demonstrating its optimality.

Given an approximation to the optimal portfolio policy $\tilde{\theta}_t$, one can compute the corresponding approximation to the value function, $\tilde{V}_t$, defined as the conditional expectation of the utility of terminal wealth, under the portfolio policy $\tilde{\theta}_t$. We then define

$$\tilde{\eta}_t := -W_t \left( \frac{\partial W \tilde{V}_t}{\partial W \tilde{V}_t} \right) \Sigma^{-1}_W \tilde{\theta}_t - \left( \frac{\partial W \tilde{V}_t}{\partial W \tilde{V}_t} \right)^{-1} \Sigma^{-1}_W \left( \frac{\partial W X \tilde{V}_t}{\partial W X \tilde{V}_t} \right)$$

(5.3.8)

where $\partial W$ denotes the partial derivative with respect to $W$, and $\partial W X$ and $\partial W W$ are corresponding second partial derivatives. We then define $\tilde{\eta}_t$ as a solution to (5.3.3b).

In the special but important case of a CRRA utility function the expression for $\tilde{\eta}_t$ simplifies. In the case of a CRRA utility function, for a given trading strategy, $\tilde{\theta}_t$, the corresponding value function is of the following form

$$\tilde{V}_t = g(t, X_t) \frac{W_t^{1-\gamma}}{1-\gamma}$$

Hence, the market price of risk in the dual problem simplifies to

\[ \text{See HKW (2006) who motivate this definition of } \tilde{\eta}_t. \]
\[
\tilde{\eta}_t = \gamma \Sigma_p \tilde{\theta}_t - \frac{\Sigma_{\eta}}{V_t} \left( \frac{\partial \tilde{V}_t}{\partial X_t} \right) = \gamma \Sigma_p \tilde{\theta}_t - \frac{\Sigma_{\eta}}{g(t, X_t)} \left( \frac{\partial g(t, X_t)}{\partial X_t} \right)
\]  \tag{5.3.9}

where \(\gamma\) is the relative risk aversion coefficient of the utility function, and one only needs to compute the first derivative of the value function with respect to the state variables, \(X_t\), to evaluate the second term in (5.3.9). This simplifies numerical implementation, since it is easier to estimate first-order than second-order partial derivatives of the value function. In the case of the static trading strategy, the analytic expression of Proposition 16 will enable us to compute an analytic expression for the partial derivatives. For the GBH trading strategy we can compute the value function and its derivatives analytically. These calculations are given in Appendix B. But for more general strategies e.g., myopic and others, we don’t have an analytical solution for the value function and its derivatives.

Obviously, \(\tilde{\eta}_t\) is a candidate for the market price of risk in the fictitious market. However, there is no guarantee that \(\tilde{\eta}_t\) and the corresponding process \(\tilde{v}_t\) belong to the feasible set \(D\) defined by (5.3.2). In fact, for many important classes of problems the support function \(\delta(\nu_t)\) may be infinite for some values of its argument. We therefore look for a price-of-risk process \(\tilde{\eta}_t \in D\) that is “close” to \(\tilde{\eta}_t\) by formulating a simple quadratic optimization problem. Depending on the portfolio constraints, this problem may be solved analytically. Otherwise, we solve it numerically at each discretization point on each simulated path of the underlying SDE’s. The lower bound is then computed by simulating the given portfolio strategy. The same simulated paths of the SDE’s are then used to estimate the upper bound given by (5.3.5). At each discretization point on
each simulated path we solve a quadratic optimization problem to find the appropriate
\( \tilde{\eta}_t \in D \). See HKW for further details.

It is worth mentioning that when HKW were computing the upper bounds corre-
sponding to the static and myopic strategies, they only used the first term in the right-
hand-side of (5.3.9) as an expression for the second term was unavailable. While the
resulting bounds were still valid upper bounds, they were not the precise bounds as
prescribed by their algorithm. In this chapter, the knowledge of Proposition 16 means
that we use both terms\(^{20}\) on the right-hand-side of (5.3.9) to derive the upper bound
corresponding to the static strategy. For the particular model and parameters of Section
5.4, it turns out that there is almost no discernable difference between the two. There
is no guarantee, however, that this will always be the case.

### 5.4 Numerical Results

We use the same\(^{21}\) model specification as that of HKW who in turn specify their model
as a continuous time version of the market model in Lynch (2001). In particular, our
model dynamics are as specified in (5.2.1), but now we assume that there are three risky
securities and one state variable so that \( N = 4 \) and \( M = 1 \). We assume the drift of the
asset returns, \( \mu(X_t) \), is an affine function of the single state variable, \( X_t \), which follows
a mean reverting Ornstein-Uhlenbeck process. Hence, it is an incomplete market model.

\(^{20}\)As in HKW, we continue to omit the second term in (5.3.9) when computing the upper bound corresponding to the myopic strategy. We can still conclude, however, that the myopic strategy is very close to the optimal as the lower and upper bounds are very close to each other. Haugh and Jain (2007) show how cross-path regressions and pathwise estimators can be used to efficiently estimate the second term in (5.3.9) for the myopic strategy.

\(^{21}\)By using the same model specification, we can also compare the performance of the static upper bound that we compute using both terms from the right-hand-side of (5.3.9) with the upper bound computed in HKW that only used the first term.
The asset return dynamics satisfy

\[ r_t = r \]

\[ dP_t = P_t[(\mu_0 + X_t \mu_1)dt + \Sigma_P dB_t^P] \]

\[ dX_t = -kX_t dt + \Sigma_X dB_t^P. \]  

(5.4.1)

The first equation gives the risk free rate which is assumed to be constant in our numerical results. The second equation specifies the dynamics of the three traded risky securities. The diffusion matrix \( \Sigma_X \) is of size 1 by 4 and coincides with last row of matrix \( \Sigma_P \). The vectors \( \mu_0 \) and \( \mu_1 \) define the drift vector for the risky securities. The third equation specifies the dynamics of the state variable, \( X_t \), whose initial value is set to zero in all of the numerical examples.

Lynch considered two choices for the state variable: (i) the dividend yield and (ii) the term spread. The dividend yield captures the rate at which dividends are paid out as a fraction of the total stock market value. The term spread is the difference in yields between twenty year and one month Treasury securities. Both of these predictive variables are normalized to have zero mean and unit variance. Lynch also considered two sets of risky assets: (i) portfolios obtained by sorting stocks on their size and (ii) portfolios obtained by sorting stocks on their book-to-market ratio. The two choices of risky assets and the two choices of the predictive variable result in four sets of calibrated parameter values. These are reported in Table 5.1. We set the risk-free rate, \( r \), equal to 0.01 throughout.

As mentioned earlier, we assume that the utility function is of the constant relative risk aversion (CRRA) type so that \( U(W) = W^{(1-\gamma)/(1-\gamma)} \). We consider three values
for the relative risk aversion parameter $\gamma = 1.5, 3, \text{ and } 5$. We consider two values for the time horizon: $T = 5$ and $T = 10$ years. When simulating the SDE's we use 100 discretization points per year.

We consider three types of market constraints: (1) the base case where the agent does not face any trading constraints (2) the agent faces no-short-sales and no-borrowing constraints and (3) the agent faces no-short-sales constraints. In the first case we evaluate the static, myopic and GBH strategies by computing their value functions. The value functions are computed analytically in the case of the static and GBH strategies, and numerically by simulating the SDE’s, in the case of the myopic strategy. Since these trading strategies are feasible dynamic trading strategies, their value functions constitute valid lower bounds on the value function of the optimal dynamic trading strategy. We also report the value of this optimal value function as it can be computed explicitly using the results of Kim and Omberg (1996).

Though the optimal value function is available, we also use the three sub-optimal strategies to compute upper bounds on this optimal value function. There are two reasons for doing this. First, ours is the first study that can compute the exact upper bounds prescribed by the algorithm of HKW and it would be interesting to see how they vary with the quality of the lower bounds. Moreover, in the case of the static strategy, it is of interest to see how the upper bound here compares with the upper bound reported in HKW. Second, when dynamic constraints are imposed the optimal value function is no longer available and so it is necessary to compute upper bounds in order to determine

---

22This is only the case for the static and GBH strategies, but see Haugh and Jain (2007) for the myopic strategy.
how far the sub-optimal strategies are from optimality. Since we therefore need to report upper bounds when trading constraints are imposed, for the sake of consistency we do the same even when there are no trading constraints.

For each of the three sub-optimal strategies, their associated upper bounds are computed by simulating the underlying SDE's. At each discretization point on each simulated path, we solve a simple quadratic optimization problem in order to solve for the market-price-of-risk process in the associated fictional market. See Section 5.3 and HKW for further details.

When trading constraints are imposed, we again use the static and myopic strategies to compute lower and upper bounds on the true optimal value function. There is nothing new here over and beyond what is already presented in HKW. However, the principal goal of this section is to compare the optimal GBH strategy with the static and myopic strategies. Recall that when trading constraints are imposed we assume that the agent can purchase the optimal GBH wealth from an unconstrained agent in the market place. We therefore display the results for the optimal GBH strategy alongside the lower and upper bounds for the constrained static and myopic strategies.

In all of our results, we report the expected utility as the continuously compounded certainty equivalent return, $R$. The value of $R$ corresponding to a value function, $V_0$, is defined by $U(W_0e^{RT}) = V_0$.

---

23 Either analytically or numerically, depending on whether or not trading constraints were imposed.

24 Except for how we computed the upper bound associated with the static strategy as mentioned earlier.

25 While the GBH strategy could be used to compute a valid upper bound for the value function of optimal constrained dynamic strategy, we do not bother to do so as the upper bound would not correspond to any feasible sub-optimal strategy.
5.4.1 Incomplete Markets

We first consider the case of incomplete markets where there are no trading constraints. Tables 5.2 and 5.3 report the estimates of the expected utility under the static, myopic, and GBH portfolio strategies as well as their corresponding upper bounds on the optimal dynamic trading strategy. As demonstrated by HKW, the myopic strategy outperforms the static strategy in that the former has a higher lower bound. Perhaps surprisingly, however, we see that the upper bound generated by the static strategy is now superior, i.e. lower, than the upper bound generated by the myopic strategy. This occurs because the static upper bound is generated using both terms in the right-hand side of (5.3.9) whereas the myopic upper bound used only the first term. In fact, HKW showed that in the case of incomplete markets, the static and myopic upper bounds will coincide when they are both generated using only the first term of (5.3.9).

Confirming the results of Corollary 1, we see that the GBH strategy always outperforms the optimal static strategy, significantly so in the cases of parameter sets 1 and 3. It is no surprise that the optimal myopic strategy generally outperforms the optimal GBH strategy as the former strategy can take explicit advantage of the variability in the state variable, $X_t$. However, it is quite surprising that in the case of the third parameter set, we see that the GBH strategy outperforms the myopic strategy when $T = 10$ and $\gamma = 3$ or $\gamma = 5$. The upper bound computed from the GBH strategy is generally comparable to the upper bound generated from the myopic strategy.
5.4.2 No Short-Sales and No Borrowing Constraints

Table 5.4 reports the results for when short sales and borrowing are prohibited. We see that the performance of the static and myopic strategies often deteriorates considerably. This is particularly true for parameter sets 3 and 4, and not surprisingly, is further pronounced for lower values of risk aversion. Of particular interest is how the GBH strategy performs in relation to the myopic strategy. We see in the cases of parameter sets 1 and 3 that the myopic strategy still outperforms the GBH strategy, though not by a significant amount. In the case of parameter sets 3 and 4, however, the GBH portfolio significantly outperforms the myopic strategy. The out-performance is on the order of four or five percentage points per annum in the case of $\gamma = 1.5$, and one to two percentage points otherwise. This is significant and it is clear that constrained investors would easily prefer to purchase the optimal GBH strategy rather than implementing a dynamic constrained strategy. Note that in Table 5.4 we do not display an upper bound generated by the GBH strategy. While it is straightforward to construct such an upper bound, we emphasize again that there does not exist a self-financing trading strategy that generates the GBH terminal wealth when trading constraints are imposed. It would be necessary to purchase such a portfolio from unconstrained agents in the market.

5.4.3 No Short-Sales Constraints

Table 5.5 reports the results when only a no short-sales constraint is imposed. For this problem, the quadratic optimization problem that we must solve at each discretization point on each simulated path needs to be solved numerically. This is in contrast to We have not discussed the specific details of these quadratic optimization problems in this chapter. HKW describes these problems in some detail and how the precise problem depends on the portfolio.
the earlier two cases where the quadratic optimization problem had an analytic solution.

Due to the increased computational burden when solving for the upper bound, we only consider \( T = 5 \) in this case.

We make the same conclusions as we did for the case where no short-sales and no borrowing constraints were imposed. The myopic strategy still outperforms the GBH strategy using parameter sets 1 and 2 but the GBH strategy outperforms under parameter sets 3 and 4. The extent of the GBH strategy’s out-performance (one or two percentage points per annum) is not as great since the myopic strategy is now less constrained. However, investors who are free to borrow but are still constrained by the inability to short-sell would still clearly prefer to purchase the GBH portfolio.

5.5 Optimizing the Upper Bound

In this section we determine the static strategies\(^{27}\) that minimize the upper bound assuming the same price dynamics of Section 5.4. We consider only the case of incomplete markets here and the minimum is taken over the set of all static strategies. We will also show that the optimal static strategy, \( \theta^{\text{static}} \), given by equation (5.2.8) is not the upper bound-minimizing static strategy.

As we saw in Section 5.3, the value function (5.3.5) in the fictitious market provides an upper bound for the optimal value function of the original problem. Computing the value function in (5.3.5) requires the state-price density process, \( \pi_t^{(\nu)} \), in the fictitious market. This is given by equation (5.3.4) where equation (5.3.3) defines the risk-free interest rate and market price-of-risk process in the fictitious market.

\(^{27}\)As we shall see, there is not a single strategy that minimizes the upper bound.
We now derive a semi-closed form expression for the upper bound of equation (5.3.5) assuming it is generated by a given static strategy, $\theta$, when markets are incomplete. As in Section 5.4, we assume there are three risky securities and one state variable, $X_t$. The drift of the asset returns, $\mu(X_t)$, satisfies $\mu(X_t) = \mu_0 + X_t \mu_1$, an affine function of $X_t$.

For a given static policy, $\theta$, the market price-of-risk in the dual problem is given by equation (5.3.9) which can be determined analytically using Proposition 16. In particular, we obtain

$$\tilde{\eta}_t = \gamma \Sigma^T \theta - \Sigma^T (1 - \gamma) \theta^T \frac{\mu_1}{k} (1 - \exp(-k(T - t))). \quad (5.5.1)$$

Recall that $\tilde{\eta}_t$ is only a candidate market price-of-risk process. In the case of incomplete markets, as we assume here, the risk-free rate process and the market price of risk in the fictitious market (see HKW for further details) actually satisfy

$$r_{t}^{(w)} = r$$

$$\eta_{t,i}^{(w)} = \eta_{t,i} \quad i = 1, 2, 3$$

$$= \tilde{\eta}_{t,i} \quad i = 4 \quad (5.5.2)$$

where

$$\eta_t = \Sigma P_t^{-1} (\mu_0 + X_t \mu_1 - r)$$

is the market price-of-risk process in the original problem. Note that $\eta_t^{(w)}$ is a $(4 \times 1)$ vector and that it depends upon the static strategy, $\theta$, only through its fourth component. This last component is given by
\[ \eta_{t,A}^{(c)} = -\Sigma^{\top}(4)(1 - \gamma)\theta^{\top}\frac{H_1}{k}(1 - \exp(-k(T - t))) \quad (5.5.3) \]

The upper bound is given by

\[ UB = \frac{W_0^{1 - \gamma}}{1 - \gamma} E_0 \left[ \frac{\pi^{(\nu, t - 1)}_T}{\gamma} \right] \quad (5.5.4) \]

Hence computing the expectation in above expression we get

\[
E_0 \left[ \frac{\pi^{(\nu, t - 1)}_T}{\gamma} \right] = E_0 \left[ \exp \left( \gamma - \frac{1}{\gamma} \left( -rT - \frac{1}{2} \left( \int_0^T \sum_{i=1}^{\nu} (\eta^{(\nu)})^2 dt + \int_0^T (\eta^{(\nu)})^2 dt \right) \right) 
- \left( \int_0^T \sum_{i=1}^{\nu} (\eta^{(\nu)})_d dB_t^{(4)} + \int_0^T (\eta^{(\nu)}) dB_t^{(4)} \right) \right] \quad (5.5.5) \]

The FOC conditions to compute the least upper bound can be written as

\[
E_0 \left[ \frac{\partial \pi^{(\nu, t - 1)}_T}{\partial \theta(j)} \right] = E_0 \left[ \frac{\pi^{(\nu, t - 1)}_T}{\gamma} \left( \gamma - \frac{1}{\gamma} \left( \int_0^T (\eta^{(\nu)}) \frac{\partial (\eta^{(\nu)})}{\partial \theta(j)} dt + \int_0^T \frac{\partial (\eta^{(\nu)})}{\partial \theta(j)} dB_t^{(4)} \right) \right) \right] = 0 
(5.5.6) \]

where \( j = 1, 2, 3 \). The above condition can be simplified to

\[
E_0 \left[ \frac{\partial \pi^{(\nu, t - 1)}_T}{\partial \theta(j)} \right] = E_0 \left[ \frac{\pi^{(\nu, t - 1)}_T}{\gamma} \left( \int_0^T \Sigma^{\top}(4)(1 - \gamma)\theta^{\top}\frac{H_1}{k}(1 - \exp(-k(T - t)))^2 dt 
- \int_0^T (1 - \exp(-k(T - t)) dB_t^{(4)} \right) \right] = 0 \quad (5.5.7) \]

which can be reexpressed as

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Thus the static strategy which provides a least upper bound should satisfy the above condition. From the above result we can also conclude that there are infinite number of static strategies which provides a least upper bound.

This is interesting because blah blah

It is worth mentioning at this point that we will not report the minimal statically-generated upper bound where the minimum is taken over all static trading strategies as described in Section 5.5. This is because the upper bound generated by \( \theta^{\text{static}} \), while not minimal, is very close to minimal in practice. In particular, our (unreported) numerical experiments show that the upper bound generated by \( \theta^{\text{static}} \) is often within just 1 or 2 basis points of the minimal statically-generated upper bound. Moreover, we never saw this difference exceeding 10 basis points.

### 5.6 Conclusions and Further Research

For a particular class of security price dynamics, we obtained a closed-form solution for the terminal wealth and expected utility of the classic constant proportion or static trading strategy. We then used this solution to study in further detail the portfolio evaluation approach recently proposed by Haugh, Kogan and Wang (2006). In particular, we solved for the more theoretically satisfying upper bound on the optimal value function that was originally proposed by HKW. We also used this result to minimize the upper bound over the class of static trading strategies and showed that the optimal static
strategy, \( \theta^{\text{static}} \), does not minimize the upper bound.

For the same class of security price dynamics, we solved for the optimal GBH strategy and showed that in some circumstances it is comparable, in terms of expected utility, to the optimal dynamic trading strategy. Moreover, when the optimizing agent faces dynamic trading constraints such as no-short sales or no-borrowing constraints, the optimal GBH strategy can often significantly outperform the optimal constrained dynamic trading strategy. This has implications for investors when: (i) a dynamic trading strategy is too costly or difficult to implement in practice and (ii) when the optimal GBH portfolio can be purchased from an unconstrained agent. We also concluded that the optimal GBH strategy is superior to \( \theta^{\text{static}} \) in that it achieves a higher expected utility.

There are several possible directions for future research. First, it would be interesting to extend the analysis to other security price dynamics. Are there other price processes, for example, where moderately risk averse investors with long time horizons might prefer the GBH strategy to the optimal myopic strategy? We saw this to be the case with Parameter Set 3, even when dynamic trading constraints were not imposed.

Another direction for future research is to continue the study of duality methods for evaluating suboptimal portfolio strategies. We were able to use Proposition 16 to compute the upper bounds for the static and GBH strategies originally proposed by HKW. Our results showed that in some circumstances the myopic strategy outperformed the GBH strategy yet failed to provide a tighter upper bound. This could be due to the fact that the upper bound computed from the myopic strategy could only use the first term in the right-hand-side of (5.3.9) as the value function and its derivatives were not
available to compute the second term. Haugh and Jain (2007) seek to resolve this issue by using regression methods and path-wise Monte Carlo estimators to estimate the second term in (5.3.9) for various strategies.

Finally, it would be interesting to develop primal-dual style algorithms for finding good sub-optimal policies. Haugh, Kogan and Zhu (2007) is a very basic attempt in this direction in that approximate dynamic programming (ADP) methods are used to construct trading strategies that are then evaluated using the dual-based portfolio evaluation approach. Their algorithm does not constitute a primal-dual algorithm, however, in that the dual formulation is not used to construct the trading strategy. We believe this could be a particularly profitable direction for future research.
Table 5.1: Calibrated model parameters

<table>
<thead>
<tr>
<th>Parameter set 1</th>
<th>k</th>
<th>( \mu_0 )</th>
<th>( \mu_1 )</th>
<th>( \Sigma_P )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.366</td>
<td>0.081</td>
<td>0.034</td>
<td>0.186 0.000 0.000 0.000</td>
</tr>
<tr>
<td></td>
<td>0.110</td>
<td>0.059</td>
<td>0.228</td>
<td>0.083 0.000 0.000 0.000</td>
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<tr>
<td></td>
<td>0.130</td>
<td>0.073</td>
<td>0.251</td>
<td>0.139 0.069 0.000 0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.000</td>
<td>-0.741</td>
<td>-0.037 -0.060 0.284</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter set 2</th>
<th>k</th>
<th>( \mu_0 )</th>
<th>( \mu_1 )</th>
<th>( \Sigma_P )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.671</td>
<td>0.081</td>
<td>0.046</td>
<td>0.186 0.000 0.000 0.000</td>
</tr>
<tr>
<td></td>
<td>0.110</td>
<td>0.070</td>
<td>0.227</td>
<td>0.082 0.000 0.000 0.000</td>
</tr>
<tr>
<td></td>
<td>0.130</td>
<td>0.086</td>
<td>0.251</td>
<td>0.139 0.069 0.000 0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.000</td>
<td>-0.017</td>
<td>0.149 0.058 1.725</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter set 3</th>
<th>k</th>
<th>( \mu_0 )</th>
<th>( \mu_1 )</th>
<th>( \Sigma_P )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.366</td>
<td>0.142</td>
<td>0.065</td>
<td>0.256 0.000 0.000 0.000</td>
</tr>
<tr>
<td></td>
<td>0.109</td>
<td>0.049</td>
<td>0.217</td>
<td>0.054 0.000 0.000 0.000</td>
</tr>
<tr>
<td></td>
<td>0.089</td>
<td>0.049</td>
<td>0.207</td>
<td>0.062 0.062 0.000 0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.000</td>
<td>-0.741</td>
<td>0.040 0.034 0.288</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter set 4</th>
<th>k</th>
<th>( \mu_0 )</th>
<th>( \mu_1 )</th>
<th>( \Sigma_P )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.671</td>
<td>0.142</td>
<td>0.061</td>
<td>0.256 0.000 0.000 0.000</td>
</tr>
<tr>
<td></td>
<td>0.109</td>
<td>0.060</td>
<td>0.217</td>
<td>0.054 0.000 0.000 0.000</td>
</tr>
<tr>
<td></td>
<td>0.089</td>
<td>0.067</td>
<td>0.206</td>
<td>0.062 0.062 0.000 0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>0.000</td>
<td>-0.017</td>
<td>0.212 0.096 1.716</td>
</tr>
</tbody>
</table>

The four sets of model parameters correspond to: (1) size sorted portfolios and the dividend yield as a state variable; (2) size sorted portfolios and the term spread as a state variable; (3) book-to-market sorted portfolios and the dividend yield as a state variable; (4) book-to-market sorted portfolios and the term spread as a state variable. Parameter values are based on the estimates in Tables 1 and 2 of Lynch (2001).
Table 5.2: Lower and upper bounds in incomplete markets - I.

<table>
<thead>
<tr>
<th>Parameter set 1</th>
<th>Parameter set 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( T = 5 )</td>
</tr>
<tr>
<td></td>
<td>( \gamma = 1.5 )</td>
</tr>
<tr>
<td>( L^B )</td>
<td>7.49</td>
</tr>
<tr>
<td>( U^B )</td>
<td>(7.47, 7.51)</td>
</tr>
<tr>
<td>( L^Bm )</td>
<td>9.44</td>
</tr>
<tr>
<td>( U^Bm )</td>
<td>(9.40, 9.49)</td>
</tr>
<tr>
<td>( L^{Bbh} )</td>
<td>9.35</td>
</tr>
<tr>
<td>( U^{Bbh} )</td>
<td>(9.32, 9.37)</td>
</tr>
<tr>
<td>( L^V )</td>
<td>9.36</td>
</tr>
<tr>
<td>( U^V )</td>
<td>(9.31, 9.41)</td>
</tr>
<tr>
<td>( L^{Vbh} )</td>
<td>9.43</td>
</tr>
<tr>
<td>( U^{Vbh} )</td>
<td>(9.35, 9.49)</td>
</tr>
<tr>
<td>( L^U )</td>
<td>9.44</td>
</tr>
<tr>
<td>( U^U )</td>
<td>(9.41, 9.50)</td>
</tr>
<tr>
<td>( L^{Vb} )</td>
<td>9.36</td>
</tr>
<tr>
<td>( U^{Vb} )</td>
<td>(9.31, 9.41)</td>
</tr>
<tr>
<td>( L^{Vbh} )</td>
<td>9.43</td>
</tr>
<tr>
<td>( U^{Vbh} )</td>
<td>(9.35, 9.49)</td>
</tr>
<tr>
<td>( L^V )</td>
<td>9.44</td>
</tr>
<tr>
<td>( U^V )</td>
<td>(9.41, 9.50)</td>
</tr>
</tbody>
</table>

This Table reports the results for parameter set 1 and 2. The parameter sets are defined in Table 1. The rows marked \( L^B \), \( L^Bm \) and \( L^{Bbh} \) report estimates of the expected utility achieved by using the static portfolio strategy, myopic portfolio strategy and generalized buy and hold (GBH) portfolio strategy respectively. Expected utility is reported as a continuously compounded certainty equivalent return. Approximate 95% confidence intervals are reported in parentheses. The rows marked \( U^B \), \( U^Bm \) and \( U^{Bbh} \) report the estimates of the upper bound on the true value function computed from the static, myopic and generalized buy-and-hold (GBH) portfolio strategies respectively. The row marked \( V^U \) reports the optimal value function for the problem.
Table 5.3: Lower and upper bounds in incomplete markets - II.

<table>
<thead>
<tr>
<th></th>
<th>$T = 5$</th>
<th>$T = 10$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$\gamma = 1.5$</td>
<td>$\gamma = 3$</td>
</tr>
<tr>
<td>$\gamma = 1.5$</td>
<td>$\gamma = 3$</td>
<td>$\gamma = 5$</td>
</tr>
<tr>
<td>$L^B$</td>
<td>14.65 (14.62, 14.68)</td>
<td>15.03 (15.01, 15.05)</td>
</tr>
<tr>
<td>$U^B$</td>
<td>16.79 (16.73, 16.85)</td>
<td>17.78 (17.74, 17.82)</td>
</tr>
<tr>
<td>$L^m$</td>
<td>16.64 9.88 (9.86, 9.90)</td>
<td>17.47 (17.44, 17.49)</td>
</tr>
<tr>
<td>$U^m$</td>
<td>16.81 (16.76, 16.87)</td>
<td>17.62 (17.58, 17.64)</td>
</tr>
<tr>
<td>$U^{gbh}$</td>
<td>16.78 (16.68, 16.87)</td>
<td>17.80 (17.76, 17.82)</td>
</tr>
<tr>
<td>$V^u$</td>
<td>16.79 10.32 7.06</td>
<td>16.79 10.32 7.06</td>
</tr>
</tbody>
</table>

This Table reports the results for parameter set 3 and 4. The parameter sets are defined in Table 1. The rows marked $L^B$, $L^m$ and $L^{gbh}$ report estimates of the expected utility achieved by using the static portfolio strategy, myopic portfolio strategy and generalized buy-and-hold (GBH) portfolio strategy respectively. Expected utility is reported as a continuously compounded certainty equivalent return. Approximate 95% confidence intervals are reported in parentheses. The rows marked $U^B$, $U^m$ and $U^{gbh}$ report the estimates of the upper bound on the true value function computed from the static, myopic and generalized buy-and-hold (GBH) portfolio strategies respectively. The row marked $V^u$ reports the optimal value function for the problem.
<table>
<thead>
<tr>
<th>Parameter set 1</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>LB</strong></td>
<td>7.38</td>
<td>4.63</td>
<td>3.37</td>
<td>7.56</td>
<td>4.84</td>
<td>3.42</td>
</tr>
<tr>
<td></td>
<td>(7.36, 7.40)</td>
<td>(4.62, 4.64)</td>
<td>(3.27, 3.28)</td>
<td>(7.55, 7.57)</td>
<td>(4.83, 4.85)</td>
<td>(3.42, 3.43)</td>
</tr>
<tr>
<td><strong>UB</strong></td>
<td>8.15</td>
<td>5.84</td>
<td>4.18</td>
<td>8.45</td>
<td>6.84</td>
<td>4.77</td>
</tr>
<tr>
<td></td>
<td>(8.12, 8.18)</td>
<td>(5.79, 5.90)</td>
<td>(4.08, 4.28)</td>
<td>(8.43, 8.47)</td>
<td>(6.42, 6.54)</td>
<td>(4.64, 4.94)</td>
</tr>
<tr>
<td><strong>LBm</strong></td>
<td>8.36</td>
<td>5.29</td>
<td>3.75</td>
<td>8.68</td>
<td>5.93</td>
<td>4.22</td>
</tr>
<tr>
<td></td>
<td>(8.31, 8.41)</td>
<td>(5.24, 5.35)</td>
<td>(3.70, 3.80)</td>
<td>(8.65, 8.72)</td>
<td>(5.78, 5.87)</td>
<td>(4.17, 4.27)</td>
</tr>
<tr>
<td><strong>UBm</strong></td>
<td>8.93</td>
<td>6.57</td>
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The four parameter sets are defined in Table 1. The rows marked **LB**, **LBm** and **Vgbh** report estimates of the expected utility achieved by using the static, myopic and GBH portfolio strategies, respectively. Expected utility is reported as a continuously compounded certainty equivalent return. Approximate 95% confidence intervals are reported in parentheses. The rows marked **UB** and **UBm** report the estimates of the upper bound on the true value function.
Table 5.5: Lower and upper bounds in no short sales case.

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The four parameter sets are defined in Table 1 and the problem horizon is $T = 5$ years. The rows marked $LB^s$, $LB^m$ and $V_{PB}$ report the estimates of the expected utility achieved by using the static, myopic and GBH portfolio strategies, respectively. Expected utility is reported as a continuously compounded certainty equivalent return. Approximate 95% confidence intervals are reported in parentheses. The rows marked $UB^s$ and $UB^m$ report the estimates of the upper bound on the true value function.
Bibliography


Appendix A

Proofs for Chapter 2

A.1 Proof of Proposition 5

The discrete variance strike can be derived as follows. Applying Itô’s lemma to $\ln(S_t)$ in equation (4.2.5) we get

$$d(\ln S_t) = \left( r - \frac{1}{2} \nu_t \right) dt + \sqrt{\nu_t} \left( \rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right) \quad (A.1.1)$$

Integrating equation (A.1.1) from $t_i$ to $t_{i+1}$ squaring and taking expectations in the risk neutral measure we get

$$E \left[ \ln \left( \frac{S_{t_{i+1}}}{S_{t_i}} \right) \right]^2 = E \left[ \int_{t_i}^{t_{i+1}} \left( r - \frac{1}{2} \nu_t \right) dt + \int_{t_i}^{t_{i+1}} \sqrt{\nu_t} \left( \rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right) \right]^2$$

$$= E \left[ (r \Delta t - \int_{t_i}^{t_{i+1}} \frac{1}{2} \nu_t dt)^2 + \left( \int_{t_i}^{t_{i+1}} \sqrt{\nu_t} \left( \rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right) \right)^2 \right]$$

$$+ 2 \left( r \Delta t - \int_{t_i}^{t_{i+1}} \frac{1}{2} \nu_t dt \right) \left( \int_{t_i}^{t_{i+1}} \sqrt{\nu_t} \left( \rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right) \right)$$

Applying Itô’s isometry rule on second term and simplifying other terms we get the following:
The variance process has the following properties:

\[ \mathbb{E}(v_t) = \exp(-kt)(v_0 - \theta) + \theta \]

\[ \mathbb{E}(v_t v_s) = \sigma_v^2 \exp(-k(t+s)) \left( \frac{\exp(ks) - 1}{k}(v_0 - \theta) + \frac{\exp(2ks) - 1}{2k}(\theta) \right) + \exp(-k(t+s))(v_0 - \theta)^2 + \exp(-kt)(v_0 - \theta)\theta + \exp(-ks)(v_0 - \theta)^2 + \theta^2 \]

\[ \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2) \right] = 0 \]

Using properties (A.1.3) we solve for the last term in equation (A.1.2)

\[ \mathbb{E} \left[ \left( \int_{t_i}^{t_{i+1}} v_t dt \right) \left( \int_{t_i}^{t_{i+1}} \sqrt{v_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2) \right) \right] = \mathbb{E} \left[ \left( \int_{t_i}^{t_{i+1}} v_t dt \right) \left( \int_{t_i}^{t_{i+1}} \sqrt{v_t} \rho dW_t^1 \right) \right] + \mathbb{E} \left[ \left( \int_{t_i}^{t_{i+1}} v_t dt \right) \left( \int_{t_i}^{t_{i+1}} \sqrt{v_t} \sqrt{1-\rho^2}dW_t^2 \right) \right] \]

The 2nd expectation in equation (A.1.4) is zero and first term can be rewritten using (4.2.6) as,
\[
E \left[ \frac{\rho}{\sigma_v} \left( \int_{t_i}^{t_{i+1}} v_t dt \right) (v_{i+1} - v_i - \int_{t_i}^{t_{i+1}} \kappa (\theta - v_t) dt) \right] \\
= E \left[ \frac{\rho}{\sigma_v} \left( \int_{t_i}^{t_{i+1}} v_t dt \right) (v_{i+1} - v_i) \right] - \frac{\rho \kappa}{\sigma_v} \Delta t E \left( \int_{t_i}^{t_{i+1}} v_t dt \right) + \frac{\rho \kappa}{\sigma_v} E \left( \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} v_t v_s dt ds \right)
\]

\text{(A.1.5)}

Next, we compute the first term in equation (A.1.5):

\[
E \left[ \frac{\rho}{\sigma_v} \left( \int_{t_i}^{t_{i+1}} v_t dt \right) (v_i) \right] = \frac{1 - \exp(-\kappa t_i)}{2\kappa^2} \left( \sigma^2_v (v_0 - \theta) - \exp(-\kappa t_i) \right) + \exp(-\kappa t_i) (v_0 - \theta) \Delta t + \theta^2 \Delta t
\]

\text{(A.1.6)}

\[
E \left[ \frac{\rho}{\sigma_v} \left( \int_{t_i}^{t_{i+1}} v_t dt \right) (v_{i+1}) \right] = \frac{\exp(-\kappa t_{i+1}) \sigma^2_v (v_0 - \theta)}{2\kappa^2} \left( \kappa \Delta t - \exp(-\kappa t_i) + \exp(-\kappa t_{i+1}) \right) \\
+ \frac{\exp(-\kappa t_{i+1}) \sigma^2_v \theta}{2\kappa^2} \left( \exp(\kappa t_{i+1}) - \exp(\kappa t_i) + \exp(-\kappa t_{i+1}) - \exp(-\kappa t_i) \right) \\
+ \frac{\exp(-\kappa t_{i+1}) (v_0 - \theta)^2}{\kappa} \left( - \exp(-\kappa t_{i+1}) + \exp(-\kappa t_i) \right) \\
+ \frac{(v_0 - \theta) \theta}{\kappa} \left( - \exp(-\kappa t_{i+1}) + \exp(-\kappa t_i) \right) \\
+ \exp(-\kappa t_{i+1}) (v_0 - \theta) \theta \Delta t + \theta^2 \Delta t
\]

\text{(A.1.7)}

Subtracting equation (A.1.6) from (A.1.7) and simplifying we get,
\[
E\left[ \frac{\rho}{\sigma_v} \left( \int_{t_i}^{t_{i+1}} v dt \right) (v_{t_{i+1}} - v_{t_i}) \right] = \frac{\sigma_v^2 v_0}{\kappa^2} \left( \exp(-\kappa t_{i+1}) (1 + \kappa \Delta t - \exp(\kappa \Delta t)) \right) \\
- \left( \frac{\sigma_v^2 \theta}{2\kappa^2} + \frac{(v_0 - \theta)^2}{\kappa} \right) \left( \exp(-2\kappa t_{i+1}) (1 - \exp(-\kappa \Delta t))^2 \right) \\
+ (v_0 - \theta) \theta \Delta t \left( \exp(-\kappa t_{i+1}) (1 - \exp(-\kappa \Delta t)) \right)
\]
(A.1.8)

Summing equation (A.1.8) from time 0 to time \( n - 1 \) we get

\[
\sum_{i=0}^{n-1} E\left[ \frac{\rho}{\sigma_v} \left( \int_{t_i}^{t_{i+1}} v dt \right) (v_{t_{i+1}} - v_{t_i}) \right] = \frac{\sigma_v^2 v_0}{\kappa^2} \left( 1 + \kappa \Delta t - \exp(\kappa \Delta t) \right) \left( \frac{1 - \exp(-\kappa \Delta t)}{-1 + \exp(\frac{\kappa \Delta t}{\kappa})} \right) \\
- \left( \frac{\sigma_v^2 \theta}{2\kappa^2} + \frac{(v_0 - \theta)^2}{\kappa} \right) \left( \frac{1 - \exp(-2\kappa \Delta t)}{-1 + \exp(\frac{2\kappa \Delta t}{\kappa})} \right) \\
+ (v_0 - \theta) \theta \Delta t \left( \frac{1 - \exp(-\kappa \Delta t)}{-1 + \exp(\frac{\kappa \Delta t}{\kappa})} \right) \left( 1 - \exp(-\kappa \Delta t) \right)
\]
(A.1.9)

Next, we compute the last term in equation (A.1.5):

\[
\sum_{i=0}^{n-1} E \left[ \left( \int_{t_i}^{t_{i+1}} v dt \right)^2 \right] = \sum_{i=0}^{n-1} E \left( \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} v_t v_s dt ds \right) \\
= \sum_{i=0}^{n-1} \text{Var} \left( \int_{t_i}^{t_{i+1}} v_s ds \right) + \sum_{i=0}^{n-1} \left( E \left( \int_{t_i}^{t_{i+1}} v_s ds \right)^2 \right)
\]
(A.1.10)

Now, we compute the both terms on right hand side of equation (A.1.10),
\[ \sum_{i=0}^{n-1} \text{Var} \left( \int_{t_i}^{t_{i+1}} v_3 ds \right) = \sum_{i=0}^{n-1} \frac{\sigma_v^2 e^{-2\kappa(\Delta t)}}{\kappa^3} \left( e^{2\kappa(\Delta t)} - 2e^{\kappa(\Delta t)}\kappa(\Delta t) - 1 \right) \left( v_0 - \theta \right) \exp \left( \frac{-i\kappa T}{n} \right) 
\]
\[ + \frac{\sigma_v^2 e^{-2\kappa(\Delta t)}}{2\kappa^3} \left( 4e^{\kappa(\Delta t)} - 3e^{2\kappa(\Delta t)} + 2e^{2\kappa(\Delta t)}\kappa(\Delta t) - 1 \right) \left( v_0 - \theta \right) \left( 1 - \exp(-\kappa T) \right) \]
\[ = \frac{\sigma_v^2 e^{-2\kappa(\Delta t)}}{\kappa^3} \left( e^{2\kappa(\Delta t)} - 2e^{\kappa(\Delta t)}\kappa(\Delta t) - 1 \right) \left( v_0 - \theta \right) \frac{1 - \exp(-\kappa T)}{1 + \exp \left( \frac{\kappa T}{n} \right)} \]
\[ + \frac{\sigma_v^2 e^{-2\kappa(\Delta t)}}{2\kappa^3} \left( 4e^{\kappa(\Delta t)} - 3e^{2\kappa(\Delta t)} + 2e^{2\kappa(\Delta t)}\kappa(\Delta t) - 1 \right) \left( v_0 - \theta \right) \]  
(A.1.11)

\[ \sum_{i=0}^{n-1} \left( E \left( \int_{t_i}^{t_{i+1}} v_3 ds \right) \right)^2 = \sum_{i=0}^{n-1} \left( \theta \Delta t + (1 - \exp(-\kappa \Delta t)) \frac{E(v_{t_i}) - \theta}{\kappa} \right)^2 
\]
\[ = n(\theta \Delta t)^2 + (1 - \exp(-\kappa \Delta t)) \frac{2\theta \Delta t(v_0 - \theta)}{\kappa} \sum_{i=0}^{n-1} \exp \left( \frac{-i\kappa T}{n} \right) 
\]
\[ + \left( 1 - \exp(-\kappa \Delta t) \right)^2 \sum_{i=0}^{n-1} E(v_{t_i} - \theta)^2 \]  
(A.1.12)

\[ \sum_{i=0}^{n-1} E(v_{t_i} - \theta)^2 = \sum_{i=0}^{n-1} E(v_{t_i})^2 - n\theta^2 - 2\theta(v_0 - \theta) \sum_{i=0}^{n-1} \exp \left( \frac{-i\kappa T}{n} \right) \]  
(A.1.13)

\[ \sum_{i=0}^{n-1} E(v_{t_i})^2 = \left( (v_0 - \theta)^2 - \frac{(v_0 - \theta)^2}{\kappa} \frac{\sigma_v^2 \theta}{2\kappa} \right) \sum_{i=0}^{n-1} \exp \left( \frac{-i2\kappa T}{n} \right) 
\]
\[ + \left( 2(v_0 - \theta)\theta + \frac{(v_0 - \theta)^2}{\kappa} \right) \sum_{i=0}^{n-1} \exp \left( \frac{-i\kappa T}{n} \right) + \theta^2 + \frac{\sigma_v^2 \theta}{2\kappa} \]  
(A.1.14)

Using equations (A.1.13) and (A.1.14) we can compute (A.1.12). Using equations (A.1.12) and (A.1.11) we can compute (A.1.10). Now, the fair discrete variance strike is the expectation of the discrete realized variance.
\[ K_{\text{var}}^* (n) = E[V_d(0, n, T)] = E_0 \left[ \sum_{i=0}^{n-1} \frac{(\ln(S_{i+1}^0))^2}{(n-1)\Delta t} \right] \]

Dividing equation (A.1.2) on both sides by \((n-1)\Delta t\) and summing from time 0 to time \(n-1\) we get

\[
E[V_d(0, n, T)] = \frac{T^2}{n-1} + \frac{\sum_{i=0}^{n-1} E\left( \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} v_t v_s dt ds \right)}{4(n-1)\Delta t}
- \frac{r E[\int_0^T v_t dt]}{(n-1)} + \frac{n E[\int_0^T v_t dt]}{T(n-1)} + \frac{\rho \sigma \theta E[\int_0^T v_t dt]}{(n-1)\sigma_v}
- \frac{\sum_{i=0}^{n-1} E\left( \rho \sigma v \left( \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} v_t v_s dt ds \right) \right)}{\sigma_v(n-1)\Delta t}
- \frac{\sum_{i=0}^{n-1} E\left( \frac{\sigma}{\sigma_v} \left( \int_{t_i}^{t_{i+1}} v_t dt \right) (v_{t_{i+1}} - v_t) \right)}{\sigma_v(n-1)\Delta t}
\]

(A.1.15)

This equation can be simplified as

\[
E[V_d(0, n, T)] = \frac{1}{T} E[\int_0^T v_t dt] + \frac{E[\int_0^T v_t dt]}{T(n-1)} + \frac{T^2}{n-1} - \frac{r E[\int_0^T v_t dt]}{(n-1)}
+ \frac{\rho \sigma \theta E[\int_0^T v_t dt]}{(n-1)\sigma_v}
+ \frac{\sum_{i=0}^{n-1} E\left( \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} v_t v_s dt ds \right)}{(n-1)\Delta t} \left( \frac{1}{4} - \frac{\rho \sigma}{\sigma_v} \right)
- \frac{\sum_{i=0}^{n-1} E\left( \frac{\sigma}{\sigma_v} \left( \int_{t_i}^{t_{i+1}} v_t dt \right) (v_{t_{i+1}} - v_t) \right)}{\sigma_v(n-1)\Delta t}
\]

(A.1.16)

The last term in equation (A.1.16) is given by equation (A.1.9). The second last term in equation (A.1.16) is given by equation (A.1.10). All terms except the first one on right hand side in expression (A.1.16) is of the order \(O(\Delta t)\) or \(O(1/n)\). The first term on the right hand side is the fair continuous variance strike, i.e., \(\frac{1}{T} E[\int_0^T v_t dt] = K_{\text{var}}^*\). It is given by equation (3.2.7). Hence, the discrete variance strike can be represented in the following way:
\[ K_{\text{var}}^*(n) = K_{\text{var}}^* + g(r, \rho, \sigma_v, \kappa, \theta, n) \quad (A.1.17) \]

where

\[
g(r, \rho, \sigma_v, \kappa, \theta, n) = \frac{r^2 T}{n - 1} + \frac{1}{T} E \left( \int_0^T v_t \, dt \right) \left( \frac{1}{(n - 1)} - \frac{r T}{(n - 1)^2} + \frac{\rho \kappa \theta T}{(n - 1)\sigma_v} \right) + \sum_{i=0}^{n-1} E \left( \int_{t_i}^{t_{i+1}} v_t \, dt \right) \left( \frac{1}{4} - \frac{\rho \kappa}{\sigma_v} \right) \frac{\sigma_v}{(n - 1)\Delta t} + \frac{\sum_{i=0}^{n-1} E \left[ \frac{2}{\sigma_v} \int_{t_i}^{t_{i+1}} v_t \, dt \right]}{\sigma_v(n - 1)\Delta t} \quad (A.1.18)\]

\[ g(r, \rho, \sigma_v, \kappa, \theta, n) = O \left( \frac{1}{n} \right) \quad (A.1.19) \]

Hence,

\[ K_{\text{var}}^*(n) \longrightarrow E \left[ \frac{1}{T} \int_0^T v_t \, dt \right] = K_{\text{var}}^* \quad \text{as} \quad \Delta t \to 0 \quad \Box \]

**A.2 Proof of Proposition 10**

The discrete variance strike can be derived as follows. Applying Itô's lemma to \( \ln(S_t) \) in equation (4.2.7) and integrating from \( t_i \) to \( t_{i+1} \) gives

\[
\ln \left( \frac{S_{i+1}}{S_i} \right) = \int_{t_i}^{t_{i+1}} (r - \lambda m - \frac{1}{2} v_t) \, dt + \int_{t_i}^{t_{i+1}} \sqrt{v_t}(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2) + \ln \left( \prod_{j=1}^{n_i} Y_j \right) \quad (A.2.1)
\]
where $n_j$ is number of jumps in the stock price during time $t_{i+1} - t_i$. Squaring equation (A.2.1), summing from time 0 to time $n - 1$, dividing on both sides by $(n - 1)\Delta t$ and taking expectation under the risk neutral measure we get

\[
E \left[ \sum_{i=0}^{n-1} \frac{1}{(n-1)\Delta t} \ln \left( \frac{S_{t_{i+1}}}{S_{t_i}} \right)^2 \right] \\
= E \sum_{i=0}^{n-1} \frac{1}{(n-1)\Delta t} \left[ \left( (r - \lambda m)\Delta t - \int_{t_i}^{t_{i+1}} \frac{1}{2} v_t dt \right)^2 + \left( \int_{t_i}^{t_{i+1}} \sqrt{\nu_t} (\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2) \right)^2 \\
+ 2 \left( (r - \lambda m)\Delta t - \int_{t_i}^{t_{i+1}} \frac{1}{2} v_t dt \right) \left( \int_{t_i}^{t_{i+1}} \sqrt{\nu_t} (\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2) \right) \right] \\
+ \left( \sum_{j=1}^{n_j} \ln Y_j \right)^2 + 2 \left( (r - \lambda m)\Delta t - \int_{t_i}^{t_{i+1}} \frac{1}{2} v_t dt \right) \left( \sum_{j=1}^{n_j} \ln Y_j \right) \\
+ 2 \left( \int_{t_i}^{t_{i+1}} \sqrt{\nu_t} (\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2) \right) \left( \sum_{j=1}^{n_j} \ln Y_j \right) \\
\] (A.2.2)

The first two lines of expressions on the right hand side of equation (A.2.2) can be computed using Proposition 13 and it is equal to

\[
\frac{1}{T} E \left[ \int_0^T v_t dt \right] + g(r - \lambda m, \rho, \sigma_v, \kappa, \theta, n) \\
\] (A.2.3)

and using equation (A.1.19) we get

\[
g(r - \lambda m, \rho, \sigma_v, \kappa, \theta, n) = O \left( \frac{1}{n} \right) \\
\] (A.2.4)

The last two lines of expressions on the right hand side of equation (A.2.2) can be computed using Proposition 14 and it is equal to

\[
\lambda(a^2 + b^2) + \lambda^2 a^2 T + \lambda \left( 2(r - \lambda m)T - E[\int_0^T v_t dt] \right) \\
\] (A.2.5)

where $E[\int_0^T v_t dt]$ is given by equation (3.2.7). Hence using equations (A.2.2), (A.2.3)
and (A.2.3) fair discrete variance strike is given by following:

\[ K_{\text{var}}^*(n) = E \left[ V_d(0, n, T) \right] = E \left[ \sum_{i=0}^{n-1} \frac{1}{(n-1)\Delta t} \ln \left( \frac{S_{i+1}}{S_i} \right) \right]^2 \]

\[ = \frac{1}{T} E \left[ \int_0^T v_t dt \right] + \lambda(a^2 + b^2) + g(r - \lambda m, \rho, \sigma_v, \kappa, \theta, n) \]

\[ + \frac{\lambda(a^2 + b^2) + \lambda^2 a^2 T + \lambda a \left( 2(r - \lambda m)T - E[\int_0^T v_t dt] \right)}{n - 1} \quad (A.2.6) \]

Hence, the fair discrete variance strike can be represented in the following way:

\[ K_{\text{var}}^*(n) = K_{\text{var}}^* + h(r, \rho, \sigma_v, \kappa, \theta, m, b, n) \quad (A.2.7) \]

where

\[ K_{\text{var}}^* = \theta + \frac{v \theta - \theta}{\kappa T} (1 - e^{-\kappa T}) + \lambda(a^2 + b^2) \quad (A.2.8) \]

and

\[ h(r, \rho, \sigma_v, \kappa, \theta, m, b, n) = g(r - \lambda m, \rho, \sigma_v, \kappa, \theta, n) \]

\[ + \frac{\lambda(a^2 + b^2) + \lambda^2 a^2 T + \lambda a \left( 2(r - \lambda m)T - E[\int_0^T v_t dt] \right)}{n - 1} \quad (A.2.9) \]

\[ h(r, \rho, \sigma_v, \kappa, \theta, m, b, n) = O \left( \frac{1}{n} \right) \quad (A.2.10) \]
Appendix B
Proofs for Chapter 4

B.1 VIX Index-Log Contract

In this appendix we show that the square of the theoretical VIX ($\tilde{\text{VIX}}$) value is equal to the negative value of the payoff of a log contract. In the SV model it is same as the one month continuous variance swap rate as shown by Carr and Wu (2006). But in the SVJ model the value of negative of a one month log contract and the the one month continuous variance swap rate are different and hence the square of the theoretical VIX value is different from the one month continuous variance swap rate.

Neuberger (1994) showed that a log contract payoff can be replicated statically using call and put options.

\[
\ln \left( \frac{S_T}{S_t} \right) = \frac{S_T - S_t}{S_t} - \int_{S_t}^{\infty} \frac{1}{K^2} (S_T - K)^+ dK - \int_0^{S_t} \frac{1}{K^2} (K - S_T)^+ dK \quad (B.1.1)
\]

In equation (B.1.1) $S_T$ refers to the index price at maturity $T$ and $S_t$ can be any constant. In particular if we choose $S_t$ to be equal to the forward value, $F_t = S_t e^{r(T-t)}$, of the index at time $t$ maturing at time $T = t + \tau, \tau = 30/365$ and taking expectation.
under the risk neutral measure and multiplying by $2/\tau$ on both sides we get

$$
\frac{2}{\tau} E^Q_t \left[ \ln \left( \frac{S_{t+\tau}}{F_t} \right) \right] = \frac{2}{\tau} E^Q_t \left[ \frac{S_{t+\tau} - F_t}{F_t} - \int_{F_t}^{\infty} \frac{1}{K^2} (S_{t+\tau} - K)^+ dK - \int_0^{F_t} \frac{1}{K^2} (K - S_{t+\tau})^+ dK \right]
$$

$$
= -\frac{2}{\tau} \left[ \int_{F_t}^{\infty} \frac{e^{rt}}{K^2} C_t(K) dK + \int_0^{F_t} \frac{e^{rt}}{K^2} P_t(K) dK \right]
$$

(B.1.2)

The market VIX definition in equation (4.2.1) represents a discretization of the equation (B.1.2) and summation over a finite range of strikes. The extra term $(F_t/K_0 - 1)^2$ in the VIX definition adjusts for the in-the-money call option used at $K_0 \leq F_t$. This is shown in Carr and Wu (2006). Hence, from equation (B.1.2) the market VIX value in the limit $(\Delta K \to 0, K_{\text{min}} \to 0, K_{\text{max}} \to \infty)$ approaches theoretical VIX ($\widetilde{\text{VIX}}$) (4.2.4).

The square of the theoretical VIX is equal to the value of negative of $2/\tau$ times the value of log contract payoff $\ln(S_{t+\tau}/F_t)$.

$$
\widetilde{\text{VIX}}_t^2 = -\frac{2}{\tau} E^Q_t \left[ \ln \left( \frac{S_{t+\tau}}{F_t} \right) \right] = -\frac{2}{\tau} \left[ \int_{F_t}^{\infty} \frac{e^{rt}}{K^2} C_t(K) dK + \int_0^{F_t} \frac{e^{rt}}{K^2} P_t(K) dK \right]
$$

(B.1.3)

where $\tau = 30/365$ (i.e., 30 calendar days). In continuous diffusion models (e.g., the SV model), the square of the VIX index level is also equal to the expectation of continuous realized variance under the risk neutral measure $Q$ or one month continuous variance swap rate.

$$
\widetilde{\text{VIX}}_t^2 = -\frac{2}{\tau} E^Q_t \left[ \ln \left( \frac{S_{t+\tau}}{F_t} \right) \right] = E^Q_t \left[ 1 \int_t^{t+\tau} v_s ds \right] = E^Q_t \left[ \frac{2}{\tau} \left( \int_t^{t+\tau} \frac{dS_u}{S_u} - \ln \frac{S_{t+\tau}}{S_t} \right) \right]
$$

(B.1.4)
B.2 Historical VIX Levels

Our SPX options data is from the Chicago Mercantile Exchange (CME) and not from the CBOE. We use our options data in the empirical testing of VIX futures pricing and computing the historical profit and loss from investing in variance swaps and VIX futures. To check if our options data is consistent with the CBOE options data we compute the historical VIX levels. From our SPX options data we compute the VIX level using the market VIX formula (4.2.2) from January, 2004 until July, 2005. Figure B.1 shows the VIX market level (VIX-Market) and VIX value from our options data set (VIX Options Data). The left plot shows both time series of VIX and their difference. The right plot shows the histogram plot of difference between two VIX series. We plot VIX market level minus VIX from our options data set. Table B.1 shows the statistics of the difference between two series. Results show that average value of difference between VIX market and from our options data set is about negative 11 basis points. The small difference between two series implies that on an average our data is consistent with the CBOE data but there are some differences in data sets.

Table B.1: Difference between VIX Market and VIX from our options data.

<table>
<thead>
<tr>
<th></th>
<th>Abs. Average</th>
<th>Max</th>
<th>Min</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>-0.11</td>
<td>2.05</td>
<td>-1.44</td>
</tr>
</tbody>
</table>

This table shows the different statistics of the difference between VIX market level and from our options data.
Figure B.1: The left plot shows the VIX time series from January 2004 to July 2005. It shows VIX market series and VIX computed from our options data set and their differences. The right plot shows the histogram of differences for the same time period.
Appendix C

Proofs for Chapter 5

C.1 The Static Strategy and Proof of Proposition 16

Proof of Proposition 16

Using (5.2.1) and applying Itô’s lemma to \( \ln P_T \) we obtain

\[
\ln P_T = \ln P_0 + \int_0^T \left( \mu_P(X_t) - \frac{1}{2} \text{diag}(\Sigma_P \Sigma_P^T) \right) dt + \int_0^T \Sigma_P dB_t^P. \tag{C.1.1}
\]

The wealth dynamics for a static trading strategy, \( \theta \), satisfy

\[
d\frac{W_t}{W_t} = \left[ (1 - \theta^T \mathbf{1}) r + \theta^T \mu_P(X_t) \right] dt + \theta^T \Sigma_P dB_t. \tag{C.1.2}
\]

A simple application of Itô’s lemma to \( \ln W_T \) implies

\[
W_T = W_0 \exp \left( \int_0^T \left( (1 - \theta^T \mathbf{1}) r + \theta^T \mu_P(X_t) \right) dt - \frac{1}{2} \left( \theta^T \Sigma_P \Sigma_P^T \theta \right) T + \theta^T \Sigma_P B_T^P \right) \tag{C.1.3}
\]

Substituting (C.1.1) into (C.1.3) we obtain
\[ W_T = W_0 \exp \left( (1 - \theta^T \mathbf{1}) r T + \frac{1}{2} \theta^T (\text{diag}(\Sigma_p \Sigma_p^T) - \Sigma_p \Sigma_p^T \theta) T + \theta^T \left[ \ln \frac{P_T}{P_0} \right] \right) \]  
\[(C.1.4)\]

as desired. \(\square\)

Under the price dynamics assumed in Section 5.4, it is easy to see that under the physical probability measure, \(P\), the terminal security prices, \(P_T\), are multivariate log-normally distributed. In particular, \(Y := \ln(P_T) \sim N(\mu_Y, \Sigma_Y)\) where

\[ \mu_Y = \left( \mu_0 - \frac{1}{2} \text{diag}(\Sigma_p \Sigma_p^T) \right) T + \ln P_0 + \frac{\mu_1 X_0}{k} (1 - \exp(-kT)) \]

and

\[ \Sigma_Y = (\Sigma_p \Sigma_p^T) T + \mu_1 \mu_1^T \Sigma_X \Sigma_X^T \left( \frac{T}{k^2} + \frac{1 - \exp(-2kT)}{2k^3} - \frac{2(1 - \exp(-kT))}{k^3} \right) \]
\[ + \left( \mu_1 (\Sigma_p \Sigma_X^T) + (\Sigma_p \Sigma_X^T) \mu_1^T \right) \left( \frac{T}{k} - \frac{(1 - \exp(-kT))}{k^2} \right). \]  
\[(C.1.5)\]

It therefore follows that \(W_T\) in (C.1.4) is also log-normally distributed. As a result, assuming CRRA utility it is straightforward to obtain an analytic expression for the value function corresponding to any static strategy as well as its derivatives. These terms can then be used to obtain an upper bound on the value function for the optimal dynamic trading strategy as described in Section 5.3 and, in further detail, in HKW.

C.2 Generalized Buy and Hold Strategy and Value Function

We now expand on the steps outlined in Section 5.2.4.
1. **Solving the SDE**: The security price dynamics are as specified in (5.4.1) so that $P_t$ is a 3-dimensional price process and $X_t$ is a scalar state variable process. The market-price-of risk\(^1\) process, $\eta_t$, is a 4-dimensional process satisfying

$$\Sigma_P \eta_t = (\mu_0 + X_t \mu_1 - r1) \tag{C.2.1}$$

and the corresponding $Q$-Brownian motion satisfies

$$dB_t^P = dB_t^Q - \eta_t dt. \tag{C.2.2}$$

The first three components of $\eta_t$ are uniquely\(^2\) determined by (C.2.1) and the fourth component, $\eta_t^{(4)}$, is unconstrained. Under any risk neutral measure, $Q$, defined by (C.2.1) and (C.2.2), the security price processes satisfy $dP_t = P_t [\mu dt + \Sigma_P dB_t^Q]$.

It immediately follows that $\ln(P_T) \sim N(\mu_Q, \Sigma_Q)$ under any risk neutral measure, $Q$, where

$$\mu_Q = \ln P_0 + \left( r - \frac{1}{2} \text{diag}(\Sigma_P \Sigma_P^T) \right) T \tag{C.2.3}$$

$$\Sigma_Q = \Sigma_P \Sigma_P^T T. \tag{C.2.4}$$

The state variable, $X_t$, is easily seen to satisfy

$$X_t = X_0 e^{-kt} + e^{-kt} \int_0^t e^{ks} \Sigma_X dB_s^P \tag{C.2.5}$$

with

$$E(X_t) = X_0 e^{-kt}$$

$$\text{Var}(X_t) = \frac{\Sigma_X \Sigma_X^T}{2k} (1 - e^{-2kt}). \tag{C.2.6}$$

---

\(^1\)See Duffie (1996) for example.

\(^2\)See Table 1 where we assumed, without loss of generality, that $\eta_t^{(4)}$ does not influence the first three rows of $\Sigma_P$. 

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Setting \( Y_t := \ln P_t \) (C.2.5) and a standard application of Itô’s lemma then yield

\[
Y_t = Y_0 + \int_0^t \left( \mu_0 - \frac{1}{2} \text{diag}(\Sigma_P \Sigma_P^T) + \mu_1 X_0 e^{-k s} \right) ds + \int_0^t \Sigma_P dB_s^P + \int_0^t \mu_1 \frac{(1 - e^{k(s-t)})}{k} \Sigma_X dB_s^P.
\]

Under the empirical measure, \( P \), it therefore follows that \( Y_T = \ln P_T \sim \mathcal{N}(\mu_Y, \Sigma_Y) \)

where

\[
\mu_Y = \left( \mu_0 - \frac{1}{2} \text{diag}(\Sigma_P \Sigma_P^T) \right) T + \ln P_0 + \frac{\mu_1 X_0}{k} (1 - e^{-k T})
\]

\[
\Sigma_Y = (\Sigma_P \Sigma_P^T) T + \mu_1 \Sigma_X \Sigma_X^T \left( \frac{T}{k^2} + \frac{1 - e^{-2kT}}{2k^3} - \frac{2(1 - e^{-kT})}{k^3} \right)
\]

\[
+ \left( \mu_1 (\Sigma_P \Sigma_P^T) + (\Sigma_P \Sigma_P^T) \mu_1^T \right) \left( \frac{T}{k^2} - \frac{(1 - e^{-kT})}{k^2} \right).
\] (C.2.7)

In particular, its PDF satisfies

\[
f_{P|b_1,b_2,b_3}^P = \frac{1}{(2\pi)^{3/2} |\Sigma_Y|^{1/2}} \exp \left( -\frac{1}{2} (\ln b - \mu_Y)^T \Sigma_Y^{-1} (\ln b - \mu_Y) \right) \] (C.2.8)

\( P_T \) has the same density under \( Q \) with the obvious replacement of \( \mu_Y \) and \( \Sigma_Y \) with \( \mu_Q \) and \( \Sigma_Q \).

### 2. Computing the Conditional State Price Density

A state price density (SPD) process, \( \pi_t \), satisfies

\[
\pi_t = e^{-r_t dQ/d\bar{P}} = e^{-r_t} \exp \left( -\int_0^t \eta_s dB_s^P - \frac{1}{2} \int_0^t ||\eta_s||^2 ds \right)
\] (C.2.9)

where \( \eta_s \) is any market price-of-risk process satisfying (C.2.1). We need to compute the conditional state price density, \( \pi_t^b \), where we condition on the terminal security prices.

In particular, we wish to solve for the time \( T \) conditional state price density

\[
\pi_T^b = \mathbb{E}_0^P \left( \pi_T(w) \mid P_T^{(l)}(\omega) = b_i, \ i = 1, \ldots, N \right)
\] (C.2.10)
where \( \omega \) represents samples of the underlying Brownian motions. It is worth mentioning that while there are infinitely many SPD processes, \( \pi_t \), corresponding to each solution of (C.2.1), it is easy to check we can use any such process on the right-hand-side of (C.2.10) and still obtain the same\(^3\) conditional state price density, \( \pi_T^b \).

Using (C.2.9) we can confirm it satisfies

\[
\pi_T^b = \frac{e^{-rT}f_{P_1,P_2,P_3}(b_1,b_2,b_3)}{f_{P_1,P_2,P_3}(b_1,b_2,b_3)} = \frac{e^{-rT}|\Sigma_Y|^\frac{1}{2}}{|\Sigma_Q|^\frac{1}{2}} \exp \left( \frac{1}{2} \left( \ln b - \mu_Y \right)^\top \Sigma_Y^{-1} (\ln b - \mu_Y) - (\ln b - \mu_Q)^\top \Sigma_Q^{-1} (\ln b - \mu_Q) \right). \tag{C.2.11}
\]

3. Computing the Optimal GBH Wealth

We then use static martingale approach to solve for the optimal GBH strategy assuming CRRA utility. In particular we solve

\[
V_0^{gbh} = \sup_{W_T} E_0^P \left[ \frac{W_T^{1-\gamma}}{1-\gamma} \right] \tag{C.2.12}
\]

subject to \( E_0^P [\pi_T^b W_T] = W_0 \).

Note that because of our use of the conditional state price density, \( \pi_T^b \), in (C.2.12), we did not need to explicitly impose the constraint that \( W_T \) be a function of only the terminal security prices. This constraint will be automatically satisfied. The problem in (C.2.12) can be solved using standard static optimization techniques and we obtain

\[
W_T^{gbh} = \frac{W_0(\pi_T^b)^{-\frac{1}{\gamma}}}{E_0^P \left[ (\pi_T^b)^{-\frac{1}{\gamma}} \right]}, \tag{C.2.13}
\]

\(^3\)This is consistent with our earlier observation that all European options prices can be uniquely determined despite the market incompleteness.

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We call the trading strategy that replicates \(W_t^{gbh}\) the generalized buy and hold (GBH) trading strategy. We also immediately obtain

\[
V_0^{gbh} = \frac{W_0^{1-\gamma}}{1-\gamma} \left( E_0^P \left[ \left( \pi_t^b((w)) \right)^{\frac{1-\gamma}{\gamma}} \right] \right)^\gamma.
\] (C.2.14)

While requiring some computation, it is straightforward to show that the expectation in (C.2.14) satisfies

\[
E_0^P \left[ \left( \pi_t^b \right)^{\frac{1-\gamma}{\gamma}} \right] = \exp \left( -r T \frac{(\gamma-1)}{\gamma} - \frac{\delta}{2} \right) \left( \frac{|\Sigma_Y|}{|\Sigma_Q|} \right) \left( \frac{\Sigma}{|\Sigma|} \right)^{\alpha} \sqrt{\frac{|\Sigma|}{|\Sigma_Y|}}.
\] (C.2.15)

where \(\alpha = (\gamma - 1)/(2\gamma)\), and \(\Sigma_Q\) and \(\Sigma_Y\) are given in equations (C.2.4) and (C.2.7), respectively. The parameters \(\delta\) and \(\Sigma\) are solutions to

\[
\Sigma^{-1} = (1 - 2\alpha) \Sigma_Y^{-1} + 2a \Sigma_Q^{-1}
\]

\[
\mu^T \Sigma^{-1} = (1 - 2a) \mu_Y^T \Sigma_Y^{-1} + 2 \alpha \mu_Q^T \Sigma_Q^{-1}
\] (C.2.16)

\[
\delta = (1 - 2a) \mu_Y^T \Sigma_Y^{-1} \mu_Y + 2 \alpha \mu_Q^T \Sigma_Q^{-1} \mu_Q - \mu^T \Sigma^{-1} \mu.
\] (C.2.17)

4. Determining the Value Function at all Intermediate Times

It is straightforward to generalize (C.2.14), (C.2.15) and (C.2.16) to obtain

\[
V_t^{gbh} = E_t^P \left[ \left( W_T \right)^{1-\gamma} \right]
\]

\[
= W_0^{1-\gamma} \left( \exp \left( -\frac{\delta + \delta(1-\gamma)}{2} + rT(1-\gamma) \right) \left( \frac{|\Sigma_Y|}{|\Sigma_Q|} \right) \left( \frac{\Sigma}{|\Sigma_Y|} \right)^{\gamma} \left( \frac{\Sigma}{|\Sigma_Q|} \right)^{\frac{1-\gamma}{\gamma}} \right) \frac{1}{(1-\gamma)}
\] (C.2.18)
where \( \tilde{\delta}_t \) and \( \Sigma_t \) satisfy

\[
\Sigma_t^{-1} = \Sigma_Y^{-1} + (-2a) \Sigma_Y^{-1} + 2a \Sigma_Q^{-1}
\]

\[
\mu_t^T \Sigma^{-1}_t = \mu_Y^T \Sigma_Y^{-1} + (-2a) \mu_Y^T \Sigma_Y^{-1} + 2a \mu_Q^T \Sigma_Q^{-1}
\]

\[
\tilde{\delta}_t = \mu_Y^T \Sigma_Y^{-1} \mu_Y + (-2a) \mu_Y^T \Sigma_Y^{-1} \mu_Y + 2a \mu_Q^T \Sigma_Q^{-1} \mu_Q - \mu_t^T \Sigma_t^{-1} \mu_t.
\]

5. Determining the Replicating Trading Strategy

We now briefly describe how to obtain the replicating strategy for the optimal GBH wealth, \( W^{\text{gbh}}_T \), given by (C.2.13). The martingale property of a state-price density process implies

\[
\pi_t W_t = E_t^P [\pi_T W_T] = \frac{W_0 E_t^P \left[ \pi_T (\pi_T^2)^{-1} \right]}{E_0^P \left[ (\pi_T^2)^{-1} \right]}.
\]  

(C.2.19)

Using (C.2.1), (C.2.9) and (C.2.10) to substitute for and \( \omega_t \) in (C.2.19), we can evaluate the expectations in (C.2.19) to obtain

\[
W^{\text{gbh}}_t = \exp \left( \tau t + \frac{\delta}{2} - \frac{\delta_t}{2} \right) \left( \sqrt{\frac{\Sigma_t}{\Sigma_t Q}} \right) \left( \sqrt{\frac{\Sigma_t}{\Sigma_t Q}} \right)
\]

(C.2.20)

where \( \Sigma_t \) and \( \delta_t \) satisfy

\[
\Sigma_t^{-1} = \frac{\Sigma_Y^{-1}}{\gamma} + \frac{\Sigma_Q^{-1}}{\gamma} + \Sigma_t^{-1}_Q
\]

\[
\mu_t^T \Sigma_t^{-1} = \frac{\mu_Y^T \Sigma_Y^{-1}}{\gamma} - \frac{\mu_Q^T \Sigma_Q^{-1}}{\gamma} + \mu_t^T \Sigma_t^{-1}_Q
\]

\[
\delta_t = \frac{\mu_Y^T \Sigma_Y^{-1} \mu_Y}{\gamma} - \frac{\mu_Q^T \Sigma_Q^{-1} \mu_Q}{\gamma} + \mu_t^T \Sigma_t^{-1} \mu_Q - \mu_t^T \Sigma_t^{-1} \mu_t.
\]

(C.2.21)

The terms \( \mu_Q \) and \( \Sigma_Q \) appearing in (C.2.21) are the mean vector and variance-covariance matrix of \( \ln P_T \) under \( Q \), conditional on time \( t \) information. In particular, given time \( t \)
information, we have \( \ln P_T \sim N(\mu_Q, \Sigma_Q) \) under any risk neutral measure, \( Q \), where

\[
\begin{align*}
\mu_Q &= \ln P_t + \left( r - \frac{1}{2} \text{diag}(\Sigma P \Sigma P^T) \right) (T - t) \\
\Sigma_Q &= \Sigma P \Sigma P^T (T - t).
\end{align*}
\] (C.22)

We can then apply Itô's lemma to \( W^{gh}_t = f(t, P^{(1)}_t, P^{(2)}_t, P^{(3)}_t) \) to obtain

\[
dW_t = \left[ \frac{\partial f}{\partial t} + \sum_{j=1}^{j=3} \sum_{i=1}^{i=3} \frac{\partial^2 f}{\partial P^{(i)}_t \partial P^{(j)}_T} \right] dt + \frac{\partial f}{\partial P^{(1)}_t} dP^{(1)}_t + \frac{\partial f}{\partial P^{(2)}_t} dP^{(2)}_t + \frac{\partial f}{\partial P^{(3)}_t} dP^{(3)}_t.
\] (C.23)

From (C.2.20), we can see that \( W^{gh}_t \) depends on \( P_t \) only through \( \delta_t \). Hence,

\[
\frac{\partial W^{gh}_t}{\partial P^{(i)}_t} = \frac{\partial f}{\partial P^{(i)}_t} = -\frac{W^{gh}_t}{2} \frac{\partial \delta_t}{\partial P^{(i)}_t}.
\] (C.24)

If \( \theta^{(i)}_t \) is the proportion of \( W^{gh}_t \) invested\(^4\) in the \( i^{th} \) security at time \( t \) then

\[
\theta^{(i)}_t = \frac{-P^{(i)}_t}{2} \frac{\partial \delta_t}{\partial P^{(i)}_t}.
\] (C.25)

6. Compute Lower and Upper Bounds on True Optimal Value function

When the agent does not face dynamic trading constraints then the expected utility, \( V^{gh}_0 \), can be attained by following the trading strategy outlined above. \( V^{gh}_0 \), which we have obtained in closed form, is therefore a lower bound on the optimal value function associated with the true optimal dynamic trading strategy.

The same strategy and it associated value function process, \( V^{gh}_t \), can then be used to obtain an upper bound on the optimal value function. This is done by simulating the SDE's for the price processes, \( P_t \), the state variable process, \( X_t \), and the wealth process, \( W_t \), and following the algorithm outlined in HKW and summarized in Section 5.3.

\(^4\)1 - \( \sum \theta^{(i)}_t \) is then the fraction invested in the cash account.