Essays on Financial Markets

Chang-Yong Ha

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ABSTRACT

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The paper in the first chapter analyzes the optimal trading behavior of the informed traders in multiple security markets where cross investment is allowed as a strategy choice. Covariance measure is extensively utilized as the representation of asymmetric information structure between the traders to obtain a fairly simple form of equilibrium characterization. The result shows, among other things, that, for two securities having non-zero correlation, there exists a unique Nash equilibrium at which trading in both markets is optimal for each trader. Unlike previous research the covariance structure of the liquidation values of the stocks is shown to be critical in understanding the behavior of the heterogeneously informed traders. Part of the results in previous research is confirmed to be limiting cases of our model and important market characteristics are examined and compared.

In the second chapter I examine how the market maker's pricing scheme affects the equilibrium characteristics and the behavior of the informed traders of the dual markets where the informational asymmetry is extant. The particular pricing rule considered in this study impounds the information contained in the order flows of both markets. We show that the suggested pricing strategy counterbalances effectively, and perfectly indeed, the insiders' attempts to further exploit their private information by combining it with the publicly available information.

The market maker has an incentive to simultaneously price multiple securities. In particular, the market maker can be more effective in warding the insiders off the arbitrage profits by choosing the securities with higher correlations.

I further extend the model to a dynamic setting to show there exists a unique sequential equilibrium whose characteristics are generalizations of the static game. I show that
the market characteristics in equilibrium depend not only on the market priors but on the stock value correlation. That is, the correlation serves as informational discount factor in characterizing the equilibrium. As the trading interval becomes uniformly small the sequential equilibrium converges to a continuous auction equilibrium. Several key findings from convergence results include that the equilibrium market depth in general needs not be constant. The analysis shows that it may well be decreasing as the trading approaches to the terminal time.

Much of the term structure modelers' efforts has focused on mitigating the trade-off between matching the various moments of yields in conditional sense. When we deal with the multiple markets that are not tightly integrated, this tension could well be amplified. In fact, recent evolutions in international financial market have observed diverse heteroskedastic co-movement of the two country interest rates in which negative conditional correlations are not rare. The paper in the third chapter shows that the proposed model offers a much enhanced flexibility in characterizing the interest rates and yield behavior in international market whereas the existing affine class cannot accommodate this feature without violating the positivity of the nominal interest rates. I suggest an international extension of the quadratic term structure model, which incorporates both common and local state variables, as an alternative that can allow for sign-switching correlations among interest rates. Using Markov Chain Monte Carlo (MCMC), I estimate the model and the latent state variables. Empirical performance of the international quadratic model (IQTSM) in explaining conditional correlations of the two country yields are examined using a panel of the US and Japanese yield data observed monthly from January 1985 to May 2002.

Although my model has much more parsimonious structure, employing only half the number of state variables used in our benchmark model, this study shows that IQTSM captures the yield behavior in international market as well as those in existing literature. The implications on forward premium anomaly are also derived endogenously from the estimated model and compared with the results from the benchmark model. The behavior of risk premium salient in the US-Japan exchange rate data is captured as well with the model that I propose in this chapter as the term structure model employed in the existing
front line research. The quadratic structure of risk premium and admissibility of negative correlations are expected to be another set of important ingredients to accounting for time series behavior of bond yields and currency prices.
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abridged - note of thanks. I dedicate this thesis to my parents.
To My Parents
Chapter 1

Multiple Market Insiders Trading Under Asymmetric Information Structure: Effects on Equilibrium and Market Characteristics
1.1 Introduction

For the past few decades, various facets of information have been central to the researchers' interest in financial economics. Included in their list are numerous topics such as 'financial market's informational efficiency', 'equilibrium effects of diverse information', 'informational aggregation', and so on. Those researchers' concern in information seems to be condensed by the following question:

*how and how much does information in its different contents and structures affect the characteristics of financial markets?*

Consequently, the role of private information, in general, and its asymmetric structure, in particular, have been extensively examined by researchers in financial economics. The purpose of the study in this chapter is to develop a model to analyze the following questions:

- How does the behavior of informed traders change if they are allowed to make use of the publicly available information for their demand decision?
- What changes are, then, to be made to the equilibrium characteristics documented in the previous research?
- Do their insider profits increase or decrease? How do they interact with the correlation of the liquidation values?
- What about the trading volume and price volatility?
1.1.1 Overview

Starting from the early 1980s there has been a booming trend in the area called *market microstructure of financial markets*. The market environment such as institutional structure and trading process has become increasingly complex, and subsequently, empirical research has documented various newly discovered market phenomena. Since many of those new observations didn't seem to be adequately analyzed by the then-standard theoretical paradigms, the researchers in financial economics began to adopt some new approaches to the market behavior including the trading details. Central to the researchers' concern lies the behavior of the *asymmetrically informed* agents, who seem to be much closer to the investors in the real markets than those postulated in the previous economic settings. The *competitive rational expectations equilibrium* developed by Grossman appears to be a natural choice for making sense of what's being observed and evidenced in empirical research.

Unlike the Walrasian competitive equilibrium, where the price or the pricing system has the main role of clearing the market, the competitive rational expectations equilibrium assigns another, yet equally important, role to price, that is, conveying information to the different classes of market participants. Numerous studies were followed in line with this newly developed analytical paradigm in an attempt to explain various aspects of financial marketplace.

Despite the powerful charm of *fully revealing* equilibria, however, the standard rational expectations equilibrium was often plagued by one of its main modeling stances that the agents are price-takers. Grossman (1980) show, for example, that under certain conditions the financial markets can not be efficient in the *strong* sense. In addition, the competitive rational expectations equilibrium is hard to obtain analytically.

Subsequent research shows that modeling investors as price-takers for *fully revealing*
equilibria requires full revelation of information among agents, which is sort of self-fulfilling prophecy.\textsuperscript{3} Since an agent with exclusive information can not be considered a price-taker due to her capability of strategic allocation if there exists a finite number of traders in the market. Insofar as its modeling legitimacy is closely linked to the total mass, or the total number of the investors in the market, some of the research, such as Kyle (1989) came to focus on price formation process. While supporting the notion that prices convey some information, this line of research also finds it inappropriate to premise the perfect competition if there exists informational asymmetry among agents. While analysis of many issues related to the financial markets produces results that are not sensitive to the underlying assumptions, it is often not the case with many others, in particular, with the issues related to the mechanism and process of trading and its organizational aspect.

This is the juncture at which the second generation of research in competitive rational expectations equilibrium with asymmetric information began. There are largely two different branches of research here, one of which is pioneered by Glosten and Milgrom (1985). Arguably the most-often cited study in this area along with Kyle (1985), Glosten and Milgrom (1985) highlight the issue of 'quality of information', examining bid-ask spread in a specialist market based upon the idea that it can be a purely informational phenomenon. They also show that if the arriving trader is definitely better informed than the market maker, it is possible that there are no bid and ask prices for which the specialist earns zero expected profits, which leads to the break-down of the market due to too severe the adverse selection faced by the specialist. They further note that the market break-down will persist unless the informational asymmetry between the specialist and the informed trader is alleviated. They conjecture in the paper that it may not be the case if the specialist is monopolistic, rather than competitive. Using a specific pricing scheme, Glosten (1989) formally show that the monopolistic specialist will remove the possible market discontinuity. Numerous studies each, ever since, have adopted specific parameterizations of Glosten and

\textsuperscript{3}See Postlewaite and Schmeidler (1986)
Milgrom (1985) to examine specific issues.  

The other line of research was initiated with Kyle (1985), wherein the price is an increasing function of the net order flow executed in a batch, not in a sequence. He shows that the informational asymmetry extant in the market provides an incentive for the insider to trade hiding behind the trades initiated by liquidity need. The information is revealed gradually in his dynamic setting while the pricing parameter tends to be constant as the trading interval becomes smaller. This simple elegant auction model has also produced a number of extensions and variants, among which are Foster and Viswanathan (1990, 1996), Subrahmanyam (1991a, 1991b), Admati and Pfleiderer (1988), Holden and Subrahmanyam (1992), Vayanos (2001, 2006), and Back, Cao, and Willard (2000) to name a few.

Utilizing similar trading structure to the one in Kyle (1985), these papers examine various aspects of equilibrium characteristics as well as their models' compatibility with the empirical observations. Admati and Pfleiderer (1988) look into the concentrated trading patterns pronounced in intraday transaction data. They examine the interaction between strategic informed traders and strategic liquidity traders in a simple dynamic context for a single security, in which all the informed traders receive the same signal. Their refinement of the type of traders called it noise traders in Kyle (1985) into two different types, namely, discretionary liquidity traders and non-discretionary liquidity traders allows the liquidity traders of the former type to concentrate their trading in an attempt to minimize their transaction cost. Given that, it is intuitive that the informed traders are also found to trade more actively in periods when liquidity trading is concentrated. The single security market setting is extended by, among others, Subrahmanyam (1991), who attempted to explain the fast growing popularity of stock index futures, or various baskets of security. Focusing on how the behavior of liquidity traders is affected by introducing a security bas-

\[\text{See Diamond and Verrecchia (1987), Easley and O'Hara (1987), Jacklin, Kleidon, and Pfleiderer (1990) for a few related studies.}\]
ket, this static model of trading game shows that the liquidity traders have an incentive to prefer trading in baskets of securities. Note that this further ramification by Subrahmanyan (1991) of the liquidity traders in his study makes them much more rational and discreet than the informed traders in their trading behavior.

More in line with our study in spirit is Foster and Viswanathan (1996), who examine the effect of heterogeneous information structure among the informed traders on the various aspects of market behavior at its equilibrium. Their study is one of the first comprehensive attempts to explore the differentially informed traders in a more realistic trading game framework. Avoiding the regress in expectations problem by somewhat oversimplified parametrization, they manage to show that the results obtained in models with identically informed traders are to be modified substantially.

While those studies obtain many intuitive findings, their majority assume rather simple information structure of homogeneity. That is, the informed traders in their papers are constrained to obtain exactly the same information or, at best, different pieces of information that have the same structure.

More importantly, none of the previous studies consider a financial marketplace where the insiders (or the informed traders, for that matter) are allowed to freely use the publicly available information in addition to their private information so as to maximize whatever welfare they are expected to maximize. To the extent that we want to model the informed traders as rational profit maximizing agents we need to let them allowed to use the information readily available to all the market participants. This is one of the major motivations of the present study. Furthermore, in all the previous research the informed traders do not have at their disposal the strategy option of cross-market investment even when it is available. Specifically, most trading models in this area posit that even in the face of additional investment opportunities in multiple securities the agents with private information about a particular security do not even consider utilizing all information available to them to achieve
the maximum welfare. Otherwise the informed traders are allowed to invest in the given basket of securities, other than their own market. It is interesting to note that all the informed traders so far are strictly confined in each respective market about which they have private information or show no interest in other trading opportunities available to make as much profit as they can. This seems partly because models of trading game, in dynamic setting especially, can get complicated too quickly to provide the equilibrium characteristics in analytical expressions. This type of simplification obviously offers enhanced model tractability and yet provides a room for generalizations. Thus our study is meaningful in that the model analyzed in the present study allows diverse information of heterogeneous structures which influences the equilibrium market characteristics and optimal behavior of the informed traders. In addition, using the security prices' covariance structure to represent the noisiness of the inferior information in a market evolves into fairly simple form of equilibrium characterization.

1.1.2 Brief Summary of the Paper: Structure and Results

The remainder of this article is organized as follows. Sections 2 and 3.1 describe and present a basic model of multiple market trading game. Our model is structured to emphasize the traders' interaction in determining their trading strategies and to represent the informational asymmetry via correlation between the security price innovations. In section 3.2 a unique Nash equilibrium for the informed traders trading in both markets is obtained and characterized. Each informed trader's optimal strategy and the liquidity characteristics are shown to depend upon the correlation between their information about the liquidation values of the asset prices. Section 4 investigates the informed traders' optimal trading behavior in each security market. The result shows that the asymmetric information structure posited in our trading game influences the optimal trades of the informed traders in opposite directions depending on the relative quality of information. In addition, in case
where $\rho = 1$, the resulting equilibrium is identical to Subrahmanyam (1991). If $\rho$ goes to 0, our equilibrium becomes parallel to single-period Kyle (1985) of two individual markets.

We consider in section 4 the liquidity characteristic vis-a-vis various information structures including those in previous studies dealing with this issue. It is shown that the liquidity of a market is not monotonically increasing or decreasing in $\rho$, a measure of the magnitude of the correlation. An analysis is made for this result that in the presence of asymmetric information structure part of the noise component of the inferior information is not reflected in the price, which is opposed to the market' persistent and consistent recognition of the noise introduced by noisy signals having homogeneous information structure.

Expected profits of the informed are examined in section 6 which provides the gross expected profit of each informed trader as sum of profits from trading individual security markets. We know from the results in section 6 that the informed traders' expected profits do not depend upon the exogenous parameter values and ,in fact, are the same as far as the security prices are perfectly correlated. Their expected profits are affected by $\rho$ value in the exactly the same fashion as the market liquidity, which indicates that their profits are monotone increasing in depth parameters. As a consequence of this result, a strategy implication is provided for noise traders.

In section 7, price informativeness is measured by the variance of security price movement conditional on the net demand or the price. The remaining variance after observing the demand or price information is shown to decrease in the magnitude of the correlation and thus the informativeness of the security price increases as the degree of the correlation goes up. And in section 8 we conclude.

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5Please refer to the section of the paper for single market with many insiders.
1.2 Model

In this section, we consider a model of an economy where two risky assets are simultaneously traded over a single period. For each security, there exists one informed trader who has the information about its liquidation value with perfect precision. While an informed trader has a perfect information about either one of the two securities, no direct information is available for that insider about the other security. However, he can be thought of as having a noisy signal of the other security since the two securities, say 1 and 2, would certainly have a certain covariance structure which is assumed to be common knowledge in this paper. We also assume no other informational source is available for the security prices.

For notational convenience, we denote the informed trader 1(2) to have the perfect information about the security 1(2). Henceforth the market for security 1(2) will be called the own market of investor 1(2) and the market for security 2(1) will be called the other market of investor 1(2).

With the Investment in the market other than his own at his disposal, each informed trader may wish to realize his trade in either security or in both referring to his investment decision criteria. Thus we present a model which is, in spirit, similar to Kyle (1984, 1985) and Foster and Viswanathan (1996) to characterize the trading strategy of the informed traders and the resulting equilibria. For the securities, 1 and 2, trading occurs at time 0 and each security is liquidated at time 1.

The values of the securities at time 0 are denoted by and respectively and their liquidation values at time 1 are given by:

\[
S_1 = \bar{S}_1 + \beta_1 \gamma + \varepsilon_1 \quad \text{(1.2.1)}
\]
\[
S_2 = \bar{S}_2 + \beta_2 \gamma + \varepsilon_2 \quad \text{(1.2.2)}
\]
Note that $\epsilon_1, \epsilon_2$, and $\gamma$ are independently distributed normal random variables, each having a mean of zero. $\beta\gamma$ and $\epsilon$ can be interpreted as systematic and security specific components of security value innovation, respectively. $\beta$ is the sensitivity of security value to the common factor $\gamma$. The structure of security price formation shown in the equations (1) and (2) is a common knowledge to all informed traders. From the equations (1) and (2), we obtain the following result:

\[
\begin{pmatrix}
S_1 \\
S_2
\end{pmatrix}
\sim MVN \left( \begin{pmatrix}
\bar{S}_1 \\
\bar{S}_2
\end{pmatrix}, \begin{pmatrix}
\beta_1^2 \sigma_\gamma^2 + \sigma_\epsilon^2 & \beta_1 \beta_2 \sigma_\gamma^2 \\
\beta_1 \beta_2 \sigma_\gamma^2 & \beta_2^2 \sigma_\gamma^2 + \sigma_\epsilon^2
\end{pmatrix} \right)
\] (1.2.3)

At time 0, each informed trader observes the exact value of $\beta\gamma + \epsilon$ of the security to be traded in his specialty market and takes position in his specialty market based on his private information. He receives the information in sum, knowing, though, the structure of the components of the information he receives. Notice that information about the covariance between $S_1$ and $S_2$ allows each insider to infer the liquidation value of the 'other' security at time 0. So he may wish to enter the other market to place an order.

There are two other types of traders in each security market: uninformed noise traders and competitive market makers. Noise traders trade only for their liquidity purpose and have no discretion over their order sizes. Thus the net amounts of liquidity demand for each security 1 and 2 are random, and denoted by $z_1$ and $z_2$, respectively. Liquidity trades contain no information per se about the payoffs of the securities, that is, $z$ is independent of $\gamma$ and $\epsilon$ and normally distributed around zero. Market makers set prices efficiently to earn a zero profit conditional on information they have about the quantities traded.

---

6Henceforth we omit the subscripts 1 and 2 when a statement is applicable to both securities.
1.3 Trading Equilibrium

1.3.1 Review of the Trading Strategies

Compared to models of single asset economy, ours provides the informed traders with enhanced flexibility for their investment strategies. Presumably each informed trader will submit an order in his specialty market since he knows he will not get nailed by other traders. On the other hand, he would think it may also be profitable to enter the other market because he possesses noisy information about the liquidation value of the other security. Each rational insider, who knows the counterpart reasons the same way, will then evaluate the option to compete in the other market. In evaluating, an insider will explicitly condition his strategy choices on possible actions the other insider may take and vice versa.

Although Subrahmanyam (1991), unlike many others, deals with both individual securities and a basket as multiple investment opportunities, an insider's trading decisions for multiple markets in his research are basically separate decision problems. In addition, an insider does not have to care about others' actions in deciding his optimal trading strategy. But the welfares of the two informed traders in this paper are mutually affecting via strategy interaction and hence their choices of optimal strategy are interdependent. Trading in this paper is, in fact, modeled as a noncooperative game theoretic one, which mainly distinguishes our study from previous ones.

Probably, most parallel to our study is Foster and Viswanathan (1996), which highlights the effects of informational heterogeneity on the trading behavior of informed traders in a dynamic context. However, our study is quite different from Foster and Viswanathan (1996) in various ways. First, while their study is motivated by the heterogeneous information extant in the market, the cause of informational asymmetry is purely attributed to the differential signals with which the informed traders are exogenously endowed at the beginning of the initial trading. This is largely due to the single market setting posited in their paper, which naturally raises the following question: How and where do the informed traders obtain the information about the correlations among their noisy signals so as to
assess the other traders' behavior?

On the other hand, our model brings the inference mechanism of the informed traders to the marketplace, thereby not only rendering the informed traders truly rational, profit-maximizing agents but also articulating the source of informational asymmetry for each market. Secondly, in an attempt to avoid the regress in expectations problem and obtain a sufficient statistic for the trading history, they impose rather too strict and unrealistic assumptions on the market and its participants. It is shown in the following sections that the model proposed in our study, quite to the contrary, requires only a minimal set of assumptions in deriving various equilibrium characteristics including the behavior of the informed traders. Our study in this chapter not only confirms and enriches the insight provided by Foster and Viswanathan (1996) in a single-period trading game context, but also serves as a flexible cornerstone for further analyses in dynamic setting, which will be done in the following chapter. Here, we are primarily concerned with the Nash equilibria of the trading game that our model defines among traders. Notice that two insiders' unanimous decision of 'not entering' the other market falls in with single asset economy paradigm various results of which have been well documented such as in Kyle (1985). Since each informed trader is endowed with two possible strategies, i.e., sticking in the specialty market, and entering both markets, we are now ready to look closely into the trading strategy of informed traders when they both choose to enter the other markets.

1.3.2 Trading Equilibrium When Both are in Both Markets

In each market, trading takes place in two steps. In step one, each of the two informed traders receives his private information about the liquidation value of the security to be traded in his specialty market and submits his market order. He also places an order in the other market based on the noisy signal obtained from comovement of prices of two securi-

7See the Model section of Foster and Viswanathan (1996) for more details
Noise traders in each market are constrained to trade a exogenously given random order size, which is not based on maximizing behavior. Informed traders do not observe the orders from noise traders at the time they submit their own market orders. In step two, the market makers for each market absorb the net order flow, denoted by \( w \), and set a price expecting to earn zero profits. They cannot distinguish the orders of informed traders from those of noise traders. Therefore, the prices set by the market makers for each market at time 0 are, respectively,

\[
P_1 = E(S_1 | \omega_1) \quad (1.3.1)
\]
\[
P_2 = E(S_2 | \omega_2) \quad (1.3.2)
\]

where \( \omega_1 \) and \( \omega_2 \) are net order flows for securities 1 and 2, respectively. And the market makers are assumed to use a linear pricing rule to set prices:

\[
P_1 = \bar{S}_1 + \lambda_1 \omega_1 \quad (1.3.3)
\]
\[
P_2 = \bar{S}_2 + \lambda_2 \omega_2 \quad (1.3.4)
\]

Notice that \( \frac{1}{\lambda} \) is a measure of the "liquidity" or the "depth" of the market for the corresponding security. A low \( \lambda \) means the market is more liquid or deeper in the sense that the cost of a given trade is low.\(^{9}\)

\(^{8}\) Note that with this specification of trading game each asset market is allowed to have two different types of informed traders. In fact, in each market the informed trader who sees the liquidation value of a security is equivalent to the insider in Kyle (1985), and the informed trader who infers the liquidation value via comovement of security prices is similar to one having a noisy signal in Admati and Pfleiderer (1988), and Subrahmanyam (1991).

\(^{9}\) In this study we will sometimes consider the properties of equilibrium by looking into the behavior of the informed trader 1 if equivalent analysis is to be made for the other informed trader 2.
not know the strategy the other informed trader chooses since he has no access to the total amount of order flow or private information the other insider has. But each insider takes as given the optimal decision rule and order quantities of the counterpart in maximizing his expected profit while explicitly taking into account the effects that the quantities he chooses are expected to have on the other informed trader's strategy.

Let \( x_{kl} \) denote informed trader \( k \)'s demand for security \( l \) for \( k, l = 1, 2 \). The following lemma describes a Nash equilibrium that obtains when both informed traders simultaneously trade in both markets.\(^{10}\)

Lemma 1. If both informed traders simultaneously trade in both markets, then, in equilibrium, market orders submitted by each informed trader:

\[
\begin{align*}
x_{11} &= \frac{\beta_1 \gamma + \varepsilon_1}{\lambda_1} \left[ \frac{2 - \rho^2}{4 - \rho^2} \right] \\
x_{12} &= \frac{\beta_1 \gamma + \varepsilon_1}{\lambda_1} \left[ \frac{\theta_{12}}{4 - \rho^2} \right] \\
x_{22} &= \frac{\beta_2 \gamma + \varepsilon_2}{\lambda_2} \left[ \frac{2 - \rho^2}{4 - \rho^2} \right] \\
x_{21} &= \frac{\beta_2 \gamma + \varepsilon_2}{\lambda_2} \left[ \frac{\theta_{21}}{4 - \rho^2} \right]
\end{align*}
\]

The equilibrium values of \( \lambda_1 \) and \( \lambda_2 \) are given by

\[
\begin{align*}
\lambda_1 &= \sqrt{\frac{[\beta_1^2 \sigma_1^2 + \sigma_{e_1}^2][4 - 3\rho^2 + \rho^4]}{\text{var}(z_1)(4 - \rho^2)^2}} \\
\lambda_2 &= \sqrt{\frac{[\beta_2^2 \sigma_2^2 + \sigma_{e_2}^2][4 - 3\rho^2 + \rho^4]}{\text{var}(z_2)(4 - \rho^2)^2}}
\end{align*}
\]

where \( \theta_{12} = \frac{\text{cov}(S_1, S_2)}{\text{var}(z_1)} \) and \( \theta_{21} = \frac{\text{cov}(S_1, S_2)}{\text{var}(z_2)} \) and \( \rho = \text{correlation coefficient between } S_1 \) and \( S_2 \).

\(^{10}\)The proofs of lemmas and propositions are provided in the Appendix.
First note that the optimal trading behavior of each informed trader is influenced by the correlation between the liquidation values of two securities. Equations (1.3.5) - (1.3.8) show the equilibrium market orders are determined not only by the exogenous uncertainty parameters such as variances of liquidity trade and the security price innovation but by the correlation between $S_1$ and $S_2$. It seems interesting, however, to point out that, with all exogenous parameters being equal, each informed trader's optimal order quantity in his specialty market is only affected by the magnitude of the comovement of security prices whereas the direction of comovement does matter to his decision on the market order in the other market.

The intuition behind this result is straightforward: For example, let's consider the informed trader 1's optimal trading behavior. Suppose he observes the realization of $(\beta_1 \gamma + \varepsilon_1)$ of a certain positive value at time 0 and the price changes of securities 1 and 2 are perfectly negatively correlated. Then he expects that the informed trader 2 observes a decrease in the value of $S_2$, or $(\beta_2 \gamma + \varepsilon_2)$ of a negative value. Subsequently, the informed trader 1 knows that the informed trader 2 would expect an increase in the value of $S_1$ or a positive value of $(\beta_1 \gamma + \varepsilon_1)$ and determine to place a purchasing order for security 1. Therefore, the informed trader 1 will explicitly take into account this positive demand of the informed trader 2 in maximizing his expected profit. This will lead to less aggressive trading of the informed trader 1. The result would be the same if the security price changes are perfectly positively correlated as long as the informed trader 1 observes the same value of $(\beta_1 \gamma + \varepsilon_1)$. On the other hand, the informed trader 1 observing $(\beta_1 \gamma + \varepsilon_1)$ of positive value would postulate$(\beta_2 \gamma + \varepsilon_2)$ to be positive if $S_1$ and $S_2$ are positively correlated and thus submit a purchasing order for security 2. If the informed trader 1 would end up with negative demand for security 2 since he expects $(\beta_2 \gamma + \varepsilon_2)$ to be negative.

Lemma 1 tells us that each of the equilibrium market orders for a security submitted
by both informed traders (e.g., $x_{11}$ and $x_{21}$) is proportional to the amount of noise trading of that security ceteris paribus. In addition, equilibrium market orders for both securities 1 and 2 submitted by an informed trader (e.g., $x_{12}$ and $x_{22}$) increase monotonically in the liquidation value of the security that the informed trader knows with perfect precision at time 0.

In relation to the pricing parameters $\lambda_1$ and $\lambda_2$, it is clear from the equations (1.3.9) and (1.3.10) that the adverse selection faced by the market makers is alleviated when there is more noise trading (i.e., higher $\text{var}(z)$) and when market makers' informational disadvantage is less severe (i.e., lower $(\beta z^2 + \sigma^2)$). As with the equilibrium market orders by informed traders, both market depth parameters $\lambda_1$ and $\lambda_2$ depend on the correlation between security price innovations of $S_1$ and $S_2$. Thus in the following sections we closely look into how the comovement of security prices affects various aspects of the trading equilibrium specified in lemma 1.

### 1.4 Comovement of Security Prices and Optimal Trading Behavior

Our economy consists of two risky assets having a certain covariance structure, which may or may not contain certain trivial terms. The robustness of the model we investigate in this paper lies in its characterization (or representation) of the economy since our model, in fact, encompasses all types of economy where two risky securities are arbitrarily chosen to be simultaneously traded. It will be shown that some results in previous research are nested by ours for specific value of $\rho$.

To avoid notational clutter, we define
\[ \varphi_{kl} = \sqrt{\frac{\beta_k^2 \sigma^2 + \sigma_{\epsilon_k}^2}{\text{var}(z_t)}}, \quad k \& l = 1, 2 \]
\[ \eta_s = \frac{(2 - \rho^2)}{\sqrt{4 - 3\rho^2 + \rho^4}} \quad \text{and} \]
\[ \eta_0 = \frac{\rho}{\sqrt{4 - 3\rho^2 + \rho^4}} \]

Plugging the values of \(\lambda_1\) and \(\lambda_2\) into the equilibrium order amounts in lemma 1, we obtain the following result:

\[ x_{11} = \frac{\eta_s}{\varphi_{11}} (\beta_1 \gamma + \varepsilon_1) \quad (1.4.1) \]
\[ x_{12} = \frac{\eta_0}{\varphi_{12}} (\beta_1 \gamma + \varepsilon_1) \quad (1.4.2) \]
\[ x_{22} = \frac{\eta_s}{\varphi_{22}} (\beta_2 \gamma + \varepsilon_2) \quad (1.4.3) \]
\[ x_{21} = \frac{\eta_0}{\varphi_{21}} (\beta_2 \gamma + \varepsilon_2) \quad (1.4.4) \]

The following result concerns how the security price comovement affects the optimal trades.

**Lemma 2**

For \(\forall -1 \leq \rho \leq 1\)

1. \(\frac{\partial \eta_s}{\partial \rho} < 0; \frac{\partial x_{kk}}{\partial \rho} < 0\) if \((\beta_k \gamma + \varepsilon_k) > 0\) and \(\frac{\partial x_{kk}}{\partial \rho} > 0\) if \((\beta_k \gamma + \varepsilon_k) < 0\) \quad (1.4.5)

2. \(\frac{\partial \eta_0}{\partial \rho} > 0; \frac{\partial x_{kl}}{\partial \rho} > 0\) if \((\beta_k \gamma + \varepsilon_k) > 0\) and \(\frac{\partial x_{kl}}{\partial \rho} < 0\) if \((\beta_k \gamma + \varepsilon_k) < 0\) \quad (1.4.6)

The lemma 2 states that if \((\beta \gamma + \varepsilon)\) is positive, each informed trader’s optimal demand for the security about which he has a perfect information decreases in the ‘magnitude’ of the
correlation between security prices, and if $(\beta \gamma + \varepsilon)$ is negative, it increases in the magnitude of the correlation. But note from lemma 1 that $x_{11}$ and $x_{22}$ have the same sign as $(\beta_1 \gamma + \varepsilon_1)$ and $(\beta_2 \gamma + \varepsilon_2)$, respectively. It implies that an informed trader observing a negative value of $(\beta \gamma + \varepsilon)$ will decrease the absolute size of his market order as the magnitude of correlation increases just as he does when he observes a positive value of $(\beta \gamma + \varepsilon)$. The lemma 2.2 indicates that an informed trader’s optimal demand for the security of which he has a noisy signal increases in the correlation between security prices if he knows the price of the security to be traded in his specialty market will go up at time 1 and it will decrease in the correlation if he knows the price will go down at time 1. Since, unlike $x_{11}$ and $x_{22}$, $x_{12}$ and $x_{21}$ are monotonically increasing or decreasing in $r$ depending upon the values of $(\beta_1 \gamma + \varepsilon_1)$ and $(\beta_2 \gamma + \varepsilon_2)$, respectively, the lemma 2.2 dictates that an informed trader will submit a market order of larger absolute size in the other market as the magnitude of the correlation increases. Therefore, from the above discussion, we readily obtain the following proposition.

**Proposition 1**

*Regardless of the values of $(\beta \gamma + \varepsilon)$, higher degree of comovement of security prices leads to each informed trader’s less aggressive trading in his specialty market, and more aggressive trading strategy in the other market. On the other hand, lower degree of comovement of security prices results in each informed trader’s more aggressive trading strategy in his specialty market and less aggressive trading strategy in the other market.*

The proposition 1 specifies how the comovement of security prices affects the equilibrium trading strategy of the informed traders. The behavior of the informed traders implied by the above proposition conforms with the fact that the rational informed traders in this non-cooperative trading game are endowed with common knowledge of covariance structure.
With higher value of $\rho$, for example, the informed trader 1 would expect that he has a fairly precise information about the liquidation value of security 2, and that the informed trader 2 is also well equipped with precise information about the security 1. Thus, the informed trader 1 will take an aggressive position in the market for security 2, while he shrinks his market order for security 1 considering the aggressive trading behavior the counterpart would take in the security 1's market. The same rationale can be applied to the trading behavior of the informed trader 2. On the other hand, if $\rho$ assumes a low value, each insider knows his information about the security to be traded in the other market tends to be more noisy, and hence his trading behavior in the market will become passive.

Notice that while the value of $\rho$ describes the manner in which the security prices move together, it can also be interpreted in terms of the magnitude of informational asymmetry lying in the market. It is because in our economic setting $\rho$ characterizes the information structure of the economy. That is, if the $\rho$ value gets closer to positive or negative unity, the information each informed trader has about the security price in a market will become more homogeneous. On the other hand, if the value of $\rho$ converges to zero, the informational asymmetry between two informed traders will be maximized since then one informed trader is a perfect insider and the other one would reduce to be a pure noise. Consequently, the proposition is restated as below;

*If informational asymmetry of a market increases, the informed trader with superior information will take more aggressive position and the informed trader with inferior information will take less aggressive position in placing a market order. The opposite is true if the informational asymmetry decreases.*

It is important to note that our equilibrium characterization is equivalent to single-shot Kyle (1985) if $\rho = 0$ and is consistent with Admati and Pfleiderer (1988) and Subrahmanyam (1991) if $\rho = 1$. Therefore, part of the results obtained in those studies can be viewed as
special cases of our study. Note, however, that, in general, \( x_{11} \) is not equal to \( x_{21} \), nor is \( x_{22} \) equal to \( x_{12} \) if the security prices are perfectly correlated and the volume of noise trades are the same for both markets. This is because the sensitivities to common shock \( \gamma \) of the two stocks are not, in general, the same.

Following proposition describes the optimal trading behavior of the informed traders and the resulting equilibrium when the information of the traders about each security are perfectly uncorrelated.

**Proposition 2**

*If \( \rho = 0 \), then the unique Nash equilibrium of our trading game is that each informed trader concentrates the trading in his own specialty market, and does not submit a market order in the other market.*

The above proposition states that although the informed traders are allowed to trade in the other market, they refuse to enter the market if the two securities of the economy are perfectly uncorrelated. In this case, each market comes to have only a single informed trader in it and the equilibrium reduces to be identical to the single period Kyle (1985). The intuition behind this result can be interpreted as follows: From the fact that the two risky

\[ \text{Lemma 1 in Subrahmanyam (1991) is identical (for } N = 2 \text{) to our result when } r = 1. \text{ This is noteworthy because each insider in our study, even with perfect correlation, can not precisely predict the security price movement in the other market whereas the informed of Subrahmanyam (1991) all have perfect information. Therefore, any piece of information which has a linear relationship with the true fact represents a perfect information in a single period model as well as multi-period setting.} \]

\[ \text{2. Notice that our equilibrium characterization is not identical to those researches even when pieces of information each market's insiders have are perfectly correlated. We will consider this in section 5.} \]

\[ \text{12In this case the cross-market demands depends on (the reciprocal of) the ratio between the two common factor sensitivities. Also note that in case of symmetric market to be examined in the next chapter, the optimal demands for own market and cross-market are identical.} \]
assets of the economy are perfectly uncorrelated, each rational informed trader knows that if he gets in the other market, he will merely play a role of 'noise' to provide liquidity in the market. Recall that we postulated no other informational source is available. Therefore, he expects with certainty to be outwitted by the other informed trader and thus decides not to make an entry into the market. It is evident from proposition 2 if the asymmetry of information in market reaches its maximum the model of this paper allows, then the informed trader with inferior information reduces to be a pure noise trader. Realizing this fact, the informed trader refuses to trade in the market since he knows he should get nailed by the other informed trader.

1.5 Asymmetric Information Structure and Market Depth

This section is concerned with how different information structures influence the liquidity of the market. Note that a change in the number of informed traders represents a change in the structure and content of information being utilized in the market they belong to. Thus our interest focuses on the general effect of informational asymmetry at its various levels on the liquidity characteristic. While our analysis confirms the findings of previous works, it also put forth an idea that a market's response to the noise characteristic of imprecise information varies depending upon different information structures the informed traders have.

Past research indicates that as long as all informed traders have the same information, whether it is of perfect precision or not, an increase in the number of informed traders intensifies competition between them, leading to a decrease in the adverse selection faced by the market makers, and hence a decrease in $\lambda$. When informed traders observe different pieces of information, an increase in their number also means that more private information is actually generated in the market as a whole. So diverse information effect dominated to lead to an increase in $\lambda$. 
While those studies produce some intuitive results on the relationship between the number of informed traders and market liquidity, their analysis is not universal in light of the information structure instituted for the traders. That is, their traders are restricted to observe the same signal, or, at best, different values of signals which all have the same configuration. In essence, their information is \textit{homogeneous in its structural sense}. Due to this constraint, the market is only allowed to have either perfectly identical or different pieces of information which have a certain \textit{positive} correlation.\textsuperscript{13} Thus our paper is attempting in this section to extend their analysis to comprehend a wider range of informational asymmetry. Table (1.1) summarizes, within the context of our model, the results as to the market depth parameter $\lambda$ for the information structures typically incorporated in the previous research.

Two facts are noteworthy from the table for further analysis:

1. Change in the number of the informed and the information structure of the market are interdependent in affecting the market liquidity.

2. The market recognizes explicitly the noise property of imprecise information in such a way to lower the value of $\lambda$, and thus increase the liquidity of market.

The above facts will serve a base for examining the characteristic of the market depth in our study. The following lemma shows how the liquidity parameter responds to the varying levels of informational asymmetry.

\textit{Lemma 3}

For $k = 1, 2$

\[
\frac{\partial \lambda_k}{\partial \rho^2} < 0 \quad \text{if} \quad 0 < \rho^2 < \frac{4}{5} ; \quad \frac{\partial \lambda_k}{\partial \rho^2} > 0 \quad \text{if} \quad \frac{4}{5} < \rho^2 < 1 \quad (1.5.1)
\]

\textsuperscript{13} In fact, using the table (1.1), we can calculate the correlation between heterogeneous noisy signals, which is \(\frac{\beta \sigma_1 \sigma_2 + \sigma_1^2 + \sigma_2^2}{\beta \sigma_1 \sigma_2 + \sigma_1^2 + \sigma_2^2}\). Note that this value falls between 0 and 1.
First note that the lemma 3 indicates the market liquidity is neither monotonically increasing nor decreasing in the value of $\rho$. This result seems somewhat counterintuitive if we only consider that the positive effect of diverse information decreases and the negative competition effect increases in $\rho^2$ (or $|\rho|$), both in monotone fashion. As indicated in Table (1.1), however, the market is consistently responsive to noise characteristic of imprecise information to reflect it in the $l$ value. The variance term of noisy signals ($\sigma^2$) in the liquidity parameters captures the market’s recognition of the noise component. Thus any noisy signal coming into the market affects the liquidity characteristic not only through competition or informational diversity but through the noise trade volume itself they bring in. The comparative statics in Table (1.1) show the adverse selection faced by the market makers is alleviated as the trader’s information becomes noisier, and this is captured by the proportional decrease in $l$. Therefore it is evident that when an informed trader in the market owns an imprecise information, salient is the market’s detection of the noise characteristic contained in his information.

Now we turn to the two informed traders as are structured in this paper. We view the noisy informed trader as entering the market in which there exists a perfect insider. Our concern is then how the market responds to his entry via adjusting its pricing rule in the presence of the perfect insider. Since, with a small value of $\rho$, the two informed traders in a market are likely to depend on different pieces of information, we posit that as $\rho$ converges to 0, competition effect becomes trivial, and hence the market entry of an informed trader with a noisy signal will only elevate the informational diversity and noisiness in the market. On the other hand, as $\rho$ goes to unity, competition between traders becomes severe whereas the noise and the informational diversity will be minimized.\footnote{From the perspective of factor model of security pricing, small value of $\rho$ can be alternatively interpreted: i.e., the security prices are determined, in most part, by the firm-specific factor and the least affected by}
converges to 0, the equilibrium $\lambda$ also converges to the pricing parameter in one period Kyle (1985). To compare this result with the case of single noisy informed trader suggests the noise effect of the imprecise information be, to a certain degree, offset by some other factors. Let us define the effective noise as the noise component of the imprecise information the market explicitly reflects in its pricing rule, and the ineffective noise as that the market does not. Then, in line with the above argument, the following result explains the market's response to the noise effect of the imprecise signal.\(^{15}\)

**Proposition 3**

For each $\rho$, the lower bound of ineffective noise of the inferior information is approximated by:

$$[\lambda|_{\rho} - \lambda|_{\rho=1}]$$

(1.5.2)

The proposition roughly specifies the minimal amount of the noise characteristic that remains unreflected in the market pricing system when the two informed traders have information of a certain level of asymmetry expressed in the $\rho$ value. As the informational asymmetry gets mitigated, the remaining effective noise of the inferior information will be diluted to be null at $\rho = 1$. Recall that, irrespective of the signal values they observe, the signal informed traders' noise completely and consistently recognized by the market as long as their signals have the same information structure. To the contrary, the above proposition shows that if the traders rely on the information of different structures, part of the noise brought about by the inferior information is resolved with no effect on market price. And

---

\(^{15}\) That is, we may interpret the value $[\lambda|_{\rho} - \lambda|_{\rho=1}]$ as the amount of noise absorbed by perfect information or asymmetric information structure without pricing effect.
this may be a partial explanation of the market entry of a signal informed trader in the presence of a perfect insider.

For each value of \( p \), the liquidity parameter \( \lambda \) reflects the net effect of competition, informational diversity and the market's recognition of noise property of inferior information. Thus for \( \rho^2 < 4/5 \), reduced informational asymmetry bring about an increase in the market liquidity via their combined effects, it is interesting to point out that for \( \rho^2 < 4/5 \), \( \lambda \) is increasing as \( \rho^2 \) rises. Since we expect, without loss of generality, that the effects of competition and informational diversity are both monotone, the increasing \( \lambda \) for \( \rho^2 < 4/5 \) suggests that the remaining noise property of the signal recognized by the market is evaporated precipitously enough to fully compensate the negative net effect of reduced diversity and augmented competition on the liquidity. That is, as \( \rho^2 \) gets larger, the noisy signal is accredited by the market as more relevant information in an increasing fashion.

### 1.6 Expected Profits of Informed Traders

In this section, we examine the expected profits of informed traders in case where they trade in both markets, and the effect of informational asymmetry on each trader's profit. Furthermore, we analyze the equilibrium properties in the non-cooperative game theoretic framework. It will be shown that each informed trader's expected profit is affected by \( \rho \) in the exactly the same way as the liquidity characteristic is affected. Also, our equilibrium is show to be a dominant strategy equilibrium making both informed traders worse off by entering the other market. So far we have seen that the equilibrium demands and pricing parameters depend on the value of \( \rho \). Therefore, we are concerned with how the expected profits are influenced by the various degrees of security price comovement.

We begin by defining for \( k = i, j \) and \( l = i, j \).

\[
E(\pi_{k}) = \text{insider k's expectation of gross profit from trading in both markets.}
\]

\[
E(\pi_{kl}) = \text{insider k's expectation of gross profit from trading in market l.}
\]
The following lemma describes the gross expected profits to each informed trader given the level of informational asymmetry.

**Lemma 4** Given the \( \rho \) value, the gross expected profit of each informed trader is respectively given by

\[
E(\pi_1) = \xi_1 \sqrt{[\beta_2^2 \sigma_1^2 + \sigma_{z_1}^2]\text{var}(z_1)} + \xi_2 \sqrt{[\beta_2^2 \sigma_2^2 + \sigma_{z_2}^2]\text{var}(z_2)}
\]

\[
E(\pi_2) = \xi_1 \sqrt{[\beta_2^2 \sigma_1^2 + \sigma_{z_2}^2]\text{var}(z_2)} + \xi_2 \sqrt{[\beta_2^2 \sigma_2^2 + \sigma_{z_1}^2]\text{var}(z_1)},
\]

where \( E(\pi_1) = E(\pi_{11}) + E(\pi_{12}) \), \( E(\pi_2) = E(\pi_{22}) + E(\pi_{21}) \) and

\[
\xi_1 = \frac{(2 - \rho^2)^2}{(4 - \rho^2)^2}, \quad \xi_2 = \frac{\rho^2}{(4 - \rho^2)^2}.
\]

From the above lemma, we find that each informed trader’s expected profit depends not only on the exogenous parameters for both markets but on the correlation between the asset price movements. Note that if \( \rho = 0 \), his expected profit becomes identical to simple shot Kyle (1985), and if \( \rho = 1 \), it is identical to the two insiders in Subrahmanyam (1991).

The following corollary 4.1 is an immediate consequence of lemma 4.

**Corollary 4.1** If the two securities are perfectly correlated, then the two informed traders’ expected profits are the same regardless of the price volatilities and noise trading volumes.

The result in corollary 4.1 is interesting in that the expected profits of the informed
traders are the same in each asset market if $\rho = 1$. Although their equilibrium market orders are, in general, not the same, the two informed traders are expected to earn the same profit in each market if the security prices are perfectly correlated. To examine how the information structure influences each informed trader’s expected profit, we provide the following lemma.

Lemma 5.16

1. \[\frac{\partial E(\pi_{kk})}{\partial \rho^2} < 0 \text{ and } \frac{\partial E(\pi_{kl})}{\partial \rho^2} > 0\] (1.6.3)

2. \[\frac{\lambda_1}{\lambda_2} = \frac{\beta_2^2 \sigma_1^2 + \sigma_2^2}{\beta_2^2 \sigma_2^2 + \sigma_1^2},\]

\[\frac{\partial E(\pi_k)}{\partial \rho^2} < 0 \text{ for } 0 < \rho^2 < \frac{4}{5} \text{ and } \frac{\partial E(\pi_k)}{\partial \rho^2} > 0 \text{ for } \frac{4}{5} < \rho^2 < 1\] (1.6.4)

, where $k \& l = 1, 2, \ k \neq l$

Lemma 5.1 states that each informed trader’s expected profits from trading in his specialty market and in the other market are respectively increasing and decreasing in the ‘magnitude’ of informational asymmetry. The intuition behind this observation is straightforward: For example, the rational informed trader 1 having common knowledge of covariance structure expects his counterpart, trader 2, to trade aggressively in the security 1’s market as the degree of security price comovement raises. Based upon this postulation, the trader 1 tends to take less aggression in his specialty market. But he also knows his noisy information about the liquidation value of security 2 becomes more reliable so that he wants to trade more aggressively in the market for security 2.

16 Note this condition is originated from; $[(\beta_2^2 \sigma_1^2 + \sigma_2^2) \text{ var}(z_1)] = [(\beta_2^2 \sigma_2^2 + \sigma_1^2) \text{ var}(z_2)]$. Hence the condition can be rewritten as $\frac{\lambda_1}{\lambda_2} = \frac{\text{ var}(z_1)}{\text{ var}(z_2)}$. 

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From lemma 5.2, which is parallel to lemma 3, we find that informed traders' expected profits decrease in $p_2$ to reach their minimum at $p_2 = 4/5$ and increase in $p_2$ thereafter. This finding suggests that there exists a certain threshold of correlation over which the increased competition effect more than just offsets the marginal benefit obtained from the improved accuracy of information about the liquidation value. As is the case for the market liquidity, this is, in most part, due to the combined effects of competition, informational diversity and noise factor that our asymmetric information structure provides for the economy. Until the noisy signals that the informed traders have become fairly decent, improved informational quality only offers the net effect of friction to the informed traders. As $p_2$ converges to unity, however, the market promptly identifies the noisy signal as relevant information, and thus their profits tend to increase. Note that lemma 5.2, together with lemma 3, suggests that informed traders' expected profits increase monotonically in both liquidity parameters $\lambda$ if the ratio of market depth for security 1 to that for security 2 is equal to market to the ratio of variances of security price innovation. The result in lemma 5 also renders an interesting strategy implication for noise traders described in the corollary 5.1.

**Corollary 5.1**  
*Assuming the cost minimization is the only decision criterion of liquidity traders, they are better off to trade, whenever possible, in the markets where the security prices have a correlation toward either $\sqrt{\frac{4}{5}}$ or $-\sqrt{\frac{4}{5}}$*

Our model is so structured as to equalize what noise traders are expected to lose with what informed traders together are expected to profit. Therefore, the noise traders would have an incentive to judiciously determine what stocks they want to trade to satisfy their liquidity need. The corollary 5.1 characterizes part of the results in Subrahmanyam (1991)  

---

17 This result, in fact, captures the idea of economics literature, which point out abundance in information sometimes lower the total welfare of the agents.
in a different context.\textsuperscript{18}

### 1.7 Price Informativeness

The idea is widely accepted that equilibrium prices in competitive security markets reflect the information possessed by various traders (Diamond and Verrcchia (1987)). In this section, we are concerned with the effect of additional information coming into the market on the informativeness of prices. Our concern also lies in the dependency of the informativeness on the magnitude of price comovement or informational asymmetry. As measure of the informativeness of prices, we refer to $\text{var}(\tilde{S}|\tilde{P})$, which is the variance of liquidation value conditional on its price. This conditional variance measures the ex-post uncertainty of the liquidation value that remains after price has been observed. That is, the smaller value of this conditional variance corresponds to more revealing price system.

**Proposition 5**

\begin{align*}
\forall -1 < \rho < 1 \\
1. \quad \text{var}(S_k|P_k) &= \left[\frac{2-\rho^2}{4-\rho^2}\right] \left(\beta_k^2 \sigma_r^2 + \sigma_{\epsilon_k}^2\right) \quad \forall k = i, j \quad (1.7.1) \\
2. \quad \frac{\partial[\text{var}(S_k|P_k)]}{\partial \rho^2} &< 0 \quad (1.7.2)
\end{align*}

The above proposition 5 points out two facts:

1. Additional information coming into the market enhances the price informativeness.

\textsuperscript{18}In fact, the proposition 1 of our paper naturally indicates one of the main results in Subrahmanyam (1991) without reference to too complex conditions.
2. the informativeness of prices increases in the degree of correlation between the liquidation values of two securities.

Note that \( \text{var}(S_i|P_t) = \text{var}(S_i)/2 \) if \( \rho = 0 \), which is equivalent to Kyle (1985). \( \text{var}(S_i|P_t) \) in case of \( \rho = 0 \) suggests that no additional information exists that the market can absorb in its price. This conditional variance is monotone decreasing in \( \rho \), making half the unconditional \( \text{var}(S_i) \) its upper-bound. When the liquidation values of two securities are perfectly correlated, the price informativeness is at its maximum. In addition, our measure of the informativeness is not affected by the level of noise trading, \( \sigma_z^2 \). In sum, any additional price of information entering the market enhances the informativeness of the price unless it is perfectly uncorrelated with the perfect information and this positive effect would be more prominent as the information becomes more symmetric in our setting.

### 1.8 Conclusion

In this paper we have analyzed a simplified model of multiple security markets in which each trader rely on the information of different values and structures. Rendering some results of previous research limiting cases, this study attempts to shed some additional lights on the trading behavior of the informed traders in competitive security markets. Covariance structure of two arbitrarily chosen risky securities is extensively utilized to establish the relationship between the degree of informational asymmetry and derive various equilibrium characteristics. It is shown that the security markets can have different types of equilibria depending upon the correlation between the security value innovations. The resulting equilibrium is shown to be a dominant strategy equilibrium unless \( \rho = 0 \).

Our findings suggest also that the magnitude of the asymmetry or the security price comovement affects in the opposite direction the optimal trading strategies and the expected profits of the informed traders in each market. We have confirmed that the response of a
security market to the information in aggregate being used in the market is different from
that analyzed in previous research in terms of its consequent effect on the liquidity char­
acteristic. Moreover, as the information the traders have becomes more homogeneous, the
inferior information is recognized by the market as relevant and reliable one precipitously
enough to raise the value of λ. Finally, the informativeness of price increases as additional
noisy signal comes into the market unless the security prices are perfectly uncorrelated and
the increase in their correlation leads to the enhanced informativeness.

This paper can be extended in several directions for future research. The dual market
setting assumed in this work may be developed into multiple market paradigm to include
higher degree of informational asymmetry. Also we may reexamine the issues raised here in
the dynamic context of repeated game to see how the behavior of informed traders and the
market characteristics change. Because this study focuses on the optimal trading behavior
of differentially informed traders given the conventional price formation scheme postulated
in the previous research, and because the welfare of the market as a whole, or of the traders
who have no access to the private information is as important an issue, it may well be in
our another immediate concern to consider the following questions:

Is there an alternative pricing scheme that the market maker wants to exercise so as to min­
imize the insiders’ profits? If so, how will the behavior of the insiders and the equilibrium
market characteristics change?

The next paper attempts to address those questions.
Table 1.1: Cross Examination of Effects of Different Information Structures and the Number of Informed Traders on Market Depth

<table>
<thead>
<tr>
<th>Information Structure</th>
<th>( \lambda (\text{= market depth parameter}) )</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>N perfect insiders</td>
<td>( \frac{1}{N+1} \sqrt{\frac{N}{\text{var}(\hat{\gamma})}} )</td>
<td>0.5 0.4714</td>
</tr>
<tr>
<td>N informed with same noisy signals</td>
<td>( \frac{1}{N+1} \sqrt{\frac{N}{\text{var}(\hat{\gamma})}} )</td>
<td>0.3536 0.3333</td>
</tr>
<tr>
<td>N informed with different noisy signals</td>
<td>( \frac{1}{(N+1)\sigma_1^2+2\sigma_2^2} \sqrt{\frac{N}{\text{var}(\hat{\gamma})}} )</td>
<td>0.3536 0.4</td>
</tr>
</tbody>
</table>

* \( S \) = liquidation value of a security: \( \text{var}(\hat{\gamma}) = \sigma_2^2 = \beta \sigma_1^2 + \sigma_2^2 \)

* \( \hat{\gamma} \) = noisy information = \( \delta + \epsilon \) \( \epsilon \) is iid \( N(0, \sigma_2^2) \) and \( \delta \) is independent \( \mu \).

* \( N \) = number of informed traders in a market.

* All exogenous parameter values are taken to be unity for the above example:

  i.e., \( \beta \sigma_1^2 + \sigma_2^2 = \sigma_2^2 = \text{var}(\hat{\gamma}) = 1 \)
Table 1.2: Strategies and Payoffs of Insiders

| player 2 | payoff | player 1 | |
|----------|--------|----------|
| (1,2)    | not enter | enter | |
| not enter | 1. $\Psi_1$ | 1. $\Psi_1 + \xi_2 \Psi_2$ | |
|         | 2. $\Psi_2$ | 2. $\xi_1 \Psi_2$ | |
| enter    | 1. $\xi_1 \Psi_1$ | 1. $\xi_1 \Psi_1 + \xi_2 \Psi_2$ | |
|         | 2. $\Psi_2 + \xi_2 \Psi_1$ | 2. $\xi_1 \Psi_2 + \xi_2 \Psi_1$ | |

\[
*\Psi_k = \frac{\sqrt{\beta^2 \sigma^2 + \sigma^2_k \text{var}(z_k)} - 2}{2}, \quad k = 1, 2
\]

\[
\xi_1 = \frac{(2 - \rho^2)^2}{(4 - \rho^2)\sqrt{4 - 3\rho^2 + \rho^4}}
\]

\[
\xi_2 = \frac{\rho^2}{(4 - \rho^2)\sqrt{4 - 3\rho^2 + \rho^4}}
\]
Chapter 2

Multiple Market Insiders Trading With Asymmetric Information: Equilibrium Characteristics and Implications to Market Maker's Pricing Scheme
The Securities and Exchange Commission has begun a broad examination into whether Wall Street bank employees are leaking information about big trades to favored clients, like hedge funds, in an effort to curry favor with those clients, executives at Wall Street banks said. The inquiry, these people said, seems aimed at determining how pervasive insider trading, or the illegal use of market-moving nonpublic information, may be on Wall Street... Trading ahead of client orders, or front-running, has long been an issue on Wall Street...

- New York Times February 6, 2007
2.1 Introduction

In our previous work \(^1\) we considered the dual markets characterized by informational asymmetry and examined the behavior of the informed traders when they are given an additional strategy choice of placing an order for the security in the \textit{other} market. At equilibrium, it was shown to be optimal that each informed trader places an order for both stocks. The magnitude of the correlation of stock value innovations is a key determinant for the absolute order amount for each insider’s own market whereas the direction of the correlation is important in determining the nature of the order he places in the \textit{other} market. Important market characteristics including the market depth and price informativeness also depend on the relationship between the value innovation processes of the two assets. The equilibrium analysis in our previous paper suggests that each insider can make more profit by exploiting his private information by combining it with publicly available information.

Faced with the insiders, who are equipped with such informational advantage, it is in the immediate interest of market makers and the market as a whole, to ensure to have a pricing scheme in the working that could minimize the insiders’ profit. This is an important issue because increased insiders’ profit might well lead to higher cost of equity, which is in stark conflict with the major economic role of stock markets, i.e., to provide effective financing channels for corporations.\(^2\)

\(^1\)Refer to the study in the previous chapter.

\(^2\)Besides the lines of research explained in the previous chapter, where we mainly focus on the market equilibrium characteristics including price and depth, there are other research streams related to the issue of insider trading. While the trading of the people, who have the information not publicly available, has been extremely pervasive historically, their identities are elusive as much(Khanna (1997)). For the regulatory aspect see Bhattacharya and Daouk (2002): insider trading law and cost of equity, Fishman and Hagerty (1995):mandatory disclosure and liquidity, John and Narayanan (1997):market manipulation and regulations, and Khanna (1997):prohibition of insider trading and non-mandatory disclosure. For the effects of insider trading on financial markets, see Fishman and Hagerty (1992):insider trading and the efficiency of stock prices, Lakonishok and Lee (2001):insider trading and stock return predictability, Masson and Madhavan
Thus the major concern of this study is how an alternative pricing strategy proposed in this paper changes the equilibrium characteristics of the markets and how effectively it can diminish the market's loss caused by the insiders' information under the pricing scheme posited in the previous chapter.

In this paper we mainly addresses the following questions:

"How will the equilibrium market characteristics change if the market maker adopts an alternative pricing strategy as opposed to the conventional pricing rule? Is there a particular pricing scheme the market maker prefers to exercise?"

Our analysis suggests that the proposed pricing scheme of the market maker is more effective than the conventional pricing rule in that

1. insiders are constrained to their own markets. That is, they each optimally choose not to place an order in the other market.

2. the profits of each insiders under the proposed alternative pricing scheme are strictly lower than those under the conventional pricing rule.

3. the demand order volatility of insiders are always lower under the proposed pricing scheme.

The above result implies that the market maker has an incentive to simultaneously price both securities, particularly those with high correlation of stock value innovations and to consider the order flow information from both markets in doing so. The results analytically obtained from the symmetric market setting are extended, using numerical analysis,

to confirm that the result holds for more general asymmetric markets.
We also consider the dynamic trading game where the insiders are allowed to trade multiple
times and its equilibrium characteristics are examined based on the model in continuous
time. Our dynamic trading model suggests that the restrictions imposed on the pricing
parameter, $\lambda$, is to be relaxed if the informational asymmetry is extant in the markets, and
that the publicly available information of stock value correlation is a key determinant for
characterizing the dynamic equilibrium of trading game with differentially informed traders.

The remainder of the paper is organized as follows. In sections 2.1 to 2.4, we describe
the basic model structure including the newly proposed pricing scheme of the market maker.
Important equilibrium market characteristics are presented for the general multiple mar­
ket trading game. Section 2.5 treats the trading game in symmetric markets, where the
equilibrium is derived in completely closed-form. Various aspects of symmetric equilibrium
is examined and compared with the results obtained in the previous chapter. We conduct
the numerical analysis in section 2.6, which suggests that the result obtained in symmetric
market equilibrium is robust to the underlying structure of the markets. The subsequent
interpretation and analysis based on the numerical result are offered in this section. In
section 2.7 we extend the model to allow the market participants to trade multiple times so
as to examine how the optimal behavior of the heterogeneously informed traders change.
Section 2.8 provides convergence result, which demonstrates the equilibrium characteristics
obtained in the previous literature is the limiting cases of our model. Section 2.9 concludes
and considers future research.
2.2 Trading Equilibrium

The pricing strategy considered in this study incorporates the order flow information from both markets for pricing each security. The motivation of this alternative pricing scheme comes from the observation that the order amount that an informed trader places in the other market contains a certain level of information about what the insider knows about the future price of the security in his own market since he determines the optimal order amount in the other market based on the price innovation of his own security and its correlation with the other security. Also when an informed trader determines the optimal order he places in his own market, he does so in conjunction with the optimal order to be placed in the other market. Thus the demand an informed trader places in his own market partly impounds the information about the price innovation of the security in his other market and vice versa. By incorporating the information contained in all the demand quantities the new pricing scheme is shown to effectively counter the insiders' attempt to exploit the market, which we will look into more closely below.

2.2.1 Trading Equilibrium

As in the first part of the paper, the two securities for markets 1 and 2 have the following value innovation process at the end of the trading games.

\[
S_1 = \bar{S}_1 + \beta_1 \gamma + \epsilon_1 \\
S_2 = \bar{S}_2 + \beta_2 \gamma + \epsilon_2
\]  

(2.2.1)

Specifically, given the structure of the model, the innovation process is specified to be bivariate normal as follows:
The market maker now sets the prices of the securities as follows:

\[
\begin{bmatrix}
S_1 \\
S_2
\end{bmatrix} \sim N \left( \begin{bmatrix}
\bar{S}_1 \\
\bar{S}_2
\end{bmatrix}, \begin{pmatrix}
\beta_1 \sigma_1^2 + \sigma_z^2 & \beta_1 \beta_2 \sigma_z^2 \\
\sigma_z^2 & \beta_2 \sigma_2^2 + \sigma_z^2
\end{pmatrix} \right)
\] (2.2.2)

The market maker now sets the prices of the securities as follows:

\[
P_1 = \bar{S}_1 + \lambda_1 \omega_1 + \lambda_2 \omega_2
\]
\[
P_2 = \bar{S}_2 + \lambda_2 \omega_1 + \lambda_2 \omega_2
\]

, where \( \omega_1 = x_{11} + x_{21} + z_1 \) and \( \omega_2 = x_{12} + x_{22} + z_2 \)

Note that the pricing rule specified above incorporates the information contained in the order flows of both security. This alternative pricing strategy of the market maker is what distinguishes this study from the one in the previous chapter. The following lemma shows that the equilibrium demands of each insider are still a function of the liquidity characteristic of the market and the stock value correlation, albeit they become more complex.

**Lemma 1:** Given the pricing rule of the market maker as specified above, the equilibrium demands of the insiders are

\[
\begin{align*}
x_{11} & = A_{11}(\beta_1 \gamma + \varepsilon_1) \\
x_{12} & = A_{12}(\beta_1 \gamma + \varepsilon_1) \\
x_{21} & = A_{21}(\beta_2 \gamma + \varepsilon_2) \\
x_{22} & = A_{22}(\beta_2 \gamma + \varepsilon_2)
\end{align*}
\] (2.2.4)
where

\[ A_{22} = \left[ -2\lambda_{11}((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22}) + \lambda_{21}((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22}) \\
+2\lambda_{11}(\lambda_{12}^2 + \lambda_{12}\lambda_{21} + \lambda_{21}^2 - 3\lambda_{11}\lambda_{22} + \lambda_{11}\lambda_{21}(-2\lambda_{22} + \lambda_{12}\theta_{12}))\theta_{12} \\
-\theta_{12}(\lambda_{22}^3 + \lambda_{11}\lambda_{22}(\lambda_{12} - \lambda_{11}\theta_{12}) + \lambda_{11}\lambda_{21}(-2\lambda_{22} + \lambda_{12}\theta_{12}))\theta_{12}^2 \\
/ \left[ ((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22})^2 - ((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22})(\lambda_{12}^2 + \lambda_{21}^2 - 2\lambda_{11}\lambda_{22})\theta_{12}\theta_{21} \\
+(\lambda_{12}\lambda_{21} - \lambda_{11}\lambda_{22})^2\theta_{12}^2\theta_{21}^2 \right] \right] \tag{2.2.5} \]

\[ A_{21} = \left[ (\lambda_{12} + \lambda_{21})^3 - 4\lambda_{11}(\lambda_{12} + \lambda_{21})\lambda_{22} - (\lambda_{22}((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22}) \\
+(\lambda_{12} + \lambda_{21})(\lambda_{12}^2 + \lambda_{12}\lambda_{21} + \lambda_{21}^2 - 3\lambda_{11}\lambda_{22})\theta_{12}\theta_{21} \\
+\theta_{12}(\lambda_{22}^3 - \lambda_{11}\lambda_{22}) - \lambda_{12}\lambda_{22}(\lambda_{21} + \lambda_{11}\theta_{12}) + \lambda_{12}^2(\lambda_{22} + \lambda_{21}\theta_{12})\theta_{12}^2 \\
/ \left[ ((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22})^2 - ((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22})(\lambda_{12}^2 + \lambda_{21}^2 - 2\lambda_{11}\lambda_{22})\theta_{12}\theta_{21} \\
+(\lambda_{12}\lambda_{21} - \lambda_{11}\lambda_{22})^2\theta_{12}^2\theta_{21}^2 \right] \right] \tag{2.2.6} \]

The above lemma shows that under the new pricing rule each equilibrium demand order is affected by all four pricing parameters as well as the market parameters including the correlation of the terminal values. Evidently, the market maker attempts to utilize the order flow information of the other market in setting the price of the stock in each market. Thus it is in the best interest of the informed traders to take into account the now-modified pricing strategy of the market maker in placing the orders. What is not so obvious is how

\[ ^3 \text{Note } \theta_{12} \text{ and } \theta_{21} \text{ are the regression coefficients of } \beta_2 \gamma + \varepsilon_2 \text{ on } \beta_1 \gamma + \varepsilon_1 \text{ and } \beta_1 \gamma + \varepsilon_1 \text{ on } \beta_2 \gamma + \varepsilon_2, \text{ respectively. That is, } \theta_{12} (\theta_{21}) = \frac{\beta_1 \beta_2 \sigma^2_\gamma + \sigma^2_\varepsilon}{\beta_1^2 \sigma^2_\gamma + \sigma^2_\varepsilon} \left( \frac{\beta_1 \beta_2 \sigma^2_\gamma + \sigma^2_\varepsilon}{\beta_1^2 \sigma^2_\gamma + \sigma^2_\varepsilon} \right). \]

The solution for \( A_{11} \) and \( A_{22} \) are symmetric to the expressions (2.2.5) and (2.2.6). Note that the denominators of all four coefficients are the same.

\[ ^4 \text{Setting } \rho = 0 \text{ above for check, the equilibrium demands specified above reduce to the original solutions in the paper with } \rho \text{ replaced with } 0. \]
those underlying market parameters affect the optimal decisions of market participants. As can be seen in the next section, equilibrium demands and equilibrium pricing parameters are all intertwined in a rather complex fashion. We'll look into this aspect of equilibrium dynamics further in the next section and put forth several ways to get around the difficulty, still not sacrificing the insight into the result.
2.3 Equilibrium Market Parameters $\lambda s$

Since both the market maker and the informed traders are assumed to be risk-neutral, the equilibrium price of each security is the unbiased predictor of its liquidation value. Thus the market efficiency condition requires the following relationship.

$$P_k = E[S_k | \omega_k], \quad k = 1, 2$$

where

$$P_1 = \tilde{S}_1 + \lambda_{11}(x_{11} + x_{21} + z_1) + \lambda_{12}(x_{12} + x_{22} + z_2),$$

$$P_2 = \tilde{S}_2 + \lambda_{21}(x_{11} + x_{21} + z_1) + \lambda_{22}(x_{12} + x_{22} + z_2)$$

Given the condition specified above, the following result provides the expressions for the equilibrium pricing parameters.

**Lemma 2:** The equilibrium market depths have the following (essentially) closed form expressions. \(^5\)

$$\lambda_{11} = \frac{\text{cov}(S_1, \omega_1)\text{var}(\omega_2) - \text{cov}(S_1, \omega_2)\text{cov}(\omega_1, \omega_2)}{\text{var}(\omega_1)\text{var}(\omega_2) - \text{cov}(\omega_1, \omega_2)^2} \quad (2.3.1)$$

$$\lambda_{12} = \frac{\text{cov}(S_1, \omega_2)\text{var}(\omega_1) - \text{cov}(S_1, \omega_1)\text{cov}(\omega_1, \omega_2)}{\text{var}(\omega_1)\text{var}(\omega_2) - \text{cov}(\omega_1, \omega_2)^2} \quad (2.3.2)$$

Note that

$$\frac{\lambda_{12}}{\lambda_{11}} = \frac{\text{cov}(S_1, \omega_2)\text{var}(\omega_1) - \text{cov}(S_1, \omega_1)\text{cov}(\omega_1, \omega_2)}{\text{cov}(S_1, \omega_1)\text{var}(\omega_2) - \text{cov}(S_1, \omega_2)\text{cov}(\omega_1, \omega_2)}$$

Likewise,

$$\frac{\lambda_{21}}{\lambda_{22}} = \frac{\text{cov}(S_2, \omega_1)\text{var}(\omega_2) - \text{cov}(S_2, \omega_2)\text{cov}(\omega_1, \omega_2)}{\text{cov}(S_2, \omega_2)\text{var}(\omega_1) - \text{cov}(S_2, \omega_1)\text{cov}(\omega_1, \omega_2)}$$

\(^5\)See the appendix for detailed expressions. The expressions for $\lambda_{22}$ and $\lambda_{21}$ are symmetric to the expressions 2.3.1 and 2.3.2, respectively.
Unlike the model in the previous chapter, it is not possible to directly obtain the closed-form expressions for pricing parameters or optimal order amounts as explicit functions of underlying market characteristics and the correlation of the stock value innovations. This is mainly because the depth parameters and the equilibrium demands are all interdependent upon one another in a highly non-linear manner. In order to gain some insight into the seemingly complicated equilibrium dynamics we adopt both modeling and simulation approaches to getting around this problem.

In section 5 we consider a model for the symmetric markets wherein the volatilities of both common and security-specific shocks are the same across the markets. This simplification of the model leads to the symmetric pricing scheme of the market maker. And we obtain fairly simple closed-form expressions for the equilibrium demands and pricing parameters of the symmetric dual markets. Numerical analysis confirms that all the equilibrium dynamics obtained for the symmetric markets are preserved in case of more general, asymmetric market structures as well.
2.4 Equilibrium Profit and Pricing Parameters

2.4.1 Equilibrium Profits of Informed Traders

In this section we present the expressions for the expected profits of each informed trader. As with their equilibrium demands the expected profits of the informed traders are non-linear functions of underlying market parameters. Note the stock price correlation is implicit in the expressions for optimal demands and market depth. We showed in the preceding chapter that under the original pricing rule it was optimal for both insiders to place an order for the security in their respective the other markets. We also showed that their combined total expected profit obtained from placing orders in both markets were strictly smaller than that they would have obtained if they had colluded to stay in their respective own markets. The following lemma provides the expected profits of the informed traders.

Lemma 3: At equilibrium, the informed traders are expected to earn positive profits, on average, from placing orders in their own market and the other market, respectively. In particular,

\[
\Pi_{11} = A_{11}[(\beta_1^2\sigma_e^2 + \sigma_e^2)(1 - A_{11}A_{11} - A_{12}A_{12}) - \beta_1\beta_2\sigma_e^2(A_{21}A_{11} + A_{22}A_{12})]\]  
\[\text{(2.4.1)}\]

\[
\Pi_{12} = A_{12}[(\beta_1^2\sigma_e^2 + \sigma_e^2)(-A_{11}A_{21} - A_{12}A_{22}) + \beta_1\beta_2\sigma_e^2(1 - A_{21}A_{21} - A_{22}A_{22})]\]  
\[\text{(2.4.2)}\]

The equilibrium profits of the other informed trader can be calculated in a similar fashion. Thus

\[
\Pi_{22} = A_{22}[(\beta_2^2\sigma_e^2 + \sigma_e^2)(1 - A_{22}A_{22} - A_{21}A_{21}) - \beta_1\beta_2\sigma_e^2(A_{12}A_{22} + A_{11}A_{21})]\]  
\[\text{(2.4.3)}\]

\[
\Pi_{21} = A_{21}[(\beta_2^2\sigma_e^2 + \sigma_e^2)(-A_{22}A_{12} - A_{12}A_{11}) + \beta_1\beta_2\sigma_e^2(1 - A_{12}A_{12} - A_{11}A_{11})]\]  
\[\text{(2.4.4)}\]
2.4.2 Conditions on Equilibrium Pricing Rule

An equilibrium is a pair \((P, X)\) such that \(P\) is a rational pricing rule, given \(X\), and \(X\) is an optimal trading strategy, given \(P\). As such, certain restrictions are to be imposed on the pricing parameters \(\lambda\) as to ensure that \(X\) is, in fact, an optimal trading strategy, given a pricing rule. For the pricing scheme that we consider in this section, the restrictions on the pricing parameters can be derived from the second order condition (SOC) of each insider’s optimization problem as follows. They also shed some light on how the new pricing strategy effectively counterbalances the insiders’ attempt to exploit the markets.\(^6\)

\[
\lambda_{11} > 0 \tag{2.4.5}
\]
\[
|\lambda_{12} + \lambda_{21}| < 2\sqrt{\lambda_{11} \cdot \lambda_{22}} \tag{2.4.6}
\]

First note that the conditions (2.4.5) and (2.4.6) together dictate that \(\lambda_{22}\) is also positive. While pricing parameters \(\lambda_{11}\) and \(\lambda_{22}\) should strictly be positive, the pricing parameters for cross-market demands, \(\lambda_{12}\) and \(\lambda_{21}\) can take negative values. The support of cross-market pricing parameters for profit function extends to the negative values such that the condition in equation (2.4.6) is satisfied. It is due to the fact that the security prices could be negatively correlated.

The new pricing strategy being examined hinges on the observation that the cross-market demand contains certain amount of information, if noisy in most cases, about the price innovation of the security being priced by the scheme. As the security prices become more correlated, the signal will get more precise and the information from the order flows of the other market will become more and more reliable. In case where the stock prices get more and more negatively correlated, however, a negative demand in the other market, for

\(^6\)Refer to the appendix for the conditions presented below.
instance, well signals a price hike of the security in specialty market. Therefore, the price system needs to reverse the demand information from the other market in order to properly reflect that in the price, which is made possible by switching the signs of $\lambda_{12}$ and $\lambda_{21}$. 
2.5 Equilibrium Characterization of Symmetric Markets

In this section we'll show analytically how the insiders’ optimal trading strategies and market maker’s rational pricing rule are characterized for the symmetric markets. Since we focus primarily on the interaction between the informed traders' profit-seeking behavior and the correlation of the stock price innovations under the new pricing rule, it is in our immediate concern to explore how the proposed pricing scheme changes the optimal behavior of insiders and its dynamics in conjunction with the information about the price correlation.

The markets are called symmetric if the following conditions hold:

1. each component shocks of the value innovation processes of the securities have the same second moments,

2. liquidity trades have the same second moments, and

3. The pricing rule of the market maker for each market is a linear functional that incorporates the demand order information from both markets.

Given the definition stated above, the following proposition provides the equilibrium demands of informed traders and the optimal pricing rule of market maker. 7

---

7See the Appendix for proof
Proposition 1:

1. Under the new pricing rule the optimal trading strategies of informed traders are as follows.

\[
x_{11} = \frac{\lambda(2 - \rho^2) - \lambda_c \rho}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \cdot (\beta\gamma + \epsilon_1) \tag{2.5.1}
\]

\[
x_{22} = \frac{\lambda(2 - \rho^2) - \lambda_c \rho}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \cdot (\beta\gamma + \epsilon_2) \tag{2.5.2}
\]

\[
x_{12} = \frac{\rho\lambda - (2 - \rho^2)\lambda_c}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \cdot (\beta\gamma + \epsilon_1) \tag{2.5.3}
\]

\[
x_{21} = \frac{\rho\lambda - (2 - \rho^2)\lambda_c}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \cdot (\beta\gamma + \epsilon_2) \tag{2.5.4}
\]

2. The equilibrium pricing parameters are as follows.

\[
\lambda = \sqrt{\frac{\beta^2\sigma_\gamma^2 + \sigma_\epsilon^2}{\sigma_\epsilon^2}} \left[ \frac{2 - \rho^2}{4 - \rho^2} \right] \tag{2.5.5}
\]

\[
\lambda_c = \sqrt{\frac{\beta^2\sigma_\gamma^2 + \sigma_\epsilon^2}{\sigma_\epsilon^2}} \left[ \frac{\rho}{4 - \rho^2} \right] \tag{2.5.6}
\]

Note that \( \forall \rho^2 \in [0,1] \) the pricing parameter \( \lambda \) is lower than \( \lambda_1 \), the pricing parameter under the original pricing rule.

Proposition 2:

Under the new pricing rule the informed traders can not take advantage of the insiders.
information to exploit the market other than his own. Specifically,

\[ x_{11} = \sqrt{\frac{\sigma_z^2}{\beta^2 \sigma_\gamma^2 + \sigma_\epsilon^2}} \cdot (\beta \gamma + \varepsilon_1) \quad (2.5.7) \]
\[ x_{22} = \sqrt{\frac{\sigma_z^2}{\beta^2 \sigma_\gamma^2 + \sigma_\epsilon^2}} \cdot (\beta \gamma + \varepsilon_2) \quad (2.5.8) \]
\[ x_{12} = x_{21} = 0 \quad (2.5.9) \]

The results presented above show that the new pricing rule completely counterbalances the informed traders' attempt to yield additional profits utilizing both the insiders' and the publicly available information. The new pricing rule sets the security prices such that each informed trader optimally determines not to enter the other market, thus rendering him back to a monopolistic insider for a single market. Also the order amount placed in each market under the new pricing rule is uniformly smaller than that placed under the original pricing rule, which is shown in the following result.

**Corollary 1**: Let \( X_1^N \) and \( X_1^O \) denote the total demand of each insider under the new and old schemes, respectively. Then

\[ \frac{X_1^N}{X_1^O} = \frac{\sqrt{4 - 3\rho^2 + \rho^4}}{(2 + \rho - \rho^2)} \quad (2.5.10) \]

Note that the right-hand side of the above equation is strictly less than unity for \( \rho \in (0, 1) \). Based on the results presented in the previous section, we can further look into the new equilibrium prices:

---

Note, however, that each insider's decision is still affected by the competition in the other market.
Corollary 2: Let $P_1^N$ and $P_1^O$ denote the pricing strategies under the new and old schemes, respectively. Then

$$p_1^N - p_1^O = \sqrt{\frac{\sigma^2}{\beta^2 \sigma^2 + \sigma^2}}\left[ z_1 \left( \frac{(2 - \rho^2) - \sqrt{4 - 3\rho^2 + \rho^4}}{4 - \rho^2} \right) + z_2 \left( \frac{\rho^2}{4 - \rho^2} \right) \right]$$ (2.5.11)

Note that the difference in the two prices is a function of volumes of noise trades alone. Unlike the original pricing rule, the new pricing scheme at equilibrium does not incorporate the information about the cross-market demand that would be part of the pricing equation under the original scheme. The new price, however, fully compensates the absence of this cross-market term by introducing the information about the order flow of the other market. Therefore, even if the new pricing strategy quarantines the insiders in their own market, the information contained in the new price about the insiders' trading behavior is equivalent to that contained in the original pricing rule.

Since the market maker can not distinguish between trades, additional noise trades information impounds in the price.

The volatility of equilibrium price, however, does not increase, nor its informativeness deteriorates as the results presented below suggest.
2.5.1 Volatility of Equilibrium Demand

The following result shows that the equilibrium order placed by each informed trader is much less volatile under the new pricing rule, which is largely attributable to the new pricing rule containing each insider in her own market.

Let $x_{11}$ and $x_{12}$ denote the equilibrium demands under the original pricing rule of insider 1 for securities in his own market and the other market, respectively. Further, $\sigma^2_{xO} =: \text{var}(x_{11} + x_{12})$ under the original pricing scheme. Likewise $\sigma^2_{xN} =: \text{var}(x_{11} + x_{12})$ under the new pricing scheme. Then

$$\sigma^2_{xN} = \sigma^2_{xO} \left[ \frac{4 - 3\rho^2 + \rho^4}{4 + \rho^2 - \rho^4} \right]$$

(2.5.12)

Note that the right hand side of the above equation decreases with $\rho$, making the demand volatility under the original scheme an upper bound of volatility under the new scheme. When the stock prices are perfectly correlated, the total order flow placed by each insider is only as half volatile as that under the original pricing rule. From Proposition 2 it is evident that the volatility of each insider's demand is the same as the noise trade volume of corresponding market.

2.5.2 Price Volatility and Informativeness

The following lemma shows that unconditional volatilities of equilibrium prices under both pricing rules are identical.

Lemma 4: Let $\sigma^2_{P_1O}$ and $\sigma^2_{P_1N}$ denote the unconditional volatilities of equilibrium prices under the original and new pricing rules, respectively. Then

$$\sigma^2_{P_1N} = \sigma^2_{P_1O} = (\beta^2 \sigma^2_{\gamma} + \sigma^2_{\eta}) \left[ \frac{2}{4 - \rho^2} \right]$$

(2.5.13)
As the stock prices are more and more correlated, the equilibrium price gets less and less volatile given the variance of stock price innovations. It is noteworthy that while the equilibrium demands of insiders are not affected by the stock price correlation, the pricing parameters $\lambda$ and the unconditional volatility of price are well dependant on the correlation.

The following lemma shows that although the new pricing rule lacks the information about the liquidation value to be inferred from the cross-market demand, the same amount of uncertainty of the security value as under the original pricing rule is resolved during the round of trade.

**Lemma 5**: Let $\Sigma_1 \equiv \text{var}(S_1|P_1) = \text{conditional variance of terminal value of stock 1 given the amount of order flow}.$

\[
\Sigma_1 = (\beta^2 \sigma_\gamma^2 + \sigma_z^2) \left[ \frac{2 - \rho^2}{4 - \rho^2} \right]
\]  \hspace{1cm} (2.5.14)

Under the new pricing scheme the insiders *optimally* determine not to place an order in the other market. So the market maker can not utilize the informational content to be obtained from the cross-market demand. However, the new scheme fully recovers the price informativeness that would have attained under the old scheme by incorporating the demand information in the other market into the price. Note that while the amount of information revealed during trade is independent of all other parameters such as noise trades volumes, the informativeness of price under new pricing scheme depends on the stock price correlation. It is striking given the fact that the equilibrium order amount placed by each informed trader is independent of the correlation structure of security prices under the new pricing strategy.
2.5.3 Insiders' Profits

In this section we examine one of the major concerns of this study: 'How much profits do the insiders make under the alternative pricing scheme?' The following result shows that while the insiders' profits are still functions of the underlying market characteristics as well as the correlation, the insiders, who are not governed by new pricing scheme, under-perform across all values of the terminal value correlation.

**Proposition 3**: Under the new pricing rule the informed traders can only make, at best, the profit which he could have made under the original pricing scheme. Also the market maker has an incentive to simultaneously price multiple securities. In particular, the market maker can be more effective in minimizing the insiders' profits by choosing the securities with higher correlation.

The expected profits of informed traders are

\[ \Pi_1 = \Pi_2 = \sqrt{(\beta^2 \sigma_1^2 + \sigma_2^2) \sigma_2^2 \left[ \frac{2 - \rho^2}{4 - \rho^2} \right]} \]  \hspace{1cm} (2.5.15)

**Corollary**: The performance of insiders under the alternative pricing relative to that under the old pricing is monotone decreasing in the stock value correlation. Specifically,
\[
\frac{\Pi^N}{\Pi^O} = \frac{2 - \rho^2}{\sqrt{4 - 3\rho^2 + \rho^4}}
\] (2.5.16)

Alternatively,
\[
\Pi^O - \Pi^N = \sqrt{(\beta^2\sigma_1^2 + \sigma_2^2)} \sigma_2^2 \left[ \frac{\sqrt{4 - 3\rho^2 + \rho^4} - (2 - \rho^2)}{4 - \rho^2} \right]
\] (2.5.17)

Note that the insiders' profit under the new pricing rule relative to the profit under the original scheme decreases with the correlation of stock prices. Furthermore, the amount of profit, in excess of the profit earned under the new pricing rule, is larger for the markets where the stock value processes are more volatile and the liquidity need is higher. As the information about the order flow of the other market becomes more reliable, the market maker tends to rebalance the pricing weights in order to take that information into account in setting the stock price of each insider's own market. This pricing mechanism is so structured as to suppress the potential gain an insider could make by entering the other market down below the loss that would be incurred in his own market. By exercising the new pricing strategy the market maker not only alleviates the adverse selection problem extant in both markets but also effectively cages the informed traders in their respective own markets. The figures 1 and 2. show the expected profits of informed traders under new pricing scheme as fraction of the profits earned under the original scheme. As will see in the next section, the results from numerical analysis confirm what the symmetric market equilibrium dictates analytically.

At equilibrium each insider optimally decides not to place an order in the other market. Hence his behavior gets less aggressive in that his total demand is always less than the total demand he would have placed under the original scheme. Further, the pricing parameter
for own market demand is smaller with the new pricing than with the original one. This is because the information contained in the own market order flow now consists exclusively of the insider information (and noise trades). The alleviated adverse selection allows the market maker to lower the price for that part. On the other hand, however, the total pricing parameter, i.e., the sum of the two pricing parameters for each market, assuming the identical stock value innovations, will always be larger under the new scheme, which results in the lower expected profit of the insiders.
2.6 Numerical Analysis

2.6.1 Reparametrization for the Analysis

Now that we've obtained the equilibrium demands and pricing parameters $\lambda$s, we like to look further into how the underlying parameters interact to affect the characteristics of the market and investors' optimal behavior. This section has two distinct, though closely related, purposes. First, we present the analysis of equilibrium/comparative statics, much of which has been given a cursory glance. We particularly aim to disentangle the fairly complex equilibrium characterization and thus obtain further insight into the interdependence between stock price correlation and different pricing schemes. Second, we'll show that the various equilibrium characteristics/comparative statics that we obtain from the numerical analysis of general, asymmetric markets are fully in line with what symmetric markets results, in their closed-form expressions, dictate.

To this end, first note that the sensitivity of common shock $\beta_i$, $i = 1, 2$ and its volatility ($\sigma_j^2$) always work together in affecting the equilibrium characteristics.\(^9\)

Since $\beta_1\gamma \sim N(0, \beta_1^2\sigma_\gamma^2)$, we can define

$$\beta_1^* = \beta_1\sigma_\gamma$$  \hspace{1cm} (2.6.1)

Then, $\beta_i\gamma$, the systematic component of price innovation can be in the following form\(^{10}\):

$$\beta_i\gamma = \beta_i^*\gamma^*,$$ where $\gamma^* \sim N(0, 1)$ and $i = 1, 2$.

Thus we effectively set $\sigma_\gamma^2 = 1$.

Again, this reparametrization is allowed due to the way the underlying parameters affect the equilibrium characteristics. Let us drop the asterisk from the newly defined sensitivity

\(^9\)Please refer to the expressions for equilibrium demands, market depths, $\theta_{12}$ and $\theta_{21}$.

\(^{10}\)Note $\theta_{12}^* \triangleq \frac{\sigma_1^2\sigma_{\gamma}^2\gamma^2}{\beta_1^2\sigma_\gamma^2 + \sigma_{\gamma}^2} = \theta_{12}$. Likewise, $\theta_{21}^* = \theta_{21}$. 

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parameter. Then stock price innovation correlation

\[
\rho = \frac{\beta_1 \beta_2}{\sqrt{(\beta_1^2 + \sigma_{e_1}^2)(\beta_2^2 + \sigma_{e_2}^2)}}
\]

It is clear that the idiosyncratic shock is inversely related to the stock price correlation. If \(\sigma_{e_1}^2 \downarrow 0\) and \(\sigma_{e_2}^2 \downarrow 0\), then \(\rho \to 1\). Also if the price innovation of a security is induced by each component of innovation in equal amount, \(\rho = \frac{1}{2}\).

Let's set \(\sigma_{e_1}^2 = c_1 \beta_1^2\) and \(\sigma_{e_2}^2 = c_2 \beta_2^2\), where \(c_1\) and \(c_2 \in \mathbb{R}^+\) are, respectively, variance ratios of component idiosyncratic shock to component common shock of market 1 and market 2. With this additional parametrization \(\rho^2\) reduces to \(\frac{1}{(1+c_1)(1+c_2)}\). The volatility parameters of the noise trading volume in each market can also be reparameterized so as to facilitate the numerical analysis: \(\sigma_{n_i}^2 = \beta_i^2 \cdot d_i\), \(i = 1, 2\).

### 2.6.2 Numerical Results

How do the equilibrium characteristics behave under the new pricing rule as the underlying market parameters change their values? How effectively can the alternative pricing scheme newly proposed in this chapter protect the market from being exploited by the informed traders? We want to gain some insight into those questions in this section by closely looking into the results obtained from the numerical analysis.

First, our base model is characterized such that all the underlying parameters of each market are symmetric. There are two main reasons that we start from this market setting: 1) to obtain a reference comparative statics to which we compare the results of more complex market setting. 2) to check if all the numerical optimization and the subsequent equilibrium calculations are correctly programmed, for we've derived all the corresponding analytic expressions for equilibrium characteristics in the previous chapter. Then we consider the general asymmetric structures of the markets, where the underlying parameters are
different across the markets. Starting from various initial values, the numerical optimization converges to produce identical equilibrium characterization for each set of underlying market parameters.

The numerical analysis conducted in this section reveals that the trading equilibrium follows a very consistent pattern in terms of the effects of underlying parameters on the equilibrium characteristics. Specifically, under various characterizations of market structure the performances of both insiders have been substantially undermined. No matter whether the underlying parameters change symmetrically or asymmetrically, the insiders' profits from both markets are much lower than under the original pricing scheme.

The order amount that an informed trader places in the other market contains a certain level of information about what the insider knows about the future price of the security in his own market since he determines the optimal order amount in the other market based on the price innovation of his own security and its correlation with the other security. Also when an informed trader determines the optimal order he places in his own market, he does so in conjunction with the optimal order to be placed in the other market. It is in this sense that his demand in his own market partly impound the information about the price innovation of the security in his other market. It is by incorporating the information contained in all the demand quantities that the new pricing scheme effectively counters the insiders’ attempt to exploit the markets. We present the effects of changes in the underlying parameters on equilibrium characteristics in more details below.

2.6.2.1 All Parameters Change - Symmetric vs. Asymmetric

Looking into the possible market characterizations, the simplest but the most immediate question we want to ask would be how the equilibrium market depths and the insiders’ expected profits change under the new pricing strategy. The tables (2.1) and (2.2) respectively present the results on the effects that symmetric and asymmetric changes in market charac-
teristics bear on the equilibrium characteristics. The symmetric changes in all parameters in each market induce no changes in equilibrium characteristics such as equilibrium demands and pricing parameters. Additional adverse selection problem induced by increased volatility of the price innovation perfectly cancels off by more noise trades. Yet it is interesting to note that the expected profits of insiders of both markets are consistently lower under the new pricing strategy than those under the original one. While the equilibrium pricing rule and the insiders' optimal demands are unaffected by the proportional changes in all parameters, the insiders' expected profits exponentially increase. This is because the uncertainty in the liquidation value as well as the noise trade volume increase hand in hand and the its correlation is held constant.

When the changes are single-handed, the consequent equilibrium shows more interesting pattern. As the values of the parameters of the market 1 proportionately increase while those for the market 2 are held constant, $\lambda_{11}$ marginally increases while all the other pricing parameters decreases in a more salient manner. In particular, the depth parameter $\lambda_{22}$ goes down as the price innovation of security 1 gets more volatile. The informational disadvantage the market maker has for the market 2 is getting smaller relative to the market 1, which triggers lowering $\lambda_{22}$. The same changes in the parameters of market 1 would have caused no change in market maker's pricing strategy under the original pricing rule (i.e., $\lambda_1$) since the improvement in adverse selection of the market maker would be exactly wiped off by increased informational disadvantage. Under the new pricing scheme, however, the market maker takes into account this increased volatility of the market to adjust the pricing parameter $\lambda_{11}$ upward. The decrease in $\lambda_{12}$ is the result of the order flow of market 2, whose adverse selection problem is relatively alleviated. The equilibrium demand $X_{11}$ slightly increases in response to the heightened volatility profile of the market 1. More notable is that the demand $X_{21}$ rises rapidly.

Table (2.1) suggests another interesting aspect of the behavior of the insiders at equilibrium. Although the sets of market parameters postulated in the table dictates a positive correlation of the same magnitude, the insider 1 decides to short the stock in market 2. As
the uncertainty in the terminal value increases in market 1, the insider 2 has an incentive to place a larger demand order in market 1, which is evidenced in Table (2.1). In response to this behavior of her competition, the insider 1 wants to pull off his fund from his own market, anticipating a decrease in the equilibrium price in such a way that her expected profit is maximized.11

2.6.2.2 Volatility of Common Shocks: $\beta$

There is nothing unusual when we observe the pricing parameters $\lambda_{11}$ and $\lambda_{22}$ move in the opposite direction in response to the increased volatility of common component of price innovation. This observation is sensible and consistent with the previous result, for the change in $\beta$ alone only enhance the informational disadvantage that the market maker faces. What is striking about results here is that the newly adopted pricing strategy virtually drives insiders out of business given the market structure considered in this subsection. Consequently, their expected profits in their respective 'other' markets are almost zero and even go further down below it in many cases, where their profits do not even barely hit 1% of what they would have made under the old pricing scheme. Detailed numerical results are shown in tables (2.3) and (2.4).

2.6.2.3 Volatility of Idiosyncratic Shocks: $\sigma^2$

The immediate implication of lessened volatility of idiosyncratic shock is the heightened correlation of price innovations. In fact, our main focus in this subsection is how the varying degrees of stock price correlation interact with the equilibrium characteristics of the markets. Our major finding is that as the magnitude of stock price correlation takes upward path, pricing parameters and optimal order flows continue to increase their values. As the stock prices tend to move more and more in lockstep, the adverse selection problem faced by the market maker is alleviated, which results in a decrease in pricing parameters. With a higher correlation the insiders place higher amounts of orders. It is noteworthy that while their

---

11Note that the behavior of the insider 1 here might well indicate that he uses a manipulation strategy.
order amounts increase with the correlation (or decrease with idiosyncratic shocks), their expected profits monotonically decrease with the level of correlation. This observation is more salient under the alternative pricing scheme as evidenced in Table (2.3). That is, the rate at which the insiders' profits decrease is faster with the new pricing rule, which suggests that the alternative pricing strategy is more effective than the original pricing strategy in protecting the market from being exploited by the insiders, especially when the stocks are highly correlated. The effects of asymmetric changes in security-specific shock are parallel to the effects explained above for symmetric changes, though with a lesser degree. The numerical analysis shown in (2.3) also confirms the results suggested by symmetric market equilibrium presented in the previous section.

2.6.2.4 Volatility of Noise Trades, $\sigma^2$

The increased volume of noise trade in market 1 single-handedly induces higher level of equilibrium demand for stock 1 by the market's insider while there is little effect on the other insider's order for his own security. Note that this unilateral increase in liquidity trades induce decrease in the cross-market pricing parameters in the same direction and magnitude. Therefore, the more the liquidity trades in either market, the lesser the weights the market maker gives to all of the cross-market demands. Quite to the contrary, however, the liquidity trades bear opposing effects on cross-market demands of the insiders.

As the noise trades volume increases in a market, both insider have an incentive to place more orders in that market while divesting from the other. For the symmetric markets it is corroborated again in Table (2.6) that each insider stays in their respective own market. Tables (2.5) and (2.6) present more detailed results. Again, the expected profit under the new pricing rule is smaller than that under the original scheme.

As seen from the analysis presented above, the new pricing strategy adopted by the market maker seems to be more effective, and consistently so, in shielding the market from the

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12 Please refer to Table (2.4) for asymmetric changes in security-specific shock.
informed traders who attempt to exploit the market using their monopolistic informational advantage.

The new scheme can do so by incorporating into the prices the additional information contained in the order flow of the security in the other market with much improved flexibility.


2.7 Sequential Equilibrium

In this section we generalize the static model examined in the preceding sections to the one in which successive rounds of trading take place sequentially. The resulting dynamic model is structured such that the equilibrium prices reflect the information set available at each trading incident and the informed traders determine the optimal demands, taking into account, at each trading, their effect on prices of both securities in the future auctions as well as in the current one. We formulate each insider’s problem as a dynamic programming problem in the subsection below.

2.7.1 Model

We assume there are $N$ trading times in total, with $t_n$ denoting the time at which the $n$th auction takes place.

Let $\bar{x}_n^i$ denote the aggregate position of the insider $i$ after the $n$th trading, and $\Delta \bar{x}_n^i = \bar{x}_n^i - \bar{x}_{n-1}^i$. So $\Delta \bar{x}_n^i$ denotes the order quantity placed by informed trader $i$ at the $n$th auction. Also we denote the market clearing price at the $n$th auction $\bar{p}_n^i$.

Then the respective positions of informed traders 1 and 2 after the $n$th trading are given by

$$\bar{x}_n^1 = X_n(\bar{p}_1, \ldots, \bar{p}_{n-1}, S_1) \quad (2.7.1)$$

$$\bar{x}_n^2 = X_n(\bar{p}_1, \ldots, \bar{p}_{n-1}, S_2) \quad (2.7.2)$$

where $X_n = (X_n^1, X_n^2)'$, a 2 by 1 vector of some measurable function $X_n$ and

$p_k = (p_k^1, p_k^2)$, for $k = 1, 2, \ldots, n-1$. 

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The equilibrium price $\hat{p}_n$ is determined by the following price functional:

$\hat{p}_n^1 = P_n(\Delta \hat{x}_1^{11} + \Delta \hat{x}_1^{21} + \Delta \hat{u}_1^1, \ldots, \Delta \hat{x}_n^{11} + \Delta \hat{x}_n^{21} + \Delta \hat{u}_n^1, S_1)$ (2.7.3)

$\hat{p}_n^2 = P_n(\Delta \hat{x}_1^{12} + \Delta \hat{x}_1^{22} + \Delta \hat{u}_1^1, \ldots, \Delta \hat{x}_n^{12} + \Delta \hat{x}_n^{22} + \Delta \hat{u}_n^2, S_2)$ (2.7.4)

We define the sequential equilibrium as follows:

**Equilibrium:** A sequential equilibrium is a pair $(X, P)$ such that the following conditions hold.

1. $(\Delta \hat{x}_k^1)_{k=1}^{N'}$: optimal trade for informed trader 1, where $\Delta \hat{x}_k^1 = (\Delta \hat{x}_k^{11}, \Delta \hat{x}_k^{12})$

2. $(\Delta \hat{x}_k^2)_{k=1}^{N'}$: optimal trade for informed trader 2, where $\Delta \hat{x}_k^2 = (\Delta \hat{x}_k^{21}, \Delta \hat{x}_k^{22})$

2. $\hat{p}_n^1 = E\{S_1 | \Delta \hat{x}_1^{11} + \Delta \hat{x}_1^{21} + \Delta \hat{u}_1^1, \ldots, \Delta \hat{x}_n^{11} + \Delta \hat{x}_n^{21} + \Delta \hat{u}_n^1\}$

$\hat{p}_n^2 = E\{S_2 | \Delta \hat{x}_1^{12} + \Delta \hat{x}_1^{22} + \Delta \hat{u}_1^2, \ldots, \Delta \hat{x}_n^{12} + \Delta \hat{x}_n^{22} + \Delta \hat{u}_n^2\}$

We conjecture pricing rules for the market 1 and 2 and the value functions of insiders 1 and 2 as follows.

1. $\hat{p}_n^1 = \hat{p}_{n-1}^1 + \lambda_n(\Delta \hat{x}_n^{11} + \Delta \hat{x}_n^{21} + \Delta \hat{u}_n^1)$

$\hat{p}_n^2 = \hat{p}_{n-1}^2 + \lambda_n(\Delta \hat{x}_n^{12} + \Delta \hat{x}_n^{22} + \Delta \hat{u}_n^2)$

2. $E[\hat{\Pi}_n^1 | \psi_{n-1}^1] = A_{n-1}(\delta_1 - p_{n-1}^1)^2 + B_{n-1}E[(\delta_2 - p_{n-1}^2)^2 | \psi_{n-1}^1] + C_{n-1}$

$E[\hat{\Pi}_n^2 | \psi_{n-1}^2] = A_{n-1}(\delta_2 - p_{n-1}^2)^2 + B_{n-1}E[(\delta_1 - p_{n-1}^1)^2 | \psi_{n-1}^2] + C_{n-1}$

Thus the value function consists of three different components: 1) profit contribution made by utilizing the insider information, 2) profit contribution made by utilizing the publicly available information, and 3) the contribution by the noise trades. The information set of
each insider includes the monopolistic insider information about the liquidation value of his own stock as well as the publicly observed information about all the past price schedules. Then we obtain the following result for sequential equilibrium.

**Proposition 4:**

There exists a unique recursive linear equilibrium, which is characterized by the difference equation system specified below.

\[
\begin{align*}
\Delta x^{11}_n &= a_n (\delta_1 - p_{n-1}^1) \\
\Delta x^{12}_n &= a_n^c \rho (\delta_1 - p_{n-1}^1) \\
\Delta x^{21}_n &= a_n (\delta_2 - p_{n-1}^2) \\
\Delta x^{22}_n &= a_n^c \rho (\delta_2 - p_{n-1}^2) \\
\Delta \tilde{p}^1_n &= \lambda_n (\Delta \tilde{x}^{11}_n + \Delta \tilde{x}^{21}_n + \Delta \tilde{u}^1_n) \\
\Delta \tilde{p}^2_n &= \lambda_n (\Delta \tilde{x}^{12}_n + \Delta \tilde{x}^{22}_n + \Delta \tilde{u}^2_n) \\
\Sigma^1_n &= \text{var}(S_1 | \Delta \tilde{x}^{11}_1, \Delta \tilde{x}^{21}_1, \Delta \tilde{u}^1_1, \ldots, \Delta \tilde{x}^{11}_n, \Delta \tilde{x}^{21}_n, \Delta \tilde{u}^1_n) \\
\Sigma^2_n &= \text{var}(S_2 | \Delta \tilde{x}^{12}_1, \Delta \tilde{x}^{22}_1, \Delta \tilde{u}^2_1, \ldots, \Delta \tilde{x}^{12}_n, \Delta \tilde{x}^{22}_n, \Delta \tilde{u}^2_n)
\end{align*}
\]

Given \(\Sigma^1_0\) and \(\Sigma^2_0\), \(A_n, B_n, C_n, A_n^c, B_n^c, C_n^c, a_n, b_n, a_n^c, b_n^c, \Sigma^1_n, \Sigma^2_n\) are unique solutions to the difference equation system specified below:

\[
\begin{align*}
A_{n-1} &= a_n (1 - \lambda_n a_n - \lambda_n a_n^c \rho^2) + a_n^c \rho^2 (1 - \lambda_n a_n^c - \lambda_n a_n) + A_n (1 - \lambda_n a_n)^2 \\
&- 2A_n \lambda_n a_n^c \rho^2 (1 - \lambda_n a_n) + B_n \lambda_n^2 a_n^c \rho^2 - 2B_n \lambda_n a_n \rho (1 - \lambda_n a_n) \\
B_{n-1} &= A_n \lambda_n^2 a_n^c \rho^2 + B_n (1 - \lambda_n a_n)^2 \\
C_{n-1} &= C_n + (A_n + B_n \rho) \lambda_n^2 a_n^c \rho \Delta t
\end{align*}
\]
\[ A_{n-1}^c = a_n(1 - \lambda_n a_n - \lambda_n a_n^c \rho^2) + a_n^c \rho^2(1 - \lambda_n a_n^c - \lambda_n a_n) + A_n^c(1 - \lambda_n a_n)^2 \]

\[ -2A_n^c\lambda_n a_n^c \rho^2 (1 - \lambda_n a_n) + B_n^c\lambda_n^2 a_n^c \rho^2 - 2B_n^c\lambda_n a_n^c \rho^2 (1 - \lambda_n a_n) \tag{2.7.16} \]

\[ B_{n-1}^c = A_n^c\lambda_n^2 a_n^c + B_n^c(1 - \lambda_n a_n)^2 \tag{2.7.17} \]

\[ C_{n-1}^c = C_n^c + (A_n^c + B_n^c)\lambda_n^2 \sigma_u^2 \Delta t \tag{2.7.18} \]

\[ a_n = \frac{(1 - \lambda_n a_n^c \rho)(1 - 2\lambda_n A_n)}{2\lambda_n(1 - \lambda_n A_n)} \tag{2.7.19} \]

\[ a_n^c = \frac{\rho (1 - \lambda_n a_n)(1 - 2\lambda_n B_n \rho)}{2\lambda_n(1 - \lambda_n B_n \rho)} \tag{2.7.22} \]

\[ \Sigma_1 = \Sigma_{n-1}(1 - \lambda_n \Delta t(\beta_n^a + \beta_n^ac \rho^2)) \tag{2.7.23} \]

\[ \Sigma_2 = \Sigma_{n-1}(1 - \lambda_n \Delta t(\beta_n^a + \beta_n^ac \rho^2)) \tag{2.7.24} \]

\[ \lambda_n = \frac{\Sigma_1^2(\beta_n^a + \beta_n^ac \rho^2)(1 - \lambda_n \Delta t(\beta_n^a + \beta_n^ac \rho^2))}{\Delta t(\beta_n^ac \rho^2(1 - \rho^2) \rho^2 \Sigma_1^2 + (1 - \lambda_n \Delta t(\beta_n^a + \beta_n^ac \rho^2)) \sigma_u^2} \tag{2.7.25} \]

2.7.2 Analysis of Equilibrium

First note that virtually all market characteristics are interrelated in complex ways, which is a natural consequence from the enlarged opportunity set allowed for each trader. Still it is interesting to note that the amount of information the insider has decreases over time at a certain rate. The rate depends, however, on both \( \beta^a \) and \( \beta^{ac} \). That is, the insiders reveal their information faster with an incremental rate of \( \lambda \Delta t \beta^{ac} \rho \) when they have additional
investment opportunity. And the rate depends on how accurate the information inferred about the cross-market is.

The equilibrium pricing parameter is also shown to be affected by the stock value correlation and the convergence result shows that at each given time \( t \) it increases with the sum of the trading intensity of his own market and the discounted intensity of the cross-market. So in our model the informational asymmetry works as sort of a discount factor for information. The expressions for demand coefficients, \( a_n \) and \( a_n^c \) in equations (2.7.20) and (2.7.22) imply certain regularity conditions imposed on the coefficients of the value function of each insider. More analysis will be made clear in the convergence section that follows.
2.8 Equilibrium Dynamics in Continuous-Time Case

In this section, we study the equilibrium characterization in the continuous-time case, where the time between trades becomes small. We consider the continuous time case since the time interval between trade can be very small in real-world financial markets. Moreover, we can determine the Nash equilibrium in closed form when both the order flows and the time between trades are small. We show that our model not only accommodates the convergence results in the previous literature but also spans the equilibrium parameter space by introducing the informational diversity. The central idea of the following results is that the equilibrium market characteristics may well be changing depending upon the level of informational asymmetry as well as the market priors.

For the exposition in this section we reproduce the functional form of the optimal demands of the informed traders.

\[
\begin{align*}
\Delta \tilde{x}_{n}^{11} &= \alpha_n (\delta_1 - p_{n-1}^1) =: \beta_n^\alpha (\delta_1 - p_{n-1}^1) \Delta t_n \\
\Delta \tilde{x}_{n}^{12} &= \alpha_n^c (\delta_1 - p_{n-1}^1) =: \beta_n^{ac} (\delta_1 - p_{n-1}^1) \Delta t_n \\
\Delta \tilde{x}_{n}^{22} &= \alpha_n (\delta_2 - p_{n-1}^2) =: \beta_n^\alpha (\delta_2 - p_{n-1}^2) \Delta t_n \\
\Delta \tilde{x}_{n}^{21} &= \alpha_n^c (\delta_2 - p_{n-1}^2) =: \beta_n^{ac} (\delta_2 - p_{n-1}^2) \Delta t_n
\end{align*}
\]

The following lemma shows how the pricing parameter and the informativeness of the current price are related. It is noteworthy that the information about the order amounts placed by the insiders is reflected in the price with different weights. The market, at equilibrium, incorporates only a portion of the cross-market order into the current price. This informational discounting is another market mechanism to minimize the adverse selection faced by the market makers.
Lemma 6:

\[
\lambda_t = \frac{\sum_1^t \left( \beta_t^a + \beta_t^{ac} \rho^2 \right)}{\sigma_u^2} \quad (2.8.1)
\]

\[
\Sigma_t' = -\lambda_t^2 \sigma_u^2 \quad (2.8.2)
\]

The above lemma, along with the regularity conditions imposed on the coefficients of the value function demand coefficients, gives the following result in lemma 2.

Lemma 7: The coefficients of the value function have the following properties.

As \( \Delta t \to 0 \)

\[
A'_t = \rho^2 \beta_t^{ac} = A'_t \quad (2.8.3)
\]

\[
B'_t = \beta_t^a = B'_t \quad (2.8.4)
\]

\[
C'_t = (A_t + B_t \rho^2) \lambda_t \sigma_u^2 = C'_t \quad (2.8.5)
\]

Although the apparent forms of the derivatives of coefficients of the profit function specified above seem to imply that the rate of contribution of the insider information to profit increases with the correlation between stock value processes, we are not able to conclude that's indeed the case until we obtain the expression for the optimal order rates of each insider. The following lemma shows how the demand coefficients should behave as the interval of trades becomes smaller.
Lemma 8.

\[ \forall t \in [0,1), \text{ and } \forall \rho^2 \in [0,1]: \]

\[ \beta_t^a = \beta_{t}^{ac} \] (2.8.6)

\[ A_t = A_0 + \int_0^t (\rho^2 \beta_{s}^{ac}) \, ds \] (2.8.7)

\[ B_t = B_0 + \int_0^t (\beta_{s}^{a}) \, ds \] (2.8.8)

In the previous section we show that the value function of each insider has the two major components: the combined contribution of insider information (the difference between the liquidation value and the current price, i.e., \( \delta - p_{n-1} \)), and the informed guess about the variance of the liquidation value of the other stock. The above lemma implies that the marginal rate of contribution of each component to the insiders' expected profit increases over time. And the rates depend on the cross-market order rates weighted by the correlation (\( \rho^2 \beta_{t}^{ac} \)). Note also that if the stock value processes are uncorrelated, then the demand rate for cross-market becomes trivial and the insiders’ profit function recovers the constant coefficients for both components. Moreover, the cross-market order rate becomes identical to the own market order rate if the stock value processes are perfectly correlated. Thus the result in the above lemmas corroborates the standard finding in the previous literature as a limiting case of our analysis. We'll elaborate on this point in more details below to show that the functional forms of the important equilibrium characteristics derived in the past literature are exactly the ones to be reproduced as special cases of our model.

From Lemma 8, we establish that \( \beta^a \) linearly depends on \( \beta^{ac} \). Therefore, it suffices to find the functional form of \( \beta_{t}^{ac} \), which is specified in Proposition 2 below.
Proposition 5:

Given $\Sigma_0$, the conditional variance of the terminal value of each stock follows the process specified below:

\[ \forall t \in [0,1) \]

\[ \Sigma_t^i = \Sigma_0^i \left( \frac{\lambda_t}{\lambda_0} \right)^{\frac{1 + \rho^2}{2\rho}} \]

\[ = \Sigma_0^i \left[ 1 - \frac{\rho^2 - \rho + 1}{1 + \rho^2} \Sigma_0^i \lambda_0^2 \right]^{\frac{1 + \rho^2}{\rho^2 - 4\rho + 1}} \tag{2.8.10} \]

This proposition states that the conditional variance is completely characterized by the equilibrium pricing parameter and the stock value correlation at any given time $t \in [0,1)$. Furthermore, as the remaining uncertainty about the terminal value of the stock becomes smaller, the informational contents in the order flow get sparse. Consequently, the market tends to incorporate the order flow information into the price at a lesser degree. The relationship shown in the above proposition captures the idea. The corollary stated below is an immediate result from the fact that the uncertainty in the liquidation value is to be completely resolved at the terminal time. \(^{13}\)

Corollary

The equilibrium pricing parameter at the beginning of the trading period takes the following simple form:

\[ \lambda_0 = \sqrt{\left[ \frac{1 + \rho^2}{\rho^2 - 4\rho + 1} \right] \frac{\Sigma_0}{\sigma_a^2}} \tag{2.8.11} \]

\(^{13}\)Note also that for the two stocks with zero correlation our $\lambda_0$ coincides with the initial pricing parameter for the single market in the past literature (refer to Kyle (1985), for example.)
We formally show in the following proposition that the equilibrium depth parameter $\lambda$ indeed uniformly decreases over time $t \in [0, 1)$ in a non-linear fashion.

**Proposition 6:**

Given $\lambda_0$ and $\Sigma_0$ the equilibrium market depth decreases with time at a rate $\kappa$ as specified below:

$$
\lambda_t = \left[\lambda_0^{(1-L)} - (1 - L) \kappa t\right]^{1/L}
$$

where $\kappa = -\frac{2\rho \sigma^2 - \lambda_0^{1+\rho^2}}{\Sigma_0(1 + \rho^2)}$

$$
1 - L = \frac{\rho^2 - 4\rho + 1}{2\rho}
$$

Note that from the equation for $\lambda_t$, both sides coincide when $t = 0$. The intuition behind the monotone decreasing pricing parameter is two-fold. First, as the trading approaches to the liquidation time, the information contained in the asset price becomes sparse, which is reflected in the equilibrium price by way of decreasing the value pricing parameter. Also note that each market accommodates diverse asymmetric information about the future asset price, either one of the insider cannot exercise the "destabilizing" trading strategy as if every player except himself is a noise trader. In other words, neither insider wants to engage in the destabilizing strategy knowing that the other insider in the market would want to do it, thus making the strategy not profitable. The above proposition also suggests that it is not until the last trading time that the uncertainty about the terminal values of the stock prices is completely resolved. The reason is again that each insider has an incentive...
to reveal his information as late as possible.

Corollary
The pricing parameter can be rewritten as follows:

\[
\lambda_t = \lambda_0 \left[ 1 - \frac{\rho^2 - 4 \rho + 1}{1 + \rho^2} \frac{\sigma_{w}^2}{\Sigma_0 \lambda_0^2} t \right]^{\frac{2 \rho}{\rho^2 - 4 \rho + 1}} \quad (2.8.13)
\]

Proposition 7
Given \( \lambda_0 \) and \( \Sigma_0 \), the order rates for both markets depend not only on the initial market priors but also on the correlation between the stock values for \( t \in [0,1) \):

\[
\beta_t = \beta_t^{ac} = \frac{\sigma_{d}^2}{\Sigma_0 (1 + \rho^2)} \lambda_0 \left[ 1 - \frac{\rho^2 - 4 \rho + 1}{1 + \rho^2} \frac{\sigma_{w}^2}{\Sigma_0 \lambda_0^2} t \right]^{\frac{1 - \rho^2}{\rho^2 - 4 \rho + 1}} \quad (2.8.14)
\]

This proposition suggests that the order rate for the insider's own market serve as the upper bound of the order rate for the cross market for \( t \in [0,1) \) and the cross-market investment rate increases linearly with the stock value correlation.\(^{14}\) This result is intuitive because the insiders don't seem to trade aggressively enough in the market he has the perfect foresight about. One reason for this result may have to do with the insiders' strategy to delay the revelation of the private information they have all the way until the last trading incident. Since the early revelation of the private information would send too much signal

\(^{14}\)Note, for example, that the cross-market order for insider 1 is \( dX_t^{12} := a_t \rho (\delta_1 - p_1^t) = \beta_t^{ac} \sigma (\delta_1 - p_1^t) dt \)
about the terminal value to the other insider and the market as a whole, each insider has an
incentive to reveal his information as late as possible. This observation is also foreshadowed
in lemma 1, which shows that the marginal value of the sure insider information to the
insiders' expected profit increases as it approaches to the terminal time. In contrast to Kyle
(1985), where the monopolistic insider reveals his information slow, though, steady, the
insiders in our study can not perfectly hide behind the noise traders due to the other insid­
ers' vigilant attempt to exploit his information by combining it with the publicly available
information. Likewise the informed traders take position aggressively in the cross-market
since the equilibrium pricing rule tends to discount the informational content of the order
amount placed by the insider from the other market depending upon the publicly available
correlation structure. Knowing that, therefore, each insider has an incentive to trade in the
other market aggressively. That is, under the current pricing scheme, the adverse selection
faced by the market maker becomes severe as the trading interval is getting smaller. This
observation is also confirmed in the following proposition from another perspective.

**Proposition 8:**

*Given the results specified above we recover the standard equilibrium market characteristics
in the existing literature as special cases of our model:*

*For \( \rho = 0, \)*

\[
\lambda_t = \sqrt{\frac{\Sigma_0}{\sigma_t^2}} \quad \text{for } t \in [0, 1] \tag{2.8.16}
\]

\[
\Sigma_t = \Sigma_0(1 - t) \tag{2.8.17}
\]

\[
\beta_t = \frac{\sigma_0^2}{\Sigma_0} \left[ 1 - \frac{\sigma_0^4}{\Sigma_0} \lambda_0^4 t \right]^{-1} = \frac{\sqrt{\Sigma_0}}{1 - t} \tag{2.8.18}
\]

The result in proposition 8 shows that as in the case of static trading game, the con­
vergence result in the previous research for single market equilibrium can be generalized
to the multiple market framework where the informational asymmetry is parameterized
by stock value correlation. The proposition confirms also that the insight acquired from the single market dynamics in continuous-time setting can be enriched much further by introducing asymmetric structure of information across the markets and by allowing the informed traders to be truly rational in the sense that they would quite naturally want to utilize all information available to achieve their well-defined goal for trading.
2.9 Conclusion

In this paper we considered how an alternative pricing system changes the behavior of heterogeneously informed traders who seek insiders profits in multiple markets. The proposed pricing strategy is more effective than the conventional one postulated in the existing literature in that the insiders’ profit seeking activity is significantly limited even if they are given an additional strategy option of cross-market investment. The new pricing scheme was shown to turn each informed trader back to a monopolistic insider for her own market behaviorally while consistently suppressing the market’s loss below the amount that would otherwise have been paid by the liquidity traders under the original pricing strategy. Our analysis also suggests the heightened efficiency of the new pricing rule in other important aspects of financial markets such as lower demand volatility and pricing parameters. The numerical analysis corroborates that across various characterizations of the market the newly proposed pricing rule has almost identical effects on the behavior of the insiders and various market characteristics of the asymmetrically structured markets.

When the insiders are allowed, under the conventional pricing rule, to trade as often as they want, it is revealed, the equilibrium behavior of the markets and their traders continues to depend on the degree of informational asymmetry. Yet, it is shown that the pricing parameter needs not be constant over the entire trading period as it is in the past research if differentially informed traders have the cross-market investment option at their disposal. As is the case of static game\textsuperscript{15}, the equilibrium characteristics including the optimal behavior of the informed traders obtained in the present study comprehends the results in the existing literature as limiting cases. So this paper provides a richer characterization of the market in both static and dynamic contexts while proposing a new pricing scheme of market maker as an improved measure for protecting the market from the insiders and thus reducing the transaction cost to be paid for liquidity purposes in financial markets.

\textsuperscript{15}See previous chapter

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The present study touches on the welfare theme of the traders having no insider information - only at second hand, however. That is, the welfare analysis for liquidity traders is done indirectly by analyzing the behavior of insiders. This is mainly because the traders with liquidity purpose is modeled in our current study as random noise order coming into the markets. So a natural extension to this study is to model the liquidity traders as being *strategic* in their trading behavior with a precisely defined objectives. The traders of *discretionary* sort in the past research are an example of the modeling in this line. Our immediate future research may well be about it.
Table 2.1: Changes In Equilibrium Market Depth, Equilibrium Demand, and Expected Profit Induced by Asymmetric Changes in Market Parameters

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Table 2.2: Changes In Equilibrium Market Depth, Equilibrium Demand, and Expected Profit Induced by Symmetric Changes in Market Parameters

| b1  | 0.2 | 0.4 | 0.6 | 0.8  | 1   |
| b2  | 0.2 | 0.4 | 0.6 | 0.8  | 1   |

Equilibrium Market Depth

| \( \lambda_1 \) | 0.065997 | 0.065997 | 0.065997 | 0.065997 | 0.065997 |
| \( \lambda_2 \) | 0.065997 | 0.065997 | 0.065997 | 0.065997 | 0.065997 |
| \( \lambda_{12} \) | 0.018856  | 0.018856  | 0.018856  | 0.018856  | 0.018856  |
| \( \lambda_{21} \) | 0.018856  | 0.018856  | 0.018856  | 0.018856  | 0.018856  |

Equilibrium Demand

| \( x_{11} \) | 2.121320  | 2.121320  | 2.121320  | 2.121320  | 2.121320  |
| \( x_{22} \) | 2.121321  | 2.121321  | 2.121321  | 2.121321  | 2.121321  |
| \( x_{12} \) | 0.000000  | 0.000000  | 0.000000  | 0.000000  | 0.000000  |
| \( x_{21} \) | 0.000000  | 0.000000  | 0.000000  | 0.000000  | 0.000000  |

Expected Profit

Under New Pricing Scheme

| \( \pi_{11} \) new | 2.375879  | 4.233784  | 6.509663  |
| \( \pi_{12} \) new | 2.375879  | 4.233784  | 6.509663  |

Under Old Pricing Scheme

| \( \pi_{11} \) old | 2.470951  | 4.392802  | 6.863753  |
| \( \pi_{12} \) old | 2.470951  | 4.392802  | 6.863753  |

Percentage of (New \( ps/old\) \( ps \))

| New \( \pi_{1/old} \) \( \pi_1 \) | 0.9615  | 0.9615  | 0.9615  | 0.9615  | 0.9615  |
| New \( \pi_{2/old} \) \( \pi_2 \) | 0.9615  | 0.9615  | 0.9615  | 0.9615  | 0.9615  |

Other parameters used

| \( \sigma_0 \) | 1  |
| \( \sigma_1 \) | 1  |
| \( \sigma_2 \) | 1  |
| \( \theta_{11} \) | 100 |
| \( \theta_{21} \) | 100 |
| \( \sigma_{11} \) | 4  | 10  | 36  | 64  | 100 |
| \( \sigma_{22} \) | 4  | 4   | 4   | 4   | 4   |
| \( \theta_{12} \) | 0.04 | 0.16 | 0.36 | 0.64 | 1    |
| \( \theta_{21} \) | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 |
| \( \theta_{22} \) | 0.5  | 0.25 | 0.106 | 0.125 | 0.1  |
| \( \theta_{11} \) | 0.5  | 1   | 1.5  | 2   | 2.5  |
| \( \rho \) | 0.5  | 0.5  | 0.5  | 0.5  | 0.5  |
| \( b_1 * \gamma + c_1 \) | 0.3  |
| \( b_2 * \gamma + c_2 \) | 0.3  |
Table 2.3: Changes In Equilibrium Market Depth, Equilibrium Demand, and Expected Profit induced by Asymmetric Idiosyncratic Shock

<table>
<thead>
<tr>
<th></th>
<th>c1</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>c2</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

### Equilibrium Market Depth

<table>
<thead>
<tr>
<th></th>
<th>( \ell_1 )</th>
<th>( \ell_2 )</th>
<th>( \ell_{12} )</th>
<th>( \ell_{21} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ell_1 )</td>
<td>0.03878</td>
<td>0.04569</td>
<td>0.05147</td>
<td>0.05654</td>
</tr>
<tr>
<td>( \ell_2 )</td>
<td>0.03878</td>
<td>0.04060</td>
<td>0.04200</td>
<td>0.04809</td>
</tr>
<tr>
<td>( \ell_{12} )</td>
<td>0.03074</td>
<td>0.02707</td>
<td>0.02549</td>
<td>0.02450</td>
</tr>
<tr>
<td>( \ell_{21} )</td>
<td>0.03074</td>
<td>0.02707</td>
<td>0.02549</td>
<td>0.02450</td>
</tr>
</tbody>
</table>

### Equilibrium Demand

<table>
<thead>
<tr>
<th></th>
<th>( x_{11} )</th>
<th>( x_{22} )</th>
<th>( x_{12} )</th>
<th>( x_{21} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_{11} )</td>
<td>2.86039</td>
<td>2.81036</td>
<td>-0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>( x_{22} )</td>
<td>2.86039</td>
<td>2.81036</td>
<td>-0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>( x_{12} )</td>
<td>0.00000</td>
<td>-0.00000</td>
<td>-0.00000</td>
<td>-0.00000</td>
</tr>
<tr>
<td>( x_{21} )</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

### Expected Profit

#### Under New Pricing Scheme

<table>
<thead>
<tr>
<th></th>
<th>( \pi_{11} )</th>
<th>( \pi_{12} )</th>
<th>( \pi_{1\text{ new}} )</th>
<th>( \pi_{21} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_{11} )</td>
<td>0.15514</td>
<td>0.17966</td>
<td>0.20102</td>
<td>0.23020</td>
</tr>
<tr>
<td>( \pi_{12} )</td>
<td>0.00000</td>
<td>-0.00028</td>
<td>-0.00115</td>
<td>-0.00478</td>
</tr>
<tr>
<td>( \pi_{1\text{ new}} )</td>
<td>0.15514</td>
<td>0.17966</td>
<td>0.20102</td>
<td>0.23020</td>
</tr>
<tr>
<td>( \pi_{21} )</td>
<td>0.15514</td>
<td>0.17966</td>
<td>0.20102</td>
<td>0.23020</td>
</tr>
</tbody>
</table>

#### Under Old Pricing Scheme

<table>
<thead>
<tr>
<th></th>
<th>( \pi_{1\text{ old}} )</th>
<th>( \pi_{2\text{ old}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_{1\text{ old}} )</td>
<td>0.19624</td>
<td>0.20865</td>
</tr>
<tr>
<td>( \pi_{2\text{ old}} )</td>
<td>0.19624</td>
<td>0.20865</td>
</tr>
</tbody>
</table>

#### Percentage of (New ps/old ps)

<table>
<thead>
<tr>
<th></th>
<th>( \pi_{1\text{ new}}/\pi_{1\text{ old}} )</th>
<th>( \pi_{2\text{ new}}/\pi_{2\text{ old}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_{1\text{ new}}/\pi_{1\text{ old}} )</td>
<td>0.7905</td>
<td>0.8474</td>
</tr>
<tr>
<td>( \pi_{2\text{ new}}/\pi_{2\text{ old}} )</td>
<td>0.7905</td>
<td>0.8474</td>
</tr>
</tbody>
</table>

#### Other parameters used

<table>
<thead>
<tr>
<th></th>
<th>( b_1 )</th>
<th>( b_2 )</th>
<th>( d_1 )</th>
<th>( d_2 )</th>
<th>( \sigma_1^2 )</th>
<th>( \sigma_2^2 )</th>
<th>( \theta_{11} )</th>
<th>( \theta_{22} )</th>
<th>( \theta_{12} )</th>
<th>( \theta_{21} )</th>
<th>( \theta )</th>
<th>( \lambda_1 + \gamma + \epsilon_1 )</th>
<th>( \lambda_2 + \gamma + \epsilon_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_1 )</td>
<td>0.2</td>
<td>0.2</td>
<td>100</td>
<td>100</td>
<td>1</td>
<td>4</td>
<td>0.00400</td>
<td>0.00400</td>
<td>0.90909</td>
<td>0.90909</td>
<td>0.90909</td>
<td>0.90909</td>
<td>0.90909</td>
</tr>
<tr>
<td>( b_2 )</td>
<td>0.2</td>
<td>0.2</td>
<td>100</td>
<td>100</td>
<td>1</td>
<td>4</td>
<td>0.00400</td>
<td>0.00400</td>
<td>0.90909</td>
<td>0.90909</td>
<td>0.90909</td>
<td>0.90909</td>
<td>0.90909</td>
</tr>
<tr>
<td>( d_1 )</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>1</td>
<td>4</td>
<td>0.00400</td>
<td>0.00400</td>
<td>0.90909</td>
<td>0.90909</td>
<td>0.90909</td>
<td>0.90909</td>
<td>0.90909</td>
</tr>
<tr>
<td>( d_2 )</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>1</td>
<td>4</td>
<td>0.00400</td>
<td>0.00400</td>
<td>0.90909</td>
<td>0.90909</td>
<td>0.90909</td>
<td>0.90909</td>
<td>0.90909</td>
</tr>
</tbody>
</table>

\( \lambda_1 + \gamma + \epsilon_1 = 0.3 \)  
\( \lambda_2 + \gamma + \epsilon_2 = 0.3 \)
Table 2.4: Changes In Equilibrium Market Depth, Equilibrium Demand, and Expected Profit induced by Symmetric Idiosyncratic Shock

<table>
<thead>
<tr>
<th>c1</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>c2</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>

**Equilibrium Market Depth**

<table>
<thead>
<tr>
<th></th>
<th>(\lambda_{11})</th>
<th>(\lambda_{12})</th>
<th>(\lambda_{13})</th>
<th>(\lambda_{14})</th>
<th>(\lambda_{15})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c_1)</td>
<td>0.06000</td>
<td>0.08410</td>
<td>0.12102</td>
<td>0.14087</td>
<td>0.15772</td>
</tr>
<tr>
<td>(c_2)</td>
<td>0.06000</td>
<td>0.08410</td>
<td>0.12102</td>
<td>0.14087</td>
<td>0.15772</td>
</tr>
<tr>
<td>(c_3)</td>
<td>0.01886</td>
<td>0.01270</td>
<td>0.01028</td>
<td>0.00887</td>
<td>0.00703</td>
</tr>
<tr>
<td>(c_4)</td>
<td>0.01886</td>
<td>0.01270</td>
<td>0.01028</td>
<td>0.00887</td>
<td>0.00703</td>
</tr>
</tbody>
</table>

**Equilibrium Demand**

| \(x_{11}\) | 2.12132 | 1.50000 | 1.22474 | 1.06066 | 0.94868 |
| \(x_{21}\) | 2.12132 | 1.50000 | 1.22474 | 1.06066 | 0.94868 |
| \(x_{12}\) | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| \(x_{22}\) | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |

**Expected Profit**

Under New Pricing Scheme

| \(\pi_{11}\) | 0.26399 | 0.39365 | 0.48647 | 0.56347 | 0.63087 |
| \(\pi_{12}\) | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| \(\pi_{1, new}\) | 0.26399 | 0.39365 | 0.48647 | 0.56347 | 0.63087 |
| \(\pi_{21}\) | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| \(\pi_{2, new}\) | 0.26399 | 0.39365 | 0.48647 | 0.56347 | 0.63087 |

Under Old Pricing Scheme

| \(\pi_{1, old}\) | 0.27455 | 0.39691 | 0.48821 | 0.56458 | 0.63167 |
| \(\pi_{2, old}\) | 0.27455 | 0.39691 | 0.48821 | 0.56458 | 0.63167 |

Percentage of (New \(\pi_1/\)old \(\pi_1\))

| New \(\pi_1/\)old \(\pi_1\) | 0.9615 | 0.9918 | 0.9964 | 0.9980 | 0.9987 |
| New \(\pi_2/\)old \(\pi_2\) | 0.9615 | 0.9918 | 0.9964 | 0.9980 | 0.9987 |

Other parameters used

| \(b_1\) | 0.2 |
| \(b_2\) | 0.2 |
| \(d_1\) | 100 |
| \(d_2\) | 100 |
| \(\sigma_1\) | 1 |
| \(\sigma_2\) | 4 |
| \(\sigma_{12}\) | 4 |
| \(\sigma_{11}\) | 0.04 | 0.12 | 0.2 | 0.28 | 0.36 |
| \(\sigma_{22}\) | 0.04 | 0.12 | 0.2 | 0.28 | 0.36 |
| \(\theta_{1}\) | 0.50000 | 0.25000 | 0.16667 | 0.12500 | 0.10000 |
| \(\theta_{2}\) | 0.50000 | 0.25000 | 0.16667 | 0.12500 | 0.10000 |
| \(\rho\) | 0.50000 | 0.25000 | 0.16667 | 0.12500 | 0.10000 |
| \(b_1 + \gamma + c_1\) | 0.3 |
| \(b_2 + \gamma + c_2\) | 0.3 |
Table 2.5: Changes In Equilibrium Market Depth, Equilibrium Demand, and Expected Profit induced by Asymmetric Noise Trading Volumn

<table>
<thead>
<tr>
<th>d1</th>
<th>50</th>
<th>70</th>
<th>90</th>
<th>110</th>
<th>130</th>
</tr>
</thead>
<tbody>
<tr>
<td>d2</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Equilibrium Market Depth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{11}$</td>
</tr>
<tr>
<td>$\lambda_{22}$</td>
</tr>
<tr>
<td>$\lambda_{12}$</td>
</tr>
<tr>
<td>$\lambda_{21}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Equilibrium Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{11}$</td>
</tr>
<tr>
<td>$x_{22}$</td>
</tr>
<tr>
<td>$x_{12}$</td>
</tr>
<tr>
<td>$x_{21}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expected Profit</th>
</tr>
</thead>
</table>

Under New Pricing Scheme
| $\pi_{11}$ | 0.18044 | 0.23090 | 0.25943 | 0.27686 | 0.30984 |
| $\pi_{12}$ | 0.00539 | 0.00192 | 0.00057 | -0.00051 | -0.00141 |
| $\pi_{12}$ new | 0.19014 | 0.22271 | 0.25100 | 0.27635 | 0.29953 |
| $\pi_{22}$ | 0.26367 | 0.26390 | 0.26398 | 0.26398 | 0.26394 |
| $\pi_{22}$ new | -0.00261 | -0.00160 | -0.00054 | 0.00054 | 0.00161 |
| $\pi_{31}$ | 0.26106 | 0.26230 | 0.26344 | 0.26452 | 0.26555 |
| $\pi_{21}$ new | 0.26840 | 0.27117 | 0.27349 | 0.27556 | 0.27745 |

Under Old Pricing Scheme
| $\pi_{11}$ old | 0.20021 | 0.23309 | 0.26152 | 0.28694 | 0.31013 |
| $\pi_{21}$ old | 0.26840 | 0.27117 | 0.27349 | 0.27556 | 0.27745 |

<table>
<thead>
<tr>
<th>Percentage of (New ps/old ps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>New $\pi_{11}$/old $\pi_{11}$</td>
</tr>
<tr>
<td>New $\pi_{21}$/old $\pi_{21}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Other parameters used</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_1$</td>
</tr>
<tr>
<td>$b_2$</td>
</tr>
<tr>
<td>$c_1$</td>
</tr>
<tr>
<td>$c_2$</td>
</tr>
<tr>
<td>$\sigma_1^2$</td>
</tr>
<tr>
<td>$\sigma_2^2$</td>
</tr>
<tr>
<td>$\sigma_1^2_{\pi_1}$</td>
</tr>
<tr>
<td>$\sigma_2^2_{\pi_1}$</td>
</tr>
<tr>
<td>$\sigma_1^2$</td>
</tr>
<tr>
<td>$\sigma_2^2$</td>
</tr>
<tr>
<td>$\theta_{12}$</td>
</tr>
<tr>
<td>$\theta_{21}$</td>
</tr>
<tr>
<td>$\rho$</td>
</tr>
<tr>
<td>$b_1 \cdot \gamma + c_1$</td>
</tr>
<tr>
<td>$b_2 \cdot \gamma + c_1$</td>
</tr>
</tbody>
</table>
Table 2.6: Changes In Equilibrium Market Depth, Equilibrium Demand, and Expected Profit induced by Symmetric Noise Trading Volume

<table>
<thead>
<tr>
<th>d1</th>
<th>50</th>
<th>70</th>
<th>90</th>
<th>110</th>
<th>130</th>
</tr>
</thead>
<tbody>
<tr>
<td>d2</td>
<td>50</td>
<td>70</td>
<td>90</td>
<td>110</td>
<td>130</td>
</tr>
</tbody>
</table>

Equilibrium Market Depth

<table>
<thead>
<tr>
<th>d</th>
<th>λ11</th>
<th>λ22</th>
<th>λ12</th>
<th>λ21</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.0933</td>
<td>0.0933</td>
<td>0.0267</td>
<td>0.0267</td>
</tr>
<tr>
<td>70</td>
<td>0.0788</td>
<td>0.0788</td>
<td>0.0254</td>
<td>0.0254</td>
</tr>
<tr>
<td>90</td>
<td>0.0695</td>
<td>0.0695</td>
<td>0.0186</td>
<td>0.0186</td>
</tr>
<tr>
<td>110</td>
<td>0.0629</td>
<td>0.0629</td>
<td>0.0179</td>
<td>0.0179</td>
</tr>
<tr>
<td>130</td>
<td>0.0578</td>
<td>0.0578</td>
<td>0.0165</td>
<td>0.0165</td>
</tr>
</tbody>
</table>

Equilibrium Demand

<table>
<thead>
<tr>
<th>x</th>
<th>x11</th>
<th>x12</th>
<th>x21</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>1.5000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>70</td>
<td>1.7748</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>90</td>
<td>2.0124</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>110</td>
<td>2.2248</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>130</td>
<td>2.4186</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Expected Profit

Under New Pricing Scheme

<table>
<thead>
<tr>
<th>π1</th>
<th>π2</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>0.18667</td>
</tr>
<tr>
<td>12</td>
<td>0.22087</td>
</tr>
<tr>
<td>21</td>
<td>0.25044</td>
</tr>
<tr>
<td>22</td>
<td>0.27687</td>
</tr>
</tbody>
</table>

Under Old Pricing Scheme

<table>
<thead>
<tr>
<th>π1</th>
<th>π2</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>0.18667</td>
</tr>
<tr>
<td>12</td>
<td>0.22087</td>
</tr>
<tr>
<td>21</td>
<td>0.25044</td>
</tr>
<tr>
<td>22</td>
<td>0.27687</td>
</tr>
</tbody>
</table>

Percentage of (New ps/old ps)

| New π1/old π1 | 0.9615 | 0.9615 |
| New π2/old π2 | 0.9615 | 0.9615 |

Other parameters used

| β1 | 0.2 |
| β2 | 0.2 |
| σγ | 1 |
| σγ | 1 |
| σ2γ | 2 |
| σ2γ | 2 |
| σ2γ | 2.8 |
| σ2γ | 3.6 |
| σ2γ | 4.4 |
| σ2γ | 5.2 |
| σ2γ | 0.04 |
| σ2γ | 0.04 |
| β1 | 0.5 |
| β1 | 0.5 |
| β1 | 0.5 |
| β1 | 0.3 |
| β1 | 0.3 |

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Chapter 3

International Stochastic Comovement of Interest Rates and Its Implications on Forward Premium Puzzle
3.1 Introduction

Understanding the dynamics of the correlations of term structures across different countries are critically important for various reasons. First, it is a useful ingredient for decision-making on the maturity structure of bond portfolios. Fund managers in global fixed-income security market need to decide target risk exposure and hedge undesirable risk in consideration of co-movement of different countries' interest rates over different horizon. Also a judicious prediction of the co-movement is an utmost importance for global fixed-income hedge funds and proprietary trading desks of investment banks as observed in the near collapse of Long Term Capital Management. Secondly, industry firms who have an access to global financing markets, should consider not only the future evolution of the domestic interest rate but also that of the alternative foreign interest rate in order to decide the optimal financing source. Finally, the term structure of correlation is also important in valuing currency options. Amin and Jarrow (1991) suggest that the relevant volatility in a currency option reflect the volatilities of the two country interest rates and their correlation as well as the volatility of the exchange rate.

The existing literature on affine class models dictates an affine relationship between the yields and underlying factors in each country. While their studies mainly focus on their implications for the determination of the exchange rate, the forward premium anomaly, and the valuation of currency derivatives, the issue of cross-country correlation of the interest rates seems to be rarely touched. Our extensive analysis of the correlation structure reveals, however, that the affine models have an innate theoretical limitation; they can only generate non-negative correlations and thus cannot allow for sign-switching cross-country correlations without violating the positivity of the nominal interest rates in both countries. This is a major drawback of the affine models especially when they are extended to an international setup.

For example, figure 3.1 illustrates the historical time-series evolution of the interest rates of US and Japan. Heuristic sketch of the co-movement of the interest rates of these
countries suggests that they are positively correlated in most of the time. This feature may be attributable to global business cycle, an alignment of their monetary policies in dealing with inflation and depression, contagion of non-fundamentals and so forth. However, as is observed in 1990's, this global positive coherence in the interest rates is no more evident. In particular, the US and Japanese interest rates have recently moved clearly in the opposite directions. Thus allowing for sign-switching heteroskedastic cross-country correlations is an important ingredient that theoretical models are desired to possess. This feature is a critical challenge to the validity of the affine models, which begs for an alternative model wherein the sign-switching property is admissible.

This paper explores an extension of quadratic term structure models to better explain -both theoretically and empirically- the dynamics of the correlation structure of the two country interest rates over different time spans. The quadratic term structure models refer to the representative non-affine models including Longstaff (1989), Beaglehole and Tenney (1991), Beaglehole and Tenney (1992) Constantinides (1992), and Ahn (2004) among others. These models determine the yields or log-bond prices as a quadratic function of underlying state variables. In this paper, after investigating the implied correlation structure of affine class models we show that our quadratic model can generate sign-switching cross-country interest rate correlations without violating the positivity of the interest rates. It turns out that this desirable feature is generated by the fact that the positivity of the interest rate is governed by the (positive semi-definite) quadratic function itself whereas the sign-switching correlation structure is embodied mainly by the admissible stochastic differential equations of the underlying state variables. We also investigate term structure of cross-correlations with different maturities implied by the quadratic model and compares it to that implied by a popular affine model.

Given the two-country term structure dynamics, we derive the implied stochastic process of the exchange rate using the technique developed by Backus, Foresi, and Telmer (2001). Since the cross-country term structures and the exchange rate are endogenously

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1See Nielsen and Saá-Requejo (1993), Bakshi and Chen (1997), and Backus, Foresi, and Telmer (2001)
determined within the model, the issue of the forward premium puzzle, which documents the tendency for the currency of the high interest rate to appreciate, emerges as a natural consequence. We adopt the Metropolis-Hastings within Gibbs methodology combined with extended Kalman filtering to estimate a three- and four-factor cross-country quadratic models.

Specifically, we assume that there is one (and two) common factors and one local factor for each country. The local factors are allowed to capture idiosyncratic local movement of each country's term structure dynamics. This scheme is suggested as an important innovation over existing common-factor models such as Backus, Foresi, and Telmer (2001). Ahn (2004) shows that existence of local factors is a necessary condition for the gains from investing in foreign bonds. We use time-series and cross-sectional data on US and Japan: six-month and five-year US and Japanese Treasury securities and their exchange rate. This setup is an over-identified system since there are four state variables which are proposed to explain seven cross-sectionals. Thus the estimation scheme is distinguishable from Dai and Singleton (2000), which is based on an exactly identified system.

The remainder of this paper is organized as follows. Section 2 provides implications of the affine and quadratic models for the cross-sectional term structure of correlations. In section 3 we discuss the data and MCMC method that we use for an empirical analysis. The empirical results are also presented in section 4. Section 5 concludes and discusses future research.

3.2 Motivation for International QTSM

Affine models have been the major workhorse in the asset pricing literature both in continuous and discrete time settings. The popularity of affine class mainly stems from the analytic tractability through which all moments of bond yields and forward rates are provided in essentially closed form. Since Duffie and Kan (1996) identified the complete affine class, numerous studies have been conducted both in modeling and empirical arenas. Despite the predominant position in term structure field, however, the empirical performances of the
affine models are *mixed* at best. ²

With repeated partial successes the researchers began to try about three things in an attempt to improve the empirical performances of their models. First, They began to explore the potential benefits that can be obtained by relaxing the restrictive forms of risk parameters. It was only possible after some adeptly recognized the flexibility about the form of prices of risks which an affine model was allowed to have as long as the state variables follow affine diffusion under the risk-neutral probability distribution. (See, for example, Duffie (2002) and Duarte (2004).)

Another thread of research has focused on incorporating the variables with economic bearing into the interest and yield equations. (Piazzesi (2005)), and Ang and Piazzesi (2003)). The last line of attempts to seek out the improved empirical performances of asset pricing models work on the form of yield equation directly. Characterizing yields as non-linear functions of state variables initially came out as a natural consequences of empirical observations.³

In fact the call for the need for non-linear term structure models originated from within the affine class. Dai and Singleton (2000) strongly suggest that negatively correlated diffusions are central to the model's abilities to match the volatility structure and non-normality of changes in bond yields. However, none of the models in affine class can achieve the goal without violating the positivities of interest rates. In other words, all the models with some empirical successes in affine class should endure positive probabilities of offering negative interest. As we'll see in section 4, the affine family of term structure models can not simultaneously allow for negative correlations among the state variables and require that the interest rate be strictly positive. Since the correlated square root models are theoretically incapable of generating negative correlations, as Dai and Singleton (2000) concluded, they

²Please refer to Chen and Scott (1993), Pearson and Sun (1994), Duffie and Singleton (1997), Dai and Singleton (2000), and Backus, Foresi, and Telmer (2001) to name a few.

³Please refer to Beaglehole and Tenney (1992), Constantinides (1992), Longstaff (1989), and Ahn, Dittmar, and Gallant (2002)
are not consistent with the historical behavior of U.S. interest rates. As we deal with the asset price behavior in world market, this drawback of affine class models that can not generate negative correlations between underlying state variables could well be heightened unless the two local markets are tightly integrated to the extent that the bonds are priced as if they were all domestic securities. Hence we tend to suffer the similar problems in explaining the multi-country yield behavior, yet with much greater magnitude.

On the other hand, the extant trade-off between conditional volatility and conditional correlation is to be easily resolved in the class of QTSM. While the interest rate and bond prices of QTSM retain heteroskedastic conditional volatilities as CIR(Cox, Ingersoll, and Ross (1985)) models, the QTSM does not hamper the flexibility in specifying the conditional correlations among the state variables. It is due to the fact that the time-varying conditional volatilities are induced by the structure of the yield equations, not from the state processes as in affine class. We explore this aspect in more details in section 4.

For the empirical performances of respective classes of term structure models see Brandt and Chapman (2002). Using 45 years of monthly data, they conclude that the canonical three-factor Gaussian-quadratic model dominates the three factor essentially affine models at matching the economic moments.

3.3 Characterization of QTSM in International Setting

In this section we introduce a version of international QTSM under no arbitrage by directly specifying the time series process of the nominal stochastic discount factor as below.

Assumption 1. The time series process of the stochastic discount factor\(^4\) of each country is characterized by the following SDE\(^5\).

\(^4\)We define the pricing kernel here as the minimum variance pricing kernel for each country

\(^5\)The superscript \(i\) is used for country index; \(i = d\) (domestic), \(f\) (foreign)
where $dW_t^i$ and $A_t^i$ are, respectively, $k^i$-vectors of standard Brownian motion and market prices of risk of the SDF of country $i$. Let $A_t^i(j)$ be the $j$-th element of $A_t^i$. Then it takes the following affine form of the state variables $V_t^i$:

$$
A_t^i(j) = \lambda_0^i(j) + \lambda_1^i(j) V_t^i. \tag{3.3.2}
$$

where $V_t^i$ is $k^i$-vector of state variables of country $i$. 

**Assumption 2.** The nominal instantaneous interest rate of country $i$ is a quadratic function of the state variables:

$$
r_t^i = \alpha^i + \beta^i V_t^i + \Psi^i V_t^i, \tag{3.3.3}
$$

where $\alpha^i$ is a scalar constant, $\beta^i$ is a $k^i$-vector and $\Psi^i$ is a symmetric matrix of dimension $k^i$. We assume that $\alpha^i \geq 0$, $\beta^i = 0$, and $\Psi^i$ is positive semi-definite with the diagonal elements equal to 1 for identification.

**Assumption 3.** There is an $N_c + N_d + N_f$ dimensional vector of latent state variables $V(t) \equiv [X_t^f, Y_t, Z_t]^T$ that follow a multivariate Gaussian process with mean reversion:

$$
dV_t = \kappa^p (\theta^p - V_t) dt + \Sigma dW_t^M \tag{3.3.4}
$$

where $w(t)$ is an $N_c + N_d + N_f$ dimensional standard Brownian Motion, $\kappa^p \in \mathbb{R}^{M \times M}$, and the eigenvalues of $\kappa^p$ are strictly positive.

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$^6k^d = N_c + N_d$ and $k^f = N_c + N_f$, where $N_c$ is the number of common factors and $N_d$ and $N_f$ are the numbers of domestic and foreign local factors.
Assumption 4. The equivalent martingale measure is defined by

$$\frac{dQ}{dP} = \exp \left( - \int_0^T \Lambda(V(u))'dw(u) - \frac{1}{2} \int_0^T \Lambda(V(u))'\Lambda(S(u))du \right) dP \quad (3.3.5)$$

where $\Lambda(V) = \lambda_0 + \lambda_1 V$. We assume the usual Novikov condition:

$$E \left[ \exp \left( \frac{1}{2} \int_0^T |\Lambda(V(u))|^2 \right) \right] < \infty \quad (3.3.6)$$

Then the process defined below follows a standard Brownian Motion under the new measure $Q$ by the Girsanov theorem.

$$dw^Q(t) = dw(t) + \Lambda(V(t))dt \quad (3.3.7)$$

Under the risk-neutral measure $Q$ the underlying state variables satisfy the following SDE.

$$dV(t) = \kappa^Q(\theta^Q - V(t))dt + \Sigma dw^Q(t) \quad (3.3.8)$$

where

$$\kappa^Q = \kappa + \Sigma \lambda_1 \quad (3.3.9)$$

$$\theta^Q = \kappa^{-1}(\kappa \theta - \Sigma \lambda_0) \quad (3.3.10)$$

Under the aforementioned assumptions the price of zero coupon bond at time $t$, which pays $1$ at time $t + \tau$ was shown to be a exponential quadratic function of the unobserved state variables:

$$P(\tau, V_t) = -\exp(\alpha^2 \tau)A(\tau)\exp\left[ B(\tau)'V_t + V_t'C(\tau)V_t \right] \quad (3.3.11)$$

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7To avoid the notational clutter we drop the country index $i$ whenever it is clear.

8we drop the superscript for country index to avoid the notational clutter.
where $A(\tau), B(\tau),$ and $C(\tau)$ are the solution to the following ODEs.

\[
\frac{\partial C(\tau)}{\partial \tau} = 2C(\tau)\Sigma \Sigma' C(\tau) + (C(\tau) (\Sigma \Sigma' \eta_i - \kappa^p) + (\Sigma \Sigma' \eta_j - \kappa^p)') C(\tau)) - \Psi \quad (3.3.12)
\]

\[
\frac{\partial B(\tau)}{\partial \tau} = 2C(\tau)\Sigma \Sigma' B(\tau) + (\Sigma \Sigma' \eta_i - \kappa^p)' B(\tau) + 2C(\tau) (\kappa^p \theta^p + \Sigma \Sigma' \eta_0) - \beta \quad (3.3.13)
\]

\[
\frac{\partial A(\tau)}{\partial \tau} = \text{tr} [\Sigma \Sigma' C(\tau)] + \frac{1}{2} B(\tau)' \Sigma \Sigma' B(\tau) + B(\tau)' (\kappa^p \theta^p - \Sigma \Sigma' \eta_0) - \alpha, \quad (3.3.14)
\]

with the initial conditions $A(0) = 1$, $B(0) = 0_N$, and $C(0) = 0_{N \times N}$.

The yield-to-maturity is then

\[
y(\tau, V_t) = \frac{1}{\tau} [\alpha^i \tau - \ln A(\tau) - B(\tau)' V_t - V_t' C(\tau) V_t]
\]

### 3.4 The Stochastic Correlation of Interest Rates

In this section, we consider the cross-country interest rate correlations implied by the affine models and the quadratic models. We first set up the world economy which is suitable for our analysis. We assume that the world economy is represented by the augmented filtered probability space $(\Omega, \mathcal{F}, F, P)$, where $F = \{\mathcal{F}_t\}_{0 \leq t \leq T}$. Further, for simplicity, we assume that there are two countries: domestic and foreign countries. Throughout the rest of the paper, we use subscript (superscript) $d$ for the domestic currency (asset) and $f$ for the foreign currency (asset). We assert that there exists a positive state priced density process or pricing kernel $M_j^d(t)$ in each country, which defines the fundamental valuation equation governing $(t, T)$ where $T > t$:

\[
x_j(t) = E_t^P \left[ \frac{M_j^d(T)}{M_j^d(t)} x_j(T) \right], \quad (3.4.1)
\]
where \( x_j^t(t) : [0, \infty) \times \Omega \rightarrow \mathbb{R}^+ \) is the price of an \( j \) country asset denominated in currency \( j = d \) or \( f \). Note that \( E^P_t[\cdot] \) means the conditional expectation operator defined over \( \mathcal{F}_t \), the information possessed by agents at time \( t \) under the true measure \( P \). We refer to \( M_j^t(t, T) = M_j^T(T)/M_j^t(t) \) as the stochastic discount factor or pricing kernel conditional on \( \mathcal{F}_t \) which is used to value county \( j \) assets denominated in country \( j \)'s currency.

We assume that \( M_j^t(t, T) \ (j = d, f) \) is the nominal stochastic discount factor. Constantinides (1992) shows that the nominal stochastic discount factor can be written as:

\[
M_j^t(t, T) = \frac{\frac{\partial U_j^t(T)/\partial C_j^t(T)}{\partial U_j^t(t)/\partial C_j^t(t)}}{1 + \Pi_j^t(t, T)},
\]

where \( U_j^t(\cdot) \) is the Von Neumann-Morgenstern utility function of the representative agent in country \( j \) which maps his/her consumption \( C_j(t) \) onto preference domains and \( \Pi_j^t \) stands for the inflation of country \( j \). Therefore the nominal stochastic discount factor is the intertemporal marginal rate of substitution adjusted for inflation of country \( j \). Although we can derive our analysis below from a general equilibrium and pre-specified inflation process, we will directly specify the stochastic process of the stochastic discount factors without loss of generality, following the argument of Constantinides (1992), Nielsen and Saá-Requejo (1993), and Backus, Foresi, and Telmer (2001).

### 3.4.1 Affine Models

Affine models are currently the most popular term structure models, which postulate the yields as affine functions of the underlying state variables. This class encompasses representative term structure models such as Vasicek (1977) and Cox, Ingersoll, and Ross (1985). A recent study of Duffie and Kan (1996) summarizes the underlying assumptions regarding the affine models and propose the maximally flexible affine models. These models are attractive in the sense that they provide closed-form expression for bond prices and other contingent claims on the interest rates. Since two stochastic processes of the underlying
state variables, i.e., gaussian and square root processes are admissible, they have affluent flexibility in the modeling sense.

However, as Dai and Singleton (2000) suggest there exist some drawbacks of the affine models. That is, for the affine model to be admissible theoretically and empirically, it has to generate three properties: The first one is the positivity of the nominal interest rate. It is well known that the nominal interest rate cannot be negative because of an embedded option not to lend as demonstrated by Black (1992). Only the models with square root factors can satisfy this property, but not any model that incorporate gaussian factors. The second property is the ability to generate sufficient heteroskedasticity in volatilities of interest rates. It is empirically well documented that the interest rates exhibit strong heteroskedasticity in their volatility even beyond the interest rate level effect. While gaussian factors cannot contribute to generating stochastic volatility of the interest rates, the square root factors can generate heteroskedasticity through the level effect of the underlying state variables. It is in these regards that the square root models (i.e., CIR models) are more appealing than Gaussian factor models. The last property is closely related to recent empirical observation, which dictates the case where the underlying affine state variables could be negatively correlated. Gaussian factors can generate negative correlations among the state variables since they are free from zero bound condition. But this desirable property of the gaussian factor models comes at the cost of not imposing non-negativity of the interest rate. In contrast, the square root factor models cannot generate negative correlations due to their regularity condition that enforces the positivity of the interest rate. Therefore, a major theoretical drawback of the models in the affine class is that they are subject to a trade-off between these three essential properties that any realistic term structure model should satisfy. An increase in the number of square root factors limits the flexibility of the affine models in specifying conditional correlations among factors while giving more flexibility in generating heteroskedasticity in the volatilities of yields. Backus, Foresi, and Telmer (2001)

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9On the other hand, the real interest rate can be negative when economic agents expect sufficiently severe depression.
examine a variety of the existing affine models and develop their own in international context in order to solve for the forward premium puzzle. They suggest that the affine models, without violating the positivity of the interest rate, is difficult to solve for the forward premium puzzle. We show that the square root models which guarantee the positivity of the interest rate cannot generate negative cross-country correlations of the interest rates. Since the admissibility of the square root models relies upon non-negative correlations among the state variables, the correlation structure of the state variables will corroborate our argument at the cost of losing analytical tractability. Thus we assume orthogonal state variables without loss of generality. This is equivalent to the affine model that Backus, Foresi, and Telmer (2001) extensively investigate.

Following Ahn (2004) and Backus, Foresi, and Telmer (2001), we assume that the time-series processes of the stochastic discount factors of the two countries are governed by the following stochastic differential equations:

\[
\frac{dM^j(t)}{M^j_d(t)} = - \left[ \sum_{i=1}^{N_c} x_i(t) + \sum_{i=1}^{N_l} y_i(t) \right] dt + \sum_{i=1}^{N_c} \sigma^j_{x_i} \sqrt{x_i(t)} dw^j_{x_i}(t) + \sum_{i=1}^{N_l} \sigma^j_{y_i} \sqrt{y_i(t)} dw^j_{y_i}(t) \tag{3.4.2}
\]

\[
\frac{dM^j(t)}{M^j_f(t)} = - \left[ \sum_{i=1}^{N_l} \alpha_i x_i(t) + \sum_{i=1}^{N_f} z_i(t) \right] dt + \sum_{i=1}^{N_l} \sigma^j_{\alpha_i} \sqrt{x_i(t)} dw^j_{\alpha_i}(t) + \sum_{i=1}^{N_f} \sigma^j_{z_i} \sqrt{z_i(t)} dw^j_{z_i}(t) \tag{3.4.3}
\]

where \( <dw^h_{i},dw^j_{k}> = 0 \) \( \forall h, i, k, l \). Note that there are three sets of state variables which govern the time-series evolution of the stochastic discount factors in both countries. First, there are \( N_c \) of \( x_i(t) \)'s, which we call common factors. They are called such since they are the only common state variables across the two stochastic differential equations. On the other hand, we call \( y_i(t) \)'s domestic local factors since they affect the stochastic evolution of the domestic discount factor exclusively. Similarly we call \( z_i(t) \)'s foreign local factors. These three sets of the state variables are assumed to follow the square root processes:

\[
dx_i(t) = \kappa_{x_i}(\theta_{x_i} - x_i(t)) dt + s_{x_i} \sqrt{x_i(t)} dw_{x_i}(t) \tag{3.4.4}
\]

\[
dy_i(t) = \kappa_{y_i}(\theta_{y_i} - y_i(t)) dt + s_{y_i} \sqrt{y_i(t)} dw_{y_i}(t) \tag{3.4.5}
\]

\[
dz_i(t) = \kappa_{z_i}(\theta_{z_i} - z_i(t)) dt + s_{z_i} \sqrt{z_i(t)} dw_{z_i}(t). \tag{3.4.6}
\]

We assume that the state variables are orthogonal to each other, i.e., \( <dw_{h_{i}},dw_{k_{i}}> = 0 \).
In addition, we assume that

\[ \langle dw^i_h, dw^i_k \rangle = \begin{cases} \rho_{hi} dt & \text{if } h = k \text{ and } i = l \\ 0 & \text{otherwise} \end{cases} \]

The above assumption is made to incorporate non-trivial risk premium associated with the state variables without loss of generality.

From the martingale property of the stochastic discount factor, it can be shown that the negative of the drift of the stochastic discount factor is the interest rate of the corresponding country:

\[
\begin{align*}
    r_d(t) &= \sum_{i=1}^{N_c} x_i(t) + \sum_{i=1}^{N_d} y_i(t) \\
    r_f(t) &= \sum_{i=1}^{N_c} \alpha_i x_i(t) + \sum_{i=1}^{N_f} z_i(t)
\end{align*}
\]

Thus the common factors \( \{x_i(t)\}_{i=1,\ldots,N_c} \) captures common movement of the interest rates of the two countries whereas the idiosyncratic movement of each country will be determined by \( \{y_i(t)\}_{i=1,\ldots,N_d} \) and \( \{z_i(t)\}_{i=1,\ldots,N_f} \).

Applying the Ito's lemma to equations (3.4.7), and (3.4.8) yields the following stochastic differential equations of the interest rate of each country:

\[
\begin{align*}
    dr_d(t) &= \left[ \sum_{i=1}^{N_c} \kappa x_i(\theta_{x_i} - x_i(t)) + \sum_{i=1}^{N_d} \kappa y_i(\theta_{y_i} - y_i(t)) \right] dt \\
    &\quad + \sum_{i=1}^{N_c} s x_i \sqrt{x_i(t)} dw_{x_i}(t) + \sum_{i=1}^{N_d} s y_i \sqrt{y_i(t)} dw_{y_i}(t) \\
    dr_f(t) &= \left[ \sum_{i=1}^{N_c} \alpha_i \kappa x_i(\theta_{z_i} - x_i(t)) + \sum_{i=1}^{N_f} \kappa z_i(\theta_{z_i} - z_i(t)) \right] dt \\
    &\quad + \sum_{i=1}^{N_c} \alpha_i s x_i \sqrt{x_i(t)} dw_{x_i}(t) + \sum_{i=1}^{N_f} s z_i \sqrt{z_i(t)} dw_{z_i}(t)
\end{align*}
\]

which indicates that the stochastic interest rates are pulled toward the steady-state means of \( \sum_{i=1}^{N_c} \theta_{x_i} + \sum_{i=1}^{N_d} \theta_{y_i} \) (domestic) and \( \sum_{i=1}^{N_c} \alpha_i \theta_{x_i} + \sum_{i=1}^{N_f} \theta_{z_i} \) (foreign). Then the
instantaneous variances of the interest rate changes can be represented as:

\[
\Var_d(t) = \sum_{i=1}^{N_c} s_{x_i}^2 x_i(t) + \sum_{i=1}^{N_f} s_{y_i}^2 y_i(t), \quad \Var_f(t) = \sum_{i=1}^{N_c} \alpha_{fi}^2 s_{x_i}^2 x_i(t) + \sum_{i=1}^{N_f} s_{z_i}^2 z_i(t).
\]

The volatilities of the interest rates are represented as the sum of the common factors and the relevant local state variables. Thus, the volatilities of the interest rates are increasing functions of the levels of the state variables including the common and local variables. The covariance between the interest rates is

\[
\Cov_{df}(t) = \sum_{i=1}^{N_c} \alpha_i s_{x_i}^2 x_i(t),
\]

which is an increasing function of the common state variables only. Thus the corresponding correlation coefficient between the two interest rates can be written as:

\[
\rho_{df}(t) = \frac{\sum_{i=1}^{N_c} \alpha_i s_{x_i}^2 x_i(t)}{\sqrt{\left(\sum_{i=1}^{N_c} s_{x_i}^2 x_i(t) + \sum_{i=1}^{N_f} s_{y_i}^2 y_i(t)\right) \left(\sum_{i=1}^{N_c} \alpha_i^2 s_{x_i}^2 x_i(t) + \sum_{i=1}^{N_f} s_{z_i}^2 z_i(t)\right)}}
\]

(3.4.9)

Now we are ready to establish the proposition regarding the feasible range of the cross-country correlation coefficient of the interest rates.

**Proposition 1** The feasible ranges of the cross country correlation of interest rates can be summarized as follows:

- When there do not exist any local state variables, the admissible range of the interest rates implied by the affine multi-state variable model is

  \[
  \sup_{\{x_i\}} \rho_{df}(t) = 1 \quad \text{and} \quad \inf_{\{x_i\}} \rho_{df}(t) > 0,
  \]

  where > is defined in a strict sense.

- When there exist local state variables, the admissible range of the interest rate implied by the affine model is

  \[
  \sup_{\{x_i\}} \rho_{df}(t) = 1 \quad \text{and} \quad \inf_{\{x_i\}} \rho_{df}(t) = 0.
  \]
**Proof** First the supremum of the correlation coefficient will be obtained when at least one common state variable converges to an infinity, i.e., \( x_i \to \infty \) \( \forall i = 1, 2, \ldots, N_c \). This is true regardless of an existence of local factors.

When there does not exist local factors, the infimum of the correlation coefficient is strictly positive. The covariance term is non-negative, and is strictly positive for non-trivial values of the state variables. The possible minimum value of the covariance is zero (when zero is accessible) or a neighbor of zero (when zero is not accessible) when all of the state variables are so. When all of the state variables converge to zero, the covariance as well as the variance converge to zero. At these values of the state variables, \( \rho_{df} = \sum_{i=1}^{N_c} (\alpha_i s_i^2) / \left[ \sqrt{\sum_{i=1}^{N_c} s_i^2} \sqrt{\sum_{i=1}^{N_c} \alpha_i^2 s_i^2} \right] > 0 \). In contrast, in the presence of local factors, \( \rho_{df} \to 0 \) when any local state variable approach infinity. _QED._

**Corollary 1** Suppose there do not exist any local state variables, and the number of common factors is two. Then,

\[
\inf_{x_1, x_2} \rho_{df} = \begin{cases} 
1 & \text{if } \alpha_1 s_{x_1}^2 = \alpha_2 s_{x_2}^2 \\
0 < \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} < 1 & \text{otherwise}
\end{cases}
\]

**Proof** When there are two common factors only, the square of equation (3.4.9) can be rewritten as (square is a monotone transformation, which does not affect the result of the optimization):

\[
\rho_{df}^2 = \frac{(\alpha_1 s_{x_1}^2 x_1 + \alpha_2 s_{x_2}^2 x_2)^2}{(s_{x_1}^2 x_1 + s_{x_2}^2 x_2) (s_{x_1}^2 \alpha_1^2 x_1 + s_{x_2}^2 \alpha_2^2 x_2)}. \tag{3.4.10}
\]

The first order condition with respect to \( x_1 \) and \( x_2 \) can be written as:

\[
\frac{\partial \rho_{df}^2}{\partial x_1} = \frac{(\alpha_1 s_{x_1}^2 - \alpha_2 s_{x_2}^2)^2 (\alpha_1^2 s_{x_1}^2 x_1 - \alpha_2^2 s_{x_2}^2 x_2)}{(s_{x_1}^2 x_1 + s_{x_2}^2 x_2) (\alpha_1^2 s_{x_1}^2 x_1 + \alpha_2^2 s_{x_2}^2 x_2)^2} \quad \frac{\partial \rho_{df}^2}{\partial x_2} = \frac{(\alpha_1 s_{x_1}^2 - \alpha_2 s_{x_2}^2 \alpha_2)^2 (\alpha_1^2 s_{x_1}^2 x_1 + \alpha_2^2 s_{x_2}^2 x_2)}{(s_{x_1}^2 x_1 + s_{x_2}^2 x_2) (s_{x_1}^2 \alpha_1^2 x_1 + s_{x_2}^2 \alpha_2^2 x_2)^2}.
\]

If \( \alpha_1 s_{x_1}^2 = \alpha_2 s_{x_2}^2 \), the first order conditions are always satisfied regardless of the values of the common state variables. In that case, substituting \( \alpha_1 s_{x_1} = \alpha_2 s_{x_2} \) into equation (3.4.10)
yields \( p_{df} = 1 \). Otherwise, there will be multiple solutions which can be delineated as a 'ray' in three dimension:

\[
x_2 = \frac{\alpha_1 s^2_{\text{X}} x_1}{\alpha_2 s^2_{\text{X}}}
\]

Substituting this into equation (3.4.10) yields the desired result. \( QED \)

Proposition 1 states that when there are no local factors across the two countries, the correlation coefficient of the interest rate is strictly positive, and there exists a lower bound. The lower bound is mainly determined by the dispersion of the sensitivities of the foreign interest rate to the common factors. This intuition can be highlighted by investigating the two common factor case. The lower bound, \( 2\sqrt{\alpha_1 \alpha_2}/(\alpha_1 + \alpha_2) \) is a decreasing function of \( |\alpha_1 - \alpha_2| \). Notice that the sensitivity of the domestic interest rate to the common factors is set to 1. Thus, the lower bound is not affected by the scales of \( \alpha_1 \) and \( \alpha_2 \), but their difference. In other words, if the foreign interest rate is more (less) sensitive to the common factors than domestic interest, the lower bound would still be close to 1 as long as the foreign interest rate is more sensitive to the factors equally. What is important is not the magnitude of the sensitivity, but their dispersion. The popular affine models such as Nielsen and Saá-Requejo (1993), Saá-Requejo (1993), and Bakshi and Chen (1997) designate strictly positive lower bounds on the cross-country correlation coefficients. Therefore, it is an interesting testable hypothesis whether this lower bound is consistent with observed data; it may be the case that the estimated parameters of these models assign too high lower bounds, which are counterfactual.

In addition, Proposition 1 states that an inclusion of local state variables greatly contribute to the flexibility in correlation coefficients. In this case, the lower bound on the correlation is 0, which is either inaccessible or reflecting, depending on the boundary conditions on the state variables. Since the local factors affect volatilities of the interest rates but not the covariance, the link between the factor volatilities and the covariance is loosened in this case. However, the correlation coefficient cannot be negative because the square-root factors cannot take negative values. Thus Ahn (2004)'s local factor model cannot
accommodate sign-switching cross-correlation of the interest rates.

### 3.4.2 Quadratic Models

In this section, we investigate the quadratic models for a potential to overcome the drawbacks of the affine models in determining the cross-correlation of the interest rates. The quadratic models which designate yields as a quadratic function of the underlying state variables goes back to the double square root model of Longstaff (1989). Beaglehole and Tenney (1992) contribute to the models in this line by detecting the mistakes of the double square root model in its boundary condition with respect to the interest rate. Ahn, Dittmar, and Gallant (2002) further generalize this model by incorporating various correlation structure among the state variables and the shape of market price of risk. This section also investigates the quadratic model with the same factor orthogonality as in the affine section to highlight its implication for the cross-country correlation structure, making the first analysis to extend the non-affine term structure model to an international setup.

We assume that the time-series processes of the stochastic discount factor of the two countries are governed by the following two stochastic differential equations:

\[
\frac{dM_d(t)}{M_d(t)} = - \left[ \alpha_d + \sum_{i=1}^{N_d} x_i(t)^2 + \sum_{i=1}^{N_d} y_i(t)^2 \right] dt
\]  
\[\text{(3.4.11)}\]

\[
- \sum_{i=1}^{N_d} \left( \sigma_{x0i}^d + \sigma_{x1i}^d x_i(t) \right) dw_i(t) - \sum_{i=1}^{N_d} \left( \sigma_{y0i}^d + \sigma_{y1i}^d y_i(t) \right) dw_i(t)
\]  
\[\text{(3.4.12)}\]

\[
\frac{dM_f(t)}{M_f(t)} = - \left[ \alpha_f + \sum_{i=1}^{N_d} \beta_i (x_i(t) + \pi_i)^2 + \sum_{i=1}^{N_f} z_i(t)^2 \right] dt
\]  
\[\text{(3.4.13)}\]

\[
- \sum_{i=1}^{N_d} \left( \sigma_{x0i}^f + \sigma_{x1i}^f x_i(t) \right) dw_i(t) - \sum_{i=1}^{N_f} \left( \sigma_{z0i}^f + \sigma_{z1i}^f z_i(t) \right) dw_i(t)
\]  
\[\text{(3.4.14)}\]

Similar to the affine term structure model that we have investigated in the previous section, there are three sets of state variables that govern the time-series evolution of the stochastic discount factors in both countries: common factors, \( \{x_i\} \), domestic local factors, \( \{y_i\} \) and foreign local factors \( \{z_i\} \). Unlike the affine models, the quadratic models specify the drift of
each country’s stochastic discount factor as a quadratic functions where its instantaneous volatility is driven by an affine function of the relevant state variables. The stochastic processes of the underlying state variables are specified as the Ornstein-Uhlenbeck process:

\[
\begin{align*}
    dx_i(t) &= \kappa_{x_i}(\theta_{x_i} - x_i(t))dt + s_{x_i}dw_{x_i}(t) \\
    dy_i(t) &= \kappa_{y_i}(\theta_{y_i} - y_i(t))dt + s_{y_i}dw_{y_i}(t) \\
    dz_i(t) &= \kappa_{z_i}(\theta_{z_i} - z_i(t))dt + s_{z_i}dw_{z_i}(t).
\end{align*}
\] (3.4.15) (3.4.16) (3.4.17)

We assume the same correlation structure of the Brownian motions as for the affine model.

Under these assumptions, the martingale property of the stochastic discount factor allows us to determine the interest rate of each country:

\[
\begin{align*}
    r_d(t) &= \alpha_d + \sum_{i=1}^{N_c} x_i(t)^2 + \sum_{i=1}^{N_d} y_i(t)^2 \\
    r_f(t) &= \alpha_f + \sum_{i=1}^{N_c} \beta_i(x_i(t) + \pi_i)^2 + \sum_{i=1}^{N_f} z_i(t)^2.
\end{align*}
\] (3.4.18) (3.4.19)

Note that the interest rate of each country has a lower bound of \(\alpha_d\) and \(\alpha_f\) respectively.

Under the assumption that these parameters along with \(\beta_i > 0 \forall i = 1, \ldots, N_c\), the interest rates of the two countries are guaranteed to be positive given positive definite specification of quadratic form.

Applying Ito’s lemma along with equations (3.4.15)-(3.4.17), we obtain the stochastic differential equation of the interest rate of each country:

\[
\begin{align*}
    dr_d(t) &= \left[ \sum_{i=1}^{N_c} s_{x_i}^2 + \sum_{i=1}^{N_d} s_{y_i}^2 + 2\kappa_{x_i}(\theta_{x_i} - x_i(t))x_i(t) + 2\kappa_{y_i}(\theta_{y_i} - y_i(t))y_i(t) \right] dt \\
    &\quad + 2\sum_{i=1}^{N_c} s_{x_i}x_i(t)dw_{x_i}(t) + 2\sum_{i=1}^{N_d} s_{y_i}y_i(t)dw_{y_i}(t) \\
    dr_f(t) &= \left[ \sum_{i=1}^{N_c} \beta_i s_{x_i}^2 + \sum_{i=1}^{N_d} s_{y_i}^2 + 2\beta_i \kappa_{x_i}(\theta_{x_i} - x_i(t))(x_i(t) + \pi_i) + 2\kappa_{y_i}(\theta_{y_i} - y_i(t))y_i(t) \right] dt \\
    &\quad + 2\sum_{i=1}^{N_c} \beta_i s_{x_i}(x_i(t) + \pi_i)dw_{x_i}(t) + 2\sum_{i=1}^{N_d} s_{y_i}y_i(t)dw_{y_i}(t)
\end{align*}
\]

Thus the drift of the interest rate as well as its instantaneous variance is governed by the quadratic function of the underlying state variables. This makes a stark difference from
affine models, which specify them as an affine function. Specifically, the instantaneous variances of the interest rate changes are given as:

\[ \text{Var}_d(t) = 4 \left[ \sum_{i=1}^{N_c} s_{x_i}^2 x_i(t)^2 + \sum_{i=1}^{N_d} s_{y_i}^2 y_i(t)^2 \right], \quad \text{Var}_f(t) = 4 \left[ \sum_{i=1}^{N_c} \beta_i^2 s_{x_i}^2 (x_i(t) + \pi)^2 + \sum_{i=1}^{N_f} s_{z_i}^2 z_i(t)^2 \right], \]

and the corresponding covariance is

\[ \text{Cov}_d(t) = 4 \sum_{i=1}^{N_c} \beta_i s_{x_i}^2 x_i(t)(x_i(t) + \pi_i), \]

which leads to the cross-country correlation coefficient of the interest rates:

\[ \rho_{df}(t) = \frac{\sum_{i=1}^{N_c} \beta_i s_{x_i}^2 x_i(t)(x_i(t) + \pi_i)}{\sqrt{\left[ \sum_{i=1}^{N_c} s_{x_i}^2 x_i(t)^2 + \sum_{i=1}^{N_d} s_{y_i}^2 y_i(t)^2 \right] \left[ \sum_{i=1}^{N_c} \beta_i^2 s_{x_i}^2 (x_i(t) + \pi)^2 + \sum_{i=1}^{N_f} s_{z_i}^2 z_i(t)^2 \right]}} \]

**Proposition 2:** The quadratic model can generate a feasible range of the cross-country correlation of the interest rates such that

- If there exists only a single common factor without any local factor,

\[ \rho_{df}(t) = \begin{cases} -1 & \text{min}(0, -\pi) < x_i < \max(0, -\pi) \\ 1 & \text{otherwise} \end{cases} \]

- If there exists either multiple common factors or any local factor,

\[ \sup_{\{x_i\}} \rho_{df}(t) = 1 \quad \text{and} \quad \inf_{\{x_i\}} = -1 \]

- **Proof** See the Appendix C

### 3.4.3 Term Structure of Correlations: Affine Models

In this section, we extend the analysis of cross-country correlation to term structure of correlations, the association between the time-to-maturities of bonds and the cross-country correlations. This analysis will provide an important implication for the management of
global fixed-income portfolios. In this section, we will investigate the affine model, which will be followed by the quadratic model in next section.

From equations (3.4.2) and (3.4.3), we can convert the stochastic differential equations of the state variables under the risk-neutral measures:

\[
\begin{align*}
    dx_i(t) &= \left(\kappa_{x_i} + \lambda_{x_i} \right) x_i(t) \ dt + s_{x_i} \sqrt{x_i(t)} \ dw_x(t) \text{ under the } Q^*_d \\
    dx_i(t) &= \left(\kappa_{x_i} + \lambda_{x_i} \right) x_i(t) \ dt + s_{x_i} \sqrt{x_i(t)} \ dw_x(t) \text{ under the } Q^*_f \\
    dy_i(t) &= \left(\kappa_{y_i} + \lambda_{y_i} \right) y_i(t) \ dt + s_{y_i} \sqrt{y_i(t)} \ dw_y(t) \text{ under the } Q^*_d \\
    dz_i(t) &= \left(\kappa_{z_i} + \lambda_{z_i} \right) z_i(t) \ dt + s_{z_i} \sqrt{z_i(t)} \ dw_z(t) \text{ under the } Q^*_f,
\end{align*}
\]

where \(Q^*_d\) and \(Q^*_f\) are the equivalent martingale measures with denomination of domestic currency and foreign currency respectively. Remember that the filtration \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) could progress in different ways with domestic and foreign denomination. Then \(Q^*_d\) and \(Q^*_f\) may not be unique in the presence of local factors of the other country, but they designate the same stochastic differential equation of the state variables since the markets are locally complete with respect to common and local shocks.

With these relevant properties of interest rates, we can derive the solutions for discount bond prices and their implications for the cross-country behavior of term structures. The definition of the stochastic discount factor of each country, (3.4.1) yields the following expression for the time \(t\) price of a country \(k\) bond which pays \$1 unit of country \(j\) currency at maturity date, \(t + \tau\),

\[
P_j(t, t + \tau) = E^F_t \left[ \frac{M^j(t + \tau)}{M^j(t)} \right].
\]

We analyze the cross-country term structure of correlations between bond returns. Even though the term structure of correlations between bond returns is more important for portfolio management and hedging strategies, its undesirable feature is that it is hard to get bond return data, especially of foreign countries. However, we will show below that the term structure of correlations between yields yield the same correlation structures as the correlation between bond returns. This is desirable since the term structure of correlations between yields is more suitable for an empirical analysis.
The solution for bond prices and their corresponding yield-to-maturities, \( yt_k(t, t + \tau) \) are derived in Cox, Ingersoll, and Ross (1985) or Chen and Scott (1993):

\[
P_d(t, \tau) = \prod_{i=1}^{N_d} A^d_{x_i} \prod_{i=1}^{N_f} A^d_{y_i} \exp \left[ \sum_{i=1}^{N_d} B_{x_i} x_i(t) + \sum_{i=1}^{N_f} B_{y_i} y_i(t) \right]
\]

\[
P_f(t, \tau) = \prod_{i=1}^{N_d} A^f_{x_i} \prod_{i=1}^{N_f} A^f_{y_i} \exp \left[ \sum_{i=1}^{N_d} B_{x_i} x_i(t) + \sum_{i=1}^{N_f} B_{y_i} y_i(t) \right],
\]

where

\[
A^d_{x_i} = \left[ \frac{2\gamma_{x_i}^d \exp \left( \frac{(\kappa_{x_i} + \lambda_{x_i}^d + \gamma_{x_i}) \tau}{2} \right)}{(\kappa_{x_i} + \lambda_{x_i}^d + \gamma_{x_i})^2 (\exp(\gamma_{x_i} \tau) - 1) + 2\gamma_{x_i}^d} \right]^{2\alpha_{x_i} s_{x_i}/\sigma_{x_i}^2}
\]

\[
B_{x_i}^d = \frac{2\alpha_{x_i}^d (\exp(\gamma_{x_i}^d - 1))}{(\kappa_{x_i} + \lambda_{x_i}^d + \gamma_{x_i})^2 (\exp(\gamma_{x_i}^d \tau) - 1) + 2\gamma_{x_i}^d}
\]

\[
\gamma_{x_i}^d = \sqrt{\left(\kappa_{x_i} + \lambda_{x_i}^d\right)^2 + 2\alpha_{x_i}^d s_{x_i}^2}
\]

where \( v = x, y \) or \( z; i = N_c, N_d \) or \( N_f; j = d \) or \( f \); \( \alpha^d = 1 \) if \( j = d \) and \( \alpha^f = \alpha \) if \( j = f \) when \( v = x \). Based on these solutions and from Ito’s lemma, we can express the time-series process of bonds in the two countries:

\[
dP_d(t, \tau) = \left[ r_d(t) - \sum_{i=1}^{N_d} \lambda_{x_i}^d B_{x_i}^d(\tau) x_i(t) - \sum_{i=1}^{N_f} \lambda_{y_i}^d B_{y_i}^d(\tau) y_i(t) \right] dt
\]

\[
- \sum_{i=1}^{N_d} B_{x_i}^d s_{x_i} \sqrt{x_i(t)} dw_{x_i}(t) - \sum_{i=1}^{N_f} B_{y_i}^d s_{y_i} \sqrt{y_i(t)} dw_{y_i}(t) \quad (3.4.22)
\]

\[
dP_f(t, \tau) = \left[ r_f(t) - \sum_{i=1}^{N_d} \lambda_{x_i}^f B_{x_i}^f(\tau) x_i(t) - \sum_{i=1}^{N_f} \lambda_{y_i}^f B_{y_i}^f(\tau) y_i(t) \right] dt
\]

\[
- \sum_{i=1}^{N_d} B_{x_i}^f s_{x_i} \sqrt{x_i(t)} dw_{x_i}(t) - \sum_{i=1}^{N_f} B_{y_i}^f s_{y_i} \sqrt{y_i(t)} dw_{y_i}(t) \quad (3.4.23)
\]

The term premia of the domestic and foreign bonds are \(- \sum_{i=1}^{N_d} \lambda_{x_i}^d B_{x_i}^d(\tau) x_i(t) - \sum_{i=1}^{N_f} \lambda_{y_i}^d B_{y_i}^d(\tau) y_i(t)\) and \(- \sum_{i=1}^{N_d} \lambda_{x_i}^f B_{x_i}^f(\tau) x_i(t) - \sum_{i=1}^{N_f} \lambda_{y_i}^f B_{y_i}^f(\tau) y_i(t)\), respectively.

On the other hand, the stochastic differential equations for the yields, \( y_d(t, \tau) \) and
\( y_{t_f}(t, \tau) \) can be represented as:\(^{10}\)

\[
dy_d(t, \tau) = \frac{1}{\tau} \left[ y_{t_d}(t, \tau) - \sum_{i=1}^{N_d} \left\{ 1 - \frac{1}{2}(B_{z_i}(\tau))^2 s_{z_i}^2 - \lambda_{z_i}^d B_{z_i}(\tau) \right\} x_i(t) \right. \\
- \sum_{i=1}^{N_d} \left\{ 1 - \frac{1}{2}(B_{y_i}(\tau))^2 s_{y_i}^2 - \lambda_{y_i}^d B_{y_i}(\tau) \right\} y_i(t) \bigg] dt \\
+ \sum_{i=1}^{N_d} \frac{B_{z_i}(\tau)}{\tau} s_{z_i} \sqrt{x_i(t)} dw_{z_i}(t) + \sum_{i=1}^{N_d} \frac{B_{y_i}(\tau)}{\tau} s_{y_i} \sqrt{y_i(t)} dw_{y_i}(t) \quad (3.4.24)
\]

\[
dy_f(t, \tau) = \frac{1}{\tau} \left[ y_{t_f}(t, \tau) - \sum_{i=1}^{N_f} \left\{ 1 - \frac{1}{2}(B_{z_i}(\tau))^2 s_{z_i}^2 - \lambda_{z_i}^f B_{z_i}(\tau) \right\} x_i(t) \right. \\
- \sum_{i=1}^{N_f} \left\{ 1 - \frac{1}{2}(B_{y_i}(\tau))^2 s_{y_i}^2 - \lambda_{y_i}^f B_{y_i}(\tau) \right\} y_i(t) \bigg] dt \\
+ \sum_{i=1}^{N_f} \frac{B_{z_i}(\tau)}{\tau} s_{z_i} \sqrt{x_i(t)} dw_{z_i}(t) + \sum_{i=1}^{N_f} \frac{B_{y_i}(\tau)}{\tau} s_{y_i} \sqrt{y_i(t)} dw_{y_i}(t) \quad (3.4.25)
\]

Since the diffusion terms of \( dP_d/P_d \) (\( dP_f/P_f \)) and \( dy_d \) (\( dy_f \)) are equivalent after scaling, the cross-country correlations of bond returns are exactly identical to those of yields:

\[
\rho_{y_d}(t, \tau) = \frac{\sum_{i=1}^{N_d} B_{z_i}^d s_{z_i}^2 x_i}{\sqrt{\sum_{i=1}^{N_d} (B_{z_i}^d)^2 s_{z_i}^2 x_i^2 + \sum_{i=1}^{N_d} (B_{y_i}^d)^2 s_{y_i}^2 y_i^2}} \sqrt{\sum_{i=1}^{N_f} (B_{z_i}^f)^2 s_{z_i}^2 x_i^2 + \sum_{i=1}^{N_f} (B_{y_i}^f)^2 s_{y_i}^2 y_i^2}}
\]

It is obvious that the correlation is non-negative.

**Proposition 3** The feasible ranges of the cross country correlation of yield-to-maturities can be summarized as follows:

- When there do not exist any local state variables, the admissible range of the correlation coefficient between yields implied by the affine multi-state variable model is 

\[ \sup_{z_i} \rho_{y_d}(t, \tau) = 1 \text{ and } \inf_{z_i} > 0. \]

- When there exist local state variable, the admissible range of the interest rate implied by the affine model is

\[ \sup_{z_i} \rho_{y_d}(t, \tau) = 1 \text{ and } \inf_{z_i} = 0. \]

\(^{10}\)The proof is in Appendix A.
Proof: The proof is an extension of Proposition 1.

### 3.4.4 Quadratic Model

This section explores the cross-country correlation of term structure implied by the quadratic term structure model. (3.4.12) and (3.4.14) can be used to convert the stochastic differential equations of the state variables under the risk-neutral measures implied by the quadratic term structure model:

\[
\begin{align*}
    dx_i(t) &= \left[ \kappa_{x_i} \theta_x - \lambda_{x_1}^d - \left( \kappa_{x_i} + \lambda_{x_1}^d \right) x_i(t) \right] dt + s_{x_i} dw_{x_i}(t) \text{ under the } Q_d^* \\
    dy_i(t) &= \left[ \kappa_{y_i} \theta_y - \lambda_{y_0}^d - \left( \kappa_{y_i} + \lambda_{y_0}^d \right) y_i(t) \right] dt + s_{y_i} dw_{y_i}(t) \text{ under the } Q_d^* \\
    dz_i(t) &= \left[ \kappa_{z_i} \theta_z - \lambda_{z_0}^d - \left( \kappa_{z_i} + \lambda_{z_0}^d \right) z_i(t) \right] dt + s_{z_i} dw_{z_i}(t) \text{ under the } Q_d^*
\end{align*}
\]

Using this measure transformation, we can use the result of Ahn, Dittmar, and Gallant (2002) under the assumption of orthogonality among the state variables for a solution for bond prices:

\[
P_d(t, \tau) = \exp\left(-\alpha_d \tau\right) \prod_{i=1}^{N_x} A_{x_i}^d(\tau) \prod_{i=1}^{N_y} A_{y_i}^d(\tau) \exp\left[-\sum_{i=1}^{N_x} \left\{ B_{x_i}^d(\tau)x_i(t) + C_{x_i}^d(\tau)x_i(t)^2 \right\} \right] \\
- \sum_{i=1}^{N_x} \left\{ B_{y_i}^d(\tau)y_i(t) + C_{y_i}^d(\tau)y_i(t)^2 \right\} \]

(3.4.26)

\[
P_f(t, \tau) = \exp\left[-\left( \alpha_f + \sum_{i=1}^{N_x} \beta_i \tau_i \right) \right] \prod_{i=1}^{N_x} A_{x_i}^f(\tau) \prod_{i=1}^{N_y} A_{y_i}^f(\tau) \exp\left[-\sum_{i=1}^{N_x} \left\{ B_{x_i}^f(\tau)x_i(t) + C_{x_i}^f(\tau)x_i(t)^2 \right\} \right] \\
- \sum_{i=1}^{N_x} \left\{ B_{y_i}^f(\tau)y_i(t) + C_{y_i}^f(\tau)y_i(t)^2 \right\} \]

(3.4.27)
where

\[
A_{v_i} = \exp \left[ \frac{\beta_{v_i}^2}{(\gamma_{v_i})^2} \right] \left[ (\gamma_{v_i} + \kappa_{v_i}' + \lambda_{v_i1}) \pi_{v_i} + \left( \kappa_{v_i}' \theta_{v_i} - \lambda_{v_i0} \right) \right] \left[ (\gamma_{v_i} - \kappa_{v_i}' - \lambda_{v_i0}) \pi_{v_i} - \left( \kappa_{v_i}' \theta_{v_i} - \lambda_{v_i0} \right) \right] \frac{1}{\beta} \left[ (\gamma_{v_i} + \lambda_{v_i1} + \gamma_{v_i}) (\exp(2\gamma_{v_i} \tau) - 1) + 2\gamma_{v_i} \right]^{-\frac{1}{2}}
\]

\[
D_{v_i} = 2 \left[ 2\gamma_{v_i} - \left( \kappa_{v_i}' + \lambda_{v_i1} + \gamma_{v_i} \right) \right] \left[ (\gamma_{v_i} - \kappa_{v_i}' - \lambda_{v_i0}) \pi_{v_i} - \left( \kappa_{v_i}' \theta_{v_i} - \lambda_{v_i0} \right) \right] \left[ (\gamma_{v_i} + \lambda_{v_i1} + \gamma_{v_i}) (\exp(2\gamma_{v_i} \tau) - 1) + 2\gamma_{v_i} \right]^{-\frac{1}{2}}
\]

\[
B_{v_i} = \frac{2\beta_{v_i}^2 \exp(\gamma_{v_i} \tau) - 1}{\gamma_{v_i} \left[ (\gamma_{v_i} + \lambda_{v_i1} + \gamma_{v_i}) \right] (\exp(2\gamma_{v_i} \tau) - 1) + 2\gamma_{v_i}}
\]

\[
C_{v_i} = \frac{\beta_{v_i}}{2 \left( \gamma_{v_i} + \lambda_{v_i1} + \gamma_{v_i} \right) (\exp(2\gamma_{v_i} \tau) - 1) + 2\gamma_{v_i}}
\]

Notice that \( \beta_{v_i} = \beta \) and \( \pi_{v_i} = \pi \) when \( v = x \) and \( j = f \); otherwise \( \beta_{v_i} = 1 \) and \( \pi_{v_i} = 0 \). Thus the domestic bond price formula is identical to Ahn, Dittmar, and Gallant (2002) with restrictions of \( \beta_{v_i} = 1 \) and \( \pi_{v_i} = 0 \) for all \( v \) and \( i \). Applying Ito's lemma and using the bond solutions yields the following expression for the stochastic process of bonds in the two
countries:

\[
dP_d(t, \tau) = \left[ r_d(t) - \sum_{i=1}^{N_d} \left\{ \lambda_{i0}^d + \lambda_{i1}^d x_i \right\} \left\{ B_{i1}^d(\tau) + 2C_{i1}^d(\tau)x_i \right\} \right] dt \\
- \sum_{i=1}^{N_d} \left\{ \lambda_{i0}^d + \lambda_{i1}^d y_i \right\} \left\{ B_{i1}^d(\tau) + 2C_{i1}^d(\tau)y_i \right\} dt \\
- \sum_{i=1}^{N_d} \left\{ B_{i1}^d(\tau) + 2C_{i1}^d(\tau)x_i \right\} s_{x_i} dw_{x_i}(t) - \sum_{i=1}^{N_d} \left\{ B_{i1}^d(\tau) + 2C_{i1}^d(\tau)y_i \right\} s_{y_i} dw_{y_i}(t)
\]

\[
dP_f(t, \tau) = \left[ r_f(t) - \sum_{i=1}^{N_f} \left\{ \lambda_{i0}^f + \lambda_{i1}^f z_i \right\} \left\{ B_{i1}^f(\tau) + 2C_{i1}^f(\tau)z_i \right\} \right] dt \\
- \sum_{i=1}^{N_f} \left\{ \lambda_{i0}^f + \lambda_{i1}^f z_i \right\} \left\{ B_{i1}^f(\tau) + 2C_{i1}^f(\tau)y_i \right\} dt \\
- \sum_{i=1}^{N_f} \left\{ B_{i1}^f(\tau) + 2C_{i1}^f(\tau)z_i \right\} s_{z_i} dw_{z_i}(t) - \sum_{i=1}^{N_f} \left\{ B_{i1}^f(\tau) + 2C_{i1}^f(\tau)y_i \right\} s_{y_i} dw_{y_i}(t)
\]

Unlike the affine model, the term premia of the domestic and foreign bonds are quadratic functions of the state variables. The stochastic differential equations for the yields, \( y_{td}(t, \tau) \) and \( y_{tf}(t, \tau) \), can be represented as:

\[
dy_{td}(t, \tau) = \frac{1}{\tau} \left[ -\alpha_d \tau + \sum_{i=1}^{N_d} \left\{ -\ln A_{i1}^d(\tau) + \frac{1}{2} s_{x_i}^2 B_{i1}^d(\tau) x_i(t) + \left( 1 + 2s_{x_i}^2 C_{i1}^d(\tau) x_i(t) + \lambda_{x_i} \right) B_{i1}^d(\tau) x_i(t) \right\} \\
- \left( \tau - 2s_{x_i}^2 C_{i1}^d(\tau) x_i(t) - 2\lambda_{x_i} C_{i1}^d(\tau) x_i(t) - C_{i1}^d(\tau) x_i(t)^2 \right) \right] dt \\
+ \sum_{i=1}^{N_d} \left\{ \ln A_{i1}^d(\tau) + \frac{1}{2} s_{x_i}^2 B_{i1}^d(\tau) x_i(t) + \left( 1 + 2s_{x_i}^2 C_{i1}^d(\tau) x_i(t) + \lambda_{x_i} \right) B_{i1}^d(\tau) x_i(t) \right\} B_{i1}^d(\tau) x_i(t) \\
- \left( \tau - 2s_{x_i}^2 C_{i1}^d(\tau) x_i(t) - 2\lambda_{x_i} C_{i1}^d(\tau) x_i(t) - C_{i1}^d(\tau) x_i(t)^2 \right) \right] dt \\
+ \sum_{i=1}^{N_d} \left\{ B_{i1}^d(\tau) + 2C_{i1}^d(\tau) x_i(t) \right\} s_{x_i} dw_{x_i}(t) + \sum_{i=1}^{N_d} \left\{ B_{i1}^d(\tau) + 2C_{i1}^d(\tau) y_i(t) \right\} s_{y_i} dw_{y_i}(t).
\]

Thus the cross-sectional correlation coefficient between bond returns or between yields is equal to:

\[
\rho_{td}(t, \tau) = \frac{\sum_{i=1}^{N_d} \left( B_{i1}^d(\tau) + 2C_{i1}^d(\tau) x_i(t) \right) \left( B_{i1}^d(\tau) + 2C_{i1}^d(\tau) x_i(t) \right)}{\left( \sum_{i=1}^{N_d} \left( B_{i1}^d(\tau) + 2C_{i1}^d(\tau) x_i(t) \right)^2 + \sum_{i=1}^{N_d} \left( B_{i1}^d(\tau) + 2C_{i1}^d(\tau) y_i(t) \right)^2 \right)^{-1/2}} \cdot \frac{\sum_{i=1}^{N_d} \left( B_{i1}^d(\tau) + 2C_{i1}^d(\tau) x_i(t) \right)^2 + \sum_{i=1}^{N_d} \left( B_{i1}^d(\tau) + 2C_{i1}^d(\tau) y_i(t) \right)^2}{\left( \sum_{i=1}^{N_d} \left( B_{i1}^d(\tau) + 2C_{i1}^d(\tau) x_i(t) \right)^2 + \sum_{i=1}^{N_d} \left( B_{i1}^d(\tau) + 2C_{i1}^d(\tau) y_i(t) \right)^2 \right)^{-1/2}}.
\]
Proposition 4  The feasible ranges of the cross country correlation of yield-to-maturities can be summarized as:

- **If there exists only a single common factor without any local factor:**

\[ \rho_{df}(t, \tau) = 1. \]

- **If there exist either multiple common factors or any local factor:**

\[ \sup_X \rho_{df}(t, \tau) = 1 \text{ and } -1 < \inf_X \rho_{df}(t, \tau) < 0. \]

- **Proof** The proof is a straightforward extension of Proposition 2
3.5 Data and Estimation

In this section I mainly present the Bayesian estimation by Markov Chain Monte Carlo. Efficient Method of Moment estimation and the results are provided in the Appendix section.

3.5.1 Description of the Data

The term structure data to estimate the international QTSM consist of estimated zero coupon Treasury bond yields for the United States and Japan from January 1985 to May 2002 on a monthly basis.\(^{11}\)

Based on the data on bond prices, coupon rates, and redemption dates a cubic spline is fitted to the yields of each country. No extrapolation is used in the estimation. \(^{12}\)

3.5.2 Bayesian Estimation by MCMC

In this section we propose an alternative estimation technique based on Markov Chain Monte Carlo (MCMC). The simulation-based Bayesian approach has various advantages in estimating term structure models. In fact, a growing number of studies take the Bayesian approach to estimating continuous time processes with discretely observed data of augmented data set. Among others are Jones (2006), Elerian and Shephard (2001), Eraker (2001), and Sanford and Martin (2005). For a detailed exposition of MCMC for continuous-time financial econometrics refer to Johannes and Polson (2006). \(^{13}\)

---

\(^{11}\)I appreciate that Yihong Xia generously provided the data set for this study.

\(^{12}\)We do not use interpolation of the data either. Note, however, that data augmentation, in many cases, could increase the efficiency of estimation in terms of the convergence rate as Polson and Müller (2003) show.

\(^{13}\)For more detailed theoretical foundation refer to Pedersen (1995).
3.5.3 State Space Representation

3.5.3.1 State Equation

We approximate the continuous time specification in (3.3.4) using Euler scheme,

\[ V_{t+\Delta t} = \kappa \theta \rho \Delta + (1 - \kappa \Delta) V_t + \Sigma \sqrt{\Delta} \varepsilon_{t+\Delta t} \]

(3.5.1)

3.5.3.2 Observation Equation

The cross section of observed yields at time \( t \) is a \( (k \times 1) \) vector

\[
y(\tau, t) = \frac{1}{\tau_1} [a \tau_1 - \ln A(\tau_1) - B(\tau_1)'V_t - V_t' C(\tau_1) V_t] \\
\vdots \\
\frac{1}{\tau_k} [a \tau_k - \ln A(\tau_k) - B(\tau_k)'V_t - V_t' C(\tau_k) V_t]
\]

(3.5.2)

Now we define the observation equation as

\[ y(\cdot, t) = m_0 + m_1 V_t + m_2 \text{vec}(V_t' \otimes V_t') + \varepsilon_{.,t}, \]

(3.5.3)

where \( \otimes \) is the Kroneker product, and \( m_1 \) is \( N \times N \) and \( m_2 \) is \( N \times N^2 \) coefficient matrices, respectively. Note \( m_2 \) is such that i-th element of \( [m_2 \text{vec}(V_t' \otimes V_t')] \) is equal to \( V_t' C(\tau_i) V_t \). \( \varepsilon_{.,t} \) follows a multivariate normal distribution:

\[
\varepsilon_{.,t} = \begin{bmatrix}
\varepsilon_{1,t} \\
\varepsilon_{2,t} \\
\vdots \\
\varepsilon_{N,t}
\end{bmatrix},
\]

(3.5.4)

where \( \text{var}(\varepsilon_{i,t}) \sim N(0, \sigma_i) \)

(3.5.5)
3.5.4 Bayesian Inference and MCMC

It is well documented that Markov Chain Monte Carlo (MCMC) methods are particularly useful for continuous-time finance models. (see Johannes and Polson (2006)) We use MCMC methods to estimate the model based on the state-space representation. Unlike Efficient Method of Moments of Gallant and Tauchen (1996), we do not need an auxiliary model to fit the actual data.

A Bayesian approach to estimating continuous time process with discretely observed data is presented in a number of studies, among which are Jones (2003) and Eraker (2001). Its theoretical foundation is laid by Pedersen (1995), who proves that a stochastic differential equation for which the likelihood function is unknown or intractable, can be estimated using approximate conditional Gaussian process, provided that the time interval is sufficiently small. We adopt the mixture of Gibbs and Metropolis-Hastings algorithms since some parameters/factors do not follow a known conditional distribution.

In M-H algorithm, a proposed value of a parameter or a factor is drawn from a standard distribution with \( q(y|x) \) called proposal density, whose \( x \) is the current value of the parameter state variable. Each draw from the proposal density is taken with the accept probability calculated as

\[
\alpha(x, y) = \min \left\{ \frac{\pi(y)q(x|y)}{p(x)q(y|x)}, 1 \right\},
\]

(3.5.6)

If the accept rate is set equal to \( \alpha \) specified above, the detailed balance condition is satisfied. So the proposal density combined with the accept rate makes the successive draws subject to stationery density in the long run.

Detailed sampling algorithm of parameters and state variables are presented in Appendix.
3.5.5 Estimation Results

We start the estimation by simulation studies. An artificial data set of the same dimension as the real data was created using reasonable values of model parameters. We use the first two sample moments to generate the data. The estimation with artificial data provide us with the insurance against programming errors, incorrect conditionals, poorly mixing Markov chains, and improper priors (Johannes and Polson (2006)). The results based on the simulated data show that all the parameter estimates converge to their true values fairly quickly. 95% numerical confidence intervals all contains the values of parameters with which the data set was created.

3.5.5.1 Estimation of 3 factor International QTSM

We then proceed to estimate independent 3 factor QTSM, where there is one common state variable that affects the yield processes of both countries and two local factors each of which only governs either country's economy. The estimation of this model is critical in that it not only serves as our base model for the whole estimation scope but provides a valuable starting point from which more complex models are estimated.

The estimates of model parameters and model-implied moments are presented in Table 3.4 and Table 3.5, respectively. The values in parentheses are 95% numerical confidence limits. All parameter estimates are within the confidence limits also.\(^\text{14}\) Note that the QTSM with independent state variables fit the first moments of both US and Japan yield data reasonably well, capturing the upward sloping yield curves of both countries. However,\(^\text{14}\)

\(^{14}\)This is also the case with the correlated QTSM. Hence we report the standard errors for the correlated model.
it shows some difficulties in matching the higher moments. Particularly, the volatility of US long-term yield is not adequately matched by the independent QTSM. The long-term US yield volatility in actual data is only slightly lower than that of 6 month yield. While the estimates of US volatilities are downward biased, quite the opposite is true for Japan yields, partly because of the extremely low rates of yields with little movement. Such incapability of the independent model is amplified in matching the cross-correlations among the yields. Some of the implied cross-correlations have the opposite sign also. Therefore, although the QTSM has the innate improved capability of capturing various time-series relationships between yield data, our result shows that the model which does not allow a flexible relationship among the state variables may not be too useful to accurately capturing the yield behavior in the international/common vs local framework. Thus our estimation result adds to the substantial evidence in the classical estimation literature (for affine models mostly) that we need the factor correlation to adequately describe the behavior of yield data. It is also consistent with the result in (Ahn, Dittmar, and Gallant (2002)), who shows that the independent model severely underperforms for the US yields than the correlated QTSM. It is interesting to note that although the yield series are highly persistent, the auto-correlations of the state variable processes are not as severe. Since the parameter $\kappa$ affects the unconditional variance of the state variables while governing the mean reverting tendency, it seems to be the case with the independent QTSM is not capable of completely matching the two different kinds of second moments at once.

For the estimation of the correlated QTSM, we use the parameter estimates obtained from the independent model. The Bayesian estimation technique adopted for this study, Metropolis within Gibbs, proves more robust to the different starting values than the classical estimation although sometimes the simulations fail to cover the entire support of the parameter space. As with the classical estimation, it helps to assign reasonable set of initial values to avoid the potential problem while sustaining the simulation under control. We provide the
estimation result in Table 3.6. After the burn-in periods\textsuperscript{15} we first generate the unobserved state variables using extended Kalman filtering. After the unobserved state variables are simulated using Metropolis-Hastings, we generate two sets of parameters separately according to each parameter's relevance to the physical measure and the equivalent martingale measure. Then, based on the factor estimates and the corresponding set of hyperparameters we calculate the implied term structure at each given time. The model-implied yields series shown in the figures 3.44 to 3.7 are obtained by repeating the process 50,000 times.

Our parameterizations for estimating the correlated QTSM is to set the matrices for mean reversion and prices of risks equal to the lower triangular matrix. Although the independent model is parameterized based on Ahn, Dittmar, and Gallant (2002), the estimation behavior of correlated model is not as good with the suggested parameterizations for the international model with both common and local factors. We, therefore pin down the first diagonal element of $\Psi^f$ equal to one, which improves the estimation behavior substantially. While the volatilities of Japan yields are higher than those of US, it is noteworthy that the conditional variance of the foreign local state variable is significantly lower than that of the domestic local state variable. This observation is due to the mean reversion parameter of foreign local variable, which tends to compensate the relatively mitigated conditional factor volatility. Our estimation strategy is to generate each of the hyperparameters after classifying them into parameters that are subject to physical measure and to risk-neutral measure. The Table 3.6, therefore, reports both the generated parameter estimates and the implied market prices of risks. The numbers in parenthesis is the numerical standard error of each parameter. Table 3.7 presents various moments that our correlated model dictates.

The improvements from the independent QTSM are evident:

1. The implied mean yields leave little differences from the actual yields.

2. Most of the second moments for each series are fairly close to the data counterpart.

The sign and directional tendency are all consistent with the data.

\textsuperscript{15}the burn-in periods vary according to the results obtained from previous runs. They fall mostly between 50,000 and 200,000
3. All cross-correlations implied by the model are in the matching signs and their magnitudes are also line with the data.

Note the only exception to the analysis stated above is autocorrelation between the US yields. We conjecture that it is mainly due to the near nonstationary behavior of both Japanese yields.

Figure 3.3 shows that the implied yield series match the actual yield behavior fairly well. We also provide in Figures 3.4 to 3.7, which illustrate the convergence behavior of parameter estimates in terms of various moments of yield series implied by the parameters generated in each run. The figures all show that all the model-implied yields converge to the limiting values in a fairly stable manner.

3.6 Exchange Rate and QTSM

The tendency for high interest currency to appreciate, known as the forward premium anomaly, has been one of the most puzzling features of currency prices (Backus, Foresi, and Telmer (2001)). Numerous studies have attempted to produce this prominently persistent feature of the currency market with little success.

Roughly speaking, there have been three different approaches to this persistent feature of the foreign exchange market: 1) Traditional asset pricing model-based approach (Frankel and Engel (1984), Bekaert, Hodrick, and Marshall (1997)) 2) Statistical approach (Hansen and Hodrick (1983), Domowitz and Hakkio (1985)), and 3) Term structure model-based approach (Nielsen and Saá-Requejo (1993), (Saá-Requejo (1993), Backus, Foresi, and Telmer (2001), Ahn (2004), Brennan and Xia (2006)).

Our study in the section examines whether the quadratic term structure model which incorporates the local state variables in addition to the common state variables is capable of generating the anomalous feature of the currency market.
Fama (1984), among others, provides one of the most insightful analyses of the puzzle, connecting the anomalous behavior of foreign exchange rate to *time-varying risk premium*. The famous *Fama decomposition* suggests that any asset pricing model that attempts to produce the behavior consistent with the foreign exchange data should necessarily be able to meet the following two conditions:

- the implied risk premium on currency must be negatively correlated with its expected rate of depreciation.
- it must have greater variance.

Note that the conditions stated above implies that a model with constant risk premium specification can not generate the forward premium anomaly. Backus, Foresi, and Telmer (2001) shows in the affine yield model framework that the necessary conditions provided by Fama (1984) impose such strong restrictions on the structure and parameter values of affine models that they have difficulty accounting for the anomaly. Specifically, such models should either allow for some positive probability of negative interest rates or for asymmetric effects of state prices on interest rates of different countries. And they also show restrictions imposed by either way bring significant shortcomings into the model itself.

More recently, Brennan and Xia (2006) examine the issue using a panel of exchange rates of ten countries whose capital markets are assumed to be integrated.\(^\text{16}\) Despite the success in retrieving Fama's (1984) necessary conditions, substantial evidence for forward premium puzzle still remains to be resolved, they report. So our task in this section is to see how well the US-Japan currency price behavior is captured by our model that we estimated in the previous section.

\(^\text{16}\)the underlying state variables in their study can be viewed as *hybrid* of more traditional latent factors and observed macro variables. Their judicious choice of term structure variables is, in part, attributable to their success in retrieving, though marginally, Fama's (1984) necessary conditions.
3.6.1 Exchange Rate Dynamics and the Forward Premium

Our modeling strategy is in line with previous research that derives currency price behavior from the term structure model they employ. In this section we derive the stochastic process of exchange rate in general terms and show what specification our IQTSM (International QTSM) implies as to the conditional mean and volatility of the exchange rate movement.

In the absence of arbitrage there exists a pricing kernel for any numeraire that prices all asset payoffs in terms of that numeraire. Based on the pricing kernel defined in section 3.3, the following relation holds at any given time $t$, and for any maturity $\tau$:

$$M_f^t = E_t(M_f^{t+\tau} x_{t+\tau}^f)$$

(3.6.1)

$$M_f^t = E_t(M_f^{t+\tau} x_{t+\tau}^f),$$

(3.6.2)

where $x_{t+\tau}$ is the gross return realized between time $t$ and $t + \tau$.

If the return on an asset of the country $i$ can be freely converted to the purchasing power of another country $j$, then the following relation also holds from the definition of a pricing kernel:

$$M_i^t = E_t \left[ M_i^{t+\tau} x_{t+\tau}^i \frac{S_{t+\tau}^j}{S_t^i} \right],$$

(3.6.3)

where $S_t$ is the currency $i$ price of currency $j$.

Defining $M_f^t$ and $M_i^t$ respectively as the minimum-variance pricing kernel for domestic and foreign countries without resorting to the complete market assumption, we obtain the following result:

$$M_f^t = M_i^t S_t$$

(3.6.4)

For more detail see Saá-Requejo (1993). Derivation in discrete time can be found in Backus, Foresi, and Telmer (2001).
The relation in equation (3.6.4) implies that

\[ d \ln S_t = d \ln \left( \frac{M_t}{M_{t-1}} \right) \]

\[ = \left[ (r^{d}_t - r^{f}_t) + 0.5 \left( \Lambda^{d}_t \Lambda^{d}_t - \Lambda^{f}_t \Lambda^{f}_t \right) \right] dt + \left[ \Lambda^{d}_t dW^{d}_t - \Lambda^{f}_t dW^{f}_t \right] \tag{3.6.6} \]

Based on the stochastic process for the logarithm of exchange rate stated above we derive the expressions for exchange rate risk premium and the expected change in the spot exchange rate, denoted as \( r_{pt} \) and \( q_t \), respectively, as follows:

**Lemma**

\[ r_{pt} = -\frac{1}{2} \left[ (\Lambda^{d}_0 \Lambda^{d}_0 - \Lambda^{d}_0 \Lambda^{f}_0) + 2(\psi^{d}_t \Lambda^{d}_1 \Lambda^{d}_0 - \psi^{f}_t \Lambda^{f}_1 \Lambda^{f}_0) \right] \]

\[ q_t = \left[ (\alpha^{d}_d - \alpha^{f}_d) + ((\beta^{d}_d + \lambda^{d}_0 \lambda^{d}_1) \psi^{d}_t - (\beta^{f}_d + \lambda^{f}_0 \lambda^{f}_1) \psi^{f}_t) \right] \]

\[ + (\psi^{d}_t \psi^{d}_t - \psi^{f}_t \psi^{f}_t) \]

\[ + (\psi^{d}_t \lambda^{d}_1 \lambda^{d}_1 - \psi^{f}_t \lambda^{f}_1 \lambda^{f}_1) \tag{3.6.8} \]

Note that both the exchange rate premium and the expected spot rate change are determined as quadratic functions of state variables and prices of risk parameters. Thus our IQTSM ensures a richer characterization of foreign exchange risk premium and the average spot rate movement.

### 3.6.1.1 Implications on Forward Premium Puzzle and Robustness

In this section we investigate what implications our international quadratic term structure model bears on the forward premium puzzle.

If our model adequately specifies the foreign exchange risk premium, then the residual variations in spot exchange rate which the forward premium captures in excess of the effect explained by incorporating the state variables from our risk premium specification into the
forward premium regression will have the coefficient equal to one.

We first run the standard forward premium regression to ensure the stylized fact documented in the existing literature as follows\(^{18}\):

\[
\Delta S_t = \ln S_{t+\Delta t} - \ln S_t = \alpha_0 + \alpha_1 (\ln F_t - \ln S_t) + \varepsilon_{t+\Delta t} \tag{3.6.9}
\]

Table 3.11 presents the result, which shows that the data used in this study show the forward premium anomaly. The Japan-US exchange rate regression produces the coefficient of forward premium equal to -1.46 with adjusted \(R^2\) less than 1%.

We next include in the regression the additional variables that affect the exchange risk premium to see how the newly incorporated variables change the effect of the forward premium variable on the spot rate movement. The following equation is estimated for that purpose\(^{19}\):

A. Mean Equation:

\[
\Delta S_t = \ln S_{t+\Delta t} - \ln S_t = c_0 + c_1 (\ln F_t - \ln S_t) + c_2 x_{t}^c + c_3 x_{t}^{dl} + c_4 x_{t}^{fl} + c_5 x_{t}^{c} x_{t}^{dl} + c_6 x_{t}^{c} x_{t}^{fl} + c_7 c_{t}^2 + c_8 x_{t}^{dl^2} + c_9 x_{t}^{fl^2} + \varepsilon_{t+\Delta t}. \tag{3.6.10}
\]

B. Variance Equation:

\[
\sigma_{s,t}^2 = c_{11} + c_{12} \sigma_{s,t-1}^2 + c_{13} \sigma_{x,t-1}^2 + c_{14} x_{t}^c + c_{15} x_{t}^{dl} + c_{16} x_{t}^{fl} + c_{17} x_{t}^{c} x_{t}^{dl} + c_{18} x_{t}^{c} x_{t}^{fl} + c_{19} x_{t}^{dl^2} + c_{20} x_{t}^{fl^2} + \varepsilon_{t+\Delta t}. \tag{3.6.11}
\]

Table 3.12(Panel A) shows the additional variables in the regression equation pull the

\(^{18}\)The standard errors are calculated using Newey-West HAC standard errors truncated at lag 5.

\(^{19}\)The standard errors are calculated using Bollerslev-Wooldrige robust standard errors truncated at lag 5.
coefficient of forward premium term \((c_1)\) toward unity\(^{20}\), the value which is implied by the risk premium literature. This result suggests that the risk premium specification derived in our IQTSM captures the risk premium component in the spot exchange movement better than the model put forth in their work.

Note that we also adopt the GARCH(1,1) characterization for the conditional volatility of spot rate change because we want to examine whether the additional conditional volatility induced by our model would fully explain the volatility of spot rate. Panel B of Table 3.12 shows that the lagged conditional variance term is still significant\(^{21}\).

Lastly, we look into Fama's decomposition suggested by our model estimated in the previous section. The Table 3.13 indicates that the risk premium and the expected depreciation rate is almost perfectly negatively correlated. However the standard deviation of the former (0.4226 \%) is slightly smaller than that of the latter (0.4352\%). Thus the requirement suggested by Fama (1984) is marginally not satisfied. Note the extremely close proximity between our sample statistics and the benchmark results in Brennan and Xia (2006). This is quite an interesting and significant result in the sense that the two studies are different in almost every aspect of modeling as well as the estimation techniques adopted albeit they use the same data set. Furthermore, compared to the existing literature in this area, our model is much more parsimonious in terms of the number of state variables and data series that are used to account for the behavior of the yields and exchange rate. Still our model is shown to perform as well as the front-line research with the same concern. This result is particularly meaningful when we consider that we have witnessed from the past research very little success to adequately explain the behavior of Japan yields and corresponding exchange, which is largely due to the externality such as government intervention and deflation. Therefore, it will be interesting to explore whether a more generalized version

\(^{20}\)When the additional exchange risk premium variables are included in the forward premium regression, the coefficient of forward premium term changed from -1.46 (\(c_1\) in Table 3.11) to -0.29 (\(c_1\) in Table 3.12, whereas the corresponding coefficient deteriorates from -1.46 to -2.216 (Table 10 in Brennan and Xia (2006))

\(^{21}\)However, the regression result from the standard GARCH specification shows that the US-Japan exchange rate doesn’t show the GARCH behavior during the period studied in this paper, which is unusual
of our model examined in this paper can produce the results that better explain what we observe in the other foreign exchange markets than the existing literature. That will be the next subject of our future research in this line.
3.7 Conclusion

We have investigated in this paper whether the quadratic term structure model postulated in international context adequately captures the US and Japan yield behavior. Also we have examined the implications of the model on the long-lasting puzzle in foreign exchange market. Our model is unique in that it explicitly incorporates additional state variables, called local factors, which exclusively affect bond payoffs of each country. Our study is the first attempt to propose the IQTSM (international Quadratic Term Structure Model) with both common and local factors as an alternative to the existing term structure models to better account for the behavior of bond yields and foreign exchange rates. In spite of much more parsimonious structure our model produced the results compatible with previous studies. The implications on forward premium anomaly was also derived endogenously from the estimated model and compared with the results from the benchmark model. The behavior of risk premium salient in the US-Japan exchange rate data was captured nearly as well with our proposed model as with the term structure model employed in the current research in front line.

It will be interesting and meaningful, therefore, to expand our model such that additional state variables and data series for other countries are employed so as to look into whether the IQTSM is consistently a better alternative for describing the yield/exchange rate dynamics being observed in the international markets.
Table 3.1: Summary Statistics (Zero-Coupon Bond Yield)

The monthly observations for U.S. and Japanese Treasury rates. The rates are annualized middle quotes, estimated zero coupon Treasury bond yields for the United States and Japan from January 1985 to May 2002 on a monthly basis. Based on the data on bond prices, coupon rates, and redemption dates a cubic spline is fitted to the yields of each country. No extrapolation is used in the estimation. I greatly appreciate the generous provision of this data set by Yihong Xia.

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<th>6 month</th>
<th>5 year</th>
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<td>Japan</td>
<td>US</td>
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<tr>
<td>Std. Dev. (% per year)</td>
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</tr>
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<td>Autocorrelation</td>
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<td>Japan(6 mo.)</td>
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<td>1</td>
</tr>
<tr>
<td>US(1 yr.)</td>
<td>0.97</td>
<td>0.37</td>
</tr>
<tr>
<td>US(1 yr.)</td>
<td>0.72</td>
<td>0.73</td>
</tr>
</tbody>
</table>
Table 3.2: Parameter Estimates of QTSMs

The results of specification tests with the score generator \{1, 0, 4, 1, 4, 3, 0, 0\} are presented. The model with the independent local factors as well as common factors serves the benchmark case. The estimation results obtained from this model are used as starting values for the correlated common-local factor setting. \(Z\) statistic for the QTSM with independent common/local factor strongly suggests a rejection of the restrictions imposed on the specification of the prices of risk, which renders the need to look into the QTSM whose unobserved state variables are correlated.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Estimates (S.E.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Independent</td>
</tr>
<tr>
<td>(\alpha_d)</td>
<td>0.0761 (0.0161)</td>
</tr>
<tr>
<td>(\alpha_f)</td>
<td>0.0394 (0.0168)</td>
</tr>
<tr>
<td>(\beta)</td>
<td>0.1064 (0.0709)</td>
</tr>
<tr>
<td>(\pi)</td>
<td>1.1744 (0.0751)</td>
</tr>
<tr>
<td>(\kappa_x)</td>
<td>0.9510 (1.029)</td>
</tr>
<tr>
<td>(\kappa_y)</td>
<td>0.6276 (1.286)</td>
</tr>
<tr>
<td>(\kappa_z)</td>
<td>0.2782 (0.0187)</td>
</tr>
<tr>
<td>(\theta_x)</td>
<td>0.0107 (0.0806)</td>
</tr>
<tr>
<td>(\theta_y)</td>
<td>0.0084 (0.210)</td>
</tr>
<tr>
<td>(\sigma_x)</td>
<td>0.0369 (0.0076)</td>
</tr>
<tr>
<td>(\sigma_y)</td>
<td>0.0049 (0.0552)</td>
</tr>
<tr>
<td>(\sigma_z)</td>
<td>0.0020 (0.327)</td>
</tr>
<tr>
<td>(\sigma_u)</td>
<td>0.0060 (0.873)</td>
</tr>
<tr>
<td>(\lambda_{x0}^{d})</td>
<td>0.0078 (0.012)</td>
</tr>
<tr>
<td>(\lambda_{x0}^{f})</td>
<td>0.0014 (0.020)</td>
</tr>
<tr>
<td>(\lambda_{y0})</td>
<td>0.0104 (0.146)</td>
</tr>
<tr>
<td>(\lambda_{z0})</td>
<td>0.0119 (0.327)</td>
</tr>
<tr>
<td>(\lambda_{x1}^{d})</td>
<td>-0.0040 (0.009)</td>
</tr>
<tr>
<td>(\lambda_{x1}^{f})</td>
<td>-0.0030 (0.008)</td>
</tr>
<tr>
<td>(\lambda_{y1})</td>
<td>0.0521 (0.146)</td>
</tr>
<tr>
<td>(\lambda_{z1})</td>
<td>0.0091 (0.327)</td>
</tr>
<tr>
<td>(\sigma_{\epsilon_1}^{d})</td>
<td>0.1568 (0.228)</td>
</tr>
<tr>
<td>(\sigma_{\epsilon_2}^{d})</td>
<td>0.2915 (0.426)</td>
</tr>
<tr>
<td>(\sigma_{\epsilon_1}^{f})</td>
<td>0.0897 (0.130)</td>
</tr>
<tr>
<td>(\sigma_{\epsilon_2}^{f})</td>
<td>0.0937 (0.135)</td>
</tr>
<tr>
<td>(\chi^2)</td>
<td>625.44 (625.44)</td>
</tr>
<tr>
<td>(df)</td>
<td>35</td>
</tr>
<tr>
<td>(z)</td>
<td>70.57</td>
</tr>
</tbody>
</table>

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Table 3.3: Diagnostics of QTSM: 10414300 Score

<table>
<thead>
<tr>
<th>SNP coefficient</th>
<th>Score</th>
<th>Standard error</th>
<th>t-statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>A(2)</td>
<td>-2.55169</td>
<td>3.28875</td>
<td>-0.776</td>
</tr>
<tr>
<td>A(3)</td>
<td>-15.86003</td>
<td>3.34325</td>
<td>-4.744</td>
</tr>
<tr>
<td>A(4)</td>
<td>18.957</td>
<td>3.35279</td>
<td>5.654</td>
</tr>
<tr>
<td>A(5)</td>
<td>-10.97868</td>
<td>3.62378</td>
<td>-3.03</td>
</tr>
<tr>
<td>A(6)</td>
<td>31.25142</td>
<td>4.9736</td>
<td>6.283</td>
</tr>
<tr>
<td>A(7)</td>
<td>10.38827</td>
<td>5.25118</td>
<td>1.978</td>
</tr>
<tr>
<td>A(8)</td>
<td>43.04299</td>
<td>5.44486</td>
<td>7.905</td>
</tr>
<tr>
<td>A(9)</td>
<td>20.00016</td>
<td>6.55251</td>
<td>3.052</td>
</tr>
<tr>
<td>A(10)</td>
<td>11.17311</td>
<td>12.40719</td>
<td>0.901</td>
</tr>
<tr>
<td>A(11)</td>
<td>-48.60113</td>
<td>12.76587</td>
<td>-3.807</td>
</tr>
<tr>
<td>A(12)</td>
<td>21.82816</td>
<td>12.56179</td>
<td>1.738</td>
</tr>
<tr>
<td>A(13)</td>
<td>-56.36729</td>
<td>15.31816</td>
<td>-3.68</td>
</tr>
<tr>
<td>A(14)</td>
<td>207.6218</td>
<td>30.51562</td>
<td>6.804</td>
</tr>
<tr>
<td>A(15)</td>
<td>79.55903</td>
<td>31.04763</td>
<td>2.562</td>
</tr>
<tr>
<td>A(16)</td>
<td>189.36569</td>
<td>30.01239</td>
<td>6.31</td>
</tr>
<tr>
<td>A(17)</td>
<td>54.59076</td>
<td>36.68689</td>
<td>1.488</td>
</tr>
<tr>
<td>ψ(1)</td>
<td>-105.93443</td>
<td>47.26016</td>
<td>-2.242</td>
</tr>
<tr>
<td>ψ(2)</td>
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<td>54.05214</td>
<td>-9.259</td>
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<tr>
<td>ψ(3)</td>
<td>-50.20935</td>
<td>36.50773</td>
<td>-1.375</td>
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<td>ψ(4)</td>
<td>205.23247</td>
<td>35.68009</td>
<td>5.752</td>
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<td>ψ(5)</td>
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<td>26.83509</td>
<td>35.2419</td>
<td>0.761</td>
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<td>-67.41032</td>
<td>35.8238</td>
<td>-1.882</td>
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<td>ψ(9)</td>
<td>29.92186</td>
<td>42.47339</td>
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<td>78.91425</td>
<td>47.71954</td>
<td>1.654</td>
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<td>11.60407</td>
<td>30.27605</td>
<td>0.383</td>
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<tr>
<td>ψ(12)</td>
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<td>31.82973</td>
<td>-0.742</td>
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<tr>
<td>ψ(13)</td>
<td>186.92263</td>
<td>31.82973</td>
<td>3.201</td>
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<td>ψ(15)</td>
<td>42.55448</td>
<td>35.58872</td>
<td>1.196</td>
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<tr>
<td>ψ(16)</td>
<td>-73.10907</td>
<td>37.30828</td>
<td>-1.96</td>
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<tr>
<td>ψ(17)</td>
<td>42.20209</td>
<td>43.62304</td>
<td>0.967</td>
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<tr>
<td>ψ(18)</td>
<td>114.45363</td>
<td>49.56344</td>
<td>2.309</td>
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<td>ψ(19)</td>
<td>26.01021</td>
<td>31.61939</td>
<td>0.823</td>
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<tr>
<td>ψ(20)</td>
<td>-29.6578</td>
<td>33.18677</td>
<td>-0.894</td>
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</table>

To be continued on the next page
<table>
<thead>
<tr>
<th>SNP coefficient</th>
<th>Score</th>
<th>Standard error</th>
<th>t-statistic</th>
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</thead>
<tbody>
<tr>
<td>$\tau(1)$</td>
<td>507.467</td>
<td>55.122</td>
<td>9.206</td>
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<td>$\tau(2)$</td>
<td>125.449</td>
<td>29.153</td>
<td>4.303</td>
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<td>$\tau(3)$</td>
<td>-605.320</td>
<td>58.705</td>
<td>-10.311</td>
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<tr>
<td>$\tau(4)$</td>
<td>152.597</td>
<td>42.507</td>
<td>3.59</td>
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<tr>
<td>$\tau(5)$</td>
<td>111.546</td>
<td>43.681</td>
<td>2.554</td>
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<tr>
<td>$\tau(6)$</td>
<td>0.598</td>
<td>43.624</td>
<td>0.014</td>
</tr>
<tr>
<td>$\tau(7)$</td>
<td>-192.420</td>
<td>34.534</td>
<td>-5.572</td>
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<tr>
<td>$\tau(8)$</td>
<td>-7.523</td>
<td>37.693</td>
<td>-0.2</td>
</tr>
<tr>
<td>$\tau(9)$</td>
<td>69.469</td>
<td>35.218</td>
<td>1.973</td>
</tr>
<tr>
<td>$\tau(10)$</td>
<td>11.614</td>
<td>1.586</td>
<td>7.321</td>
</tr>
<tr>
<td>$\tau(11)$</td>
<td>-14.351</td>
<td>2.244</td>
<td>-6.395</td>
</tr>
<tr>
<td>$\tau(23)$</td>
<td>0.662</td>
<td>1.707</td>
<td>0.389</td>
</tr>
<tr>
<td>$\tau(36)$</td>
<td>3.567</td>
<td>2.398</td>
<td>1.404</td>
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<tr>
<td>$\tau(50)$</td>
<td>7.511</td>
<td>1.696</td>
<td>4.426</td>
</tr>
<tr>
<td>$\tau(63)$</td>
<td>-9.393</td>
<td>1.533</td>
<td>-6.127</td>
</tr>
<tr>
<td>$\tau(76)$</td>
<td>0.744</td>
<td>1.441</td>
<td>0.514</td>
</tr>
<tr>
<td>$\tau(90)$</td>
<td>3.400</td>
<td>2.058</td>
<td>1.652</td>
</tr>
<tr>
<td>$\tau(91)$</td>
<td>9.376</td>
<td>1.438</td>
<td>6.52</td>
</tr>
<tr>
<td>$\tau(103)$</td>
<td>-15.138</td>
<td>1.772</td>
<td>-8.541</td>
</tr>
<tr>
<td>$\tau(116)$</td>
<td>1.887</td>
<td>1.286</td>
<td>1.467</td>
</tr>
<tr>
<td>$\tau(130)$</td>
<td>7.181</td>
<td>1.959</td>
<td>3.657</td>
</tr>
<tr>
<td>$\tau(131)$</td>
<td>11.025</td>
<td>1.527</td>
<td>7.219</td>
</tr>
<tr>
<td>$\tau(143)$</td>
<td>-15.608</td>
<td>1.601</td>
<td>-9.747</td>
</tr>
<tr>
<td>$\tau(156)$</td>
<td>3.871</td>
<td>1.600</td>
<td>2.419</td>
</tr>
<tr>
<td>$\tau(170)$</td>
<td>8.323</td>
<td>2.168</td>
<td>3.857</td>
</tr>
</tbody>
</table>
Table 3.4: Parameter Estimates (Independent 3 Factor IQTSM)

For each complete iteration, the parameters are generated using Metropolis-within-Gibbs based on simulated unobserved state variables. All the parameter estimates are calculated as the sample averages of 50,000 iterations after 100,000 burn-in period.

<table>
<thead>
<tr>
<th>State variable process</th>
<th>Parameter Estimate 1</th>
<th>Parameter Estimate 2</th>
<th>Parameter Estimate 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{11}$</td>
<td>1.504</td>
<td>$\theta_1$ 0.01574</td>
<td>$\sigma_1^2$ 0.0564</td>
</tr>
<tr>
<td></td>
<td>(0.4813, 2.6345)</td>
<td>(-0.1053, 0.1145)</td>
<td>(0.0477, 0.0654)</td>
</tr>
<tr>
<td>$k_{22}$</td>
<td>0.9459</td>
<td>$\theta_2$ -0.1853</td>
<td>$\sigma_2^2$ 0.0060</td>
</tr>
<tr>
<td></td>
<td>(0.2043, 1.7037)</td>
<td>(-0.2662, -0.1312)</td>
<td>(0.0042, 0.0086)</td>
</tr>
<tr>
<td>$k_{33}$</td>
<td>1.46</td>
<td>$\theta_3$ 0.1277</td>
<td>$\sigma_3^2$ 0.0682</td>
</tr>
<tr>
<td></td>
<td>(0.7534, 2.1944)</td>
<td>(0.0252, 0.2212)</td>
<td>(0.0616, 0.0754)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Short rate equation</th>
<th>Parameter Estimate 1</th>
<th>Parameter Estimate 2</th>
<th>Parameter Estimate 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_d$</td>
<td>0.7846</td>
<td>$\pi$ -1.3009</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0281, 2.5246)</td>
<td>(-1.4197, -1.0787)</td>
<td></td>
</tr>
<tr>
<td>$\alpha_f$</td>
<td>0.7846</td>
<td>$\beta$ 0.613</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0079, 0.9290)</td>
<td>(0.0188, 2.5583)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Market price of risk</th>
<th>Parameter Estimate 1</th>
<th>Parameter Estimate 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{01}$</td>
<td>0.0042</td>
<td>$L_{11}$ 0.5313</td>
</tr>
<tr>
<td></td>
<td>(0.011035, 0.0302636)</td>
<td>(0.4526, 0.6226)</td>
</tr>
<tr>
<td>$L_{02}$</td>
<td>0.0004</td>
<td>$L_{22}$ 1.7509</td>
</tr>
<tr>
<td></td>
<td>(-0.0319, -0.03002134)</td>
<td>(1.7040, 1.7983)</td>
</tr>
<tr>
<td>$L_{03}$</td>
<td>0.0000</td>
<td>$L_{33}$ 3.0391</td>
</tr>
<tr>
<td></td>
<td>(-0.01524, 0.0130)</td>
<td>(2.9352, 3.0992)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Measurement error</th>
<th>Parameter Estimate 1</th>
<th>Parameter Estimate 2</th>
<th>Parameter Estimate 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_1$</td>
<td>0.1532</td>
<td>$\varepsilon_2$ 0.2287</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.1144, 0.2028)</td>
<td>(0.1298, 0.3519)</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon_3$</td>
<td>0.1658</td>
<td>$\varepsilon_4$ 0.0381</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.1291, 0.2120)</td>
<td>(0.0194, 0.0634)</td>
<td></td>
</tr>
</tbody>
</table>

Measurement error: $\times 10^{-3}$

$\alpha_d$ and $\alpha_f$: $\times 10^{-3}$

$\beta$: $\times 10^{-4}$
Table 3.5: Model Implied Moments (Independent States Variables)

For each complete iteration, we simulate the unobserved state variables using Extended Kalman Filter and Metropolis-Hastings algorithm. The model-implied moments are then calculated based on the state variables and hyperparameters generated at each iteration. All the moments presented in the table are reported as the sample averages of 50,000 series of each moment after 100,000 burn-in period.

<table>
<thead>
<tr>
<th></th>
<th>6 month</th>
<th></th>
<th>5 year</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Japan</td>
<td>US</td>
<td>Japan</td>
<td>US</td>
</tr>
<tr>
<td>Mean (% per year)</td>
<td>4.26</td>
<td>6.09</td>
<td>4.08</td>
<td>6.66</td>
</tr>
<tr>
<td>Std. Dev. (% per year)</td>
<td>3.08</td>
<td>2.26</td>
<td>1.87</td>
<td>1.28</td>
</tr>
<tr>
<td>Autocorrelation</td>
<td>0.8864</td>
<td>0.7532</td>
<td>0.9272</td>
<td>0.8643</td>
</tr>
<tr>
<td>Correlation</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Japan(6 mo.)</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>US(6 mo.)</td>
<td>-0.2099</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Japan(1 yr.)</td>
<td>0.8932</td>
<td>-0.1959</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>US(1 yr.)</td>
<td>-0.0121</td>
<td>0.3813</td>
<td>0.0599</td>
<td>1</td>
</tr>
</tbody>
</table>

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Table 3.6: Parameter Estimates (Correlated 3 Factor IQTSM)

For each complete iteration, the parameters are generated using Metropolis-within-Gibbs based on simulated unobserved state variables. All the parameter estimates are calculated as the sample averages of 50,000 iterations after 100,000 burn-in period.

<table>
<thead>
<tr>
<th>state variable process</th>
<th>$k_{11}$</th>
<th>$\theta_{1}$</th>
<th>$\sigma^2_{11}$</th>
<th>$k_{21}$</th>
<th>$\theta_{2}$</th>
<th>$\sigma^2_{22}$</th>
<th>$k_{22}$</th>
<th>$\theta_{3}$</th>
<th>$\sigma^2_{33}$</th>
</tr>
</thead>
<tbody>
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<td>0.5207</td>
<td>0.0393</td>
<td>0.0242</td>
<td>1.049</td>
<td>0.0136</td>
<td>1.288</td>
<td>6.357</td>
<td>0.0131</td>
<td>0.059</td>
</tr>
<tr>
<td></td>
<td>(0.1954)</td>
<td>(0.0517)</td>
<td>(0.0053)</td>
<td>(0.9259)</td>
<td>(0.0481)</td>
<td>(0.0462)</td>
<td>(0.8422)</td>
<td>(0.1171)</td>
<td>(0.0255)</td>
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<tr>
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<table>
<thead>
<tr>
<th>short rate equation</th>
<th>$\alpha_d$</th>
<th>$\alpha_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0187</td>
<td>0.0003076</td>
</tr>
<tr>
<td></td>
<td>(0.00278)</td>
<td>(0.0001678)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>measurement error</th>
<th>$\varepsilon_1$</th>
<th>$\varepsilon_2$</th>
<th>$\varepsilon_3$</th>
<th>$\varepsilon_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.7349</td>
<td>1.808</td>
<td>1.341</td>
<td>1.915</td>
</tr>
<tr>
<td></td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

Measurement error: $10^{-4}$
<table>
<thead>
<tr>
<th>market price of risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{11}'$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$L_{21}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$L_{22}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$L_{31}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$L_{33}$</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Yield Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{1}^{Qd}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$\mu_{2}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$\mu_{3}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$\mu_{1}^{Qf}$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$\kappa_{33}^{Q}$</td>
</tr>
<tr>
<td>$\kappa_{11}^{Qf}$</td>
</tr>
</tbody>
</table>

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Table 3.7: Model Implied Moments (Correlated States Variables)

For each complete iteration, we simulate the unobserved state variables using Extended Kalman Filter and Metropolis-Hastings algorithm. The model-implied moments are then calculated based on the state variables and hyperparameters generated at each iteration. All the moments presented in the table are reported as the sample averages of 50,000 series of each moment after 100,000 burn-in period.

<table>
<thead>
<tr>
<th></th>
<th>6 month</th>
<th>5 year</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Japan</td>
<td>US</td>
</tr>
<tr>
<td>Mean (% per year)</td>
<td>3.0197</td>
<td>6.2988</td>
</tr>
<tr>
<td>Std. Dev. (% per year)</td>
<td>2.1927</td>
<td>1.2765</td>
</tr>
<tr>
<td>Autocorrelation</td>
<td>0.9381</td>
<td>0.6940</td>
</tr>
<tr>
<td>Correlation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Japan(6 mo.)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>US(6 mo.)</td>
<td>0.5674</td>
<td>1</td>
</tr>
<tr>
<td>Japan(5 yr.)</td>
<td>0.8917</td>
<td>0.6842</td>
</tr>
<tr>
<td>US(5 yr.)</td>
<td>0.6627</td>
<td>0.9047</td>
</tr>
</tbody>
</table>
Table 3.8: Summary Statistics: Foreign Exchange Rate and Forward Premium

- Spot rate change and forward premium are calculated as $100 \times \ln(S_{t+\Delta t}/S_t)$ and $100 \times \ln(S_{t+\Delta t}/S_t)$, respectively. $\Delta$ is set equal to 1 month. $S_t$ is expressed as dollar price of Japanese Yen.

- Forward and spot exchange rates are obtained from Datastream. The sample period is from January 1985 to May 2002 (209 monthly observations for each series). All the rates are the values of the second day of each month.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Autocorrelation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Change in Spot Rate</td>
<td>0.3167</td>
<td>3.67</td>
<td>0.032</td>
</tr>
<tr>
<td>Forward Premium</td>
<td>0.2424</td>
<td>0.2625</td>
<td>0.494</td>
</tr>
</tbody>
</table>
Table 3.9: Principal Component Analysis (Benchmark Affine Model)

- For Panel A: US yields with maturities 6 mo., 1 yr., 2 yr., 3 yr., 5 yr., 7 yr., and 10 yr are used.
- For Panel B: Japan yields with maturities 6 mo., 1 yr., 2 yr., 3 yr., 5 yr., and 7 yr. are used.
- For panel C: All US and Japan yields are used.

<table>
<thead>
<tr>
<th>Panel A: US Yields</th>
<th>comp 1</th>
<th>comp 2</th>
<th>comp 3</th>
<th>comp 4</th>
<th>comp 5</th>
<th>comp 6</th>
<th>comp 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalue</td>
<td>6.0306</td>
<td>0.5333</td>
<td>0.4280</td>
<td>0.0080</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>Variance Prop.</td>
<td>0.8615</td>
<td>0.0762</td>
<td>0.0611</td>
<td>0.0011</td>
<td>0.0001</td>
<td>0.0000</td>
<td></td>
</tr>
<tr>
<td>Cumulative Prop.</td>
<td>0.8615</td>
<td>0.9377</td>
<td>0.9988</td>
<td>1.0000</td>
<td>1.0000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Japan Yields</th>
<th>comp 1</th>
<th>comp 2</th>
<th>comp 3</th>
<th>comp 4</th>
<th>comp 5</th>
<th>comp 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalue</td>
<td>5.9176</td>
<td>0.0642</td>
<td>0.0171</td>
<td>0.0010</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>Variance Prop.</td>
<td>0.9863</td>
<td>0.0107</td>
<td>0.0029</td>
<td>0.0002</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>Cumulative Prop.</td>
<td>0.986</td>
<td>0.9970</td>
<td>0.9998</td>
<td>1.0000</td>
<td>1.0000</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C: US/Japan Yields</th>
<th>comp 1</th>
<th>comp 2</th>
<th>comp 3</th>
<th>comp 4</th>
<th>comp 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalue</td>
<td>9.8637</td>
<td>2.3102</td>
<td>0.5068</td>
<td>0.2615</td>
<td>0.0366</td>
</tr>
<tr>
<td>Variance Prop.</td>
<td>0.7588</td>
<td>0.1777</td>
<td>0.0390</td>
<td>0.0201</td>
<td>0.0028</td>
</tr>
<tr>
<td>Cumulative Prop.</td>
<td>0.7588</td>
<td>0.9365</td>
<td>0.9754</td>
<td>0.9956</td>
<td>0.9984</td>
</tr>
</tbody>
</table>
Table 3.10: Principal Component Analysis (IQTSM)

US and Japan yield series with maturities equal to 6 month and 5 years are used for the analysis. The data period is consistent with our benchmark model to facilitate the direct comparison.

<table>
<thead>
<tr>
<th></th>
<th>comp 1</th>
<th>comp 2</th>
<th>comp 3</th>
<th>comp 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalue</td>
<td>3.0095</td>
<td>0.8255</td>
<td>0.1485</td>
<td>0.0166</td>
</tr>
<tr>
<td>Variance Prop.</td>
<td>0.7524</td>
<td>0.2064</td>
<td>0.0371</td>
<td>0.0041</td>
</tr>
<tr>
<td>Cumulative Prop.</td>
<td>0.7524</td>
<td>0.9587</td>
<td>0.9959</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
Table 3.11: Forward Premium Regression: Benchmark

\[ \Delta S_t = \ln S_{t+\Delta t} - \ln S_t = \alpha_0 + \alpha_1 (\ln F_t - \ln S_t - t) + \varepsilon_{t+\Delta t} \]

The standard errors are calculated using Newey-West HAC standard errors truncated at lag 5.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
<th>(Adj.) $R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_0$</td>
<td>0.0067</td>
<td>0.0030</td>
<td>2.2289</td>
<td>0.027</td>
<td>0.0109</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>-1.4600</td>
<td>0.8963</td>
<td>-1.6289</td>
<td>0.1049</td>
<td>0.0061</td>
</tr>
</tbody>
</table>

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Table 3.12: Forward Premium Regression with Model-implied Risk Premium

For Panel A: \( \Delta S_t = \ln S_{t+\Delta t} - \ln S_t = c_0 + c_1 (\ln F_t - \ln S_t) + c_2 X_t^a + c_3 X_t^{dl} + c_4 X_t^{fl} + c_5 X_t^e X_t^{dl} + c_6 X_t^e X_t^{fl} + c_7 X_t^{a^2} + c_8 X_t^{dl^2} + c_9 X_t^{fl^2} + \epsilon_{t+\Delta t} \).

The standard errors are calculated using Bollerslev-Wooldridge robust standard errors truncated at lag 5.

For Panel B: \( \sigma^2_{x,t} = c_{11} + c_{12} \epsilon_{t-1}^2 + c_{13} \sigma^2_{x,t-1} + c_{14} X_t^a + c_{15} X_t^{dl} + c_{16} X_t^{fl} + c_{17} X_t^e X_t^{dl} + c_{18} X_t^e X_t^{fl} + c_{19} X_t^{a^2} + c_{20} X_t^{dl^2} + c_{21} X_t^{fl^2} + \epsilon_{t+\Delta t} \).

Panel A: Mean Equation

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob. (Adj.)</th>
<th>( R^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_0 )</td>
<td>-0.0033</td>
<td>0.0057</td>
<td>-0.5861</td>
<td>0.5578</td>
<td>0.0109</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>-0.2891</td>
<td>1.0912</td>
<td>-0.2650</td>
<td>0.7910</td>
<td>(-0.0511)</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>-0.0535</td>
<td>0.0629</td>
<td>-0.8517</td>
<td>0.3944</td>
<td>0.3944</td>
</tr>
<tr>
<td>( c_3 )</td>
<td>-0.0119</td>
<td>0.0569</td>
<td>-0.2099</td>
<td>0.8337</td>
<td>0.8337</td>
</tr>
<tr>
<td>( c_4 )</td>
<td>-0.0524</td>
<td>0.0610</td>
<td>-0.8603</td>
<td>0.3896</td>
<td>0.3896</td>
</tr>
<tr>
<td>( c_5 )</td>
<td>0.1512</td>
<td>0.2062</td>
<td>-0.7333</td>
<td>0.4634</td>
<td>0.4634</td>
</tr>
<tr>
<td>( c_6 )</td>
<td>0.0593</td>
<td>0.1665</td>
<td>-0.3563</td>
<td>0.7216</td>
<td>0.7216</td>
</tr>
<tr>
<td>( c_7 )</td>
<td>0.1953</td>
<td>0.1660</td>
<td>1.1766</td>
<td>0.2393</td>
<td>0.2393</td>
</tr>
<tr>
<td>( c_8 )</td>
<td>0.0211</td>
<td>0.0620</td>
<td>1.2411</td>
<td>0.2148</td>
<td>0.2148</td>
</tr>
<tr>
<td>( c_9 )</td>
<td>0.2220</td>
<td>0.2126</td>
<td>1.0443</td>
<td>0.2964</td>
<td>0.2964</td>
</tr>
</tbody>
</table>

Panel B: Variance Equation

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_{11} )</td>
<td>0.0011</td>
<td>0.0000</td>
<td>122.88</td>
<td>0.0000</td>
</tr>
<tr>
<td>( c_{12} )</td>
<td>-0.0581</td>
<td>0.0410</td>
<td>-1.4185</td>
<td>0.1560</td>
</tr>
<tr>
<td>( c_{13} )</td>
<td>0.5365</td>
<td>0.0908</td>
<td>5.9065</td>
<td>0.0000</td>
</tr>
<tr>
<td>( c_{14} )</td>
<td>-0.0004</td>
<td>0.0020</td>
<td>-0.1756</td>
<td>0.8606</td>
</tr>
<tr>
<td>( c_{15} )</td>
<td>0.0002</td>
<td>0.0018</td>
<td>0.0998</td>
<td>0.9205</td>
</tr>
<tr>
<td>( c_{16} )</td>
<td>-0.0004</td>
<td>0.0021</td>
<td>-0.1703</td>
<td>0.8648</td>
</tr>
<tr>
<td>( c_{17} )</td>
<td>-0.0009</td>
<td>0.0060</td>
<td>-0.1520</td>
<td>0.8792</td>
</tr>
<tr>
<td>( c_{18} )</td>
<td>0.0009</td>
<td>0.0056</td>
<td>0.1553</td>
<td>0.8766</td>
</tr>
<tr>
<td>( c_{19} )</td>
<td>-0.0017</td>
<td>0.0048</td>
<td>-0.3587</td>
<td>0.7198</td>
</tr>
<tr>
<td>( c_{20} )</td>
<td>-0.0062</td>
<td>0.0129</td>
<td>-0.4848</td>
<td>0.6278</td>
</tr>
<tr>
<td>( c_{21} )</td>
<td>-0.0029</td>
<td>0.0069</td>
<td>-0.4210</td>
<td>0.6738</td>
</tr>
</tbody>
</table>

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Table 3.13: Variability of the forward premium - risk premium vs expected depreciation rate

The values parentheses are the benchmark model (Brennan and Xia (2006)) counterpart of the statistics. Note the close proximity of the corresponding metrics obtained from the two models, which are different with respect to factor structure, yield equation, and data used. The means of risk premium and expected change in spot rate $i$ are not reported in their study.

<table>
<thead>
<tr>
<th></th>
<th>Std. dev.</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$rp_t$</td>
<td>0.4326</td>
<td>-1.0000</td>
</tr>
<tr>
<td></td>
<td>(0.4412)</td>
<td>(-0.9985)</td>
</tr>
<tr>
<td>$q_t$</td>
<td>0.4352</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.4354)</td>
<td></td>
</tr>
</tbody>
</table>
The monthly observations for U.S. and Japanese 3 month Treasury rates. The rates are annualized middle quotes, estimated zero coupon Treasury bond yields for the United States and Japan from January 1985 to May 2002 on a monthly basis.

Figure 3.1: 3 Month Treasury Yields of U.S. and Japan
Figure 3.2: Variance Decomposition: IQTSM-independent
For each complete iteration, we simulate the unobserved state variables using Extended Kalman Filter and Metropolis-Hastings algorithm. The model-implied series of U.S. and Japan yields are then calculated based on the state variables and hyperparameters generated at each iteration. 100,000 burn-in period is used.

Figure 3.3: Model-Implied Yield Series
For each complete iteration, we simulate the unobserved state variables using Extended Kalman Filter and Metropolis-Hastings algorithm. The model-implied series of U.S. and Japan yields are then calculated based on the state variables and hyperparameters generated at each iteration. All the implied moments are calculated based on 50,000 iteration after 100,000 burn-in period.

Figure 3.4: Series of Model-Implied Moment: Mean
For each complete iteration, we simulate the unobserved state variables using Extended Kalman Filter and Metropolis-Hastings algorithm. The model-implied series of U.S. and Japan yields are then calculated based on the state variables and hyperparameters generated at each iteration. All the implied moments are calculated based on 50,000 iteration after 100,000 burn-in period.

Figure 3.5: Series of Model-Implied Moment: SD
For each complete iteration, we simulate the unobserved state variables using Extended Kalman Filter and Metropolis-Hastings algorithm. The model-implied series of U.S. and Japan yields are then calculated based on the state variables and hyperparameters generated at each iteration. All the implied moments are calculated based on 50,000 iteration after 100,000 burn-in period.

Figure 3.6: Series of Model-Implied Moment: Autocorrelation
For each complete iteration, we simulate the unobserved state variables using Extended Kalman Filter and Metropolis-Hastings algorithm. The model-implied series of U.S. and Japan yields are then calculated based on the state variables and hyperparameters generated at each iteration. All the implied moments are calculated based on 50,000 iteration after 100,000 burn-in period.

Figure 3.7: Series of Model-Implied Moment: Cross-correlation
'Metropolis within Gibbs' algorithm is used to simulate the hyperparameters in transition equation.

Figure 3.8: Parameters Generated by MCMC: Transition Equation
'Metropolis within Gibbs' algorithm is used to simulate the hyperparameters in measurement equation.

Figure 3.9: Parameters Generated by MCMC: Measurement Equation
'Metropolis within Gibbs' algorithm is used to simulate the hyperparameters in measurement errors.

Figure 3.10: Parameters Generated by MCMC: Measurement Error Variance
Appendices

A  Chapter 1

A.1  Proof of Lemma 1

1) Equilibrium Demand: The informed trader i's objective function for security i is

\[ E[x_{11}(S_i - P_i)|\beta_1 \gamma + \epsilon_1] \]  \hspace{1cm} (A-1)

where

\[ S_i = \bar{S}_i + \beta_1 \gamma + \epsilon_i \]
\[ P_i = S_i + \bar{\lambda}_1(\omega) \]

\[ = \bar{S}_i + \beta_1(x_{11} + x_{21} + z_1) \]

Let's assume the informed trader i conjectures that the order of informed trader j on security i is given by

\[ \delta_2 E[\beta_1 \gamma + \epsilon_1|\beta_2 \gamma + \epsilon_2] \]

That is,

\[ x_{21} = \delta_2 E[\beta_1 \gamma + \epsilon_1|\beta_2 \gamma + \epsilon_2] \]
It is well known that the conditional expectation of the right hand side of the above equation is
\[
\frac{\beta_1\beta_2\sigma_\gamma^2}{\beta_2^2\sigma_\gamma^2 + \sigma_\varepsilon^2} (\beta_2\gamma + \varepsilon_2^2)
\]

For notational convenience, let
\[
\theta_{21} = \frac{\beta_1\beta_2\sigma_\gamma^2}{\beta_2^2\sigma_\gamma^2 + \sigma_\varepsilon^2}
\]  
(A-2)

Then
\[
x_{21} = \delta_2 \theta_{21} (\beta_2\gamma + \varepsilon)
\]

Thus he maximizes
\[
E[x_{11}(\beta_1\gamma + \varepsilon_1 - \lambda_1(x_{11} + x_{21} + \varepsilon_1))][\beta_1\gamma + \varepsilon_1]
\]
\[
= x_{11}(\beta_1\gamma + \varepsilon_1) - \lambda_1 x_{11} \delta_2 \theta_{21} \theta_{12} (\beta_1\gamma + \varepsilon_1)
\]

Maximizing this expression with respect to \(x_{11}\) yields
\[
x_{11} = \frac{\beta_1\gamma + \varepsilon_1}{2} \left[ \frac{1}{\lambda_1} - \delta_1 \theta_{12} \theta_{21} \right]
\]  
(A-3)
On the other hand, security i informed trader will also maximize the following expected profit for security j.

\[
E[x_{12}(S_2 - P_2)|\beta_1 \gamma + \epsilon_1]
\]

\[
= E[x_{12}((\beta_2 \gamma + \epsilon_2) - \lambda_2(x_{22} + x_{12} + z_2)|\beta_1 \gamma + \epsilon_1]
\]

\[
= x_{12} \left[ \theta_{12}(\beta_1 \gamma + \epsilon_1) - \frac{\lambda_2}{2} \left[ \frac{1}{\lambda_2} - \delta_1 \theta_{12} \theta_{21} \right] E[\beta_2 \gamma + \epsilon_2|\beta_1 \gamma + \epsilon_1] \right] - \lambda_2 x_{12}^2
\]

\[
= x_{12} \theta_{12}(\beta_1 \gamma + \epsilon_1)[1 - \frac{1}{2} + \frac{1}{2} \lambda_2 \delta_1 \theta_{12} \theta_{21}] - \lambda_2 x_{12}^2
\]

\[
.: x_{12} = \frac{\theta_{12}(\beta_1 \gamma + \epsilon_1)[1 + \lambda_2 \delta_1 \theta_{12} \theta_{21}]}{4\lambda_2}
\] (A-4)

Setting this equal to \( \delta_1 \theta_{12}(\beta_1 \gamma + \epsilon_1) \) gives

\[
\delta_1 = \frac{1 + \lambda_2 \delta_1 \theta_{12} \theta_{21}}{4\lambda_2}
\]

\[
.: 4\lambda_2 \delta_1 = 1 + \lambda_2 \delta_1 \theta_{12} \theta_{21}
\]

\[
.: \delta_1 = \frac{1}{\lambda_1(4 - \theta_{12} \theta_{21})}
\]

Similarly,

\[
\delta_2 = \frac{1}{\lambda_1(4 - \theta_{12} \theta_{21})}
\]
Notice that $\theta_{12}\theta_{21} = \rho^2$, where $\rho$ = correlation coefficient between $\tilde{S}_1$ and $\tilde{S}_2$.

\[
\therefore \delta_1 = \frac{1}{\lambda_2(4 - \rho^2)} \tag{A-5}
\]

\[
\delta_2 = \frac{1}{\lambda_1(4 - \rho^2)} \tag{A-6}
\]

Substituting (A-5) and (A-6) for (A-3) and (A-4) yield $x_{11}$ and $x_{12}$. $x_{22}$ and $x_{21}$ can be obtained in the same way.

2) Pricing Parameter $\lambda$ :

Market efficiency condition dictates

\[
P_i = E[S_1|\omega_1] \tag{A-7}
\]

where $P_1 = \tilde{S}_1 + \lambda_1\omega_1$

\[
E[S_1|\omega_1] \Rightarrow E[S_1|\lambda_1\omega_1] = \left( \tilde{S}_1 + \frac{\text{cov}(S_1, \lambda_1\omega_1)}{\text{var}(\lambda_1\omega_1)} \lambda_1\omega_1 \right) \tag{A-8}
\]

\[
\text{cov}(S_1, \lambda_1\omega_1) = (\beta_1^2\sigma_\gamma^2 + \sigma_{\varepsilon_1}^2) \left[ \frac{2}{4 - \rho^2} \right] \tag{A-9}
\]

\[
\text{var}(\lambda_1\omega_1) = \left[ \frac{2 - \rho^2}{4 - \rho^2} \right]^2 (\beta_1^2\sigma_\gamma^2 + \sigma_{\varepsilon_1}^2) + \left[ \frac{\theta_{21}}{4 - \rho^2} \right]^2 (\beta_2^2\sigma_\gamma^2 + \sigma_{\varepsilon_2}^2)
\]

\[
+ \left[ \frac{2\theta_{21}}{4 - \rho^2} \right] \left[ \frac{2 - \rho^2}{4 - \rho^2} \right] \beta_1\beta_2\sigma_\gamma^2 + \lambda_1^2\text{var}(z_1) \tag{A-10}
\]
Notice that second and third terms of right hand side of the above equation are respectively,
\[ \rho^2(\beta_1^2 \sigma_\gamma^2 + \sigma_{\epsilon_1}^2) \quad \text{and} \quad \frac{2\rho^2(2 - \rho^2)}{(4 - \rho^2)}(\beta_1^2 \sigma_\gamma^2 + \sigma_{\epsilon_1}^2) \]

From the equations (A-6) and (A.1), \( \text{cov}(S_1, \lambda_1 \omega_1) = \text{var}(\lambda_1 \omega_1) \)
\[ \therefore \lambda_1^2 = \left( \frac{\beta_1^2 \sigma_\gamma^2 + \sigma_{\epsilon_1}^2}{\text{var}(z_1)} \right) \left( \frac{4 - 3\rho^2 + \rho^4}{[4 - \rho^2]^2} \right) \]

Similarly,
\[ \lambda_2^2 = \left( \frac{\beta_2^2 \sigma_\gamma^2 + \sigma_{\epsilon_2}^2}{\text{var}(z_2)} \right) \left( \frac{4 - 3\rho^2 + \rho^4}{[4 - \rho^2]^2} \right) \]

A.2 Proof of Lemma 2

It is sufficient to show lemma 2 holds for the informed trader \( \text{i} \) since the proof for the informed trader \( \text{j} \) is symmetric.

1) \( \frac{\partial x_{11}}{\partial \rho^2} = \frac{\partial}{\partial \rho^2} \left( \beta_1 \gamma + \varepsilon_1 \frac{\partial \eta_s}{\partial \rho^2} \right) \)

Let \( t = \rho^2 \) for \( \rho^2 \leq 1 \)

Then \( \frac{\partial \eta_s}{\partial \rho^2} = \frac{\eta_s}{t} = \frac{-(t + 2)}{(4 - 3t + t^2)(\sqrt{4 - 3t + t^2})} \)
\[ \therefore \frac{\partial \eta_s}{\partial \rho^2} < 0 \forall \rho^2 < 1 \]
Notice that \( \varphi_{11} > 0 \); \( \therefore \) Lemma 2.1 holds.

\[
2) \frac{\partial x_{12}}{\partial \rho} = \frac{\beta_{11} + \varepsilon_1 \frac{\partial \eta_{10}}{\partial \rho}}{\varphi_{12}} \frac{\partial \eta_{0}}{\partial \rho} = \frac{\delta - 2\rho^3}{(4 - 3\rho^2 + \rho^4)^{\frac{1}{2}} \sqrt{4 - 3\rho^2 + \rho^4}}
\]

\( \therefore \frac{\partial \eta_{10}}{\partial \rho} > 0 \quad \forall \quad \rho^2 \leq 1 \)

A.3 Proof of Lemma 3

For the market for security \( k, k = i, j \);

Let \( \eta_\lambda = \frac{4 - 3\rho^2 + \rho^4}{[4 - \rho^2]^2} \)

Then \( \frac{\partial \lambda_k^2}{\partial \rho^2} = \varphi_{kk} \frac{\partial \eta_\lambda}{\partial \rho^2} \)

\( \frac{\partial \eta_\lambda}{\partial \rho^2} = \frac{[4 - \rho^2][5\rho^2 - 4]}{[4 - \rho^2]^4} \)

\( \therefore \frac{\partial \eta_\lambda}{\partial \rho^2} < 0 \quad \text{if} \quad \rho^2 < \frac{4}{5} \quad \text{and} \quad \frac{\partial \eta_\lambda}{\partial \rho^2} > 0 \quad \text{if} \quad \rho^2 > \frac{4}{5} \)

Notice that \( \operatorname{sign} \left( \frac{\partial \eta_\lambda}{\partial \rho^2} \right) = \operatorname{sign} \left( \frac{\partial \lambda_k}{\partial \rho^2} \right) \quad \text{for} \quad k = i, j \)

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A.4 Proof of Lemma 4

It will suffice to show the expected profit of the informed trader $I$:

$$\pi_1 = \pi_{11} + \pi_{12}$$

1) $$\pi_{11} = E \left[ \frac{\beta_1 \gamma + \epsilon_1}{\lambda_1} \left( \frac{2 - \rho^2}{4 - \rho^2} \right) \left( \beta_1 \gamma + \epsilon_1 \right) - \lambda_1 (x_{11} + x_{21} + z_1) \right]$$ (A-11)

Let $\sigma_{\xi_1}^2$ denote $\beta_1^2 \sigma_\gamma^2 + \sigma_\epsilon_1^2$ (A-12)

Then $$\pi_{11} = \frac{1}{\lambda_1} \left[ \frac{2 - \rho^2}{4 - \rho^2} \right] \left[ \sigma_{\xi_1}^2 - \lambda_1 E \left[ \frac{\beta_1 \gamma + \epsilon_1}{\lambda_1} \left[ \frac{2 - \rho^2}{4 - \rho^2} \right] + \theta_{21}(\beta_2 \gamma + \epsilon_2) \right] \right]$$

$$= \frac{1}{\lambda_1} \left[ \frac{2 - \rho^2}{4 - \rho^2} \right] \left[ \frac{2}{4 - \rho^2} \right] \sigma_{\xi_1}^2 - \left[ \frac{\theta_{21}}{4 - \rho^2} \right] \left( \beta_1 \beta_2 \sigma_\gamma^2 \right)$$ (A-13)

Note that $\theta_{21}(\beta_1 \beta_2 \sigma_\gamma^2) = \rho^2 \sigma_{\xi_1}^2$ (A-14)

Plugging (A-13) and (A-14) in lemma 1 into the equation (A-13), we obtain

$$\pi_{11} = \frac{(2 - \rho^2)^2 \sqrt{[\beta_1^2 \sigma_\gamma^2 + \sigma_\epsilon_1^2][\text{var}(z_1)]}}{(4 - \rho^2) \sqrt{4 - 3 \rho^2 + \rho^4}}$$

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2) Let $\sigma_{Z_2}^2$ denote $\beta_2^2\sigma_\gamma^2 + \sigma_{\varepsilon_2}^2$

Then

\[
\pi_{12} = E\left[\frac{\theta_{12}(\beta_1\gamma + \varepsilon_1)}{\lambda_2(4 - \rho^2)} \left[ (\beta_2\gamma + \varepsilon_2) - \lambda_2 \left[ \frac{\beta_2\gamma + \varepsilon_2}{\lambda_2} \left[ \frac{2 - \rho^2}{4 - \rho^2} \right] + \frac{\theta_{12}(\beta_1\gamma + \varepsilon_1)}{\lambda_2(4 - \rho^2)} \right] \right] \right]
\]

\[
= \frac{\theta_{12}}{\lambda_2(4 - \rho^2)} \left[ (\beta_1\beta_2\sigma_\gamma^2) \left[ \frac{2}{4 - \rho^2} \right] - \frac{\theta_{12}\sigma_{\varepsilon_1}^2}{4 - \rho^2} \right]
\]

\[
= \frac{1}{\lambda_2(4 - \rho^2)^2} \left[ 2\theta_{12}\beta_1\beta_2\sigma_\gamma^2 - \theta_{12}\sigma_{\varepsilon_1}^2 \right]
\]

\[
= \frac{1}{\lambda_2(4 - \rho^2)^2} \left[ 2\rho^2\sigma_{Z_2}^2 - \rho^2\sigma_{Z_2}^2 \right] \tag{A-15}
\]

Plugging $\lambda_2$ in lemma 1 into (A-15), we obtain

\[
\pi_{12} = \frac{\rho^2\sqrt{[\text{var}(Z_2)]/[\beta_2^2\sigma_\gamma^2 + \sigma_{\varepsilon_2}^2]}}{(4 - \rho^2)\sqrt{4 - 3\rho^2 + \rho^4}}
\]

A.5 Proof of Lemma 5

1) Lemma 5.1:

Notice that $\text{sign} \left( \frac{\partial E(\pi_{kk})}{\partial \rho^2} \right) = \text{sign}(\xi_1)$ and $\text{sign} \left( \frac{\partial E(\pi_{kl})}{\partial \rho^2} \right) = \text{sign}(\xi_2)$

Let $t = \rho^2$ and $\tau = \sqrt{4 - 3\rho^2 + \rho^4}$

Then

\[
\frac{\partial \xi_1}{\partial t} = \frac{-(2 - t)[4(4 - t)(4 - 3t + t^2) + (2 - t)(-20 + 17t - 4t^2)]}{[2(4 - t)^2\tau^3]} \tag{A-16}
\]
Numerator of (A-16) = \( (t - 2)(53t^2 - 10t + 24) \) \( (A-17) \)

Second term of the right hand side of (A.18) is positive for all \( \rho^2 \geq 0 \).

2) Lemma 5.2

\[
\text{sign} \left( \frac{\partial E(\pi_k)}{\partial \rho^2} \right) = \text{sign} \left( \frac{\partial (\xi_1 + \xi_2)}{\partial \rho^2} \right)
\]

\[
\frac{\partial (\xi_1 + \xi_2)}{\partial \rho^2} = \frac{\partial \left( \frac{\sqrt{A - 3t + \rho^2}}{4 - t} \right)}{\partial t}
\]

\[
= \frac{-4 + 5t}{2\tau(4 - t)^2}
\]

\( \therefore \) if \( \rho^2 \geq \frac{3}{8} \) (A.19) is positive and if \( \rho^2 < \frac{3}{8} \) (A-18) is negative.

A.6 Sketch of Proof of Proposition 4

Let \( R_1(N) = \) reduction in \( \lambda \) due to noise component of imprecise information
\( C_1(N) = \) reduction in \( \lambda \) due to competition among the informed
\( D_1(N) = \) augmentation in \( \lambda \) due to diverse information effect
, where \( N = \) number of informed traders and \( I = p, o, e \)

( The subscripts, 'p', 'o', and 'e' that are used for the following parameters, denote perfect information, homogeneous signal, and heterogeneous signal, respectively. )

And also let \( \lambda|_\rho \) = liquidity parameter obtained from our economic setting when the correlation is \( \rho \).
$R|_\rho$ = noise effect of the imprecise signal on $\lambda_p(1)$ when the signal informed trader enters the market in which there exists a perfect insider.

Then

$R|_\rho = \lambda_p(1) - \lambda|_\rho$

$R_o(1) = \lambda_p(1) - \lambda_o(1)$

$\lambda_e(2) = \lambda_e(1) + D_e(2) - [R_e(2) - R_e(1)]$

We assume $[R_e(2) - R_e(1)]$, the incremental reduction in $\lambda$, is trivial. Then the maximum diversity effect on $\lambda$ of two informed traders observing different information is

$Max[\lambda_e(2)] = Max[\lambda_e(2) - \lambda_e(1) + [R_e(2) - R_e(1)]]$

$Max[\lambda_e(2)] = Max[\lambda_e(2)] - \lambda_e(1)$

$Max[\lambda_e(2)] = \lim_{\sigma^2 \to 0} \lambda_p(2) = \frac{\sqrt{2}}{3} = \lambda_p(2)$

$\therefore Max[D_e(2)] = \lambda_p(2) - \lambda_e(1)$

Let $\Delta R(2)$ = the portion of noise of the imprecise signal that the market disregards in pricing the asset.

Then, from equation (A-26),

$Min[\Delta R(2)] = \lambda_p(1) - \lambda_o(1) - \lambda_p(1) = \lambda|_\rho - \lambda_p(2) + \lambda_e(1)$

$= \lambda|_\rho - \lambda_p(2)$

Notice that $\lambda_o(1) = \lambda_e(1)$.

From the lemma 1, $\lambda_p(2) = \lambda|_{\rho=1}$. 

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A.7 Proof of Proposition 5

For k and l = 1, 2,
let \( x_{kl}(I, J) \) = equilibrium demand of informed trader k for security l when the traders 1 and 2 when to take the actions I and J respectively.

\[
\begin{align*}
I \text{ and } J &= \begin{bmatrix} 
N: \text{not entering the other market} \\
E: \text{entering the other market}
\end{bmatrix}
\end{align*}
\]

Then,

\[
\begin{align*}
x_{11}(E, N) &= \frac{\beta_1 \gamma + \varepsilon_1}{2\lambda_1} \\
x_{12}(E, N) &= \frac{\theta_{12}(\beta_1 \gamma + \varepsilon_1)}{x_2(4 - \rho^2)} \\
x_{21}(E, N) &= 0 \\
x_{22}(E, N) &= \frac{\beta_2 \gamma + \varepsilon_2}{\lambda_2} \left[ \frac{2 - \rho^2}{4 - \rho^2} \right]
\end{align*}
\]

Likewise,

\[
\begin{align*}
x_{11}(N, E) &= \frac{\beta_1 \gamma + \varepsilon_1}{\lambda_1} \left[ \frac{2 - \rho^2}{4 - \rho^2} \right] \\
x_{12}(N, E) &= 0 \\
x_{21}(N, E) &= \frac{\theta_{21}(\beta_2 \gamma + \varepsilon_2)}{\lambda_2(4 - \rho^2)} \\
x_{22}(N, E) &= \frac{\beta_2 \gamma + \varepsilon_2}{2\lambda_2}
\end{align*}
\]
We know $x_{kl}(N,N)$ and $x_{kl}(E,E)$ for $k$ and $l = i,j$.

\[
\begin{align*}
\therefore \pi_1(E,N) & > \pi_1(N,N) \quad \text{and} \quad \pi_1(E,E) > \pi_1(N,E) \quad (A-28) \\
\pi_2(E,N) & > \pi_2(N,N) \quad \text{and} \quad \pi_2(E,E) > \pi_2(N,E) \quad (A-29)
\end{align*}
\]

From (A-28) and (A-29), 'entering the other market' is the dominant strategy for both traders.

A.8 Proof of Proposition 5

1. For security 1:

\[
P_1 = \tilde{P}_1 + \lambda_1 (x_{11} + x_{21} + z_1) = \tilde{P}_1 + \lambda_1 \omega_1
\]

\[
\text{var} [S_1 | P_1] = \text{var} [S_1 | \lambda_1 \omega_1]
\]

Then

\[
\begin{align*}
\text{cov}(S_1, \lambda_1 \omega_1) & = (\beta_1^2 \sigma^2_\gamma + \sigma^2_{\xi_1}) \left[ \frac{2}{4 - \rho^2} \right] \quad (A-30) \\
\text{var}(\lambda_1 \omega_1) & = \left( \beta_1^2 \sigma^2_\gamma + \sigma^2_{\xi_1} \right) \left[ \frac{8 - 2 \rho^2}{4 - \rho^2} \right] \quad (A-31) \\
\text{var}(S_1) & = \beta_1^2 \sigma^2_\gamma + \sigma^2_{\xi_1} \quad (A-32)
\end{align*}
\]

\[
\therefore \rho^2 = \frac{2}{4 - \rho^2}
\]

\[
\therefore \text{var} [S_1 | P_1] = \left( \beta_1^2 \sigma^2_\gamma + \sigma^2_{\xi_1} \right) \left[ \frac{2 - \rho^2}{4 - \rho^2} \right]
\]
2. Notice that

\[
\text{sign} \left[ \frac{\partial [\text{var}(S_1)|P_1)]}{\partial \rho^2} \right] = \text{sign} \left[ \frac{\partial \left( \frac{2-\rho^2}{4-\rho^2} \right)}{\partial \rho^2} \right]
\]

\[
\frac{\partial \left( \frac{2-\rho^2}{4-\rho^2} \right)}{\partial \rho^2} = \frac{-2}{(4-\rho^2)^2} < 0 \quad \forall \rho^2 < 1
\]

\[
\therefore \text{Monotonically decreasing in } \rho^2.
\]
B Chapter 2

B.1 Equilibrium Demand

Lemma 1:

We provide below a brief derivation of equilibrium demands of each insiders:

Let’s conjecture that equilibrium demands of informed traders have the following form:

\[ x_{11} = A_{11}(\beta_1 \gamma + \varepsilon_1) \]
\[ x_{12} = A_{12}(\beta_1 \gamma + \varepsilon_1) \quad \text{(B-1)} \]
\[ x_{21} = A_{21}(\beta_2 \gamma + \varepsilon_2) \]
\[ x_{22} = A_{22}(\beta_2 \gamma + \varepsilon_2) \]

Investor 1’s problem

\[
\max E[x_{11}(S_1 - P_1) + x_{12}(S_2 - P_2)|\beta_1 \gamma + \varepsilon_1] \quad \text{(B-2)}
\]

Let \( \Psi_1 \) = objective function specified above

\[
\Psi_1 = E\left[ x_{11} ( \beta_1 \gamma + \varepsilon_1 - \lambda_{11} (x_{11} + x_{21}) - \lambda_{12} (x_{12} + x_{22})) + x_{12} ( \beta_2 \gamma + \varepsilon_2 - \lambda_{21} (x_{11} + x_{21}) - \lambda_{22} (x_{12} + x_{22})) |\beta_1 \gamma + \varepsilon_1 \right]
\]

\[
= E\left[ -\lambda_{11} x_{11}^2 + (\beta_1 \gamma + \varepsilon_1) x_{11} (1 - \lambda_{11} A_{21} \theta_{12} - \lambda_{12} A_{12} - \lambda_{12} A_{22} \theta_{12} - \lambda_{21} A_{12}) + x_{12} (\beta_2 \gamma + \varepsilon_2 - \lambda_{21} x_{21} - \lambda_{22} (x_{12} + x_{22})) |\beta_1 \gamma + \varepsilon_1 \right]
\]

Note that \( \theta_{12} (\theta_{21}) \) is \( E[\beta_2 \gamma + \varepsilon_2 | \beta_1 \gamma + \varepsilon_1] \left( E[\beta_1 \gamma + \varepsilon_1 | \beta_2 \gamma + \varepsilon_2] \right) \)

\( \Rightarrow x_{11} = \frac{\beta_1 \gamma + \varepsilon_1}{2 \lambda_{11}} \left[ 1 - \lambda_{11} \theta_{12} A_{21} - (\lambda_{12} + \lambda_{21}) A_{12} - \lambda_{12} \theta_{12} A_{22} \right] \quad \text{(B-3)} \)
\( x_{22} = \frac{\beta_2 \gamma + \varepsilon_2}{2 \lambda_{22}} \left[ 1 - \lambda_{22} \theta_{21} A_{12} - (\lambda_{12} + \lambda_{21}) A_{21} - \lambda_{21} \theta_{21} A_{11} \right] \quad \text{(B-4)} \)
Also from $\frac{\partial \Psi}{\partial x_{12}}$, we get

$$x_{12} = \frac{\beta_1\gamma + \epsilon_1}{2\lambda_{22}} \left[ \theta_{12} - (\lambda_{12} + \lambda_{21})A_{11} - \lambda_{21}\theta_{12}A_{21} - \lambda_{22}\theta_{12}A_{22} \right]$$  \hspace{1cm} (B-5)

$$x_{21} = \frac{\beta_2\gamma + \epsilon_2}{2\lambda_{11}} \left[ \theta_{21} - (\lambda_{12} + \lambda_{21})A_{22} - \lambda_{12}\theta_{21}A_{12} - \lambda_{11}\theta_{21}A_{11} \right]$$  \hspace{1cm} (B-6)

From equation (3) to (7);

$$A_{21} = \frac{1}{2\lambda_{11}} \left[ \theta_{21} - (\lambda_{12} + \lambda_{21})A_{22} - \lambda_{12}\theta_{12}A_{12} - \lambda_{11}\theta_{21}A_{11} \right]$$  \hspace{1cm} (B-7)

$$A_{12} = \frac{1}{2\lambda_{22}} \left[ \theta_{12} - (\lambda_{12} + \lambda_{21})A_{11} - \lambda_{21}\theta_{12}A_{21} - \lambda_{22}\theta_{12}A_{22} \right]$$  \hspace{1cm} (B-8)

$$A_{11} = \frac{1}{2\lambda_{11}} \left[ 1 - \lambda_{11}\theta_{12}A_{21} - (\lambda_{12} + \lambda_{21})A_{12} - \lambda_{12}\theta_{12}A_{22} \right]$$  \hspace{1cm} (B-9)

$$A_{22} = \frac{1}{2\lambda_{22}} \left[ 1 - \lambda_{22}\theta_{21}A_{12} - (\lambda_{12} + \lambda_{21})A_{21} - \lambda_{21}\theta_{21}A_{11} \right]$$  \hspace{1cm} (B-10)

The following solutions are obtained after somewhat tedious calculation.\(^{22}\) \(^{23}\)

$$A_{22} = \left[ -2\lambda_{11}((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22}) + \lambda_{21}((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22}) \right]$$

$$+ 2\lambda_{11}\lambda_{12}^2 + \lambda_{12}\lambda_{21} + \lambda_{21}^2 - 3\lambda_{11}\lambda_{22})\theta_{12}$$

$$-\theta_{12}(\lambda_{12}^3 + \lambda_{11}\lambda_{22}(\lambda_{12} - \lambda_{11}\theta_{12}) + \lambda_{11}\lambda_{21}(-2\lambda_{22} + \lambda_{12}\theta_{12}))\theta_{12}^2$$

$$\sqrt{\left[ ((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22})^2 - ((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22})((\lambda_{12}^2 + \lambda_{21}^2 - 2\lambda_{11}\lambda_{22})\theta_{12}\theta_{21}$$

$$+ (\lambda_{12}\lambda_{21} - \lambda_{11}\lambda_{22})^2\theta_{12}^2\theta_{21}) \right]}$$  \hspace{1cm} (B-11)

\(^{22}\)The solution for $A_{11}$ and $A_{22}$ are symmetric to equations (10) and (11), respectively. Note that the denominators of all four coefficients are the same.

\(^{23}\)setting $\rho = 0$ above for check, the equilibrium demands specified above reduce to the original solutions in the paper with $\rho$ replaced with 0.
\[ A_{21} = \left( (\lambda_{12} + \lambda_{21})^3 - 4\lambda_{11}(\lambda_{12} + \lambda_{21})\lambda_{22} - (\lambda_{22}((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22}) + (\lambda_{12} + \lambda_{21})(\lambda_{12}^2 + \lambda_{12}\lambda_{21} + \lambda_{21}^2 - 3\lambda_{11}\lambda_{22})\theta_{12}\theta_{21} + \theta_{12}(\lambda_{22}(\lambda_{21}^2 - \lambda_{11}\lambda_{22}) - \lambda_{12}\lambda_{22}(\lambda_{21} + \lambda_{11}\theta_{12}) + \lambda_{12}^2(\lambda_{22} + \lambda_{21}\theta_{12}))\theta_{21}^2 \right) \left( (\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22})^2 - ((\lambda_{12} + \lambda_{21})^2 - 4\lambda_{11}\lambda_{22})(\lambda_{12}^2 + \lambda_{21}^2 - 2\lambda_{11}\lambda_{22})\theta_{12}\theta_{21} + (\lambda_{12}\lambda_{21} - \lambda_{11}\lambda_{22})^2\theta_{12}\theta_{21} \right) \] 

(B-12)

B.2 Pricing Parameters

B.2.1 Derivation of Equilibrium Pricing Parameters

Lemma 2:

Note that the equilibrium price of security 1 is affected by demands from both markets.

\[ E[S_1|\omega_1, \omega_2] = E[S_1|\lambda_{11}\omega_1, \lambda_{12}\omega_2] \] 

(B-13)

\[
\begin{pmatrix}
Z_1 \\
Z_2
\end{pmatrix} 
\sim \text{MVN} \left( 
\begin{pmatrix}
\mu_1 \\
\mu_2
\end{pmatrix}, 
\begin{pmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix}
\right)
\]

\[
Z_1|Z_2, \Psi_{t-1} \sim N(\mu_{1|2}, \Sigma_{1|2})
\]

where \( \mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(Z_2 - \mu_2) \)

\( \Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \)
\[
\begin{pmatrix}
S_1 \\
\omega
\end{pmatrix}
\sim \text{MVN}
\begin{pmatrix}
\begin{pmatrix}
\delta_1 \\
0
\end{pmatrix}
& 
\begin{pmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix}
\end{pmatrix}
\]

where

1. \( \omega = (\lambda_{11}, \lambda_{12})^T \)
2. \( \Sigma_{11} = \text{var}(S_1) = \beta_1^2 \sigma^2 \)
3. \( \Sigma_{12} = (\text{cov}(S_1, \lambda_{11}), \text{cov}(S_1, \lambda_{12})) = \Sigma'_{21} \)

\[
E[S_1|\lambda_{11}, \lambda_{12}] = \delta_1 + \Sigma_{12} \Sigma^{-1}_{22} \omega, \quad \text{where} \quad \omega = (\lambda_{11}, \lambda_{12})^T
\]

\[
\begin{aligned}
\Sigma^{-1}_{22} &= \frac{1}{\text{var}(\lambda_{11}) \text{var}(\lambda_{12}) - \text{cov}(\lambda_{11}, \lambda_{12}) \text{cov}(\lambda_{12}, \lambda_{11})} \\
& \quad \times \begin{bmatrix}
\text{var}(\lambda_{12}) & -\text{cov}(\lambda_{12}, \lambda_{12}) \\
-\text{cov}(\lambda_{11}, \lambda_{12}) & \text{var}(\lambda_{11})
\end{bmatrix}
\end{aligned}
\]

\[
\begin{aligned}
\Sigma_{12} \Sigma^{-1}_{22} &= \frac{1}{|\Sigma_{22}|} \left[ \text{cov}(S_1, \lambda_{11}) \text{var}(\lambda_{12}) - \text{cov}(S_1, \lambda_{12}) \text{cov}(\lambda_{12}, \lambda_{11}) \\
& \quad + \lambda_{11} \lambda_{12} \left[ \text{cov}(S_1, \lambda_{11}) \text{var}(\lambda_{12}) - \text{cov}(S_1, \lambda_{12}) \text{cov}(\lambda_{12}, \lambda_{11}) \right] \\
& \quad + \lambda_{12} \lambda_{11} \left[ \text{cov}(S_1, \lambda_{11}) \text{var}(\lambda_{12}) - \text{cov}(S_1, \lambda_{12}) \text{cov}(\lambda_{12}, \lambda_{11}) \right] \right]
\end{aligned}
\]

\[
E[S_1|\lambda_{11}, \lambda_{12}] = \delta_1 + \frac{1}{|\Sigma_{22}|} \\
\quad \times \left[ \lambda_{11} \lambda_{12} \left( \text{cov}(S_1, \omega_1) \text{var}(\omega_2) - \text{cov}(S_1, \omega_2) \text{cov}(\omega_2, \omega_1) \right) \\
\quad + \omega_1 \omega_2 \left( \text{cov}(S_1, \omega_1) \text{var}(\omega_2) - \text{cov}(S_1, \omega_2) \text{cov}(\omega_2, \omega_1) \right) \right]
\]

Note that \(|\Sigma_{22}| = \lambda_{11}^2 \lambda_{12}^2 (\text{var}(\omega_1) \text{var}(\omega_2) - \text{cov}(\omega_1, \omega_2)^2)\)
\[ \Sigma_{12} \Sigma_{22}^{-1} \omega = \omega_1 \times \frac{\text{cov}(S_1, \omega_1) \text{var}(\omega_2) - \text{cov}(S_1, \omega_2) \text{cov}(\omega_1, \omega_2)}{\text{var}(\omega_1) \text{var}(\omega_2) - \text{cov}(\omega_1, \omega_2)^2} \]  
(B-14)

\[ + \omega_2 \times \frac{\text{cov}(S_1, \omega_2) \text{var}(\omega_1) - \text{cov}(S_1, \omega_1) \text{cov}(\omega_1, \omega_2)}{\text{var}(\omega_1) \text{var}(\omega_2) - \text{cov}(\omega_1, \omega_2)^2} \]  
(B-15)

The equilibrium market depths have the following closed form expressions.

\[ \lambda_{11} = \frac{\text{cov}(S_1, \omega_1) \text{var}(\omega_2) - \text{cov}(S_1, \omega_2) \text{cov}(\omega_1, \omega_2)}{\text{var}(\omega_1) \text{var}(\omega_2) - \text{cov}(\omega_1, \omega_2)^2} \]

\[ \lambda_{12} = \frac{\text{cov}(S_1, \omega_2) \text{var}(\omega_1) - \text{cov}(S_1, \omega_1) \text{cov}(\omega_1, \omega_2)}{\text{var}(\omega_1) \text{var}(\omega_2) - \text{cov}(\omega_1, \omega_2)^2} \]
B.2.2 Component Second Moments of Pricing Parameters

In this section we detail all the second moments used for the expression of λs.

(i) \( \text{cov}(S_1, \lambda_{11}\omega_1) = \text{cov}(\beta_1\gamma + \varepsilon_1, \lambda_{11}(x_{11} + x_{21} + z_1)) \)

\[ = \lambda_{11}\text{cov}(\beta_1\gamma + \varepsilon_1, (x_{11} + x_{21} + z_1)) \]

\[ = \lambda_{11}[\text{cov}(\beta_1\gamma + \varepsilon_1, A_{11}(\beta_1\gamma + \varepsilon_1)) + \text{cov}(\beta_1\gamma + \varepsilon_1, A_{21}(\beta_2\gamma + \varepsilon_2))] \]

\[ = \lambda_{11}A_{11}(\beta_1^2\sigma_\gamma^2 + \sigma_\varepsilon_1^2) + \lambda_{11}A_{21}\text{cov}(\beta_1\gamma + \varepsilon_1, \beta_2\gamma + \varepsilon_2) \]

\[ = \lambda_{11}A_{11}(\beta_1^2\sigma_\gamma^2 + \sigma_\varepsilon_1^2) + \lambda_{11}A_{21}\beta_1\beta_2\sigma_\gamma^2 \]

\[ = \lambda_{11}[A_{11}(\beta_1^2\sigma_\gamma^2 + \sigma_\varepsilon_1^2) + A_{21}\beta_1\beta_2\sigma_\gamma^2] \quad (B-16) \]

(ii) \( \text{cov}(S_1, \lambda_{12}\omega_2) = \text{cov}(\beta_1\gamma + \varepsilon_1, \lambda_{12}(x_{12} + x_{22} + z_2)) \)

\[ = \lambda_{12}\text{cov}(\beta_1\gamma + \varepsilon_1, (x_{12} + x_{22} + z_2)) \]

\[ = \lambda_{12}[\text{cov}(\beta_1\gamma + \varepsilon_1, A_{12}(\beta_1\gamma + \varepsilon_1)) + \text{cov}(\beta_1\gamma + \varepsilon_1, A_{22}(\beta_2\gamma + \varepsilon_2))] \]

\[ = \lambda_{12}A_{12}(\beta_1^2\sigma_\gamma^2 + \sigma_\varepsilon_1^2) + \lambda_{12}A_{22}\beta_1\beta_2\sigma_\gamma^2 \]

\[ = \lambda_{12}[A_{12}(\beta_1^2\sigma_\gamma^2 + \sigma_\varepsilon_1^2) + A_{22}\beta_1\beta_2\sigma_\gamma^2] \quad (B-17) \]

(iii) \( \Sigma_{22} = \begin{pmatrix} \text{var}(\lambda_{11}\omega_1) & \text{cov}(\lambda_{11}\omega_1, \lambda_{12}\omega_2) \\ \text{cov}(\lambda_{12}\omega_2, \lambda_{11}\omega_1) & \text{var}(\lambda_{12}\omega_2) \end{pmatrix} \)

(iv) \( \text{var}(\lambda_{11}\omega_1) = \lambda_{11}^2\text{var}(x_{11} + x_{21} + z_1) \)

\[ = \lambda_{11}^2[A_{11}^2(\beta_1^2\sigma_\gamma^2 + \sigma_\varepsilon_1^2) + A_{21}^2(\beta_2^2\sigma_\gamma^2 + \sigma_\varepsilon_2^2) \]

\[ + 2A_{11}A_{21}(\beta_1\beta_2\sigma_\gamma^2 + \text{var}(z_1)] \quad (B-18) \]
\[(v) \text{cov}(\lambda_1\omega_1, \lambda_2\omega_2) = \lambda_1\lambda_2\text{cov}(\omega_1, \omega_2) = \lambda_1\lambda_2\text{cov}(x_{11} + x_{21}, x_{12} + x_{22}) = \lambda_1\lambda_2[A_{11}A_{12}(\beta_1^2\sigma^2_\gamma + \sigma^2_{\epsilon_1}) + A_{11}A_{22}\beta_1\beta_2\sigma^2_\gamma + A_{21}A_{12}\beta_1\beta_2\sigma^2_\gamma + A_{21}A_{22}(\beta_2^2\sigma^2_\gamma + \sigma^2_{\epsilon_2})] = \lambda_1\lambda_2[A_{11}A_{12}(\beta_1^2\sigma^2_\gamma + \sigma^2_{\epsilon_1}) + (A_{11}A_{22} + A_{21}A_{12})\beta_1\beta_2\sigma^2_\gamma + A_{21}A_{22}(\beta_2^2\sigma^2_\gamma + \sigma^2_{\epsilon_2})] \quad (B-19)\]

\[(vi) \text{var}(\lambda_2\omega_2) = \lambda_2^2\text{var}(x_{12} + x_{22} + z_2) = \lambda_2^2[A_{12}^2(\beta_1^2\sigma^2_\gamma + \sigma^2_{\epsilon_1}) + A_{12}^2(\beta_2^2\sigma^2_\gamma + \sigma^2_{\epsilon_2}) + 2A_{12}A_{22}(\beta_1\beta_2\sigma^2_\gamma) + \text{var}(z_2)] \quad (B-20)\]

**B.3 Equilibrium Profits of Informed Traders**

*Lemma 3:*

\[\Pi_{11} = E[x_{11}((\beta_1\gamma + \epsilon_1) - \lambda_{11}(x_{11} + x_{21} + z_1) - \lambda_{12}(x_{12} + x_{22} + z_2))] = E[A_{11}(\beta_1\gamma + \epsilon_1)((\beta_1\gamma + \epsilon_1) - \lambda_{11}(A_{11}(\beta_1\gamma + \epsilon_1) + A_{21}(\beta_2\gamma + \epsilon_2) + z_1) - \lambda_{12}(A_{12}(\beta_1\gamma + \epsilon_1) + A_{22}(\beta_2\gamma + \epsilon_2) + z_2))]

\[= A_{11}(\beta_1^2\sigma^2_\gamma + \sigma^2_{\epsilon_1}) - [A_{11}^2\lambda_{11}(\beta_1^2\sigma^2_\gamma + \sigma^2_{\epsilon_1}) + A_{11}A_{21}\lambda_{11}\beta_1\beta_2\sigma^2_\gamma + A_{11}A_{12}A_{12}\lambda_{12}\beta_1\beta_2\sigma^2_\gamma] + A_{11}A_{12}\lambda_{12}(\beta_2^2\sigma^2_\gamma + \sigma^2_{\epsilon_1}) + A_{11}A_{22}\lambda_{12}\beta_1\beta_2\sigma^2_\gamma]

\[= (\beta_1^2\sigma^2_\gamma + \sigma^2_{\epsilon_1})[A_{11} - A_{11}^2\lambda_{11} - A_{11}A_{12}\lambda_{12} - \beta_1\beta_2\sigma^2_\gamma] + A_{11}A_{21}\lambda_{11}\beta_1\beta_2\sigma^2_\gamma + A_{11}A_{22}\lambda_{12}\beta_1\beta_2\sigma^2_\gamma]

\[= A_{11}[(\beta_1^2\sigma^2_\gamma + \sigma^2_{\epsilon_1})(1 - A_{11}^2\lambda_{11} - A_{12}\lambda_{12}) - \beta_1\beta_2\sigma^2_\gamma(A_{21}\lambda_{11} + A_{22}\lambda_{12})] \quad (B-21)\]
\[ \Pi_{12} = E\left[ x_{12}(\beta_2 \gamma + \epsilon_2) - \lambda_{21}(x_{11} + x_{21} + z_1) - \lambda_{22}(x_{12} + x_{22} + z_2) \right] \]

\[ \begin{align*}
&= E\left[ A_{12}(\beta_1 \gamma + \epsilon_1)((\beta_2 \gamma + \epsilon_2) - \lambda_{21}(A_{11}(\beta_1 \gamma + \epsilon_1) + A_{21}(\beta_2 \gamma + \epsilon_2) + z_1) \\
&\quad - \lambda_{22}(A_{12}(\beta_1 \gamma + \epsilon_1) + A_{22}(\beta_2 \gamma + \epsilon_2) + z_2)) \right] \\
&= A_{12}[(\beta_1 \sigma_\gamma + \sigma_\epsilon_1)(-A_{11} \lambda_{21} - A_{12} \lambda_{22}) + \beta_1 \beta_2 \sigma_\gamma^2 (1 - A_{21} \lambda_{21} - A_{22} \lambda_{22})](B-22)
\end{align*} \]

The equilibrium profits of the other informed trader can be calculated in a similar fashion.

This concludes the proof of lemma 3. ■

**B.4 Feasible Range of Equilibrium Pricing Parameters**

Let \( \Pi_i \triangleq \Pi_{11} + \Pi_{12} \).

\[ \begin{align*}
\frac{\partial^2 \Pi_i}{\partial x_{11}^2} &= -2\lambda_{11} \\
\frac{\partial^2 \Pi_i}{\partial x_{12}^2} &= -2\lambda_{22} \\
\frac{\partial^2 \Pi_i}{\partial x_{11} \partial x_{12}} &= -\lambda_{12} - \lambda_{21} \\
\frac{\partial^2 \Pi_i}{\partial x_{12} \partial x_{11}} &= -\lambda_{21} - \lambda_{12}
\end{align*} \]

In order to have a well-defined optimal trading strategy for informed trader \( i \) the second-order partial derivatives of profit function \( \Pi_i \) with respect to \( x_{11} \) and \( x_{12} \) should be negative. Also \( |H| \), the determinant of the Hessian of profit function \( \Pi_i \), should be positive. The restrictions on the pricing parameters specified in equations (20) and (21) are thus obtained. We obtain the same results from the conditions imposed on \( \Pi_2 \).
B.5 Equilibrium Demands of Insiders for Symmetric Markets

B.5.1 Optimal Trading Strategy

Let us denote \( \delta_1 = \beta \gamma + \varepsilon_1 \) and \( \delta_2 = \beta \gamma + \varepsilon_2 \).

Set \( \sigma_{\varepsilon_1^2} = \sigma_{\varepsilon_2^2} =: \sigma_{\varepsilon}^2 \).

Then

\[
E[\delta_2|\delta_1] = \frac{\beta^2 \sigma_{\varepsilon}^2}{\beta^2 \sigma_{\varepsilon}^2 + \sigma_{\varepsilon}^2} \delta_1 = \rho \delta_1,
\]

\[
E[\delta_1|\delta_2] = \frac{\beta^2 \sigma_{\varepsilon}^2}{\beta^2 \sigma_{\varepsilon}^2 + \sigma_{\varepsilon}^2} \delta_1 = \rho \delta_2,
\]

where \( \rho \) is the correlation between the stock price innovations.

Let \( \lambda \triangleq \lambda_{11} = \lambda_{22} \) and \( \lambda_c \triangleq \lambda_{12} = \lambda_{21} \).

From the FOC of the objective function we obtain the following conditions:

\[
\begin{align*}
\delta_1 - 2x_{12} \lambda - 2\lambda x_{11} - \lambda E[x_{21}|\delta_1] - \lambda_c E[x_{22}|\delta_1] &= 0 \quad \text{(B-27)} \\
\rho \delta_1 - 2x_{12} \lambda - 2\lambda_c x_{11} - \lambda_c E[x_{21}|\delta_1] - \lambda E[x_{22}|\delta_1] &= 0 \quad \text{(B-28)}
\end{align*}
\]

\[
\Rightarrow (x_{11} + x_{22}) = \frac{\delta_1 (1 + \rho)}{2(\lambda + \lambda_c)} - \frac{E[x_{21}|\delta_1] + E[x_{22}|\delta_1]}{2}
\]

Let \( x_{11} + x_{22} =: A \delta_1 \) and \( x_{21} + x_{22} =: A \delta_2 \)

\[
A \delta_1 = \frac{(1 + \rho) \delta_1}{2(\lambda + \lambda_c)} - \frac{A \rho \delta_1}{2}
\]

\[
\Rightarrow A = \left[ \frac{(1 + \rho)}{(2 + \rho)} \right] \frac{1}{\lambda + \lambda_c}
\]
So the total net orders of informed traders 1 and 2 are respectively,

\[ x_{11} + x_{12} = \left[ \frac{1 + \rho}{2 + \rho} \right] \frac{1}{\lambda + \lambda_c} \cdot \delta_1 \]

\[ x_{21} + x_{22} = \left[ \frac{1 + \rho}{2 + \rho} \right] \frac{1}{\lambda + \lambda_c} \cdot \delta_2 \]

Let \( \phi \) denote \( \frac{1 + \rho}{2 + \rho} \).

Plugging the above expressions for \( x_{12} \) and \( x_{21} \) into the equation (B-27), we obtain:

\[ 0 = \delta_1 - 2\lambda_c \left( \frac{\phi}{\lambda + \lambda_c} \delta_1 - x_{11} \right) - 2\lambda x_{11} - \lambda \left( \frac{\rho \cdot \phi}{\lambda + \lambda_c} \delta_1 - E[x_{22} | \delta_1] \right) - \lambda_c E[x_{22} | \delta_1] \]

\[ = \delta_1 \left( 1 - \frac{2\lambda_c \phi}{\lambda + \lambda_c} \right) - 2(\lambda - \lambda_c) x_{11} - \frac{\lambda \rho \cdot \phi}{\lambda + \lambda_c} \delta_1 + (\lambda - \lambda_c) E[x_{22} | \delta_1] \]

\[ = \left( \frac{\lambda + \lambda_c - 2\lambda_c \phi}{\lambda + \lambda_c} \right) \delta_1 - 2(\lambda - \lambda_c) x_{11} - \frac{\lambda \rho \cdot \phi}{\lambda + \lambda_c} \delta_1 + (\lambda - \lambda_c) E[x_{22} | \delta_1] \]  

(B-29)

Thus

\[ 2(\lambda - \lambda_c) x_{11} = \delta_1 \left[ \frac{\lambda(1 - \rho \phi) + \lambda_c(1 - 2\phi)}{\lambda + \lambda_c} \right] + (\lambda - \lambda_c) E[x_{22} | \delta_1] \]  

(B-30)

Now set \( x_{11} = B\delta_1 \) and \( x_{22} = B\delta_2 \).

Then

\[ B = \frac{\lambda(1 - \rho \phi) + \lambda_c(1 - 2\phi)}{(2 - \rho)(\lambda^2 - \lambda_c^2)} \]

\[ \Rightarrow x_{11} = \frac{\lambda(1 - \rho \phi) + \lambda_c(1 - 2\phi)}{(2 - \rho)(\lambda^2 - \lambda_c^2)} \cdot \delta_1 \]

\[ = \left[ \frac{\lambda(2 - \rho^2) + \lambda_c \rho}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \right] \cdot \delta_1 \]  

(B-31)
Note that the equilibrium demands $x_{11}$ and $x_{22}$ are always positive due to the second order condition.

\[
x_{12} = \delta_1 \left[ \frac{(2 - \rho)(\lambda - \lambda_c)\phi - \lambda(1 - \rho\phi) - \lambda_c(1 - 2\phi)}{(2 - \rho)(\lambda + \lambda_c)(\lambda - \lambda_c)} \right]
\]

\[
= \delta_1 \left[ \frac{\lambda(2\phi - 1) - \lambda_c(1 - \rho\phi)}{(2 - \rho)(\lambda^2 - \lambda_c^2)} \right]
\]

\[
= \left[ \frac{\rho\lambda - (2 - \rho^2)\lambda_c}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \right] \cdot \delta_1
\]

Likewise, $x_{22}$ and $x_{21}$ are obtained as presented below.

\[
x_{22} = \left[ \frac{\lambda(2 - \rho^2) + \lambda_c\rho}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \right] \cdot \delta_2 \quad (B-33)
\]

\[
x_{21} = \left[ \frac{\rho\lambda - (2 - \rho^2)\lambda_c}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \right] \cdot \delta_2 \quad (B-34)
\]

### B.5.2 Equilibrium Pricing Rule

From the demand expressions derived in the last section,

\[
\omega_1 = \left[ \frac{\lambda(2 - \rho^2) + \lambda_c\rho}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \right] \cdot \delta_1 + \left[ \frac{\rho\lambda - (2 - \rho^2)\lambda_c}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \right] \cdot \delta_2 + z_1 \quad (B-35)
\]

\[
\omega_2 = \left[ \frac{\rho\lambda - (2 - \rho^2)\lambda_c}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \right] \cdot \delta_1 + \left[ \frac{\lambda(2 - \rho^2) + \lambda_c\rho}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \right] \cdot \delta_2 + z_2 \quad (B-36)
\]

Market efficiency condition dictates $\lambda \omega_1 + \lambda_c \omega_2 = E [\delta_1 | \omega_1, \omega_2]$. 

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\[
E[\delta_1\omega_1] = \left[ \frac{\lambda(2 - \rho^2) + \lambda_c \rho}{c} \right] \cdot E[\delta_1^2] + \left[ \frac{\rho \lambda - (2 - \rho^2) \lambda_c}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \right] \cdot E[\delta_1 \delta_2]
\]

\[
E[\delta_1\omega_2] = \left[ \frac{\rho \lambda - (2 - \rho^2) \lambda_c}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \right] \cdot E[\delta_2^2] + \left[ \frac{\lambda(2 - \rho^2) + \lambda_c \rho}{(4 - \rho^2)(\lambda^2 - \lambda_c^2)} \right] \cdot E[\delta_2 \delta_1]
\]

Let \( N_1 \triangleq \lambda(2 - \rho^2) + \lambda_c \rho \), \( N_2 \triangleq \rho \lambda - (2 - \rho^2) \lambda_c \), \( D \triangleq \rho \lambda - (2 - \rho^2) \lambda_c \)

\[
\Rightarrow \quad \lambda E[\omega_1^2] + \lambda_c E[\omega_1\omega_2]
\]

\[
= \lambda \left[ \left( \frac{N_1}{D} \right)^2 E[\delta_1^2] + \left( \frac{N_2}{D} \right)^2 E[\delta_2^2] + \left( \frac{2N_1N_2}{D^2} \right) E[\delta_1 \delta_2] + \sigma_1^2 \right] + \lambda_c \left[ \left( \frac{N_1N_2}{D} \right) E[\delta_1^2] + \left( \frac{N_1N_2}{D} \right) E[\delta_2^2] + \left( \frac{N_1^2N_2^2}{D^2} \right) E[\delta_1 \delta_2] \right]
\]

\[
\lambda E[\omega_1\omega_2] + \lambda_c E[\omega_2^2]
\]

\[
= \lambda \left[ \left( \frac{N_1N_2}{D^2} \right)^2 E[\delta_1^2] + \left( \frac{N_1N_2}{D^2} \right)^2 E[\delta_2^2] + \left( \frac{2N_1N_2}{D^2} \right) E[\delta_1 \delta_2] \right] + \lambda_c \left[ \left( \frac{N_2}{D} \right)^2 E[\delta_1^2] + \left( \frac{N_2}{D} \right)^2 E[\delta_2^2] + \left( \frac{2N_1N_2}{D^2} \right) E[\delta_1 \delta_2] + \sigma_2^2 \right]
\]
Set $E[\delta_1^2] = \beta^2 \sigma_1^2 + \sigma_2^2 = E[\delta_2^2] =: \Sigma$, and

$E[\delta_1 \delta_2] = \beta^2 \sigma_1^2 = \rho \Sigma$.

Then

$$
\lambda E[\omega_1^2] + \lambda_c E[\omega_1 \omega_2]
= \lambda \sigma_1^2 + \frac{\Sigma}{D^2} \left[ \lambda \left( \left( N_1^2 + N_2^2 \right) \rho + 2N_1 N_2 \right) + \lambda_c \left( 2N_1 N_2 + \rho \left( N_1^2 + N_2^2 \right) \right) \right]
= \lambda \Phi + \frac{1}{D^2} \left[ \lambda \left( \left( N_1^2 + N_2^2 \right) \rho + 2N_1 N_2 \right) + \lambda_c \left( 2N_1 N_2 + \rho \left( N_1^2 + N_2^2 \right) \right) \right]
= \frac{N_1}{D} \Sigma + \frac{N_2}{D} \rho \Sigma
= \frac{\Sigma}{D} \left( N_1 + N_2 \rho \right)
$$

(B-45)

It follows from the equations $(B-45)$ and $(B-46)$:

$$
\frac{(1 + \rho)^2}{2 + \rho} (\lambda + \lambda_c) = \Phi (\lambda + \lambda_c)^3 + \lambda \frac{(1 + \rho)^3}{(2 + \rho)^2} + \lambda_c \frac{(1 + \rho)^3}{(2 + \rho)^2}
$$

(B-47)

$$
\frac{(1 - \rho)^2}{2 - \rho} (\lambda - \lambda_c) = \Phi (\lambda - \lambda_c)^3 + \lambda \frac{(1 - \rho)^3}{(2 - \rho)^2} + \lambda_c \frac{(1 - \rho)^3}{(2 - \rho)^2}
$$

(B-48)
Under the new pricing scheme the equilibrium pricing parameters are given as below.

\[ \lambda = \left[ \frac{2 - \rho^2}{4 - \rho^2} \right] \sqrt{\frac{\beta^2 \sigma^2_2 + \sigma^2_z}{\sigma^2_z}} \] 

(B-49)

\[ \lambda_c = \left[ \frac{-\rho}{4 - \rho^2} \right] \sqrt{\frac{\beta^2 \sigma^2_2 + \sigma^2_z}{\sigma^2_z}} \] 

(B-50)

### B.6 Sequential Equilibrium

Let \( \phi^1_{n+1} : = E[\pi^1_n | \bar{p}_1, \bar{p}_2, ..., \bar{p}_{n-1}, S_1] \)

\[ = A_{n-1}(\delta_1 - p^1_{n-1})^2 + B_{n-1}\rho E[\{(\delta_2 - p^2_n)^2 | F_n, \delta_2\} + C^1_n \] 

(B-51)

\[ \phi^2_{n+1} : = E[\pi^2_n | \bar{p}_1, \bar{p}_2, ..., \bar{p}_{n-1}, S_2] \]

\[ = A_{n-1}(\delta_2 - p^2_{n-1})^2 + B_{n-1}\rho E[\{(\delta_1 - p^1_n)^2 | F_n, \delta_2\} + C^2_n \] 

(B-52)

It suffices to consider the optimization problem of insider 1:

\[ \Pi^1_1 = E[\Delta x^1_n ((\delta_1 - p^1_{n-1}) - \lambda_n(\Delta x^1_n + \Delta x^{21}_n + \Delta u^1_n)) \]

\[ + \Delta x^1_n (\delta_2 - p^2_{n-1}) - \lambda_n(\Delta x^1_n + \Delta x^{22}_n + \Delta u^2_n)) \]

\[ + A_n((\delta_1 - p^1_{n-1}) - \lambda_n(\Delta x^{11}_n + \Delta x^{21}_n + \Delta u^1_n))^2 \]

\[ B_n\rho E[(\delta_2 - p^2_{n-1}) - \lambda_n(\Delta x^{12}_n + \Delta x^{22}_n + \Delta u^2_n))^2 | F_n, \delta_1] \]

\[ + C^1_n | F_{n-1}, \delta_1] \] 

(B-53)

Differentiating \( \phi^1_n \) with respect to \( \Delta x^1_n \) and \( \Delta x^2_n \), we obtain the optimal demands of own market and cross-market respectively:
\[ \Delta x_n^{11} = \frac{(\delta_1 - p_{n-1}^1)[(1 - \lambda_n a_n^c \rho^2)(1 - 2\lambda_n A_n)]}{2\lambda_n(1 - \lambda_n A_n)} \]  
\[ \Delta x_n^{12} = \frac{\rho(\delta_1 - p_{n-1}^1)[(1 - \lambda_n a_n)(1 - 2\lambda_n B_n \rho)]}{2\lambda_n(1 - \lambda_n B_n \rho)} \]  

\[ \phi_n^1 = E[a_n(\delta_1 - p_{n-1}^1)[(\delta_1 - p_{n-1}^1) - \lambda_n(a_n(\delta_1 - p_{n-1}^1) + a_n^c \rho(\delta_2 - p_{n-1}^1) + \Delta u_n^1)] 
+ a_n^c \rho(\delta_1 - p_{n-1}^1)[(\delta_2 - p_{n-1}^2) - \lambda_n(a_n \rho(\delta_1 - p_{n-1}^1) + a_n(\delta_2 - p_{n-1}^2) + \Delta u_n^2)] 
+ A_n[(\delta_1 - p_{n-1}^1) - \lambda_n(a_n(\delta_1 - p_{n-1}^1) + a_n^c \rho(\delta_2 - p_{n-1}^2) + \Delta u_n^1)]^2 
+ B_n \rho E[(\delta_2 - p_{n-1}^2)^2|F_n, \delta_1] + c_n|F_{n-1}, \delta_1] \]  

\[ \therefore \phi_n^1 = (\delta_1 - p_{n-1}^2)^2[a_n(1 - \lambda_n a_n - \lambda_n a_n^c \rho^2) + a_n^c \rho^2(1 - \lambda_n a_n - \lambda_n a_n^c)] \]  

\[ \therefore A_{n-1} = a_n(1 - \lambda_n a_n - \lambda_n a_n^c \rho^2) + a_n^c \rho^2(1 - \lambda_n a_n - \lambda_n a_n^c) \]  

\[ -2\lambda_n B_n a_n^c \rho^2(1 - \lambda_n a_n) + a_n(1 - 2\lambda_n A_n a_n^c \rho^2 + A_n \lambda_n^2 a_n^c \rho^2) \]  

\[ B_{n-1} = A_n \lambda_n^2 a_n^c \rho^2 + B_n \rho(1 - 2\lambda_n a_n + \lambda_n^2 a_n) \]  

\[ C_{n-1} = (A_n + B_n \rho)\lambda_n^2 a_n^c \Delta t + c_n \]
B.6.1 Lemma 8

The regularity condition for equilibrium demands suggests:

\[ A_t' = \beta_t' \rho \]  \hspace{1cm} (B-70)

B.6.2

The following derivations involve all the remaining results in continuous time case.

\[ A_t' = -\frac{\lambda_t'}{2\lambda_t^2} = \rho \beta_t'^{ac} \quad \text{and} \quad \lambda_t' = -2\rho \lambda_t^3 \beta_t'^{ac} \]  \hspace{1cm} (B-71)

from the regularity condition for equilibrium order amounts.

\[ \lambda_t = \frac{\Sigma_i(1 + \rho^2)\beta_t'^{ac}}{\sigma_t^2} \]  \hspace{1cm} (B-72)

\[ \lambda_t = \left[ \frac{1 + \rho^2}{2\rho} \right] \frac{\Sigma_i \lambda_t'}{\Sigma_t'} \]  \hspace{1cm} (B-73)

\[ \Sigma_t = \left[ \frac{1 + \rho^2}{2\rho} \right] \lambda_t \]  \hspace{1cm} (B-74)

\[ : \Sigma_t = \Sigma_0 \left( \frac{\lambda_t'}{\lambda_0} \right)^{1+\rho^2} \]  \hspace{1cm} (B-75)
\[
\lambda_t = \frac{(1 + \rho^2)}{\sigma^2} \beta_{acc} \lambda_0 \left( \frac{\lambda_t}{\lambda_0} \right)^{\frac{1 + \rho^2}{2\rho}}
\]  
(B-76)

\[
\therefore \beta^{ac}_{t} = \frac{\sigma^2}{(1 + \rho^2) \lambda_0} \left( \frac{1 + \rho^2}{2\rho} \right) \lambda_t \left( \frac{2\rho - 1 - \rho^2}{2\rho} \right)
\]  
(B-77)

\[
\therefore \beta^{ac}_{t} = \frac{\sigma^2}{(1 + \rho^2) \lambda_0} \left( \frac{1 + \rho^2}{2\rho} \right) \lambda_t \left( \frac{2\rho - 1 - \rho^2}{2\rho} \right)
\]  
(B-78)

Now let us turn to the equilibrium market depth parameter:

It was shown that

\[
\lambda_t' = \frac{-2\rho \lambda_t^2 \beta_{tac}}{(1 + \rho^2) \lambda_0} \left( \frac{6\rho - 1 - \rho^2}{2\rho} \right)
\]  
(B-79)

\[
\therefore \lambda_t' = \frac{-2\rho \sigma^2 \lambda_0^{\frac{1 + \rho^2}{2\rho}}}{(1 + \rho^2) \lambda_0} \left( \frac{6\rho - 1 - \rho^2}{2\rho} \right)
\]  
(B-80)

\[
\kappa := \frac{-2\rho \sigma^2 \lambda_0^{\frac{1 + \rho^2}{2\rho}}}{(1 + \rho^2) \lambda_0}
\]  
(B-81)

\[
L := \left( \frac{6\rho - 1 - \rho^2}{2\rho} \right)
\]  
(B-82)

Then \( \kappa t = (1 - L)^{-1} \left( \lambda_t^{1-L} - \lambda_0^{1-L} \right) \)

(B-83)

(B-84)

\[
\therefore \lambda_t = \left[ \lambda_0^{1-L} + (1 - L)\kappa t \right]^{\frac{1}{1-L}}
\]  
(B-85)

\[
= \lambda_0^{1-L} \left[ 1 - \left( \frac{(1 - L) - 2\rho \sigma^2 \lambda_0^{\frac{1 + \rho^2}{2\rho}} - (1 - L)}{(1 + \rho^2) \lambda_0} \right)^t \right]^{\frac{1}{1-L}}
\]  
(B-86)

\[
= \lambda_0 \left[ 1 - \left( \frac{(\rho^2 - 4\rho + 1) \sigma^2 \lambda_0^{\frac{1 + \rho^2}{2\rho}}}{(1 + \rho^2) \lambda_0^{\frac{1 + \rho^2}{2\rho}}} \right)^{\frac{2\rho}{2\rho - 4\rho + 1}} \right]
\]  
(B-87)

\[
\therefore \Sigma_t = \Sigma_0 \left[ 1 - \left( \frac{(\rho^2 - 4\rho + 1) \sigma^2 \lambda_0^{\frac{1 + \rho^2}{2\rho}}}{(1 + \rho^2) \lambda_0^{\frac{1 + \rho^2}{2\rho}}} \right)^{\frac{2\rho}{2\rho - 4\rho + 1}} \right]
\]  
(B-88)

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For propositions 5 and 6:

Now using the results obtained above, we can show that:

\[
\beta_{t} = \frac{\sigma_{t}^{2}}{(1 + \rho^{2})\Sigma_{0}} \lambda_{0}^{1 - \rho_{0}^{2}} \left[ \lambda_{0}^{1 - L} + (1 - L)\lambda t \right]^{-\frac{(1 - \rho_{0}^{2})}{\rho_{0}^{2} - 4\rho + 1}} 
\]

(B-89)

\[
= \frac{\sigma_{t}^{2}}{(1 + \rho^{2})\Sigma_{0}} \lambda_{0} \left( 1 - \frac{\rho^{2} - 4\rho + 1}{1 + \rho^{2}} \frac{\sigma_{t}^{2}}{\Sigma_{0}} \lambda_{0}^{2} t \right) 
\]

(B-90)

Note that for the two stocks whose value processes are uncorrelated,

\[
\beta_{t|\rho=0} = \frac{\sigma_{t}^{2}}{\Sigma_{0}} \lambda_{0} \left( 1 - \frac{\sigma_{t}^{2}}{\Sigma_{0}} \lambda_{0}^{2} t \right)^{-1} 
\]

(B-91)

\[
= \frac{\sigma_{t}^{2}}{\Sigma_{0} - \sigma_{t}^{2} \lambda_{0}^{2} t} = \frac{1}{(1 - t)\sqrt{\Sigma_{0}}} 
\]

(B-92)

Also

\[
\lambda_{t|\rho=0} = \lambda_{0} \quad \forall t \in [0, 1] 
\]

(B-93)

\[
\Sigma_{t|\rho=0}\Sigma_{0} \left( 1 - \frac{\sigma_{t}^{2}}{\Sigma_{0}} \lambda_{0}^{2} t \right) = \Sigma_{0}(1 - t) 
\]

(B-94)

We can also obtain the same result by noting that

\[
\Sigma_{t|\rho=0} = \Sigma_{0}(1 - t) 
\]

(B-95)

\[
\Sigma_{t|\rho=0} = \Sigma_{0}(1 - t) 
\]

(B-96)

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Thus we show that the equilibrium market parameters in this study such as the optimal demands, pricing parameters and conditional variances, all nest the results in the previous literature as special cases of our model.
Chapter 3

C.1 Proof: the SDEs of Yields

Affine Model: Applying Ito’s Lemma yields the following SDE of the domestic yield:

\[
\frac{dy_{td}}{dy_{td}} = \left[ \sum_{i=1}^{N_c} \left\{ \frac{1}{2} \frac{\partial^2 y_{td}}{\partial x_i^2} \kappa_{x_i}(\theta_{x_i} - x_i) - \frac{\partial y_{td}}{\partial x_i} \right\} + \sum_{i=1}^{N_v} \left\{ \frac{1}{2} \frac{\partial^2 y_{td}}{\partial y_i^2} \kappa_{y_i}(\theta_{y_i} - y_i) - \frac{\partial y_{td}}{\partial y_i} \right\} \right] dt \\
+ \sum_{i=1}^{N_c} \frac{\partial y_{td}}{\partial x_i} s_{x_i} \sqrt{\kappa_{x_i}} dW_{x_i} + \sum_{i=1}^{N_v} \frac{\partial y_{td}}{\partial y_i} s_{y_i} \sqrt{\kappa_{y_i}} dW_{y_i},
\]

(C-1)

while the partial derivatives are

\[
\frac{\partial y_{td}}{\partial v_i} = \frac{B_{v_i}(\tau)}{\tau}, \quad \frac{\partial^2 y_{td}}{\partial v_i^2} = 0
\]

(C-2)

\[
\frac{\partial y_{td}}{\partial \tau} = \frac{1}{\tau} \left[ \ln A_{v_i}^{d}(\tau) - B_{v_i}(\tau) v_i - A_{v_i}^{d}(\tau) B_{v_i}(\tau) \tau x_i \right].
\]

(C-3)

Note that the fundamental partial differential equation for bond prices associated with a state variable \(v_i\) can be broken down into two ordinary differential equation:

\[
\frac{A_{v_i}^{d}(\tau)}{A_{v_i}(\tau)} = -\kappa_{v_i} \theta B_{v_i}(\tau), \quad B_{v_i}^{d}(\tau) = 1 - \frac{1}{2} \left( B_{v_i}(\tau) \right)^2 - \left( \kappa_{v_i} + \lambda_{v_i}^{d} \right) B_{v_i}(\tau).
\]

Therefore, equation (C-3) can be rewritten as:

\[
\frac{\partial y_{td}}{\partial \tau} = \frac{1}{\tau^2} \sum_{i=1}^{N_c+N_v} \left[ \ln A_{v_i}^{d}(\tau) - B_{v_i}(\tau) v_i + \kappa_{x_i} \theta_{x_i} B_{v_i}(\tau) \tau - \left( 1 - \frac{1}{2} \left( B_{v_i}(\tau) \right)^2 \right) s_{x_i} \left( \kappa_{x_i} + \lambda_{x_i}^{d} \right) B_{v_i}(\tau) \right] \tau x_i.
\]

Substituting this as well as equations (C-2) into equation (C-1), equation (C-1) can be written as:

\[
\frac{dy_{td}}{dy_{td}} = \left[ \sum_{i=1}^{N_c} \left( \frac{B_{v_i}(\tau)}{\tau} \kappa_{x_i}(\theta_{x_i} - x_i) \right) \right. \\
- \left. \frac{1}{\tau^2} \left\{ \ln A_{x_i}(\tau) - B_{x_i}(\tau) x_i + \kappa_{x_i} \theta_{x_i} B_{x_i}(\tau) + \left( 1 - \frac{1}{2} \left( B_{x_i}(\tau) \right)^2 s_{x_i} \left( \kappa_{x_i} + \lambda_{x_i}^{d} \right) B_{x_i}(\tau) \right) \tau x_i \right\} \right] \\
+ \sum_{i=1}^{N_v} \left( \frac{B_{v_i}(\tau)}{\tau} \kappa_{x_i}(\theta_{x_i} - x_i) \right) \\
- \left. \frac{1}{\tau^2} \left\{ \ln A_{x_i}(\tau) - B_{x_i}(\tau) x_i + \kappa_{x_i} \theta_{x_i} B_{x_i}(\tau) + \left( 1 - \frac{1}{2} \left( B_{x_i}(\tau) \right)^2 s_{x_i} \left( \kappa_{x_i} + \lambda_{x_i}^{d} \right) B_{x_i}(\tau) \right) \tau x_i \right\} \right] dt \\
+ \sum_{i=1}^{N_c} \frac{B_{x_i}(\tau)}{\tau} s_{x_i} \sqrt{\kappa_{x_i}} dW_{x_i} + \sum_{i=1}^{N_v} \frac{B_{v_i}(\tau)}{\tau} s_{v_i} \sqrt{\kappa_{v_i}} dW_{v_i}.
\]
Arranging the terms in square bracket yields the desired results for $dy_{td}$.

Applying the similar procedure results in the expression for $dy_{tf}$ after adjusting for $\alpha$ such that

$$\frac{\partial y_{tf}}{\partial \tau} = \frac{1}{\tau^2} \sum_{i=1}^{N_t+N_s} \left[ \ln A_i^f(\tau) - B_i^f(\tau)v_i + \alpha \frac{1}{2} \left( B_i^f(\tau) \right)^2 s_i^2 - \left( \alpha + \lambda v_i \right) B_i^f(\tau) \right] x_i \tau x_i.$$ 

C.2 Proposition 2: Cross-country Correlation of Interest Rates

1) The first part of the proposition is evident.

2) For the second part, let us consider the following form of correlation. We first define the following variables to avoid the notational clutter.

Let

\begin{align*}
c_1 &= \sum_{i=1}^{N_t} s_i^2 \cdot x_i(t)^2 \\
c_2 &= \sum_{i=1}^{N_s} \beta_i^2 \cdot s_i^2 \cdot (x_i(t) + \pi)^2 \\
L_1 &= \sum_{i=1}^{N_d} s_{y_i}^2 \cdot y_i(t)^2 \\
L_2 &= \sum_{i=1}^{N_f} s_{z_i}^2 \cdot z_i(t)^2
\end{align*}

Hence

$$\rho = \frac{c_1 c_2 + L_1 L_2}{\sqrt{(c_1^2 + L_1^2)(c_2^2 + L_2^2)}} \quad (C-8)$$

Hence

$$c_2^2 L_1^2 + c_1^2 L_2^2 - 2c_1 c_2 L_1 L_2 \geq 0 \quad (C-9)$$

$$(c_1 L_2 - c_2 L_1)^2 \geq 0 \quad (C-10)$$

∴ $\rho = 1$ is achieved at $c_1 L_2 = c_2 L_1$.

∴ $\rho = 1$ if $c_1 c_2 + L_1 L_2 > 0$ and $c_1 L_2 = c_2 L_1$ \hspace{1cm} (C-11)

$\rho = -1$ if $c_1 c_2 + L_1 L_2 < 0$ and $c_1 L_2 = c_2 L_1$ \hspace{1cm} (C-12)
C.3 Bond Prices Implied by the Quadratic Model

From the orthogonality of the state variables, the fundamental PDE of a bond price can be separated into fundamental PDEs associated with each state variables. Here we explore a fundamental PDE associated with a common factor \( x_i(t) \) for a foreign bond price since that PDE can be reduced to other PDEs with relevant restrictions. For simplicity, we suppress the subscript.

\[
\frac{1}{2} \frac{\partial^2 P}{\partial x^2} s_x^2 + \frac{\partial P}{\partial x} \kappa (\theta - x) - \frac{\partial P}{\partial \tau} = P \left[ \alpha + \beta (x + \pi)^2 + \frac{\partial P}{\partial x} (\lambda_{x_0} + \lambda_{x_1} x) \right]. \tag{C-13}
\]

We will conjecture the solution form by

\[ P(t, \tau) = A(\tau) \exp \left[ -B(\tau)x - C(\tau)x^2 \right]. \]

We compute the relevant derivatives based on this conjectured form of the solution, and put them into (C-13):

\[
0 = \frac{1}{2} s_x^2 [B(\tau) + 2C(\tau)^2 - 2C(\tau)] - (B(\tau) + 2C(\tau)x) (\kappa \theta - \lambda_{x_0} - (\kappa + \lambda_{x_1})x) - \frac{A'(\tau)}{A(\tau)} \\
+ B'(\tau)x + C'(\tau)x^2 - [\alpha + \beta x^2 + 2\beta \pi x + \beta \pi^2],
\]

which can be rewritten as:

\[
0 = x^2 [2C(\tau)^2 s_x^2 + 2(\kappa + \lambda_{x_1})C(\tau) + C'(\tau) - \beta] \\
+ x \left[ 2B(\tau)C(\tau)s_x^2 - (2C(\tau)(\kappa \theta - \lambda_{x_0})B(\tau)) + B'(\tau) - 2\beta \pi \right] \\
+ \frac{1}{2} s_x^2 B(\tau)^2 - s_x^2 C(\tau) - (\kappa \theta - \lambda_{x_0})B(\tau) - \frac{A'(\tau)}{A(\tau)} + B'(\tau)^2 - \alpha - \beta \pi^2.
\]

Since the above equation is true for any value of \( x \), we can break down the above PDE into three ODEs:

\[
2C(\tau)^2 s_x^2 + 2(\kappa + \lambda_{x_1})C(\tau) + C'(\tau) - \beta = 0 \\
2B(\tau)C(\tau)s_x^2 - (2(\kappa \theta - \lambda_{x_0})C(\tau) + (\kappa + \lambda_{x_1})B(\tau) + B'(\tau) - 2\beta \pi = 0 \\
\frac{1}{2} s_x^2 \left( B(\tau)^2 - 2C(\tau) \right) - (\kappa \theta - \lambda_{x_0})B(\tau) - \alpha - \beta \pi^2 - \frac{A'(\tau)}{A(\tau)} = 0.
\]

ODE for \( C(\tau) \): The ODE for \( C(\tau) \) is
\[ 2C(\tau)^2 s_x^2 + 2(\kappa + \lambda_{x_0})C(\tau) + C'(\tau) - \beta = 0, \]

which can be rewritten as:

\[ C'(\tau) = \beta - 2s_x^2 C(\tau)^2 - 2(\kappa + \lambda_{x_1})C(\tau). \]

In order to solve for \( C(\tau) \), we rearrange the terms and take integral:

\[- \int_{0}^{\tau} \frac{1}{2s_x^2 C(\tau)^2 + 2(\kappa + \lambda_{\tau})C(\tau) - \beta} ds = \tau + K,\]

where \( K \) is a constant. Solving for the integral leads to:

\[- \int_{0}^{\tau} \frac{1}{2s_x^2 C(\tau)^2 + 2(\kappa + \lambda_{\tau})C(\tau) - \beta} ds = \frac{1}{2\gamma} \ln \left[ \frac{C(\tau) + \frac{\kappa + \lambda + \gamma}{2s_x^2}}{C(\tau) + \frac{\kappa + \lambda - \gamma}{2s_x^2}} \right] = \tau + K,\]

where \( \gamma = \sqrt{(\kappa + \lambda_{x_1})^2 + 2\beta s_x^2} \). Taking logarithmic transformation in both sides of the equation yields:

\[ \frac{C(\tau) + \frac{\kappa + \lambda + \gamma}{2s_x^2}}{C(\tau) + \frac{\kappa + \lambda - \gamma}{2s_x^2}} = \exp(2\gamma \tau) + K. \]

Since the bond price at its maturity date is 1, \( C(0) = 0 \), which results in \( K = \frac{\kappa + \lambda + \gamma}{\kappa + \lambda - \gamma} \).

Therefore,

\[ C(\tau) = \frac{\frac{\kappa + \lambda + \gamma}{2s_x^2} \left[ \exp(2\gamma \tau) - 1 \right]}{1 - \frac{\kappa + \lambda + \gamma}{\kappa + \lambda - \gamma} \exp(2\gamma \tau)}. \]

Arranging the terms leads to the desired result.

**ODE for \( B(\tau) \):** The ODE for \( B(\tau) \) is given as:

\[ 2B(\tau)C(\tau)s_x^2 - 2(\kappa \theta - \lambda_{x_0})C(\tau) + (\kappa + \lambda_{x_1})B(\tau) + B'(\tau) - 2\beta \pi = 0, \]

which can be rewritten as:

\[ B'(\tau) + B(\tau) \left[ 2C(\tau)s_x^2 + (\kappa + \lambda_{x_1}) \right] = 2\kappa \theta C(\tau) + 2\beta \pi. \]

Using integration rule:

\[ B(\tau) = \frac{\int e^\int p(\tau) d\tau q(\tau) d\tau}{e^\int p(\tau) d\tau}, \]
where

\[ p(\tau) = 2C(\tau) s_2^2 + (\kappa + \lambda x_1) \]

\[ q(\tau) = 2(\kappa \theta - \lambda x_0) C(\tau) + 2\beta \pi. \]

Thus first we will solve for \( \int p(\tau) d\tau \):

\[ \int p(\tau) d\tau = \int [2C(\tau) s_2^2 + (\kappa + \lambda x_1)] d\tau, \]

where

\[ \int C(\tau) d\tau = \int \frac{\beta (e^{2\gamma \tau} - 1)}{(\kappa + \lambda x_1 + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma} d\tau \]

\[ = \frac{\beta}{(\kappa + \lambda x_1 + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma} \left[ \frac{1}{\kappa + \lambda x_1 + \gamma} - \frac{1}{2\gamma} \right] \]

\[ = \frac{\beta}{(\kappa + \lambda x_1 - \gamma)(\kappa + \lambda x_1 + \gamma)} \left[ \frac{(\kappa + \lambda x_1 + \gamma) \tau - \ln[(\kappa + \lambda x_1 + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma]}{(\kappa + \lambda x_1 - \gamma)(\kappa + \lambda x_1 + \gamma)} \right], \]

which leads to

\[ \exp \left( \int p(\tau) d\tau \right) = \left[ (\kappa + \lambda x_1 + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma \right] e^{-\gamma \tau}. \]

Secondly

\[ \int e^{\int p(\tau) d\tau} q(\tau) d\tau = \int \frac{[(\kappa + \lambda x_1 + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma]}{e^{\gamma \tau}} [2(\kappa \theta - \lambda x_0) C(\tau) + 2\beta \pi] d\tau \]

\[ = 2(\kappa \theta - \lambda x_0) \beta \int \frac{(e^{2\gamma \tau} - 1)}{e^{\gamma \tau}} d\tau + 2\beta \pi \int \frac{(\kappa + \lambda x_1 + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma}{e^{\gamma \tau}} d\tau \]

\[ = \frac{2\beta}{\gamma e^{\gamma \tau}} \left[ (\kappa \theta - \lambda x_0) + (\kappa + \lambda x_1 - \gamma) \pi + (\kappa \theta - \lambda x_0 + (\kappa + \lambda x_1 + \gamma) \pi) e^{2\gamma \tau} \right]. \]

Thus \( B(\tau) \) is

\[ B(\tau) = \frac{\int e^{\int p(\tau) d\tau} q(\tau) d\tau}{\int e^{\int p(\tau) d\tau} d\tau} \]

\[ = \frac{2\beta}{\gamma e^{\gamma \tau}} \left[ (\kappa \theta - \lambda x_0) + (\kappa + \lambda x_1 - \gamma) \pi + (\kappa \theta - \lambda x_0 + (\kappa + \lambda x_1 + \gamma) \pi) e^{2\gamma \tau} \right] + K \]

\[ = \frac{2\beta}{\gamma e^{\gamma \tau}} \left[ (\kappa \theta - \lambda x_0) + (\kappa + \lambda x_1 + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma \right] e^{-\gamma \tau} \]

\[ = \frac{2\beta}{\gamma e^{\gamma \tau}} \left[ (\kappa \theta - \lambda x_0) + (\kappa + \lambda x_1 + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma \right] e^{-\gamma \tau} \]

(C.14)
Imposing $B(0) = 0$ yields $K = -\frac{2\beta}{\gamma} [2(\kappa\theta - \lambda_{x_0}) + 2(\kappa + \lambda_{x_1})\pi]$. Substituting the solution for $K$ into (C-14) results in the desired result.

ODE for $A(\tau)$: The ODE for $A(\tau)$ is given as

$$\frac{1}{2} s_0^2 (B(\tau)^2 - 2C(\tau)) - (\kappa\theta - \lambda_{x_0})B(\tau) - (\alpha + \beta\pi^2) - \frac{A'(\tau)}{A(\tau)} = 0.$$ 

Thus,

$$\ln A(\tau) = \frac{1}{2} s_0^2 \int B(\tau)^2 d\tau - \frac{s_0^2}{2} \int C(\tau) d\tau - (\kappa\theta - \lambda_{x_0}) \int B(\tau) d\tau - (\alpha + \beta\pi^2) \tau + K.$$

We have already computed $\int C(\tau) d\tau$. Thus here we will concentrate on $\frac{1}{2} s_0^2 \int B(\tau)^2 d\tau - (\kappa\theta - \lambda_{x_0}) \int B(\tau) d\tau$.

First,

$$\frac{1}{2} \beta^2 s_0^2 \int \frac{4 (e^{\gamma \tau} - 1)^4 (\kappa\theta - \lambda_{x_0})^2}{\gamma^2 [(\kappa + \lambda_{x_0} + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma]^3} d\tau - \beta (\kappa\theta - \lambda_{x_0}) \int \frac{2(\kappa\theta - \lambda_{x_0}) (e^{\gamma \tau} - 1)^2}{\gamma [(\kappa + \lambda_{x_0} + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma]} d\tau$$

$$+ \frac{s_0^2}{2} \int \frac{2\beta^2 (e^{\gamma \tau} - 1)^3 \pi [(\kappa\theta - \lambda_{x_0}) \{(\kappa + \lambda_{x_1} + \gamma) (e^{\gamma \tau} - 1) + 2\gamma\}^2]}{\gamma^2 [(\kappa + \lambda_{x_0} + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma]^3} d\tau$$

$$+ \frac{s_0^2}{2} \int \frac{4\beta^2 (e^{\gamma \tau} - 1)^3 \pi [(\kappa\theta - \lambda_{x_0}) \{(\kappa + \lambda_{x_1} + \gamma) (e^{\gamma \tau} - 1) + 2\gamma\}]}{\gamma^3 [(\kappa + \lambda_{x_0} + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma]} d\tau$$

$$- (\kappa\theta - \lambda_{x_0}) \int \frac{2\beta (e^{\gamma \tau} - 1) \pi [(\kappa\theta - \lambda_{x_0}) \{(\kappa + \lambda_{x_1} + \gamma) (e^{\gamma \tau} - 1) + 2\gamma\}^2]}{\gamma [(\kappa + \lambda_{x_0} + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma]} d\tau$$

First,

$$\frac{1}{2} \beta^2 s_0^2 \int \frac{4 (e^{\gamma \tau} - 1)^4 (\kappa\theta - \lambda_{x_0})^2}{\gamma^2 [(\kappa + \lambda_{x_0} + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma]^3} d\tau - \beta (\kappa\theta - \lambda_{x_0}) \int \frac{2(\kappa\theta - \lambda_{x_0}) (e^{\gamma \tau} - 1)^2}{\gamma [(\kappa + \lambda_{x_0} + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma]} d\tau$$

$$\frac{\beta (\kappa\theta - \lambda_{x_0})^3}{\gamma^2 (\kappa + \lambda_{x_0} + \gamma)} + \frac{2\beta^2 (\kappa\theta - \lambda_{x_0})^2 (\kappa + \lambda_{x_1}) [2(\kappa + \lambda_{x_1}) (1 - e^{\gamma \tau}) - \gamma^2 - 2e^{\gamma \tau}] }{\gamma^3 [(\kappa + \lambda_{x_0} + \gamma) (e^{2\gamma \tau} - 1) + 2\gamma]}$$
and

\[ s^2 \int \frac{2\beta^2 (e^{\gamma \tau} - 1)^2 \pi^2 \left[ (\kappa \theta - \lambda_x) \left( (\kappa + \lambda_x + \gamma) (e^{\gamma \tau} - 1) + 2\gamma \right)^2 \right]}{\gamma^2 \left[ (\kappa + \lambda_x + \gamma) (e^{\gamma \tau} - 1) + 2\gamma \right]^2} d\tau \]

\[ = \frac{2\beta^2}{\gamma^2} \left[ \beta s^2 e^{\gamma \tau} + \frac{(\gamma - \kappa - \lambda_x)(\kappa + \lambda_x)}{\gamma^2 \left[ (\kappa + \lambda_x + \gamma) (e^{\gamma \tau} - 1) + 2\gamma \right]} \right] \]

\[ = \frac{2\beta^2}{\gamma^2} \left[ \frac{4\beta^2 (e^{\gamma \tau} - 1)^2 \pi \left[ (\kappa \theta - \lambda_x) \left( (\kappa + \lambda_x + \gamma) (e^{\gamma \tau} - 1) + 2\gamma \right)^2 \right]}{\gamma^2 \left[ (\kappa + \lambda_x + \gamma) (e^{\gamma \tau} - 1) + 2\gamma \right]^2} d\tau \]

\[ = \frac{4\beta \pi (\kappa \theta - \lambda_x) \left( 2 (1 - e^{\gamma \tau}) (\kappa + \lambda_x)^2 + e^{\gamma \tau} \gamma^2 - 2(\kappa + \lambda_x) \gamma + (\kappa + \lambda_x + \gamma) \frac{\gamma^2}{2} \right)}{\gamma^2 \left[ (\kappa + \lambda_x + \gamma) (e^{\gamma \tau} - 1) + 2\gamma \right]} + \ln \left[ (\kappa + \lambda_x + \gamma) (e^{\gamma \tau} - 1) + 2\gamma \right] \]

\[ = -\frac{(\kappa \theta - \lambda_x)}{\gamma} \int 2\beta (e^{\gamma \tau} - 1) \pi \left[ (\kappa \theta - \lambda_x) \left( (\kappa + \lambda_x + \gamma) (e^{\gamma \tau} - 1) + 2\gamma \right) \right] d\tau \]

\[ = \frac{2\beta \pi (\kappa \theta - \lambda_x) \left( \frac{e^{\gamma \tau} \sqrt{\kappa + \lambda_x + \gamma}}{\sqrt{\gamma - \kappa - \lambda_x}} \right)}{\gamma} + \ln \left[ (\kappa + \lambda_x + \gamma) (e^{\gamma \tau} - 1) + 2\gamma \right] \]

Making sum of the above expressions and imposing \( A(0) = 1 \) results in the desired result.
C.4 Estimation by EMM

Description of the Data:

The weekly data for U.S. and Japanese Eurocurrency rates with maturities of 3 month and 1 year are obtained from Datastream. For exchange rate data. There are 1066 observations for each series of euro rates, which covers the period extending from Aug. 02, 1978 to Dec. 30, 1998. The rates are annualized middle quotes. Tables 1 and 2 provide the summary statistics. Issue of temporal aggregation and consequent choice of data has to be mentioned (cf: Bekaert 1997). A notable feature of the data is the negative correlations between EuroYen and Eurodollar rates for both 3 month and 1 year. The distributional properties of the yields change also show variations over time.

Estimation Method and Procedure:

In order to estimate the model we employ the Efficient Method of Moments (EMM) proposed by Duffie and Singleton (1993) and Gallant and Tauchen (1996). The EMM is a natural choice since our dynamic non-linear model has unobserved state variables, and thus standard statistical methods, both classical and Bayesian, are not usually applicable. Also we don’t want to be too much concerned about the issue of discretization bias. This is largely due to the fact that the exact likelihood of the entire state vector can not be obtained nor the integration required to eliminate observables is feasible. While determining the likelihood of the nonlinear dynamic system that has unobservables is infeasible, simulating the evolution of state vector is often quite practicable, which does underlie the estimation with EMM, an efficient variant of the Simulated Method of Moments (SMM).

In EMM estimation we first summarize the data by using quasi-maximum likelihood to project the observed data onto a transition density that is close approximation to the true data generating process. This transition density is called the auxiliary model and its score is called the score generator for EMM. A Hermite series representation of the transition density of the observable process is suggested as a convenient general purpose auxiliary model in this connection.

Once a score generator is in hand, given a parameter setting for the system, one may use simulation to evaluate the expected value of the score under the stationary density of the system and compute a chi-squared criterion function. A non-linear optimizer is used to find the parameter setting that minimizes the criterion.

If the auxiliary model encompasses the true data generating process, then quasi maximum likelihood estimates become sufficient statistics and EMM is fully efficient (Gallant and Tauchen (1996)). If the auxiliary model is a close approximation to the data generating process, then one can expect the efficiency of EMM to be close to that of maximum likelihood (Gallant and Long (1997); Tauchen (1997)). Another advantage over the classical Method of Moment estimation is that unlike the classical Method of Moment estimation, we can apply standard model selection criteria such as the Schwarz BIC (Bayes information criterion)(Schwarz (1978)) and Akaike AIC information criterion (Akaike (1969))

In what follows, we will briefly discuss steps involved in EMM estimation in conjunction with our model.

**Projection Step : SNP Density Estimation**

EMM estimates a model by matching moments implied by the system to moments implied by the transition density for observables. To this end we first need to find the best moment function that the Euro- currency data we’re using dictates. The requisite tran-
sition density of the respective interest rates is determined by projecting the data onto the Hermite polynomial of the innovation density. Then the resulting semi-nonparametric (SNP) density of the observed interest variables is derived as a location-scale transform of the Hermite expansion. Density parameters are then computed using the quasi-maximum likelihood. We resort to the Schwarz BIC criterion (Schwarz (1978)) to determine on the optimal number of parameters including the first and second moments of the interest rates. The score of the auxiliary model obtained turns out to be the best moment function in light of the properties that the estimators of the model parameters have as explained above.

The particular Hermite expansion employed in order to approximate the innovation density of the data generating process has the following form

\[
h_K(z|x) = \frac{P^2(z, x)\phi(z)}{P^2(u, x)\phi(u)du}, \quad \text{(C-15)}
\]

where

\[
P(z, x) = \sum_{|\alpha|=0}^{K_x} \left( \sum_{|\beta|=0}^{K_x} a_{\alpha\beta} x^\beta \right) z^\alpha
\]

and \(\phi(z)\) denotes the density function of the multivariate Gaussian distribution with mean zero and the identity as its variance-covariance matrix. The location-scale transformation \(y = Rz + \mu\), where \(R\) is an upper triangular matrix and \(\mu\) is an \(M\)-vector, completes the definition of the SNP density\(^{25}\).

\[
f_K(y|x, \theta) = |det(R_x)|^{-1}h_K[R_x^{-1}(y - \mu_x)|x], \quad \text{(C-16)}
\]

The parameter vector \(\theta\) that characterizes SNP density comprises three different subgroups:

\(^{25}\) Similarly, we can expand the square root of the stationary joint density of the data generating process in a Hermite series, which gives SNP density that approximates the transition density of the system (see Gallant, Hsieh, and Tauchen (1991))
1) the coefficients of the polynomial $\mathcal{P}(z, x)$.

2) the conditional mean parameters in the Gaussian part of SNP.

3) the conditional variance parameters in the Gaussian part of SNP.

The SNP density is flexible enough to accommodate data series with different properties such as fat, t-like tails, conditional heteroskedasticity, ARCH, GARCH, etc. Those different characterizations are embodied in the following parameter structure of SNP,

$$(L_u, L_g, L_r, L_p, K_z, I_z, K_x, I_x)$$

$L_p$ denotes the number of lags in the $x$ part of the polynomial $\mathcal{P}(z, x)$, $L_u$ the number of lags in the conditional mean of $y_t$, and $L_r$ and $L_g$ represent the numbers of lags in the ARCH and GARCH specification parameters, respectively. $K$ variables denote the degree of the polynomials in $x$ or $z$ in $\mathcal{P}$ and $I$ indicates the degree of suppressed interactions.

The SNP parameters of each candidate model is estimated by minimizing

$$s_n(\theta) = -(1/n) \sum_{t=1}^{n} \log[f(y_t|x_{t-1}, \theta)].$$

Various specifications of SNP density are then evaluated based on some model selection criterion. We use the Schwarz Bayes information criterion computed as

$$BIC = s_n(\hat{\theta}) + (1/2)(p \theta/n) \log(n)$$

As Gallant and Tauchen (1996) suggested, we adopt an upward fitting strategy which dictates to estimate the parameters of the SNP density at each level and then use the estimates as starting values of more richly parameterized SNP at the next level until an adequate model is determined. Based on the BIC criterion specified above we determine that our preferred model fit is represented by $\{L_u, L_g, L_r, L_p, K_z, I_z, K_x, I_x\} = \{1, 0, 4, 1, 4, 3, 0, 0\}$. Thus the conditional means of the yields are to be fully characterized by one lag of the data, and the conditional volatilities show an ARCH behavior of order 4. The departure from
normality or semi-parametric nature of the data is also reflected in $K_z$, which is a positive value for the data we use. $I_z = 3$ indicates that the interactions are suppressed among the highest three order terms. The data do not seem to ask to incorporate lags of the process in modelling the coefficients of the Hermite polynomial. The specification obtained in this paper reconfirms the usual nature of term structure data, which is found in other studies. We transformed the data using a spline transformation for the reason exploited in Gallant and Tahoken. There exists two types of potential problem in estimating the term structure model using EMM routine:  

i) fixed points can be introduced into the established location function $\mu_x$ that lead to unrealistic simulations. and ii) during the course of simulation, the SDE may generate explosive data at parameter settings that must be examined.

**Estimation Step**

Now that a score generator is in hand, we use simulation to generate a data set given the parameter settings of the term structure models we investigate. The scores of the fitted SNP density serve as moment conditions and the models are estimated by minimizing the quadratic form specified below.

$$
\hat{\rho}_n = \arg \min \, m'(\rho, \tilde{\theta}_n)(\tilde{\Gamma}_n)^{-1}m(\rho, \tilde{\theta}_n)
$$

where

$$
m'_n(\rho) = \frac{1}{N} \sum_{t=1}^{N} \tilde{\psi}_f(\hat{y}_{t-L}, \cdots, y_{t-1}, \hat{y}_0)
$$

and $\tilde{\Gamma}_n$ is the weighting matrix.

Specification tests of the models are performed using asymptotics of the estimator, i.e., under the null hypothesis that the stationary density represented by the model is correct,
then

\[ L_0 = n \times m'(\rho, \tilde{\theta}_n)(\tilde{\Gamma}_n)^{-1}m(\rho, \tilde{\theta}_n) \sim \chi^{2}_{p_\theta - p_\rho} \]

where \( p_\theta \) and \( p_\rho \) denote the dimensions of SNP density and model parameters, respectively.

It was shown (Gallant and Tauchen (1996)) that if the auxiliary model encompasses the true data generating process, then quasi maximum likelihood estimate becomes sufficient statistics and EMM is fully efficient. Also one can expect the efficiency of EMM to be close to that of maximum likelihood as long as the auxiliary model is a close approximation to the data generating process (Gallant (1997); Tauchen (1997)).

Estimation of the Models

We first estimate the QTSM with all the common and local factors that are independent. The parameter estimates thus obtained are then compared to those of the correlated factor QTSM. Table 3 shows the estimation results along with the specification tests for respective models.

EMM Specification Tests

The results of specification tests with the score generator \( \{1, 0, 4, 1, 4, 3, 0, 0\} \) are presented in Table 3. The model with the independent local as well as common factors serves the benchmark case. The estimation results obtained from this model are used as starting values for the correlated common-local factor setting. \( \tilde{z} \) statistic for the QTSM with independent common/local factor strongly suggests a rejection of the restrictions imposed on the specification of the prices of risk, which renders the need to look into the QTSM whose unobserved state variables are correlated.

For each individual parameter, we first note that the parameters for lower bounds of instantaneous short rate are all positive as expected. Also the lower bound for Japanese
interest rate is confirmed to be smaller than that for US rate.

While common state variable appears to be more persistent, none of them shows any sign of non-stationarity. Which is partly due to the specification of measurement equation. Constant term of the prices of risk term is starkly larger for the common state variable than both local variables. It is a strong indication that the common state variable may well be the predominant determinant of Japanese term structure. We will get to this point in the reprojection section.

We can obtain further insight into the model performance by looking at the scores of best fitted SNP score generator. The parameter estimates of score generator and t-statistics are provided in table 4. As with the US domestic yield data, most of the parameters which govern the coefficients of Hermite polynomial for the mean of the VAR are significant (12 out of 17). However, unlike the previous study to estimate QTSM's with the domestic model (for example, Ann, Dittmar and Gallant), the fitted scores with respect to each SNP parameter indicate the QTSM is able to capture ARCH type behavior extant in the international term structure data.

**Reprojection of the Unobserved State Variables**

Based on EMM estimates, we construct the filtered estimates of latent factors with extended Kalman filter. Essentially, we express our models in state space forms and then, after applying the first order Taylor approximation to the measurement equation, we utilize the usual Kalman filtering to the transformed model. A brief exposition about extended Kalman filtering used in for our model is presented in appendix C. More thorough treatment can be found on Anderson and Moore(1979).

Table 6 confirms our that common factor is indeed important to explain the joint movement of U.S. and Japan yields over time. we use the EMM parameter estimates and the fitted values of the state variables to compute the variances of eurocurrency rates and their decomposition. We find that almost 80 percent of the U.S. and 97 percent of the Japan
term structures can be explained by one common factor. In particular, Japan's local factor does not seem to have any explanatory power. Figure 2 provides similar evidence on the importance of the common factor.

However, these results should be interpreted with caution. Recently, several international quadratic term structure papers such as Inci and Lu (2004) and Leippold and Wu (2002) find that we might need more than three correlated factors to explain the international term structures. Furthermore, Inci and Lu (2004) argue that exchange rate must be added to the measurement equations to describe data accurately.

Therefore, we are currently conducting Bayesian estimation also of the general models to explain international term structure and exchange rate movements together. We expect that bayesian approach with parameter uncertainty can resolve some empirical anomalies or at least provides us with new perspective. For instance, by careful prior sensitivity analysis or Bayes factor comparison, we might check what kind of prior belief is consistent with the current seemingly incomprehensible empirical phenomenon.
C.5 Extended Kalman filtering

Measurement equation:

\[ z_k = h(x_k, k) + v_k \]  \hspace{1cm} (C-17)

Transition equation:

\[ x_k = f(x_{k-1}, k - 1) + w_k \]  \hspace{1cm} (C-18)

Where: \( f \) and \( h \) are nonlinear functions of the state \( x \) and \( w_k \) and \( v_k \) are gaussian noises.

If we know an unbiased estimate \( \hat{x}_{k-1|k-1} \) of the state at the instant \( k - 1 \), we can develop \( f(x_{k-1}, k - 1) \) in a first order Taylor series approximation around \( \hat{x}_{k-1|k-1} \).

\[ x_k = f(\hat{x}_{k-1|k-1}, k - 1) + f_x(\hat{x}_{k-1|k-1}, k - 1)e_{k-1|k-1} \]  \hspace{1cm} (C-19)

Where: \( e_{k-1|k-1} = x_{k-1} - \hat{x}_{k-1|k-1} \) and \( f_x = \frac{\partial f}{\partial x} \).

Likewise, before the arrival of the measurement \( z_k \) the best estimate of \( x_k \) is its mathematical expectation: \( \hat{x}_{k|k-1} = E[x_k] = f(\hat{x}_{k-1|k-1}, k - 1) \) We can now develop the first order Taylor series approximation of \( h(x_k, k) \) around the new estimate: \( \hat{x}_{k|k-1} \)

\[ z_k = h(\hat{x}_{k|k-1}, k) + h_x(\hat{x}_{k|k-1}, k)e_{k|k-1} \]  \hspace{1cm} (C-20)

After the arrival of the measurement \( z_k \) the best estimate of \( x_k \) is:

\[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k[z_k - h(\hat{x}_{k|k-1}, k)] \]  \hspace{1cm} (C-21)

Where \( K_k = P_{k|k-1}h_x^T[h_x; P_{k|k-1}h_x^T + R_k + 1]^{-1} \) and \( P_{k|k-1} = E[e_{k|k-1}e_{k|k-1}^T] = f_x.P_{k-1|k-1}.f_x^T + Q_k \)

The new error covariance matrix is:

\[ P_{k|k} = E[e_{k|k}e_{k|k}^T] = [I - K_k.h_x].P_{k|k-1}[I - K_k.h_x]^T + K_k.R_k.K_k^T \]  \hspace{1cm} (C-22)
Where $Q_k$ and $R_k$ are the covariances of the process and measurement noises respectively. By repeatedly applying above techniques, we can easily construct filtered estimates of latent factors.

In the current quadratic model, we have the following measurement equation specifications

$$Z_k = \begin{pmatrix} \text{yield(jpn, 3m)} \\ \text{yield(usa, 3m)} \\ \text{yield(jpn, 1y)} \\ \text{yield(usa, 1y)} \end{pmatrix}$$  \hspace{1cm} (C-23)

$$h(x_k, k) = M + H * X_k + J * X_k^2$$  \hspace{1cm} (C-24)

Where:

1. $i$th element of $M$: $\alpha_i + \beta_i \pi_i^2 - \ln A_{\text{common}}^i(\tau)/\tau - \ln A_{\text{local}}^i(\tau)/\tau$

2. $i$th element of $H * X$: $(B_{\text{common}}^i(\tau)/\tau) * X_{\text{common}}(k) + (B_{\text{local}}^i(\tau)/\tau) * X_{\text{local}}(k)$

3. $i$th element of $J * X^2$: $(C_{\text{common}}^i(\tau)/\tau) * X_{\text{common}}^2(\tau) + (C_{\text{local}}^i(\tau)/\tau) * X_{\text{local}}^2(k)$

4. $i = \{jpn, usa\}$ and $\tau = \{3/12, 1\}$

Therefore, we need to get the first order Taylor approximation of $J * X_k^2$ around the estimate $X_{k|k-1}$

$$J * X_k^2 = J * \dot{X}_{k|k-1}^2 + 2 * \dot{X}_{k|k-1} * (X_k - \dot{X}_{k|k-1})$$

And our transition equation has the following linear form.

$$X_k = \kappa * \theta + (1 - \kappa * \Delta) * X_{k-1} + \Sigma * \sqrt{\Delta} \varepsilon_k$$  \hspace{1cm} (C-25)
C.6 Sampling Algorithm

1. The Observed State Variables
The measurement equation of QTSM is not in latent factors. So we use the drawing methodology suggested in previous literature under reasonable regularity condition, it can be shown that

\[ \Delta t^{-\frac{1}{2}}(\hat{x}_t - \frac{1}{2}(\hat{x}_{t-1} + \hat{x}_{t+1}) \sim N(0, \frac{1}{2}\sigma^2_{t-1}) \]

as \( \Delta t \to 0 \). (See Eraker 2001 for details)

So we have approximately

\[ \hat{x}_t | \hat{x}_{t-1}, \hat{x}_{t+1} \sim N\left(\frac{1}{2}(\hat{y}_{t-1} + \hat{y}_{t+1}), \frac{1}{2}\gamma^2_{t-1}, \Delta t\right) \]

2. Drift parameters of state equation \( \kappa_i \) and \( \theta_i \).
We adopt the random walk M-H algorithm.

\[ \alpha = \min \left\{ \frac{P(Y | X, \gamma_i, \kappa_i, \theta_i)}{P(Y | X, \gamma_i, \kappa_i', \theta_i')}, 1 \right\} \]

3. Diffusion parameters of state equation \( \Sigma \)

\[ \alpha = \min \left\{ \frac{P(Y | X, \gamma_i, \Sigma_{ii})}{P(Y | X, \gamma_i, \Sigma_{ii}', \Sigma_{ii}'')}, 1 \right\} \]

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Bibliography


