Contracting and Information Sharing in Supply Chain Management

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ABSTRACT

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Starting with the aim of an actual contract implementation, this thesis contributes to the supply chain contracting literature at various levels in vertically differentiated settings.

We first identify the economic distortions that arise when a manufacturer sells vertically differentiated products through a retailer and propose several coordinating contracts in both monopolistic and competitive settings. We later derive the equilibrium when a gray market emerges and show the efficiency of wholesale pricing in the existence of gray markets. We also discuss how the performance of the supply chain contracts studied earlier in the literature starts to change in these settings. We then identify the inefficiencies in multi-supplier one-manufacturer settings where the manufacturer’s decision is what quality to choose from each supplier. We propose various contracts to mitigate these inefficiencies. Finally, considering an inventory model with advance supply information, where the supply information is modeled as dynamic forecasts of capacity availability, we characterize the optimal policy for such systems and develop a heuristic and derive the operating environments under which information is more valuable.
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Chapter 1

Introduction

In supply chain management, a key challenge is how to align the economic incentives between different organizations so that they all act in the best interest of the overall supply chain. This has been successfully studied in OM literature. On the other hand, the planning processes that used to be performed within the four walls of a company are replaced by more anticipatory business models that involve collaboration and information sharing between downstream and upstream supply chain partners. In parallel to these developments in industry, the literature on collaboration and information sharing has advanced in several directions. While the first three essays of this dissertation contribute to the contracting literature at different levels, the last essay contributes to the collaboration and information sharing literature.

The first three essays are interrelated and are motivated by a sponsored research project with a leading semi-conductor manufacturer. The firm designs, produces and sells several product families of CPUs (processors) and chip sets. These products are targeted for desktop, server and mobile businesses. Each product family consists of CPUs whose
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Performance (used interchangeably with "quality") ranges from highest to lowest. These are vertically differentiated parts where the consumer always prefers high quality over low given everything else is the same. The firm has two primary customer segments: (1) OEMs: Large original equipment manufacturers like Dell, HP IBM and (2) The channel: distributors and resellers. The channel business has special strategic importance to the firm as it represents a competitive counterbalance to large OEMs who are currently financially not healthy. Based on these, our sponsor wanted to focus our research efforts on better understanding and improving its channel business, specifically targeting the trading relationship between the firm and its distributors.

The problem analyzed in the first essay is that within a given family, different performance levels are produced as a result of a "binning" process during production, which results in a random (but predictable) yield of parts that are able to run at different clock speeds. Hence, the cost of producing different quality levels are not that different, yet the firm is able to charge significantly higher prices for high quality parts. For example, a 2.8GHz CPU may be priced 100% higher than a 1.8GHz CPU from the same family. As a result, margins at the higher end of the product line are much larger. But since the firm attempts to capture this value through a simple wholesale pricing scheme, distributors face significantly higher costs for high quality parts and do not enjoy nearly the same margin advantage selling them. Hence the economics of the two parties are poorly aligned. While the manufacturer's incentive is to maximize its gross margin and encourage selling up in the product line without sacrificing too much from overall volume, the distributor's incentive is to sell as many units as possible. To summarize, using the firm's terminology, the manufacturer's incentive is to "sell-up" and "sell-through", while the distributor's
incentive is just to “sell fast”. This tension between the manufacturer and the distributor creates distortion and incentive conflicts in the channel. The analysis in this essay first identifies these inefficiencies in the system. Several contracts are designed to restore these inefficiencies and align the economics between the manufacturer and its distributor.

While the analysis is rooted in the specific sponsor firm and industry, the issue we address is generic. On a theoretical level, this work makes unique contributions. In general, the coordination literature has not addressed selling in a vertical differentiation setting, and economic models of vertical differentiation almost all assume direct selling without any intermediary. The work in the first essay tries to fill this gap in the literature. The practical insights gained throughout this project also offer a contribution to the existing literature on contracting and supply chain coordination.

As a result of our analysis, a customized version of a revenue sharing program was proposed to help improve the overall supply chain profit and help the manufacturer. However, the program was not piloted due to several objections from different departments. The primary concern was the presence of gray markets. A gray market refers to those genuine brand owner’s products that are sold by unauthorized retailers. The objection was that some of the contracts and incentive programs proposed would fail to meet their objectives because of this gray market factor. While this resistance was unfortunate, it provided the motivation for the second essay of the dissertation. The main goal of this part of the dissertation is to understand the impact of a gray market on the decisions that the supply chain partners make and the performance of common supply chain contracts studied earlier in the literature and explore new contracts that can be more effective. Toward that main goal, we create a model with several non-competing authorized retailers
with uncertain demand for a given period and first characterize the equilibrium gray market price based on a market-clearing model. The decentralized and centralized system in a gray market environment are compared and it is shown that the wholesale pricing contract itself is "almost" coordinating. A partial buy-back contract that results in perfect coordination for this system is studied as well. With this essay, we show that the type of business environment the firms operate in dictates which contract may be preferable over another. In this case, the existence of a gray market makes wholesale pricing contract an "acceptable" contract while it also requires the modification of standard contract forms.

Earlier, we said that the channel business consisted of distributors and resellers. The third essay of the dissertation is motivated by a problem observed between the manufacturer and its resellers. A reseller builds no-brand computers, which compete with larger OEMs. Therefore they need to be quite cost-effective. They source different components such as hard-disk, CPU and memory from different suppliers and assemble them into a computer. When deciding on the bill-of-material for the computer, they need to choose a quality level for each component. For example, they could pick a 2.8GHz CPU, 140GB hard disk and 1GB memory. The trade-off the reseller faces is this: The quality levels of components determines the overall quality (performance) of the computer, and as is common in all vertically differentiated products, this overall value will be higher as individual quality levels increase. However, since suppliers charge more for higher quality parts, the total cost increases with quality as well. The semi-conductor manufacturer is just one of several suppliers to resellers and, to increase profits, they want to encourage sell-up within their product line. On the other hand, there are suppliers providing other components who also want to sell higher quality parts. The result is a problem of competition among components
for share of the total design budget.

We assume that the main trade-off here can be studied as a strategic design problem in which we maximize the gap between the total product value and its cost. Based on a model that translates individual component quality levels into a final end-product quality, we first define the reseller's strategic design problem. Each supplier, in turn needs to decide on a price per quality level of its component. As they increase their wholesale prices, their margin increases, but the quality level demanded by the manufacturer will decrease. We characterize the strategic interaction among these suppliers and show that the quality levels that the reseller demands from each supplier is lower in a decentralized system than those in a centralized system. We then identify potential contracts and mechanisms such as revenue sharing and quality-price discounting that restore efficiency. The existing work on supply chain coordination and contracting focuses on price and quantity decisions and not on quality or product design as we do in this essay. Although bundling and product design are relevant research streams in terms of the basic decision making, to the best of our knowledge none of the prior work in this area looks at the incentive conflicts and coordination issues associated with such decision making in multiple-supplier, single-manufacturer environments. Our work makes unique contributions to these literature streams as well.

While our sponsor firm faces competition from other semi-conductor companies, competition has not been included in our essays for two reasons. First, the firm has significant market power in its core products and hence a monopoly model is a reasonable approximation. Second, while including manufacturer competition would be a desirable extension, it would make the models significantly more difficult. We will comment throughout the
essays on where we think the monopoly assumption may impose significant limitations in terms of our results and insights.

Given the fact that the biggest challenge of supply chain management is the uncertainty on both sides of the demand-supply equation, supply chain implementations in industry place almost equal emphasis on both the demand and supply collaboration. By demand collaboration, we refer to companies sharing their downstream demand information. Similarly, suppliers upstream in the supply chain typically have better information about the state of their facilities and their potential ability to deliver any orders in the immediate future. Such information, when provided to the downstream supply chain partner, has the potential to improve decision making. We refer to this kind of information exchange as supply collaboration. In the academic literature, there have been several papers that investigate the sharing of demand information and companies investing in acquiring "advance demand" information. However, the literature on upstream information sharing is relatively under explored.

This final essay analyzes a single-stage supply chain where the upstream supplier provides advance supply information in the form of capacity forecasts to the downstream partner. In this work, the advance supply information is modeled as the supplier's capacity. At the beginning of every period, the supplier provides the manufacturer with its forecasted capacity availability for a specified number of future periods. In order to focus on the advance supply information aspect, we use a simple single stage inventory model with one-period lead time, full backlogging, linear holding and backlogging costs and i.i.d. demand, but most of our results can be relaxed as usual. We first analyze the structural properties of the optimal solution. We show that state-dependent base-stock policies are
optimal for the inventory problem with advance supply information. We then focus on developing easily computable and implementable heuristic policies and then identifying the operating environments in which information is most valuable for the downstream partner.
Chapter 2

Literature Review

Vertical differentiation has been an important research area in the field of economics and marketing. Researchers have investigated the manner in which products of different quality levels compete in the marketplace. Likewise, supply chain and distribution channel coordination has attracted the attention of researchers in the field of both operations and marketing for a long time. These are the two relevant research streams that are common for the first three essays. Hence, we will review this related literature here and provide any other remaining essay-specific literature under each chapter.

In the vertical differentiation literature, "quality" generally refers to the level of some attribute in which a higher level is always preferred to a lower level. For example, everything else being equal, a higher resolution camera is preferable to a lower resolution one, a faster processor is preferable to a slower one, etc. This is in contrast to markets with horizontally differentiated products where there is no ordering with respect to the level of attribute. For example, not everyone would prefer a red over a blue shirt.

Mussa and Rossen (1978) were the first to consider a monopolist choosing quality
positions to serve a market of heterogeneous customers. Moorthy (1984) investigates the same problem with a different model, emphasizing the fact that consumers self-select the product they purchase; if lower quality products are sufficiently attractive, higher end consumers may find it beneficial to buy the lower quality product rather than buying the higher quality one targeted at them. Therefore, while the firm provides the top valuation segment with its preferred quality, it distorts the quality of the lower level segment.

These basic models have been extended to consider oligopolies competing on quality. Gabszewicz and Thisse (1979) look into the effect of competition in differentiated industries. Similarly, Gal-Or (1983) investigates the impact of increased competition on the quality levels and on the prices of the products when firms choose both the quantity and quality of their products. Shaked and Sutton (1982) consider an oligopolistic market where each firm chooses both the quality and the price of its product. They analyze the problem at three stages where in the first stage each firm observes which firms have entered and which have not. In the second stage, each firm chooses the quality of its product and in final stage each firm chooses prices. Moorthy (1988) investigates product and price competition in a duopoly and as in earlier papers finds out that each firm should differentiate its product from its competitor.

One distinction among the papers in this research stream has been the assumption on cost structures. Mussa and Rossen (1978), Gabszewicz and Thisse (1979), Moorthy (1984), Ronnen (1991) all assume that variable cost of production is independent of quality (There is fixed cost that increases with quality). Moorthy (1988) is the first to explicitly include variable cost that is increasing and convex in quality. Desai (2001) and Rhee (1996) use similar models.
Some of the other papers in this area look into the same problem by adding an attribute other than quality to the product. For example, Rhee (1996) investigates the effect of heterogeneity along an unobservable attribute (such as brand) on both quality and price equilibrium in a two-stage game framework and find that firms may offer products of identical qualities in equilibrium. Vonderbosch and Weinberg (1995) also extend one dimensional vertical differentiation to two dimensions and analyze product and price differentiation. A significant finding of theirs is that, unlike the one-dimensional vertical differentiation model, firms do not tend toward maximum differentiation; they tend to choose positions that represent maximum differentiation in one dimension and minimum differentiation on the other dimension.

Another extension in this area has been the research on damaged goods where a lower quality product is manufactured by damaging the main higher quality product. Laser printers that are provided in fast and slow speeds that are identical otherwise, and software with different levels of functionalities are common examples of such products. Since the cost of producing these quality levels are almost the same, this is a special case of earlier research with a specialized cost structure. Deneckere and McAfee (1996) identify conditions under which introducing a damaged lower quality product is profitable for a monopolist under two different market segmentation assumptions. All this literature assumes that the firms sell these vertically differentiated parts directly to the market and do not identify any incentive conflicts that could arise as a result of any intermediaries. Villas-Boas (1998) considers an intermediary when selling vertically differentiated goods, but their focus is on quality selection within a product line and they do not analyze coordination issues and potential competition scenarios.
After Spengler (1950)'s seminal paper on double marginalization, there has been a sequence of papers in the economics, marketing and operations literature on channel coordination. Channel coordination, in essence, involves optimizing the joint performance of the supply chain and then allocating the gains among the various parties.

Jeuland and Shugun (1983) study coordination issues in a bilateral monopoly and derives an optimal discount pricing policy. McGuire and Staelin (1983) investigate the optimality of forward integration in a duopolistic retail market. This literature concentrates on deriving the terms of trade that generate channel coordination. Lariviere and Porteus (2001) explore a price-only contract. Quantity-flexibility contracts (Tsay and Lovejoy (1999)) and sales-rebate contracts (Taylor 2002) have also been shown to coordinate the channel in this setting. In a price-setting newsvendor model, Bernstein and Federgruen (2005) study a price-discount contract and demonstrate that it is a coordinating contract. Emmons and Gilbert (1998) study a model that incorporates price sensitive end-consumer demand in a one period return model. In a similar setting, Cachon and Lariviere (2005) and Dana and Spier (2001) study coordinating revenue sharing contracts and show its advantages and limitations compared to other contracts. For a one supplier multiple retailer setting, Chen et al. (2001) show that a non-traditional discounting scheme that requires non-periodic fees and a discount that depends on sales volume, quantity and frequency of orders, coordinates the channel. Chen (2007) discusses the procurement strategy of a buyer from multiple suppliers who have private information about their cost structure. Both the price and the quantity decisions need to be made. The optimal strategy is to design a supply contract with payment scheme for each potential quantity and choose the highest bid supplier at the end of an auction and let the supplier decide the optimal quantity. Lariviere
Chapter 3

Efficient Channel Contracting for Vertically Differentiated Products

As mentioned in the introduction, this essay analyzes a model of the channel where the manufacturer makes a high (H) and low (L) quality product which are pre-determined. The manufacturer prices and sells these products to a distributor which in turn prices and sells these to a market with consumers that have heterogeneous valuations for quality.

In this environment, the economic distortions are identified in two dimensions. The first one is the well-known “double marginalization” phenomenon, which basically undermines the “sell through” objective of the manufacturer by causing retail prices to be higher overall than is channel optimal. The second distortion is more novel; it is that the price gap between high and low quality products increases even further when products are sold through an intermediary. This second distortion is ultimately what degrades the “sell-up” incentive in the channel.
While the manufacturer could potentially help alleviate this problem by changing the quality of its product line, e.g., increase or decrease the quality gap between H and L products, this option is not considered here, mainly because the part quality is largely fixed by the characteristics of the semi-conductor manufacturing process. Choosing quality levels is well studied in the vertical differentiation literature for firms selling directly. Because in our case quality differences are essentially fixed, the main focus is on ways to restore the distortions through potential contractual arrangements and other mechanisms. Starting with a single distributor setting, several potential contracts are studied, which create different incentives for different parts within a given product line and show which ones can potentially align the economics of the firm and its distributors and improve channel performance. Since the manufacturer has a large network of distributors, this model is then extended to multiple distributors to understand the effect of competition.

3.1 Overview of Relevant Literature

In the field of economics and marketing, vertical differentiation has been an important research area. Researchers have investigated the manner in which products of different quality levels compete in the marketplace. On the other hand, supply chain and distribution channel coordination has attracted the attention of researchers in the field of both operations and marketing. This essay is at the intersection of both research streams and has been reviewed in chapter 2.
3.2 Model

Our model is modest but it has sufficient detail to study the potential economic distortions in selling vertically differentiated products and how to mitigate them with contractual arrangements. We consider a manufacturer and a distributor which sells the manufacturer's products. We take one product family and two parts that belong to that family. Throughout, we will use High (H) and Low (L) to denote the quality levels of these two parts. We assume that the distributor faces a market with heterogeneous customers. The manufacturer was not really sure how much the customer valued the brand vs. the individual performance of the CPU together with the price the distributor charges. In order to reflect that in the model, we assume that the consumer valuation for the products has 2 components:

1. Brand/Family valuation \( (R) \) which is the same for both products

2. Quality valuation \( (v) \) which increases linearly with the quality of the product

Normalizing the quality for the H-product to 1 and the L-product to \( \gamma \), consumer's overall valuation would be \( R + v \) and \( R + \gamma v \) for H and L-products respectively. First, we introduce the following notation and then describe the sequence of events:

- \( p_i \): Selling price at the consumer market per unit for product \( i = H, L \)
- \( w_i \): Manufacturer's price to the distributor per unit for product \( i = H, L \)
- \( c_i \): Manufacturer's production cost per unit for product \( i = H, L \)
- \( N \): Total market size for this family of products
- \( d_i \): Demand generated by the distributor for product \( i = H, L \)
CHAPTER 3. EFFICIENT CHANNEL CONTRACTING FOR VERTICALLY DIFFERENTIATED PRODUCTS

\( \nu \): Consumer valuation for quality which is assumed to have Uniform Distribution over \((0, 1)\)

All market parameters are assumed to be known. The sequence of events is as follows:
The Manufacturer announces the wholesale prices for both the \( H \) and the \( L \) product. The distributor decides how to price these two products which in turn determines the demand \( d_H \) and \( d_L \) for both \( H \) and \( L \) products respectively. This quantity demanded is built and shipped by the manufacturer which has no capacity constraint.

A comment on the assumptions so far is in order. Most of the supply chain coordination literature in operations is motivated by single products whose demand is stochastic, which makes the ordering decision and the associated inventory cost the key concerns in those models. Our work is primarily motivated by high-tech supply chains which have three important features: First, semiconductor manufacturers tend to make an aggregate forecast at the product family level rather than at the part level since family level forecasts are generally very accurate. They plan and position their supply chain according to these forecasts and build the products as part-level demand is realized. That part level demand is what we are referring to by \( d_H \) and \( d_L \) which can be met quickly because of the nature of this forecasting and positioning process. Secondly, most manufacturers provide parts to their downstream supply chain partners in a consignment agreement so that the downstream partner (the distributor in our case) is not overly concerned with inventory holding costs. Indeed, our sponsor provides regular inventory assistance programs to help reduce holding costs for the distributors. As for the manufacturer's capacity, capacity in such manufacturing environments is generally allocated in advance for a product family based on its aggregate forecast which, as mentioned, is quite accurate. Therefore capacity
CHAPTER 3. EFFICIENT CHANNEL CONTRACTING FOR VERTICALLY DIFFERENTIATED PRODUCTS

for individual parts generally does not become a problem as the difference between the production time of a high and a low performance part is negligible.

Consistent with this high-tech supply chain structure and to maintain our focus on segmentation of the market and the associated dynamics and problems of planning and selling two vertically differentiated products, we assume all demand generated by the distributor for individual parts can fully be satisfied and sold.

The distributor needs to set prices \( p_H \) and \( p_L \) for both \( H \) and \( L \) products which will in turn determine demand \( d_H \) and \( d_L \). If we call \( v_H \) and \( v_L \) the valuation of threshold customers, the prices \( p_H \) and \( p_L \) need to be set such that the type \( v_H \) will be indifferent between buying \( H \) and \( L \) products and \( v_L \) will be indifferent between buying the \( L \) product and not buying. Assuming the utility of not buying is zero:

\[
\begin{align*}
p_L &= R + \gamma v_L & \text{(3.1)} \\
R + v_H - p_H &= R + \gamma v_H - p_L & \text{(3.2)}
\end{align*}
\]

Equation 3.1 says that a \( v_L \) type customer would gain zero utility by buying the \( L \)-product and equation 3.2 says that a \( v_H \) type customer would gain the same utility if he had switched to the \( L \) product and paid \( p_L \). From 3.2, \( p_H = R + v_H - \gamma(v_H - v_L) \)

Now, the above price setting problem can be viewed as finding the threshold customer types. Due to uniform distribution assumption, the demand generated as a result is:

\[
\begin{align*}
d_H &= N(1 - v_H) \\
d_L &= N(v_H - v_L)
\end{align*}
\]
Let $d = d_H + d_L$

Our assumptions are:

A1. $0 < c_L \leq c_H < R$
A2. $c_H - c_L < 1 - \gamma$
A3. $R - c_L < \gamma$

The first assumption says that producing H type products is more expensive than producing the L types, and the brand value alone is higher than either production cost, so there is some profit to be gained by selling both types. The second assumption says that the increase in production cost from $H$ to $L$ types should not be more than the increase in quality valuation of these two products, which ensures that the manufacturer makes a higher margin on the $H$ product. The third assumption ensures some of the market is uncovered. Under these assumptions which generally reflect our sponsor's business environment, it is profitable for the integrated channel to sell both $H$ and $L$ products in positive quantities.
3.3 Single Distributor Case

In this section, we assume there is a single distributor for the manufacturer and it operates in a monopolistic environment. While this is not true in reality of course, starting with a single distributor for the initial analysis helps isolate and understand the key phenomenon. Furthermore, it roughly approximates a geography where there exists a major distributor with market power.

We first look at the performance of the centralized solution to give us the first best outcome as a baseline for comparison. We then proceed to analyze the status quo wholesale pricing mechanism under decentralization. Observing the inefficiency of wholesale pricing, we then analyze several alternative channel coordinating mechanisms in section 3.3.3. Channel coordination is achieved when the performance of the integrated channel is replicated by the decentralized supply chain. To achieve this, the terms of the contract must be specified to induce the distributor to behave in the way that is optimal for the integrated channel. We want to understand how channel coordination can be achieved. When coordination is achieved, we are interested in how total profit is allocated between the two parties, which is an indicator of whether or not coordination can be feasibly implemented.

3.3.1 Centralized Solution

We begin by examining the scenario in which the manufacturer and the distributor are under the same ownership. The performance of this "centralized solution" will serve as a benchmark against which we compare the performance of the decentralized system where the distributor is independent.
Let superscript C represent the values associated with the centralized solution and let \( \Pi \) be the profit of the system. As a result, the profit maximization problem for the centralized system is:

\[
\Pi^C(v_H, v_L) = \max_{v_H, v_L} \{(R + v_H - \gamma(v_H - v_L) - c_H)N(1 - v_H) + (R + \gamma v_L - c_L)N(v_H - v_L)\}
\]

Solving for the above equation, we get the demand generated for both products together with the total profit of the centralized solution.

### 3.3.2 Decentralized Solution

Assuming that the manufacturer is the Stackelberg leader, we analyze the wholesale pricing game between the manufacturer and the distributor. In this section, we will use the superscript D to represent the values for the decentralized solution and the subscript d and m to represent the distributor and the manufacturer respectively. Here, we assume that the manufacturer announces the wholesale prices first. In response, the distributor prices both products in the market and generates demand \( d_H^D \) and \( d_L^D \) which is satisfied by the unconstrained manufacturer. The manufacturer optimizes its own system i.e. decides on wholesale prices \( w_H \) and \( w_L \) knowing that it will in return get \( d_H^D \) and \( d_L^D \).

In this decentralized system, the distributor’s problem is the same as the integrated channel except that the production cost \( c_H \) and \( c_L \) are replaced with \( \omega_H \) and \( \omega_L \):

\[
v_H^D = \frac{(1-\gamma) + \omega_H - \omega_L}{2(1-\gamma)}, v_L^D = \frac{\gamma - R + \omega_L}{2\gamma}
\]
\[
d_H^D = N\left(\frac{1}{2} - \frac{\omega_H - \omega_L}{2(1-\gamma)}\right); d_L^D = N\left(\frac{R(1-\gamma) + \omega_H \gamma - \omega_L}{2\gamma(1-\gamma)}\right)
\]
In order for the distributor to sell both products and generate positive demand, we assume that the difference between wholesale prices satisfies $w_H - w_L < 1 - \gamma$. Imposing this constraint gives us the manufacturer’s problem:

$$
\pi_m^{D}(v_H, v_L) = \max_{w_H, w_L} \left\{ (w_H - c_H)N\left(\frac{1}{2} - \frac{w_H - w_L}{2(1 - \gamma)}\right) + (w_L - c_L)N\left(\frac{R}{2\gamma} + \frac{w_H \gamma - w_L}{2\gamma(1 - \gamma)}\right) \right\}
$$

s.t.

$$
w_H - w_L < 1 - \gamma
$$

$$
w_L > R - \gamma
$$

Solving for the manufacturer’s problem we get: $w_H^* = \frac{R + 1}{2} + \frac{R}{2} \gamma, w_L^* = \frac{R + \gamma}{2} + \frac{\gamma}{2}$

Our first result is on the inefficiency due to decentralization (All proofs for the results in this essay are in appendix A):

**Proposition 3.1**  

a) $p_H^D > p_H^C$ and $p_L^D > p_L^C$

b) $p_H^D - p_L^D > p_H^C - p_L^C$

c) Total demand for the channel is determined by the L-product wholesale price $w_L$
which leads to an inflated difference in the price of the low and high quality products; this kind of channel distortion is specific to this problem. Together with this and the double marginalization effect on both products, there are actually three economic distortions that arise in this problem. The combined effect of these causes a decrease in not only total demand but also the “sell-up” achieved by the distributor. Finally, part c) says that it is really the $L$-product wholesale price that determines the total demand while $w_H$ determines how much of that total demand is for the $H$-product. A manufacturer concerned with just increasing its market share would focus more on pricing the low quality part. In general, what we observe here is that the economics of the two parties are not well coordinated.

Next, we want to understand how the magnitude of the sell-up inefficiency derived in part (b) changes.

**Proposition 3.2** Let’s define $\delta_D = p_H^D - p_L^D$ and $\delta_C = p_H^C - p_L^C$. We then have $\delta_D - \delta_C = (1 - \gamma) - (c_H - c_L)$.

We observe that the magnitude of the distortion is higher when the production process is such that the incremental increase in quality with an increase in cost is much higher. In industries such as the semi-conductor industry as studied in this essay, this observation is quite valid which means that the magnitude of the inefficiencies would be high and that any contract that would restore these would be very useful.

In the next section, we analyze coordinating mechanisms that eliminate these distortions.
3.3.3 Contracts

This section first considers channel coordination with revenue sharing contract. We then analyze other types of potential contracts which our company sponsor asked us to investigate.

Revenue Sharing Contracts

A well-known implementation of revenue sharing is the case of Blockbuster Inc. (See Cachon and Lariviere (2005) for details.) In a revenue sharing arrangement, the distributor keeps a certain portion of the total revenue and gives the remainder back to the manufacturer. In return, manufacturers provide products at a variable cost that is closer to their manufacturing cost. In Blockbuster’s case, the main motivation behind revenue sharing was increasing product availability. Our focus, in contrast, is ensuring that downstream channel partners have the same sell-through and sell-up incentive as the upstream manufacturer.

We assume that before the distributor decides on selling prices $p_H$ and $p_L$, the manufacturer and the distributor agree on a revenue sharing contract with three parameters. The first two are the wholesale prices $w_H$ and $w_L$ per unit that the distributor will pay. The second, $\lambda$, is the distributor’s share of revenue generated from each unit, the remaining $1-\lambda$ going to the manufacturer. Hence, we can write the profit functions for the problem as:

$$
\pi_d(v_H, v_L) = \lambda \left[ (R + v_H - \gamma(v_H - v_L))N(1 - v_H) + (R + \gamma v_L)N(v_H - v_L) \right] - w_HN(1 - v_H) - w_LN(v_H - v_L)
$$
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\[ \pi_m(w_H, w_L) = (1 - \lambda) \left\{ \left( R + v_H - \gamma(v_H - v_L) - c_H \right) N(1 - v_H) + \left( R + \gamma v_L - c_L \right) N(v_H - v_L) \right\} + \\
\] 
\[ w_H N(1 - v_H) + w_L N(v_H - v_L) - c_H N(1 - v_H) - c_L N(v_H - v_L) \]

\[ \Pi(v_H, v_L) = \pi_m + \pi_d \]

Recall, \( \left\{ v^C_H, v^C_L \right\} \) are the maximizers of \( \Pi \) i.e. the total channel profit if it was managed centrally. We then have:

**Theorem 3.1** Consider a revenue sharing contract with parameters \( w_H = \lambda c_H, w_L = \lambda c_L \) and \( \lambda \in (0, 1] \). Then, \( \left\{ v^C_H, v^C_L \right\} \) are the optimal threshold valuations for the distributor, and the contract coordinates the channel and allocates the profit according to \( \lambda \).

In the wholesale pricing scheme we analyzed in section 3.3.2, the manufacturer adds a margin to the production cost of both products when selling to the distributor. Similarly, the distributor adds its own margin (which is a fixed percentage in several industries) and determines a market price for both products. This upward pressure on the prices as the products move downstream is the main reason for channel distortions. If the manufacturer provided both products at unit production cost this would make the distributor take the same action as in a centralized system; however this can basically be viewed as transferring the company to the distributor since all the profit will stay with the downstream partner. Revenue sharing spans these two extremes. It allows the downstream partner, the distributor in our case, to be the \( \lambda \) percent owner of the entire channel – paying for \( \lambda \) percent of the production cost and keeping \( \lambda \) percent of the total revenue generated. Hence, it is in distributor's best interest to increase the total profit. Even though the structure of the contract is similar to what has been studied earlier in the literature, here we show that
revenue sharing ensures that the distributor has the exact same sell-up and sell-through incentive as the centralized system i.e. helps restore different set of incentives.

Figure 3.2: Percent profit improvement under *wholesale pricing* for both the manufacturer and the distributor as amount of upgrade increases

To get a more realistic sense of the impact of revenue sharing, we took a sample from our sponsor's quarterly data to compare its current wholesale pricing with a potential revenue sharing arrangement. We took four products in a family whose quality ranged from lowest to highest and obtained some base demand levels. We then came up with different scenarios in which the demand figures are changed by assuming a percentage of a lower performing part demand is upgraded to the next high performing part in the family. We created several such upgrade scenarios and calculated the percent profit improvement for both the manufacturer and the distributor. The graph in figure 3.2 shows the profits for the current process of wholesale pricing, where the horizontal axis represents the upgrade scenario and the amount of upgrading increases along this axis. Obviously,
the profit increases for both as there is more upgrading, but the figure confirms that the manufacturer has a greater incentive to sell-up (which is an upgrade on the consumer side) since that has a much greater impact on its profit compared to that of the distributor. The second graph in figure 3.3 is created using the exact same scenarios, but with a revenue sharing arrangement; note that the manufacturer's and the distributor's incentives are perfectly aligned in this case.

Every contract design is evaluated based also on its practical feasibility. One important parameter in a revenue sharing environment is $\lambda$. As the outside opportunity cost for the distributor under consideration increases, this percentage will likely have to increase. In other words, it should be set such that the distributor will not deviate from what is best for the channel. We discuss implementation challenges associated with revenue sharing and other contracts in more detail in section 3.5.

Figure 3.3: Percent profit improvement under revenue sharing for both the manufacturer and the distributor as amount of upgrade increases
We have not explored how revenue sharing would work when there is effort required by the distributor to induce sell-up. Prior literature in supply chain contracting shows that revenue sharing is not coordinating when there is private effort required of the seller (Cachon and Lariviere (2005)), and we would expect a similar outcome in our case.

**Average Selling Price (ASP)**

One of the mechanisms we were asked to study was how an average selling price (ASP) based sales would affect the distributor incentives. As explained briefly in the introduction section, under this mechanism, the manufacturer keeps track of the average selling price of the distributor for a given quarter for a certain product family. At the beginning of the quarter, a target ASP $a$ is set and announced to the distributor. At the end of the quarter, the realized ASP $r$ is checked and if it is greater than the target ASP, the manufacturer gives away a fixed percentage ($\lambda$) of the revenue realized from this difference back to the distributor. Our main interest here was whether or not such a mechanism could motivate the distributor to sell up in the market without sacrificing from volume. Writing the profit function for the distributor as:

$$\pi_d(v_H, v_L) = \begin{cases} 
R(v_H, v_L) - C(v_H, v_L) + \lambda(r - a)N(1 - v_L) & r \geq a \\
R(v_H, v_L) - C(v_H, v_L) & r < a 
\end{cases}$$

where $R(v_H, v_L) = (R + v_H - \gamma(v_H - v_L))N(1 - v_H) + (R + \gamma v_L)N(v_H - v_L)$ and $\gamma = \frac{R(v_H, v_L)}{N(1 - v_L)}$ and $C(v_H, v_L) = w_HN(1 - v_H) + w_LN(v_H - v_L)$ and $r = \frac{R(v_H, v_L)}{N(1 - v_L)}$

We can then say that for a fixed $\lambda$:

**Proposition 3.3**  

a) The total demand $d$ is non-increasing in $a$.

b) The target ASP has no effect on $d_H$. 
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We studied ASP based sales to understand its effect on the two main objectives of the manufacturer. The first part of the proposition says that the ASP mechanism degrades the sell-through effect and the second part of the proposition, contrary to intuition, tells us that setting an ASP for the distributor does not really create a sell-up incentive. The main reason is that increasing the realized ASP on the distributor’s side can be done by decreasing the overall volume and increasing the prices $p_H$ and $p_L$ and that is exactly what the above mechanism does. This really does not help achieve the main objectives of the manufacturer.

Other Contracts

In this section, we briefly examine other contracts our sponsor either used in the past or suggested as possible coordinating mechanisms. The first such contract is a selective target rebate.

In this type of contract, a rebate is offered selectively on the product line. In order to achieve the sell-up objective, we design it such that the manufacturer offers a rebate $r$ only for the $H$ product if its demand exceeds a threshold $t$. The $L$ product is sold at a wholesale price that supports the sell-through objective of the manufacturer. This is based on the observation made above that $w_L$ determines total demand.

In this setting, the two transfer payments from the distributor to the manufacturer are:

$$T_H = \{w_H N(1 - v_H) - (N(1 - v_H) - t)r \} \quad N(1 - v_H) > t$$

$$w_H(1 - v_H) \quad \quad N(1 - v_H) \leq t$$

$$T_L = \{w_L N(v_H - v_L)\}$$

Writing the profit functions with these transfer payments, we have:
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Theorem 3.2 Consider the selective target rebate contract with rebate \( r \) and the set of parameters:

\[ w_H = r + c_H ; w_L = c_L \] where

\[ t \leq t_0. \] The profit allocations are as:

\[ \pi_d = \Pi^C - tr \]
\[ \pi_m = tr \text{ and} \]
\[ t_0 = N \left[ \frac{1}{2} - \frac{r}{4(1-\gamma)} - \frac{c_H - c_L}{2(1-\gamma)} \right] \text{ and } r < (1 - \gamma) - (c_H - c_L) \]

and the channel is coordinated under this contract.

Intuitively, the manufacturer is making margin \( r \) on the \( t \) units and giving away the volume above \( t \) at production cost \( c_H \). However, as the manufacturer increases the margin \( r \), the threshold \( t \) has to decrease, otherwise it becomes unprofitable for the distributor to use the rebate option. That sets an upper bound on \( t \) based on the opportunity profit of simply purchasing at the wholesale prices. In other words, the manufacturer’s incentive to place a threshold only on the \( H \) product is to push the distributor toward selling up. However, this offer stops providing an incentive once the threshold is too high. Note that, contrary to what has been studied earlier in the literature for single product problems, there are a total of three distortions that the contract needs to restore to achieve coordination. In this case, setting \( w_L = c_L \) ensures restoring only one of them while the other parameters in the contract help restore the remaining two. With this and the other contracts studied in this section, we provide one way to achieve this, even though there could be more.

Another observation we made about this contract is that the profits can not be allocated completely arbitrarily as in revenue sharing contract. Actually, there we argued that the \( \lambda \) must be set considering the opportunity profit of the supplier \( \bar{\pi} \). When we consider the
same argument, the upper bound on the threshold $t$ needs to be changed to
\[
\min \left\{ \frac{NC-N'}{r}, N \left[ \frac{1}{2} - \frac{r}{4(1-\gamma)} - \frac{c_H - c_L}{2(1-\gamma)} \right] \right\}.
\]

Let's define $1 - \gamma$ as the quality gap between $H$ and $L$ products. If we define the manufacturer's share of the contract as $\frac{\pi m(H/L)}{F(H/L)}$, this value is only 50% for simple wholesale price contract under any quality gap i.e. it is independent of the product line design. For some fixed cost values ($c_L = 0.01; c_H = 0.03$ and $R = 0.06$) that also satisfies the assumptions, based on our numerical study, we observe that this share for the selective target rebate contract can be around 85% when the quality gap is as high as 95%. However it significantly drops as the quality gap decreases and reaches to that of the wholesale pricing contract when the quality gap is 57%. Even though the selective target rebate is a coordinating contract, the manufacturer would prefer to implement it for a product family where the quality gap between the parts is wide. The $L$ part cannibalizes sales of the $H$ part when the quality of these parts is close and therefore the distributor decreases the quantity of $H$ parts it sells. In that case, the manufacturer needs to provide more incentive to the distributor. However, as the difference in quality between these two parts increases and the cannibalization effect decreases, the consumers in the higher segment would not consider buying an $L$ part as easily which creates its "own market". This means that there is less need for sell-up incentives from the manufacturer. Hence, when $1 - \gamma$ is high, the manufacturer can afford to set a higher threshold ($t_0$) as well as a higher rebate $r$.

Another contract we analyzed is the quantity discount contract that have been studied in several other contexts in prior literature. We focus on how to modify and apply this kind of contract to our problem of channel coordination with vertically differentiated products.
Based on earlier observations we made for the selective target rebate contract, we similarly narrow down our focus and assume that the quantity discount only be used for the $H$ product, while wholesale pricing is used for the $L$ product. This approach provides a valid comparison of both contracts in the end. Specifying $w_H = W - w d_H$, the profit functions are:

$$
\pi_d = (R + v_H (1 - \gamma) + \gamma v_L - W + wN(1 - v_H))N(1 - c_H) + (R + \gamma v_L - wL)N(v_H - v_L)
$$

$$
\pi_m = (W - wN(1 - v_H) - c_H)N(1 - v_H) + (wL - c_L)N(v_H - v_L)
$$

We then have:

**Theorem 3.3** Consider the quantity discount contract with the set of parameters:

$$
\begin{align*}
\pi_d &= \pi_c - mN(1 - \frac{c_H - c_L}{1 - \gamma}) \\
\pi_m &= mN(1 - \frac{c_H - c_L}{1 - \gamma})
\end{align*}
$$

and the channel is coordinated under this contract.

The wholesale price for the lower quality product set at the production cost ensures the same sell-through objective, while the discounted wholesale price for the $H$ product targets the sell-up incentive. The margin $m$ needs to be bounded such that the distributor profit is greater than its opportunity profit $\bar{\pi}$. The design of this contract is quite similar to the selective target rebate contract. However, the profit allocation of the quantity discount
contract is much more favorable for the manufacturer. We again observe that the manufacturer can afford to provide less incentive for the distributor as the quality gap increases. While the degree of incentive as measured by the threshold $t_0$ in the target rebate contract, was increasing in the quality gap $1 - \gamma$, here the measure is the discount term $w$ decreases as the quality gap widens, i.e. the manufacturer could provide less of a discount for a product family containing parts whose quality range is quite different. However, in the case of a small quality gap, even though the discount term has to be higher, the manufacturer can still get most of the channel profit by counterbalancing this effect with an increased margin $m$. This was not possible in the target rebate contract where the margin had to be bounded by an expression which was decreasing in the quality gap.

The next contract we study is a bundling contract, in which the manufacturer sells a mix of products in a bundle at a single price. With two vertically differentiated products as in our case, the manufacturer can bundle $x$ units of the $H$ product and $y$ units of the $L$ product and sell is at a bundle price $w_B$. When the distributor buys $Q$ bundles, it will have $Qx$ units of the $H$ product and $Qy$ units of the $L$ product to sell. If the manufacturer’s objective is to push the market toward selling higher quality parts, it can design the bundle such that the distributor ends up getting, and therefore selling, more $H$ products than it would if they were sold separately.

The manufacturer first needs to determine the bundle design, i.e. $x$ and $y$ together with a wholesale price $w_B$, such that when the distributor orders $Q$ units of the bundle, the channel reaches the same $H$ and $L$ product sales in the market as the centralized system.

**Theorem 3.4** There is a coordinating bundling contract $(x^*, y^*)$ with
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\[ w_B^* = \min \left( \frac{R(x+y+x+(1-y)y-X_H-Y_L)}{2(x^2+2(1-y)x+y+1-y^2)} \right), \eta, \beta \) where the distributor orders \( Q = \frac{(1+x)^2(1-y)R+2}{2(x^2+y^2+y(1-x))} \) with \( \eta = Ry + Rx + x + (1 - y)y \) and \( \beta = Ry + y(1 - y)(d_C^C + 1 - 2d_H^C) \)

However, the bundling contract does not let the manufacturer allocate profits in its best interest, leaving most of the profit with the distributor, which is clearly undesirable from the manufacturer’s standpoint.

### 3.4 Competing Distributors

In actuality, our manufacturer had to sell its products through several distributors and resellers that were roughly similar, at least within the same geography. Therefore, they were interested in understanding the effects of competition among distributors and how it might impact the contractual arrangements we analyzed. In this section, we assume that the manufacturer is selling its product line to multiple competing identical distributors. We also will assume that the products are not further differentiated at the distributors and that any cost they incur is normalized to zero.

In this setting, if the distributors enter into Bertrand price competition, then at the equilibrium they will each end up selling the products at a price equal to their cost (i.e. the wholesale price), leaving them with zero profit. This level of competition is arguably too extreme. Therefore, we will instead analyze Cournot quantity competition where each distributor \( i \) orders \( q_H^i \) and \( q_L^i \) of \( H \) and \( L \) products respectively, which collectively determine the prices \( p_H \) and \( p_L \) in the market. Let \( (q_H^i, q_L^i) = [(q_H^1, q_L^1), \ldots (q_H^n, q_L^n)] \) be the vector of quantities and \( \pi^i(q_H^i, q_L^i) \) the distributor \( i \)'s profit where \( i=1 \ldots n \).

In a competitive setting the collection of all \( (q_H^i, q_L^i) \) determine \( p_H \) and \( p_L \). Following the
model used for single distributor case, (replacing $v_H$ and $v_L$ with their corresponding $q_H$ and $q_L$ values) and normalizing the market size to one, we have:

$$p_H = R + 1 - q_H^i - \gamma q_L^i - \sum_{j \neq i} q_H^j - \gamma \sum_{j \neq i} q_L^j$$

$$p_L = R + \gamma(1 - q_H^i - q_L^i - \sum_{j \neq i} q_H^j - \sum_{j \neq i} q_L^j)$$

First, we will analyze the distributor’s Cournot game under a general linear wholesale pricing scheme ($w_H, w_L$). With this we have:

$$\pi_d(q_H^i, q_L^i) = p_H q_H^i + p_L q_L^i - w_H q_H^i - w_L q_L^i$$
Define \( \mu^H_i(q^1_H, q^1_L, q^2_H, q^2_L, \ldots, q^N_H, q^N_L) = \frac{\partial v^H_i}{\partial q^H_i} \) and \( \mu^L_i(q^1_H, q^1_L, q^2_H, q^2_L, \ldots, q^N_H, q^N_L) = \frac{\partial v^L_i}{\partial q^L_i} \). We assume:

A4. There is a compact set \( K \) of \( R^{2N} \) such that for \( (q^1_H, q^1_L, q^2_H, q^2_L, \ldots, q^N_H, q^N_L) \in R^{2N} \setminus K \),
\[ \mu^H_i(q^1_H, q^1_L, q^2_H, q^2_L, \ldots, q^N_H, q^N_L) < 0, \quad \mu^L_i(q^1_H, q^1_L, q^2_H, q^2_L, \ldots, q^N_H, q^N_L) < 0 \] \( \forall i \) meaning that the industry output is bounded.

A5-a. \( w_H - w_L < 1 - \gamma \)

A5-b. \( w_L - \gamma w_H < R(1 - \gamma) \)

Assumptions A5-a and A5-b guarantee a non-degenerate Cournot equilibrium.

**Theorem 3.5** For a fixed \( w \), in the distributors game under Cournot competition:

a) There exists a unique symmetric Nash equilibrium \( (q^*_H, q^*_L) \)

b) The Nash equilibrium \( (q^*_H, q^*_L) \) is locally stable

If the system was managed centrally and \( \Pi^C_i(q^i_H, q^i_L) \) is the profit from this centrally managed distributor \( i \), then the total system profit is:

\[
\Pi^C(q^*_H, q^*_L) = \sum_{i=1}^{N} \Pi^C_i(q^i_H, q^i_L)
\]

where:

\[
\Pi^C_i = (R + 1 - q^i_H - \gamma q^i_L - \gamma \sum_{j \neq i} q^j_H) q^i_H + (R + \gamma(1 - q^i_H - q^i_L - \gamma \sum_{j \neq i} q^j_H - \sum_{j \neq i} q^j_L)) q^i_L - c_H q^i_H - c_L q^i_L
\]

Let \( (\tilde{q}^H_L, \tilde{q}^L_L) = [(q^{10}_H, q^{10}_L), (q^{20}_H, q^{20}_L), \ldots, (q^{1n}_H, q^{1n}_L)] \) be the optimal quantities for the above system.

In the following section, we analyze some of the potential mechanisms that achieve the
performance of the centrally managed channel and in section 3.4.2, we study the combined effect of wholesale pricing and competition.

3.4.1 Coordinating Contracts

In the case of a monopoly distributor, we showed that revenue sharing was a coordinating contract and helped the manufacturer achieve its sell-up and sell-through objectives. The same is true when there are several competing distributors:

**Theorem 3.6** The revenue sharing contract with the following set of parameters:

\[
\begin{align*}
\omega_H^i &= \lambda_i(c_H + Q_H^i + \gamma Q_L^i) \\
\omega_L^i &= \lambda_i(c_L + \gamma(Q_H^i + Q_L^i)) \\
\end{align*}
\]

where \( Q_H^i = \sum_{j=1}^n q_H^i \) and \( Q_L^i = \sum_{j=1}^n q_L^i \) coordinates the channel with profit allocations:

\[
\begin{align*}
\pi_H^i &= \lambda_i(\Pi_H^i(q_H^i, q_L^i) + \pi_L^i) \\
\pi_L^i &= \sum_{i=1}^n (1 - \lambda_i)(\Pi_L^i(q_H^i, q_L^i) + \pi_H^i) - \pi \text{ with } \pi = (Q_H^i + \gamma Q_L^i)q_H^i + \gamma(Q_H^i + Q_L^i)q_L^i
\end{align*}
\]

The percentage share \( \lambda \) needs to be bounded by a term \( \frac{\Pi_H(q_H^i, q_L^i)}{\Pi_L(q_H^i, q_L^i) + \pi} \) to make sure that the manufacturer's profit is greater than zero. These \( \lambda \)'s can be made equal across the distributors by taking into account these different bounds. With symmetric centralized solution \( (q_H^0, q_L^0) \) for all distributors, which is the expected outcome when distributors are identical, the wholesale prices offered will be identical which makes its implementation across the distribution channel feasible and in conformance with antitrust laws. Otherwise, when the above coordinating contract parameters require different values for different distributors,
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due to the uniform pricing strategy of the sponsor, this would not be a policy that they could implement.

The selective target rebate is coordinating in the case of a single distributor. With multiple competing distributors, it is again a coordinating contract with the right contract parameters:

**Theorem 3.7** The selective target rebate contract with rebate \( r \) and the following set of parameters:

\[
\begin{align*}
\bar{w}_H^i & = (c_H + r + Q_H^i + yQ_L^i) \\
\bar{w}_L^i & = (c_L + y(Q_H^i + Q_L^i))
\end{align*}
\]

where the threshold \( t_i \leq t_{i0} \) coordinates the channel with profit allocations:

\[
\begin{align*}
\pi_d^i & = \Pi_d^i(q_H^i, q_L^i) - \pi_r^i - t_ir \\
\pi_m & = \sum_{i=1}^n (\pi_i^r + t_ir) with \\
\pi_r^i & = (r + Q_H^i + yQ_L^i)q_H^i + y(Q_H^i + Q_L^i)q_L^i and \\
t_{i0} & = 2(q_H^i - r) + q_L^i - (Q_H^i + yQ_L^i) - c_H + (R + 1 - q_H^i - yq_L^i - \sum_{j \neq i} q_H^j - y \sum_{j \neq i} q_L^j)
\end{align*}
\]

Similar to the revenue sharing contract, the above contract parameters are identical for a centrally managed channel with equal \((q_H^{i0}, q_L^{i0})\).

Lastly, we consider the quantity discount contract which was again shown to be coordinating in the monopoly case with appropriate parameters, can be designed to be coordinating with competing distributors as well:
Theorem 3.8 The quantity discount contract with discount term \( w \) and the following set of parameters:

\[
w_H^i = W_i - w_H^i q_H^i \text{ where } W_i = (c_H + m + Q_{H}^i + \gamma Q_{L}^i) \\
w_L^i = c_L + \gamma (Q_{H}^i + Q_{L}^i)
\]

where the discount contract term \( w_i = \frac{m_i}{2n_H} \) coordinates the channel with profit allocations:

\[
\pi_d^i = \Pi_d^i(q_H^i, q_L^i) - \pi_Q^i \\
\pi_m = \sum_{i=1}^{n} \pi_Q^i \text{ with } \\
\pi_Q^i = \left( \frac{m}{2} + Q_{H}^i + \gamma Q_{L}^i \right) q_H^i + \gamma (Q_{H}^i + Q_{L}^i) q_L^i
\]

In the next section, we explore the effect of competition under their current wholesale pricing scheme and then evaluate all potential mechanisms.

3.4.2 Wholesale Pricing

In this section, we will explore how competition affects the overall supply chain efficiency and the manufacturer’s profit when the manufacturer offers just a wholesale price contract with \( w_H \) and \( w_L \), which was also studied in the single distributor case. For this wholesale pricing game, the quantities ordered by the distributor are:
\[
q_H = \frac{1}{n+1} - \frac{w_H - w_L}{(1-\gamma)(n+1)} \\
q_L = \frac{R}{\gamma(n+1)} + \frac{w_H}{(1-\gamma)(n+1)} - \frac{w_L}{\gamma(n+1)(1-\gamma)}
\]

\[i = 1\ldots n\]

The manufacturer’s problem will then be:

\[
\pi_m(w_H, w_L) = \max_{w_H, w_L} \left\{ (w_H - c_H) \frac{n}{n+1} (1 - \frac{w_H - w_L}{1-\gamma}) + (w_L - c_L) \frac{n}{n+1} \left( \frac{R}{\gamma} + \frac{w_L}{1-\gamma} - \frac{w_L}{\gamma(1-\gamma)} \right) \right\}
\]

The manufacturer will offer wholesale prices:

\[
w_H^* = \frac{R+1}{2} + \frac{c_H}{2} \\
w_L^* = \frac{R + \gamma}{2} + \frac{c_L}{2}
\]

These are surprisingly the same wholesale prices as in monopoly case. Furthermore, they are independent of \(n\) which is consistent with previous literature (Tyagi (1999)). If we denote the total demand generated by all distributors by \(d_H\) and \(d_L\), we capture the efficiency results in the below proposition:

**Proposition 3.4**

a) \(\lim_{n \to \infty} d_H = d_H^C; \lim_{n \to \infty} d_L = d_L^C\) where \(d_H^C\) and \(d_L^C\) are the \(H\) and \(L\) demand in centralized solution.

b) \(\lim_{n \to \infty} \pi_m(w_H, w_L) = \Pi^C\)

This proposition shows that as the number of Cournot competitors increases, the efficiency of the system improves, leaving most of the profit with the manufacturer. This result
suggests it may be a better strategy to use wholesale pricing in geographies where there are many small competing distributors since that would lead naturally to the coordinating outcome. More complex contracts, such as revenue sharing, are consequently more advantageous in markets where there is a large dominant distributor with market power.

3.5 Implementation Challenges

Our objective was to design a contract that i) would align the economics of our sponsor and its distributors and ii) could be feasibly implemented. In this section, we evaluate the contracts we studied from a practical standpoint and summarize the reactions and concerns of company executives. We start with revenue sharing:

1. Revenue Sharing: This contract was presented as a “profit partnership program” during our formal and informal discussions throughout the company. The analysis and our recommendations were taken quite favorably by the sponsor and it generated the greatest internal support. However, we heard many concerns from different departments within the company as we communicated our ideas. These included:

i. Gray market impact: As explained earlier, the manufacturer does not have much control over the gray (open) market and they were concerned that any kind of new program could exacerbate the problem. Since our recommendation involved lowering wholesale prices significantly, their main concern was that this would encourage distributors to divert high
end parts to the gray market. We suggested as a possible enforcement mechanism that they offer the program with a “trigger strategy” threat to withdraw it if diversion was detected. Still, there was significant doubt as to how such an arrangement could affect open market dynamics.

ii. High administrative cost: This is typical concern for all revenue sharing implementations and was true for our case as well. The manufacturer currently did not have visibility into sales at its distributors. While requiring some development, this was among the more minor concerns since it was seen as something that they could resolve with modest effort and investment.

iii. “Best terms” contractual arrangements: Our sponsor had contracts with major OEMs which required them to provide the best available pricing at any point in time. There was concern that a revenue sharing arrangement would imply lower pricing to distributors than OEM pricing under certain conditions. For some in the finance department, this was a “show-stopper”. Others felt that if the contract offered the right incentives, it should be made available to OEMs as well.

iv. Forecasting accounts receivable: If a program like profit partnership was implemented, the credit department believed that forecasting accounts receivable would become a major issue since the revenue the sponsor would collect would have two components, one of which was paid at the time of sale and the other portion coming from the revenue the distributor would make at the end of the quarter. This would make it harder to forecast
sales and receivables.

v. Credit risk: Charging a small wholesale price up front and waiting to receive a revenue share after parts were sold created a significant credit risk. Since the wholesale prices was lowered significantly under the program, this meant that the riskier distributors would become even more risky. While a concern, it was felt that appropriate limits on lines of credit could be used to manage this risk.

According to theorem 3.1, to achieve full coordination, wholesale prices have to be even lower than the production cost. We realized that we had to modify the contract design to mitigate the business risks identified above. Ultimately, we recommended that there be two payments by the distributor:

1. An advance payment (same for all parts within the family): Covers production cost and a portion of the revenue share.
2. An after-sales payment: Remaining portion of the revenue share at the time of sales or in 90 days whichever comes first.

These modifications to the original contract eased some of the credit related concerns of our sponsor.

2. Other Contracts:

The ASP mechanism was ruled after we observed that it did not lead to centralized outcome. Furthermore, it has many similarities with what the manufacturer was already doing and it could not be fully adopted across all distributors.

The selective target rebate contract is an asymmetric contract with different terms for
different parts within the product line. Moreover, the targets had to be adjusted depending on the characteristics of each distributor. Moreover, the fact that it resembled several other bonus programs already tried was another disadvantage, since several managers felt it would be seen as 'just another sales gimmick' at the distributor level.

The quantity discount contract, on the other hand, creates an incentive for the distributor to buy at volume discounts then sell on the gray market. As a result, it was again not a strong candidate in terms of implementation.

Among all the alternatives, revenue sharing ultimately proved to be the most promising and generated the greatest internal interest due to its simplicity and the fact that it directly addressed the core incentive issues. It was also robust and didn't require detailed knowledge of the distributors' cost and demand information. Positioning it as a profit partnership program helped communicate the concept internally and to distributors. Initial discussions with a key distributor were also positive; they were eager to participate in a pilot implementation. Despite this promising feedback and and many proposal, studies and internal meetings to flesh out the practical details of the concept, ultimately senior managers opted not to pursue a pilot program. Concerns about the gray market impact, the potential treat to OEM relationships, the financial risk and the implementation complexity and cost, collectively, were simply too great to give executives sufficient comfort about prototyping the program. In addition, the firm's sales growth and margins were improving due to a new generation of products, so the organization overall felt less of a need to modify its long-standing pricing and trading practices. In short, revenue sharing was simply too much of a radical change from the status quo and the business risks were judged to outweigh the potential rewards.
Chapter 4

Supply Chain Efficiency and Contracting in the Presence of Gray Market

Along the crowded counters of Bi-Rite Photo in midtown Manhattan, bargain hunters contend not only with the usual bewildering selection of cameras and lenses, but also with a choice of prices for the same item; the popular Nikon FE-2 camera, for example, costs either $279.50 or $239.50. The first buys a camera backed by an authorized U.S. Nikon distributor; for the lower price, a buyer gets the same product but with only Bi-Rite’s guarantee (Time (1985)). Even though products sold through these supply chains are genuine brand products, their sales are unauthorized by the brand manufacturer. This alternative channel is referred to as gray market or parallel channels.

There are various reasons why manufacturers are concerned about gray markets. First,
they occur in many industries. They exist for small-ticket goods such as electronic compo­
nents, watches, fashion goods, cosmetics as well as big-ticket ones including automobiles
and heavy equipment. They are observed in markets ranging from beauty aids to prescrip­
tion drugs. Second, the magnitude of gray market sales is quite substantial. A study of
manufacturers of health and beauty aids determined that gray market sales amounted to
20% of authorized sales in some markets and as much as 50% of authorized sales in others
(Antia et al). The total value of the products distributed in the US through gray market
channels is estimated to exceed 5-10 billion. This includes an estimated 30% of high-quality
camera sales. These goods are sold across the country by such giant chains as Kmart and
Montgomery Ward, as well as by local specialty outlets. As an example on industrial side,
Caterpillar excavators imported abroad sell in the U.S. for between $85,000 and $215,000,
15 % less than an American-made model.

Why do gray markets exist? With globalization, products sold today by multi-national
companies in different countries are quite similar. However, applying competitive pricing
strategies for the same product in different markets, leads to price differentials and arbitrage
opportunities for unauthorized gray market brokers that can exploit this opportunity. And
reduced international tariffs and transaction barriers in logistics and communications all
enable such arbitrage. Moreover, gray markets are perfectly legal, unlike black markets
in which counterfeit products are sold which is illegitimate. Beyond differential pricing,
exchange rate fluctuations and inability of a firm to synchronize demand and supply in
different markets also encourages gray markets. That is why, for example, gray markets
boomed when the dollar was strong.

A group called the Coalition to Preserve the Integrity of American Trademarks (COPIAT),
whose members include Revlon, Inc., Nikon Inc., the Seiko Corporation of America, Waterford Crystal, Inc. and dozens of other American companies, consider gray marketers "free riders" because such unauthorized retailers exploit the significant investment and goodwill of U.S. trademark owners. Furthermore, prolonged existence of gray markets can damage both brand reputation and the manufacturer's relationship with its authorized distribution network.

For most companies, the gray markets are already a fact of life and strategies to reduce their impact can only be reactive. For example, increasing the perceived quality of the product sold at authorized channels is one such strategy. A new warranty and after-sales service are services that cannot be matched by unauthorized retailers and hence increases the quality of the authorized good. Alternatively select dealers can match gray market prices, though this cannot be a long-term strategy. Promoting limitations of the gray market products is another alternative. These are all examples of reactive strategies once a gray market exists. Companies can also adopt more proactive strategies, such as strategic pricing across different markets so that gray markets do not form in the first place. Another such proactive strategy is to design products that are different for different markets. Lobbying and influencing legal standards are also possible responses.

There can be cases where gray markets help a manufacturer. For example, they can help segment customers and reach previously untapped markets resulting in incremental sales, something which may be difficult to achieve with an existing distribution channel. Gray markets can also be helpful in meeting demand when supply shortage emerges in local markets.

Gray markets are indeed a complex phenomenon. For this essay, we analyze a model of
gray markets, where we assume that a manufacturer has a network of authorized retailers in a given country (or geography or region) and there exist unauthorized retailers who create the “potential” or “means” for a gray market to form. While the price differentials between different markets are one important reason for gray markets, in this essay, we choose to focus on another reason: The diversion of products from authorized to unauthorized retailers. Indeed, not all gray-market products are imported. For example, 47 St. Photo in New York offers an American-made IBM personal-computer package for two different prices. The higher price buys a machine that comes with IBM’s standard 90-day warranty and service. The lower price gets the same computer but with three months of protection from 47 St. Photo, not IBM. The retailer obtains the lower-priced IBM machines from authorized dealers who sell their excess inventories. Like many companies, IBM refuses to honor warranties on products not bought through its regular dealers (Time (1985)).

If gray markets survive for a long period of time and if the manufacturer does not actively protect its authorized retailers (as we observe in the above example), these authorized retailers often participate in gray market activity by diverting excess inventory. This also serves as a hedge against the uncertainty of demand. This is the environment we model in this essay. In short, we focus on the domestic gray market and inventory diversion within this market.

We assume that unauthorized retailers are purchasing excess stock from authorized retailers and selling to consumers that prefer the gray market product because of its lower price. The authorized retailers decide on how much to purchase from the manufacturer realizing that their demand is uncertain and that a fraction of that demand will be lost to gray market based on the market clearing price of the gray market product, the authorized
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retail price and the perceived quality gap between the two products sold through each channel. That clearing price is determined based on the total supply of excess inventory diverted from authorized to unauthorized retailers and the total gray market demand. We ignore any source of supply from parallel imports in our model, even though it can be incorporated. Our goals are to understand:

1. How gray markets change the decisions supply chain partners make and the resulting impact on their profits.

2. The gray market equilibrium and how market conditions and retailer or manufacturer decisions determine it.

3. How the economic incentives and supply chain contracts provided by the manufacturer to its authorized retailers are influenced by the presence of gray markets and how they in turn affect the gray market itself.

4.1 Overview of Relevant Literature

The literature on gray markets/parallel imports is somewhat limited. Most research focuses more on pricing rather than quantity and inventory. Ahmadi and Yang (2000) propose a two-country, three-stage model to study strategies for a manufacturer. In the third stage, and in the higher priced country where parallel imports have entered, they characterize
the resulting market segmentation. A profit-maximizing parallel importer sets price and quantity in the second stage after observing the manufacturer’s prices in both countries. In the first stage, the manufacturer anticipates the possible occurrence of a parallel import and makes a coordinated pricing decision to maximize the global supply chain profit. Through a Stackelberg game, they solve for the optimal pricing strategy in each scenario. They conclude that parallel imports may help the manufacturer extend the global reach of its product and even boost its global profit. Assmus and Wiese (1995) present a framework to select the right approach to gray market threats by coordinating price-setting decisions based on the subsidiary’s local resources and the complexity of the product market. Through examples from their sample of companies that have dealt with gray markets, they show how price coordination methods have been implemented. Myers (1999), through a survey of export managers of manufacturing firms, coupled with a series of qualitative interviews, investigates how organizational specific, and market specific factors drive gray market activity, and in turn explores the effect of unauthorized distribution on export performance. Several factors are proposed to evaluate the potential of unauthorized distribution in a firm’s export markets, namely the centralization of decision making, the degree to which the product is standardized, channel integration, and channel control. Furthermore, the effects of gray market activity on strategic versus economic performance is shown to be significantly different. Each of these issues is discussed in detail, along with the implications for export managers. K.D. Antia and Fisher (2006) examine whether and how enforcement deters gray marketing. The results from a field survey of manufacturers and an experimental design suggest that by itself enforcement severity has no impact; deterrence results only when the multiple facets of enforcement are used in combination. Duhan and Sheffet (1998)
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describe how and why gray marketing occurs in the context of legal and illegal (shadow) marketing activities. The regulatory and judicial decisions relating to gray marketing activities are reviewed and the implications of an upcoming Supreme Court ruling on gray marketing are discussed. Chaudhry and Walsh (1995) discuss the gray market in European Union within pharmaceutical industry context. Ahmadi and Carr (2002) consider the effect of inventory ordering in an uncertain demand environment where the excess inventory is diverted to unauthorized channels. However, demand for each channel is assumed to be independent random variables. They characterize the optimal ordering quantity and conditions under which a manufacturer and retailer would be better off.

Another relevant literature stream is on secondary markets, as studied by Lee and Whang (2002), such as on-line exchanges formed by several firms where each firm can buy or sell their excess inventory. Note that in such environments, the demand each firm faces does not change by the existence of such exchanges. Therefore, retailers are always better off with such markets unlike a gray market which has a negative impact on the authorized retailers' demand.

4.2 Model

We consider a manufacturer that sells its product to consumers through its authorized retailers. We assume that there are \( n \) such non-competing retailers. Each retailer \( i \) faces an independent and identically distributed demand \( D^i \). We assume that there is one selling season and the retailers make an ordering decision at the beginning of the period. It is assumed throughout the paper that unauthorized retailers already exist in this environment.
They provide the “means” to form a gray market by buying the excess inventory from authorized retailers and selling it to consumers in the lower end of the market. We do not take into account the unauthorized retailer’s profit maximization decision and assume that they make a negligible (i.e. zero) profit in this exchange. We assume that the manufacturer produces exactly the amount required by the retailers and does not hold any inventory. Furthermore, we assume that the manufacturer never takes direct actions to encourage gray market activity, such as selling to gray market. We consider an MSRP type of environment, where the prices retailers charge do not change. The assumption that the retail price is fixed is also consistent with prior supply chain contracting literature in which demand is stochastic and the primary decision of interest is the quantity supplied and produced.

While the gray market provides an opportunity for authorized retailers to get rid of excess inventory, it also poses a threat since some of the consumers that would normally buy from authorized retailers switch to the gray market product sold by unauthorized channels. An environment where no such gray market exists is possible if the manufacturer takes necessary actions to derive unauthorized retailers out of business. Throughout the essay we will assume that this is possible creating an environment which we will refer to as a “no gray market” environment. We then compare and contrast manufacturer’s and retailer’s position and strategies to those under “gray market” conditions.

Here is the notation for our model:

\[ p_a: \text{Authorized retailer's market price} \]
\[ p_G: \text{Market-clearing gray market price} \]
\[ w: \text{Manufacturer's wholesale price} \]
CHAPTER 4. SUPPLY CHAIN EFFICIENCY AND CONTRACTING IN THE PRESENCE OF GRAY MARKET

\( c \): Manufacturer's production cost

\( Q_G^i \): Quantity ordered by authorized retailer \( i \) from the manufacturer in the presence of gray market \( i = 1,...,n \)

\( Q_0^i \): Quantity ordered by authorized retailer \( i \) from the manufacturer when there is no gray market \( i = 1,...,n \)

\( v \): Consumer valuation

\( y \): Quality (discount factor of the valuation) of the gray market product perceived by customers

\( D^i \): Random Demand that each authorized retailer \( i \) faces

\( D_G^i \): Authorized retailer \( i \)'s demand in the presence of gray market

\( D_0^i \): Authorized retailer's demand when there is no gray market

\( n \): Number of authorized retailers

\( \mu_D \): Mean demand

\( \sigma_D \): Standard deviation of the demand

\( F, f \): Cdf and pdf of \( D^i \) for each \( i = 1,...,n \)

\( \Pi_R^i \): Retailer \( i \)'s profit

\( \Pi_M \): Manufacturer's profit

We assume that consumer valuation \( v \) is uniformly distributed with support \((0, v)\). As explained earlier, because unauthorized retailers do not provide the after-sales or warranty service that authorized retailers do, the perceived quality of the products sold in the gray market is less than those sold by authorized retailers. Normalizing the quality of the product sold by authorized retailers to 1, \( y < 1 \) represents the quality of unauthorized gray
market products in our model. We thus have a market segmented as if the manufacturer provided two vertically differentiated products. We will first describe how the market is segmented in the presence of a gray market and then analyze the retailer’s problem under both environments.

4.2.1 Demand Model

When there is no gray market, the authorized retailer i’s demand is:

\[ D_i^j = D^j P(v > p_s) = D^j (1 - \frac{p_s}{\theta}) \]  

(4.1)
When there is gray market, consumers need to make a purchasing decision comparing both authorized and unauthorized products on the basis of price ($p_a$ and $p_G$ respectively) and quality ($1$ and $\gamma$). Assuming linear dependence of valuation and quality, the resulting demand for authorized retailers is determined based on the below incentive compatibility and individual rationality constraints for both. An authorized product is purchased by a consumer with valuation $v$ when:

1. $v - p_a > \gamma v - p_G$
2. $v > p_a$

Similarly, the gray market product is purchased by a consumer when:

1. $\gamma v - p_G > v - p_a$
2. $\gamma v > p_G$

Consumers are segmented as shown in figure 4.2. The consumer who is indifferent between buying the gray market and the authorized retailer product has threshold valuation $v_H = \frac{p_a - p_G}{1 - \gamma}$, and similarly the consumer who is indifferent between buying the gray market product and not buying has threshold valuation $v_L = \frac{p_G}{\gamma}$. Consumers whose valuation

![Figure 4.2: Segmentation of the market](image-url)
is above \( v_H \) determine the demand for each authorized retailer \( i \) in the presence of gray market as follows:

\[
D_i^G = D^i(1 - \frac{p_u - p_G}{(1 - \gamma)\theta})
\]  

(4.2)

\( i = 1, \ldots, n \)

which means that each authorized retailer loses a portion of its demand determined by 

\[
\left( \frac{\gamma p_u}{(1 - \gamma)\theta} - \frac{p_G}{(1 - \gamma)\theta} \right) > 0.
\]

The total demand for the gray market products is then:

\[
D_G^{Tot} = \frac{1}{\theta(1 - \gamma)}(p_u - \frac{p_G}{\gamma}) \sum D_i
\]  

(4.3)

In order for both products to be sold in this market, our first condition is:

\[
C1: p_G < \gamma p_u
\]

### 4.2.2 Retailer's Problem

For a given manufacturer’s wholesale price \( w \), the retailer’s problem when there is no gray market is:

\[
\Pi_r(Q') = \max_{Q_e} \left\{ p_u E[\min(D^i, Q')] - wQ_e \right\}
\]

which is the newsvendor problem with solution \( Q^{e0} = (1 - \frac{p_u}{\theta})F^{-1}(\frac{w - p_u}{p_G}) \)

However, in the presence of gray market, the retailer now has an alternate revenue opportunity from selling its excess inventory to the gray market at price \( p_G \):
CHAPTER 4. SUPPLY CHAIN EFFICIENCY AND CONTRACTING IN THE PRESENCE OF GRAY MARKET

\[ \Pi_R(Q_G) = \max_{Q_G} \left\{ p_s E[\min(D_G^l, Q_G^l)] - wQ_G^l + p_G E[(Q_G^l - D_G^l)^+] \right\} \]

Together with the market-clearing gray market price \( p_G^* \), we want to determine the equilibrium retailer ordering quantity \( Q_G^* \) in the next section.

4.2.3 Manufacturer's Problem

In a no gray-market environment, the manufacturer's problem is to decide on the optimal wholesale price \( w_0^* \) knowing that it will in return receive order quantities of \( Q_0^l = (1 - \frac{E_p}{c})F^{-1}(\frac{p_s - w_0}{p_s}) \) from each retailer. Based on the symmetry, the manufacturer's profit maximization problem is:

\[ \Pi_M(w_0) = \max_{w_0} \left\{ (w_0 - c)Q_0^l \right\} \]

In the presence of gray market, the formulation of the problem stays the same:

\[ \Pi_M(w_G) = \max_{w_G} \left\{ (w_G - c)Q_G^l \right\} \]

The difference is that now the manufacturer has to decide on the optimal wholesale price \( w_G \) knowing that it gets \( Q_G^l \), which in turn leads to the development of gray market based on the excess supply as explained above. This problem will be studied in further detail under section 4.3.2.
4.3 Wholesale Price Equilibrium

4.3.1 Gray Market Equilibrium for a Given Wholesale Price

We have already derived the total demand for the gray market product as in 4.3. The gray market clearing price $p_C$ is determined by equating the total supply coming from excess inventory of all authorized retailers and the total gray market demand as in the below equation:

$$
\sum_i (Q_i^c - D_i^c)^* = \frac{1}{\delta (1 - \gamma)} (p_a - \frac{p_C}{\gamma}) \sum_i D_i^c
$$

(4.4)

We refer to the above equation as the equilibrium equation. Note that for each sample path $\omega$ of demand realizations, we get a market-clearing price $p_C(\omega)$, so $p_C$ itself is random. However, we assume our environment represents a very large geography (such as a country or even a larger region) with dispersed, non-competing authorized retailers and unauthorized retailers (which are “invisible” in our model). This leads to the following assumption:

**Large market assumption:** The number of authorized retailers $n$ is large, so for aggregate quantities the law of large number holds.

To derive the equilibrium result, we further assume:

**C2:** \( \frac{E[Q^*-D^*]}{\mu D} < \frac{1}{(1 - \gamma \theta)} \)

**C3:** $p_a < (1 - \gamma)^{\theta}$
In the above condition C2, \( D \) is the same random demand observed by each retailer, just without the market segmentation fraction and \( Q^* \) is the optimal ordering quantity in a pure newsvendor solution for demand \( D \). This condition ensures the existence of positive equilibrium gray market price. Condition C3, is required to make sure that authorized retailers have a market even for smaller values of \( p_C \). This is reasonable since we would never expect to see only gray market products in a market without authorized products.

**Theorem 4.1** Under the large market assumption:

a) The equilibrium equation (4.4) becomes:

\[
E[(Q_C - D_C)^+] = \frac{1}{\theta(1 - \gamma)} (p_a - p_G) \mu_D
\]

b) For a given equilibrium market-clearing price \( p^*_G \), each retailer \( i \)'s optimal ordering quantity is:

\[
Q^*_C = (1 - \frac{p_a - p^*_G}{(1 - \gamma) \theta} \mu_D)^{-1} (\frac{p_a - w}{p_a - p_G})
\]

\( i = 1...n \)

c) Under conditions (1),(2) and (3), the equilibrium market-clearing price \( p^*_G \) exists and is characterized by equations 4.5 and 4.6

All proofs for the results in this essay are in appendix B.

Note that if the manufacturer competition was to be included and if there is a quality difference between the original products of both manufacturers, that would lead to four vertically differentiated products: The original (high quality) product sold by the authorized retailers of both manufacturers and the gray market products (low quality) of again both manufacturers sold by unauthorized retailers. While deriving the equilibrium results would get more difficult, one can see that the gray market demand for the "higher quality"
manufacturer would be negatively affected as consumers in the lower end of the market, has the option of buying the original product or the gray market product of the “lower” quality manufacturer, depending on their valuation.

4.3.1.1 Comparative Statics

We next examine how the quality of the gray market product, the wholesale price manufacturer charges and the heterogeneity of the consumer valuations have different kinds of impact on the equilibrium market clearing gray market price $p^*_G$.

**Theorem 4.2**

a) $p^*_G$ decreases with $\bar{v}$
b) $p^*_G$ decreases with manufacturer wholesale price $w$
c) $p^*_G$ decreases with quality $\gamma$

When $\bar{v}$ is large, there is more demand for the authorized retailers’ products, the high end segment grows. This can be interpreted in two ways. Authorized retailers tend to stock more because of this increase in demand. This in turn increases their excess inventory overall which increases the gray market supply leading to a price decrease.

Another interesting result comes from observing how the wholesale price changes the equilibrium outcome. It is well known that double marginalization results in a decrease in a retailer’s order quantity, adversely affecting overall supply chain profits. However, as the wholesale price decreases to the production cost, while this has positive effect on the quantity decision, it also decreases the equilibrium gray market price. That decrease in $p^*_G$, as explained earlier, has a negative impact on the retailer as it decreases the demand and the potential additional revenue opportunity at the same time. Therefore, while eliminating
double marginalization in a gray market environment still helps restore the distortion in quantity decisions, at the same time, it increases gray market activity, so it is unclear how the combination ultimately affects each supply chain partner. And, any remedy to correct double marginalization is subject to similar concerns. This observation also could explain why high-fashion branded apparel manufacturers add a big mark-up when selling to retailers compared to less branded products. For a particular scenario, we see in the below chart that as the wholesale price increases, the effect of double marginalization is immediate in a no-gray market environment, whereas there is an “indifference” range in a gray market environment, that is the negative effect is not immediately observable until the wholesale price increases dramatically.

![Figure 4.3: The effect of double marginalization on retailer order quantity](image)
A change in the perception of the quality ($\gamma$) of the gray market product is possible via a change in the quality of the original product sold by authorized retailers. In our model, since that is normalized to 1, we can think of a decrease in the quality of the gray market product as an increase in the quality gap $(1 - \gamma)$ between the original product and the gray market product. Manufacturers can potentially engage in activities to increase the perceived quality of their products sold by authorized retailers such as adding additional services, which implies an indirect decrease in the relative quality of the gray market product. According to our result in Theorem 4.2, this results in a decrease in $p^*_G$. However, the net effect of a decrease in $p^*_G$ and an increase in $(1 - \gamma)$ on demand or the quantity decision of the retailer is not that clear. Hence, the effort and the cost of such a quality increase for the authorized retailer’s product may not be justified.

### 4.3.1.2 Bounds and Approximations

Note that the excess inventory from each retailer $i$ determines the total available supply in the market. Replacing that total supply by a smaller amount gives us a higher equilibrium price. We will use this observation in the below result to derive two different upper bounds for $p_G$. Gallego (1992) shows that for any random variable $D$ with finite mean $\mu_D$ and standard deviation $\sigma_D$ and for some real number $Q$:

$$E[(D - Q)^+] \leq \sqrt{\sigma_D^2 + (Q - \mu_D)^2 - (Q - \mu_D)}$$

(4.7)

The expression $E[(D - Q)^+]$ in equation 4.7 can be rearranged to get a lower bound for the supply side ($E[(Q - D)^+]$) of the equilibrium equation. Replacing $E[(Q - D)^+]$ with this lower bound in equation 4.5, we have the following result:
CHAPTER 4. SUPPLY CHAIN EFFICIENCY AND CONTRACTING IN THE PRESENCE OF GRAY MARKET

Proposition 4.1  

a) Equation (5) can now be bounded as:

\[
3(Q - \mu_D) - \frac{\sigma_D^2 + (Q - \mu_D)^2}{2} \leq \frac{1}{\theta(1 - \gamma)} \frac{(p_a - p_G)}{1 - \gamma} D
\]

b) Let the resulting equilibrium gray market price be $p_G$. Then $p_G^* < p_G$

Instead of using 4.7 to derive a bound for excess inventory, if we simply replace each $E[(Q - D)^+]$ by $E[Q - D]$ conjecturing that the negative part of that expression occurs very infrequently among a large number of retailers gives us another bound:

Proposition 4.2  

Assume that demand $D'$ is uniformly distributed with support $(0, D)$ in addition to the new excess supply formulation as explained above. The resulting gray market price $p_G$ is determined in closed form according to:

\[
\frac{1}{p_a - p_G} + \frac{p_G}{2\gamma \theta m} = A
\]

where $m = (p_a - \omega)$ and $A = \frac{1}{2m} + \frac{1}{(1 - \gamma) \theta}$

b) $p_G^* < p_G$

We ran several experiments finding $p_G^*$ based on equation 4.5 and $p_G$ based on equation 4.8 and observed that actually $p_G^*$ approximates $p_G$ very well for large $n$, as we assumed earlier. Some of these cases are illustrated in the below table where we see that the average error is around 0.12%.

Now that we have an understanding of the equilibrium and what that means in terms of retailer's action for a given wholesale price $\omega$, we next look more closely at how the manufacturer is affected and what it can do to respond to gray markets. We will first analyze its wholesale pricing decision in section 4.3.2.
4.3.2 Stackelberg Equilibrium

In this section, we assume an environment where the manufacturer anticipates the emergence of gray market and the retailer’s ordering decision accordingly. We first want to understand how the manufacturer reacts and “changes” its wholesale pricing. When there is no gray market, the wholesale price $w_0$ is:

$$w_0 = p_a - p_a F(Q_o/\beta)$$  \hspace{1cm} (4.9)
where $\beta = (1 - \frac{\gamma}{\gamma})$. Based on equation 4.6, when the manufacturer anticipates the gray market, the wholesale price $w_G$ that it charges is:

$$w_G = p_a - p_a F(Q_G / \xi) + p_G F(Q_G / \xi)$$

(4.10)

where $\xi = (1 - \frac{\gamma - p_G}{1-\gamma})$.

From equations 4.9 and 4.10, we see that the manufacturer allows a mark-down from the authorized retailer's market price $p_a$, determined by the second component for both $w_G$ and $w_o$. However, in the presence of gray market, we observe from equation 4.10 that it also adds a mark-up that is proportional to $p_G$. This decision, in a way, penalizes the authorized retailer because of its alternate revenue opportunity from selling into the gray market. On the other hand, since $\xi < \beta$, for the same quantity $Q$, the mark-down manufacturer applies in the presence of gray market is more than what it is without a gray market. This can be seen as a reward the manufacturer provides to the retailer because of the demand threat he is facing from unauthorized goods in the gray market.

When the manufacturer's problem is converted to a quantity-setting problem based on equation 4.10, it becomes:

$$\Pi_M(Q) = \max_Q \left[ (p_a - p_a F(Q / \xi) + p_G F(Q / \xi) - c)Q \right]$$

(4.11)

We will assume the following conditions:

C4: The demand distribution has IGFR property
CHAPTER 4. SUPPLY CHAIN EFFICIENCY AND CONTRACTING IN THE PRESENCE OF GRAY MARKET

C5: \( p''_G \leq 0 \)

**Theorem 4.3** a) The first order condition for manufacturer's problem in 4.11 can be written as:

\[
(1 - F(Q/\xi) [1 - g(Q/\xi)(1 - v)] = \frac{c - p'_{G} + \eta}{p_a - p'_{G}}
\]

where \( v = \xi (Q/\xi) \); \( g(x) = \frac{f(x) + x}{1 - F(x)} \) and \( \eta(Q) = -p_G' F(Q/\xi)Q \)

b) Under conditions C4 and C5, the manufacturer's problem is unimodal and there exists a unique equilibrium \( Q^*_G \).

### 4.4 Efficiency of Wholesale Pricing

In earlier sections, we observed that an increase in wholesale price does not always have a negative effect in a gray market environment. As we observe in figure 4.3, as the wholesale price increases, the effect of double marginalization on retailer's quantity decision is not as immediate in a gray market environment as in no-gray market environment. This tells us that the wholesale pricing used in a decentralized system in gray market settings may be quite efficient. This is the main idea explored in this section.

We begin by formulating the profit function for the centralized system. Since we are only considering the gray market environment, we will drop \( G \) from our notation. Let \( \Pi^C_{Tot} \) be the total system profit in a centralized system, which assumes that the retailer buys \( Q^i \) from the manufacturer at cost \( c \) and the gray market price is determined by:

\[
\sum_i (Q^i - D^i)^+ = \frac{1}{\delta(1 - \gamma)}(p_a - \frac{p_G}{\gamma}) \sum_i D^i,
\]

(4.12)
where the realized demand $D^i$ taking into account the segmentation from the gray market product for each retailer $i$ is again determined according to equation 4.2. In this environment, the total system profit in a centralized system is:

$$\Pi^C_{Tot} = \sum_i \left\{ p_a E[min(D^i, Q^i)] - cQ^i + p_G(\sum_i (Q^i - D^i)^+)E[(Q^i - D^i)^+] \right\},$$

where the gray market price $p_G$ is a function of the excess inventory as in decentralized system. Our large market assumption in this centralized system again ensures that equation 4.12 can be simplified to:

$$E[(Q^i - D^i)^+] = \frac{1}{\vartheta(1 - \gamma)(p_a - p_G)}\mu_D$$

which says that the excess inventory - and hence the gray market price $p_G$ - is independent of individual demand realizations. Since retailers in our model are assumed to have symmetric structure, we will assume that $Q^i = Q$ which makes the formulation for the centralized system as in below two equations:

$$\Pi^C_{Tot}(Q) = n \left\{ p_a E[min(D, Q)] - cQ + p_G(Q)E[(Q - D)^+] \right\}$$

$$E[(Q - D)^+] = \frac{1}{\vartheta(1 - \gamma)}(p_a - p_G)\mu_D$$

The main difference compared to the decentralized system is that now $p_G$ depends on the quantity decision $Q$. Considering the decentralized system we studied earlier, the total system profit $\Pi^D_{Tot}$, which is the sum of manufacturer's and all retailers' profit, becomes:

$$\Pi^D_{Tot} = \sum_i \left\{ p_a E[min(D^i, Q^i)] - wQ^i + p_G^i E[(Q^i - D^i)^+] \right\} + (w - c) \sum_i Q^i$$
which can be rewritten:

\[ \Pi_{tot}^D = \sum_i \left( p_a E[\min(D', Q')] - cQ' + p_G^r E[(Q' - D')^+] \right) \]

and considering the closed-form solution we had found in the equilibrium for \( Q' \) for all \( i = 1...n \), the system equations for the decentralized system are:

\[
\Pi_{tot}^D(Q) = n \left( p_a E[\min(D, Q)] - cQ + p_G^r E[(Q - D)^+] \right) \tag{4.15}
\]

\[
Q = (1 - \frac{p_a - p_G^r}{(1 - \gamma)^\beta})^{-1} \left( \frac{p_a - w}{p_a - p_G^r} \right) \tag{4.16}
\]

\[
E[(Q - D)^+] = \frac{1}{\beta(1 - \gamma)} (p_a - p_G^r) \mu_D \tag{4.17}
\]

Let the optimum centralized quantity be \( Q^* = \arg \max \Pi_{tot}^C \), and denote the quantity decision from decentralized system by \( Q(w) \), making the dependence on the wholesale price \( w \) explicit. We then have the following result:

**Proposition 4.3** There exists a \( w^* > c \) such that \( Q(w^*) = Q^* \)

We know that a wholesale pricing contract with \( w = c \) coordinates the channel, but leaves the manufacturer with zero profit in supply chains where there is no gray market. The above result says that there exists a coordinating wholesale pricing contract that leaves the manufacturer with positive profit. Let \( \Pi_M^e = (w^* - c)Q(w^*) \), i.e. the manufacturer’s profit when it provides a wholesale pricing contract with \( \{w^*\} \). As we discussed in section 4.3.2, \( \Pi_M^e \) is manufacturer’s optimal profit under a wholesale pricing contract (in a decentralized system) when it anticipates a gray market.
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By changing the authorized retailer price $p_a$ and $y$, we ran a total of 551 experiments and while we confirm our main result and that the total decentralized system profit reaches the performance of the centralized system profit under a wholesale pricing contract with parameter $\{w^\ast\}$, we also observe that even though the manufacturer has positive profit, it is worse off compared to the optimal profit under a wholesale pricing contract in a decentralized system. Let $k$ denote the fixed payment from the retailer to the manufacturer of $k = \Pi^o_M - \Pi^C_M$. To design a contract that the manufacturer would be willing to offer, we modify the wholesale pricing contract as follows:

**Theorem 4.4** Consider the contract with parameters $\{w^\ast, k\}$. In equilibrium, each retailer $i$ orders:

$$\left(1 - \frac{p_a - p^*_G}{(1 - y)\theta}\right)P^{-1}\left(\frac{w^*}{p_a - p^*_G}\right) = Q^* \forall i = 1...n$$

where equilibrium $p^*_G$ is determined according to equation 4.17 i.e. the contract achieves coordination making the manufacturer indifferent and retailers better off.

Based on the experiments we ran, we wanted to understand the magnitude of the inefficiency in a gray market environment and compare it to that of no gray market environment. Note that $\Pi^C_{Tot}$ refers to the total centralized system profit and $\Pi^D_{Tot}$ refers to the total decentralized system profit. We computed the average of these values over all the results. The results are presented in Table 4.4:

Note that while the inefficiency in a no gray market environment is close to 25% on the average, it is only on average 5.14% in a gray market environment. This confirms our earlier observation that double marginalization does not hurt the total channel profit in a gray market environment as much as it would in a no-gray market environment. This is mainly due to the effect of the wholesale price on the gray market price and the positive
CHAPTER 4. SUPPLY CHAIN EFFICIENCY AND CONTRACTING IN THE PRESENCE OF GRAY MARKET

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Table 4.2: Average System Efficiency Loss

The impact of a higher gray market price on the total supply chain profit.

As we mentioned above, \( \Pi_{M}^{C} \) is the manufacturer's optimal profit when the manufacturer offers the coordinating wholesale pricing contract in both gray and no-gray market environments. To understand the magnitude of the difference between \( \Pi_{M}^{C} \) and \( \Pi_{M}^{e} \) in both environments, we computed the average profit values below in table 4.4:

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<th>Loss %</th>
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Table 4.3: Average Profit Impact of Coordinating Wholesale Price

Note that by offering the coordinating wholesale pricing contract with \( w^* \), the manufacturer, on the average, loses only 10.1 % of its optimal decentralized profit in a gray market environment. On the other hand, we observe that this loss is 100 % in regular no gray market environments since the only coordinating wholesale pricing contract is \( w = c \) which leaves zero profit with the manufacturer.
4.5 Coordinating Supply Chain Contract

A supply chain contract is generally designed so that it induces the performance of the theoretical best outcome i.e. the centralized system. The goal with this approach is to eliminate double marginalization in the decentralized system. In the previous section, we showed the efficiency of the decentralized system. Furthermore, we also showed that there exists a coordinating wholesale pricing contract. In this section, we continue in this direction and introduce a new contract which is a modified version of the buy-back contract that has been studied in the literature.

Partial Buy-Back

It is known that returns with a full refund or no returns at all lead to non-coordinating outcomes in supply chains where there is no gray market. On the other hand, returns with correct partial refunds lead to coordinating outcomes. In a supply chain with gray markets, the key issue is to control how much of the left-over inventory at authorized retailers goes to the gray market. If a buy-back contract results in full returns, it completely eliminates the gray market. However, a contract that explicitly specifies the percentage $a$ of left over inventory that each retailer can return to the manufacturer at a specified price $b$, allows one to control the gray market and is a practical program to communicate and execute among authorized retailers.

With the three-parameter partial buy-back contract $(w, b, a)$, for each retailer $i$, we get:

$$\Pi_{R}^{i} = \left[ p_{d}E[\min(D, Q)] - wQ + (1 - a)\rho_{c}E[(Q - D)^{+}] + abE[(Q - D)^{+}] \right]$$  (4.18)
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with:

\[
(1 - a)E[(Q - D)^+] = \frac{1}{\gamma(1 - \gamma)}(p_a - \hat{p}_G)\mu_D \\
\Pi_M = (w - c)Q - abE[(Q - D)^+] 
\]

Recall \( Q^* \) is the optimum centralized quantity. As we observe from equation 4.19, \( \hat{p}_G \) depends on both \( Q \) and \( a \), but for the next result, we are interested only in \( \hat{p}_G = \hat{p}_G(Q^*) \):

**Theorem 4.5** a) Define \( \delta = (1 - a)p_G + ab \). For a given pair of \((a, b)\) such that \( \delta = p_G(Q^*) \), there exists a \( w \) such that each retailer \( i \) orders:

\[
(1 - \frac{p_a - \hat{p}_G}{1 - \gamma}E(\frac{p_a - w}{p_a - \delta}) = Q^* \forall i = 1...n \tag{4.21}
\]

i.e. the contract with \((a, b, w)\) achieves coordination

b) Let's define:

\[
Lb(\alpha) = \frac{p_G(Q^*) - (1 - a)p_G}{\alpha} \\
Ub(\alpha) = \frac{(p_a - p_G)^* + (1 - a)p_G - c)\gamma(1 - \gamma) - \gamma(1 - \gamma)\mu_D}{1 - \gamma} \\
\text{where } \xi = 1 - \frac{p_a - p_G}{(1 - \gamma)\mu_D}
\]

For a given \( \alpha \) such that \( Lb(\alpha) < Ub(\alpha) \), if we choose \( a, b \) such that \( Ub(\alpha) = b \), then there exists a \( w \) satisfying 4.21 and the resulting contract achieves coordination making manufacturer indifferent and retailers better off.

4.6 Robustness of Supply Chain Contracts under Gray Market

Other than wholesale pricing, manufacturers can potentially offer several other contracts and economic incentive to retailers. This is quite common in industry and has been well studied in the literature. However, external business factors such as gray market activity, have mostly been left out of the contracting literature. As mentioned in the
introduction section, this was exactly the situation that we faced when it came time to pilot our proposed contracting scheme; the program was ultimately rejected due to several such external (and organizational) business factors, the primary one being the gray market and the fear that some of the contracts and incentive programs proposed would not achieve their objective because of the incentives it would great to divert more inventory to the gray market. This raises the larger question: What risks to contracting are posed by the existence (or emergence) of a gray market? Which contract types are robust and which are most vulnerable to gray market activity?

To answer these questions, we evaluate several contracts by comparing what we call a \textit{naive} manufacturer, who disregards the potential for gray market, with a \textit{strategic} manufacturer, who anticipates the gray market and its effects. We chose this strategy since we believe it reflects many businesses in which gray markets are not anticipated or factored correctly into decision making. Our comparison is also representative of a business environment where gray markets are currently not a widespread concern but have the potential to develop. Here, our goal is to compare manufacturer’s profit levels under gray market conditions to those without a gray market.

We consider the same coordinating contracts studied in the literature. We mainly analyze how the incentives and contracts are distorted when a gray market emerges. Starting with the wholesale pricing contract, we will go through four such contracts and compare and contrast the performance and robustness of all, mainly from manufacturer’s standpoint. We will also try to understand how these contracts affect the gray market itself.
4.6.1 Wholesale Pricing Contract

When there is no gray market, the problem reduces to a known wholesale pricing problem, which also can be seen as a quantity-setting problem. Denote the wholesale price obtained from this problem as \( w_0 \), which we call the naive optimal wholesale price. Since the authorized retailer’s decisions are identical, we will drop the \( i \) from the notation in the rest of the essay.

**Proposition 4.4** Under the naive optimal wholesale price \( w_0 \):

a) The optimal retailer order quantity under gray market \( Q^*_G \), is less than its order quantity in no gray market environment \( Q^*_n \), when the gray market price \( p^*_G \) is less than a threshold \( \tau(p^*_G) \) where

\[
\tau(p^*_G) = \left( (e(p^*_G) - 1)(1 - \frac{p^*_G}{p_A}) + \gamma \right) \theta \] with \( e(p^*_G) = \frac{F^{-1}(\frac{m}{1 - \gamma})}{1 - \gamma p^*_G/p_A} \) and \( m = \frac{p_A - w_0}{p_A} \)

b) \( \tau(p^*_G) \) is monotonically decreasing in \( p^*_G \)

c) There exists a sufficiently large \( \theta^* \) above which the condition in part (a) is met and the manufacturer is worse off

Given the retailer’s ordering decision \( Q \), the manufacturer’s profit is simply:

\[
\Pi_M = (w - c) \cdot Q
\]

As a result, for a given wholesale price \( w_0 \), the profits are lower or higher depending on \( Q \) itself. From proposition 4.4, we can see that there can be a case when a gray market emerges in which retailers order less than they normally would under no gray market. Under such a scenario, the manufacturer’s profits suffer. Note that, for authorized retailers, a gray market is both a threat due to its demand-stealing effect and an additional revenue opportunity.
to sell to the lower end of the market. As $p_G^*$ goes down, more demand switches to unauthorized products and the alternate revenue stream by selling excess inventory also go down. Each of these two negative effects from lower values of $p_G^*$ potentially lead retailers to order less compared to the environment with no gray market. However, the opposite is true when the condition is reversed i.e. manufacturer might be better off when gray market emerges. This shows the robustness of the wholesale pricing contract to gray market. Note that this is not a comment we make for the other contracts we study in this section.

Next, we want to understand how the wholesale price and the quantity retailers order change when the manufacturer is strategic i.e. takes into account the potential for gray market. Particularly, we want to observe whether they increase or decrease and the magnitude of manufacturer's profit improvement. To make the notation clear, let's denote the retailer's equilibrium ordering quantity by $Q_G^*$ and $Q_G^{**}$ when the manufacturer is naive and strategic respectively, both under gray market environment. To be able to answer some of these questions, we first fixed $(y, \bar{v}, c)$ and chose a uniform distribution for $D$ and changed $p_a$ to compare $Q_G^{**}, Q_G^{***}, w_G$ and $w_0$ as well as the manufacturer's profit levels $(\Pi_M^*, \Pi_M^{**})$ when again it is strategic and naive respectively.

We observe that the profit for the manufacturer improves on average by 3.16 % and we also observe that $w_G$ is always higher than $w_0$ and $Q_G^{**}$ is always less than $Q_G^{***}$. Next, we run similar scenarios by changing just the quality level of the gray market product $(\gamma)$.

The magnitude of the profit improvement in these scenarios reaches as high as 35 % and is approximately 11 % on the average. As in our earlier scenarios, we also observe that $w_G$ is always higher than $w_0$ and $Q_G^*$ is always less than $Q_G^{**}$. A higher gray market price means
authorized retailers lose less of their demand to unauthorized retailers while simultaneously increasing the profit of selling its excess stock. An increase in wholesale price has an adverse affect on a retailer's quantity decision because of double marginalization effect. On the other hand, that increase in wholesale price results in higher gray market price with the two positive effects argued above. Therefore, when the manufacturer is strategic and takes into account the gray market, it balances the negative affect from double marginalization (as a naive manufacturer would ) with the positive effects from an increase in gray market price. As a result, the wholesale price ($w_G$) is set at higher.

Table 4.4: Comparison of Strategic vs. Naive Manufacturer

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Table 4.5: Comparison of Strategic vs. Naive Manufacturer: Increasing $\gamma$ value

4.6.2 Revenue Sharing Contract

In a revenue sharing contract, the manufacturer offers its product at a much cheaper price in exchange for a certain percentage ($\lambda$) of the revenue that the retailer generates. In order for the revenue sharing contract to be coordinating, the wholesale price has to be set as $w = \lambda c$ where $c$ is the production cost (Please refer to Cachon and Lariviere (2005) for further details). Therefore, we assume that the manufacturer offers a revenue sharing contract to its authorized retailers with $[\lambda, w]$ where $w = \lambda c$ and expects to reach a profit level $\Pi_r^o$. Throughout this section, let the superscripts "r" represent values under revenue sharing contract; "w" under wholesale pricing contract; "o" under no gray market and "g" under gray market environment. Assuming $\lambda$ is set such that the retailer gets the same profit
as it does under a wholesale pricing contract, the retailer’s and manufacturer’s profits respectively are:

\[ \Pi^W_R = \lambda \left( p_a E[\min(D_o, Q_o)] - cQ_o \right) \]

\[ \Pi^W_M = (1 - \lambda) \left( p_a E[\min(D_o, Q_o)] - cQ_o \right) \]

where \( Q_o \) is the optimal centralized quantity. Note that there is a risk in revenue sharing; a retailer can buy extra product at a lower marginal cost and sell it directly to the gray market. To analyze this, let the retailer’s profit under such a revenue sharing contract and in the presence of gray market be denoted \( \Pi^g_R \). We then have:

\[ \Pi^g_R = \lambda \left( p_a E[\min(D'_G, Q'_G)] - cQ'_G \right) + p'_G E[(Q'_G - D'_G)^+] \]

Note, the retailer now has a second revenue opportunity which is not subject to revenue sharing i.e. manufacturer cannot get a portion of this revenue stream. In this case, the manufacturer experiences a different sales level (determined by \( D_G \) and \( Q_G \)) resulting in a different profit level \( \Pi^g_M \) as given by:

\[ \Pi^g_M = (1 - \lambda) \left( p_a E[\min(D'_G, Q'_G)] - cQ'_G \right) \]

Revenue sharing contracts coordinate the supply chain by making retailer’s profit a fraction of the centralized system profit. Therefore, it is in retailer’s best interest to take actions that is also optimal for the whole system profit. When there is no external factor such as a gray market, this works quite well. However, in a gray market environment, the retailer’s profit is not a fraction of the whole system profit. The additional revenue stream that comes from selling the products to the gray market stays with the retailer. Therefore, the retailer may tend to emphasize the revenue stream and increase its order quantity. However, this results
in a decrease in the gray market price, which in turn decreases the demand of authorized retailers. The decrease in sales and increase in production cost ultimately have a negative affect on the manufacturer’s profit. We formalize these observations in the following result:

**Proposition 4.5**  
\( a) \ p^r_G < p^w_G \)  
\( b) \ \Pi^w_M < \Pi^o_M \)

The above result says that the equilibrium gray market clearing price under a revenue sharing contract is always less than what it is under a wholesale pricing contract. Furthermore, the manufacturer’s profit under a gray market is always less than it is without no gray market (We earlier showed that this is not the case for a wholesale pricing contract, where the manufacturer’s profit might improve in a gray market environment). For several scenarios, we numerically compared both manufacturer and retailer’s profits under gray market and no-gray market environments with wholesale pricing and revenue sharing contracts. The results are shown in Table 4.6.2

Comparing columns (1) and (2) in table 4.6.2, we see that the manufacturer’s profit declines in revenue sharing contracts when a gray market emerges, as we proved in proposition 4.5. By comparing columns (1) and (4), we see that manufacturer prefers revenue sharing over wholesale pricing, as expected, when there is no gray market. This is not the case when there is a gray market. In these scenarios, the manufacturer is better off with wholesale pricing in a majority of the cases, but not all. We also see that with wholesale pricing, the manufacturer would prefer to be in a gray market environment in a majority of the scenarios. This suggests that there are business environments where gray markets actually benefit the manufacturer. From these scenarios, we see that under
CHAPTER 4. SUPPLY CHAIN EFFICIENCY AND CONTRACTING IN THE PRESENCE OF GRAY MARKET  

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Table 4.6: Comparison of Manufacturer and Retailer Profits: Revenue Sharing vs. Wholesale Pricing

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revenue sharing, the retailer is always better off when there is a gray market, and as we argued and proved, the manufacturer is worse off. This may help explain why revenue sharing is more common in certain industries than in others.

4.6.3 Buy-Back Contract

In a coordinating buy-back contract, the manufacturer offers a contract with parameters $\{\bar{w}, b\}$ where $b$ is a per unit payment the manufacturer gives to the retailer for any excess inventory he might have. As we did above, we assume that the manufacturer provides the same coordinating buy-back contract it would in a no-gray-market environment. Following
a similar notation, we will denote values under buy-back contract with superscript "b" and use the same abbreviations \((w, o, g)\) as we did in the earlier section to refer to values under wholesale pricing contract and no-gray market environments. In a coordinating contract the buy-back rate is set as: 

\[ b = \frac{\hat{w}-c}{\hat{w}-c} \]

Note that the contract is coordinating for a continuum of parameters. When the retailer takes the contract, orders \(Q_o\), realizes demand \(D_o\) and returns all its excess inventory back to the manufacturer, the manufacturer’s profit is:

\[
\Pi_{M}^{b} = (\hat{w} - c)Q_o - bE[(Q_o - D_o)^+] 
\]

whereas the retailer’s profit is:

\[
\Pi_{R}^{b} = [p_{s}E[\min(D_o, Q_o)] - \hat{w}Q_o + bE[(Q_o - D_o)^+]] 
\]

However, in a gray market environment, the retailer now has the choice of diverting products to unauthorized retailers. In this case, the retailer orders \(Q_G^b\) realizing again a demand of \(D_G^b\) where:

\[
\Pi_{M}^{b} = (\hat{w} - c)Q_G^b 
\]

\[
\Pi_{R}^{b} = [p_{s}E[\min(D_G^b, Q_G^b)] - \hat{w}Q_G^b + p_G^bE[(Q_G^b - D_G^b)^+]] 
\]

When the retailer diverts its excess inventory to the gray market, the buy-back arrangement turns into a simple wholesale pricing contract with wholesale price \(w = \hat{w}\), the wholesale pricing contract that we are referring to by the superscript \(w\). Note that \(p_G^w\) is the gray market price realized under wholesale price \(\hat{w}\). Let \(\Pi_M^w\), be the profit that the manufacturer will realize under this contract. This has interesting consequences as summarized in the following result.
Chapter 4. Supply Chain Efficiency and Contracting in the Presence of Gray Market

Proposition 4.6 When:

(a) When \( b < p^w_G \), \( \Pi^b_M = \Pi^{W^G}_{M} \) and \( p^G_b = p^w_G \); \( Q^G_b = Q^w_G \)

(b) When \( b > p^w_G \), gray market will not emerge and \( \Pi^b_M = \Pi^{W^G}_{M} \)

(b) When \( b = p^w_G \), manufacturer's profit is indeterminate

The above result says that under the buy-back contract specified above, when the retailers divert their excess inventory, the resulting gray market clearing price is the same as the one under wholesale pricing contract with wholesale price \( w \). Furthermore, if the buy-back rate is high enough, the retailers prefer to give the remaining inventory back to the manufacturer instead of selling to unauthorized retailers in the gray market. That is because without the gray market, the authorized retailer demand is higher, and as long as the extra revenue opportunity provided by selling excess inventory is higher as well, then authorized retailers are better off not participating in gray market. When there is no gray market, the contract is coordinating and the manufacturer realizes the exact same profit he would expect to receive, i.e. \( \Pi^b_M \). And when a gray market emerges, its profit is now the same as that under wholesale pricing with wholesale price \( w = w \). We numerically compare some of these results under different business scenarios as summarized in table 4.6.3:

Column (1) confirms our finding in proposition 4.6; the manufacturer realizes either \( \Pi^b_M \) (column (2)) when there is no gray market or \( \Pi^{W^G}_{M} \) (column (3)) when the retailer participates in the gray market. In that sense, there is no surprise for the manufacturer. If manufacturer does not want their authorized retailers to participate in the gray market, a buy-back contract with the right parameters might be a good option; it is also a "safe"
4.6.4 Quantity-Discount Contract

There could be different forms of a quantity-discount contract that a manufacturer could provide to its authorized retailers. For our analysis, we assume the manufacturer charges the retailer a per unit wholesale price \( w(q) = (c + \frac{m}{q}) \) which also can be seen as fixed fee plus variable price. In this setting, the retailer’s profit is:

Table 4.7: Comparison of Manufacturer and Retailer Profits: Buy-back vs. Wholesale Pricing

contract from a profit standpoint even if gray market emerges.
CHAPTER 4. SUPPLY CHAIN EFFICIENCY AND CONTRACTING IN THE PRESENCE OF GRAY MARKET

\[ \Pi_{K}^{q} = \{p_{o}E[\min(D_{o}, Q_{o})] - cQ_{o} - m] \}
\]

resulting in a coordinating contract while guaranteeing \( \Pi_{M}^{q} = m \), which is set such that \( m > \Pi_{M}^{g} \). However, in a gray market environment, we then have:

\[ \Pi_{K}^{g} = \{p_{o}E[\min(D_{o}^{g}, Q_{o}^{g})] - cQ_{o}^{g} - m + p_{G}^{g}E[(Q_{G}^{g} - D_{G}^{g})^+] \}\]

If we denote the manufacturer’s profit under this contract by \( \Pi_{M}^{g} \), we then have the following result:

**Proposition 4.7**

a) \( p_{G}^{g} < p_{G}^{w} \)

b) \( \Pi_{M}^{g} = m \) regardless of gray market existence

We did a similar numerical comparison for various business scenarios as summarized in table 4.6.4. We confirm proposition 4.7 by comparing columns (1) and (2). This shows that the quantity discount contract guarantees the same profit level regardless of the existence of a gray market. In other words, from a manufacturer’s profit standpoint, it is very robust.
Table 4.8: Comparison of Manufacturer and Retailer Profits: Quantity-Discount vs. Wholesale Pricing

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Chapter 5

Product Quality Selection:

Contracting in One Manufacturer

Multiple Supplier Environment

Designing a product that consists of several different components each of which has different quality offerings is a key challenge for a manufacturer. This is a common problem in many industries. Car manufacturers need to make decisions on the performance of the engine, the quality of the interior seating materials, the suspension and brake systems etc. A similar problem presents itself in aerospace manufacturing, consumer electronics and even home remodeling. Each industry is characterized by a situation where the end product quality is a function of the quality of its many individually supplied components. As pointed out in the introduction section, this was a problem encountered during a sponsored research project with a leading semi-conductor manufacturer. Hence, the problem
is described based on their business, though the model and the overall issues studied are quite general.

5.1 Overview of Relevant Literature

After Spengler (1950) identified the double marginalization phenomenon, there have been several papers in economics, marketing and operations on channel coordination. The work in this area can be categorized into two groups: i) One-supplier and one or multiple retailers and ii) Multiple suppliers and one manufacturer (Assemble-To-Order).

Our setup falls under the second category, assembly type systems. Bernstein and DeCroix (2006) seek to understand whether insights from previous research on centralized systems apply to assembly type environments. Because assembly systems involve issues of horizontal balance (component inventory levels) as well as vertical coordination (sufficient inventory to meet demand), the impact of decentralized management of such systems is not clear. They consider two versions of the problem. One with local and one with echelon base-stock policies. Nash equilibrium exists for both and they show that competing firm’s base-stock levels are economic complements. A coordinating contract is provided.

Gurnani and Gerchak (2007) consider an assembly system with random component yield. Each supplier decides on quantity and the manufacturer penalizes suppliers with poor delivery performance. They analyze the conditions under which system coordination is achieved while respecting participation constraints. Gerchak and Wang (2003) again consider an assembly system where the final product has stochastic demand with known distribution. Before demand is realized, firms need to decide on capacity. In the cen-
CHAPTER 5. PRODUCT QUALITY SELECTION: CONTRACTING IN ONE MANUFACTURER MULTIPLE SUPPLIER ENVIRONMENT

tralized system, they find out that the optimal capacity depends on the cost structure. Under a decentralized system, two distinct setups are considered depending on whether the manufacturer or the supplier is the leader. They find a unique equilibrium with identical suppliers in the latter form of the game. Gerchak and Wang (2004) compare revenue sharing contracts with wholesale price contracts in an assembly system with random demand. Wang (2006) considers $n$ different suppliers providing $n$ products that are perfectly complimentary and are sold to a retailer with a price sensitive uncertain demand. In a game where manufacturers set both price and quantity, under multiplicative demand model, they fully characterize equilibria and derive closed form performance measures. A consignment sales contract with revenue sharing is also analyzed. Carr and Karmarkar (2005) consider decentralized assembly systems with price sensitive but deterministic demand. Corbett and Karmarkar (2001) consider competition in serial supply chains with deterministic demand.

Another relevant research stream is on bundling, which has been extensively studied in economics and marketing. Bundling helps the seller extract value from a given set of goods since it provides a form of price discrimination. Adams and Yellen (1976), R.L.Schmalensee (1976), McAfee et al. (1989), Hanson and Martin (1989) and Bakos and Brynjolfsson (1999), Bakos and Brynjolfsson (2000) are prominent papers in this area. In terms of extracting value by putting parts together, our work is similar. However, unlike bundling, each part is not used individually once it is sold. Our work is relevant to another set of literature on product design and positioning.

The product positioning problem is to find a location of a product attribute space so that it will be closest to the largest number of consumers' ideal points. The first discussion on
the topic of optimal product positioning is by Shocker and Srinivasan (1974). They modeled the problem in the context of an MDS joint space of existing brands and consumer's ideal points. Other papers using this approach include Hauser and Simmie (1981) and Zufryden (1981). Zufryden (1977) was the first to propose a formal representation of the product design problem in a conjoint context. Green et al. (1981) took a different approach using a consumer choice simulator. These papers were followed by Zufryden (1982), Green and Krieger (1985), Dobson and Kalish (1988), Kohli and Sukumar (1990) and Green and Krieger (1996).

5.2 Model

We consider a manufacturer who procures and assembles \( N \) components from \( N \) different suppliers to make a final product. The utility of the product to consumer is determined by the aggregate quality, which is a function of quality levels \( (x_i) \), of each component. In this model, the manufacturer seeks to maximize the gap between the total product value and its total cost (We assume throughout that utility is measured in dollar terms, that is in terms of a consumer’s willingness to pay). Once that gap is maximized, the product price determines how much of that value stays with the consumer and how much is retained by the manufacturer. The price decision also determines the total demand for the product. However, we assume this is a separate problem which is solved after the design problem. Consider the manufacturer’s strategic design problem where the primary selection decision is component quality levels which in turn determines the final product quality (We show how demand and quality decisions can be incorporated in a separate setting as in section
6).

Let $x_i$ represent the quality level for component $i$, $i=1...N$ and $X = [x_1, x_2, ..., x_N]$ denote the vector of quality levels. $w_i$ is the wholesale price and $c_i$ is the production cost per quality level for component $i$. We assume that $w_i \in [c_i, w_i^H]$. We also assume that the quality level $x_i$'s are normalized and scaled for each supplier with an upper bound. For example, if the product is a computer, then the $x_i$ could be the processing speed of the CPU or the size of the memory. Given the quality levels of each component, the final product quality will be a function of $X$. Note that we are considering the key components that make up the final product and are interested in cases where the quality of this final product is low when the quality of even one of these components is low. This aspect of our model is motivated by the business environments we discussed above. For example, when the engine of an airplane has low quality, the plane is low quality plane regardless of the quality levels of its other key components. When this kind of logic is factored into the model, the manufacturer would tend to balance the quality levels as opposed to making one component extremely high quality and the rest extremely low. On the other hand, the incremental contribution of component quality levels for the final product should start to diminish as the component quality levels increase. For example, a computer would not function without its CPU and even the slowest one would be sufficient to make it work; at that level an increase in the speed of the CPU would contribute significantly to the overall performance of the computer. However, after a certain level, whether it is, for example, 2.6 GHz or 2.8GHz, additional speed does not contribute as much since other limitations (in this case, memory for example) become the performance bottleneck.
CHAPTER 5. PRODUCT QUALITY SELECTION: CONTRACTING IN ONE MANUFACTURER MULTIPLE SUPPLIER ENVIRONMENT

We use a Cobb-Douglas function to model this relation and designate the final product quality by \( R(X) \) where:

\[
R(X) = A(x_1^{\beta_1}x_2^{\beta_2} \ldots x_N^{\beta_N})
\]

\( R(X) \) is the monetary utility, that is the value a consumer is willing to pay for the final product. For a given constant \( A \) and a vector of \( \beta = [\beta_1, \beta_2, \ldots, \beta_N] \), the sum of \( \beta_i \)'s determine how individual component qualities are translated into the total product quality. There are three cases:

1. \( \sum_i \beta_i = 1 \): This would represent the case of "constant returns to quality". For example, increasing each component quality by 20% would increase the final product quality by 20%.

2. \( \sum_i \beta_i < 1 \): This would represent the case of "decreasing returns to quality". Continuing with the above example, increasing each component quality by 20% in this scenario, would increase the final product quality by less than 20%.
3. $\sum_i \beta_i > 1$: This would represent the case of “increasing returns to quality” and similarly increasing each component quality by 20% in this scenario, would increase the final product quality by more than 20%.

Note that due to the multiplicative structure of the $R(X)$ function, the total product quality will be low if one of the component quality levels is low, which says that the manufacturer needs to balance the quality levels across all components. As argued above, the sum of $\beta_i$s, determine how individual component qualities are translated into the total product quality. Since it is most natural to assume the contribution of component quality levels toward the final product diminishes as the component quality levels increase, we assume that $\beta_i$s are such that they meet the decreasing returns to quality condition i.e. $\sum_i \beta_i < 1$. With this representation, we can also see the problem as the competition of components within a product and refer to $\beta_i$ as the “component quality power” going forward.

While this is a quite stylized model with continuous quality variable and linear wholesale prices, we believe that it is rich enough to study the type of economic distortions that could arise in such systems and understand what kind of mechanisms could restore them, which is the main focus of our work. On the other hand, we believe that this is a more representative model if the product line consists of several different quality levels as opposed to a few. Similarly, our choice of product value function ($R(X)$) is more representative of the environments that motivated this research i.e. products whose overall quality is strongly dependent on the lowest quality of its key component. Nevertheless, we use a different
utility function for a variant of this problem in section 5.5.

Note that in this model, suppliers decide on a wholesale price schedule anticipating the quality the manufacturer will demand for that wholesale price, so this model would not be appropriate for a business environment in which the manufacturer has the power to lead the strategic interaction.

5.2.1 The Manufacturer's Quality Selection Problem

The manufacturer seeks high overall product quality. However, since suppliers charge more for better quality components, the manufacturer needs to trade-off the total cost with the total quality of the final product. Hence, the manufacturer problem can be stated as:

\[
\Pi_m(X) = \max_X \left( A(x_1, x_2, ..., x_N) - \sum_i w_i x_i \right)
\]  

(5.1)

Solving this maximization problem, the manufacturer will determine the quality levels \(x_i\) for each component required to produce the final product. On the other hand, each supplier \(i\) needs to decide on its wholesale price \(w_i\) which will determine its margin per quality level. We consider this problem next.

5.2.2 Supplier's Wholesale Pricing Problem

Based on manufacturer's problem, we know that as the supplier increases its wholesale price, the quality level demanded by the manufacturer decreases. Let \(x_i(w_i, w_{-i})\) denote the demand for supplier \(i\)'s quality as a function of \(w_i\) and \(w_{-i} = (w_1, w_2, ..., w_{i-1}, w_{i+1}, ..., w_N)\). Then we can write the supplier's problem as:
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\[ \Pi_i(w_i, w_{-i}) = \max_{w_i} \{ (w_i - c_i)x_i(w_i, w_{-i}) \} \] (5.2)

where \( c_i \) is the per unit quality cost for supplier \( i \). Note that the quality level demanded by the manufacturer depends not only on its own wholesale price but also the wholesale prices of all other suppliers. We characterize this wholesale pricing game among suppliers in the next section.

5.3 Analysis

In this section, we begin by identifying inefficiencies that take place in a decentralized system with wholesale pricing. We then analyze mechanisms and potential contracts that eliminate these inefficiencies.

5.3.1 System Inefficiencies

Consider the performance of a centralized system, in which the manufacturer sources all the components internally, i.e. it owns all the suppliers. How would the component quality decisions compare to those of a decentralized system where each party independently maximizes its profit?

Denote:

- \( x_i^C \): Quality level the manufacturer selects for component \( i = 1...N \) in a centralized system
- \( x_i^D \): Quality level the manufacturer selects for component \( i = 1...N \) in a decentralized system

\[ X^C = [x_1^C, x_2^C, ..., x_N^C] \]
\[ X^D = [x_1^D, x_2^D, ..., x_N^D] \]
The solution $X^C$ to the centralized system is determined by:

$$
\Pi_C(X) = \max_{X^C} \left\{ A(x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_N^{\beta_N}) - \sum_i c_i x_i \right\}
$$

and the solution $X^D$ to the decentralized system is determined by:

$$
\Pi_D(X) = \max_{X^D} \left\{ A(x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_N^{\beta_N}) - \sum_i w_i x_i \right\}
$$

To make the exposition simpler, we present the analysis for the case of $N = 2$ suppliers. Though, the results generalize easily to the case of $N > 2$ suppliers. Our first main result is as follows (All proofs for the results in this essay are in Appendix C):

**Theorem 5.1**

a) $x_i^C > x_i^D$ for $i = 1, 2$ That is the quality of each component in the centralized system is higher than that in the decentralized system.

b) The total utility ($R(X)$) of the product in the decentralized system is less than that of centralized system

The first result resembles the classical double marginalization result of Spengler (1950). This is due to the margin the suppliers add to their wholesale prices in the decentralized system, which decreases the quality levels the manufacturer demands. This result shows what undermines the "sell-up" incentive of the suppliers. To illustrate this inefficiency loss, we tested different scenarios by changing the "$\beta$" values (keeping the total of $\beta$s at a constant value). Figure 5.2 shows that the loss in quality for a given supplier can be quite substantial.

The second result directly follows from the first. As argued earlier, since the utility $R(X)$ determines the revenue the manufacturer can generate, this second result says there
5.3.1.1 Price Competition Among Suppliers

Let $\Pi_i(w_i, w_{-i}) = \max_{w_i} \{ (w_i - c_i)x_i(w_i, w_{-i}) \}$ denote the profit of supplier $i$. The following theorem characterizes the strategic interaction among the suppliers:

Theorem 5.2  

a) $\Pi_i(w_i, w_{-i})$ is quasi-concave in $w_i$

b) There exists dominant strategy equilibrium $w^*$ in the suppliers' wholesale pricing game
Let's take supplier $i$ to understand this interaction. The other suppliers' wholesale price decision affects supplier $i$'s profit through its quality level $x_i$. As other suppliers increase their wholesale prices, the quality level $x_i$ will start to decrease. However, due to the multiplicative nature of the utility model, as shown in the Appendix, $\Pi_i(w_i, w_{-i})$ becomes an affine transformation of other supplier's wholesale prices. Therefore the equilibrium is the strongest possible i.e. the dominant strategy type.

5.3.2 Contracts

In this section, we will study two mechanisms that result in perfect coordination.

Revenue Sharing

A well-known implementation of revenue sharing is Blockbuster Inc. Blockbuster agreed
to pay its suppliers a portion of its rental income in exchange for a reduction in unit price of tapes. This reduced the break-even point for a tape and allowed Blockbuster to purchase more tapes (See Cachon and Lariviere (2005) and Dana and Spier (2001) for further details on this).

In a revenue sharing context, the manufacturer keeps a certain portion of the total revenue, and agrees to give the remaining revenue back to the suppliers. While the main objective of the Blockbuster revenue sharing contract was increasing quantity, our focus is ensuring the manufacturer has the correct incentive to select quality levels.

**Theorem 5.3**

a) Consider a contract with an agreed on revenue percentage $\lambda_0$ for the manufacturer and a vector of $(\lambda_1, \lambda_2)$ such that $\sum_i \lambda_i = 1 - \lambda_0$, that distributes the remaining revenue among the two suppliers, and wholesale prices set as:

$$w_i = \lambda_i c_i, i = 1, 2$$

Then under this contract, the channel is coordinated and the profit is allocated according to $\lambda_0$ and $(1 - \lambda_0)$

b) There exists $\lambda_0$ and $(\lambda_1, \lambda_2)$ which makes all players better off.

In a decentralized system, suppliers add a margin to the production cost when selling to the manufacturer. This results in a larger marginal cost for quality for the manufacturer, negatively affecting the sell-up incentive of the suppliers. As a result, the manufacturer will select lower quality components. If the suppliers could all provide their components at unit production cost, this would incentivize the manufacturer to take the same action as in a centralized system, but this is equivalent to transferring the firm to the other party. Revenue sharing is in between these two scenarios. Under revenue sharing, the
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manufacturer is \( \lambda_0 \) percent owner of the entire channel, since it gets \( \lambda_0 \) percent of the total revenue generated and pays for \( \lambda_0 \) percent of the production cost. Therefore, it is in manufacturer's best interest to increase the total channel profit. This ensures that the manufacturer has the exact same quality incentives as the centralized system. For \( \beta_1 = 0.2 \), \( \beta_2 = 0.5 \) and \( c_1 = c_2 = 1 \), we ran several different scenarios using different \( \lambda_0 \) and \( \lambda_i \) values and determined the percentage profit improvement for all parties under revenue sharing. Some of these scenarios are summarized in Figure 5.4.

![Figure 5.4: Profit Improvement Scenarios under Revenue Sharing](image)

As a practical matter, it would be more natural for the manufacturer to offer this kind of a contract. Depending on its bargaining power in the overall channel, the manufacturer could dictate the percentages that the suppliers get based on their opportunity profit. The difficulty in implementing this contract is the administrative burden of tracking the revenue the manufacturer collects. Hence, we next propose another mechanism that is based on a two-part payment and only requires that the supplier is able to observe the
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manufacturer’s end product quality.

Quality-Price Discounting

The main idea with this mechanism is that suppliers tie their wholesale prices to the end product utility, \( R(X) \), of the manufacturer’s end product. Alternatively, if the supplier announces a quality-price discounting schedule rather than a fixed wholesale price per unit quality, we want to understand whether it is possible, for the supplier to offer a “discount” on quality where the wholesale price decreases as the quality the manufacturer purchases increases. Let’s introduce some notation: \( \alpha \) is an agreed-on percentage that splits the total profit between the manufacturer and all the suppliers and \( k_i \) is another percentage that splits the suppliers’ share of the revenue among all suppliers.

If we denote \( \frac{R(X)}{x_i} \) as the actual per quality utility level, with the wholesale prices:

\[
w_i = (1 - \alpha)c_i + \alpha k_i \frac{R(X)}{x_i}
\]

we see that supplier \( i \)'s wholesale price decreases as the manufacturer asks for higher quality from the supplier. We assume that all parties are truthful. With that, we get our coordinating contract, as summarized in the next theorem:

**Theorem 5.4** a) Consider a contract with all suppliers having the quality-price discount parameters \( (\alpha, K) \). With the wholesale prices for each supplier \( i \) set as:

\[
w_i = (1 - \alpha)c_i + \alpha k_i \frac{R(X)}{x_i}
\]

where the vector \( K = (k_1, k_2) \) is such that \( \sum_i k_i = 1 \)

the channel is coordinated and the total profit is split according to \( \alpha \) for the manufacturer, and \( 1 - \alpha \) for the two suppliers, and supplier \( i \) receiving a fraction \( k_i \) of total supplier profits.
b) There exists \( a \) and \( K \) where all firms make more profit than they do under wholesale pricing.

In terms of the contracts studied in the literature, the Price Discount Sharing (PDS), studied by Lal et al. (1996) and Bernstein and Federgruen (2005), most resembles our contract. In PDS type agreements, the supplier discounts at a certain percentage from a suggested wholesale price based on the retailer's price discount. In this way, a supplier ties its wholesale price to the end product price the retailer sets, similar to our arrangement. Note that the above contract requires suppliers to agree on \( K \) and \( a \). These parameters of course need to be set according to the opportunity profits of each party to ensure participation.

5.4 Numerical Study

In this section, we present results of a numerical study to identify several other important managerial insights. We ran these experiments by changing the \( \beta \) values (which we also refer to as the component quality power) in two ways representing different business environments. All cases are for two suppliers.

Case I. Sum of \( \beta \)’s are constant: This scenario represents an environment where the overall utility of the product is fixed for given quality values, but the component that provides this value changes depending on the \( \beta \)’s.

We did this by increasing \( \beta_1 \) (and decreasing \( \beta_2 \)) keeping the sum at a constant value. In this way, the component that plays the "dominant" role in determining the overall product value changes, because as one \( \beta \) increases, the other one decreases. We see each set of \( \beta \)’s
as exogenous potential business scenarios (for example, as a result of technology, brand recognition etc.) and not as something that the firms can achieve by some short-term actions (e.g. pricing).

As we see from Figure 5.3 in the earlier section, the efficiency loss in total supply chain profit is greatest when the suppliers are “symmetric”, i.e. when there is no dominant component in the product. That is observed in figure 5.5 as well. This is counter intuitive, because when the β’s are close, the competition among components is more intense and one would think this might lead to wholesale prices that are closer to the marginal cost of the suppliers. However, it turns our the main factor driving efficiency here is not competition among suppliers but how the supplier’s incentive to price changes with respect to β. As β₁ increases (and β₂ decreases), supplier 1’s component becomes the dominant factor determining quality. While this gives it greater bargaining power, as we see in figure 5.6, supplier 1 charges lower prices as its β₁ increases. The reason is that the incremental
increase in quality demanded by the manufacturer from a marginal decrease in wholesale price is much higher for a dominant supplier; that is, dominant supplier faces a higher price elasticity of demand from the manufacturer. This increase in elasticity more than offsets their increased pricing power and leads to lower wholesale price, which means greater supply chain efficiency.

What happens to the profit values? Does a manufacturer do better with one dominant component or a product where components are more symmetric in terms of contributing to the product value? If we denote the sum of supplier 1’s ($\Pi_1$), supplier 2’s ($\Pi_2$) and the manufacturer’s ($\Pi_m$) profit as $\Pi_{Total}$, then we want to understand the distribution of each party’s profit within the total and how that changes as $\beta_1$ increases (and $\beta_2$ decreases). In other words, we are testing the three ratios: $\frac{\Pi_1}{\Pi_{Total}}$, $\frac{\Pi_2}{\Pi_{Total}}$ and $\frac{\Pi_m}{\Pi_{Total}}$. 

Figure 5.6: Per quality wholesale price ($\omega_1$) vs. $\beta_1$
Our observation with supplier 1 and supplier 2 are consistent with expectations. As supplier 1 becomes the dominant component provider, its share of total profit increases. The manufacturer’s profit, however, shows a different pattern. As we see from the third chart in figure 5.7, the manufacturer is better off when there is one dominant component. That is again due to what we observed in figure 5.6. As one supplier becomes dominant, its incentive is to lower wholesale prices, so the overall channel becomes more efficient and channel profits rise. This increase in total profit more than compensates for the manufacturer’s reduced share of profits.

**Case II.** $\beta_2$ is constant and $\beta_1$ increases: The experiments under this scenario reflect the
cases where the total value of the product increases as well. However, the contribution of component 1 in that overall value increases.

While the utility changed in a similar way as in earlier scenario, the share of the total profit differed. As seen in Figure 5.9, even though it is $\beta_1$ that increases while $\beta_2$ stays constant, the profit share of supplier 2 increases as well. This positive externality for supplier 2 is due to the overall increase in utility of the product that enables not only supplier 1 but also supplier 2 to obtain a larger share of the overall profit, especially for high values of $\beta_1$. Hence suppliers that provide “insignificant“ components can benefit from the presence of a “very significant“ component in the product that they are supplying. In effect, suppliers become more powerful and as a result gain a larger share of the overall profit pie. The manufacturer’s profit share shown in the third chart in the same figure, decreases as $\beta_1$ increases.
5.5 Extension

Thus far, we have assumed that the manufacturer is making a strategic design decision about the quality levels of individual components and that the manufacturer wants to maximize the difference between the product value and its cost. Deciding what price to charge and the resulting demand is assumed to be a second stage problem. However, in certain business environments, the manufacturer may want to design for a fixed price point due to competitive reasons and include the dependence of demand on quality in its decision making. In this case, the manufacturer still needs to make decisions on the individual
quality levels of each component it uses, but now we assume demand is dependent on the quality of the final product and the price is a fixed exogenous value.

Define the additional decision variables and parameters for the above setting with the following notation:

\[ p : \text{The market price for the end product (exogenous)} \]
\[ N : \text{The market size} \]
\[ \beta_i : \text{The coefficient of quality for component } i \]
\[ \beta = [\beta_1, \beta_2, \ldots, \beta_n] \]
\[ U(X) : \text{Utility from consuming an end product with quality vector } X \]
\[ e : \text{Random term for consumer utility with density } \phi \text{ and distribution } \Phi \]
\[ d(X) : \text{The total demand for the end product with quality } X. \]
\[ c(X) : \text{The total cost for the manufacturer for procuring } n \text{ components with quality levels } X \]

Note that the earlier notation representing quality \((x_i, X)\) and wholesale prices \((w_i)\) remain the same. We assume that \(\beta_i\)'s are fixed for all consumers in the market and that the consumer utility \(U(X) = \beta^T X + e\). With this:

\[ d(X) = N Pr(U(X) > p) = N Pr(\beta^T X + e > p) = N \Phi(p - \beta^T X) \quad (5.3) \]

Note that, while we assumed the utility function was in multiplicative form in the strategic problem introduced in the earlier section; here we assumed an additive form with a random term which determines the total demand for the product. There are two
main reasons for this change. First, we wanted to understand the affect of an additive utility model on our problem. Second, this model simplifies the task of adding the demand component to the objective function. We first analyze the manufacturer's problem and then turn to the supplier's problem. In this section, as noted earlier, the difference will be the addition of quality-dependent demand into the profit function.

5.5.1 Manufacturer's Problem

The problem for the manufacturer is the same as described earlier; to design the end product by choosing quality levels $x_i$ for each of the components in the final product. Again the trade-off that the manufacturer faces is that as the quality level for components goes up, both the procurement cost and the total utility (and therefore in this case the total demand $d(X)$) increases. The selling price is assumed to be exogenous. Therefore, the manufacturer needs to balance his margin and total demand, both of which are a function of quality. Denoting the manufacturer’s profit by $\Pi_m$, the manufacturer’s problem is:

$$\Pi_m(X) = \max_X \{(p - c(X))d(X)\}$$  \hspace{1cm} (5.4)

Note that a distribution (with cdf $\Phi$ and pdf $\phi$), has increasing failure rate (IFR), if its failure rate $h(u) > \frac{\phi'(u)}{\phi(u)}$.

We assume the manufacturer’s cost function $c(X)$ is jointly convex and $\Phi$ is IFR. We then have the following result:

Theorem 5.5 Under above assumptions, the manufacturer's problem (5.4) has a unique solution $X^*$.  

5.5.2 Supplier’s Problem

For each supplier $i$, the problem is the same as in the strategic problem. Each supplier has to decide on a wholesale price $w_i$ per quality level. We assume that the supplier’s cost of quality increases linearly with $c_i$. In this setting, each supplier decides on its own wholesale price $w_i$ anticipating that the manufacturer will ask for quality level $x_i(w_i, w_{-i})$ in the amount of $d(X)$ to make $d(X)$ units of final product. Hence, with $\Pi_i(X)$ denoting the profit function, supplier $i$'s problem can be written as:

$$\Pi_i(X) = \max_{w_i} \{(w_i - c_i)x_i(w_i, w_{-i})d(x_i, x_{-i})\} \quad (5.5)$$

The sequence of events is as follows:

(1) Suppliers decides on wholesale prices $w_i$ expecting that the manufacturer will ask for quality level $x(w_i, w_{-i})$ in the amount of $d(X)$ to make $d(X)$ units of the final product.

(2) The manufacturer takes these offers from $n$ suppliers and decides on quality level $X$, observes the demand $d(X)$ and orders $d(X)$ from each supplier $i$ and pays them $(w_i x_i) d(X)$

(3) Suppliers make and deliver the required amounts, the manufacturer makes and sells the final product at $p$ and collects the revenue.
5.5.3 Analysis

To simplify our analysis, we will again assume the special case of two suppliers for the above problem, although the results will are valid for the general case of multiple suppliers. We make another assumption on the distribution of the random term in our utility function: Namely, the random term $e$ has uniform distribution over $(0,1)$.

With this assumption, the manufacturer’s problem is:

$$
\Pi_m(X) = \max_X \left( (p - c(X))N(1 - p + \beta^T X) \right)
$$  \hspace{1cm} (5.6)

We then have the following result on manufacturer’s problem:

**Theorem 5.6**  
\(a\) $\Pi_m(X)$ is concave in $x_i; i=1,2$

\(b\) $x_i^* = \frac{\frac{x_i}{\mu_i}\{\eta_i\mu_i + \eta_j\eta_j\}}{\eta_i\mu_i^2 + \eta_j\eta_j^2} \quad i, j = 1, 2$

Based on the characterization of the manufacturer’s problem, we see that characterizing the suppliers’ interaction in this multiple supplier setting is extremely difficult. Therefore, in the rest of our analysis, we focus on a simpler case where the manufacturer is interested in deciding on the quality level ($x_1$) of one component and that the quality level of the other component is fixed at $x_2$.

While admittedly restrictive, this corresponds to a case where the manufacturer does not have a choice when it comes to making a quality decision for all but maybe one key component. This puts the problem in a vertical one-supplier-one manufacturer setting as
studied in the operations literature under different settings where the decision is quantity and/or price.

Including the fixed quality levels in the same earlier notation, let's define:

\[ p = p - \beta_2 x_2 \]
\[ \bar{p} = p - w_2 x_2 \]

which makes the demand function under assumption 5 as \( d(x_1) = (1 - \bar{p} + \beta_1 x_1) \) and the manufacturer's problem can be represented as:

\[
\Pi_m(x_1) = \max_{x_1} \{ (p - w_1 x_1)(1 - p + \beta_1 x_1) \} \quad (5.7)
\]

and the supplier's problem is:

\[
\Pi_s^I(x) = \max_{w_1} \{ (w_1 - c_1) x_1 (1 - \bar{p} + \beta_1 x_1) \} \quad (5.8)
\]

As we did above, our first step is to understand the distortions introduced in this environment and understand their magnitude under different scenarios. We then develop mechanisms to eliminate these distortions. We summarize our first result toward that goal in the following proposition:

**Theorem 5.7**

a) \( x_1^C > x_1^D \)
b) \( d_1^C > d_1^D \)

This result says that the quality level the manufacturer demands from the supplier is lower in a decentralized system than in a centralized system. This is again due to the fact that the supplier tries to make margin by increasing its wholesale price in a decentralized system, which increases the marginal cost of quality for the manufacturer, reducing the quality level the manufacturer demands. That is how double marginalization manifests
itself in this environment. The second result in the proposition is a direct result of the first given the demand function we use. This says the final product the manufacturer builds ends up being a lower quality product and also the total demand that it satisfies is less than the channel optimal.

Contracts

We next study the two contracts introduced in the earlier section and explore how these can help the channel achieve the performance of a centralized system. The first one is the revenue sharing contract, which was proved to be coordinating in the earlier multiple supplier one manufacturer setting. Since we have only one supplier, dropping the supplier subscript from the same notation used earlier, we can summarize the result as follows:

**Theorem 5.8**

a) Consider a contract with an agreed on percentage $\lambda$ that represents the manufacturer's share of the total revenue generated and a wholesale price charged by the supplier of $w = \lambda c$

Under this mechanism, the channel is coordinated and the profit is arbitrarily allocated according to $\lambda$ and $(1 - \lambda)$ determining manufacturer's and supplier's profit share respectively.

b) There exists a $\lambda$ where all firms make better profit than they do with wholesale pricing contract.

This result shows us that, the channel would benefit from a properly designed revenue sharing arrangement which helps the supplier sell up, eliminating the distortion that would otherwise be present in the decentralized system. The structure of the agreement is similar to before, since the supplier offers the per unit quality of the product at a given percentage of its production cost. The administrative difficulties associated with revenue sharing type agreements are still present i.e. the manufacturer needs to keep track of the revenue
generated from the product. Similarly, the percentages are constrained by the opportunity profits. Since the supplier would generally want to encourage sell-up within its product line, it is plausible that the supplier offer this contract, setting the percentage such that the manufacturer gets at least its opportunity profit.

We next explore a contract similar to the quality-price schedule we introduced above for the multiple supplier setting. The main idea is again to somehow tie the wholesale price the supplier charges to the quality the manufacturer selects and its end price. On the supplier side, this corresponds to the supplier announcing a quality-price schedule as opposed to a fixed wholesale price per unit quality. Let $a$ be the agreed-on percentage that splits the total profit between the manufacturer and the supplier.

Define $\hat{p}/x$ as the per quality price. Then the supplier’s wholesale price set as $w = (1 - a)c + a\hat{p}/x$ will change again based on the quality the manufacturer will demand. In this scheme, as the quality increases, the supplier will discount more from its wholesale price. We again assume all parties are truthful yielding our coordinating contract:

**Theorem 5.9** a) Consider the contract where the supplier sets the wholesale price as:

$$w = (1 - a)c + a\hat{p}/x$$

then the channel is coordinated and the total profit is split according to $a$ and $(1 - a)$.

b) There exists an $a$ where both players are better off than they are under wholesale pricing.

The $a$ needs to be set according to the opportunity profits of both parties to ensure participation.
Chapter 6

Supply Chain Management with Advance Supply Information

In industry, sharing of upstream supply information is an observed practice. On the other hand, as mentioned in the introduction section, academic literature has focused relatively more on the downstream demand information. In this essay, we contribute to this literature by considering an inventory system with advance supply information, where the supply information is modeled as dynamic forecasts of capacity availability. In our model, the upstream partner does not make any decisions, but simply provides the downstream partner with capacity forecasts, who optimizes her replenishment policy. We assume that the upstream partner has already agreed on sending the capacity forecast information truthfully to the inventory manager. Assuming such truthful information sharing, we address two main points throughout the chapter:

1. How can the manager utilize and integrate the advance supply information into its
replenishment decisions? Our first goal is to develop a simple to implement policy that takes advance supply information into account.

2. We would like to identify the types of operating environments under which advance supply information is most valuable, compared to a fixed base stock policy.

The capacity of the upstream source varies stochastically. The manager of the inventory system receives forecasts from the upstream source about the capacity availability of future periods within a given rolling information horizon. We refer to these forecasts as “advance supply information”. The evolution of the capacity availability forecasts are modeled via the Martingale Method of Forecast Evolution (MMFE), developed by Graves et al. (1986) and Heath and Jackson (1994). The forecasts may be the result of different processes within the organization of the upstream source. For example, the upstream source may be serving multiple downstream partners and some of its capacity in a future period may have been committed to other downstream partners. For instance, if the upstream source committed to using all its capacity for orders from downstream partner A two periods from now, its current capacity forecast available to downstream partner B for that same period will be zero. Such information can be very useful to downstream partner B. If she knows that two periods from now, the upstream source is likely to not have any capacity availability for her, she may choose to inflate her current orders somewhat to plan for this forecasted upcoming shortage. This is the main idea investigated in this essay. The capacity forecasts may also vary due to other private information of the upstream source, such as upcoming scheduled downtime due to maintenance or inspection or future staffing variations, etc. From now on, we refer to the upstream source as the supplier and the downstream partner as the
manufacturer, but the model applies to a more general setting, such as a retailer ordering from a capacitated manufacturer, or a household making purchases from a capacitated retailer.

6.1 Overview of Relevant Literature

There are three main research streams pertinent to our work. The first group of papers are on single item/single location capacitated inventory problems. Federgruen and Zipkin (1986a) and Federgruen and Zipkin (1986b) prove the optimality of modified base-stock policies for a single location inventory problem with a deterministic capacity constraint for infinite horizon problems using both the discounted and the average cost criteria. Ciarallo et al. (1994) prove the optimality of modified base-stock policies under a stochastic capacity constraint. Glasserman (1997) develops bounds and approximations for setting base-stock levels in production-inventory systems with both deterministic and stochastic capacity constraints. Kapuscinski and Tayur (1998) study a capacitated production-inventory system with periodic (cyclic) demand. Iida (2002) studies a non-stationary periodic review production-inventory system with uncertain production capacity and demand.

The second group of relevant papers are on supply chain information sharing. Gavirneni et al. (1999), Lee et al. (2000), Cachon and Fisher (2000) and Gaur et al. (2005) are some of the papers that analyze the value of downstream information sharing. However, upstream information has received limited attention in the literature. Song and Zipkin (1996) study an inventory model where the supply system evolves according to a Markovian system which determines how replenishment leadtimes change over time. They identify the
optimal policy which includes parameters that change dynamically reflecting the current supply conditions. Contrary to conventional wisdom, they show that a longer leadtime does not necessarily lead to a higher base stock levels. Chen and Yu (2005) study the value of leadtime information in a single-location inventory model with a markovian leadtime process. Through a numerical study, they show that the value of leadtime information can be significant. Jain and Moinzadeh (2005) consider a two stage supply chain where the manufacturer allows the retailer access to its inventory status. They provide an exact method for computing performance and develop a procedure for evaluating the optimal policy.

The third group of papers are on production-inventory systems with dynamic forecast updates. Graves et al. (1986), Graves et al. (1998) and Heath and Jackson (1994) are papers that explicitly address how to model an existing forecasting process in a production setting. They independently develop a model of how forecasts evolve in time, referred to as the Martingale Model of Forecast Evolution (MMFE). Gullu (1996a) and Gullu (1996b) build on this model to investigate the value of information in a production-inventory system and a depot-retailer system respectively. Toktay and Wein (2001) consider a capacitated production stage that produces a single item in a make-to-stock manner and model advance demand information via MMFE. Iida and Zipkin (2004) consider an inventory system with dynamic demand forecast updates based on the MMFE and develop a computational approach to obtain approximate solutions. Aviv (2001) shows that integrating demand forecast information into the replenishment decisions reduces, on average, the supply chain costs by 11%. Chen and Lee (2009) consider a two stage supply chain where the retailer's external demand is assumed to unfold according to an MMFE process and the
retailer shares its projected future orders with the supplier. Gallego and Ozer (2001) show that state-dependent \( (s, S) \) and base-stock policies are optimal for systems under advance demand information with and without a fixed ordering cost respectively. Other related studies include Gallego and Ozer (2003), Ozer and Wei (2004). All these studies investigate downstream demand information. We study the sharing of upstream supply information.

6.2 Model Formulation

Consider a periodic review inventory model. Periods are numbered \( 1, 2, \ldots, T \). Demand is independent and identically distributed with mean \( \mu_D \). Let \( D_t \) be the demand in period \( t \). The lead time from the supplier to the manufacturer is assumed to be instantaneous. Unsatisfied demand is fully backlogged. The objective is to minimize expected inventory holding and backlogging costs.

At the beginning of period \( t \), the manager receives a capacity forecast vector

\[
F_t = (C_{t,t+1}, C_{t,t+2}, C_{t,t+3}, \ldots, C_{t,t+N}),
\]

where \( C_{t,j} \) is the forecasted capacity availability for period \( j \) in period \( t \) and \( N \) is the information horizon. The inventory level at the beginning of period \( t \) is \( x_t \). The manager makes an ordering decision \( u_t \). Let \( y_t = x_t + u_t \). This is followed by the delivery of all or part of the order, depending on the realized capacity level \( C_t \) for period \( t \). The post delivery inventory level is \( \min(y_t, x_t + C_t) \). This is followed by the realization of demand \( D_t \). Finally, holding and backlogging costs are charged. Let \( g(\cdot) \) be the one period cost function, that is assumed to be convex and coercive\(^\text{1}\). Costs are charged at the end of the period, so the cost in period \( t \) is \( g(\min(y_t, x_t + C_t) - D_t) \). \( \alpha \) represents the

\(^\text{1}\) We say that a function \( f(x) \) is coercive in \( x \) if it goes to infinity on both sides, i.e., if \( \lim_{|x| \to \infty} f(x) = \infty \)
discount factor, where $0 < \alpha \leq 1$.

### 6.2.1 Forecast Evolution

The inventory manager observes the output of the capacity forecast activity in the form of a stream of exogenous forecasts and needs to design a replenishment policy based on these forecasts to minimize the sum of inventory holding and backlogging costs. We do not make any assumptions about how the upstream source generates these forecasts, as there can be many different ways to do this, such as previously committed orders, scheduled maintenance, future staffing schedules, etc. We assume that the result of this complex forecasting activity is the sequence of forecast vectors as modeled by the MMFE framework.

As an example, consider the situation where the upstream source is a supplier who is working with several customers, one of which is the inventory system we are considering in our model. In each period, as the supplier sees new advance demand information from its other customers (in the form of either new demand or cancellations), he updates the capacity forecast provided to the inventory system for the duration of the information horizon $N$. We refer to the difference between a vector of supply forecasts and the one that was generated in the previous period as the forecast update vector $\epsilon_t = (\epsilon_{1,t}, \epsilon_{t,t+1}, \ldots, \epsilon_{t,t+N})$. At time $t-1$, the forecast vector is $F_{t-1} = (C_{t-1,t}, C_{t-1,t+1}, C_{t-1,t+2}, \ldots, C_{t-1,t+N-1})$. At the beginning of period $t$, the new forecast vector $F_t = (C_{t,t+1}, C_{t,t+2}, C_{t,t+3}, \ldots, C_{t,t+N})$ is generated by letting $C_{t,t+i} = C_{t-1,t+i} + \epsilon_{t,t+i}$ for $i = 1, \ldots, N-1$ and letting $C_{t,t+N} = \mu_C + \epsilon_{t,t+N}$ and becomes available to the manager. The final forecast for period $t$ capacity is $C_{t-1,t}$. The realized capacity $C_t$ is the final forecast plus the update term $\epsilon_{t,t}$, i.e., $C_t = C_{t-1,t} + \epsilon_{t,t}$. This random variable is observed by the manager after delivery in period $t$. The MMFE is a descriptive
model that characterizes the resulting sequence of forecast update vectors as i.i.d. vectors with mean zero.

The MMFE model described above produces a capacity process $C_t$ that is stationary with $E[C_t]=\mu_C$ for all $t$ (we need $\mu_D < \mu_C$ for stability). At time zero, assume that the a-priori capacity forecast for all periods $t = 1, \ldots, T$ is equal to $\mu_C$. As time progresses, the forecasts within the rolling information horizon are updated. The capacity $C_t$ at time $t$ is equal to $\mu_C + e_{1-N,t} + e_{1-N+1,t} + \ldots + e_{t,t}$, where the $e_{i-j,t}$ terms are independent random variables that are being revealed successively throughout the duration from $t - N$ to $t$. In other words, the capacity $C_t$ in any given period $t$ is stochastic, but the uncertainty is resolved step by step throughout the $N$ periods before period $t$.

The MMFE model of Heath and Jackson (1994) and Graves et al. (1986) essentially assumes that a forecast represents the conditional expectation of demand (capacity in our case) given all available information, which implies that forecasts are unbiased and that forecast updates are uncorrelated over time. The model uses a multivariate normal distribution $N(0, \Sigma)$ to represent the forecast updates.

6.2.2 Analysis of the i.i.d. Capacity Case

In this section, we study the special case where capacity is i.i.d. and there is no advance supply information. This problem was studied by Ciarallo et al. (1994) and the optimality of base stock policies was shown. We briefly revisit the i.i.d. capacity case here, because it acts as a building block for our main advance supply information model. Our method for proving the optimality of base stock policies is different from that of Ciarallo et al. (1994) and is based on a lemma, which is also used in the advance supply information case. The
lemma may also be useful in other similar settings.

Let \( J_t(x_t) \) be the optimal cost to go at period \( t \) when the inventory level is \( x_t \). The dynamic programming recursion can be written as follows:

\[
J_t(x_t) = \min_{y_t \geq x_t} E_{C_t,D_t} \left[ g_t(\min(y_t, x_t + C_t) - D_t) + \alpha J_{t+1}(\min(y_t, x_t + C_t) - D_t) \right],
\]

for \( t = 1, \ldots, T \), where \( T \) is the planning horizon, \( J_{T+1}(x_{T+1}) \) is defined to be equal to 0 for all \( x_{T+1} \).

The following lemma allows us to characterize the structure of the optimal policy.

**Lemma 6.1** Suppose that \( f(y) \) is a convex function such that \( \lim_{y \to \infty} f(y) = \infty \), and \( y^* \) is a minimizer of \( f(y) \). Then

a) \( E_C[f(\min(y, x + C))] \) is quasi-convex in \( y \) for any \( x \). Furthermore, \( y^* \) is a minimizer of that function for all \( x \), i.e.

\[
E_C[f(\min(y^*, x + C))] = \min_{y \in \mathbb{R}} E_C[f(\min(y, x + C))], \forall x.
\]

b) If \( J(x) = \min_{y \geq x} E_C[f(\min(y, x + C))] \), then \( J(x) \) is convex in \( x \) and \( \lim_{x \to \infty} J(x) = \infty \).

**Proof of Lemma 6.1**

a) For a given \( x \) and \( \bar{c} \):

\[
f(\min(y, x + \bar{c})) = \begin{cases} 
f(x + \bar{c}) & \text{if } y \geq x + \bar{c} \\
f(y) & \text{if } y < x + \bar{c} \end{cases}
\]

Note that \( f \) is a convex function with a minimizer \( y^* \). If \( (x + \bar{c}) > y^* \), then the resulting function \( f(\min(y, x + \bar{c})) \) will be non-increasing up to \( y^* \) and non-decreasing after that. See Figure 6.1 b. If \( (x + \bar{c}) \leq y^* \), then the function will be non-increasing up to \( (x + \bar{c}) \) and
constant after that, as in Figure 6.1 c. The important thing to note is that in both cases, \( y^* \) is a minimum of the resulting function \( f(\min(y, x + \bar{c})) \) (Figure 6.1 b and c) and the function is quasi-convex. In other words, \( f(\min(y, x + \bar{c})) \) is quasi-convex and is minimized at \( y^* \) no matter what \( \bar{c} \) is. In general, adding two quasi convex functions does not result in a quasi convex function, but adding two quasi-convex functions that are minimized at the same point results in a quasi-convex function. Therefore, taking expectation with respect to \( C \) preserves the above property and \( E_C f(\min(y, x + C)) \) is quasi-convex in \( y \) for a given \( x \) and is minimized at \( y^* = \arg \min(f(y)) \) (Figure 6.1 d).

b) By Lemma 6.1 (a), \( E_C [f(\min(y, x + C))] \) is quasi-convex and is minimized at \( y^* \). Therefore

\[
J(x) = \begin{cases} 
E_C [f(\min(y^*, x + C))] & \text{if } x \leq y^* \\
J(x) & \text{if } x > y^*.
\end{cases}
\]

For a particular value of \( C = \bar{c} \), let

\[
J(x, \bar{c}) = \begin{cases} 
f(\min(y^*, x + \bar{c})) & \text{if } x \leq y^* \\
f(x) & \text{if } x > y^* \\
f(x + \bar{c}) & \text{for } x \leq y^* - \bar{c} \\
f(y^*) & \text{for } y^* - \bar{c} < x \leq y^* \\
f(x) & \text{for } x > y^*.
\end{cases}
\]

Then, \( J(x) = E_C [J(x, C)] \). Note that \( J(x, \bar{c}) \) is convex and coercive in \( x \) for any given \( \bar{c} \). Since taking expectation over \( C \) preserves convexity and coerciveness, the result follows.

Let

\[
f_i(z_t) = E_{D_t}[g_t(z_t - D_t) + \alpha E[J_{t+1}(z_t - D_t)]]
\]

\[
H_i(y_t, x_t) = E_C[f_i(\min(y_t, x_t + C_t))],
\]
for all $t = 0, 1, \ldots, T$. Using these definitions, DP recursion (6.1) can be written as

$$f_t(x_t) = \min_{y \geq x_t} H_t(y_t, x_t),$$

for $t = 1, \ldots, T$.

By applying Lemma 6.1 we can show the quasi-convexity of the function $H_t(y_t, x_t)$ in $y_t$ for any given $x_t$, which implies the optimality of base stock policies in the i.i.d. capacity case. The result is formalized in the following theorem.

**Theorem 6.1** For all $t = 0, \ldots, T$: 
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a) \( f_t(z_t) \) is convex in \( z_t \) and is coercive. Let \( s_t^* \) be the smallest minimizer of \( f_t(z_t) \), i.e. \( s_t^* = \min\{s|f_t(s) = \min_z f_t(z)\} \).

b) \( H_t(y_t, x_t) \) is quasi-convex in \( y_t \) for all \( x_t \), and \( s_t^* \) is a minimizer of \( H_t(y_t, x_t) \), for any \( x_t \).

Mathematically, \( H_t(s_t^*, x_t) = \min_{y_t} H_t(y_t, x_t) \), for all \( x_t \).

c) \( J_t(x_t) \) is convex in \( x_t \).

c) (Ciarello et al.) A base-stock policy is optimal for the i.i.d capacity case.

The proof of Theorem 6.1 uses Lemma 6.1 and is provided in the appendix (along with the proofs of subsequent results).

6.3 Advance Supply Information Problem

Consider now the original problem with advance supply forecasts. In this case, the state includes the forecast vector \( F_t \) for future capacity availabilities. Let \( J_t(x_t, F_t) \) be the optimal cost-to-go function. The functional equations are given by:

\[
J_t(x_t, F_t) = \min_{y_t \in \mathbb{R}} E \left[ g_t(y_t + x_t + C_t) - D_t + \alpha J_{t+1}(\min(y_t, x_t + C_t) - D_t + F_{t+1}) | F_t \right],
\]

for \( t = 0, 1, \ldots, T \), where \( J_{T+1}(x_{T+1}, F_{T+1}) = 0 \) for all \( x_{T+1} \) and \( F_{T+1} \). The expectation in (6.2) is with respect to demand \( D_t \), capacity \( C_t \) and the forecast vector \( F_{t+1} \), given \( F_t \). The forecast update vector \( \epsilon_{t+1} = (\epsilon_{t+1,t+1}, \epsilon_{t+1,t+2}, \ldots, \epsilon_{t+1,t+N+1}) \), combined with \( F_t \) determines \( F_{t+1} \). Let

\[
f_t(z_t, F_t) = E \left[ g_t(z_t - D_t) + \alpha J_{t+1}(z_t - D_t + F_{t+1}) | F_t \right],
\]

\[
H_t(y_t, x_t, F_t) = E_C \left[ f_t(\min(y_t, x_t + C_t), F_t) \right].
\]
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for \( t = 0, 1, \ldots, T \). Then, the recursion in (6.2) can be written as

\[
J_t(x_t, F_t) = \min_{y_t \geq x_t} H_t(y_t, x_t, F_t).
\]

The following theorem establishes the quasi-convexity of \( H_t(y_t, x_t, F_t) \) in \( y_t \), which implies the optimality of state dependent base stock policies.

**Theorem 6.2** For \( t = 1, \ldots, T \):

a) \( f_t(z_t, F_t) \) is convex in \( z_t \), for all \( F_t \). Let \( s^*_t(F_t) \) be the smallest minimizer of \( f_t(z_t, F_t) \), i.e.

\[
s^*_t(F_t) = \min [s | f_t(s, F_t) = \min_z f_t(z, F_t)].
\]

b) \( H_t(y_t, x_t, F_t) \) is quasi-convex in \( y_t \) for all \( x_t \) and all \( F_t \), and \( s^*_t(F_t) \) is a minimizer of \( H_t(y_t, x_t, F_t) \), for any \( x_t \) and \( F_t \). Mathematically, \( H_t(s^*_t(F_t), x_t, F_t) = \min_y H_t(y, x_t, F_t) \), for all \( x_t \) and \( F_t \).

c) \( J_t(x_t, F_t) \) is convex in \( x_t \) for all \( F_t \).

d) A state-dependent base stock policy with base stock levels \( s^*_t(F_t) \) is optimal for the advance supply information problem.

This means that the manager needs to observe the inventory level \( x_t \) and the forecast vector \( F_t \) at the beginning of each period \( t \) and if \( x_t \) is less than the state-dependent base-stock level \( s^*_t(F_t) \), she needs to order the difference \( s^*_t(F_t) - x_t \). Note that the base-stock levels depend on the entire forecast vector.

### 6.4 Easily Computable and Implementable Approximations

The previous section identifies the structure of optimal policies as state-dependent base-stock policies. There are two potentially problematic issues with the exact optimal solution.
First, to compute the state-dependent base-stock levels, one needs to solve an \( N + 1 \) dimensional dynamic program, which is impractical for all but very short information horizons. Second, once the optimal state-dependent base-stock levels are computed, there will be a different base-stock level \( S(F_t) \) for each forecast vector \( F_t \). The large number of possible base stock levels \( S(F_t) \) with no structure can be problematic for practical implementation.

In order to overcome these difficulties, in this section, we develop easily computable and implementable heuristic policies that are based on assuming a predetermined functional form for the relationship between the forecast vector \( F_t \) and the state-dependent base stock levels \( S(F_t) \). We study such policies in a setting with stationary data, under an infinite horizon average cost per period criterion.

### 6.4.1 \( K(F) \)-Dependent Base-Stock Policies

The policies we study are state-dependent base stock policies with base stock levels \( S(F_t) \) given by

\[
S(F_t) = s + K(F_t),
\]

where \( s \) is a constant, independent of the forecast vector \( F_t \) and \( K(F_t) \) is a pre-determined function. We call such a policy a \( K(F) \)-dependent base-stock policy. Given \( K(F_t) \), the only parameter of the policy is \( s \), which we also refer to as the \( K(F) \)-dependent base-stock level.

Let \( s^* \) denote the optimal \( K(F) \)-dependent base-stock level. Let \( Y_t = s + K(F_t) - x_t \), referred to as the shortfall\(^2\), and let \( Z_t = Y_t - K(F_t) \) so that \( x_t = s - Z_t \). Let \( Z_{\infty} \) be the steady-state mean of \( Z_t \),

\(^2\)Note that, contrary to common usage, \( Y_t \) here can be negative, due to fluctuations in \( F_t \), but in this case, no order is placed.
random variable that $Z_t$ converges to. The following theorem characterizes the optimal $K(F)$-dependent base stock level for any given $K(F)$ function.

**Theorem 6.3**  

(a) Under a $K(F)$-dependent base stock policy, $Z_\infty$ exists and is independent of $s$, i.e. the distribution of the random variable $Z_\infty$ that $Z_t$ converges to does not depend on the value of $s$ used.

(b) For the case where the one period cost consists of a linear holding cost with rate $h$ and a linear backlog cost with rate $b$, the optimal $K(F)$-dependent base-stock level is $s^* = F^{-1}_Z(b/b + h)$, where $F^{-1}_Z$ is the inverse cumulative distribution function of the random variable $Z_\infty$.

We see from the above theorem that in order to find $s^*$, we need the distribution of $Z_\infty$. The optimal $K(F)$-dependent base stock level $s^*$ is the critical fractile of this distribution (i.e., a percentile of the distribution). Since $Z_\infty$ is independent of $s$, one can simulate the system under some arbitrary parameter $s$ just once, to estimate the critical fractile of $Z_\infty$, and consequently find $s^*$.

### 6.4.2 Cumulative Forecast Dependent Base-Stock Policy

The versatile simulation-based method developed in the previous subsection can optimize the parameter of any $K(F)$-dependent base stock policy. Our next goal is to identify a function $K(F)$ that is simple enough to facilitate easy implementation, yet rich enough to capture the advance supply information in a reasonable way. To guide us on this path, we first analyze a deterministic version of the problem.

Consider the $T$-period problem (P) below where demand $d_i$ and capacity $c_i$ at every
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period \( i = 1 \ldots T \) are known and deterministic. Assume that the holding cost rate is \( h_i \) per unit at time \( t \) and that backlogging is not allowed. Without loss of generality, assume that there is zero initial inventory and that the problem is feasible (i.e. \( \sum_{i=1}^{j} c_i \geq \sum_{i=1}^{j} d_i \), \( \forall j = 1 \ldots T \)). Let \( x_i \) and \( u_i \) be the inventory level and order size, respectively, at time \( i \), as in the stochastic problem.

\[(P)\]

\[
\min_{u_i, x_i} \sum_{i=1}^{T} h_i x_i
\]

\[
x_i = x_{i-1} + u_i - d_i, \quad i = 1 \ldots T
\]

\[
u_i \leq c_i
\]

\[
x_i \geq 0
\]

\[
x_0 = 0
\]

For this problem, at the beginning of each period, an ordering decision \( u_i \) is made. Demand \( d_i \) is met from stock and end of period inventory \( x_i \) is updated and holding cost is charged. This problem is a special case of the well known dynamic economic lot sizing problem. Because of the simple structure we are considering here, we are able to obtain a sharper, closed form solution to the problem, as described in the following proposition.

Proposition 6.1 Let

\[
\bar{x}_T = 0
\]

\[
\bar{x}_t = \max_{x \in [t+1, T]} \left\{ \sum_{k=t+1}^{T} (d_k - c_k) \right\}^+ \quad \text{for } t = 1, \ldots, T - 1
\]

a) For problem \((P)\), the optimal ordering policy is defined as:

\[
u_i^* = \min\{c_i, d_i + \bar{x}_i\}, \forall t.
\]
b) In particular, for $t = 1$,
\[
\begin{align*}
\text{u}^*_{1} &= d_1 + \bar{x}_1 \\
&= d_1 + \max_{r\in\{2, \ldots, T\}} \left\{ \sum_{k=2}^{r} (d_k - c_k) \right\}^+.
\end{align*}
\]

The optimal order quantity given in Proposition 6.1 can be interpreted as follows. The amount ordered is either the full capacity of the period, or that period's demand, plus a possible extra term. That extra term calculates the maximum possible cumulative shortage, if any, for future periods starting from the next period.

Our purpose for studying the deterministic version of the problem was to identify a function $K(F_t)$ that would relate the forecast vector $F_t$ to the base stock levels $S(F_t)$ in a reasonable way. That $K(F_t)$ function should, as in problem (P), inflate the current order to take into account future shortages anticipated by the forecast vector $F_t$. Inspired by the solution of the deterministic version of the problem given in Proposition 6.1, and keeping the goal of a closed-form solution in mind, we believe that using the forecasted cumulative shortage within the information horizon as the $K(F_t)$ function is a reasonable way to incorporate forecasts into the replenishment decisions. $\sum_{i=1}^{N} (\mu_D - C_{t+i})$ represents the forecasted cumulative shortage/excess capacity, within the information horizon. Since $\mu_D$ is a constant, we can equivalently use the following $K(F_t)$ function:

\[
K(F_t) = -\sum_{i=1}^{N} C_{t+i}.
\]

We call the policy that uses this specific $K(F_t)$ function the "cumulative-forecast-dependent base-stock policy". Recall that such a policy would use $s + K(F_t)$ as its state dependent base stock levels. Paralleling our earlier definitions, $s$ here is the cumulative forecast dependent base-stock level and $s^*$ is the optimal cumulative forecast dependent base-stock level.
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6.4.3 Asymptotic Approximation of $s^*$

In the rest of this section, we develop a closed-form asymptotic approximation for $s^*$, in the special case of Normal demands. Let the demand be Normal with mean $\mu_D$ and variance $\sigma_D^2$.

In any given period, the system orders products to bring the inventory level up to $s + K(F_t)$, if the inventory level $x_t$ is below this level. If $x_t$ is above this level, no order is placed. Consider a related system, where if the inventory is above the level $s + K(F_t)$ (due to fluctuations in $F_t$), the excess inventory is given back to the upstream source. Let $\hat{x}_t$ be the inventory process and $\hat{Y}_t = s + K(F_t) - \hat{x}_t$ be the shortfall process of the related system. If $\hat{Y}_t$ is negative, it represents the amount given back to the upstream source. We characterize the asymptotic behavior of the shortfall process $\hat{Y}_t$ of the related system. We also show that the original shortfall process $Y_t$ is equal to the shortfall process $\hat{Y}_t$ of the related system in the limit, therefore their asymptotic behavior are equivalent.

The following proposition shows that the shortfall process $\hat{Y}_t$ of the related system has a special structure.

**Proposition 6.2** For Normal demand with mean $\mu_D$ and variance $\sigma_D^2$:

a) Under the MMFE assumption, the distribution of the capacity $C_t$ for period $t$ is $N(\mu_C, \sigma_C^2)$

with $\sigma_C^2 = e^T \Sigma e$ where $\Sigma$ is the covariance matrix of the forecast update vectors and $e$ is the column vector of 1s.

---

4In this model, we have not explicitly modeled an ordering cost, but the model can be extended to that case and if a per unit ordering cost is present, the upstream source would be willing to buy the inventory back for the same price under this related system.
b) Under the cumulative-forecast dependent base-stock policy, $\hat{Y}_t$ is a reflected random walk with step sizes $X_t = M_t - C_t$ where $M_t = D_t + \Delta_t$ with $\Delta_t = K(F_t) - K(F_{t-1})$.

c) The step sizes $X_t$ are i.i.d. with distribution $N(\mu_D - \mu_C, \sigma^2_D + \epsilon^T \Sigma \epsilon)$.

Let $Y_\infty$ be the steady state random variable that $Y_t$ converges to. Under the assumption of heavy-traffic, Siegmund (1979) provides an asymptotic approximation for the tail of a reflected random walk with i.i.d. step sizes that are normally distributed. This leads to the following proposition.

**Proposition 6.3** Under the cumulative-forecast dependent base-stock policy, as $\mu_D \to \mu_C$, $y \to \infty$ and $(\mu_D - \mu_C)y \to constant$:

$$P(Y_\infty > y) = P(\hat{Y}_\infty > y) = e^{-\beta(y+\theta)}$$

(6.5)

where $\beta = 0.583 \sqrt{\sigma^2_D + \epsilon^T \Sigma \epsilon}$ and $\theta = (2(\mu_C - \mu_D))/(\sigma^2_D + \epsilon^T \Sigma \epsilon)$.

Note that the shortfall process $\hat{Y}_t$ of the related system is a reflected random walk, but the shortfall process $Y_t$ of the original system is not. The processes $Y_t$ and $\hat{Y}_t$ differ only when $\hat{Y}_t$ becomes negative, and inventory is given back to the upstream source. However, in heavy traffic, the process $\hat{Y}_t$ becomes a reflected Brownian motion, and never becomes negative. Therefore, $\hat{Y}_t$ and $Y_t$ coincide in the limit. This is why Siegmund's asymptotic result applies to $Y_\infty$ as well as to $\hat{Y}_\infty$.

As shown in Theorem 6.3, the optimal cumulative-forecast-dependent base stock level $s^*$ is the critical fractile of the distribution of $Z_\infty$. In order to derive an expression for $s^*$, we therefore need to characterize the tail behavior of $Z_\infty$ as well.
**Proposition 6.4** Using the heavy traffic approximation (6.5) from Proposition 6.3, as \( z \to \infty \),

\[
P(Z_\infty > z) = e^{-\theta(\beta - \gamma + z)} \text{ where } \gamma = \sqrt{\frac{\theta}{2}} \left( NP + \frac{e^T \Sigma e}{\Delta} \right).
\]

If holding and backorder costs are linear with rates \( h \) and \( b \), respectively (the linear cost case),
the optimal cumulative-forecast dependent base stock level \( s^* \) is equal to \( F_{\infty}^{-1}(b/b + h) \). For
large values of the backlogging cost \( b \), the critical fractile falls on the tail of the distribution of
\( Z_\infty \). The tail behavior of \( Z_\infty \) has been characterized as in Proposition 6.4. Consequently, in
the next theorem, we are able to obtain a closed-form expression for the optimal cumulative-forecast
dependent base stock level, under heavy traffic and as the backlogging cost grows large.

**Theorem 6.4** For the linear cost case and under the heavy traffic approximation for the tail distribution of \( Z_\infty \) from Proposition 6.4, the optimal cumulative-forecast-dependent base stock level is

\[
\frac{1}{b} \ln(1 + \frac{b}{h}) - \beta + \gamma, \text{ as } b \to \infty.
\]

Toktay and Wein (2001) develop a similar approximation for the advance demand
information case. Our analysis in Section 6.4.3 partially parallels theirs for the advance
supply information problem, but there are also significant differences.

In order to understand the accuracy of the approximations, we simulated the system
and found the optimal \( s^* \) using exhaustive search for a variety of instances. Let the
average per period cost associated with the optimal \( s^* \) be \( C(s^*) \). We then simulated the
system with the asymptotically optimal cumulative-forecast dependent base stock level
given in Theorem 6.4 (referred to as \( \tilde{s} \)). Let the cost associated with this policy be \( C(\tilde{s}) \).
We considered 3 different utilization levels \( (\rho) \) with 2 different \( b/h \) levels and 3 different
correlation coefficients ($v$) for the capacity that represent positive, negatively correlated and i.i.d. capacities. The information horizon is $N = 2$. The optimality loss due to using the approximate base stock level $\bar{s}$ as opposed to the optimal $s^*$ can be calculated as $(C(\bar{s}) - C(s^*)/C(s^*)$. These optimality loss values for the various combinations of problem parameters are illustrated in Table 6.1 below.

<table>
<thead>
<tr>
<th>$b/h$</th>
<th>Correlation Coeff. $v$</th>
<th>Utilization</th>
<th>80%</th>
<th>90%</th>
<th>95%</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.3</td>
<td>2.30%</td>
<td>0.95%</td>
<td>0.15%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>4.76%</td>
<td>0.10%</td>
<td>0.06%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.3</td>
<td>33%</td>
<td>1.33%</td>
<td>0.17%</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.3</td>
<td>0.73%</td>
<td>0.57%</td>
<td>0.11%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.43%</td>
<td>0.01%</td>
<td>0%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.3</td>
<td>11.60%</td>
<td>0.70%</td>
<td>0.11%</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.1: Optimality loss of the approximated base stock level $\bar{s}$

Table 6.1 shows that $\bar{s}$ is very accurate when the manufacturer is under high load. We see that the accuracy is within 1.4% when $\rho \geq 90\%$. However, the performance deteriorates when system utilization decreases. This is expected, since our formula involves a heavy-traffic approximation. We also observe that performance is better for higher values of the ratio $b/h$, as expected from the tail approximation utilized.
6.5 Numerical Results and Managerial Insights

In the previous section, we proposed an easily computable and implementable closed form policy for the advance supply information inventory problem. Consider a setting where an IT manager for a manufacturing company has to make a decision on whether or not it is worthwhile to implement a tool that would provide advance supply information shared by an outside supplier. Given that IT departments have limited budget for only a number of implementations a year, they need to make sure that the systems and tools that are implemented provide the most value to the company. In this section, our goal is to identify the types of operating environments under which advance supply information is most valuable, compared to a fixed base stock policy.

When one talks about the value of advance supply information, the real theoretical comparison should be between an optimal policy that takes advance supply information into account (a state dependent base stock policy as in Theorem 6.2) and a policy that optimizes the system without receiving the advance supply forecasts. Given the dynamics of the system, the past capacity values carry some information about future capacity values, even in the absence of any forecast information. Therefore, the optimal policy without forecast information will be a history dependent base stock policy where the base stock levels would depend on the past capacity values. However, it is computationally very expensive to identify the state dependent base stock levels with forecast information, and the history dependent base stock levels without forecast information, due to the high dimensions of the state spaces. Neither of these policies are practical for implementation and we are focusing on easily computable and implementable policies. Our proposed
method is to use the closed form approximation for the state dependent base stock levels from Theorem 6.4 to incorporate the forecast information into replenishment decisions. The obvious alternative policy without forecast information that would be relatively easy to compute and implement is a fixed base stock policy. We use such a fixed base stock policy as a benchmark and define the value of advance supply information as the percentage difference between the costs of these two policies. We determined the exact optimal fixed base stock level via simulation. Let $C_f^*$ be the cost of the best fixed base stock policy. For each instance we considered, we also found the cost $C(s)$ of the cumulative-forecast dependent base stock policy with base-stock level $s$ derived in Theorem 6.4. The relative value of advance supply information is calculated as

$$\frac{C(s) - C_f^*}{C_f^*}.$$

To cover a broad range of potential business scenarios and operating environments, we considered four main parameters: Utilization, capacity variability, demand variability and the $b/h$ ratio to represent the service level. We ran two sets of experiments for two different information horizons ($N \in \{4, 7\}$) for a total of 960 experiments.

**Information horizon $N = 4$**

We considered normally distributed i.i.d. demand and normally distributed i.i.d capacity. We fixed the mean capacity to 20 and varied the mean demand to test different utilization levels. Four different utilization levels (80%, 90%, 95%, 98%), four different $b/h$ ratios (2, 10, 20, 100), six different coefficient of variation levels for capacity (0.1, 0.2, 0.3, 0.4, 0.5, 1), and three different coefficient of variation levels for demand (0.1, 0.5, 0.8) were considered.

We fixed the holding cost $h$ at 1 and varied the backlogging cost $b$. 
Utilization

To assess the effect of utilization on the value of advance supply information, in Figure 6.2 (a), we plotted the percentage of relative value against four different utilizations.

Figure 6.2: The value of information as a function of various problem parameters for \( N = 4 \). Each reported value is an average over instances in the corresponding set. For example, the first bar in part a) of the figure represents the average value of information over all instances with 80% utilization.

We observe that the benefit of advance supply information is minimal when the utilization is either very high or very low. Note that in an uncapacitated environment, the inventory manager can just implement a myopic policy considering only the current pe-
period and that would be optimal in a stationary environment. However, in a capacitated system, she needs to consider the possibility of future shortages and hence inflate her current period order to plan for such possible shortages. When the utilization is low, those future shortages do not occur frequently. Therefore, advance supply information is not very useful in such an environment. When the utilization is very high, such shortages occur very frequently. However, since the current period capacity is tight as well, there is no room to inflate the current order. In other words, advance supply information can not help. Therefore, the utilization level is a critical criterion to check in order to assess the value of implementing a supply collaboration tool. Such a tool is most valuable if the utilization level is medium.

**Capacity Variability**

To study the effect of capacity variability on the value of advance supply information, in Figure 6.2 (b), we plotted the relative value against six different capacity variabilities.

The main insight we gain from the analysis of Figure 6.2 (b) is that advance supply information is most beneficial in operating environments where the capacity is moderately variable. At very low levels of capacity variability, the system with advance supply information does not have much advantage over the system with no such information, since there is not much uncertainty in capacity levels anyway. On the other hand, at very high levels of capacity variability, advance supply information is valuable since it helps in resolving some of the variability in the system. However, the overall variability is so prevalent that the resolved variability with the help of advance supply information can be negligible compared to the remaining unresolved variability. Therefore, the system still needs to maintain high levels of inventory and that results in reduced relative value
CHAPTER 6. SUPPLY CHAIN MANAGEMENT WITH ADVANCE SUPPLY INFORMATION

of information. The value of information is maximized at moderate levels of capacity variability.

Companies engage in several supplier relationships these days using several contracts. What this insight tells us is that implementing a supply collaboration tool would not be that beneficial with an important and very strategic supplier which has already agreed to providing certain allocations at every period (there are contracts used in industry that provide up/down flexible limits for the capacity the supplier needs to provide and if for example the difference between these limits is very narrow, advance supply information would be of marginal benefit), since capacity variability would be low in this case. This result would be counter-intuitive for a practicing manager since firms generally consider implementing supply collaboration tools with such very strategic suppliers. Similarly, implementing a supply collaboration tool with a very unreliable supplier would again be of marginal benefit.

b/h Ratio

To assess the effect of the b/h ratio on the benefit of information, in Figure 6.2 (c), we plotted the percentage of relative value against 4 different b/h ratios.

We observe that at lower levels of the b/h ratio the value of information is low. When the shortage cost is low, advance supply information, which helps mitigate the effect of shortages is naturally less valuable. When the b/h ratio is very high, the cost of holding inventories is relatively low, meaning that high amounts of inventory are carried in both systems (with and without advance supply information) and therefore advance supply information is not as beneficial in relative terms. The value of information is higher when there is a delicate trade-off between holding and shortage costs, and that happens when
we observed that the value of information decreases as demand becomes more variable. At higher demand variability levels, even though the system prepares itself for future capacity shortages using advance supply information, a high level of safety stock is kept to protect against demand variability. The resulting high cost reduces the relative benefit of advance supply information. This is shown in Figure 6.2 (d).

To test the effect of the length of the information horizon $N$ on the value of advance supply information, we performed an additional set of experiments with $N = 7$.

**Information horizon $(N) = 7$:**

We considered four different utilization levels (80%, 90%, 95%, 98%), four different $b/h$ ratios (2, 10, 20, 100), four different coefficient of variation levels for capacity (0.1, 0.4, 0.7, 1), and three different coefficient of variation levels for demand (0.1, 0.5, 0.8). The conclusions we drew in the case of $N = 4$ about the operating environments under which advance supply information is most valuable are reconfirmed for the case of $N = 7$ as well, as illustrated in Figure 6.3. We also observe that the value of supply information is in general higher with a longer information horizon. This is an important finding, since in certain cases the length of the information horizon could be something that can be controlled.
Figure 6.3: The value of information as a function of various problem parameters for $N = 7$. Each reported value is an average over instances in the corresponding set. For example, the first bar in part a) of the figure represents the average value of information over all instances with 80% utilization.
Chapter 7

Conclusions

This dissertation started with the aim of an actual contract implementation that would align the incentives of the manufacturer’s supply chain and improve overall profit. In the end, we could not achieve this objective, despite a successful research effort to identify and analyze coordinating contracts and significant efforts to translate this theory into a practical distributor contracting program. Our theoretical results contribute to our understanding of contracting when selling vertically differentiated products. The lessons learned from the implementation efforts provide new and interesting research directions. For example, it may be that wholesale pricing is the only feasible pricing strategy for certain industries and business environments due to factors that lie beyond the models considered in the theoretical literature, such as gray markets which gave the motivation for the second essay of the dissertation. In the presence of a gray market, we observe that the contracts that have been studied earlier in the literature start to perform differently and that wholesale pricing contract itself is “almost coordinating”. The third essay is also motivated by an incentive problem observed in semi-conductor manufacturer’s channel business. We study
a multiple supplier one manufacturer environment where the manufacturer is a computer reseller that makes a quality decision for the components provided by different suppliers. After identifying the inefficiencies that would arise in such systems and the contracts that could restore them, we observe that the magnitude of the inefficiencies can be quite substantial which means that all parties may significantly benefit from a contract that helps restore efficiency. In industry, supplier collaboration, i.e. sharing of upstream supply side information, is an observed practice in supply chain implementations. Considering an inventory model with advance supply information, where the supply information is modeled as dynamic forecasts of capacity availability, in the final essay, we characterize the optimal policy for such systems and develop a heuristic and derive the operating environments under which information is more valuable.
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Appendix A

Supplement to Chapter 3: Proof of Results

Proof of Proposition 3.1

a) From equations 3.1 and 3.2, we know that \( p_H = R + v_H - \gamma (v_H - v_L) \) and \( p_L = R + \gamma v_L \)
(Note that \( \frac{\partial^2 \Pi_H(v_H, v_L)}{\partial v_H^2} = -(1 - \gamma) \); \( \frac{\partial^2 \Pi_H(v_H, v_L)}{\partial v_L^2} = -2\gamma \) and \( \frac{\partial^2 \Pi_L(v_H, v_L)}{\partial v_H^2} = \frac{\partial^2 \Pi_L(v_H, v_L)}{\partial v_L^2} = 0 \) which makes the Hessian negative-definite and the profit function concave). In the centralized system, we have:
\[
v_H^C = \frac{(1-\gamma) + v_H - v_L}{2(1-\gamma)}, \quad v_L^C = \frac{R - v_H}{2}\gamma.
\]
In the decentralized system, we have:
\[
v_H^D = \frac{(1-\gamma) + v_H - v_L}{2(1-\gamma)}, \quad v_L^D = \frac{R - v_H}{2}\gamma.
\]
Plugging in \( w_H^* \) and \( w_L^* \) in these decentralized values, we get:
\[
p_H^D = \frac{3}{4}(R + 1) + \frac{\gamma \theta}{4} \quad \text{and} \quad p_L^D = \frac{3}{4}(R + \gamma) + \frac{\gamma \theta}{4}.
\]
Similarly, \( p_H^C = \frac{1+R}{2} + \frac{\gamma \theta}{2} \) and \( p_L^C = \frac{R+\gamma}{2} + \frac{\gamma \theta}{2} \).

The first part follows by assumption 1.

b) \( p_H^D - p_L^D > p_H^C - p_L^C \) by assumption 2 which says that the value gain due to quality difference is more than the difference in cost of manufacturing.

c) Total Demand = \( N(1 - v_L) = N(\frac{1}{2} - \frac{(R-v_H)}{2\gamma}) \). Hence, total demand is not affected by \( w_H \). \( \blacksquare \)
APPENDIX A. SUPPLEMENT TO CHAPTER 3: PROOF OF RESULTS

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Proof of Proposition 3.2

The result is obtained by plugging in the appropriate \( p_H \) and \( p_L \) values from the first proposition.

Proof of Theorem 3.1

With the contract parameters as given and revenue shared as explained:

\[
\pi^D_d = \lambda [(R + v_H - \gamma(v_H - v_L))N(1 - v_H) + (R + \gamma v_L)N(v_H - v_L)] - \lambda c_H N(1 - v_H) - \lambda c_L N(v_H - v_L)
\]

\[
= \lambda [(R + v_H - \gamma(v_H - v_L) - c_H)N(1 - v_H) + (R + \gamma v_L - c_L)N(v_H - v_L)] = \lambda \Pi^C
\]

As a result, \((v_H^*, v_L^*)\) would maximize \( \pi^D_d \) as well and the profit is arbitrarily allocated according to \( \lambda \).

Proof of Proposition 3.3

For a very small \( \alpha \) and \( \lambda \), from the FOC of the \( \pi_d(v_H, v_L) \) which is concave \( \frac{\partial^2 \pi_d(v_H, v_L)}{\partial v_H^2} = -2(1 - \gamma)(1 + \lambda) \), \( \frac{\partial^2 \Pi_c(v_H, v_L)}{\partial v_H^2} = -2\gamma \) and \( \frac{\partial^2 \Pi_c(v_H, v_L)}{\partial v_L^2} = \frac{\partial^2 \Pi_c(v_H, v_L)}{\partial v_H \partial v_L} = 0 \) which makes the Hessian negative-definite, we have:

\[
v_H = \frac{(1 - \gamma)(1 + \lambda) + \omega_H - \omega_L}{2(1 - \gamma)(1 + \lambda)} \quad \text{and} \quad v_L = \frac{(1 + \lambda)(\gamma - R) + \omega_L + \lambda \alpha}{2\gamma(1 + \lambda)}
\]

\[
d_H = N(\frac{1}{2} - \frac{\omega_H - \omega_L}{2(1 - \gamma)(1 + \lambda)}) \quad \text{and total demand} \quad d = N(\frac{1}{2} + \frac{R}{2\gamma} - \frac{\omega_L + \lambda \alpha}{2\gamma(1 + \lambda)})
\]

From \( d_H \) and \( d \), both parts of the proposition follow.

Proof of Theorem 3.2

\[
\pi_d(v_H, v_L) = (R + v_H(1 - \gamma) + \gamma v_L)N(1 - v_H) + (R + \gamma v_L)N(v_H - v_L) - T_H - T_L \quad \text{where} \quad T_H \text{ and } T_L \text{ are the transfer payments:}
\]
APPENDIX A. SUPPLEMENT TO CHAPTER 3: PROOF OF RESULTS

\[ T_H = \{w_H N(1 - v_H) - N(1 - v_H - t)r\} N(1 - v_H) > t \]
\[ w_H(1 - v_H) \quad N(1 - v_H) < t \]

\[ T_L = \{w_L N(v_H - v_L)\} \]

\[ \pi_d(v_H, v_L) = T_H + T_L - c_H N(1 - v_H) - c_L N(v_H - v_L) \]

Define \( K(t) = \pi_d^f - \pi_d^h \) where \( \pi_d^h \) is the profit from not using the rebate option (i.e. using only the wholesale price without receiving the rebate). Suppose that \((v^h_H, v^h_L) = \text{argmax}(\pi_d^h)\) and \((v^f_H, v^f_L) = \text{argmax}(\pi_d^f)\). \( K(t) \) is continuous and decreasing. Hence a threshold \( t_0 \) where \( K(t_0) = 0 \) with \( t_0 \in (d^f_H, d^f_L) \) where \( d^f_H = N(1 - v^f_H) \) and similar for \( d^f_L \) Consider the FOC with \( t > t_0 \):

\[ \frac{\partial \pi_d(v_H, v_L)}{\partial v_H} = (1 - \gamma)N(1 - v_H) - N(R + v_H(1 - \gamma) + \gamma v_L) + N(R + \gamma v_L - c_L) - \frac{\partial T_L}{\partial v_H} \]
\[ \frac{\partial \pi_d(v_H, v_L)}{\partial v_L} = \gamma N(1 - v_H) + (V_H - v_L) - (R + \gamma v_L - w_L) - \frac{\partial T_L}{\partial v_L} \]

Doing the algebra, we have \( v^f_H = v^f_L \) and \( v^f_L = v^f_C \) if and only if \( w_H = r + c_H \) and \( w_L = c_L \). With this, we have \( \pi_d = \pi^C - tr \). If the distributor does not use the rebate, we have \( v_H = \frac{1}{2} + \frac{r+c_H-c_L}{2(1-\gamma)} \). \( r < (1 - \gamma) - (c_H - c_L) \) ensures non-negative \( d_H \). We also find that \( t_0 = N \left[ \frac{1}{2} - \frac{r}{4(1-\gamma)} - \frac{c_H-c_L}{2(1-\gamma)} \right] \). If we say \( \pi^f_m \) and \( \pi^h_m \) is manufacturer’s profit under rebate and no-rebate option respectively, we know that \( \pi^f_m = r t_0 > (w_H - c_H) d^f_H = r \left( \frac{1}{2} - \frac{c_H-c_L}{2(1-\gamma)} - \frac{r}{2(1-\gamma)} \right) \). Therefore, with these parameters, the distributor and the manufacturer is better off using the target rebate contract which coordinates the channel as well.
APPENDIX A. SUPPLEMENT TO CHAPTER 3: PROOF OF RESULTS

Proof of Theorem 3.3

\[ \pi_d(v_H, v_L) = (R + v_H(1 - \gamma) + \gamma v_L - W + w(1 - v_H)N)(1 - v_H)N + (R + \gamma v_L - c_L)N(v_H - v_L) \]

Setting \( W = c_H + m \) where \( m > 0 \) and based on concavity of the profit function:

\[ \frac{\delta \pi_d(v_H, v_L)}{\delta v_H} = (1 - \gamma - wN)(1 - v_H)N - (R + v_H(1 - \gamma) + \gamma v_L - c_H - m + w(1 - v_H)N)N + (R + \gamma v_L - c_L)N \]

\[ \frac{\delta \pi_d(v_H, v_L)}{\delta v_L} = -\gamma (1 - v_H)N + \gamma N(v_H - v_L) - (R + \gamma v_L - c_L)N \]

Setting \( w = \frac{m}{0.4N(1 - \gamma)} \) (\( > 0 \) with assumption (2)) and \( w_L = c_L \) and rearranging the terms leads to: \( v_H^* = v_H^C \) and \( v_L^* = v_L^C \). With these parameters, the quantity discount contract coordinates the channel with:

\[ \pi_d(v_H^C, v_L^C) = \Pi^C - m(1 - v_H^C) + w(1 - v_H^C)2N = \Pi^C - \frac{mN}{4}(1 - \frac{c_H - c_L}{1 - \gamma}) \]

\[ \pi_m(v_H^C, v_L^C) = \frac{mN}{4}(1 - \frac{c_H - c_L}{1 - \gamma}) \]

Proof of Theorem 3.4

We know that if \( q \) represents the quantity sold, \( p_H = 1 - q_H - (1 - \gamma)q_L \) and \( p_L = (1 - \gamma)(1 - q_H - q_L) \). When \( Q \) units is ordered, we have \( q_H = Qx \) and \( q_L = Qy \). Therefore:

\[ \pi_d(Q(x, y) = (R + 1 - Qx - (1 - \gamma)Qy)Qx + (R + (1 - \gamma)(1 - Qx - Qy))Qy - wQ \]

is concave in \( Q \) which gives us:

\[ Q' = \frac{(R + 1)x + (1 - \gamma + R)y - w}{2(x^2 + (1 - \gamma)xy + (1 - \gamma)y^2)} \]

(1) \[ Q'x = \frac{(R + 1)x^2 + (1 - \gamma + R)yx - wxy}{2(x^2 + (1 - \gamma)xy + (1 - \gamma)y^2)} = N\left(\frac{c_H - c_L}{2(1 - \gamma)} - \frac{R - c_L}{2y}\right) \]

(2) \[ Q'y = \frac{Rx + Ry^2 + yx + (1 - \gamma)y^2 - wxy}{2(x^2 + (1 - \gamma)xy + (1 - \gamma)y^2)} = N\left(\frac{c_H - c_L}{2(1 - \gamma)} - \frac{R - c_L}{2y}\right) \]

which places the condition on wholesale price \( w_q \leq Ry + Rx + x + (1 - \gamma)y + \eta \). Combining (1) and (2), gives us an equation of the form \( Ax^2 + Bx + C \) for a given \( y = \bar{y} \) with \( A = R + 1 - 2d_H^C - 2d_L^C \geq 0 \) where \( d_H^C \) and \( d_L^C \) are RHS of equations (1) and (2)
respectively. \[ B = (2d_L^C(1 - \gamma) - (1 - \gamma) - 1)\bar{y} - w < 0 \text{ since } b < \frac{1}{2} \text{ and } \bar{y} > 0 \] and
\[ C = w\bar{y} - R\bar{y}^2 - \bar{y}^3(1 - \gamma)(d_L^C + 1 - 2d_H^C) \]
In order to have \( x \) that meets (1) and (2), we need \( C \leq 0 \) which places the second condition for a given \( \bar{y} \): \( w_B \leq R\bar{y} + \bar{y}(1 - \gamma)(d_L^C + 1 - 2d_H^C) = \beta \). On the other hand, manufacturer's problem for a given \( x \) and \( y \) is:
\[ \pi_m(w_B) = w_BQ - Qxc_H - Qyc_L \]
Together with the constraints, we have: \( w_B^* = \min\left(\frac{R(x+y)+x+(1-\gamma)y-xc_H-yc_L}{2(x+y+y(1-\gamma)y^2)}, \eta, \beta\right) \)

**Proof of Theorem 3.5**

a) Based on Freidman (1977), the existence of NE is guaranteed under concave \( \pi_d(q_H^i, q_L^i) \)
and assumption A4. (With \( \frac{\partial^2 \pi_d(q_H^i, q_L^i)}{\partial q_H^i} = -2 ; \frac{\partial^2 \pi_d(q_H^i, q_L^i)}{\partial q_L^i} = -2y \) makes the Hessian negative-definite)

b) Under given wholesale prices, the best response functions \( r_H^i(q_H^i, q_L^*), r_L^i(q_H^i, q_L^*) \) for each distributor \( i \) is:
\[
\begin{align*}
r_H^i(q_H^i, q_L^*) &= \frac{1-w_H-2yq_H^i+y\sum_{j \neq i} q_j^i-q_H^i-R}{2} \\
r_L^i(q_H^i, q_L^*) &= \frac{y-2yq_L^i+y\sum_{j \neq i} q_j^i-q_L^i-w_L+R}{2y}
\end{align*}
\]
With the above \( 2n \) symmetric equations, we get the symmetric unique outcome \( (q_H^*, q_L^*) \) as the unique NE.

c) In an "n" firm single product Cournot competition, define \( a_i = \frac{\partial \pi_i}{\partial q_H} \) and \( b_i = \frac{\partial \pi_i}{\partial q_L} \) with following assumptions:

(i) Costs and demand are twice continuously differentable

(ii) Industry output is bounded
(iii) For all $i$, $\pi_i$ is pseudo-concave wrt. own output

(iv) All equilibrium is non-degenerate

(v) $\forall i, P' < C_i''(x^*_i)$ where $P(X)$ is the inverse demand function and $C_i$ is cost function.

In Dastidar (2000) Proposition 2 states that under conditions (i) to (v) a regular, Cournot equilibrium is always locally stable if for all $i$, either $b_i < 0$ or for all $i b_i > 0$. If we define $J$ as:

$$
\begin{bmatrix}
  a_1 & b_1 & b_1 & \ldots & b_1 \\
  b_2 & a_2 & b_2 & \ldots & b_2 \\
  b_3 & b_3 & a_3 & \ldots & b_3 \\
  & & & \ldots & \ldots \\
  b_n & b_n & \ldots & b_n & a_n
\end{bmatrix}
$$

Negative trace $\sum(s_i a_i)$ (where $s_i > 0$ is adjustment speed) is necessary and positive $|J|$ is sufficient condition for local stability of unique Cournot equilibrium and the conditions in the proposition in Dastidar (2000) ensure that. For our model, if we define:

$$
a^H_i = \frac{\partial^2 \pi_i}{\partial q^2_i}; a^L_i = \frac{\partial^2 \pi_i}{\partial q^L_i}$$

$$
b^H_i = \frac{\partial^2 \pi_i}{\partial q^H_i}; b^L_i = \frac{\partial^2 \pi_i}{\partial q^L_i}; c^L_i = \frac{\partial^2 \pi_i}{\partial q^L_i}
$$

where $a^H_i = -2; a^L_i = -2\gamma; b^H_i = -1; b^L_i = -\gamma; c^L_i = -\gamma$ which makes the marginal profit matrix $JJ=$
where the matrix $A = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$

and the matrix $B = \begin{bmatrix} b^H_1 & b^H_2 & b^H_3 & \cdots & b^H_n \\ b^L_1 & b^L_2 & b^L_3 & \cdots & b^L_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^L_1 & b^L_2 & b^L_3 & \cdots & b^L_n \end{bmatrix}$

with matrix $C =$
APPENDIX A. SUPPLEMENT TO CHAPTER 3: PROOF OF RESULTS

\[
\begin{bmatrix}
  c_1^L & c_1^r & \cdots & c_1^L \\
  c_2^L & c_2^r & \cdots & c_2^L \\
  c_3^L & c_3^r & \cdots & c_3^L \\
  \vdots & \vdots & \ddots & \vdots \\
  c_n^L & c_n^r & \cdots & c_n^L \\
\end{bmatrix}
\]

By row and column transformations and because \( c_i^L = b_i^L \) in our case, the marginal profit matrix \( JJ \) simplifies to \( J' \):

\[
\begin{bmatrix}
  AA & 0 \\
  0 & I \\
\end{bmatrix}
\]

where \( AA \) is a new symmetric matrix obtained by row transformation from \( JJ \); \( I \) is an identity matrix and 0 is the zero matrix. We know that \( |JJ'| = |JJ| = |AA| \) in such a system. In Dastidar (2000), the associated results were obtained based on the \( J \) matrix which has the same structure with our \( A \) and hence \( AA \) matrix where all \( b_i < 0 \) with \( \gamma < 1 \). Therefore, same results carry over to our model which proves local stability of the unique Cournot equilibrium.

Proof of Theorem 3.6

\[
\Pi^C(q^H, q^L) = \sum_{i=1}^{N} \Pi^C_i(q^H_i, q^L_i). \quad \text{If } (q^H_i, q^L_i) \text{ is the optimal solution for } \Pi^C(q^H, q^L) \text{ with } (q^H_0, q^L_0) \text{ representing the part of the solution for all but the } i^{th} \text{ distributor, the FOC would be:}
\]

\[
\frac{\partial \Pi^C(q^H, q^L)}{\partial q^H_i} = (R + 1 - 2q^H_i - 2\gamma q^L_i - \sum_{j \neq i} q^H_j - \gamma \sum_{j \neq i} q^L_j) - \gamma \sum_{j \neq i} q^H_j - \gamma \sum_{j \neq i} q^L_j - c_H
\]

\[
\frac{\partial \Pi^C(q^H, q^L)}{\partial q^L_i} = (R + \gamma(1 - 2q^H_i - \sum_{j \neq i} q^H_j - \sum_{j \neq i} q^L_j) - \gamma \sum_{j \neq i} q^H_j - \gamma \sum_{j \neq i} q^L_j - c_L)
\]
APPENDIX A. SUPPLEMENT TO CHAPTER 3: PROOF OF RESULTS

With the wholesale prices as given, the FOC for the $i$th distributor given that the others have $(q^i_H, q^i_L)$ is:

$$\frac{\partial \pi_i}{\partial q_H^i} = \lambda [R + 1 - 2q_H^i - 2\gamma q_L^i - \Sigma_{j\neq i} q_H^j - \gamma \Sigma_{j\neq i} q_L^j - c_H - \Sigma_{j\neq i} q_H^j - \gamma \Sigma_{j\neq i} q_L^j]$$

$$\frac{\partial \pi_i}{\partial q_L^i} = \lambda [R + \gamma (1 - q_H^i - q_L^i - \Sigma_{j\neq i} q_H^j - \Sigma_{j\neq i} q_L^j) - c_L - \gamma q_H^i - \gamma q_L^i - \gamma \Sigma_{j\neq i} q_H^j - \gamma \Sigma_{j\neq i} q_L^j]$$

which gives $(q^i_H, q^i_L)$ as the unique solution i.e. $(q^0_H, q^0_L)$ coordinates the channel. Working on the algebra with these values, the profits are:

$$\pi_d = \lambda_i (\Pi_d^i (q^i_H, q^i_L) + (Q^o_H + \gamma Q^o_L) q^o_H + \gamma (Q^o_H + Q^o_L) q^o_L)$$

$$\pi_m = \sum_{i=1}^n (1 - \lambda_i) (\Pi_d^i (q^i_H, q^i_L) + (Q^o_H + \gamma Q^o_L) q^o_H + \gamma (Q^o_H + Q^o_L) q^o_L) - ((Q^o_H + \gamma Q^o_L) q^o_H + \gamma (Q^o_H + Q^o_L) q^o_L)$$

where $Q^o_H = \sum_{j\neq i} q^j_H$ and $Q^o_L = \sum_{j\neq i} q^j_L$ which also gives the bound on $\lambda$. 

Proof of Theorem 3.7

$$\pi_d^i(q_H^i, q_L^i) = (R + 1 - q_H^i - \gamma q_L^i - \Sigma_{j\neq i} q_H^j - \gamma \Sigma_{j\neq i} q_L^j) q_H^i + (R + \gamma (1 - q_H^i - q_L^i - \Sigma_{j\neq i} q_H^j - \Sigma_{j\neq i} q_L^j)) q_L^i - T_H - T_L$$

where $T_H$ and $T_L$ are the transfer payments for distri $i$:

$$T_H^i = \{ w_H^i q_H^i - (q_H^i - \tau) r \} q_H^i > \tau$$

$$w_H^i q_H^i \quad q_H^i \leq \tau$$

$$T_L^i = \{ w_L^i q_L^i \}$$

$$\pi_s(q_H^i, q_L^i) = \sum_{n=1}^N (T_H^i + T_L^i - c_H q_H^i - c_L q_L^i)$$

$$\Pi_s(q_H^i, q_L^i) = \sum_{n=1}^N \Pi_d^i (q_H^i, q_L^i)$$

As expressed earlier, let $(q^0_H, q^0_L)$ be the optimal solution for
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$\Pi^C(q_H^i, q_L^i)$ with $(q_H^{io}, q_L^{io})$ representing the part of the solution for all but the $i$th distributor.

Define $\pi^i_d = \pi^i_d - \pi^m_d$ where $\pi^m_d$ is the profit from not using the rebate option. Suppose that $(q_H^{io}, q_L^{io}) = \text{argmax}(\pi^m_d)$ and $(q_H^{io}, q_L^{io}) = \text{argmax}(\pi^i_d)$. $K(t_i)$ is continuous and decreasing.

Hence a threshold $t_0$ where $K(t_0) = 0$ with $t_0 \in (q_H^{jo}, q_L^{jo})$. Consider the FOC with $t \geq t_0$:

$$\frac{\partial \pi^i_d}{\partial q_H} = \lambda[R + 1 - 2q_H^{io} - 2\gamma q_L^{io} - \sum_{j\neq i} q_H^{jo} - \gamma \sum_{j\neq i} q_L^{jo} - \omega_H + r]$$

$$\frac{\partial \pi^i_d}{\partial q_L} = \lambda[R + \gamma(1 - q_H^{io} - q_L^{io} - \sum_{j\neq i} q_H^{jo} - \sum_{j\neq i} q_L^{jo}) - w_L]$$

With the wholesale prices as given, the FOC for the $i$th distributor given that the others have $(q_H^{io}, q_L^{io})$ is:

$$\frac{\partial \pi^i_d}{\partial q_H} = \lambda[R + 1 - 2q_H^{io} - 2\gamma q_L^{io} - \sum_{j\neq i} q_H^{jo} - \gamma \sum_{j\neq i} q_L^{io} - c_H - \sum_{j\neq i} q_H^{jo} - \gamma \sum_{j\neq i} q_L^{jo}]$$

$$\frac{\partial \pi^i_d}{\partial q_L} = \lambda[R + \gamma(1 - q_H^{io} - q_L^{io} - \sum_{j\neq i} q_H^{jo} - \sum_{j\neq i} q_L^{jo}) - c_L - \gamma q_H^{io} - \gamma q_L^{io} - \gamma \sum_{j\neq i} q_H^{jo} - \gamma \sum_{j\neq i} q_L^{jo}]$$

which gives $(q_H^{io}, q_L^{io})$ as the unique solution i.e. $(q_H^{io}, q_L^{io})$ coordinates the channel. If the distributor does not use the rebate option with these wholesale prices, it is easy to see that the ordering quantity in this case $q_H^{io} = q_H^{io} - r$ and $q_L^{io} = q_L^{io}$. With these values, we would get:

$$\pi^i_d = \Pi_d(q_H^{io}, q_L^{io}) - \pi^i_d - r[2(q_H^{io} - r) + q_L^{io} - (Q_H^{io} + \gamma Q_L^{io}) - c_H + R + 1 - q_H^{io} - \gamma q_L^{io} - \sum_{j\neq i} q_H^{jo} - \gamma \sum_{j\neq i} q_L^{jo}]$$

where $\pi^i = q_H^{io}(Q_H^{io} + \gamma Q_L^{io}) - \gamma q_L^{io}(Q_H^{io} + \gamma Q_L^{io})$ with $Q_H^{io}$ and $Q_L^{io}$ as defined which would give us $t_0 = 2(q_H^{io} - r) + q_L^{io} - (Q_H^{io} + \gamma Q_L^{io}) - c_H + (R + 1 - q_H^{io} - \gamma q_L^{io} - \sum_{j\neq i} q_H^{jo} - \gamma \sum_{j\neq i} q_L^{jo})$.

Proof of Theorem 3.8
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With the \( W_R = W - wq_H \) type contract and \( W = c_H + m \) where \( m > 0 \):

\[
\pi_d(q_H^*, q_L^*) = (R + 1 - q_H^* - \gamma q_L^* - \sum_{j \neq i} q_H^* - \gamma \sum_{j \neq i} q_L^* - W + wq_H^*)q_H^* + (R + \gamma(1 - q_H^* - q_L^* - \sum_{j \neq i} q_H^* - \sum_{j \neq i} q_L^*) - w_L)q_L^* \]

As we did earlier, let \((q_H^0, q_L^0)\) be the optimal solution for \( \Gamma_C(q_H, q_L) \) with \((q_H^{i0}, q_L^{i0})\) representing the part of the solution for all but the \( i \)'th distributor. Then:

\[
\frac{\partial \pi_d(q_H^0, q_L^0, q_H^{i0}, q_L^{i0})}{\partial q_H} = [R + 1 - 2q_H^0 - 2\gamma q_L^0 - \sum_{j \neq i} q_H^0 - \gamma \sum_{j \neq i} q_L^0 - c_H - \sum_{j \neq i} q_H^{i0} - \gamma \sum_{j \neq i} q_L^{i0} - m + 2wq_H^0] \]

\[
\frac{\partial \pi_d(q_H^0, q_L^0, q_H^{i0}, q_L^{i0})}{\partial q_L} = [R + \gamma(1 - q_H^0 - q_L^0 - \sum_{j \neq i} q_H^0 - \sum_{j \neq i} q_L^0 - c_L - \gamma q_H^{i0} - \gamma \sum_{j \neq i} q_L^{i0} - \gamma (1 - q_H^0 - \gamma \sum_{j \neq i} q_L^{i0})] \]

would give \((q_H^{i0}, q_L^{i0})\) as the unique solution and would achieve channel coordination with \( w_i = \frac{m_i}{2q_H} \).

Proof of Proposition 3.4

a) \( d_H = \sum_{i=1}^n q_H^i = \frac{n}{n+1}(1 - \frac{w_H - w_L}{1 - \gamma}) \)

\( d_L = \sum_{i=1}^n q_L^i = \frac{n}{n+1}(\frac{R}{\gamma} + \frac{w_H}{1 - \gamma} - \frac{w_L}{1 - \gamma}) \)

Plugging in \( w_H^* \) and \( w_L^* \), we would get \( d_H = \frac{n}{n+1}(\frac{1}{2} - \frac{c_H - c_L}{2(1 - \gamma)}) \) and \( d_L = \frac{n}{n+1}(\frac{R}{2\gamma} + \frac{\gamma c_H - c_L}{2\gamma(1 - \gamma)}) \)

\[
\lim_{n \to \infty} d_H = \left( \frac{1}{2} - \frac{c_H - c_L}{2(1 - \gamma)} \right) = d_H^C
\]

\[
\lim_{n \to \infty} d_L = \left( \frac{R}{2\gamma} + \frac{\gamma c_H - c_L}{2\gamma(1 - \gamma)} \right) = d_L^C
\]

b) \( \pi_m(w_H^*, w_L^*) = \frac{n}{n+1}\left\{ \frac{R+1-c_H}{2} \left( \frac{1}{2} - \frac{c_H - c_L}{2(1 - \gamma)} \right) + \frac{\gamma c_H - c_L}{2\gamma(1 - \gamma)} \right\} \)

\( \pi_m(w_H^*, w_L^*) = \frac{n}{n+1}\Gamma_C \) which gives us the result.
Appendix B

Supplement to Chapter 4: Proof of Results

Lemma B.1

Let $X$ be a random variable with cdf $\Phi(.)$ and $Y = \alpha X$ with cdf $\tilde{\Phi}(.)$. Then $\tilde{\Phi}^{-1}(k) = \alpha \Phi^{-1}(k)$

Proof of Lemma B.1

Let $k = \tilde{\Phi}(y) = P(Y < y) = P(X < \frac{y}{\alpha}) = \Phi(\frac{y}{\alpha})$

$y = \tilde{\Phi}^{-1}(k)$ by first equation above (1)

$k = \Phi(\frac{y}{\alpha}) \Rightarrow y = \alpha \Phi^{-1}(k)$ by the last and the first equation (2)

By (1) and (2), we have: $\tilde{\Phi}^{-1}(k) = \alpha \Phi^{-1}(k) \quad \square$

Proof of Theorem 4.1

a) For $n$ retailers, equation 4.4 can be rewritten as:
\[ \frac{\sum_{i=1}^{n}(Q_G^i - D_G^i)^+}{n} = \frac{1}{\delta(1-\gamma)}(p_a - p_G)^+ \frac{\sum_{i=1}^{n} D_i}{n} \]

By law of large numbers, we have:

\[ \lim_{n \to \infty} \frac{\sum_{i=1}^{n} D_i}{n} = \mu_D \]

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n}(Q_G^i - D_G^i)^+ = E[Q_G^i - D_G^i]^+ \]

With our large \( n \) assumption and based on above results, equation 4.4 would lead to:

\[ E[Q_G^i - D_G^i]^+ = \frac{1}{\delta(1-\gamma)}(p_a - p_G)^+) \mu_D \]

b) The retailer’s profit function under gray market with equilibrium gray market price \( p_G^* \) was:

\[ \Pi_r(Q_G^i) = \max_{Q_G^i} \left[ p_a E[\min(D_G^i, Q_G^i)] - wQ_G^i + p_G^* E[(Q_G^i - D_G^i)^+] \right] \]

Due to our large \( n \) assumption, \( p_G^* \) can be seen as independent from a given authorized retailer’s quantity decision \( Q_G^i \). Let \( F \) and \( f \) be respectively the cdf and pdf for \( D_G^i \). We have then:

\[ \frac{\partial \Pi_r}{\partial Q_G^i} = pa(1 - \bar{F}(Q_G^i)) - w + p_G^* \bar{f}(Q_G^i) \]

\[ \frac{\partial^2 \Pi_r}{\partial Q_G^i^2} = (-pa + p_G^*) \bar{f} < 0 \text{ because of the relation between } pa \text{ and } p_G. \]

Based on \( \frac{\partial \Pi_r}{\partial Q_G^i} = 0 \), we get:

\[ Q_G^* = \bar{F}^{-1}(\frac{pa-w}{p_G^*}) \]

To complete the proof, we will use Lemma B.1. Note that the relation between \( D_G^i \) and \( D^i \) is the same as in provided in Lemma B.1. Hence, we get our final result:

\[ Q_G^* = (1 - \frac{p_a-p_G^*}{(1-\gamma)p_a}) \bar{F}^{-1}(\frac{pa-w}{p_G^*}) \]

c) Based on equation 4.4, let’s define:
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\[ K(p_G) = E[(Q_G^i - D_G^i)_+] - \frac{1}{\beta(1-\gamma)}(p_a - \frac{p_G}{\gamma})\mu_D \]

i) At \( p_G = 0, K(0) = (1 - \frac{p_a}{\beta(1-\gamma)})E[F^{-1}(\frac{w_a-w}{p_a}) - D]^+ - \frac{p_a\mu_D}{\beta(1-\gamma)}] \]

If we define, \( Q^0 = F^{-1}(\frac{w_a-w}{p_a}) \), we have \( K(0) = (1 - \frac{p_a}{\beta(1-\gamma)})E[Q^0 - D]^+ - \frac{p_a\mu_D}{\beta(1-\gamma)}. \) Imposing condition (2), we have \( K(0) < 0 \)

\[ \frac{\partial K(p_G)}{\partial p_G} = (1 - \frac{p_a-p_G}{(1-\gamma)\beta})E[F^{-1}(\frac{w_a-w}{p_a-p_G}) - D]^+ + \frac{1}{(1-\gamma)}E[F^{-1}(\frac{w_a-w}{p_a-p_G}) - D]^+ + \frac{p_a\mu_D}{\beta(1-\gamma)} \]

Because \( p_G < \gamma p_a \), \( \frac{\partial K(p_G)}{\partial p_G} > 0. \)

ii) \( \frac{\partial K(p_G)}{\partial p_G} \)

At \( p_G = \gamma p_a \) \( K(p_G) \geq 0 \)

\( K(p_G) \) is continuous, increasing and intersects at \( p_G = p^*_G \) between \( (0 \gamma p_a) \) which proves the last part of the theorem.

Proof of Theorem 4.2:

\[ K(p_G) = (1 - \frac{p_a-p_G}{(1-\gamma)\beta})E[F^{-1}(\frac{w_a-w}{p_a-p_G}) - D]^+ + \frac{1}{\beta(1-\gamma)}(p_a - \frac{p_G}{\gamma})\mu_D \]

a) \( \frac{\partial K(p_G)}{\partial p} = \frac{p_a-p_G}{(1-\gamma)\beta}E[F^{-1}(\frac{w_a-w}{p_a-p_G}) - D]^+ + \frac{1}{\beta(1-\gamma)}(p_a - \frac{p_G}{\gamma})\mu_D > 0 \)

Hence, if \( \bar{w}_1 > \bar{w}_2 \), then \( K(\bar{w}_1) > K(\bar{w}_2) \) resulting in a decrease in \( p^*_G \)

\[ \frac{\partial K(p_G)}{\partial w} = -(1 - \frac{p_a-p_G}{(1-\gamma)\beta})E[F^{-1}(\frac{w_a-w}{p_a-p_G}) - D]^+ \frac{1}{\beta(1-\gamma)}(p_a - \frac{p_G}{\gamma})\mu_D < 0 \]

i.e. if \( w_1 > w_2, K(w_1) < K(w_2) \) which means an increase in \( p^*_G \).

\[ \frac{\partial K(p_G)}{\partial \gamma} = \frac{p_a-p_G}{(1-\gamma)^2}E[F^{-1}(\frac{w_a-w}{p_a-p_G}) - D]^+ - \frac{1}{\beta(1-\gamma)^2}(p_a - \frac{p_G}{\gamma})\mu_D - \frac{1}{\beta(1-\gamma)^2}p_a\mu_D \frac{p_G}{\gamma} < 0 \]

An increase in \( \gamma \) would result in a decrease in \( K(p_G) \) function which would mean an increase in \( p^*_G \).

Proof of Proposition 4.1:
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a) \( E[Q - D] = E[Q - D]^* + E[Q - D]^* \).

\( E[Q - D]^* = Q - \mu_D - E[Q - D]^* \leq \frac{\sqrt{\mu_D^2 + (Q - \mu_D)^2} - (Q - \mu_D)}{2} \) based on 4.7, which would give us:

\( \frac{3}{2}(Q - \mu_D) - \sqrt{\frac{\mu_D^2 + (Q - \mu_D)^2}{2}} \leq E[Q - D]^* \). Replacing the expression in equation 4.5 with this lower bound would give the result.

b) Let's denote the resulting \( K \) function as \( \bar{K}(p_G) \) and the resulting gray market price as \( \bar{p}_G \).

We then have \( \bar{K}(p_G) \leq K(p_G) \). Having the same properties as \( K(p_G) \), we would have \( p_G^* < \bar{p}_G \)

\textbf{Proof of Proposition 4.2:}

In equation 4.5, replacing \( E[Q - D]^* \) by \( E[Q - D] \) would give us another lower bound. With the additional assumption that \( D \) is uniformly distributed with support \((0, D)\), equation 4.5 would be:

\( (1 - \frac{p_a - p_G}{(1 - \gamma)p})[F^{-1}(\frac{p_a - w}{p_a - p_G}) - \mu_D] = \frac{1}{\gamma(1 - \gamma)}(p_a - \frac{p_G}{\gamma})\mu_D \)

\( \Rightarrow \bar{D}(p_a - p_G) - \bar{D}(p_a - p_G) - \mu_D = \frac{1}{\gamma(1 - \gamma)}(p_a - \frac{p_G}{\gamma})\mu_D \)

\( \bar{D} = 2\mu_D \) and denoting \( m = p_a - w \), we have:

\( \Rightarrow \frac{2m}{p_a - p_G} - \frac{2m}{(1 - \gamma)p} - \frac{(1 - \gamma)b - p_a + p_G}{(1 - \gamma)p} = \frac{p_a - p_G}{\gamma(1 - \gamma)p} \mu_D \)

\( \Rightarrow \frac{1}{p_a - p_G} + \frac{p_G}{2\gamma bm} = \frac{1}{(1 - \gamma)p} + \frac{1}{2m} \)

Representing the right-hand side as \( A \), would give the result. Using the same arguments as in Corollary (1), the resulting gray market price \( \bar{p}_G \) is an upper bound i.e. \( p_G^* < \bar{p}_G \)

\textbf{Proof of Proposition 4.4:}

a) \( Q_G = (1 - \frac{p_a - p_G}{(1 - \gamma)p})F^{-1}(\frac{p_a - w}{p_a - p_G}) \)
APPENDIX B. SUPPLEMENT TO CHAPTER 4: PROOF OF RESULTS

$$Q_0' = (1 - \frac{P_a}{p_a})F^{-1}(\frac{P_a - \vartheta_0}{p_a})$$

$$Q_G' < Q_0' \Rightarrow ((1 - \gamma)\vartheta - p_a + p_G') < \epsilon(p_G')(1 - \frac{P_a}{p_a})(1 - \gamma)\vartheta$$

$$\Rightarrow p_G' < \epsilon(p_G')(1 - \frac{P_a}{p_a})(1 - \gamma)\vartheta + p_a - (1 - \gamma)\vartheta$$

$$\Rightarrow p_G' < [(\epsilon(p_G')(1 - \gamma) - 1)(1 - \frac{P_a}{p_a}) + \gamma]\vartheta = \tau(p_G')$$

where \(\epsilon(p_G') = F^{-1}(\frac{P_a - \vartheta_0}{p_a})/F^{-1}(\frac{p_a - \vartheta_0}{p_a} - p_G') = \frac{F^{-1}(m)}{F^{-1}(\tau)}\) with \(m = \frac{P_a - \vartheta_0}{p_a}\)

b) Let \(G = F^{-1}\) and \(g = G'.\) We would then have:

$$\frac{\partial^n(p_G)}{\partial p_G^n} = \frac{g'(p_G)}{G'(p_a - p_G)}(p_a - p_G)^2 < 0$$

c) Assume that \(p_G' > \tau(p_G').\)

$$\lim_{\vartheta \to \infty} p_G' = 0$$

$$\lim_{p_G' \to 0} \epsilon(p_G') = 1$$

$$\lim_{\epsilon(p_G') \to 1} \tau(p_G') = \gamma p_a$$

For sufficiently large \(\vartheta',\) we get: \(p_G' > \gamma p_a\) which contradicts with what we found in theorem 4.1. Hence, for \(v > \vartheta',\) condition is satisfied which means \(Q_G' < Q_0'.\)

Proof of Theorem 4.3

$$\Pi_m(Q) = [(p_a - p_a F(Q/\xi) + p_G F(Q/\xi) - c)Q]$$

$$\frac{\partial \Pi_m}{\partial Q} = p_a - p_a F(Q/\xi) + p_G F(Q/\xi) - c - p_a f(Q/\xi)Q^{'2} + p_a f(Q/\xi)Q + p_G f(Q/\xi)Q^{'2} + p_G f(Q/\xi)Q^{'2}$$

Now, add and subtract \(p_G\) to the above equation:

$$(p_a - p_G)F(Q/\xi) - (p_a - p_G) f(Q/\xi)(Q/\xi) + (p_a - p_G) f(Q/\xi)Q^{'2} + p_G f(Q/\xi)Q - c - p_G \Rightarrow$$

$$F(Q/\xi)[1 - \frac{f(Q/\xi)(Q/\xi)}{F(Q/\xi)} + \frac{f(Q/\xi)(Q/\xi)}{F(Q/\xi)}(\xi)Q^{'2}] = \frac{c - p_G}{p_a - p_G} - \frac{p_G f(Q/\xi)Q}{p_a - p_G}$$
Let's define \( g(Q/\xi) = \frac{f(Q/\xi)Q}{F(Q/\xi)} \). As argued in Lariviere and Porteus (2001), for distributions that are IGFR, \( g \) is increasing. We then have:

\[
F(Q/\xi)[1 - g(Q/\xi)(1 - \xi^Q \xi)] = \frac{c - \mu G}{p_a - \mu G} - \frac{p_G(Q/\xi)Q}{p_a - \mu G}.
\] (B.1)

which can be written as in the theorem once we define \( v = \xi^Q \xi \) and \( \eta(Q) = -p_G(Q/\xi)Q \).

From equation 4.4, we know that \( \frac{\partial \mu G}{\partial Q} < 0 \) which means \( Q/\xi \) is increasing as \( Q \) increases. We also know that \( \xi^Q \xi < 0 \). Hence \( g(Q/\xi)(1 - \xi^Q \xi) \) increases in \( Q \). As a result, the LHS in equation B.1 is decreasing in \( Q \).

Let's define \( k_G = \frac{c - \mu G}{p_a - \mu G} \) and \( u = p_a - \mu \). We then get: \( \frac{\partial \eta(Q)}{\partial Q} = -(1 - \frac{m}{\mu G - u})^2 \eta(Q) > 0 \).

Since \( \frac{\eta(Q)}{\mu G - u} \) increases as well, RHS increases with \( Q \). Note that at \( Q = 0 \), we get LHS = 1 and RHS = \( \frac{(c - \mu G)}{p_a - \mu} \) and because \( p_a > \mu \), we have LHS greater than RHS. Hence unique \( Q^* \) exists.

Proof of Proposition 4.3

\[
\frac{\partial \Pi^C}{\partial Q} = (p_a - \mu) - (p_a - \mu G(Q))F(Q) + p_G(Q) \mu (Q - D)^+
\]

Note that \( \frac{\partial \Pi^C}{\partial Q} \) does not have a special structure. Let \( Q^* = \min(Q|Q = \arg \min \Pi^C(Q)) \). We first need to compare \( Q(c) \)-the equilibrium retailer order quantity when \( w = \mu \) in a decentralized system - with \( Q^* \).

\[
\frac{\partial \Pi^D}{\partial Q} = (p_a - \mu) - (p_a - \mu G(Q))F(Q).
\]

Since the equilibrium gray market price \( \mu G \) changes with respect to \( Q \) according to the same equation and because \( p_G' < 0 \), \( \frac{\partial \Pi^D(Q)}{\partial Q} > 0 \); hence each retailer would be better off increasing its quantity (until \( Q(c) \)) which gives us: \( Q(c) > Q^* \).

As the wholesale price changes between \( (c \ p_a) \), the order quantity in decentralized system changes between \( (0 \ Q(c)) \). Due to continuity of \( Q(w) = (1 - \frac{p_a - \mu G}{(1 - \mu G)}F^{-1}(\frac{p_a - w}{p_a - \mu G})) \), there exists a \( w^* \) such that \( Q(w^*) = Q^* \).
Proof of Theorem 4.4

In this scheme, the retailer's profit is:

\[ \Pi_R(Q) = \max_{Q'} \left[ p_a E[\min(D, Q')] - w^*Q^i + p_c E[(Q' - D)^+] - k \right] \]

which would give the equilibrium retailer order quantity \( (1 - \frac{\rho - p_r}{(1 - \gamma)^p})F^{-1}(\frac{p_r - w}{\rho - p_r}) \) and based on Proposition 4.3, \( (1 - \frac{\rho - p_r}{(1 - \gamma)^p})F^{-1}(\frac{p_r - w}{\rho - p_r}) = Q^* \). Since \( \Pi_D < \Pi_C \) and because \( \Pi_M \) stays the same (based on the definition of \( k \)), retailers are better off.

Proof of Theorem 4.5

a) Following what we did for proposition 4.3, under this contract, for any \( a \) and \( b \) and when \( w = c \) with the notation \( \delta = (1 - a)p_r + ab \), we get:

\[ \frac{\partial \Pi_C}{\partial Q} = (p_a - c) - (p_a - \delta)F(Q) \]

The optimum retailer order quantity \( Q(c) = (1 - \frac{\rho - p_r}{(1 - \gamma)^p})F^{-1}(\frac{p_r - w}{\rho - p_r}) \) making the dependence on \( w = c \). We also know that:

\[ \frac{\partial \Pi_C}{\partial Q} = (p_a - c) - (p_a - p_c(Q))F(Q) + p_c(Q)^'E[(Q - D)^+] \]

If we denote \( Q^* = \min(Q \mid Q = \arg \min \Pi_C(Q)) \) as we did earlier, and let \( (a, b) \) be set such that \( \delta = p_c(Q^*) \) when \( Q = Q^* \). In that case, because \( p_c' < 0 \), \( \frac{\partial \Pi_C(Q^*)}{\partial Q} > 0 \), each retailer would be better off increasing its quantity (until \( Q(c) \)) which gives us: \( Q(c) > Q^* \). As the wholesale price changes between \( (c, p_a) \), the order quantity changes between \( (0, Q(c)) \). Due to continuity of \( Q(w) = (1 - \frac{\rho - p_r}{(1 - \gamma)^p})F^{-1}(\frac{p_r - a}{\rho - p_r}) \), there exists a \( w \) such that \( Q(w) = Q^* \), which means that the contract with \( (w, b, a) \) achieves coordination.

b) In part (a), we actually needed \( \delta > p_c(Q^*) \) to show the existence of \( \delta \) in the end. Hence, for a given \( a \) and when \( p_r \) is determined at \( Q = Q^* \), we need:
i) \( \frac{P(Q^*) - (1-a)P_G}{a} < b \)

On the other hand, if we impose the condition \( \Pi_M = (\hat{w} - c)Q^* - abE[(Q^* - D)^+] > \Pi_M^* \)
and plug in \( \hat{w} = p_a - p_G F(Q^*_G) + ab + (1 - a)p_G F(Q^*_G) \) and rearrange for the \( b \) term, we would get:

\[ b \leq \frac{(p_a - p_G F(Q^*_G) + (1-a)p_G F(Q^*_G) - c)Q^*_G - \Pi_M^*}{1 - \xi \frac{p_G}{F(Q^*_G) - D^+}} \]

where \( \xi = 1 - \frac{p_G}{(1-\gamma)^0} \). The rest of the paper follows from the same argument as in (a).

\[ \square \]

Proof of Proposition 4.5

a) \( \Pi^*_R = \lambda \left[p_a E[\min(D'_G, Q'_G)] - cQ'_G\right] + p_G E[(Q'_G - D'_G)^+] \)

\[ \frac{\partial \Pi^*_R}{\partial Q'_G} = \lambda p_a (1 - F(Q'_G)) - \lambda c + p_G F(Q'_G) \]

which would give:

\[ Q^*_G = \left(1 - \frac{p_a - p_G}{F(Q'_G)}\right)F^{-1}\left(\frac{p_a - p_G}{F(Q'_G)}\right) \]

We also know that, under wholesale pricing:

\[ Q^*_G = \left(1 - \frac{p_a - p_G}{F(Q'_G)}\right)F^{-1}\left(\frac{p_a - p_G}{F(Q'_G)}\right) \]

For the same \( p_G \) value (i.e. \( p'_G = p^{\text{wh}}_G \)):

\[ F^{-1}\left(\frac{p_a - p'_G}{p_a - p^{\text{wh}}_G}\right) > F^{-1}\left(\frac{p_a - p'_G}{p_a - p^{\text{wh}}_G}\right) \]

which gives us: \( Q^*_G > Q^{\text{wh}}_G \). If we call the resulting \( K \) function as

\[ K'(p_G) = \left(1 - \frac{p_a - p_G}{F(Q'_G)}\right)E[(Q'_G - D)^+] - \left(\frac{1}{Q^*_G}\right)(p_a - p_G) \mu_D > \right.

\( (1 - \frac{p_a - p_G}{F(Q'_G)}\right)E[(Q'_G - D)^+] - \left(\frac{1}{Q^*_G}\right)(p_a - p_G) \mu_D = K(p_G) \)

As a result, using the same argument made earlier, \( p'_G < p^{\text{wh}}_G \).

b) \( \Pi^*_M = (1 - \lambda)p_a E[\min((D'_G, Q'_G))] - cQ'_G \)

\[ \Pi^*_M = (1 - \lambda)p_a E[\min((D'_G, Q'_G))] - cQ'_G \]

Note that \( D'_G = (1 - \frac{p_a - p_G}{(1-\gamma)^0})D < (1 - \frac{p_G}{(1-\gamma)^0})D = D'_G \)

Take a value of random variable \( D = d \) and a \( Q \). We would have:
\[ D_G^* = (1 - \frac{p_a - p_G}{(1 - \gamma_P)}) \hat{d} \] and \[ D_0 = (1 - \frac{p_a}{\bar{p}}) \hat{d}. \] Suppose:

CASE I: \( Q < \{D_G^*, D_0\} \):

\[ \Pi_{M}^{G_{1}} = (1 - \lambda)(p_a - c)Q \]
\[ \Pi_{M}^{G_{0}} = (1 - \lambda)(p_a - c)Q \]
\[ \Rightarrow \Pi_{M}^{G_{1}} = \Pi_{M}^{G_{0}} \]

CASE II: \( D_G^* < Q < D_0 \):

\[ \Pi_{M}^{G_{1}} = (1 - \lambda)(p_a D_G^* - cQ) \]
\[ \Pi_{M}^{G_{0}} = (1 - \lambda)(p_a - c)Q \]
\[ \Rightarrow \Pi_{M}^{G_{1}} < \Pi_{M}^{G_{0}} \]

CASE III: \( Q > \{D_G^*, D_0\} \):

\[ \Pi_{M}^{G_{1}} = (1 - \lambda)(p_a D_G^* - cQ) \]
\[ \Pi_{M}^{G_{0}} = (1 - \lambda)(p_a D_0 - cQ) \]
\[ \Rightarrow \Pi_{M}^{G_{1}} < \Pi_{M}^{G_{0}} \]

Taking an average over all \( \hat{d} \) values would not change the results from Cases I, II and III.

Hence for \( Q, \Pi_{M}^{G_{1}} < \Pi_{M}^{G_{0}}. \)

Therefore, \( \Pi_{M}^{G_{0}}(Q_G) > \Pi_{M}^{G_{1}}(Q_G^*) \)

Proof of Proposition 4.6

a) The contract is assumed to be offered with parameters \( w, b \) where \( w = \check{w} \) and \( b = p_a(\frac{\check{w} - c}{p_a - c}). \)

The buy-back option of the contract is avoided under the given condition. Hence with participation in gray market, we have:

\[ \Pi_{R}^{b_{G}} = \left[ p_a E[\min(D_{G}^b, Q_{G}^b)] - \check{w}Q_{G}^b + p_G E[(Q_{G}^b - D_{G}^b)^+] \right] \]
With only a wholesale pricing contract where \( w = \bar{w} \), in gray market environment, we have:

\[ Q_{G}^{*} = (1 - \frac{p_{w} - p_{G}}{p_{w} - p_{G}})F^{-1}(\frac{p_{w} - \bar{w}}{p_{w} - p_{G}}) \]

as found in theorem 4.1.

Since the wholesale prices are equal in both, if we call the \( K \) function under buy-back contract as \( K^{b} \), we have \( K^{b} = K \) for a given \( p_{G} \). Therefore \( p_{w}^{b} = p_{G} \). As a result, \( Q_{G}^{w*} = Q_{G}^{b*} \)

i.e. how much the retailer orders remain the same under both contracts in gray market environment.

b) We know that the demand realized under gray market, \( D_{G}^{b} \), is less than what it is under no gray market i.e. \( D_{G}^{b} \). Hence, when \( b > p_{G}^{b} \), the revenue coming from selling excess inventory is larger. Hence in equilibrium, the gray market participation would never occur since \( p_{G}^{*} < \gamma p_{a} \) and that under the given condition, \( b \) is always higher.

c) Under the given condition, in equilibrium, since some retailers will return at rate \( b \) and since some will sell at \( p_{a}^{w} \), the manufacturer’s position \( \Pi^{b}_{M} \) is indeterminate.

Proof of Proposition 4.7

\[ Q_{G}^{w} = (1 - \frac{p_{w} - p_{G}}{p_{w} - p_{G}})F^{-1}(\frac{p_{w} - \bar{w}}{p_{w} - p_{G}}) \]

We also know that, under wholesale pricing:

\[ Q_{G}^{w} = (1 - \frac{p_{w} - p_{G}}{p_{w} - p_{G}})F^{-1}(\frac{p_{w} - \bar{w}}{p_{w} - p_{G}}) \]

For the same \( p_{G} \) value (i.e. \( p_{w}^{b} = p_{G}^{w} \)):

\[ F^{-1}(\frac{p_{w} - \bar{w}}{p_{w} - p_{G}}) > F^{-1}(\frac{p_{w} - \bar{w}}{p_{w} - p_{G}}) \]

which gives us: \( Q_{G}^{b} > Q_{G}^{w} \). If we call the resulting \( K \) function as

\[ K^{b}(p_{G}) = (1 - \frac{p_{w} - p_{G}}{p_{w} - p_{G}})E[(Q_{G}^{b} - D)^{+}] - (\frac{1}{\gamma(1 - \gamma)})(p_{w} - \frac{p_{G}}{\gamma})\mu_{D} \]

\[ (1 - \frac{p_{w} - p_{G}}{p_{w} - p_{G}})E[(Q_{G}^{w} - D)^{+}] - (\frac{1}{\gamma(1 - \gamma)})(p_{w} - \frac{p_{G}}{\gamma})\mu_{D} = K(p_{G}) \]
As a result, using the same argument made earlier, $p^g_G < p^m_G$.

b) $\Pi_M = (w - c) \cdot Q = (c + \frac{m}{Q} - c)Q = m$ regardless of the gray market existence.
Appendix C

Supplement to Chapter 5: Proof of Results

Proof of Theorem 5.1

a) From manufacturer's problem as in 5.4, and because $\beta_i < 1, \forall i$, one can easily show that $\Pi_m$ is concave in $x_i$. Based on the assumption that the upper bounds for quality levels are scaled to be very high values, we have:

$$x_1 = \left(\frac{A^*\delta_1^2}{\omega_1}\right)^{1/\beta_1}$$
$$x_2 = \left(\frac{A^*\delta_2^2}{\omega_2}\right)^{1/\beta_2}$$

Plugging in the above values, we get:

$$x_1^* = (M)^{1/n} \text{ where } M = \left(\frac{A^*\delta_1^2}{\omega_1}\right)^{1/\beta_1} \left(\frac{A^*\delta_2^2}{\omega_2}\right)^{1/((1-\beta_1)(1-\beta_2))} \text{ and } n = 1 - \frac{\beta_1\beta_2}{(1-\beta_1)(1-\beta_2)}$$
$$x_2^* = (R)^{1/n} \text{ where } R = \left(\frac{A^*\delta_2^2}{\omega_2}\right)^{1/\beta_2} \left(\frac{A^*\delta_1^2}{\omega_1}\right)^{1/((1-\beta_1)(1-\beta_2))}$$

$$x_1^C = \left(\frac{A^*\delta_1^2}{c_1}\right)^{1/\beta_1} \left(\frac{A^*\delta_2^2}{c_2}\right)^{1/((1-\beta_1)(1-\beta_2))} \right)^{1/n} > \left(\frac{A^*\delta_1^2}{c_1}\right)^{1/1-\beta_1} \left(\frac{A^*\delta_2^2}{c_2}\right)^{1/((1-\beta_1)(1-\beta_2))} \right)^{1/n} = x_1^D$$
b) Because $R(X) = A(x_1^\beta_1 x_2^\beta_2 \ldots x_n^\beta_n)$, based on part (a), we know that this value is less in decentralized systems.

Proof of Theorem 5.2

a) $\Pi_i^j(w_i, w_{-i}) = (w_i - c_i)x_i(w_i, w_{-i}), i=1,2$

We know from Theorem 5.1 a that $x_1 = \frac{a(w_2)}{w_1^{(1-\beta_1)n}}$ where $a(w_2)$ is the part that is function of $w_2$:

$$\frac{\partial \Pi_i^j}{\partial w_1} = a(w_2)(w_1)^{-1/(1-\beta_1)n}(1 - \frac{1}{(1-\beta_1)n} + \frac{\epsilon}{(1-\beta_1)n w_1})$$

$$\frac{\partial \Pi_i^j}{\partial w_1} > 0$$ with $w_1 = c_1$ and is non-increasing in $w_1$. Therefore, $\Pi_i^j$ is quasi-concave with $w_1^* = c_1(\frac{b}{b+1})$ and $b > 1$ $\forall \beta_1, \beta_2$, $b = \frac{1}{(1-\beta_1)n}$.

b) Due to the multiplicative nature of our model, the profit function for each supplier $i$ is a multiple of a function of $w_2$ and therefore, it becomes an affine transformation of the other supplier’s wholesale prices. As we also see from the $w_1^*$, the equilibrium emerges in the strongest possible sense as dominant strategy equilibrium where each supplier charges $w_i^*$ regardless of what others do.

Proof of Theorem 5.3

a) As we argued earlier, $R(X)$ in this problem represents the value of the product and hence is assumed to be proportional to the total revenue that could potentially be generated as a result. Therefore $R(X)$ will be used in the profit functions here as the revenue term. When $w_i = \lambda_0 c_i$ for $\forall i$, $\Pi_m(X) = \lambda_0 R(X) - \sum_i \lambda_0 c_i x_i = \lambda_0 (R(X) - \sum_i c_i x_i) = \lambda_0 \Pi_C(X)$. Therefore, the quality levels, the manufacturer will decide on i.e. $X^* = X^C$; hence the channel is coordi-
nated and $\Pi^*_i = \lambda_i R(X) - (1 - \lambda_i)c_i x_i$ which means $\sum_i \Pi^*_i = (\sum_i \lambda_i) R(X) - (1 - \lambda_0) \sum_i c_i x_i = (1 - \lambda_0)(R(X) - \sum_i c_i x_i) = (1 - \lambda_0) \Pi_C(X)$

b) Let's assume that we initially set the vector $(\lambda_1, \lambda_2)$ and $\lambda_0$ such that all the suppliers receive their profit as in decentralized system. We know that the overall profit generated in the channel is greater i.e. $\Pi_C(X) > \Pi_D(X)$. The manufacturer will definitely be better off. Hence, we can say that $(\lambda_1, \lambda_2)$ and $\lambda_0$ can further be adjusted to make all the players better off in this scheme.

Proof of Theorem 5.4

a) With the wholesale price $w_i = (1 - \alpha)c_i + \alpha k_i \frac{R(X)}{x_i}$, we have the manufacturer’s profit as:

$$\Pi_m(X) = R(X) - \sum_i w_i x_i = R(X) - \sum_i ((1 - \alpha)c_i + \alpha k_i \frac{R(X)}{x_i}) x_i = R(X) - (1 - \alpha) \sum_i c_i x_i$$

$$= (1 - \alpha)(R(X) - \sum_i c_i x_i) = (1 - \alpha) \Pi_C(X)$$

b) This can be argued as in the earlier theorem.

Proof of Theorem 5.5:

$$\Pi_m'(X) = \log \Pi_m(X) = \log(p - c(X)) + \log d(X).$$

We will show that $\Pi_m'(X)$ is concave. Let $c', d'$ and $\Pi_m''$ be the derivative of the cost, demand and profit functions respectively with respect to quality levels $x_i$. $\Pi_m'' = \frac{c''(X)(p-c(X)) - c'(X)^2}{(p-c(X))^2} + \frac{d''(X)d(X) - d'(X)^2}{d(X)^2}$ < 0 if

(1) $c''(X) \geq 0$ and

(2) $d''(X)d(X) < d'(X)^2$
APPENDIX C. SUPPLEMENT TO CHAPTER 5: PROOF OF RESULTS

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We assume (1). (2) is ensured by the IFR assumption on $\Phi$. Note that $d'(X) = \beta_i N\phi(p - \beta^T X)$ and $d''(X) = -\beta_i^2 N\phi'(p - \beta^T X)$. Condition (2) requires:

$$-\beta_i^2 N\phi'(p - \beta^T X)(N\Phi(p - \beta^T X)) \leq N^2 \beta_i^2 \phi'(p - \beta^T X)$$

(Let $U = p - \beta^T X \Leftrightarrow -\phi'(u)\Phi(u) < \phi^2(u) \Leftrightarrow \frac{-\phi'(u)}{\phi(u)} < \frac{\phi(u)}{\phi'(u)} \Leftrightarrow h(u) > -\frac{\phi'(u)}{\phi(u)}$)

which is known to be true for IFR distributions. Hence, $\Pi_m$ is unimodal.

Proof of Theorem 5.6:

a) With $C(X) = \sum_i w_ix_i$, the profit function for the manufacturer is $\Pi_m(X) = (p - \sum_i w_i x_i)(1 - p + \sum_i \beta_i x_i)$ and $\frac{\partial \Pi_m}{\partial x_i} = -2w_i \beta_i < 0 \forall i = 1, 2$ proving the first part.

b) Based on (a) and:

$$\frac{\partial \Pi_m}{\partial x_1} = -w_1 (1 - p + \beta_1 x_1 + \beta_2 x_2) + (p - w_1 x_1 - w_2 x_2)\beta_1 = 0$$

$$\frac{\partial \Pi_m}{\partial x_2} = -w_2 (1 - p + \beta_1 x_1 + \beta_2 x_2) + (p - w_1 x_1 - w_2 x_2)\beta_2 = 0$$

will give us:

$$x_1 = \frac{p\beta_1 - w_1 (1-p) - x_2 (w_2 \beta_2 + w_1 \beta_1)}{2w_1 \beta_1}$$

$$x_2 = \frac{p\beta_2 - w_2 (1-p) - x_1 (w_1 \beta_1 + w_2 \beta_2)}{2w_2 \beta_2}$$

Plugging in the values, we get:

$$x_i^* = \left[ \frac{\beta_i}{2w_i} \left( \frac{w_i \beta_j + w_j \beta_i}{w_i^2 \beta_j} \right) \right] i,j = 1,2$$

Proof of Theorem 5.7

a) The structure of the objective function stays the same with the additional assumption and we get $x_1 = \frac{\rho}{2w_1} - \frac{1-\rho}{2\beta_1}$. Hence $x_1^C = \frac{\rho}{2\beta_1} - \frac{1-\rho}{2\beta_1} > \frac{\rho}{2w_1} - \frac{1-\rho}{2\beta_1} = x_1^D$

b) The result simply follows from the first part and the nature of the demand function.
Proof of Theorem 5.8

In this problem, the centralized system profit would be $\Pi^C(X) = (\bar{p} - c_1 x_1)(1 - \bar{p} + \beta_1 x_1)$.

With the proposed scheme:

$\Pi_m(X) = \lambda(\bar{p}(1 - \bar{p} + \beta_1 x_1)) - \lambda c_1 x_1 = \lambda \Pi^C(X)$ and as a result, the coordination is achieved and since the profit can be allocated arbitrarily, one can find an $\alpha$ that would make each party better off.

Proof of Theorem 5.9

With the wholesale price $w = (1 - \alpha)c + \alpha \bar{p}/x$, we can now write the manufacturer's profit function:

$\Pi_m(x_1) = (\bar{p} - ((1 - \alpha)c_1 + \alpha \bar{p}/x_1)x_1)((1 - \bar{p} + \beta_1 x_1)

= (\bar{p} - \alpha \bar{p} - (1 - \alpha)c_1 x_1)((1 - \bar{p} + \beta_1 x_1)

= (1 - \alpha)(\bar{p} - c_1 x_1)(1 - \bar{p} + \beta_1 x_1)

= (1 - \alpha)\Pi^C(X)

which proves the first part

b) This part can be argued as in earlier theorem.
Appendix D

Supplement to Chapter 6: Proof of Results

Proof of Theorem 6.1

The proof of parts (a)-(c) will be done by the following induction. We start with the base case \( t = T \). \( f_T(z_T) = E_{D_T}[g_T(z_T - D_T)] \) is convex and coercive, since \( g_T() \) is convex and coercive. Let \( s^*_T = \min \{ s | f_T(s) = \min_{z_T} f_T(z_T) \} \). Based on Lemma 6.1 (a), we know that

\[
H_T(y_T, x_T) = E_{C_T} f_T(\min(y_T, x_T + C_T))
\]

is quasi-convex in \( y_T \) and \( s^*_T \) minimizes this term for any \( x_T \). Finally, \( f_T(x_T) \) is convex, by Lemma 6.1 (b). This completes the base case. Next, assume that parts (a)-(c) are satisfied for \( t + 1 \). Then \( f_t(z_t) = E_{D_t}[g_t(z_t - D_t) + \alpha J_{t+1}(z_t - D_t)] \) is convex in \( z_t \) since the single-period cost function \( g_t \) is convex and coercive, \( J_{t+1} \) is convex and coercive by the induction hypothesis and expectation with respect to demand \( D_t \) preserves these properties. Let
$s^*_t = \min\{s|f_t(s) = \min_{z_t} f_t(z_t)\}$. Based on Lemma 6.1 (a), we know that

$$H_t(y_t, x_t) = E_{C_t}[f_t(\min(y_t, x_t + C_t))],$$

is quasi-convex in $y_t$ and $s^*_t$ minimizes this term for any $x_t$. Finally, $J_t(x_t)$ is convex, by Lemma 6.1 (b), which completes the induction. Part (d) then follows by the quasi-convexity of $H_t(y_t, x_t)$.

**Proof of Theorem 6.2**

The proof is almost identical to that of Theorem 6.1, except the state now also includes the forecast vector $F_t$, resulting in a state-dependent base stock policy.

**Proof of Proposition 6.1**

a) Let

$$\bar{x}_T = 0$$

$$\bar{x}_t = [d_{t+1} - c_{t+1} + \bar{x}_{t+1}]^+, \quad \text{for } t = 1, \ldots, T - 1$$

First, we argue that any feasible solution to problem (P) must have $x_t \geq \bar{x}_t$ for all $t$. To show this, consider a solution where $x_t < \bar{x}_t$, for some $t$. Then, by the definition of $\bar{x}_t$, the total demand in $t + 1$ to $T$ minus the total capacity in $t + 1$ to $T$ is more than $x_t$. This means that even if the solution orders up to the capacity $c_k$ in every period $k = t + 1, \ldots, T$, there will be a period when demand will not be satisfied, meaning that the solution cannot be feasible.

Since any feasible solution must have $x_t \geq \bar{x}_t$ for all $t$, $\sum_{k=1}^{T} h_k \bar{x}_k$ is a lower bound on the cost of problem (P). We now show that the solution given in Proposition 6.1
achieves exactly this lower bound, and hence is optimal. In order to do this, we first show that

\[
\bar{x}_T = 0 \tag{D.1}
\]

\[
\bar{x}_t = \max_{r \in \{t+1, \ldots, T\}} \left\{ \sum_{k=t+1}^{r} (d_k - c_k) \right\}^+, \quad \text{for } t = 1, \ldots, T - 1.
\]

The proof of the equality in (D.1) is done by induction. The base case is \( t = T - 1 \). We have

\[
\bar{x}_{T-1} = \max_{r \in \{T\}} \left\{ \sum_{k=T}^{r} (d_k - c_k) \right\}^+
\]

\[
= (d_T - c_T)^+,
\]

which verifies the base case. Now, assume that the equation (D.1) is valid for \( t + 1 \), and we show its validity for \( t \).

\[
\bar{x}_t = [d_{t+1} - c_{t+1} + \bar{x}_{t+1}]^+
\]

\[
= \left[ d_{t+1} - c_{t+1} + \max_{r \in \{t+2, \ldots, T\}} \left\{ \sum_{k=t+2}^{r} (d_k - c_k) \right\}^+ \right]^+
\]

\[
= \max_{r \in \{t+1, \ldots, T\}} \left\{ \sum_{k=t+1}^{r} (d_k - c_k) \right\}^+,
\]

where the first equality follows from the induction hypothesis. This completes the induction. Now, note that under the solution given in Proposition 6.1, \( x_t = \bar{x}_t \), for all \( t \), by construction. Therefore, this solution is feasible, and achieves the lower bound, meaning that it is optimal.

b) For \( t = 1 \), we must have \( c_1 > d_1 + \bar{x}_1 \), by the feasibility of the problem. The result follows.
Proof of Theorem 6.3

a) First, let's show the existence of $Z_{\infty}$. Note that the joint process $(Z_t, F_t)$ and the process $F_t$ are both regenerative. Let's define:

$$P_{k,j}(t) = P(Z_t = k, F_t = j)$$

$$Q_j(t) = P(F_t = j)$$

Based on Theorem 6.7 in Heyman and Sobel (2004):

$$\lim_{t \to \infty} P_{k,j}(t) = p_{k,j}$$

$$\lim_{t \to \infty} Q_j(t) = q_j$$

$$\lim_{t \to \infty} P(Z_t = k) = \lim_{t \to \infty} \int_{j \in J} P(Z_t = k, F(t) = j)P(F_t = j) = \int_{j \in J} p_{k,j}q_j = r_k$$

where the third inequality can be justified by bounded convergence theorem, which proves the existence result. To show that the distribution $Z_{\infty}$ is independent of $s$:

Consider 2 different base stock levels $s$ and $s'$ and consider two parallel systems that use the two base stock levels. Assume that the two systems experience the same sequence of random demand and capacity update values. Assume that the initial inventory levels are

i) $x_0 = s + K(F_0)$

ii) $x_0' = s' + K(F_0)$,

where $x_i$ is the inventory level of the first system at time $i$ and $x'_i$ is the inventory level of the second system at time $i$. Hence, $Z_0 = Z'_0 = 0$. Given this setup, $Z_t = Z'_t$, for all $t$, where $Z_t$ and $Z'_t$ are defined for the corresponding systems. Therefore $Z_{\infty} = Z'_{\infty}$. Note that the steady state distributions $Z_{\infty}$ and $Z'_{\infty}$ do not depend on the initial conditions by ergodicity.

Hence the steady state the distribution of $Z_{\infty}$ under any base stock level is the same.
b) Let $x_\infty = s - Z_\infty$. We are trying to minimize $C(s) = hE[x_{s+1}^+] + bE[x_{s-1}^-]$. We have $C(s) = hE[s - Z_\infty^+] + bE[s - Z_\infty^-]$, which is a strictly convex function of $s$. Setting $C(s)' = 0$ yields $s^* = F_{Z_\infty}^{-1}(b/(b + h))$.

Proof of Proposition 6.2

a) Because of the MMFE assumption, at every period the capacity receives an update starting from $N$ periods earlier. Hence, the capacity at period $t$ will receive $(N + 1)$ updates starting from period $(t - N)$ on the trivial initial forecast which is the mean capacity $\mu_C$. Therefore, $C_t = \mu_C + \sum_{i=0}^N \varepsilon_{t-N+i}$. Again due to MMFE assumption that $\varepsilon_t$'s are iid, $Var(C_t) = Var(\sum_{i=0}^N \varepsilon_{t-N+i}) = \sum_{i=0}^N \varepsilon_{t-N+i}^2 = \varepsilon^T \Sigma \varepsilon$. Note that if the covariance matrix is diagonal, the $\{C_t\}$'s are iid. If not, they are correlated.

b) We know that $\hat{Y}_t = \hat{Y}_{t-1} + a_t - D_t$, where $a_t$ is the amount received from the supplier (or the amount given back to the supplier if $a_t$ is negative) at period $t$. This will in turn give us the evolution of $\hat{Y}_t$ after adding constant $s$ appropriately to the above equation: $\hat{Y}_t = \hat{Y}_{t-1} + D_t + \Delta_t - a_t$ which can also be expressed $\hat{Y}_t = \hat{Y}_{t-1} + M_t - a_t$ where $\Delta_t = K(F_t) - K(F_{t-1})$, $M_t = D_t + \Delta_t$ and $a_t = \min(\hat{Y}_{t-1}, C_t)$. This gives us: $\hat{Y}_t = \max_{1 \leq j \leq T} \sum_j X_j$ where $X_j = M_j - C_j$. Hence, $\hat{Y}_t$ is a reflected random walk.

c) $M_t = D_t + K(F_t) - K(F_{t-1}) = D_t - \sum_{i=1}^N C_{t,t+i} + \sum_{i=1}^N C_{t-1,t-1+i} = D_t + C_{t-1,t} - C_{t,t+N} - \sum_{i=1}^{N-1} \varepsilon_{t+i}$. $X_t = M_t - C_t = D_t - \mu_C - \sum_{i=1}^N \varepsilon_{t+i} = D_t - \mu_C - e^T \varepsilon_t$ where $\varepsilon_t = (e_{t,t}, e_{t,t+1}, ..., e_{t,t+N})$.

Note that both $\{\varepsilon_t\}$ (because of the MMFE assumption) and $\{D_t\}$ are iid and independent of each other. Hence $X_t$ is iid with $E[X_t] = \mu_D - \mu_C$ and $Var(X_t) = Var(D_t) + Var(\sum_{i=0}^N \varepsilon_{t+i}) = \sigma_D^2 + \sum_{i=0}^N \sum_{j=0}^N Cov(\varepsilon_{t+i}, \varepsilon_{t+j}) = \sigma_D^2 + \sum_{i=0}^N \sum_{j=0}^N e_{t+i}^T \Sigma e$ where $\Sigma$ is the
covariance matrix of the update vector $\epsilon_t$. 

Proof of Proposition 6.3

The step size $X_t = D_t - \mu_C - e^T \epsilon_t$ is Normally distributed as $N(\mu_D - \mu_C, \sigma_D^2 + e^T \Sigma e)$. The distribution of $\hat{Y}_t$ in heavy traffic follows from Siegmund (1985) who characterizes the tail of a reflected random walk with Normal i.i.d. step sizes in heavy traffic. The shortfall process for the original system is not a reflected random walk. Assume that $Y_t$ and $\hat{Y}_t$ start at the same level at time 0 (without loss of generality). The two processes will coincide, until $\hat{Y}_t$ becomes negative. However, in heavy traffic, $\hat{Y}_t$ becomes a reflected Brownian motion, which never goes below zero. Therefore, in heavy traffic, the two processes $Y_t$ and $\hat{Y}_t$ coincide.

Lemma D.1 $Z_\infty = \max[\hat{Y}_\infty + L_0, \max_{1 \leq k \leq N} L_k]$ where $L_k = k\mu_C - \sum_{i=k+1}^{N} \sum_{j=1}^{k} \epsilon_{t-N+i,t-N+j} + \sum_{i=1}^{k} \sum_{j=N+1}^{N+i} \epsilon_{t-N+i,t-N+j} + \sum_{i=1}^{N} \sum_{j=N+1}^{N+i} D_i$.

Proof of Lemma D.1. Toktay and Wein (2001) prove a similar result for a problem with advance demand information using the MMFE model. We parallel their approach here for the case of advance supply information. $\hat{Y}_t$ is a reflected random walk with step sizes $M_t - C_t$ and can be written as:

$$\hat{Y}_t = \max \left\{ \hat{Y}_{t-N} + \sum_{i=t-N+1}^{t} (M_i - C_i), \sum_{i=t-N+2}^{t} (M_i - C_i), \ldots, (M_t - C_t), 0 \right\} \quad (D.2)$$

where $M_t - C_t = D_t - \mu_C - e^T \epsilon_t$ as derived in Proposition 6.2(b). We can revise (D.2) with some algebra as:
\[
\hat{Y}_t = \max[\hat{Y}_{t-N} + \sum_{i=t-N+1}^t D_i - N\mu_C - \sum_{i=1}^N \epsilon_i^T \epsilon_{t-N+i}, \sum_{i=t-N+2}^t D_i - (N-1)\mu_C - \sum_{i=2}^N \epsilon_i^T \epsilon_{t-N+i}, ..., (D_t - \mu_C - \epsilon_t), 0] = \\
\max[\hat{Y}_{t-N} + \sum_{i=t-N+1}^t D_i - N\mu_C - \sum_{i=1}^N \sum_{j=i}^{N+i} \epsilon_{t-N+i+j}, \sum_{i=t-N+2}^t D_i - (N-1)\mu_C - \sum_{i=2}^N \sum_{j=i}^{N+i} \epsilon_{t-N+i+j}, ..., (D_t - \mu_C - \epsilon_t), 0] \\
\]

We can write \(\sum_{i=1}^N C_{t,t+i} = N\mu_C + \sum_{i=1}^N \sum_{j=N+1}^{N+i} \epsilon_{t-N+i+j}\) Substituting revised version \((M_i - C_i \text{ replaced as above})\) of (D.2) into \(x_t = s - (\hat{Y}_t + \sum_{i=1}^N C_{t,t+i})\) will give:

\[
x_t = s - \max[\hat{Y}_{t-N} + \sum_{i=t-N+1}^t D_i - N\mu_C - \sum_{i=1}^N \sum_{j=i}^{N+i} \epsilon_{t-N+i+j}, \sum_{i=t-N+2}^t D_i - (N-1)\mu_C - \sum_{i=2}^N \sum_{j=i}^{N+i} \epsilon_{t-N+i+j} + \sum_{i=1}^1 \sum_{j=N+1}^{N+i} \epsilon_{t-N+i+j}, 2\mu_C + \sum_{i=t-N+3}^t D_i - \sum_{i=3}^N \sum_{j=i}^{N+i} \epsilon_{t-N+i+j} + \sum_{i=1}^2 \sum_{j=N+1}^{N+i} \epsilon_{t-N+i+j}, ..., \]

\(N\mu_C + \sum_{i=1}^N \sum_{j=N+1}^{N+i} \epsilon_{t-N+i+j}\), which leads to

\[
\hat{x}_t = s - \max[\hat{Y}_{t-N} + L_{t0}, \max_{1 \leq k \leq N} L_{tk}] \\
\]

where \(L_{tk} = k\mu_C - \sum_{i=k+1}^N \sum_{j=i}^{N+i} \epsilon_{t-N+i+j} + \sum_{i=1}^k \sum_{j=N+1}^{N+i} \epsilon_{t-N+i+j} + \sum_{i=t-N+k+1}^t D_i\)

Thus, \(L_t = (L_{t0}, L_{t1}, ..., L_{tN})\) is an \((N+1)\) dimensional multi-variate normal vector with \(E[L_{tk}] = k\mu_C + (N-k)\mu_D, \ Var(L_{tk}) = (N-k)s_D^2 + \sum_{i=1}^k p_i^T \Sigma p_i + \sum_{i=k+1}^N v_{N+1-i}^T \Sigma v_{N+1-i}\) where \(p_i\) is \((N+1)\) dimensional column vector whose last \(i\) elements are 1. Similarly, \(v_i\) is \((N+1)\) column vector with first \(i\) elements 1 (rest are zero). The distribution of \(L_{tk}\) is independent of \(t\), therefore we will represent it with some generic random variable \(L_k\). \(L_k\) is independent of \(\hat{Y}_{t-N}\) (Because \(\hat{Y}_{t-N}\) depends on \(\epsilon_k\) and \(D_k, k \leq (t-N)\)). This gives us:

\[
\hat{x}_\infty = s - \max[\hat{Y}_{\infty} + L_{t0}, \max_{1 \leq k \leq N} L_{tk}] \\
\]

Note that \(\mu_{t0} = N\mu_D\) and \(\sigma_{t0}^2 = \epsilon^T \Sigma \epsilon\).}

Proof of Proposition 6.4
We know from Lemma D.1 that $L_k$ is normally distributed for all $k$. Based on Proposition 6.3, $Y_\infty$ has a tail that decays as an exponential with rate $\theta$. To prove the theorem, we can first write that $\lim_{z \to \infty} P(Y_\infty + L_o > z) = C_Z e^{-\theta z}$ for some $C_Z$ since in the sum of an exponential and normal random variable, the exponential term dominates as $z \to \infty$.

Given the expression of $Z_\infty$ from Lemma D.1, we can also write:

$$P(Y_\infty + L_0 > z) \leq P(Z_\infty > z) \leq P(Y_\infty + L_0 > z) + \sum_{i=1}^{N} P(L_i > z).$$

An upper bound for the tail probability of a Normal random variable $R$ is $P(R > x) \leq \frac{K}{\sqrt{2\pi}} e^{-x^2/2}$ (see Gordon (1941)) for some constant $K$. Hence: $0 \leq \lim_{z \to \infty} \frac{P(L_i > z)}{P(Y_\infty + L_0 > z)} \leq \lim_{z \to \infty} \frac{K e^{\theta z}}{z C Z e^{\theta z}} = 0$

This proves that $\lim_{z \to \infty} P(Z_\infty > z) = \lim_{z \to \infty} P(Y_\infty + L_0 > z) = \lim_{z \to \infty} C_Z e^{-\theta z}$.

Let $h(z) = \int_{-\infty}^{z} e^{\frac{t^2}{2}} \Phi\left(\frac{t}{\sigma}\right) dt$, where $\Phi(\cdot)$ is the standard normal density. Then

$$h(z) = e^{\mu \theta + \frac{1}{2} \sigma^2 \theta^2} \Phi\left(\frac{z - (\mu + \theta \sigma^2)}{\sigma}\right). \quad (D.3)$$

If $R$ is standard normal, $P(Y_\infty + L_0 < z) = P(Y_\infty + \mu L_0 + R \sigma L_0 < z)$

$$= \int_{-\infty}^{z} P(Y_\infty + \mu L_0 + r \sigma L_0 < z | R = r) \phi(r) dr \quad \text{(since $Y_\infty$ and $L_0$ are independent)}$$

$$= \int_{-\infty}^{z} \left(1 - e^{-\theta \sigma^2} e^{-\theta (z - \mu L_0 - \sigma^2 L_0 r^2)} \right) \phi(r) dr = \Phi(z) - e^{-\theta \sigma^2} e^{-\theta (z - \mu L_0 - \frac{1}{2} \sigma^2 L_0^2)} \Phi(z) \quad \text{based on equation (D.3).}$$

$$\lim_{z \to \infty} P(Z_\infty < z) = \lim_{z \to \infty} P(Y_\infty + L_0 < z) = 1 - e^{-\theta \sigma^2} e^{-\theta (z - \mu L_0 - \frac{1}{2} \sigma^2 L_0^2)}.$$ Hence:

$$\lim_{z \to \infty} P(Z_\infty > z) = \lim_{z \to \infty} e^{-\theta \sigma^2} e^{-\theta (z - \mu L_0 - \frac{1}{2} \sigma^2 L_0^2)}$$

And based on $\mu L_0 = N \mu_D$ and $\sigma^2 L_0 = \Theta L_0$, we can define $\gamma = N \mu_D + \frac{1}{2} \Theta e^T \Theta e$ which gives $\lim_{z \to \infty} P(Z_\infty > z) = \lim_{z \to \infty} C_Z e^{-\theta z}$ where $C_Z = e^{-\theta \gamma}$.

Proof of Theorem 6.4
From the heavy traffic approximation from Proposition 6.4 we have
\[
\lim_{z \to \infty} P(Z_\infty > z) = \lim_{z \to \infty} e^{-\theta(z+\beta+\gamma)}. \quad \text{We also know that } s^* = F_{Z_\infty}^{-1} \left( \frac{b}{b+h} \right) \text{ from Theorem 6.3.} \quad \text{For large values of the backlog parameter } b, \text{ we can use this approximation for the tail of the } Z_\infty \text{ distribution, which leads us to } F_{Z_\infty}(s^*) = 1 - e^{-\theta(s^*-\gamma+\beta)} = \frac{b}{b+h}. \quad \text{Therefore } s^* = \frac{1}{\theta} \ln(1 + \frac{b}{h}) - \beta + \gamma, \text{ using the heavy traffic approximation from Proposition 6.4.}
\]