Pricing and Revenue Management with Limited Market Information

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Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy under the Executive Committee of the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2008
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ABSTRACT

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Serkan S. Eren

Traditional models from the revenue management literature assume that firms have full information about the market demand and consumer preferences. This thesis studies pricing, capacity allocation and product line positioning models for a firm with limited market information using relative performance criteria and maximum entropy estimation.

In our first essay, we examine different monopoly pricing mechanisms under limited customer willingness-to-pay (WtP) information. We use the competitive ratio and the maximum regret criteria to study a dynamic pricing, a third-degree price discrimination, and a second-price sealed-bid auction setting, where customers have private WtP drawn from common distribution that is unknown to the seller. We provide closed-form solutions for the optimal pricing policies and highlight important structural insights. We show that price skimming arises naturally as a hedging mechanism against two principal risks that the firm faces: first, the risk of foregoing revenue from pricing too low, and second, the risk of foregoing sales from pricing too high. We focus on the competitive ratio criterion and the dynamic pricing setting to illustrate how learning and partial information can be incorporated. Even limited learning, e.g., market information at a few price points, leads to significant performance gains with relative performance criteria, and the resulting policies have very
good revenue performance across all distributions.

In our second essay, we study the joint problem of product line positioning and pricing for a monopolist when consumer preferences and WtP are unknown. We extend classical models of horizontal and vertical differentiation to cover this uncertainty again using the relative performance criteria. Our analysis provides insights into practices observed in many real world markets. For the horizontal differentiation case, we show that the optimal decision for both criteria is to position products at equal intervals in the attribute space and to price them identically. For the vertically differentiated case, we show that the optimal policy consists of offering a number of the highest quality versions, and that the more ambiguity over customers' taste for quality, the more versions the firm should offer.

In our last essay, we change our focus to incorporating partial information in a dynamic forecasting and optimization cycle of capacity allocation. [21] illustrate, using a two class example, that most common forecasting methods lead to divergence and degeneration of optimal policies when used jointly with optimization routines in such a cycle; and call this phenomenon the “spiral-down effect”. We propose a tractable and intuitive approach based on maximum entropy (ME) distributions that readily incorporates and apply uncensoring to censored sales data in an intuitive manner. We show that the protection levels given by our algorithm avoids “spiral down” and converge to optimal values.
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Chapter 1

Introduction

Classical models from the economics, marketing, and revenue management literature assume that firms have accurate characterizations of the market demand and consumer preferences. In practice, however, there are many settings, such as introduction of new and innovative products, where one rarely has such full and accurate information. This source of model uncertainty may lead to significant revenue loss and may be insufficiently hedged against through the use of policies that do not explicitly incorporate it in their derivation. This thesis studies these issues for a monopolist operating in settings with limited market demand and consumer preference information. In our first two essays, we study different pricing mechanisms and the joint problem of product line positioning and pricing, respectively, under limited customer willingness-to-pay and preference information using competitive ratio and maximum regret criteria. In our third essay, we focus on incorporating partial information in a dynamic forecasting and optimization cycle of capacity allocation using maximum entropy distributions.
As a motivating example, consider a monopolist firm that offers a new product to a set of risk-neutral, heterogeneous consumers, each endowed with a private willingness-to-pay (WtP or valuation), which is an independent draw from a common distribution. The market information is summarized by the number of potential consumers, i.e., the market size, and the WtP distribution. Classical models would assume that both of these elements are known to the firm, and are used to determine the expected revenue maximizing price. How should the seller approach this problem if the market size and WtP distribution were unknown? What should be the form of seller's pricing policy in that case? How should that be adjusted to take advantage of partial demand information extracted, for example, from experimenting at a few price points?

There are two natural ways to specify this type of model uncertainty that lead to different formulations and different policy recommendations. The first one is stochastic, wherein the unknown preference and WtP distributions are assumed to be drawn from a given set of possible distributions according to some known probability law. The firm's goal is to optimize her expected revenues over all possible market model realizations. Main shortcoming of this approach is that it requires detailed information on the distribution of the model uncertainty, which itself may not be available.

The second approach is to use a formulation that adopts a worst-case perspective via a max-min criterion on expected revenue or profit. That is, one treats the unknown information as being controlled by an adversary ("nature") who seeks to construct instances that minimize the firm's profits for any given decision it makes. The firm then seeks policies that produce the maximum profit against this adversary. The difficulty with this approach is that it can lead to decisions that are driven by excessively extreme and pessimistic scenarios about the unknown information, e.g.,
by setting the WtP of all consumers equal to its minimum allowed value in the above pricing example irrespective of the firm’s pricing decision. To reduce this inherent conservatism, one typically imposes constraints on the decision set of the adversary, that are either ellipsoidal (see BenTal and Nemirovski [6], and ElGhaoui and Lebret [23]), or polyhedra (see Bertsimas and Sim [10], as well as Bertsimas and Thiele [11], Perakis and Sood [63]). In a similar vein, Lim and Shanthikumar [49] suggests using a relative entropy constraint to bound the distance of the WtP distribution from a nominal one. These approaches are usually grouped under the name robust optimization. Robust optimization essentially imposes an uncertainty “budget” to the adversary in the form of additional constraints exemplified above so as to reduce the pessimistic nature of the associated solution. However, the selection of this budget is often arbitrary, and therefore, amounts to a statement about the degree of uncertainty in the information, which is not easy-to-interpret. More to the point, robust optimization is often a computational formalism and does not usually lend itself to finding analytic solutions that provide structural insights.

An alternative approach to reduce the conservatism of max-min formulations, while maintaining their appealing low informational requirements, is through the use of relative performance criteria such as competitive ratio and maximum regret. These measure the performance relative to a fully-informed decision maker. In broad terms, they are defined as follows: the firm first selects a policy $\pi$, and the adversary selects a worst-case distribution function for the unknown consumer attribute, $F(\cdot)$; in the above example, $\pi$ could be the posted price, and $F(\cdot)$ the WtP distribution. Let $R(\pi, F)$ be the actual expected revenue earned for the pair of actions $\pi$ and $F(\cdot)$, and $R(\pi^*(F), F)$ be the maximum expected revenue the firm could have extracted if she knew the selected distribution $F(\cdot)$; here $\pi^*(F)$ denotes the optimal policy if $F$ was
known. Then, the competitive ratio is given by

$$c^* = \max_{\pi} \min_{F} \frac{R(\pi, F)}{R(\pi^*(F), F)},$$

while the minimum regret is

$$r^* = \min_{\pi} \max_{F} \left[ R(\pi^*(F), F) - R(\pi, F) \right].$$

That is, in both cases, the firm strives to minimize the relative difference from the maximum revenues it could have extracted in the full-information case. To contrast, the max-min criterion takes the form

$$\max_{\pi} \min_{F} R(\pi, F).$$

Relative performance criteria implicitly constrain the actions of the adversary without having to impose additional constraints, and often result in intuitive policy recommendations. They prevent trivial choices for the adversary, such as choosing the minimal WtP for all consumers in the above example, which would result in a bad revenue outcome for the firm no matter what the policy, because the fully-informed manager would also be harmed in such instances. In that sense, they allow us to distinguish between "bad market conditions" and "bad decisions". Our belief is that these criteria might be of particular value in explaining some observed behavior in product and revenue management practice due to its proximity to common performance evaluation schemes: for example, a revenue/product manager often has to select a policy before the uncertainty of the market reveals itself. However, her own performance is usually evaluated after the selling horizon, when the uncertainty is at least partially revealed, e.g., good/bad season; and her actions are often compared to what could have been optimally done in such conditions. Therefore, it is plausible
to hypothesize that a risk averse revenue/product manager, given the opportunity, could (to some extend) choose to hedge her own performance evaluation risk. In fact, these criteria gained significant popularity in decision theory literature in economics during the last decade.

Relative performance criteria originate from and are used extensively in the statistics and the computer science literature, and have recently been applied in pricing and operations management problems. Specifically, Ball and Queyranne [4] use a competitive ratio criterion for a single-resource capacity allocation problem, while Bergemann and Schlag [8] and Perakis and Roels [61] adopt the regret criterion to study the monopolist pricing and the newsvendor problems, respectively. Lan et al. [43] generalize Ball and Queyranne's analysis and extend it to cover the regret criterion. Perakis and Roels [62] apply similar techniques for network revenue management.

Some weaker notions of adversary for relative performance criteria have also been studied. Alternatively, the “algorithm designer” (e.g. the firm in our case) has been allowed to randomize over different strategies. For example, the so-called “oblivious adversary” selects the distribution function $F(\cdot)$ without observing the firm’s pricing policy. As another example, the “adaptive online adversary” is only allowed to see the decisions of the firm in the past stages in a dynamic setting before selecting the input for the next stage. The adversary in our work corresponds to the notion of “offline adversary”. Ben-David et al. [5] examine and compare above notions of adversary as well as randomized policies. They show that randomized policies offer no benefit for the firm in the presence of an offline adversary, which implies that the optimal competitive ratios we drive for the pricing mechanisms below remain valid even when the firm is allowed randomization.

In our first two essays, we also use competitive ratio and maximum regret criteria
to study pricing and product line positioning decisions of a monopolist in settings where the firm has limited information about the WtP distribution (for the pricing problems in the first essay), and the distribution that characterizes the consumers' preferences over product variety and quality (for the product positioning problems in the second essay). First, we examine monopoly pricing mechanisms of dynamic pricing, third-degree price discrimination, and second-price sealed-bid auction under limited WtP information. We assume customers have private WtP drawn from common distribution that is unknown to the seller. We provide closed-form solutions for the optimal pricing policies and highlight important structural insights. We show that price skimming arises naturally as a hedging mechanism against two principal risks that the firm faces: first, the risk of foregoing revenue from pricing too low, and second, the risk of foregoing sales from pricing too high. We focus on competitive ratio criterion and the dynamic pricing setting in detail to illustrate how learning and partial information can be incorporated. We show that even limited learning, e.g., market information at a few price points, leads to significant performance gains with relative performance criteria, and the resulting policies have very good revenue performance across all distributions.

In our second essay, we study the joint problem of product line positioning and pricing for a monopolist when consumer preferences and WtP are unknown. We extend classical models of horizontal and vertical differentiation to cover this uncertainty again using the relative performance criteria. Our analysis provides insights into practices observed in many real world markets. For the horizontal differentiation case, we show that the optimal decision for both criteria is to position products at equal intervals in the attribute space and to price them identically. For the vertically differentiated case, we show that the optimal policy consists of offering a number of
the highest quality versions, and that the more ambiguity over customers' taste for quality, the more versions the firm should offer.

A potential practical shortcoming of an approach based on the relative performance criteria is that it may lose analytic and computational tractability as one tries to incorporate partial information about the unknown demand model primitives. In that respect, most papers adopt this framework to derive insights about the structure of good policies and the effect of ambiguity on system performance, as opposed to the computation of implementable policies. One exemption is Perakis and Roels [62], which incorporates partial demand information using the probabilistic tight bounds for mean and variance specifications derived by Bertsimas and Popescu [9]. Another is Bergemann and Schlag [8], which allows for the unknown distribution to be within a distance of a nominal distribution that could encapsulate prior information. Both papers work only with the regret criterion and do not generalize easily to the competitive ratio criterion, and neither framework seems to allow for an easy way in which to incorporate intuitive information extracted from past sales that usually translate to fractiles of the WtP distribution. We illustrate how to incorporate the latter type of demand information in a tractable way in the dynamic pricing section of our first essay. Through numerical examples, we show that relative performance criteria, combined with partial information, not only provides structural insights but also robust policies that have very good revenue performance across many market conditions.

Our models in the first two essays are mainly static in nature in that the basic problems boil down to one-stage (or two-stage for learning) dynamic games between the firm and the adversary. In a more dynamic setting, where available information is constantly updated, a different perspective may be needed to incorporate available stream of information, as the inherent complexity of the relative performance criteria
often hinder integration of such dynamic information. In this respect, we change our focus to incorporating a continuous stream of partial information in a dynamic forecasting and optimization cycle of capacity allocation in our last essay. Using a capacity allocation example with two demand classes, Cooper et al. [21] illustrate that most common forecasting methods lead to degeneration of optimal policies when used jointly with optimization in such a cycle; and call this phenomenon the “spiral-down effect”. We propose a tractable and intuitive approach based on maximum entropy (ME) distributions that readily incorporates partial demand information in the form of censored sales data. We show that the capacity controls given by the ME algorithm we propose avoids the spiral down effect and converge to the optimal values. We make use of adaptive algorithms and stochastic approximations in our analysis for two fare-classes and provide a heuristic algorithm for the multifare problem. The solution to the ME problem simultaneously applies uncensoring to the raw sales data in an intuitive and statistically sound manner.

In the remainder of this chapter, we introduce the specifics of our three essays, review the relevant literature, and provide a general outline of our results.

1.1. An essay about monopoly pricing schemes with limited demand information

Our first essay adopts relative performance criteria introduced above to study three classical pricing settings. Specifically, we look at the settings where a monopolist offers one product to a market of heterogenous consumers, each endowed with a private WtP, which is an independent draw from a common but unknown distribution. We
study three variants of the firm's pricing problem that differ in terms of their selling format. In the first one, the firm has the ability to change its price over time, and its key decision is to figure out a pricing policy (how much to charge and for how long to stay at each price point) that would perform well even though the firm does not know the underlying consumers' WtP distribution. We also analyze the effects of learning in this setting. The second variant looks at the case in which the firm can (third-degree) price discriminate its market, i.e. segment the market into subgroups that each can be charged a different price, but where the firm does not know what the representative WtP and relative size of each of these market segments are. As a special case of that model as the number of market segments grows large, we also discuss the case of first degree price discrimination, where a different price is offered to each customer. In the third variant, the firm attempts to sell one unit of the product using a second-price sealed-bid auction, again without knowing the underlying WtP distribution.

The first of the common pricing schemes we consider is the dynamic pricing. We allocate majority of our discussion to this setting, as it is the most commonly applied one in revenue management practice; and, as it provides practical extensions to learning as well as insights into the tradeoffs of the firm when faced with uncertainty. Dynamic pricing is concerned with adjusting prices to regulate demand over a finite sales horizon to maximize revenue. Price skimming is a commonly used example of such a policy in many industries like airlines, hospitality and fashion goods. Clearing excess inventory and perishable products – rather than salvaging leftover items at low value at the end of the sales horizon – has been proposed as a possible explanation for this practice; see Talluri and van Ryzin [73] for a review of this body of work. Another possible explanation for the use of dynamic pricing policies is as a hedging
mechanism in settings where demand is uncertain; see Lazear [46] for an analysis of this problem and Pashigan [59], and Pashigan and Bowen [60] for empirical evidence of this explanation. Harris and Raviv [34] showed that a price skimming policy may emerge as the optimal mechanism when demand is uncertain. Our work shows that such a policy will optimize the firm's relative revenue performance when the demand model is unknown.

In our dynamic pricing setting, the firm has the ability to change its price over time, and its key decision is to figure out a pricing policy (how much to charge and for how long to stay at each price point) that would perform well even though the firm does not know the underlying consumers' WtP distribution. An alternate interpretation of this price skimming policy is to treat the relative length of time over which a price is offered as a probability of that price point, thus interpreting the proposed price scheme as a randomizing pricing policy; this was done in Bergemann and Schlag [8].

Our model is deterministic, disregarding the stochastic variability of the sales process, e.g., due to its Poisson nature. This allows us to emphasize the effects of "first order" uncertainty introduced by not knowing the sales rate itself at a selected price point, as opposed to "second order" fluctuation due to stochastic nature of the process. Our analytical contributions are the following: 1) The competitive ratio and maximum regret optimization problems are solvable in closed form, offering a detailed description of the optimal policies for the seller and the adversary, the tradeoffs faced by the seller, and a precise characterization of the resulting revenue loss. 2) We extend our formulation and results to a two period setting that allows the seller to learn from the sales observations in period one. 3) We address the situation where the seller only has limited price experimentation capability, or has limited past
sales information. Observing the demand at a price point gives cumulative demand information above and below that price point and essentially decomposes the problem into simpler subproblems in the respective regions that are readily solvable using linear programming techniques. As a special case of practical interest, we also study the "ex-post" problem that allows the seller to take into account actual demand observation data in her price optimization and performance analysis decisions.

We highlight three observations from our work that are of potential interest. First, the worse case market scenarios for the firm, as captured by the WtP distribution selected by the adversary, correspond to homogeneous markets where all consumers have the same, yet unknown, valuation. Mathematically, this corresponds to an extreme point -unit mass- distribution whose exact position is uncertain, forcing the firm hedge against opposing risks at each price point: first, the risk of foregoing revenue from pricing too low, and second, the risk of foregoing sales from pricing too high. In response, the firm's strategy tries to hedge against this exposure.

The extreme point nature of the adversary's strategy has appeared elsewhere in the literature. One example from decision theory is Smith [71], which studies the expectation maximization problem among a set of probability distributions. He shows the equivalence of that problem to a linear program and, as a result, recovers extreme point distributions as potential solution points. Our objective function is not linear (and not even convex), and our results do not follow from Smith's observation. However, such a structure emerges in many papers because the inner optimization step can often be reduced to a quasi-convex maximization problem over the probability simplex, which admits an extreme point solution. In settings with learning or with partial demand information, the worst-case distribution retains some of its structural form by having point masses at distinct valuations, but becomes more dispersed. We
give a complete characterization of the latter and discuss several examples in Section 4.

Second, we highlight that in settings with limited or no market information it is optimal for the firm to adopt a price skimming policy to minimize the risk of lost sales and foregone revenue that could result from mis-estimating the market characteristics. To contrast, if the firm knew the customer WtP distribution, then it would be optimal to charge a static price over the entire sales horizon. This result suggests that lack of market information could offer one possible justification for the use of dynamic pricing policies. Analytically, the precise form of the resulting pricing policy ensures that both the firm and the adversary are indifferent with regard to the positioning (i.e., the representative valuation) of the market.

Third, the effect of learning is both significant and quick in the sense that even a few observations at different price points can provide considerable lift in the revenue performance of the proposed policies. Both the resulting competitive ratio, which is a worse case bound, and the actual performance relative to some underlying WtP distribution unknown to the seller, improve considerably. In the case where the seller is not restricted in the number of price points that she can experiment at, we show that it is optimal to use a price skimming policy during a “learning” period. This achieves full learning of the demand model and allows the seller to charge the optimal (full-information) price in the remainder of the sales horizon. Often, there may be practical constraints that link the firm’s pricing decision over time, e.g. retailers hesitate to increase prices after an early mark down. We show that in such settings it is still optimal to adopt a price skimming policy, but in this case the seller is willing to sacrifice performance due to the learning phase so as to retain adequate pricing flexibility in the remainder of the sales horizon.
Incorporation of partial information is typically done in a Bayesian setting under some parametric assumptions for the WtP distribution and using conjugate pairs of distributions to maintain tractability; see, e.g., Lobo and Boyd [51], Aviv and Pazgal [3], Araman and Caldentey [2], and Farias and Van Roy [25]. Assuming a parametric family of distributions for the unknown demand runs the risk of model misspecification due to the arbitrariness of that assumption. Similar to our work, another subset of literature uses non-parametric approaches, which make minimal distributional assumptions and often involve some form of an adaptive learning algorithm; see, e.g., McGill and van Ryzin [55], Huh and Rusmevichientong [36], and Eren and Maglaras [24]. An interesting recent paper in the latter set is Besbes and Zeevi [12] that studies a prototypical dynamic pricing problem in a stochastic environment. Two important insights from their work for purposes of our work is that they show that: a) in settings with long sales horizons and large market sizes, an asymptotically optimal policy in terms of its relative regret is to divide the sales horizon in two phases that are dedicated to learning and revenue optimization, respectively; and b) the uncertainty due to the stochastic nature of the demand arrival process is indeed negligible in such settings.

The second pricing scheme we study is the third degree price discrimination setting. Price discrimination is the practice of charging different people different prices for the same goods or services. In the third degree price discrimination case, the monopolist firm is capable of accurately differentiating between consumer segments through an observable attribute. Each segment pays a different price for the same product, and the segments with higher valuations pay more than those with lower valuations. Student or senior citizen discounts are common examples of this practice. A more recent application domain is in cross-selling that is done in call centers. This
is the practice that the agent attempts to cross sell some product or service after completing the handling of the original customer request. At the time of that cross selling opportunity, the call center can indeed classify the customer to a particular segment and tailor its product offering.

Pigou was the first one to formally study this practice [65], which was further studied by Robinson [67]. Since then, this setting received substantial attention in economics literature and was extended to cover more general setups. The main focus in economics literature has been the output level and the welfare impact of price discrimination with respect to a monopolist charging a single uniform price to the whole market. For a contemporary treatment and substantial recent contributions, the reader is referred to Schmalensee [56] and Varian [74] [75].

For the third degree price discrimination case, assuming that the market can be segmented on the basis of the customers' WtP, we show that it is optimal for the firm to set the price for a particular segment equal to its minimum WtP value within the segment. Moreover, if the firm has the ability to choose the market segmentation, it will set equal the (absolute or relative) sizes of the WtP intervals. This is somewhat a similar strategy to the price skimming policy above in that the firm segments the total market size into smaller subgroups by skimming through the whole valuation range. However, the adversary is strong enough to force the firm to price at the lowest WtP value within each segment. Combined with the segmentation strategy, this results in a piecewise (step-like) price menu being offered to the market.

The third pricing mechanism we consider is the the second price, sealed-bid auction. The choice of this mechanism is motivated by the fact that irrespective of the number of bidders and the underlying WtP distribution, it is still a dominant strategy for each potential buyer to bid her/his true WtP within this mechanism, thus
simplifying the analysis of the buyer's decision using relative performance criteria.

The second price auction setting we consider is a sealed-bid auction, where bidders submit written bids without knowing bids of the other people in the auction. The rules of the auction is such that the highest bidder wins, but pays a price equivalent to the second highest bid. This auction was first proposed and studied by Vickrey [76], and is therefore also named the “Vickrey auction”. In his seminal 1961 paper, Vickrey not only showed that each bidder has a dominant strategy of bidding her true value, but also established an early form of the “revenue equivalence” theorem. The second price auction is an optimal ex-post efficient pricing mechanism that requires very few assumptions, and is therefore robust in nature. Truthful bidding remains a dominant strategy (and the mechanism works) in the absence of assumptions such as “independent valuations”, “identical valuation distributions”, or “risk neutrality”, which are usually necessary for other auction types. For a contemporary treatment of auctions and literature survey, the reader is referred to McAfee and McMillan [54] and Klemperer [64].

Ambiguity in auctions is a recent topic in the literature. Lo [50] analyzes sealed bid auctions within a max-min expected utility framework for risk averse bidders. Ozdenoren [58] extends and generalizes Lo's results. Bose et. al. [16] show that the seller needs to fully insure risk averse buyers in a max-min expected utility framework. Levin and Ozdenoren [47] analyze auctions where the number of bidders is uncertain. Chen et. al. [20] consider a setting where both ambiguity averse and ambiguity loving behavior is allowed. For interesting empirical experiments involving ambiguity in auctions, we refer the reader to Sarin and Weber [68], and to Chen et. al. [20]. For competitive analysis of auction mechanisms with respect to “other selling mechanisms”, we refer the reader to Goldberg et al. [29], [30].
In the second-price auction setting, with only range information about the buyers' WtP, we show that it is optimal for the seller to set his reservation price so as to balance the risks of not selling the item with the risk of selling it at a low price just as in the dynamic pricing setting above. The resulting reservation price has similar properties to the one derived under the classical revenue maximizing setting, but, is slightly lower, reflecting risk aversion.

The outline of our first essay is as follows: Section 2.1 introduces the prototypical dynamic pricing problem with no market information and no learning. Sections 2.2 and 2.3 study two period extensions that allow for different degrees of learning. We study the third degree price discrimination and the second-price auction settings in Section 2.4.

1.2. An essay about product line positioning without market information

Positioning and pricing a product line is a central problem in marketing. (See the survey papers Dobson and Yano [22], Green and Krieger [31], Krishnan and Ulrich [41], Manez and Waterson [52], Ramdas [66].) In product line positioning, a firm must decide which different versions of a product to offer. Versions may differ in ways that appeal to the heterogeneous tastes of consumers, such as offering different colors, sizes or flavors - called horizontal differentiation - or they may differ in their level of quality and performance - called vertical differentiation. Product line positioning decisions are intimately related to the prices charged, too; because product versions closer to customers' preferences for variety or quality provide them more utility and can
therefore support higher prices. At the same time, differentiated versions of the same product are almost always substitutes, so that lower prices for one version cannibalize demand from the other versions in the product line. Hence, positioning and pricing a product line should be coordinated decisions.

The difficulty in practice is that in many cases little is known about customer preferences for various product attributes and the experimentation necessary to determine their preferences (e.g., conjoint analysis, see Green and Srinivasan [32], Green et al. [33])) may be either too costly or too time consuming and/or potentially unreliable, as in the case of an innovative product that customers have never experienced using. In such cases, firms must make product line positioning and pricing decisions with little information about customer preferences. Most of the literature on product line positioning and pricing assume the firm has full information on preference. In our work, we consider the opposite extreme in which the firm has minimal information about customer preferences. Understanding both these extremes helps build intuition because real life practice lies somewhere between the two. However, the case of limited information has received no attention to date in the research literature. Our work fills this gap. Even when firms have some preference information, it is often limited and unstructured. The issues then are filtering out and organizing the useful information, and incorporating these into the decision making process in a meaningful manner. We try to address some of these issues in our first and third essays.

We use competitive ratio and maximum regret criteria to study product line positioning and pricing decisions in a setting where the firm does not have information about the distribution that characterizes the consumers' preferences over product variety (in the case of horizontal differentiation) or quality (in the case of vertical differentiation).
Our approach to modeling horizontal product variety is based on classical locational models, in which each version of the product is mapped into a location in a product attribute space and each consumer’s location represents her most preferred values of the product attributes (her “ideal point”). The distance between a customer’s ideal point and the locations of different product versions provides a measure of their disutility from consuming a less-than-ideal product. Given a distribution of customer locations or preferences, one can then analyze issues of optimal product line design, positioning and pricing. Of course in our case, this distributional information is what is unknown.

The seminal paper in this area belongs to Hotelling [35], who considers a linear one-dimensional attribute space. Vickrey [77] is credited for formulating the circular model, which gets rid of the boundary effects of the linear model. In 1966, Lancaster [44] offered an alternative approach in which the utility of a choice is determined by a parametric function of the consumer characteristics and product attributes. This theory significantly increased interest in locational product differentiation models. Salop [69] provides the most complete treatment of this setup in his seminal paper for monopolistic competition, where he shows a symmetric equilibrium along the product space (i.e. the circle) exists with equal prices, and all the consumers in the market (i.e. around the circle) are served. Several extensions of this basic game theoretical model has been made covering different scenarios of the market structure. Further interested reader is referred to Lancaster [45], Manez and Waterson [52], and Ramdas [66].

Our work is also closely related to the product line positioning literature. The product line positioning problem itself is a complex multi-facet problem touching on marketing concerns (consumer utility and product attributes modeling, pricing and
positioning to maximize market share or revenues, etc.), operations concerns (costs of production, configuration of supply chains for variety, etc.), and engineering design concerns (product architecture, shared versus unique components, etc.). There are several survey papers structuring the literature around these three areas. (Again, see the survey papers of Dobson and Yano [22], Green and Krieger [31], Krishnan and Ulrich [41], Manez and Waterson [52], Ramdas [66].) Prior work in the literature can be roughly characterized as follows: most directly uses or is inspired by the utility framework of spatial differentiation models. Generally, models consider a finite number of possible product offerings and the aim is to choose which of these to offer with the objective of maximizing market share, revenues, or profits. However, there are a number of papers in operations literature that solely consider cost minimization or supply chain configuration. It is assumed in all this literature that customer preferences/valuations for each possible product offering and the potential market size at each such point are known or can be estimated using conjoint analysis. Moreover, most of the models do not treat price as a decision variable, though several treat it as part of the attribute space. The corresponding problems are usually mixed integer programs and exact analytical solutions are generally not available. Algorithmic solutions usually consist of greedy or simulation based heuristics (e.g., simulated annealing).

Our work is aligned more closely with the marketing view of the problem explained above. We base our model on the classical locational attributes and linear utility framework as above. However, unlike the product line positioning literature above, we try to avoid additional assumptions on top of this basic utility set up. Specifically, we allow for uncertainty/ambiguity in customer valuations, and are interested in analyzing the effects of this uncertainty on optimal policies. Our objective
is to maximize revenues; we do not consider production costs (though variable costs can be added without much change). Further, we only consider a single dimension of variety for simplicity. Unlike most of the literature, we treat both variety and price as continuous decision variables; hence, an infinite number of possible product offerings/locations is allowed. While these assumptions are highly stylized, we are able to derive analytical closed-form solutions for optimal positioning and pricing policies which provide structural insights and permit sensitivity analysis with respect to problem primitives, allowing us to gain intuitive understanding of the underlying issues related to uncertainty.

As mentioned, we also analyze the case of vertical positioning, in which all consumers have a common ordering of their preferences for different versions, i.e. every consumer agrees that a certain version $i$ is better (in terms of what literature denotes as "quality") than another version $j$. Our model of vertical positioning is also classical, and is based on the early influential work of Mussa and Rosen [57], who introduce a framework that makes use of a linear utility function of quality. This utility framework has been widely adopted by both following researchers and practitioners. In their seminal paper, Shaked and Sutton [70] use this model to show that firms can sustain positive profits even under price competition, contrary to the classical "zero-profit" result of Bertrand price competition, when they are allowed to choose the quality levels of their products under monopolistic competition. Assuming convex costs as a function of product quality, Mussa and Rosen [57], and later Maskin and Riley [53] and Kim [40], show that a firm can increase profits by offering vertically differentiated products to customers with heterogeneous tastes for quality.

These models were widely adopted as they successfully explained product differentiation seen in the traditional manufactured goods. However, the theory fell short
of explaining product differentiation, or “versioning”, for information goods. Information goods are particularly relevant in our case since most production costs are sunk development costs and thus profits maximization coincides with revenue maximization, which is the case we analyze. Early works using the linear utility function for information goods conclude that there are no gains to product differentiation for information goods and the product should be produced and supplied only at the highest quality levels to prevent cannibalization (see Bhargava and Choudhary [13], Jones and Mendelson [39], Acharyya [1]). However, empirical evidence (see Ghose and Sundararajan [28]) shows that information goods have high levels of differentiation. Only the highest quality is produced, as predicted by the above works, but the product is then degraded afterwards and offered at varying levels of quality in the market. In order to explain this practice, several models with different and more complex utility functions have been proposed, but none has been as widely accepted as the linear utility function of quality. Finally, Bhargava and Choudhary [14] demonstrated that previous insights about suboptimality of vertical differentiation are not robust, and showed that quality differentiation does indeed occur under a general class of utility functions and sunk costs assumptions. They conclude that the higher the heterogeneity in consumers’ taste for quality, the more likely vertical differentiation is and that the highest quality version is always offered in the product bundle. Other popular marketing texts (see for example, Lilien et al. [48]) also conclude that versioning is attractive when consumers are sufficiently heterogeneous.

More recently, there have been attempts to combine vertical and horizontal differentiation models by allowing firms to compete on both dimensions at the same time. There are also numerous other work that provide extensions to the above product differentiation settings which we cannot mention due to space limitations. We refer
the interested reader to the survey papers of Manez and Waterson [52], and Ramdas [66] to learn more about these.

Finally, a remark on terminology related to competition is in order. Almost all the works mentioned above study "monopolistic competition", in which different firms compete with each other across vertical and/or horizontal product attributes in the same market space under full information about customer preferences. Our work differs in two fundamental ways. First, we focus on the product line decisions of a monopolist. Second, our monopolist "competes against nature" rather than competing against other firms, in the sense that it is the unknown customer preferences that drive the firm's decision making rather than the actions of other firms. In particular, we caution the reader not to confuse the "competitive ratio" criterion with the "monopolistic competition" concept found in prior literature. The competitive ratio term used here originates from statistics and computer science literature.

Our findings and contributions can be summarized as follows: in Section 3.1, we consider the horizontal positioning of a monopolist's product line using Salop's classical circular model of spatial differentiation. For both competitive ratio and maximum regret criteria, we first derive the optimal pricing policy for a given product line with fixed attributes. We show that the optimal price vector depends on the maximum attribute difference among neighboring products. We also show that the worst-case performance decreases (i.e. the competitive ratio decreases or the regret increases) as the consumers' sensitivity for differentiating attribute increases, consumers' nominal valuations for their ideal product decreases, or the maximum attribute difference among neighboring products increases. Then, we show that for both criteria, the optimal positioning and pricing policy is to position products at equal intervals in the attribute space and to price them identically; that is, to evenly span the product
space with uniformly priced versions of the product. This type of positioning and pricing of a product line is frequently observed in practice (e.g. colors of a t-shirt, flavors of ice cream, i-Tunes downloads, etc.) and our results are consistent with these observations. Of course there are other explanations for such practices, such as operational simplicity and concern for perceived fairness. Still, there has been little formal theoretical justification of the optimality of such policies in the literature; our theory provides one explanation. It also provides closed-form pricing policies and performance measures which give insight into the economics of product positioning with minimal market information.

In Section 3.2, we study the vertical positioning of a product line using the linear utility of quality framework of Mussa and Rosen. Again, we first derive the optimal pricing policy for a fixed number of versions with given quality levels. Furthermore, consistent with Bhargava and Choudhary's results, we show that the optimal policy consists of offering some number of highest quality versions, which we call nested quality offerings; and that the number of versions offered increase as the heterogeneity and ambiguity in consumers' taste for quality increases. This result is consistent with empirical evidence and conclusions about versioning in information goods in literature. (See again Ghose and Sundararajan [28] and Lilien et al. [48].)

For both the case of horizontal and vertical positioning, we solve for the optimal policy in closed form, which provides a detailed description of the optimal pricing and positioning strategies for the firm and the resulting worst-case revenue loss. Our analysis, while stylized, provides insights into practices observed in many real world markets.
1.3. An essay about maximum entropy estimation in capacity allocation problems

One of the key challenges of revenue management systems in dynamic settings is to accurately forecast demand when one only has access to observed (censored) sales data. As is well known in the area, common uncensoring techniques and the interaction of forecasting and revenue optimization routines may prevent these systems from making optimal decisions in a dynamic setting; c.f., Boyd et al. [17] and Cooper et al. [21]. In this essay, we propose a tractable and intuitive algorithm for the dynamic (airline) capacity allocation problem based on maximum entropy (ME) distributions. We show that the proposed ME algorithm can readily incorporate censored sales data and that it leads to control decisions (in the form of capacity protection levels) that converge to optimal ones for the underlying demand distribution.

Revenue management systems consist mainly of forecasting and optimization modules that operate jointly. Main goal of the former module is to provide accurate demand forecasts. Once such a forecast is available, the optimization module provides policies to be implemented, and as a result, the new sales observation are used to update the previous forecast. We denote this iterative feedback loop as the “cycle of joint forecast and optimization”. One of the main issues in practice is the effect of optimization on future forecasts in such interactive systems, which result in possible accumulation of errors through time. In the scope of the airline capacity allocation problem we study, this can described as follows: consider a setting with two fare-classes with independent random demands that arrive sequentially in nonoverlapping intervals. The low fare demand is realized before the high fare demand; and the airline's problem is to decide on the number of seats to reserve for the random high
fare demand in the future. This form of capacity control is denoted as the protection-level. In order to find protection levels at any period, past sales data is accumulated to form demand forecasts. Note, however, that the sales data is censored and has to go through some uncensoring step first. The airline uses its forecast to optimize its protection level, implements this control, and observes the corresponding new sales, which is then used as the input for the next period’s forecasting step. However, if the airline gets an incorrect forecast at any iteration, it could lead to a wrong protection-level; and the resulting new observation would be biased as well, which could in return make the next forecast even less accurate. This bad cycle may produce protection-levels that degenerate further away from the optimal ones at each iteration. This phenomenon has been named as the “spiral-down effect”. Cooper et al. [21] analyze this effect and show that the protection levels and forecasts produced by many common algorithms degenerate consistently and converge to suboptimal levels due to joint optimization/forecasting interaction. Converging to suboptimal levels is a particularly dangerous scenario in practice, as forecasts consistently (and incorrectly) match observations, making the problem harder to detect.

At the core of the above problem lies the issue of uncensoring the sales observations. Censored demand observations provide only information about the fractiles of the underlying demand distribution. For example, if the number of available seats is $L = 50$ and the observed sales is $S = 50$, this observation corresponds to the event $\{D > 50\}$ for random demand $D$. Majority of the existing uncensoring techniques in practice consist of discarding such censored events or inflating uncensored observations heuristically. Weatherford and Polt [78] show that none of these uncensoring techniques avoid the “spiral-down” effect. Rather than trying to heuristically interpret the event $\{D \geq 50\}$ in the above example, we propose to use the framework of
Maximum entropy distributions:

Entropy is defined as $H = - \int_C f(x) \ln f(x) \, dx$ for a continuous distribution or as $H = - \sum_{j \in C} f_j \ln f_j$ for a discrete distribution where $f(x)$ and $f_j$ denote the probability density function and the probability mass distribution respectively on some set $C$. It measures disorder or randomness of a probability distribution. ME criterion provides a means of nonparametric estimation in statistics. Given prior information, ME distribution can be interpreted as the "most random", or as the distribution that is "maximally uncommitted/unbiased" with respect to unknown while satisfying given specifications and prior information (see Jaynes [38]).

Prior information for many estimation problems takes the form of observed frequencies, constraints on fractiles, and moments of certain functions. Some common estimation approaches include the maximum likelihood estimation and Bayesian updating, which entail imposing a parametric distribution to such data. However, by constraining ourselves to a specific parametric family of distributions, we introduce artificial estimation error to the problem; and at the worst-case, we might be forcing a structure that is significantly different from that of the actual distribution. ME estimation avoids this possible problem. Furthermore, many common parametric distributions are special cases of ME distributions; e.g., see Table 1.1.

Table 1.1: Common parametric distributions as special cases of ME distributions.

<table>
<thead>
<tr>
<th>Specification Type</th>
<th>ME Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range: $S = [a, b]$</td>
<td>$U [a, b]$</td>
</tr>
<tr>
<td>Mean: $\mu$</td>
<td>$\exp(1/\mu)$</td>
</tr>
<tr>
<td>Mean and variance: $\mu, \sigma$</td>
<td>$N(\mu, \sigma)$</td>
</tr>
</tbody>
</table>

ME estimation is analytically convenient as it allows for partial/prior information
to be incorporated as linear constraints to a convex optimization problem, solutions to which are readily available in closed-form. For example, for the following standard form ME estimation problem, which our formulation (4.4) can be reduced to,

$$\max_f \left\{ -\sum_j f_j \ln f_j : A f = b, \sum_j f_j = 1, f \geq 0 \right\},$$

the solution has the form

$$f_j^* = \frac{e^{\sum_{i=1}^n \lambda_i A_j}}{\sum_{j=1}^S e^{\sum_{i=1}^n \lambda_i A_j}} \quad j = 1 \ldots S,$$  \hspace{1cm} (1.1)

where \(\lambda\) denotes the vector of optimal Lagrange multipliers corresponding to constraints \(A f = b\) above. One could argue that the family of maximum entropy distributions is itself parametric according to (1.1), however, the above form is general enough to recover many parametric families of distributions; see again Table 1.1. The ME estimation also has a sound statistical basis. It can be shown that given data from a statistical experiment with unknown underlying probability measure, the distribution that maximizes entropy is the one that is most likely to produce the data at hand among the set of all consistent distributions (see Jaynes [38] for further discussion).

Despite its desirable properties, ME estimation has not received as much attention in academic research as other estimation techniques. One of the few papers that deal with “maximum entropy” in revenue management literature belongs to Bilegan et al. [15] who simply formulate a dual geometric program for the convex ME problem for capacity allocation and demonstrate how to solve it in a short paper.

We use ME estimation in the uncensoring step for the above airline capacity control setting. We keep track of the frequency of uncensored and censored observations. As noted above, the frequency of uncensored observations at any level provides only a
lower bound for the actual empirical demand distribution at that level, because some of the actual demand might have been realized as censored observations at lower levels. These can be incorporated as fractile constraints to our ME formulation in (4.4). Actually, censored observations also supply additional information (and constraints) as explained further below. Other information in the form of seasonal data or subjective opinion can be incorporated as linear constraints as well. In our analysis, we first describe our ME algorithm for the two fare-class problem in Section 4.1, prove the convergence of protection levels produced to the optimal ones for the underlying demand in Section 4.2, and finally, provide an extension of the algorithm to multifare problems in Section 4.3.

The idea behind our ME approach is that the algorithm tries to push the current control a little upwards (or downwards respectively) by changing the underlying estimate distribution a little bit (by $O(1/k)$ at each step $k$), when we have a censored (or uncensored respectively) observation. Therefore, the algorithm can be thought of moving in the right direction locally at each step. The formal analysis of such stochastic processes that are controlled locally in the right direction is given by the theory of adaptive algorithms and stochastic approximations. (See Kushner and Yin [42], Chen [19], and Benveniste et al. [7].) McGill and van Ryzin [55] propose a simple adaptive algorithm -a Robins-Monro algorithm-, which increases the protection level a little when the allocated seats sell out and reduces it a little otherwise. Similarly, a concurrent working paper by Huh and Rusmevichientong [37] provides an adaptive algorithm that adjusts protection levels by making use of stochastic online convex optimization and adjusting the protection levels based on a gradient ascent algorithm. However, these papers take the approach of directly adjusting the protection levels by bypassing the forecasting step, whereas we establish the equivalence of the joint
optimization/forecasting iteration to an adaptive algorithm when ME distributions are used in the forecasting step. In that respect, the adaptive algorithm should be seen as a mathematical tool for proving convergence in our essay, rather than the focus of our work. Our contribution can be seen as augmenting the effectiveness of legacy optimization/forecasting routines in practice by use of ME estimation.
Chapter 2

Monopoly Pricing with Limited Demand Information

In our first essay, we study the dynamic pricing, the third-degree price discrimination, and the second-price sealed-bid auction pricing schemes for a monopolist with limited WtP information using competitive ratio and maximum regret criteria. We assume customers have private WtP drawn from common distribution that is unknown to the seller. We focus more on the dynamic pricing scheme to illustrate the use of partial information. For the simplest case in Section 2.1, we solve the resulting optimization problem in closed form and provide a description of the optimal policies for the seller and the adversary. Price skimming arises naturally as a hedging mechanism against uncertainty in this setting. In Section 2.2, we analyze the effects of learning in a two period setting that allows the seller to learn from the sales observations in period one. Depending on the level of restrictions on the firm, experimenting with as many price points as possible through a price skimming policy is optimal in the learning period. In Section 2.3, we study the case where the seller only has limited
price experimentation capability or has limited past sales information. We show that partial sales information can be easily incorporated into our setting, and that even with limited learning, the resulting policies have very good revenue performance across many common demand distributions. Finally, in Section 2.4, we analyze the other two monopoly pricing schemes. We first consider the case in which the firm can (third-degree) price discriminate its market. We show that it is optimal for the firm to set the sizes of the WtP intervals equally and to set the price for a particular segment equal to its minimum WtP value within the segment. Lastly, we also study the second price sealed-bid auction. The optimal reservation price balances risks related to uncertainty in WtP and is similar to the one derived under the revenue maximizing criterion, but, is slightly lower, reflecting risk aversion.

One observation from many cases considered is that the adversary's policy often takes the form of an extreme point distribution. This follows from the fact that the inner optimization step can often be reduced to a quasi-convex maximization problem over the probability simplex, which admits an extreme point solution. This setting accentuates the two types of risk faced by the firm: first, the risk of foregoing too much revenue from pricing too low, and second, the risk of foregoing sales from pricing too high. In response, the firm's strategy focuses on hedging against this exposure.

As a side note, for some of the models studied, the adversary's best response is achieved only in the limit. In other words, the point mass of the extreme point distribution is allocated to some valuation \( v - \epsilon \) for some \( v \) and an arbitrarily small \( \epsilon > 0 \) chosen by the adversary, resulting in a competitive ratio of \( c + \delta(\epsilon) \) where \( \delta(\cdot) \) is monotone increasing with \( \delta(\epsilon) \to 0 \) as \( \epsilon \to 0 \). Consequently, the competitive ratio achieves the value \( c \) in the limit. This limiting response of the adversary will be stated in detail using arbitrarily small constants \( \delta, \epsilon \) as in the above sense wherever
2.1. Dynamic pricing with no market information

2.1.1 Problem formulation

We consider a monopolist selling a homogeneous good over a sales horizon that is normalized to have length one. The firm’s is assumed to have ample capacity. Potential customers arrive at the firm according to a deterministic arrival process with rate $\Lambda$, each with a WtP for one unit of that product, denoted by $v$, which is an independent draw from a common discrete distribution $F$ on the set $\{p_1, \ldots, p_K\}$ where $p_1 = v$ and $p_K = \bar{v}$. That is the support of the WtP distribution is an appropriate discretization of the range $[v, \bar{v}]$, e.g., in $\$1$ or $5\%$ increments. It is common to assume that the customer arrival process is Poisson in the literature with the expected revenue maximization criterion, but in the sequel we will restrict attention to a deterministic model where, in addition, customers are assumed to arrive continuously as opposed to in unit increments. This corresponds to the fluid approximation to the common Poisson (or other stochastic arrival process) setting, and allows us to emphasize the effects of “first order” uncertainty introduced by not knowing the sales rate itself at a selected price point, as opposed to “second order” fluctuation due to stochastic nature of the process. The rate $\Lambda$ can also be interpreted as the “market size.” Assuming that the price at time $t$ is equal to $p(t)$, then the sale rate at that instant is given by $\lambda(t) = \Lambda P(v \geq p(t)) = \Lambda \bar{F}(p(t))$, where $\bar{F}(\cdot) = 1 - F(\cdot)$, and the corresponding revenue rate is $p(t)\Lambda \bar{F}(p(t))$.

The firm’s goal is to maximize the total revenues accrued in $[0, 1]$. When the WtP
distribution $F$ is known, this problem reduces to a special case of the deterministic relaxation of the single-product dynamic pricing problem studied by Gallego and van Ryzin [26] (that paper considered the capacity constrained case), for which it is optimal to charge a constant price $p^* = \arg\max_i p_i \bar{F}(p_i)$ throughout the sales horizon. That is, the dynamic nature of the pricing decisions is inconsequential, and the above problem reduces to the classical monopolist pricing problem.

We consider the problem of selecting a pricing policy when the firm only knows the support $[v, \bar{v}]$ of the distribution function $F$, but not $F$ itself. Sections 2.2 and 2.3 will consider extensions that incorporate demand learning from early sales. Estimating (or bounding) the support rather than the distribution itself is much easier in practice; for example, $v$ might represent the "cost of goods sold" below which the firm is not willing to engage in trade, and $\bar{v}$ might represent the price of a superior substitute in the market.

The firm's pricing strategy is a vector $t \in \mathbb{R}^K$, where $t_i$ is the length of time over which the firm will use price $p_i$. Note that the labeling of the price points and the assumption that $p_1 = v$ and $p_K = \bar{v}$ are innocuous since it is always possible to decide not to offer some particular price $p_j$ by setting the corresponding $t_j = 0$.

Given a policy $t$ and a distribution $F$, the revenue accrued by the firm is given by

$$R(t, F) := A \sum_{j=1}^K t_j p_j F(p_j).$$

The firm selects a strategy $t$, and then an imaginary adversary selects a distribution $F$ after he observes the firm's policy $t$. The goal of the firm is to optimize its relative performance when compared to that of a fully informed player, i.e. one that could maximize its revenues with full knowledge of the distribution $F$; this is the so-called "competitive ratio" criterion.

Specifically, let $t^*(F) \in \arg\max_t R(t, F)$, be the policy that maximizes the to-
tual revenue with full information about \( F(\cdot) \), which is given by \( t_j^*(F) = 1 \) for \( j = \arg\max_i p_i \bar{F}(p_i) \) and \( t_i^*(F) = 0 \) for all \( i \neq j \). The competitive ratio (CR) problem is given by

\[
c^* = \max_t \min_F \left\{ \frac{R(t,F)}{R(t^*(F),F)} : \sum_{j=1}^{K} t_j = 1, \quad t \geq 0 \right\}.
\] (2.1)

### 2.1.2 Characterization of the optimal pricing policy

For any distribution \( F \), let \( f_j := P(p_{j+1} > v \geq p_j) \) for \( j = 1 \ldots K-1 \), \( f_K := P(v = \bar{v}) \), and \( \bar{f}_j := \sum_{k \leq j} f_k = P(v > p_j) \). This allows us to rewrite the revenue function as

\[
R(t,F) = \Lambda \sum_k f_k \sum_{j \leq k} p_j t_j = \Lambda \sum_j t_j p_j \bar{f}_j,
\]

and (2.1) as:

\[
c^* = \max_t \min_f \left\{ \frac{\sum_j t_j p_j \bar{f}_j}{\max_j \{p_j \bar{f}_j\}} : 1 = \bar{f}_1 \geq \bar{f}_2 \geq \cdots \bar{f}_K \geq 0, \quad \sum_j t_j = 1, \quad t \geq 0 \right\}.
\] (2.2)

where the denominator, \( \max_j \{p_j \bar{f}_j\} = R(t^*(F),F) \), is the maximum revenue that the firm could extract, if \( F(\cdot) \) was known, by charging the revenue maximizing price throughout the sales horizon.

The key observation that underlies the solution of (2.2) is that the objective function is quasi-concave in \( \bar{f} \), and as a result, the adversary’s problem admits an extreme point optimal solution which is easy to characterize and exploit.

**Theorem 2.1.1** Consider the dynamic pricing problem with no market information specified in (2.1), or equivalently in (2.2). The firm’s optimal policy is the following price skimming rule:

\[
t_1 = \left( K - \sum_{j=1}^{K-1} \frac{p_j}{p_{j+1}} \right)^{-1} \quad \text{and} \quad t_j = \frac{p_j - p_{j-1}}{p_j} t_1 \quad \text{for} \quad j = 2, \ldots, K.
\]

and the resulting competitive ratio is \( c^* = t_1 \).
Proof: The denominator in the objective in (2.2) is the maximum of \( K \) linear functions in \( \bar{f} \), and is therefore convex in \( \bar{f} \). The numerator in (2.2) is linear, and thus concave in \( \bar{f} \). Thus, for fixed \( t \), the adversary’s problem is one of minimizing a quasi-concave function over a polyhedron, which admits an extreme point optimal solution. The polyhedron defined by \( 1 \geq \bar{f}_2 \geq \cdots \geq \bar{f}_K \geq 0 \) has \( K \) extreme points, all of which correspond to vectors of the form \((1, 0, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, 1, 1, \ldots, 1)\). Since, for every fixed \( t \) the optimal value for the inner minimization occurs at one of these extreme points, (2.2) can be reduced to the problem:

\[
c^* = \max_t \left\{ \min_{j=1, \ldots, K} \frac{\sum_{i \leq j} p_i t_i}{p_j} : \sum_j t_j = 1, \quad t \geq 0 \right\},
\]

which, in turn, is equivalent to the linear program

\[
c^* = \max_{t, c} \left\{ c : c \leq \sum_{i \leq j} \frac{p_i}{p_j} t_i \quad \forall j, \quad \sum_j t_j = 1, \quad t \geq 0 \right\}.
\]

This LP can be solved in closed-form as follows. Consider its dual:

\[
c^* = \min_{x, y} \left\{ y : y \geq p_j \sum_{i \geq j} \frac{x_i}{p_i} \quad \forall j, \quad \sum_j x_j = 1, \quad x \geq 0 \right\}.
\]

The first step is to construct a dual feasible solution that satisfies the first set of inequality constraints with equalities. Solving \( y = p_j \sum_{i \geq j} x_i \forall j \), then \( x_K = y \) and \( x_j = \frac{p_{j+1} - p_j}{p_{j+1}} \) \( y \) for \( j = 1, \ldots, K - 1 \). Substituting these into the normalizing constraint \( \sum_j x_j = 1 \), yields \( y = \left( K - \sum_{j=1}^{K-1} \frac{p_j}{p_{j+1}} \right)^{-1} \). This dual objective value is equal to the primal objective value that corresponds to the primal feasible solution given in the body of Theorem 2.1.1. By strong duality, we conclude that the solution in the proposition is optimal for the primal problem. \( \square \)

Consequently, if the firm does not know the WtP distribution \( F \), it no longer charges a constant price, but it adopts a price skimming policy that charges each
price point for an appropriate amount of time. As mentioned earlier, an alternative interpretation is to treat the \( t_i \)'s as probabilities and \( t \) as a randomized pricing policy; see Bergemann and Schlag [8]. To gain some intuition behind this result, recall that the worst case scenario for the firm occurs when the market is homogeneous and every potential customer shares the same WtP. This setting raises two types of opposing risk for the firm at each price. First, if the firm prices too high for a significant portion of its sales horizon, it may suffer low sales when the market's WtP is low. Second, if the firm prices too low for a significant portion of its sales horizon, it may forego a significant revenue opportunity when the market's WtP is high. In both cases, the resulting competitive ratio would be low. Our analysis specifies how to balance these two effects in constructing the optimal pricing policy, which essentially makes the adversary indifferent between the extreme market scenarios that is optimal for him to choose. It is also worth comparing the above behavior against the solution to the maxmin formulation with objective \( \max_t \min_F R(t, F) \). In this case, the optimal strategy for the adversary is to put all of the probability at \( v \), while the firm would also price at \( v \) for the entire sales horizon, making the result too conservative. Actually, the revenue performance of the resulting policy is typically much higher than the competitive ratio as illustrated by the numerical examples in Sections 2.2 and 2.3.

The competitive ratio for (2.2) depends on the discretization of the grid \( \{p_1, \ldots, p_K\} \). The next result derives a lower bound for the competitive ratio that is independent of that grid.

**Proposition 2.1.1** For any price grid \( \{p_1, \ldots, p_K\} \) used of any size \( K \), the optimal competitive ratio \( c^* \) for (2.2) satisfies the following bound:

\[
c^* \geq (1 + \ln(\bar{v}/\bar{y}))^{-1} =: c^{LB}.
\]
Proof: Note that

\[
K - \sum_{j=1}^{K-1} \frac{p_j}{p_{j+1}} = 1 + \sum_{j=1}^{K-1} \frac{p_{j+1} - p_j}{p_{j+1}} = 1 + \sum_{j=1}^{K-1} \int_{p_j}^{p_{j+1}} \frac{1}{x} \, dx
\]

\[
\leq 1 + \sum_{j=1}^{K-1} \int_{p_j}^{p_{j+1}} \frac{1}{x} \, dx = 1 + \int_{v}^{\bar{v}} \frac{1}{x} \, dx = 1 + \ln(\bar{v}/v)
\]

Then, \( c^* \) satisfies

\[
c^* = \left( K - \sum_{j=1}^{K-1} \frac{p_j}{p_{j+1}} \right)^{-1} \geq (1 + \ln(\bar{v}/v))^{-1} = c^{LB}.
\]

The lower bound is achieved as the number of prices grows large and \( \{p_1, \ldots, p_K\} \) becomes a dense covering of the range \([v, \bar{v}]\). We note that Ball and Queyranne [4] recover similar results in their study of the single-resource (airline) capacity allocation problem under a competitive ratio criterion. Similarly, Bergemann and Schlag [8] obtained analogous results using the maximum regret criterion.

A somewhat crude upper bound on the competitive ratio can be obtained if we allow the firm to optimize over its offered price grid, but simultaneously restrict the adversary to also price on that grid.

**Proposition 2.1.2** For any set of price points \( v = p_1 \leq \cdots \leq p_K = \bar{v} \) and any number \( K \), the optimal competitive ratio \( c^*(p) \) for (2.2) satisfies the following bound:

\[
c^*(p) \leq (2 - v/\bar{v})^{-1} =: c^{UB}.
\]

**Proof:** To achieve the upper bound we maximize the competitive ratio derived in Proposition 2.1.1, \( c^*(p) = \left( K - \sum_{j=1}^{K-1} \frac{p_j}{p_{j+1}} \right)^{-1} \) with respect to the price vector \( p_1 \leq \cdots \leq p_K \). This is equivalent to maximizing \( \sum_{j=1}^{K-1} \frac{p_j}{p_{j+1}} \) over \( p_1 \leq \cdots \leq p_K \), which can be written as the following problem:

\[
\max_p \left\{ \frac{v}{p_2} + \sum_{i=2}^{K-2} \frac{p_i}{p_{i+1}} + \frac{p_{K-1}}{\bar{v}} : \frac{v}{p_2} \leq \cdots \leq \frac{v}{p_{K-1}} \leq \bar{v} \right\}, \quad (2.3)
\]
where we have used the assumption that \( p_1 = v \) and \( p_K = \bar{v} \). The above problem can be recast as a geometric program. Let \( e^y_i = p_j, a = ln\, v \), and \( b = ln\, \bar{v} \), and rewrite (2.3) as:

\[
\max_y \left\{ e^{a-y_2} + \sum_{i=2}^{K-2} e^{y_j-y_{j+1}} + e^{y_{K-1}-b} : c^a \leq e^{y_2} \leq \cdots \leq e^{y_{K-1}} \leq c^b \right\}.
\]

The right-hand-side constraint is equivalent to the condition \( a \leq y_2 \leq \cdots \leq y_{K-1} \leq b \), and therefore the above problem can be rewritten as

\[
\max_y \left\{ e^{a-y_2} + \sum_{i=2}^{K-2} e^{y_j-y_{j+1}} + e^{y_{K-1}-b} : a \leq y_2 \leq \cdots \leq y_{K-1} \leq b \right\}.
\]

This is again a convex maximization problem over a polyhedron that admits an extreme point solution. Moreover, there is a one-to-one correspondence between the extreme points of this polyhedron and the extreme points of the polyhedron defined by \( v \leq p_2 \leq \cdots \leq p_{K-1} \leq \bar{v} \). The latter has \( K-1 \) extreme points that can be written in terms of prices as follows:

\[
(v, \bar{v}, \bar{v}, \cdots, \bar{v}), (v, v, \bar{v}, \cdots, \bar{v}), \cdots, (v, v, \cdots, v, \bar{v}).
\]

All of these extreme points achieve the same objective function value given by \( (K - 2 + y/\bar{v}) \), and yield the optimal competitive ratio of \( c = \left[ K - (K - 2 + \frac{y}{\bar{v}}) \right]^{-1} = (2 - \frac{y}{\bar{v}})^{-1} \). Moreover, the upper bound is achieved when only the lowest and highest valuations (i.e. \( y \) and \( \bar{v} \)) are offered for periods of length \( t_y = \frac{y}{2\bar{v} - y} \) and \( t_{\bar{v}} = \frac{\bar{v} - y}{2\bar{v} - y} \), \( \square \)

The upper bound \( c^{UB} \) is crude because at least some of the performance gain reported therein is due to the pricing inflexibility imposed on the adversary, which, in turn, affects the revenues achieved by a fully informed adversary that serve as the benchmark of the competitive ratio analysis.

As intuition would suggest, as the relative difference between the lowest and the highest valuation decreases, i.e. \( \frac{y}{\bar{v}} \to 1 \), \( c^{LB} \uparrow 1 \). That is, as the aggregate
uncertainty about the market preferences decreases, the risk from pricing too low becomes negligible, and the firm's revenue approaches the one that is achieved under full knowledge of $F$.

### 2.1.3 Maximum regret

We now turn to the analysis of the maximum regret formulation of the dynamic pricing problem. In this case, the firm's objective is to minimize the maximum regret, which can be written as follows

$$\min_t \max_F \left\{ [R(t^*(F), F) - R(t, F)] : \sum_{j=1}^K t_j = 1, \ t \geq 0 \right\}.$$  

Using the definitions from the previous subsection, this can be rewritten as

$$\min_{t \geq 0} \max_f \left\{ \lambda \left[ \max_j \{p_j \bar{f}_j\} - \sum_j t_j p_j \bar{f}_j \right] : 1 = \bar{f}_1 \geq \cdots \geq \bar{f}_K \geq 0, \ \sum_j t_j = 1 \right\}. \quad (2.4)$$

The objective function above is convex in $\bar{f}$, and therefore, the inner maximization again admits an extreme point optimum. That is, the adversary will again select an extreme point distribution that places all of probability mass on one price point. This allows us to again reduce this problem to a linear program, which can be solved in closed form.

**Proposition 2.1.3** For a fixed set of price points $\{p_1, \ldots, p_K\}$, define the constants

$$\Delta_j := \frac{p_j - p_{j-1}}{p_j}, \ j = 2, \ldots, K.$$  

The optimal policy for the maximum regret problem in (2.4) is the following price skimming rule:

$$t_K = \Delta_K, \ t_j = \min \{\Delta_j, (1 - \sum_{i>j} \Delta_j)^+\} \text{ for } j = 2, \ldots, K - 1, \text{ and } t_1 = 1 - \sum_{j=2}^K t_j.$$
Denote the lowest price charged for a positive amount of time in this policy as $p_n$. Then, the optimal regret is given by

$$r(p) = \Lambda p_n \left[ (K - n) - \sum_{j=n}^{K-1} \frac{p_j}{p_j+1} \right].$$

The proof of this result is similar to that of Proposition 2.1.1 and is omitted. Similar to the bound derived in Propositions 2.1.1, we can derive an upper bound for the maximum regret case. This is summarized in the following proposition, whose proof follows similar steps to the competitive ratio case and is also omitted.

**Proposition 2.1.4** Let $v = \max\{v, \bar{v}/e\}$. Then, for any set of price points $v = p_1 \leq \cdots \leq p_K = \bar{v}$ and any number $K$, the optimal regret $r(p)$ for (2.4) satisfies

$$r(p) \leq \Lambda v \ln(\bar{v}/v) =: r_{UB}.$$

Again, the upper bound gets achieved as the number of prices grows large and the price grid $\{p_1, \ldots, p_K\}$ becomes a dense subset of the range $[v, \bar{v}]$. Observe that as $(p_{j+1} - p_j) \to 0$ in the limit, we have $t_j \to \frac{dp_j}{p_j}$. That is we can define the optimal limiting pricing policy as charging price $p$ for an amount of time $t(p) = \frac{1}{p}$ for $p \in [v, \bar{v}]$. This limiting case corresponds to the results of Bergemann and Schlag [8] for monopoly pricing. Bergemann and Schlag consider a monopolist who wants to minimize its maximum regret by charging a single price. They allow for a randomizing policy over all possible set of prices, and the adversary is allowed to choose any WtP distribution. Although, such a policy may be hard to implement in practice, it is a valid framework theoretically. Under this setting, they show that the optimal randomizing policy distribution for the firm has a probability density function $t(p) = \frac{1}{p}$ for $p \in [v, \bar{v}]$. We recover their result as a special case of our results.
in the limit. Our dynamic pricing model offers an alternative interpretation of their randomizing pricing policy.

In common with the results of the competitive ratio analysis, the worst case scenario for the firm is a homogeneous market whose representative WtP is unknown. The structure of the optimal firm's response is a price skimming policy that is also similar to the one obtained under the competitive ratio criterion, making both of these findings fairly insensitive to the particular criterion in use. The two policies, however, have some important differences. The first one is that it is no longer optimal to charge all available prices for some positive amount of time as we got in the previous subsection. Instead, the firm will start pricing at the highest possible price point, and then gradually reduce its price until the sales horizon is depleted. It is possible that some of the lower price points will never be used, as the potential revenue that they may generate is small relative to the that at the higher price points. The emphasis on absolute as opposed to relative revenue performance shifts the firm's focus towards the risk of uncaptured revenue on high market scenarios, where customers attach a high WtP to the offered product. A second consequence of the different performance criteria is that the maximum regret analysis produces an arithmetic price skimming policy, whereas the competitive ratio one produces a geometric one. The upper bound on regret is similar to the lower bound obtained under the competitive ratio analysis. As the absolute uncertainty decreases, i.e., $\bar{v} - v \rightarrow 0$, the upper bound on the maximum regret go to zero as expected.
2.2. The effect of learning

We consider a version of the dynamic pricing problem with passive learning where: a) the firm splits the sales horizon into two periods of length $\tau^1$ and $\tau^2 = 1 - \tau^1$; b) selects a pricing strategy in $[0, \tau^1]$ without any information about $F$; and c) subsequently selects a pricing strategy in $(\tau^1, 1]$ that uses fractile information extracted in $[0, \tau^1]$ for all price points charged in that period. Note that the adversary need only commit to the underlying WtP distribution at the price points where the firm chooses to experiment in period one, and can select the remaining information, i.e. unobserved specifications of the distribution, in period two. We assume that the demand measurements are noiseless, i.e. the firm observes $\hat{F}(p_j)$ instead of a random variable with that mean. The term "passive" learning indicates that the firm cannot update its information set incrementally during $[0, \tau^1]$ or $(\tau^1, 1]$; instead, it only updates its information at time $\tau^1$, and makes use of this new information in $(\tau^1, 1]$. In that sense, the first period has a dual role of learning and revenue optimization, while the second period is solely dedicated to revenue optimization. This structure is suggested due to its simplicity and potential practical appeal, which was recently proposed and analyzed in an asymptotic setting with large market size and large sales horizon by Besbes and Zeevi [12].

An alternative interpretation of our model is one where the firm sells one product through many potential stores, and where rather than dynamic pricing over time, the fractions $t^i_j$ represent the fraction of the stores that price at $p_j$ in period $i$ for $j = 1, \ldots, K$ and $i = 1, 2$. [This is also an alternate interpretation for the randomized policy of Bergemann and Schlag [8].] In that setting, the firm selects at how many stores to apply each potential price point, and then combines the information
extracted from all these stores to update its demand information and optimize its downstream pricing decisions. Gaur and Fisher [27] have studied this problem, although the emphasis in their paper was the issue of how to combine the demand information from each store taking into account the differences between the local market conditions faced by each store; this feature is not considered in our work.

An important consideration is whether the prices used during the first period constrain the firm’s pricing options in the second period. We study two variants:

a) *Unconstrained learning*: no downstream pricing constraints in period two; i.e., both markups and markdowns are allowed.

b) *Constrained learning*: the firm can only apply price markdowns in period two. This is easiest understood through the interpretation of a firm selecting how to price a product across many stores, in which case in a constrained setting, if a particular store priced at $p_j$ in period one, then the same store can only price at $p_j$ or lower in period two. This constraint -or slight variation thereof- is plausible from a practical viewpoint, and creates a clear tradeoff between the pricing decisions in the two periods. Algebraically, this constraint translates into $\sum_{j \geq k} t^1_j \geq \sum_{j \geq k} t^2_j \geq 0$ for all $k$, where $t^1_j$ and $t^2_j$ are the fractions of the intervals $[0, \tau^1]$ and $(\tau^1, 1]$ respectively dedicated to price $p_j$ and $\sum_j t^i_j = 1$ for $i = 1, 2$.

The analysis and the notation of the learning case is heavier than the single period problem of the previous section. Therefore, for brevity, we focus on only the competitive ratio criterion to analyze these two variants in the following subsections. The analysis of the maximum regret criterion would follow similar steps. As it will be apparent, an important feature of the emerging solution for both variants is that the firm chooses to experiment on all possible price points in $\{p_1, \ldots, p_K\}$ in period one, so as to price under full information in period two. Section 2.3 will study more
restricted settings where the firm can only experiment on a few price points in period one.

2.2.1 Unconstrained learning

Adopting our previous notation, we will normalize the length of each period to one but scale the market size that corresponds to each period to \( \Lambda^1 \) and \( \Lambda^2 \), where \( \Lambda^i = \Lambda \tau^i \) for \( i = 1, 2 \). This problem can be formulated as the following two-stage dynamic game:

\[
c^* = \max_{t^1} \min_{f^1} \max_{t^2} \min_{f^2} \frac{\Lambda^1 \sum_j t^1_j p_j f^1_j + \Lambda^2 \sum_j t^2_j p_j f^2_j}{(\Lambda^1 + \Lambda^2) \max_j \{p_j f^2_j\}}
\]

\[
s.t. \quad \sum_j t^i_j = 1, \quad t^i_j \geq 0 \quad i = 1, 2
\]

\[
1 = f^1_j \geq f^2_j \geq \cdots \geq f^K_j \geq 0 \quad i = 1, 2
\]

\[
t^1_j (f^2_j - f^1_j) = 0, \quad \forall j
\]

where \( t^i_j \) is the proportion of time spent at price \( j \) during period \( i \) for \( j = 1, \ldots, K \) and \( i = 1, 2 \). \( f^2_j, j = 1, \ldots, K \) are the fractiles of the WtP distribution chosen by the adversary in period two, while \( f^1_j \) represent the information revealed to the firm at the end of period one. The constraints \( t^1_j (f^2_j - f^1_j) = 0 \) force the adversary to be consistent in the choice of the WtP distribution and the information revealed to the firm in the first period. There are no constraints linking the pricing decisions in periods one and two.

We first prove that the firm tests all prices in period one. The proof is relegated to Appendix A.

**Proposition 2.2.1** Let \((t^1_*, t^2_*)\) denote the solution of (2.5). Then, \( t^1_*,j > 0 \) for \( j = 1, \ldots, K \).
This allows the firm to price under full information in period two such that

\[
\max_{t_2} \left\{ \Lambda^2 \sum_j t_j^2 p_j f_j^1 : \sum_j t_j^2 = 1, \ t^2 \geq 0 \right\} = \Lambda^2 \max_j \{p_j f_j^1\},
\]

i.e. the firm extracts the maximum possible revenue in period two, same as the adversary. Given this observation the problem reduces to

\[
\max_{t^1} \min_{f^1} \frac{\Lambda^1 \sum_j t_j^1 p_j f_j^1}{(\Lambda^1 + \Lambda^2) \max_j \{p_j f_j^1\}} + \frac{\Lambda^2}{(\Lambda^1 + \Lambda^2)} \tag{2.6}
\]

s.t. \( \sum_j t_j^1 = 1, \ t^1 \geq 0, \ 1 = f_1^1 \geq \cdots \geq f_K^1 \geq 0, \)

which is equivalent to the single period problem studied in Section 2.1. Using Theorem 2.1.1, we conclude the following:

**Theorem 2.2.1** For the two period dynamic pricing problem with “unconstrained learning” described in (2.5), the firm’s optimal decision is the following

- **Period one:** adopt a price skimming policy for which

  \[ t_1 = \left( K - \sum_{j=1}^{K-1} \frac{p_j}{p_{j+1}} \right)^{-1} \text{ and } t_j = \frac{p_j - p_{j-1}}{p_j} t_1 \text{ for } j = 2, \ldots, K. \]

- **Period two:** price at \( p_{j^*} \), where \( j^* = \arg\max_j \{p_j f_j^1\} \).

Let \( \lambda^i := \Lambda^i / (\Lambda^1 + \Lambda^2), \ i = 1, 2. \) The competitive ratio is

\[
c^* = \lambda^1 \left( K - \sum_{j=1}^{K-1} \frac{p_j}{p_{j+1}} \right)^{-1} + \lambda^2. \tag{2.7}
\]

That is the competitive ratio is a weighted average of the one for the single period problem identified in Theorem 2.1.1 and 1 with respective weights of \( \lambda^1 \) and \( \lambda^2 \). As we shrink the length of the learning period \( \lambda^1 \to 0, \lambda^2 \to 1 \) and \( c \to 1 \). This follows from the assumption that allows for perfect learning of the demand at each tested
price point irrespective of the time spent on it. In reality, as the length of the learning phase decreases, the accuracy of the firm's observations diminishes due to the inherent uncertainty of the demand realization, which may lead to an error in the estimation of the underlying WtP distribution and of the firm's downstream pricing decisions.

As mentioned earlier, Besbes and Zeevi [12] studied a stochastic variant of what we call here the "unconstrained" problem in settings where the market size is large, and showed that the firm can select the learning period to be short and still ensure that the estimation error due to the stochastic nature of the demand is asymptotically negligible. As $\lambda_1 \to 0$, the pricing policy in the first period becomes irrelevant as long as the firm does try all $K$ price points; indeed Besbes and Zeevi [12] prescribes a uniform pricing policy in period one, since its effect on the overall performance becomes negligible.

### 2.2.2 Constrained learning

As explained earlier the downstream pricing constraints can be succinctly summarized by the set of conditions $\sum_{j \geq k} t^1_j \geq \sum_{j \geq k} t^2_j$ for all $k$. The resulting formulation becomes:

$$
c^* = \max_{t^1} \min_{f^1} \max_{t^2} \min_{f^2} \frac{\Lambda^1 \sum t^1_j p_j \bar{f}^1_j + \Lambda^2 \sum t^2_j p_j \bar{f}^2_j}{(\Lambda^1 + \Lambda^2) \max_{j} \{p_j \bar{f}^2_j\}}
$$

s.t. $\sum_j t^i_j = 1$, $t^i \geq 0$ $i = 1, 2$

$1 = \bar{f}^1_1 \geq \bar{f}^1_2 \geq \cdots \geq \bar{f}^1_K \geq 0$ $i = 1, 2$

$t^1_j (\bar{f}^2_j - \bar{f}^1_j) = 0$, $\forall j$

$\sum_{j \geq k} t^1_j \geq \sum_{j \geq k} t^2_j$, $\forall k$.

We state the following result without proof (which follows the steps of Proposition
Proposition 2.2.2 Let \((t^*_j, t^*_j)\) denote the solution to (2.8). Then, \(t^*_j > 0\) for \(j = 1, \ldots, K\).

Let

\[
R(t^1, \bar{f}^1) := \max_{t^2} \left\{ \Lambda^2 \sum_j t^2_j p_j \bar{f}^1_j : \sum_{j \geq k} t^1_j \geq \sum_{j \geq k} t^2_j \quad \forall k, \quad \sum_j t^2_j = 1, \quad t^2 \geq 0 \right\}
\]

Using Proposition 2.2.2, the problem can be reduced to the following formulation:

\[
\max \min_{t^1} \frac{\Lambda^1 \sum_j t^1_j p_j \bar{f}^1_j + R(t^1, \bar{f}^1)}{\left( \Lambda^1 + \Lambda^2 \right) \max_j \{p_j \bar{f}^1_j\}} \quad (2.9)
\]

\[
s.t. \quad \sum_j t^1_j = 1, \quad t^1 \geq 0, \quad 1 = \bar{f}^1_1 \geq \bar{f}^1_2 \geq \cdots \bar{f}^1_K \geq 0.
\]

\(R(t^1, \bar{f}^1)\) reflects the revenue maximization problem of the firm in the second period under full information but with the downstream pricing constraints. It is easy to show that in period two the firm adopts the revenue maximizing price \(p_j^* = \arg\max \{p_j \bar{f}^1_j\}\) for as long as possible, while marginally satisfying the downstream pricing constraints for all prices below \(p_j^*\); at the optimal solution, \(t^2_j = t^1_j\) for \(j < j^*\) and \(t^2_{j^*} = \left(1 - \sum_{i < j^*} t^1_i\right)\). As a result, we can rewrite (2.9) as

\[
c^* = \max_{t^1} \min \frac{\Lambda^1 \sum_j t^1_j p_j \bar{f}^1_j + \Lambda^2 \left[ \sum_{i < j^*} t^1_i p_i \bar{f}^1_i + \left(1 - \sum_{i < j^*} t^1_i\right) p_j^* \bar{f}^1_j\right]}{\left( \Lambda^1 + \Lambda^2 \right) \max_j \{p_j \bar{f}^1_j\}} \quad (2.10)
\]

\[
s.t. \quad \sum_j t^1_j = 1, \quad t^1 \geq 0, \quad 1 = \bar{f}^1_1 \geq \bar{f}^1_2 \geq \cdots \bar{f}^1_K \geq 0.
\]

Theorem 2.2.2 For the two period dynamic pricing problem with “constrained learning” described in (2.8), the firm’s optimal decision is the following
Period one: adopt a price skimming policy for which

\[ t_1^j = \left(1 + \frac{p_j - p_1}{p_2} + \sum_{j=3}^{K} \frac{(p_j - p_{j-1}) \prod_{i=2}^{j-1} [p_i - \lambda^2 p_{i-1}]}{\prod_{i=2}^{j} p_i \left(\lambda^1\right)^{j-1}} \right)^{-1} \quad \text{and} \quad t_2^1 = \frac{p_2 - p_1}{p_2}, \]

\[ t_j^1 = \frac{(p_j - p_{j-1})[p_{j-1} - \lambda^2 p_{j-2}]}{(p_{j-1} - p_{j-2})p_j \lambda^1} t_{j-1}^1 \quad \text{for} \quad j = 3, \ldots, K. \]

Period two: price at \( p_j \) for \( t_j^2 = \left(1 - \sum_{i<j^*} t_i^1\right) \) and at \( p_j \) for \( t_j^1 \) for \( j < j^* \), where \( j^* = \arg\max_j \{p_j f_j^1\} \).

Let \( \lambda^i = \Lambda^i / (\Lambda^1 + \Lambda^2) \), \( i = 1, 2 \). The competitive ratio is

\[ c^* = \lambda^1 \left(1 + \frac{p_2 - p_1}{p_2} + \sum_{j=3}^{K} \frac{(p_j - p_{j-1}) \prod_{i=2}^{j-1} [p_i - \lambda^2 p_{i-1}]}{\prod_{i=2}^{j} p_i \left(\lambda^1\right)^{j-1}} \right)^{-1} + \lambda^2. \quad (2.11) \]

**Proof:** The proof makes use of the following result, which we prove in Appendix A.

**Lemma 2.2.1** For any feasible \( t^1 \), the optimal solution for the inner minimization in (2.10) occurs at an extreme point of the simplex \( 1 \geq f_2 \geq \cdots \geq f_K \geq 0 \).

There are \( K \) extreme points to consider for the inner minimization in problem (2.10) above corresponding to vectors of the form \((1,0,0,\ldots,0)\), \((1,1,0,\ldots,0)\), \ldots, \((1,1,1,\ldots,1)\). Using Lemma 2.2.1, we can rewrite (2.10) as follows:

\[ c^* = \max_t \min_{j=1-K} \left\{ \frac{\Lambda^1 \sum_{i<j} p_i t_i^1 + \Lambda^2 \left[ \sum_{i<j} p_i t_i^1 + (1 - \sum_{i<j} t_i^1) p_j \right]}{(\Lambda^1 + \Lambda^2) p_j} \right\} \]

s.t. \( \sum_j t_j^1 = 1, \quad t^1 \geq 0, \)
which, in turn, is equivalent to:

\[
\begin{align*}
    c^* &= \max_{t,c} \quad c \\
    \text{s.t. } c &\leq \frac{\Lambda^1 p_1 t_1^1 + \Lambda^2 p_1}{(\Lambda^1 + \Lambda^2)p_1} \\
    &\leq \frac{\Lambda^1 \sum_{i<j} p_i t_i^j + \Lambda^2 \left[ \sum_{i<j} p_i t_i^j + \left( 1 - \sum_{i<j} t_i^j \right) p_j \right]}{(\Lambda^1 + \Lambda^2)p_j} \quad j = 2, \ldots, K \\
    \sum_j t_j^1 &= 1, \quad t_1^1 \geq 0 .
\end{align*}
\]

This LP can be solved in closed form using its dual. This completes the proof. \( \Box \)

We make three observations. First, irrespective of whether the firm has downstream pricing constraints, it is optimal to adopt a price skimming policy which charges all prices for a positive amount of time in period one. Consequently, the optimal decision of the firm again decomposes into two parts: in the first period, the firm tests all prices in an optimal manner to learn the WtP distribution. In the second period, the firm maximizes its revenue under full information by charging \( p_j^* \) as long as possible and meeting the downstream pricing constraints marginally.

Second, the effect of downstream pricing constraints makes the firm charge higher prices in the first period, compared to the unconstrained case, to hedge against foregone revenues in the second period from not being able to charge higher prices.

Third, the constrained formulation offers a natural extension to the model studied in Besbes and Zeevi [12] in the sense that one could adopt their style of analysis to prove the asymptotic optimality of our proposed policy. In contrast to our comments after the analysis of the unconstrained learning case, the pricing policy adopted during the learning phase has a crucial effect on the overall system performance even if the length of the learning phase is shrunk to zero.
2.2.3 Discussion and numerical results

<table>
<thead>
<tr>
<th>(\lambda^1, \bar{v}/v)</th>
<th>Single Period CR</th>
<th>Const. Gain% CR</th>
<th>Unconst. Gain% CR</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.1, 2)</td>
<td>0.595</td>
<td>0.900 51%</td>
<td>0.960 61%</td>
</tr>
<tr>
<td>(0.1, 6)</td>
<td>0.372</td>
<td>0.900 142%</td>
<td>0.937 152%</td>
</tr>
<tr>
<td>(0.1, 10)</td>
<td>0.322</td>
<td>0.900 180%</td>
<td>0.932 190%</td>
</tr>
<tr>
<td>(0.4, 2)</td>
<td>0.595</td>
<td>0.706 19%</td>
<td>0.838 41%</td>
</tr>
<tr>
<td>(0.4, 6)</td>
<td>0.372</td>
<td>0.624 68%</td>
<td>0.749 101%</td>
</tr>
<tr>
<td>(0.4, 10)</td>
<td>0.322</td>
<td>0.614 91%</td>
<td>0.729 126%</td>
</tr>
<tr>
<td>(0.7, 2)</td>
<td>0.595</td>
<td>0.634 7%</td>
<td>0.717 20%</td>
</tr>
<tr>
<td>(0.7, 6)</td>
<td>0.372</td>
<td>0.461 24%</td>
<td>0.560 51%</td>
</tr>
<tr>
<td>(0.7, 10)</td>
<td>0.322</td>
<td>0.427 33%</td>
<td>0.525 63%</td>
</tr>
</tbody>
</table>

Table 2.1: Competitive ratios of single period, constrained learning, and unconstrained learning cases, and the respective gains due to learning. \( K = 20 \) prices uniformly spanning the support.

The numerical results reported next give a rough indication of the theoretical performance improvement under these two learning schemes when compared to the results in Section 2.1 with no learning. Specifically, for a fixed price grid with \( K \) price points that uniformly span the support \([v, \bar{v}]\) and given \( \lambda^1, \lambda^2 \), the solution of the single period problem with \( \Lambda = \lambda^1 + \lambda^2 \), identified by Theorem 2.1.1, is compared to the solutions of the two learning schemes identified in Theorems 2.2.1 and 2.2.2.

The main observation from these results is that the effect of learning is most pronounced in settings with higher ambiguity as measured by \( \bar{v}/v \). This is intuitive, as in these cases the risks associated with worst-case pricing are accentuated. Analytically, the competitive ratio for the unconstrained learning case is \( c_I := \lambda^1 c + \lambda^2 \), and the relative gain is \( (c_I/c - 1) = \lambda^2 (c^{-1} - 1) \). Using the lower bound \( c^{LB} = (1 + \ln(\bar{v}/v))^{-1} \) derived in Proposition 2.1.1, we see that the gain can be close to \( \lambda^2 \ln(\bar{v}/v) \), which is increasing in \( \bar{v}/v \). The same conclusions hold for the constrained learning case. The
second period problem reduces to the monopolist’s revenue maximization problem for both the unconstrained and constrained learning formulations. But while the effect of the first period revenue diminishes as $\lambda_1 \rightarrow 0$, its impact on the second period revenue does not in the case with downstream pricing constraints. Table 2.1 provides a numerical example of these gains for different parameters.

2.3. Learning with limited price experimentation

From a practical viewpoint, firms typically have a limited time and budget for learning, and as a result try to gauge the WtP distribution only at certain price points. This section extends our analysis to cover settings where the firm can experiment with only a small number of price points in the first period.

2.3.1 Single price in period one

The first case we study is one where the firm can only experiment with one price in period one, which we denote by $p_n$. It will consequently, observe the fractile of the WtP distribution at that point, denoted by $\bar{f}_n$. The adversary needs to commit to only $\bar{f}_n$ in period one and is free to choose the remainder of the distribution in period two. The firm can use its knowledge of $\bar{f}_n$ in its pricing decision for period two. This is formulated as follows:

\[
\begin{align*}
    c^* &= \max_{n \in \{1, \ldots, K\}} \min_{\bar{f}_n \in [0,1]} \max_{t} \min_{\bar{f}} \frac{\Lambda_1 p_n \bar{f}_n + \Lambda_2 \sum_j t_j p_j \bar{f}_j}{(\Lambda_1 + \Lambda_2) \max_j \{p_j \bar{f}_j\}} \\
    \text{s.t.} \quad &\sum_j t_j = 1, \quad t \geq 0 \\
    &1 = \bar{f}_1 \geq \bar{f}_2 \geq \cdots \geq \bar{f}_n \geq \bar{f}_n \\
    &\bar{f}_n \geq \bar{f}_{n+1} \geq \cdots \geq \bar{f}_K \geq 0,
\end{align*}
\]
where $t_j$ is the proportion of time spent at $p_j$ in the second period. Note that for the inner maxmin problem, $\bar{f}_n$ is a given constant rather than an optimization variable. The solution to the inner subproblem is of independent interest as it demonstrates how to price and what is the worst-case WtP distribution in settings where the firm has partial demand information in the form of a sales observation at one price point. This is extended later in Section 2.3.2 to allow for multiple such observations.

The blueprint of our analysis is to show that given $(p_n, \bar{f}_n)$, the problem of the firm decouples into two related subproblems similar to that of Section 2.1: one on the grid $\{p_1, \ldots, p_{n-1}\}$ with a probability mass of $1 - \bar{f}_n$, and the other on the grid $\{p_n, \ldots, p_K\}$ with probability mass of $\bar{f}_n$. For each subproblem, results of Section 2.1 such as the extreme point optimality for adversary's decision continue to hold. The strategy of the firm is again a price skimming policy for each subinterval $[v, p_n)$ and $[p_n, \bar{v}]$ and then $p_n$ to balance the potential revenue loss due to each subinterval.

We start our analysis by noting that the adversary's inner problem in (2.12) is one of minimizing a quasi-concave function in $\bar{f}$ over a polyhedron, as in Section 2.1.2, which admits an extreme point optimal solution. However, instead of $C(K, 1) = K$ extreme points, the partitioned constraints above admits $C(n - 1, 1) \times C(K - n + 1, 1) = (n - 1)(K - n + 1)$ extreme points. For example, one such extreme point corresponds to the $\bar{f}$ vector of the form $((1, 1, \bar{f}_n, \bar{f}_n, \ldots, \bar{f}_n), (\bar{f}_n, \bar{f}_n, \bar{f}_n, 0, 0, \ldots, 0))$, which corresponds to a point mass of size $1 - \bar{f}_n$ at price $p_{j_1}$ and a point mass of size $\bar{f}_n$ at price $p_{j_2}$ for $1 \leq j_1 < n \leq j_2 \leq K$. Exploiting this concave minimization structure, and defining

\[
c_{j_1, j_2} = \frac{\Lambda^1 p_n \bar{f}_n + \Lambda^2(\sum_{j=1}^{j_1} t_j p_j + \sum_{j=j_1+1}^{j_2} t_j p_j \bar{f}_n)}{(\Lambda^1 + \Lambda^2) \max\{p_{j_1}, p_{j_2} \bar{f}_n\}}, \quad 1 \leq j_1 < n \leq j_2 \leq K,
\]
the problem in (2.12) can be rewritten as

\[ c^* = \max_{n \in \{1, \ldots, K\}} \min_{f_n \in [0,1]} \max_{t \geq 0} \min_{1 \leq j_1 < j_2 \leq K} c_{j_1, j_2}. \tag{2.13} \]

That is, the adversary now selects two point masses, the first of size \((1 - \tilde{f}_n)\) that is positioned in \([v,p_n]\), and the second of size \(\tilde{f}_n\) that is positioned in \([p_n,\bar{u}]\).

**Proposition 2.3.1** For given \(p_n\), \(\tilde{f}_n\), and \(t\), the extreme points of the inner minimization problem \(\min_{1 \leq j_1 < j_2 \leq K} c_{j_1, j_2}\) in (2.13) is characterized by a pair of indices \((j_1, j_2)\) that correspond to positioning of the point masses \(1 - \tilde{f}_n\) at \(p_{j_1}\) with \(1 \leq j_1 < n\) and \(\tilde{f}_n\) at \(p_{j_2}\) with \(n \leq j_2 \leq K\) respectively. There exists an optimal solution to (2.13) that places \(1 - \tilde{f}_n\) probability at \(p_1\) or \(\tilde{f}_n\) probability at \(p_n\). Consequently, the optimal choice of \((j_1, j_2)\) is of the form \((j_1, n)\) or \((1, j_2)\).

**Proof:** The proof is divided into two cases. Let us first suppose that \(p_{j_1} < p_{j_2}\tilde{f}_n\) and \(j_1 > 1\) at the optimal solution. Then, the optimal ratio for fixed \(n\), \(\tilde{f}_n\), and \(t\), denoted by \(c(n, \tilde{f}_n, t)\), is

\[
c(n, \tilde{f}_n, t) = c_{j_1, j_2} = \frac{\Lambda^1 p_n \tilde{f}_n + \Lambda^2 (\sum_{j=1}^{j_1} t_j p_{j} + \sum_{j=j_1+1}^{j_2} t_j p_{j} \tilde{f}_n)}{(\Lambda^1 + \Lambda^2) p_{j_2} \tilde{f}_n} \\
\geq \frac{\Lambda^1 p_n \tilde{f}_n + \Lambda^2 (t_1 p_1 + \sum_{j=2}^{j_1} t_j p_{j} \tilde{f}_n + \sum_{j=j_1+1}^{j_2} t_j p_{j} \tilde{f}_n)}{(\Lambda^1 + \Lambda^2) p_{j_2} \tilde{f}_n} \\
= \frac{\Lambda^1 p_n \tilde{f}_n + \Lambda^2 (t_1 p_1 + \sum_{j=2}^{j_2} t_j p_{j} \tilde{f}_n)}{(\Lambda^1 + \Lambda^2) p_{j_2} \tilde{f}_n} \\
= c_{1, j_2}; \tag{2.14} \]

so \(c_{1, j_2}\) is also optimal whenever \(p_{j_1} < p_{j_2}\tilde{f}_n\).

Second, suppose that \(p_{j_1} \geq p_{j_2}\tilde{f}_n\) and \(j_2 > n\) at the optimal solution. Conse-
sequently, the optimal ratio is
\[
c(n, \tilde{f}_n, t) = c_{j_1, j_2} = \frac{\Lambda^1 p_n \tilde{f}_n + \Lambda^2 (\sum_{j=1}^{j_1} t_j p_j + \sum_{j=j_1+1}^{j_2} t_j p_j \tilde{f}_n)}{(\Lambda^1 + \Lambda^2)p_{j_1}} \geq \frac{\Lambda^1 p_n \tilde{f}_n + \Lambda^2 (\sum_{j=1}^{j_1} t_j p_j + \sum_{j=j_1+1}^{n} t_j p_j \tilde{f}_n)}{(\Lambda^1 + \Lambda^2)p_{j_1}}
\]

and \(c_{j_1, n}\) is also optimal whenever \(p_{j_1} \geq p_{j_2} \tilde{f}_n\). □

As a result the problem can be reduced to

\[
c^* = \max_{n \in \{1, \ldots, K\}} \min_{f_n \in [0,1]} \max_{t, c} \sum_{j_1, j_2} c
\]

\[
s.t. \quad c \leq c_{j_1, n}, \quad 1 \leq j_1 < n
\]

\[
c \leq c_{1, j_2}, \quad n \leq j_2 \leq K
\]

\[
\sum t_j = 1, \quad t \geq 0
\]

The next proposition characterizes the worst-case \(\tilde{f}_n\) that the firm can observe and Corollary 2.3.1 formulates the corresponding problem of choosing \(p_n\) to experiment at according to this.

**Proposition 2.3.2** For fixed \(p_n\), there exists an optimal solution of the outer minimization in (2.12) with \(\tilde{f}_n = 0\) or 1. Hence, it is sufficient to restrict attention to \(K\) extreme points where the unit probability mass is allocated to a single price.

Propositions 2.3.1 and 2.3.2 together imply that for a given price \(p_n\) in period one, the adversary's problem decomposes into two subproblems: a) an upper problem where the unit probability mass is placed at some price \(p_j \in [p_n, \bar{v}]\), and b) a lower problem where the unit mass is placed at some price \(p_j \in [\underline{y}, p_{n-1}]\). The adversary selects the solution that yields a smaller ratio. In return, the firms' problem in period one is to choose the price point \(p_n\) that balance the ratios in subproblems a) and b).
Corollary 2.3.1

Let \( c^u(n) = \max \sum_{t_j=1, t \geq 0, n \leq j \leq K} \min \frac{A_1 p_n + A_2 \sum_{j=1}^{j_2} t_j p_j}{(A_1 + A_2)p_{j_2}} \),

and \( c^l(n) = \max \sum_{t_j=1, t \geq 0, 1 \leq j \leq n} \min \frac{A_1 \sum_{j=1}^{j_1} t_j p_j}{(A_1 + A_2)p_{j_1}} \).

the competitive ratio problem in (2.12) reduces to

\[ c^* = \max_{n \in \{1, \ldots, K\}} \min \{c^u(n), c^l(n)\}. \] (2.17)

Observe that both \( c^u(n) \) and \( c^l(n) \) are equivalent to simple linear programs as demonstrated in Section 2.1.2. Furthermore, \( c^l(n) \) is directly equivalent to the single period problem in Section 2.1.2 using only prices \( p_1 = v \) to \( p_{n-1} \), and has its solution readily available.

The choice of \( p_n \) balances two types of risk: a) the firm faces the risks of lost sales and foregone revenue in period one; b) the exposure faced by the seller in each of the subintervals \([v, p_n)\) and \([p_n, v]\), which increases with the ambiguity measured by \( \frac{v}{p_n} \) and \( \frac{p_n}{v} \). As \( A_1 \) decreases, i.e., the emphasis shifts on balancing these risks in period two. Hence, as \( A_1 \to 0 \) and as \( \{p_1, \ldots, p_K\} \) becomes a dense covering of \([v, \bar{v}]\) with \( K \to \infty \), the optimal price at which to experiment is such that the relative ambiguity of the two subproblem becomes equal: specifically optimal \( p_n \) satisfies \( \frac{v}{p_n} = \frac{p_n}{\bar{v}} \), i.e. optimal \( p_n \) is the geometric mean of \( v \) and \( \bar{v} \). This result directly follows from Proposition 2.1.1.

2.3.2 Multiple prices and incorporating partial demand information

Let \( N \) be the number of price points used in the first period with \( 1 \leq N < K \), and label these prices by \( p_{i_1}, p_{i_2}, \ldots, p_{i_N} \). In this subsection we will focus on the
following practically important problem: given a testing schedule \( \{p_i, \ldots, p_{iN}\} \) and the associated fractiles \( \{\tilde{f}_{i1}, \ldots, \tilde{f}_{iN}\} \), how should the firm exploit and incorporate information into its pricing decision? As a byproduct, we will characterize the worst-case market condition based on the observed demand information, which is of interest on its own. Specifically, we will not carry through the full analysis as we did in Section 2.3.1. We illustrate below that analogues of Propositions 2.3.1 and 2.3.2 can be derived and the structure of Corollary 2.3.1 still holds. The solution, however, to the problem of selecting the optimal set of prices \( \{p_i, \ldots, p_{iN}\} \) for testing is combinatorial.
and does not seem to simplify significantly.

For a given set of indices of test prices, \( N = \{i_1, \ldots, i_N\} \), we formulate the problem as:

\[
c_N = \max_{t^1} \min_{f_{i_n} \in [0,1]} \max_{t^2} \min_{f_j} \frac{\Lambda^1 \sum_{i_n \in N} p_{i_n} \bar{f}_{i_n} t^1_{i_n} + \Lambda^2 \sum_{j = 1}^{K} t^2_j p_j \bar{f}_j}{(\Lambda^1 + \Lambda^2) \max_j \{p_j \bar{f}_j\}}
\]  
(2.18)

\[
s.t. \sum_{i_n \in N} t^1_{i_n} = 1, \quad t^1 \geq 0
\]

\[
\sum_{j = 1}^{K} t^2_j = 1, \quad t^2 \geq 0
\]

\[
1 = \bar{f}_1 \geq \cdots \geq \bar{f}_K \geq 0
\]

\[
\bar{f}_j = \bar{f}_{i_n} \quad \forall \ j = i_n \in N.
\]

Note that problem (2.18) mimics the unconstrained learning problem previously studied, which is indeed a special case of (2.18) with \( N = K \).

The essence of the single price analysis carries through in the following sense. First, the inner maxmin problem that pertains to the second period pricing problem decomposes into \( N + 1 \) subproblems in intervals \( [p_{i_1}, p_{i_2}], \ldots, [p_{i_N}, p_K] \) that can be studied using the results of Section 2: the firm uses a price skimming policy within that interval, and the adversary selects a point mass distribution for the probability that belongs to the respective interval. Second, one can “piece” together the above results to characterize the first period behavior which only requires comparing these \( N + 1 \) subproblems. Furthermore, complexity-wise, this overall method requires concentrating at only \( K \) extreme points in total. Together these results yield the solution to (2.18).

The main idea in the proof of Proposition 2.3.1 for the single price analysis is that there exists a price point yielding a maximum revenue rate for a given distribution and information constraint, and allocating the probability mass at all other intervals to
the lowest possible price can only improve the objective function of the adversary (i.e. reduce the ratio), because it potentially reduces the numerator with the denominator unchanged (see equations (2.14) and (2.15)). The same argument goes through when we have more fractile observations with the same steps for each interval. We state this in the following, the proof of which follows similar steps to Proposition 2.3.1 and is therefore omitted.

**Proposition 2.3.3** If fractile information at $N < K$ points are given, one can still restrict attention to $K$ extreme points for the inner minimization problem. For $\eta = 1 \ldots N - 1$, each extreme point is of the form $f_j = f_{i\eta} - f_{i\eta+1}$ for some $i\eta \leq j < i\eta+1$ and $f_{i\eta} = \bar{f}_{i\eta} - \bar{f}_{i\eta+1}$ for all other $n \neq \eta$.

That is we consider a price interval $[p_{i\eta}, p_{i\eta+1})$ one at a time and fix the probability mass for all other intervals at the lowest price possible, i.e. $f_{i\eta} = \bar{f}_{i\eta} - \bar{f}_{i\eta+1}$, $n \neq \eta$, while assuming the mass at this interval is at one of the price points within the interval. Consequently, it is sufficient again to concentrate on a total of $K$ extreme points.

This seemingly simple result is important because the number of extreme points can in general grow exponentially with additional information as explained before. For example, while adding a constraint on the mean of the WtP distribution will increase the number of extreme points to $O(K^2)$, fractile information can be incorporated without effectively increasing the number of extreme points to consider. Furthermore, given the fractile information the resulting competitive ratio problem for the remaining sales horizon can be solved using a simple LP formulation which will be illustrated later in this section.
Proposition 2.3.4  For any feasible $t^1$, there exists an optimal solution for the outer minimization in (2.18) which occurs at an extreme point of the simplex $1 \geq \bar{f}_2 \geq \cdots \bar{f}_K \geq 0$. In other words, the adversary chooses a distribution which allocates the unit probability mass to a single price in the first period problem.

Proof: Fix some $t^1$, let the optimal solution to the inner maximization be $t^2$ and to the inner minimization be $\bar{f}$, and denote the resulting optimal ratio by $c(t^1) := n(t^1)/d(t^1)$, where $n(t^1)$ and $d(t^1)$ denote the corresponding values of the numerator and the denominator respectively in (2.18) at the optimal solution. Also, let $j^* := \arg\max_j \{p_j \bar{f}_j\}$ be the index of the revenue maximizing price.

First, observe that there exists an optimal distribution with $\bar{f}_j = 0$ for $j > j^*$ for the inner problem. To see this, consider the constraints $1 = \bar{f}_1 \geq \bar{f}_2 \geq \cdots \bar{f}_{j^*} \geq \bar{f}_{j^*+1} \geq \cdots \bar{f}_K \geq 0$. Suppose that for any fixed values of $\bar{f}_1, \ldots, \bar{f}_{j^*}$, some of the variables $\bar{f}_{j^*+1}, \ldots, \bar{f}_K$ have positive values. Then, by reducing them to zero, we do not change the value of $p_{j^*} \bar{f}_{j^*} = \max_j \{p_j \bar{f}_j\}$, and hence the the value of the denominator in (2.18), while potentially reducing the value of the numerator. This would yield a lower competitive ratio. It follows that $\bar{f}_{j^*+1} = \cdots \bar{f}_K = 0$ for some optimal solution.

Now, we also show that there exists an optimal distribution with $\bar{f}_{j^*} = 1$ for the inner problem. Suppose that the optimal solution has $\bar{f}_{j^*} < 1$. Then increasing $\bar{f}_{j^*}$ by $\epsilon := 1 - \bar{f}_{j^*}$ would increase the numerator $n(t^1)$ at most by

$$
\epsilon \left( \Lambda^1 \sum_{i_n \leq j^*, i_n \in \mathcal{N}} p_{i_n} t^1_{i_n} + \Lambda^2 \sum_{j \leq j^*} t^2_j p_j \right),
$$

while increasing the denominator exactly by $\epsilon (\Lambda^1 + \Lambda^2) p_{j^*}$. The new competitive
ratio, denoted by $c_\epsilon$, with $\bar{f}_{j^*} = 1$ and $\bar{f}_j = 0$ for $j > j^*$, satisfies

$$c_\epsilon \leq \frac{n(t^1) + \epsilon \left( \Lambda^1 \sum_{i_0 \leq j^*, i_0 \in \mathcal{N}} p_{i_0} t_{i_0}^1 + \Lambda^2 \sum_{j \leq j^*} t_j^2 p_j \right)}{d(t^1) + \epsilon \left( \Lambda^1 + \Lambda^2 \right) p_{j^*}} \leq \frac{n(t^1)}{d(t^1)} = c(t^1), \quad (2.19)$$

which shows that setting $\bar{f}_{j^*} = 1$ is also optimal. The second inequality above follows from

$$\epsilon \left( \Lambda^1 \sum_{i_0 \leq j^*, i_0 \in \mathcal{N}} p_{i_0} t_{i_0}^1 + \Lambda^2 \sum_{j \leq j^*} t_j^2 p_j \right) \leq \frac{\Lambda^1 \sum_{i_0 \leq j^*, i_0 \in \mathcal{N}} p_{i_0} \bar{f}_{i_0} t_{i_0}^1 + \Lambda^2 \sum_{j \leq j^*} t_j^2 p_j \bar{f}_j}{(\Lambda^1 + \Lambda^2) p_{j^*} \bar{f}_{j^*}}$$

as $1 = \bar{f}_1 \geq \bar{f}_2 \geq \cdots \geq \bar{f}_{j^*}$ and $\bar{f}_j = 0$ for $j > j^*$; the inequality holds with equality if and only if $1 = \bar{f}_1 = \bar{f}_2 = \cdots = \bar{f}_{j^*}$. □

Propositions 2.3.3 and 2.3.4 together imply that for a given set $\mathcal{N}$ of first period prices chosen by the firm, the problem of the adversary again decomposes into $N + 1$ subproblems corresponding to each of the intervals $[y, p_{i_1}), [p_{i_1}, p_{i_2}), \ldots, [p_{i_N}, p_K]$. In each subproblem, the adversary positions the corresponding probability mass at a single price point that belongs the respective interval in a way that minimizes the ratio.

**Corollary 2.3.2**

Let

$$c^0 = \max_{i_1, i_2} \min_{1 \leq j_0 < n} \Lambda^2 \sum_{j \leq j_0} t_j^2 p_j \left( \Lambda^1 + \Lambda^2 \right) p_{j_0} = \frac{\Lambda^1 \sum_{i_0 \leq j_0, i_0 \in \mathcal{N}} p_{i_0} t_{i_0}^1 + \Lambda^2 \sum_{j \leq j_0} t_j^2 p_j}{(\Lambda^1 + \Lambda^2) p_{j_0}}, \quad 0 \leq j_0 < n,$$

$$c^n = \max_{i_1, i_2} \min_{i_0 \leq j_0 < i_{n+1}} \Lambda^1 \sum_{i_0 \leq j_0, i_0 \in \mathcal{N}} p_{i_0} t_{i_0}^1 + \Lambda^2 \sum_{j \leq j_0} t_j^2 p_j \left( \Lambda^1 + \Lambda^2 \right) p_{j_0} = \frac{\Lambda^1 \sum_{i_0 \leq j_0, i_0 \in \mathcal{N}} p_{i_0} t_{i_0}^1 + \Lambda^2 \sum_{j \leq j_0} t_j^2 p_j}{(\Lambda^1 + \Lambda^2) p_{j_0}}, \quad n = 1 \ldots N,$$

the competitive ratio problem in (2.18) reduces to

$$c_N = \min_{n \in \{0, \ldots, N\}} \{c^n\} \quad (2.20)$$
Once again each subproblem \( c^n \) is equivalent to a linear program. The next step would ideally be optimizing the set of test prices \( \mathcal{N} \subset \{1, \ldots, K\} \) in period one. However, this is purely a combinatorial problem that requires numerical techniques, and therefore, is left out of our discussion.

A practically important special case to the problem (2.18) above can be used to incorporate additional information available to the dynamic pricing problem of Section 2.1. The overall setting is the same, but instead of a learning period, the fractile information is assumed to be readily available for a subset \( \mathcal{N} \) of prices. This limited information could represent an expert opinion, an industry forecast, past experience, or the result of price testing. Mathematically, this problem, which is equivalent to the inner minimax formulation of (2.18) with \( \Lambda^1 = 0 \), is given by:

\[
\mathcal{e}_\mathcal{N} = \max_t \min_f \frac{\sum_{j=1}^K t_j p_j \bar{f}_j}{\max_j \{p_j \bar{f}_j\}}
\]

\[
s.t. \quad \sum_{j=1}^K t_j = 1, \quad t \geq 0
\]

\[
1 = \bar{f}_1 \geq \cdots \geq \bar{f}_K \geq 0
\]

\[
\bar{f}_j = \bar{f}_i \quad \forall \ j = i \in \mathcal{N}.
\]

Fractile information for a subset of prices \( \{p_{i_1}, \ldots, p_{i_N}\} \) can be incorporated without increasing the complexity of the problem, as explained by Proposition 2.3.3. The resulting problem can be reduced to an LP with \( K \) constraints. Each extreme point identified in Proposition 2.3.3 corresponds to a linear upper bound constraint on the
objective function of the following equivalent LP formulation:

$$\begin{align*}
\max_{t, c} & \quad c \\
\text{s.t.} & \quad c \leq \frac{t_j p_j f_{i_{\eta}} + \sum_{i_{\eta}, i_{i_{\eta}} \in N} t_{i_{\eta}} p_{i_{\eta}} f_{i_{i_{\eta}}}}{\max\{p_j f_{i_{\eta}}, \max_{i_{\eta}, i_{i_{\eta}} \in N} \{p_{i_{\eta}} f_{i_{i_{\eta}}}\}\}} \\
& \quad \eta = 1 \ldots N - 1, \quad i_{\eta} \leq j < i_{\eta+1} \\
& \quad \sum_{j=1}^{K} t_j = 1, \quad t \geq 0.
\end{align*}$$

The solution of this LP provides both a pricing policy recommendation and the corresponding optimal competitive ratio. The actual revenue performance of the policy $t$ is quite good across many common demand functions as illustrated with the numerical examples reported below, even when the fractile/sales information is available at only a few price points.

### 2.3.3 Numerical examples

We conclude this section with a set of numerical results that highlight the revenue improvement that is achieved through partial learning under our policy. Our experiments contrast the "no information" policy of Sections 2.1 to the partial information policy extracted via (2.22) if one is given fractile information at a set of price points. The fractile data was generated using four common WtP distributions, each of which corresponds to common demand model listed in Table 2.2. We restricted the WtP to the range $[v, \bar{v}]$ for each distribution. For the Normal and Gumbel distributions we extracted the mean as the midpoint of the range and that standard deviation by assuming that the range is equal to $\pm 3\sigma$. For the exponential distribution we assumed that the WtP of a typical consumer is given by $v + w$, where $w$ is an exponentially distributed in $[0, \bar{v} - v]$ and its rate parameter is selected so that the probability that $w$ lies in that range is 99.5% (this is consistent with the $\pm 3\sigma$ assumption of the
Normal distribution).

In each test case, we also compared against a policy that tries to make use of the observed fractile information by first fitting an exponential demand model to this data, and then use this model to compute a static price to be used throughout the sales horizon. The latter heuristic will, of course, turn out to be optimal in test cases that correspond to an underlying exponential WtP distribution, but its performance on the other three test cases will give a rough idea of the performance loss due to the wrong parametrization of the demand model.

<table>
<thead>
<tr>
<th>WtP distribution</th>
<th>Demand model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>Linear demand</td>
</tr>
<tr>
<td>Exponential</td>
<td>Exponential demand</td>
</tr>
<tr>
<td>Normal</td>
<td>Probit demand</td>
</tr>
<tr>
<td>Gumbel</td>
<td>Logit demand</td>
</tr>
</tbody>
</table>

Table 2.2: WtP distributions and corresponding demand models.

The three sets of results summarized in Tables 2.3 - 2.5 illustrate the performance of the policy derived using the competitive ratio analysis in a variety of settings as we varied the range of the WtP distribution, the number of test prices for which the seller has observed information, and also as we varied the ambiguity of the range information, which is captured by the ratio $\bar{v}/v$. The last example reported in Table 2.6 illustrates the performance of the pricing policy extracted form the competitive ratio analysis when only one price point is tested in period one and as we vary the positioning of the price point within the predefined range $[v, \bar{v}]$. Note that the range of WtP distribution for that last example coincides with that in Table 2.4.

There are several observations to be made. First, although the competitive ratio is conservative, the actual revenue performance of the policy across different distributions is significantly higher. Actually, as the ambiguity ratio $\bar{v}/v$ gets smaller
Table 2.3: Competitive ratio with and without fractile information, and corresponding revenue performance under common WtP distributions. \( K = 500 \) prices, price grid \([1,500]\) in increments of 1. Sales (fractile) information at 3 price points: 125, 250, and 375.

<table>
<thead>
<tr>
<th>WtP Dist.</th>
<th>CR w/o Info</th>
<th>Rev Perf</th>
<th>CR with Fractile Info</th>
<th>Rev Perf</th>
<th>Rev Perf under Exponential Fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>14.7</td>
<td>29.4</td>
<td>71.0</td>
<td>96.0</td>
<td>89.1</td>
</tr>
<tr>
<td>Exponential</td>
<td>14.7</td>
<td>39.6</td>
<td>47.1</td>
<td>90.9</td>
<td>100</td>
</tr>
<tr>
<td>Normal</td>
<td>14.7</td>
<td>25.2</td>
<td>65.0</td>
<td>87.4</td>
<td>47.7</td>
</tr>
<tr>
<td>Gumbel</td>
<td>14.7</td>
<td>24.4</td>
<td>64.8</td>
<td>85.7</td>
<td>42.4</td>
</tr>
</tbody>
</table>

Table 2.4: \( K = 100 \) prices, price grid \([1,100]\) in increments of 1. Sales (fractile) information at 5 price points: 16, 33, 50, 66, and 83.

<table>
<thead>
<tr>
<th>WtP Dist.</th>
<th>CR w/o Info</th>
<th>Rev Perf</th>
<th>CR with Fractile Info</th>
<th>Rev Perf</th>
<th>Rev Perf under Exponential Fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>19.3</td>
<td>38.2</td>
<td>80.3</td>
<td>98.1</td>
<td>91.8</td>
</tr>
<tr>
<td>Exponential</td>
<td>19.3</td>
<td>38.4</td>
<td>80.1</td>
<td>98.1</td>
<td>100</td>
</tr>
<tr>
<td>Normal</td>
<td>19.3</td>
<td>32.6</td>
<td>77.5</td>
<td>95.8</td>
<td>54.0</td>
</tr>
<tr>
<td>Gumbel</td>
<td>19.3</td>
<td>31.6</td>
<td>77.6</td>
<td>95.3</td>
<td>50.8</td>
</tr>
</tbody>
</table>

Table 2.5: \( K = 100 \) prices, price grid \([51,150]\) in increments of 1. Sales (fractile) information at 5 price points: 66, 83, 100, 116, and 133.

<table>
<thead>
<tr>
<th>WtP Dist.</th>
<th>CR w/o Info</th>
<th>Rev Perf</th>
<th>CR with Fractile Info</th>
<th>Rev Perf</th>
<th>Rev Perf under Exponential Fit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>48.3</td>
<td>85.1</td>
<td>84.7</td>
<td>98.0</td>
<td>98.0</td>
</tr>
<tr>
<td>Exponential</td>
<td>48.3</td>
<td>65.6</td>
<td>82.5</td>
<td>92.2</td>
<td>100</td>
</tr>
<tr>
<td>Normal</td>
<td>48.3</td>
<td>67.5</td>
<td>85.7</td>
<td>96.6</td>
<td>84.7</td>
</tr>
<tr>
<td>Gumbel</td>
<td>48.3</td>
<td>66.3</td>
<td>86.0</td>
<td>96.8</td>
<td>81.8</td>
</tr>
</tbody>
</table>

the performance of the policy derived from the competitive ratio analysis increases significantly, e.g. from 19.3% to 48.3% between the second and third examples. Second, partial information (and learning) significantly increases both the performance guarantee of the competitive ratio policy and the actual revenue performance across
Table 2.6: Competitive ratio and actual revenue performance as % of optimal when the firm only experiments with one price. $K = 100$ prices, price grid $[1,100]$ in increments of 1.

distributions. Third, the revenue performance of the competitive ratio policy with partial information is very good across all distributions. Significant gains are achieved even when experimenting at a small number of price points: sampling at just 3 prices out of 500 achieves in excess of 85% of the maximum achievable revenues under full information across all distributions tested even when the ambiguity ratio is very high. Decreasing the ambiguity ratio, from 500 to 100, and increasing the number of test prices slightly, from 3 to 5 points, increases the guaranteed performance to 95% across all test cases. Fourth, using partial information to fit an incorrect parametric model to the unknown distribution can lead to substantial revenue loss. For example, in Table 2.3 above, fitting an exponential distribution for the underlying Gumbel distribution results in only 42.4% of the maximum achievable revenues, whereas the competitive ratio policy can use the partial information to capture 85.7%. In fact, competitive ratio policy with partial information outperforms parametric fitting across all distributions, except for the exponential case, for which exponential fitting is optimal.

Finally, in all of our experiments we observed that the performance of the policy extracted via the competitive ratio analysis performed very well even when the test prices did not happen to fall close to the optimal price for the underlying WtP distribution. The results in Table 2.6 provide an illustration of this last comment by
focusing at the performance of the policy extracted via the competitive ratio analysis as we vary the position of a single price point for which the seller has fractile information. As a benchmark we note that the competitive ratio and actual revenue performance of the policy based on the competitive ratio analysis without the fractile information was reported in the first and second columns of Table 2.4. In addition, the optimal prices that would correspond to each of the WtP distributions that we tried were 50 for the Uniform, 19 for the Exponential, 39 for the Normal and the Gumbel distributions.

2.4. Other pricing mechanisms

2.4.1 Third degree price discrimination

We consider the problem of revenue maximization for a monopolist firm that offers a product to a market of size Λ. Each consumer has a WtP for one unit of that product, denoted by v, that is an independent draw from a common distribution F with support [v, \bar{v}]. We assume that the market can be segmented into distinct groups, e.g., through the observation of a consumer attribute or—as in the case of call centers—using consumer information that has been collected through prior interactions that the firm may have had with each consumer. This segmentation procedure separates the market, and the WtP distribution, into K contiguous groups, where segment j comprises of consumers with WtP v ∈ [v_j, v_{j+1}). Let v = \{v_1, \ldots, v_K\} and v_0 = v to reduce the notation. The firm can customize its price to each segment. For a price vector p, the firm’s revenue is given by R(v, p, F) = \sum_{j=0}^{K-1} p_j \Lambda P(v_{j+1} > v \geq p_j) + p_K \Lambda P(\bar{v} \geq v \geq p_K).
The full information problem has two parts: the first is to select its pricing vector \( p \) to optimize her revenues under a given segmentation rule \( v \), and the second is to also optimize over its choice of segments as well. The solution of the first question is easy to describe: for each segment the firm selects a price to maximize revenues from that segment that is characterized by the conditional WtP distribution on the subinterval \([v_j, v_{j+1})\), and the pricing decisions across segments are decoupled (this follows from the absence of the capacity constraint). The problem of selecting the segments within the WtP distribution can also be formulated as an optimization problem, whose tractability, however, depends on the structure of the distribution \( F \). In the sequel, we will denote by \( p^*(v) \) the optimal pricing rule given \( v \) for the competitive ratio or maximum regret problem, by \( v^* \) the optimal segmentation rule, and by \( v^*(F) \) and \( p^*(v, F) \) the revenue maximizing segmentation and price vector for given \( v \) respectively under full information about distribution \( F \). In the remainder of this section we will study the above two questions in a setting where the firm knows—or can choose—the segmentation rule \( v \) and the price vector \( p(v) \), but does not know the WtP distribution \( F \).

### 2.4.1.1 Competitive ratio

Taking the segmentation decision \( v \) as fixed to start with, the firm’s third degree price discrimination problem under the competitive ratio criterion is the following:

\[
\max_{p} \min_{p} \left\{ \frac{R(v, p, F)}{R(v, p^*(v, F), F)} : p_j \in [v_j, v_{j+1}) \text{ for } j = 0, \ldots, K - 1, \ p_K \in [v_K, \bar{v}] \right\}.
\]  

(2.23)

**Proposition 2.4.1** Given a segmentation \( v \), the optimal price vector \( p \) for (2.23) is:

\( p_j^* = v_j \) for all \( j \).
Proof: Assume by contradiction $p_j^* > v_j$ for some $j$. Then, the adversary can set everyone's valuation at $v_j$ deterministically, which would result in zero revenues, and therefore, a competitive ratio of zero. Setting $p_j^* = v_j$ for all $j$ would guarantee a positive ratio. □

The above result is somewhat expected, since both the firm's and adversary's decisions are essentially decoupled across segments, and the above strategy is the only one that will result in non-zero revenues. Consequently, the total revenue extracted by the firm is given by

$$R(v, p^*, F) = \sum_{j=0}^{K-1} v_j \Lambda P(v_{j+1} > v \geq v_j) + v_K \Lambda P(\bar{v} \geq v \geq v_K) = \Lambda \sum_{j=0}^{K} v_j f_j,$$

where $f_j = P(v_{j+1} > v \geq v_j)$ for $j = 0, \ldots, K - 1$ and $f_K = P(\bar{v} \geq v \geq v_K)$. Note that for a given segmentation rule $v$, the revenue earned by the firm under the optimal pricing policy does not depend on the particular specification of the distribution $F(\cdot)$ on the interval $[v_j, v_{j+1})$, but only on the total probability mass $f_j$ allocated to each such interval. However, the competitive ratio may potentially depend on the specifics of the distribution $F(\cdot)$ on each such interval, as different WtP distributions yield different first-best revenues. The next result characterizes the optimal competitive ratio for a given a segmentation and the WtP distribution that is selected by the adversary.

Proposition 2.4.2 Let $j^* = \arg\min_{j} \{v_j/v_{j+1}\}$. Given a segmentation $v$, the adversary's strategy is to set the WtP of every customer equal to $(v_{j^*+1} - \epsilon)$, where $\epsilon > 0$ is an arbitrarily small constant chosen by the adversary. The optimal competitive ratio for (2.23) is

$$c(v) = v_{j^*}/v_{j^*+1} + \delta,$$

(2.24)

where $\delta = \epsilon v_{j^*}/(v_{j^*+1}^2 - \epsilon v_{j^*+1})$ is also arbitrarily small.
**Proof:** By Proposition 2.4.1, the firm selects the price vector \( p^* = v \) for given \( v \).

Also, note that the full information revenue for given \( v \) satisfies

\[
R(v, p^*(v, F), F) \leq \Lambda \left( \sum_{j=0}^{K-1} \lim_{x \to v_{j+1}} \int_{v_j}^{x} v \, dF(v) + \int_{v_K}^{v^0} v \, dF(v) \right)
\]

as the right hand side of the inequality would correspond to potential maximum revenue that could be extracted if we could charge every single customer their WtP. Then,

\[
c(v) = \min_F \frac{\Lambda \sum_{j=0}^{K} v_j f_j}{R(v, p^*(v, F), F)} \geq \min_F \frac{\sum_{j=0}^{K-1} \lim_{x \to v_{j+1}} \int_{v_j}^{x} v \, dF(v) + \int_{v_K}^{v^0} v \, dF(v)}{\sum_{j=0}^{K} v_j f_j}
\]

\[
= \min_{\{f : \sum_j f_j = 1, f \geq 0\}} \frac{\sum_{j=0}^{K} v_j f_j}{\sum_{j=0}^{K} v_{j+1} f_j} \quad (2.25)
\]

\[
= \frac{v_j}{v_{j+1}} = \frac{v_{j^*}}{v_{j^*+1}}
\]

where the third equality follows from the fact that the function in (2.25) above is quasi-concave in \( f \), and therefore the minima is attained at one of the extreme points of the polyhedron defined by \( \sum_j f_j = 1, f \geq 0 \). Each of the \( K \) extreme points corresponds to a policy that allocates all probability mass at the right end point of one of the segments \( v_{j+1} \), and the minimum occurs at \( j^* \).

To complete the proof we verify that the above lower bound can be approached arbitrarily under the adversary strategy that allocates all probability mass at \( (v_{j^*+1} - \epsilon) \) for arbitrarily small \( \epsilon > 0 \); the resulting competitive ratio is \( c(v) = v_{j^*}/(v_{j^*+1} - \epsilon) = v_{j^*}/v_{j^*+1} + \delta \) for \( \delta \) specified above. \( \square \)
Consequently, the competitive ratio is arbitrarily close to $v_j/v_{j+1}$ in the limit. The next result determines the most favorable segmentation $v$ for firm. This could either be selected by the firm to be such, or it could be exogenously specified; indeed, in many settings where the firm can third degree price discriminate its market, the segments are fixed by exogenous factors (predetermined age groups, laws, etc.) that are not controllable by the firm. The optimization problem that determines the most favorable segmentation $v$ is as follows:

$$\max_{v,c} \left\{ c : c \leq v_j/v_{j+1}, \quad j = 0, \ldots, K, \quad v \leq v_1 \leq \cdots \leq v_K \leq v \right\}. \quad (2.26)$$

This formulation can be reduced to a linear program by taking logarithms and solved analytically.

**Proposition 2.4.3** The optimal competitive ratio achievable if the firm segments its market into $K$ segments is $c = (y/v)^{1/(K+1)} + \delta$, for an arbitrarily small $\delta > 0$ chosen by the adversary. The optimal segmentation $v^*$ corresponds to $v_j^* = y c^{-j}$, $j = 0, \ldots, K$, and the optimal pricing rule is $p_j^* = v_j^* = y c^{-j}$, $j = 0, \ldots, K$.

**Proof:** Taking logarithms of the variables in (2.26), and defining $\tilde{c} = \ln(c)$ and $\tilde{v}_j = \ln(v_j)$ the formulation can be reduced to the following LP

$$\max_{\tilde{v}, \tilde{c}} \left\{ \tilde{c} : \tilde{c} \leq \tilde{v}_j - \tilde{v}_{j+1}, \quad j = 0, \ldots, K, \quad \ln(y) \leq \tilde{v}_1 \leq \cdots \leq \tilde{v}_K \leq \ln(v) \right\}. $$

Solving this LP analytically using its dual, we see that the optimal dual variables for the constraints $\tilde{c} \leq \tilde{v}_j - \tilde{v}_{j+1}$ are all strictly positive, which yields $c^* = \frac{y}{v_1} = \frac{y}{v_2} = \cdots = \frac{y}{v_K}$. Solving these equalities recursively yields the result. $\square$

### 2.4.1.2 Maximum regret

The analysis of the maximum regret criterion for the third degree price discrimination problem is similar. We will start by fixing a segmentation rule $v$, and focus on the
Proposition 2.4.4  Given a segmentation \( \mathbf{v} \), the optimal price vector \( \mathbf{p} \) is given by
\[
p_j^* = \max\{v_j, v_{j+1}/2\} \text{ for all } j.
\]

Proof: Given \( \mathbf{v} \) and a price vector \( \mathbf{p} \), we first consider the adversary's strategy in a given segment \([v_j, v_{j+1})\). [To keep things brief, the argument that follows only outlines a skeleton of the main results that need to be established, which could be done following the approach of the previous subsection.] As outlined in Lemma 2.4.2 above, for a given distribution \( F(\cdot) \), the maximum revenue that can be extracted from this segment is upper-bounded by
\[
\lim_{x \to v_j^-} \int_{v_j}^{v_{j+1}} v \, dF(v) < v_{j+1} f_j.
\]
Consequently, when \( p_j = v_j \), the optimal response of the adversary is again to allocate the probability mass \( f_j \) arbitrarily close to valuation \( v_{j+1} \), that is \( P(v = v_{j+1} - \epsilon) = f_j \) for arbitrarily small \( \epsilon > 0 \), resulting in a regret of \( (v_{j+1} - v_j - \epsilon) \Lambda f_j \) for this segment. Similarly, one can show that when \( p_j > v_j \), the optimal response is to allocate the probability mass \( f_j \) arbitrarily close to either \( v_{j+1} \), resulting in a regret of \( (v_{j+1} - p_j - \epsilon) \Lambda f_j \), or to \( p_j \), resulting in a regret of \( (p_j - \epsilon) \Lambda f_j \) as no customer in segment \( j \) makes a purchase under this scenario. As a result, defining
\[
r_j(\mathbf{v}, \mathbf{p}) = \begin{cases} 
(v_{j+1} - v_j) - \epsilon & \text{if } p_j = v_j \\
\max\{(v_{j+1} - p_j), p_j\} - \epsilon & \text{if } p_j > v_j,
\end{cases}
\]
where \( \epsilon > 0 \) chosen to be arbitrarily small, the maximum regret problem can be written as
\[
r^* = \min_{\mathbf{v}} \min_{\mathbf{p}} \max_f \sum_{j=0}^{K} r_j(\mathbf{v}, \mathbf{p}) f_j
\]
subject to the constraints \( v = v_0 \leq v_1 \leq \cdots \leq v_K \leq v_{K+1} = \bar{v} \) and \( p_j(\mathbf{v}) \in [v_j, v_{j+1}) \) for \( j = 0 \ldots K - 1 \), and \( p_K(\mathbf{v}) \in [v_K, v_{K+1}] \).
Using the minimax inequality, we have

\[
    r^* = \min_v \min_p \max_f \sum_{j=0}^{K} r_j(v, p) f_j
\]

\[
    = \min_v \max_f \sum_{j=0}^{K} r_j(v, p) f_j
\]

\[
    = \min_v \max_f \sum_{j=0}^{K} \left( \min_{p_j \in [v_j, v_{j+1}]} r_j(v, p) \right) f_j, \tag{2.27}
\]

where the first step follows because the minimax inequality holds with equality since the objective function is linear in \( f \) and convex in \( p \), and the second follows from the fact that the inner minimization is separable in prices.

As a result, for a fixed \( v \), the optimal pricing policy solves the problem \( \min_{p_j \in [v_j, v_{j+1}]} r_j(v, p) \) for all \( j \). By definition of \( r_j(v, p) \), we have

\[
    \min_{p_j \in [v_j, v_{j+1}]} r_j(v, p) = \min \left\{ (v_{j+1} - v_j - \epsilon), \min_{v_{j+1} > p_j > v_j} \max \{ (v_{j+1} - p_j), p_j \} - \epsilon \right\}
\]

\[
    \geq \min \left\{ (v_{j+1} - v_j - \epsilon), \min_{v_{j+1} > p_j > v_j} \max \{ (v_{j+1} - p_j), p_j \} - \epsilon \right\}
\]

\[
    = \min \left\{ (v_{j+1} - v_j - \epsilon), \max \{ v_j, \frac{v_{j+1} - v_j}{2} \} - \epsilon \right\}
\]

\[
    = \begin{cases} 
        \frac{v_{j+1} - v_j}{2} - \epsilon, & \text{if } v_j < \frac{v_{j+1}}{2} \\
        v_{j+1} - v_j - \epsilon, & \text{if } v_j \geq \frac{v_{j+1}}{2}
    \end{cases}
\]

\[
    = \min \{ (v_{j+1} - v_j), \frac{v_{j+1} - v_j}{2} \} - \epsilon, \tag{2.28}
\]

for arbitrarily small \( \epsilon > 0 \), which is the regret from segment \( j \) that is achieved by setting \( p^*_j = \max \{ v_j, v_{j+1}/2 \} \). \( \square \)

Below, we provide the main result of the section. The proof is relegated to Appendix A.

**Proposition 2.4.5** Let \( \bar{v} = \max \{ v, \bar{v}/(K+2) \} \). The optimal regret is \( r^* = \Lambda \frac{\bar{v} - \bar{v}}{K+1} - \epsilon \) for an arbitrarily small \( \epsilon > 0 \) chosen by the adversary. The optimal segmenting
and pricing strategy spans the range \([\bar{v}, \tilde{v}]\) in a uniform manner, i.e. \(v_j^* = p_j^* = \bar{v} + \frac{\tilde{v} - \bar{v}}{K+1} j, \quad j = 1, \ldots, K,\) and the customers in the range \([y, \bar{v})\) are not served when \(y < \bar{v} \).

We see that the worst-case performance of each criterion deteriorates as a function of the relative or the absolute difference between the lowest and the highest valuations, which measures the inherent uncertainty of the underlying customer WtP distribution in each case. Under both criteria, the worst-case performance improves as the number of segments (and price points) \(K\) increases. As \(K \to \infty\), the firm can (perfectly) first-degree price discriminate the market, and the effect of market uncertainty becomes irrelevant.

The stylized analysis of this section highlighted the structure of the worse case market scenarios for the firm under which each segment is homogeneous but has an uncertain representative WtP, and where the size of each of these segments is uncertain itself. The firm’s best response is to price conservatively in each of these segments. One important aspect of the third degree price discrimination problem that has been suppressed in this formulation is on the relation between the observable consumer attribute along which the market is segmented and the corresponding WtP. We have assumed that the firm can separate the market into segments that form contiguous intervals in terms of their WtP, which need not be the case. A more elaborate model would allow these intervals to be overlapping and allow the firm to dynamically price within each segment. Given that there few or no constraints that link the pricing decisions across segments, the resulting solution would be to adopt appropriate price skimming policies within each segment; this follows from our analysis in the previous section.
2.4.2 Second price auction

Another popular sales format that has attracted increased attention lately is the use of auctions. The analysis of most auction mechanisms requires strong informational requirements such as that the bidders' valuations to be independent and identically distributed random variables drawn from some common distribution, which, moreover, must be known to all bidders and the seller. Both of these assumptions are hard to justify in many application settings, raising the question of designing auction mechanisms with low informational requirements. One way to approach this problem is to search for the mechanism that optimizes some appropriate minimax (robust) or relative performance criterion among all possible mechanisms; the related literature was briefly discussed in the introduction. Another approach, which we follow in our work, is to fix the auction mechanism itself, and to optimize over its design parameters in order to optimize the respective performance criterion. Specifically, we adopt the so called independent private value (IPV) model (see Vickrey [76], McAfee and McMillan [54]) under the assumption that the valuation distribution \( F \) is unknown to the seller and the bidders. A natural auction format to consider is that of the second price, sealed-bid auction, which has low informational requirements, and for which it is known that truthful bidding remains a dominant strategy even when the valuation distribution is unknown to the bidders; i.e., it is optimal for each bidder to bid their true valuation irrespective of what other bidders may do. Given this selling mechanism, the remaining decision for the seller is to select the reserve price, i.e., the minimum bid that the seller is willing to accept in order to allocate the good.

In more detail, we consider a seller with a single indivisible good, and \( N \) bidders trying to buy the good through a second price auction mechanism. The seller's own
valuation of the product is denoted by $v_0$. Bidder $i$ knows her own valuation $v_i$, which is assumed to be an independent draw from a common distribution denoted by $F(\cdot)$ that takes values on some support $v \in [v, \bar{v}]$. The seller sets a reserve price $v_s$, below which he is not willing to allocate the good. Each bidder $i$ submits a sealed bid $b_i$ for the good. Let $b_{[1]}$ and $b_{[2]}$ denote the highest and second highest bids, respectively. Then, in the second price auction, the highest bidder provided that $b_{[1]} \geq v_s$ gets awarded the good and pays the seller $\max(v_s, b_{[2]})$; all other bidders do not pay the seller. If $b_{[1]} < v_s$, then the good is not awarded. It is well known that it is a dominant strategy for each bidder to bid truthfully, i.e., $b_i = v_i$, even if the valuation distribution is unknown to the bidders, and potentially, is not common across all bidders (see again Vickrey [76]). Assuming that the seller knows the valuation distribution $F$, the reserve price given by

$$v_a = \arg\max\{v : v - (1 - F(v))/f(v) = 0\}$$

optimizes the seller’s expected revenue among all possible mechanisms. We let $R(v_s, F)$ denote the expected revenue achieved under this auction when the reserve price is set equal to $v_s$, and $R(v_a, F)$ be the maximum expected revenue achieved under the optimal reserve price $v_a$. Note that while the optimal reserve price is independent of the number of bidders $N$, the expected revenue $R(v_s, F)$ will depend on $N$, which captures the scarcity of the good.

The following two subsections will address the above problem in a setting where the seller does not know the valuation distribution $F$ but knows its support $[v, \bar{v}]$, under the competitive ratio and maximum regret criteria, respectively. In both cases, we assume that an imaginary adversary is selecting the valuation distribution and that this adversary knows the seller’s valuation $v_0$. Given the valuation distribution
chosen, bidders’ behavior satisfy the assumptions of the independent private value model. The seller is interested in selecting a reserve price that hedges against the worst-case valuation distribution selected by the adversary. We assume that the dynamics of the system is as follows: first, the seller selects a reserve price $v_s$ without knowing the valuation distribution of the bidders, and then, the nature selects a worst-case valuation distribution to minimize the relative performance measure used by the firm.

2.4.2.1 Competitive ratio

Note that the seller never selects a reserve price lower than her own valuation $v_0$, or the lowest possible bidder valuation $v$, as she can guarantee that much utility with probability one. That is, the seller will select a reserve price $v_s$ in the interval $[\bar{v}, \bar{v}]$, where $\bar{v} := \max\{v_0, v\}$. Given the above definitions, the competitive ratio criterion is the following:

$$c = \max_{v_s \in [\bar{v}, \bar{v}]} \min_F \frac{R(v_s, F)}{R(v_a, F)}.$$  

(2.30)

Given $v_s$, the distribution selection problem of the adversary can be viewed as first deciding on the revenue maximizing reservation price $v_a$ implied, and then selecting the worst-case distribution implying that reservation price. The revenue expressions and the competitive ratio depends on whether $v_a \geq v_s$ or $v_a < v_s$. The following lemma shows that indeed the problem can be partitioned into dealing with these two respective cases separately; the proof of the lemma is given in Appendix A.

**Lemma 2.4.1** Let $q \in (0,1)$ denote the root of the polynomial $(\bar{v} - v_0)q^N - N\bar{v}q + (N - 1)\bar{v} = 0$. For any reserve price $v_s$ set by the seller, nature’s response is one of the following two:
• select a deterministic valuation at \( v_s - \epsilon \) for some arbitrarily small \( \epsilon > 0 \),

• select a valuation distribution with \( P(v = \bar{v}) = 1 - q \), \( P(v = \bar{v}) = q \),

whichever yields a lower competitive ratio. The resulting competitive ratio of the seller is

\[
\frac{c(v_s)}{\min \left\{ \frac{v_0}{v_s - \epsilon}, \frac{\bar{v} p_a + v_s p_b + v_0 p_c}{\bar{v} p_a + \bar{v} p_b + v_0 p_c} \right\} }
\]

where \( p_a, p_b, p_c \) denotes respectively the probabilities of seeing at least two bids at \( \bar{v} \), exactly one bid at \( \bar{v} \), and no bids at \( \bar{v} \) out of \( N \) bidders under the second strategy above.

The two cases identified in the above lemma highlight the risks faced by the seller. The adversary selects either an extreme point valuation distribution concentrated arbitrarily close to \( v_s \), or a Bernoulli type random valuation that allocates considerable amount of mass at \( \bar{v} \). The former option creates a lost sales risk when \( v_s \) is high, while the latter option creates a risk of foregone revenue when \( v_s \) is low. The optimal policy of the seller is to try to balance these two risks. This is reminiscent of our analysis in §2.1. The first argument of the minimum formula for \( c(v_s) \) in (2.31) is strictly decreasing function in \( v_s \). Quantities \( q, p_a, p_b, p_c \) are independent of \( v_s \), and therefore the second argument in (2.31) is strictly increasing in \( v_s \). It follows from this monotonicity observations that \( c(v_s) \) is maximized by setting a reserve price \( v_s^* \) where these two expressions are equal.

**Proposition 2.4.6** The optimal reserve price of the seller for the competitive ratio problem solves the equation

\[
c^* = \frac{v_0}{v_s^* - \epsilon} = \frac{\bar{v} p_a + v_s^* p_b + v_0 p_c}{\bar{v} p_a + \bar{v} p_b + v_0 p_c},
\]

where \( \epsilon > 0 \) is chosen by the adversary to be arbitrarily small.
Proof: Using Lemma 2.4.1, the competitive ratio is given by

\[ c = \max_{v_s \in [\bar{v}, \tilde{v}]} \min \{ c_1(v_s), c_2(v_s) \} = \max_{v_s \in [\bar{v}, \tilde{v}]} \min \left\{ \frac{v_0}{v_s - \epsilon}, \frac{\bar{v} p_a^* + v_s p_b^* + v_0 p_c^*}{\bar{v} p_a^* + \bar{v} p_b^* + v_0 p_c^*} \right\}, \]

for an arbitrarily small \( \epsilon > 0 \) chosen by the adversary. \( c_1(v_s) = \frac{v_0}{v_s - \epsilon} \) is a strictly decreasing function of \( v_s \). Also, as \( p_a^*, p_b^*, p_c^* \) are independent of \( v_s \), \( c_2(v_s) = \frac{\bar{v} p_a^* + v_s p_b^* + v_0 p_c^*}{\bar{v} p_a^* + \bar{v} p_b^* + v_0 p_c^*} \) is a linear increasing function of \( v_s \). Therefore, the optimal reserve price occurs where these two functions have equal value. \( \square \)

Solving for \( v_s^* \) is equivalent to solving the higher order polynomial equation in Lemma 2.4.1, which is not doable in closed-form, in general. It is easily solvable numerically for any specific value for number of bidders, \( N \).

**Proposition 2.4.7** If the adversary is also allowed to choose the number of bidders \( N \in [\hat{N}, \tilde{N}] \) in the above setting, she selects it to be as small as possible, i.e. \( N^* = \hat{N} \).

**Proof:** We now consider the problem

\[ c = \max_{v_s \in [\bar{v}, \tilde{v}]} \min_{N \in [\hat{N}, \tilde{N}]} \min_F \frac{R(v_s, F)}{R(v_a, F)}. \]  \hspace{1cm} (2.32)

For fixed \( v_s \) and \( N \) the solution to the inner minimization problem is

\[
c(v_s, N) := \min \left\{ \frac{v_0}{v_s - \epsilon}, \frac{\bar{v} p_a^* + v_s p_b^* + v_0 p_c^*}{\bar{v} p_a^* + \bar{v} p_b^* + v_0 p_c^*} \right\}
= \min \left\{ \frac{v_0}{v_s - \epsilon}, \frac{\bar{v} [1 - (N(1-q)q^{N-1} + q^N)] + v_s N(1-q)q^{N-1} + v_0 q^N}{\bar{v} (1-q^N) + v_0 q^N} \right\}
\]

as characterized by Lemma 2.4.1. For fixed \( v_s \), the second term is a strictly increasing function of \( N \), and therefore \( c(v_s, N) \) is a non-decreasing function of \( N \). Consequently, the adversary's response is to select as small an \( N \) as possible in the outer minimization problem, i.e. \( N = \hat{N} \). \( \square \)
2.4.2.2 Maximum regret

The regret formulation of the above problem is given below:

\[
   r = \min_{v_a \in [v_s, v]} \max_F [R(v_a, F) - R(v_s, F)] .
\]  

(2.33)

The proof of Lemma 2.4.1 in the competitive ratio section proceed by lower bounding the actual expected revenue earned by the firm and upper bounding the maximum revenue the firm could extract if she knew about the valuation distribution. Therefore, this proof can be directly extended to the maximum regret criterion with identical steps as regret is the difference between the latter and the former. Consequently, the nature’s response is similar to the previous case, which is formalized by the following result proof of which is omitted.

**Lemma 2.4.2** For any reserve price \(v_s\) set by the seller, nature’s response is one of the following two:

- select a deterministic valuation at \(v_s - \epsilon\) for some arbitrarily small \(\epsilon > 0\),
- select a valuation distribution with \(P(v = \bar{v}) = p = N^{-1}\), \(P(v = \bar{v}) = 1 - p\), whichever yields a higher regret. The resulting regret of the seller is

\[
   r(v_s) := \max \{v_s - v_0 - \epsilon, (\bar{v} - v_s) (1 - N^{-1})^{N-1}\},
\]

for an arbitrarily small \(\epsilon > 0\) chosen by the adversary.

Again, the first part in (2.34) is strictly increasing, while the second part is a strictly decreasing linear function in \(v_s\). Thus, the optimal strategy for the seller is to set a reserve price \(v_s^*\) that equates these two terms, which yields the following result.
Proposition 2.4.8 The optimal reserve price of the seller for the maximum regret problem is

\[ v^*_s = \frac{\bar{v}(1 - N^{-1})^{N-1} + v_0}{1 + (1 - N^{-1})^{N-1}} \] with the optimal regret of \( r^* = \frac{(\bar{v} - v_0)(1 - N^{-1})^{N-1}}{1 + (1 - N^{-1})^{N-1}} - \delta, \]

for an arbitrarily small \( \delta > 0. \)

Figure 2.2: Optimal reserve prices given by Competitive Ratio and Maximum Regret criteria compared to the Revenue Maximizing reserve price as a function of number of bidders \( N, \) when valuation distribution is uniform on \([10, 20].\)

The behavior of \( v^*_s \) on the number of bidders is depicted in Figure 2.2.

Proposition 2.4.9 If the adversary is also allowed to choose the number of bidders \( N \in [N, \bar{N}] \) in the above setting, she selects it to be as small as possible, i.e. \( N^* = N. \)

Proof: We now consider the problem

\[ r = \min_{v_s \in [\bar{v}, \hat{v}]} \max_{N \in [N, \bar{N}]} \max_F [R(v_s, F) - R(v_s, F)] . \] (2.34)

For fixed \( v_s \) and \( N \) the solution to the inner maximization problem is

\[ r(v_s, N) := \max \{ v_s - v_0 - \epsilon, (\bar{v} - v_s) (1 - N^{-1})^{N-1} \} , \]
as characterized by Lemma 2.4.2. The second term is a strictly decreasing function of \( N \), and therefore \( r(v_s, N) \) is a non-increasing function of \( N \). Consequently, the adversary’s response is to select as small an \( N \) as possible in the outer maximization problem, i.e. \( N = N \), and the resulting regret is

\[
r(v_s) := \max \left\{ v_s - v_0 - \epsilon , \ (\bar{v} - v_s) \ (1 - N^{-1})^{N-1} \right\} \]

Observe that the optimal reserve price, which is a decreasing function of the number of bidders, is actually relatively insensitive to the exact value of \( N \) for both criteria as indicated by Figure 2.2. This resembles the classical second price auction setting where the reserve price is independent of the number of bidders. Actually, it is easily seen from the formula of the reserve price for the regret that its limit is

\[
\lim_{N \to \infty} v^*_s = \frac{\bar{v} + \max \ v_s}{1 + \epsilon},
\]

which has a similar structure to that of the revenue maximizing reserve price for the uniform distribution on \([v_0, \bar{v}]\), i.e. \( v^u_s := \frac{\bar{v} + v_0}{2} \), but which is slightly lower, reflecting the risk averseness or conservatism of the regret criteria. Also, note that the optimal regret is directly proportional to the term \((\bar{v} - v_0)\), which reflects the level of uncertainty in the problem.
Chapter 3

Product Line Positioning without Market Information

In this essay, we study the joint problem of product line positioning and pricing for a monopolist when consumer preferences and WtP are unknown. We extend classical models of horizontal and vertical differentiation to cover uncertainty in customer preferences using relative performance criteria. In Section 3.1, we consider the horizontal positioning of a monopolist's product line using Salop's classical circular model of spatial differentiation. We show that the optimal decision for both criteria is to position products at equal intervals in the attribute space and to price them identically. We next study the vertical positioning of a product line using the linear utility of quality framework of Mussa and Rosen in Section 3.2. We show that the optimal policy consists of offering a number of the highest quality versions, and that the optimal number of versions offered increases with the uncertainty over customers' taste for quality.

Similar to the previous essay, adversary's best response is sometimes achieved
only in the limit. However, unlike the previous essay, we do not explicitly explain these using arbitrarily small constants $\delta$, $\epsilon$ in our analysis below, as the analysis and notation are significantly heavier here. Whenever, such limiting policies come into play, we note them verbally to keep the notation simpler.

3.1. Horizontal product line positioning

We first look at the case of horizontal positioning. We begin by defining the model and the formulation of the basic competitive ratio and maximum regret problems. We then analyze each of these problems in turn.

3.1.1 Model

We use Salop’s classical model of spatial differentiation, also known as the circular Hotelling model. In it, versions of the product are differentiated along a single attribute, represented by a location on the unit circle (attribute space). Each version $j$ is represented by a location $l_j$ in this attribute space.

We assume the firm can offer at most $K$ versions of a product. Such a limitation could, for example, reflect minimum efficient economies of scale needed for production. We do not incorporate fixed or variable costs of producing up to this upper limit of $K$ versions; rather, our analysis concentrates on revenue maximization. However, it is easy to incorporate linear variable costs in our analysis simply by rescaling the nominal valuations for products. If one wanted to analyze the optimal level of variety given such costs, it is possible to solve our model for different values of $K$ and use the solutions to determine if changes in revenue offset the costs of production and warrant increasing or decreasing the level of variety. But again in what follows below,
we shall assume $K$ is fixed.

Consumer preferences are also represented by locations on the unit circle. A randomly selected customer location is denoted $l$ with distribution $F(\cdot)$. Our key assumption is that this preference distribution $F(\cdot)$ is unknown to the firm and it must therefore make its positioning and pricing decisions accounting for this missing information.

Consumers have a common deterministic nominal valuation, $v$, for the product, which represents the maximum amount they would be willing to pay if the product were at their preferred location. The market size ("number of customers"), denoted $\Lambda$, is assumed fixed and continuous. We do not claim that these parameters are known in a real-life setting; a more complete analysis would certainly allow for uncertainty in these parameters as well. However, this would make model intractable. Instead, we isolate and focus on understanding the effects of consumer preference uncertainty. Furthermore, the structure of the results we provide are independent of the specific values of $\Lambda$ and $v$, e.g., the symmetric positioning and pricing strategy holds for any value of $\Lambda$ and $v$. And often retailers do have a sense of the overall market size from past experience even though the preferences for individual versions is highly uncertain (e.g., ski jackets). One can see evidence of this in the typical retail planning process, where budgets are allocated to categories of products based on past sales and the primary uncertainty is over how to allocate that budget to different products within the category. The valuation $v$, similarly, can be thought of as a historical "price point" for the given category that is relatively well understood from historical experience even though the preference for individual version is unknown.

Consumers incur a disutility that depends on the distance the product location to their preferred location, given by $\theta |l - L_f|$ where $\theta$ is a known disutility coefficient,
which can be interpreted as the consumers' sensitivity for variety and $|l - l_j|$ denotes the distance between $l$ and $l_j$ on the unit circle. Therefore the net utility for a consumer of type $I$ buying a product located at $l_j$ offered at price $p_j$ is given by $u(l, j) := v - \theta|l - l_j| - p_j$. Note that the higher the nominal valuation, the lower the consumers' sensitivity for variety, and/or the closer the product is to the consumer's ideal point, the more they are willing to pay for it. For the distance between two versions to be equal to the directional distance, either clockwise of counter-clockwise, we assume that $K \geq 3$ and that the directional distance between any two versions is not greater than 0.5. This is really a "cosmetic" assumption: all the results in this section can be extended to the general case, but it would require additional notation and case-wise definitions of all functions.

A consumer of type $I$ between $l_j$ and $l_{j+1}$ prefers $l_j$ if $v - \theta(l - l_j) - p_j \geq v - \theta(l_{j+1} - l) - p_{j+1}$; that is if the $j$-th indifference point $x_j := \frac{p_{j+1} - p_j}{2\theta} + \frac{l_{j+1} + l_j}{2} \geq l$. Similarly a consumer of type $I$ between $l_j$ and $l_{j-1}$ prefers $l_j$ if $v - \theta(l - l_{j-1}) - p_{j-1} \leq v - \theta(l_j - l) - p_j$; that is if $x_{j-1} = \frac{p_{j} - p_{j-1}}{2\theta} + \frac{l_j + l_{j-1}}{2} \leq l$. As a result, a consumer of type
$l \in [x_{j-1}, x_j]$ prefers product $j$ to its neighboring products. Without loss of generality we denote the location of first version to be $l_1 = 0$ (thus, also $l_1 = 1$ depending on the context). Consequently, in the remainder of the section whenever $j = K$, $j + 1$ denotes 1; and $x_1 := \frac{p_2 - p_1}{2\theta} + \frac{\theta}{2}$ and $x_K := \frac{p_1 - p_K}{2\theta} + \frac{1 + \theta}{2}$ above. Observe that the coefficient $\theta$ effects the aggregate demand in the sense that higher values of $\theta$ make all consumers less willing to pay for each version irrespective of its location and price, reducing the overall demand faced by the firm.

Each consumer has a most preferred product, which yields a utility of $u(l) := \max_j \{u(l, j)\}$ for a given price vector $p$, and which is purchased iff $u(l)$ is nonnegative. Note that if a version with a relatively low price is placed close enough to a version with a high price, the low price version may be preferred over the high price version by every consumer location on the unit circle. In this case, we say that the low price version dominates the high price version.

### 3.1.2 Product line positioning and pricing decision

Having introduced the model, let us describe the firm’s decision making process. First, the firm chooses how many of the $K$ possible versions to offer, the positions of these versions, i.e. vector $l = \{l_j\}_{j=1..K}$, on the unit circle, and their prices, i.e. the vector $p$. Then, it evaluates this decision against the “worst-case” outcome of customer preferences. To do so, we imagine an adversary (nature) that selects a worst-case preference distribution $F(\cdot)$ in order to minimize the competitive ratio or maximum regret. Thus, the competitive ratio problem is

$$\max_{l, p} \min_F \frac{R(l, p; F)}{R(l^*(F), p^*(F); F)}.$$
where \((I^*(F), p^*(F))\) is the revenue maximizing strategy of the firm given the distribution \(F(\cdot)\). Here the firm seeks a policy which maximizes its revenue as a percentage of the full-information revenue under worst-case outcomes. For example, a competitive ratio of 1/2 would say the firm is guaranteed to get at least 50% of the revenue it could have gotten had it known the exact customer preference distribution. Similarly, the maximum regret problem is

\[
\min_{p,l} \max_F [R(I^*(F), p^*(F); F) - R(l, p; F)].
\]

Here the firm seeks a policy which minimizes the difference between its revenues and the full-information revenue under worst-case outcomes. For example, a maximum regret value of $1,000 would say that the revenue the firm achieves with the policy is guaranteed to be no more than $1,000 below the revenue it would have obtained with exact information on the customer preference distribution.

Our goal is to specify the optimal pricing and product line positioning strategy of the firm for these two relative performance objectives. Specifically, how many versions should the firm offer? Where should these versions be positioned on the unit circle? What prices should we charge for the different versions? We also would like to explain how the answers depend on the nominal valuation, the disutility coefficient, the maximum number of products, and their relative positions. We begin with the competitive ratio case.

### 3.1.3 Competitive ratio

We begin by assuming a fixed number of versions, \(K\), is offered with fixed attributes (locations on the unit circle), denoted by vector \(l \in [0, 1]^K\), and consider the pricing decision of the firm. We say a product \(i\) at price \(p_i\) dominates product \(j\) at price \(p_j\) if
every customer (regardless of location) prefers \( i \) to \( j \). Firstly, we have the following result:

**Proposition 3.1.1** There exists an optimal pricing policy where no version dominates any other version.

**Proof:** By contradiction assume at all optimal solutions, there exist a price, \( p_j \) which dominates its neighboring price \( p_{j+1} \), i.e. \( x_j > l_{j+1} \) or equivalently \( p_{j+1} > p_j + \theta(l_{j+1} - l_j) \). Then, for any distribution the adversary chooses, the firm earns \( p_j \) from those who choose to buy in the interval, \([x_{j-1}, x_j]\). However, decreasing the price \( p_{j+1} \) to \( p_j + \theta(l_{j+1} - l_j) \), the firm can earn \( p_j + \theta(l_{j+1} - l_j) \) from those in the interval \([x_j, l_{j+1}]\), plus \( p_j \) from those in the interval \([x_{j-1}, l_{j+1}]\) without changing any other prices or affecting revenues of other regions, which yields a revenue as at least high as the optimal revenue. Repeating this augmentation for every dominating price we achieve a price vector where there is no price strictly dominating any other. □

Thus, for the rest of the analysis, we restrict our attention only to such non-dominating pricing policies, in which case all the consumers of type \( l \in [x_{j-1}, x_j] \) prefer product version \( j \), and \( x_j \in [l_j, l_{j+1}] \) holds for all \( j \). Also, defining the lengths \( y_j := \frac{x_j - l_j}{\theta} \), we see that everyone within \( y_j \) distance of the location \( l_j \) finds that buying version \( j \) yields a nonnegative net utility. Therefore, everyone with \( l \in [k_j, z_j] \) prefers and buys product \( j \), where \( k_j \) and \( z_j \) are the points defined as \( z_j := \min(x_j, l_j + y_j) \), \( k_j := \max(x_{j-1}, l_j - y_j) \) for \( j = 2 \ldots K \), and \( z_1 := \min(x_1, y_1) \) and \( k_1 := \max(x_K, 1 - y_1) \). Consequently, the revenue of the firm can be written as

\[
R(l, p; F) = \sum_{j=1}^{K} p_j (F(z_j) - F(k_j)).
\]  

(3.1)

For the competitive ratio problem, we can further characterize the revenue functions using the following result:
Lemma 3.1.1 The firm does not enter the market unless \( \theta \max_j |l_j - l_{j-1}| \leq 2v \). If the firm chooses to enter the market, the prices are set such that the whole market is served at the optimal solution, i.e. \( x_j \leq l_j + y_j \) and \( x_{j-1} \geq l_j - y_j \) holds for all \( j \), and consequently, we have that \([z_j, k_j] = [x_j, x_{j+1}]\) for all \( j \).

Proof: First, assume that \( \theta \max_j |l_j - l_{j-1}| > 2v \). Using the definitions of \( x_j \) and \( y_j \), we see that requiring \( x_j \leq l_j + y_j \) and \( x_{j-1} \geq l_j - y_j \) for all \( j \) implies the constraints \( p_j + p_{j-1} + \theta |l_j - l_{j-1}| \leq 2v \), \( \forall j \). However, it is clear that some price \( p_j \) must be negative for some variant \( j \) (which could be thought as providing positive subsidies or rebates) to satisfy these inequalities, or otherwise (i.e. in the case \( x_j > l_j + y_j \) or \( x_{j-1} < l_j - y_j \) for some \( j \)) not all the consumers of type \( l \) will choose to buy in the corresponding preference region \([x_{j-1}, x_j]\) of variant \( j \). If the former is the case, the adversary would allocate all the probability mass of consumers to this region resulting in a negative revenue and competitive ratio. If the latter is the case, the adversary would allocate the unit probability mass to the region that is not covered, and hence, the revenue would be zero. Consequently, by entering the market when \( \theta \max_j |l_j - l_{j-1}| > 2v \), the firm can never do better that earning zero revenues and achieving a competitive ratio of zero.

Similarly, when \( \theta \max_j |l_j - l_{j-1}| \leq 2v \), if the optimal prices are such that for some \( l_j \) we have that \( x_j > l_j + y_j \) (or \( x_{j-1} < l_j - y_j \)), the adversary would again allocate all the probability mass to the interval \([l_j + y_j, x_j]\) (or \([x_{j-1}, l_j - y_j]\) respectively), resulting in zero sales, and thus, achieving a competitive ratio of zero. On the other hand, any price policy that satisfies inequalities \( x_j \leq l_j + y_j \) and \( x_{j-1} \geq l_j - y_j \) for all \( j \) guarantees a positive ratio when \( \theta \max_j |l_j - l_{j-1}| \leq 2v \). □
As a result of this lemma, we can restrict our attention to the case with $\theta \max_j |l_j - l_{j-1}| \leq 2v$ and to the pricing policies in which the firm collects $p_j$ from everybody within the interval $[x_j, x_{j+1}]$ for all $j$. For a distribution function $F(\cdot)$, define the probability mass allocated to each such interval as $f_j := F(x_{j+1}) - F(x_j)$ for $j = 1 \ldots K-1$ and $f_K := F(x_1) + 1 - F(x_K)$. Using the above result, we can characterize the worst-case distribution as follows:

**Lemma 3.1.2** The worst-case competitive ratio distribution function $F(\cdot)$ allocates the mass $f_j$ to location $l_j$ for all $j$.

**Proof:** Under a price policy that satisfies $x_j \leq l_j + y_j$ and $x_{j-1} \geq l_j - y_j$ for all $j$, note that the revenue in equation (3.1) depends only on how much probability mass exists within each interval $[x_j, x_{j+1}]$, that is on $f_j = F(z_j) - F(k_j) = F(x_{j+1}) - F(x_j)$. However, the specifics of the distribution function matters for the adversary, as the maximum revenue that can be achieved, which appears in the denominator of the competitive ratio to be minimized, depends on this distribution. On the other hand, the value of the maximum revenue is always less than $\Lambda v$. Also note that given an aggregate decision as to how much probability mass to put on each interval, i.e. the vector $f$, the adversary can achieve the maximum revenue $\Lambda v$ in the denominator of the competitive ratio by allocating each mass $f_j$ on the point $l_j$ for all $j$. Therefore, such an allocation is a dominant strategy for the adversary for all $F(\cdot)$. □

**Corollary 3.1.1** The revenue function can be written as $R(l, p; F) = \Lambda \sum_{j=1}^K p_j f_j$, and the maximum revenue the firm could extract under full information is $R(p^*(F), F) = \Lambda v$. 
As a consequence of these results, we can write down the competitive ratio problem as the following formulation

$$\max_p \min_f \left\{ \frac{\Delta \sum_j p_j f_j}{\Delta v} : p_j + p_{j-1} + \theta |l_j - l_{j-1}| \leq 2v \ \forall j, \ p \geq 0, \ \sum_j f_j = 1, \ f \geq 0 \right\},$$

where $|l_1 - l_K|$ equals to $1 - l_K$, and where constraints $p_j + p_{j-1} + \theta |l_j - l_{j-1}| \leq 2v$ impose the condition that the whole market must be served under the optimal policy of the firm. This formulation can be reduced to a linear program, and can be solved analytically as shown in the proof of the following proposition. The intuition behind the analysis is that the worst-case distributions are extreme point distributions, and the firm’s optimal response is to choose prices so that the adversary is indifferent among these extreme point distributions.

**Proposition 3.1.2** Assume $\theta \max_j |l_j - l_{j-1}| \leq 2v$ so that the firm enters the market. For a fixed number of versions $K$, and their attributes $l$, consider the following indexing of the price vector that satisfies $p_{[1]} \leq p_{[2]} \leq p_{[3]} \leq \ldots \leq p_{[K]}$, and the vector $\tilde{p}$ solving the equations $\tilde{p}_{[j]} = 2v - \tilde{p}_{[j-1]} - \theta |l_{[j]} - l_{[j-1]}|$ $\forall j$, then the optimal price vector $p$ satisfies

$$p_{[1]} = p_{[2]} = \tilde{p}_{[1]} = \tilde{p}_{[2]} = v - \frac{\theta \max_j |l_j - l_{j-1}|}{2} \text{ and } p_{[j]} \in [p_{[1]}, \tilde{p}_{[j]}] \text{ for } j = 3..K,$$

and the resulting optimal competitive ratio is

$$c = \left( 1 - \frac{\theta \max_j |l_j - l_{j-1}|}{2v} \right)^+.$$
Proof:

\[ c = \max_{p} \min_{f} \left\{ \frac{\Lambda \sum_{j} p_{j} f_{j}}{\Lambda v} : p_{j} + p_{j-1} + \theta |l_{j} - l_{j-1}| \leq 2v \ \forall j, \ p \geq 0, \ \sum_{j} f_{j} = 1, \ f \geq 0 \right\} \]

\[ = \frac{1}{v} \max_{p} \left\{ \min_{j} \{ p_{j} \} : p_{j} + p_{j-1} + \theta |l_{j} - l_{j-1}| \leq 2v \ \forall j, \ p \geq 0 \right\} \]

As the objective \( \min_{j} \{ p_{j} \} \) is homogeneous in the components of price vector \( p \), it is obvious that \( p_{[1]} + p_{[2]} + \max_{j} |l_{j} - l_{j-1}| = 2v \) at any optimal solution, and the upper bounds for other prices are given by setting \( \bar{p}_{[j]} = 2v - \bar{p}_{[j-1]} - \theta |l_{[j]} - l_{[j-1]}| \ \forall j \), which yields the result. □

Thus, we see that the minimum price is charged at two neighboring locations which are the furthest apart on the unit circle, and the competitive ratio depends on these minimum prices. The competitive ratio decreases as the consumers’ sensitivity for differentiating attribute increases, consumers’ nominal valuations for their ideal product decreases, or the maximum attribute difference among neighboring products increases.

Inspecting the above formula for the optimal competitive ratio, we see that it is maximized when the term \( \max_{j} |l_{j} - l_{j-1}| \) is minimized, which would occur when \( |l_{j} - l_{j-1}| = 1/K \) for all \( j \). In other words, we have the following result, the proof of which is straightforward and omitted:

**Proposition 3.1.3** For fixed \( K \), the optimal product line positioning and pricing decision is to locate products at equally spaced intervals along the attribute space and price them equally, in which case the optimal competitive ratio is \[ c = \left(1 - \frac{\theta}{2v K}\right)^{+}. \]
3.1.4 Maximum regret

For the maximum regret problem, we consider again the case with fixed number of versions $K$ and fixed attributes for different versions, $l \in [0,1]^K$, and focus on the pricing strategy first. It is easy to see that the worst-case distribution can again be reduced to deciding how much probability mass to allocate to the intervals $[z_j,k_j]$, $j = 1, \ldots, K$. However, unlike the competitive ratio case, it is not necessarily the case that $[z_j,k_j] = [x_j,x_{j+1}]$ for all $j$. In other words, it might be the case that not everybody chooses to buy at the optimal prices. We will denote such a pricing strategy where some consumers choose not to buy a non-spanning pricing policy. Formally, defining the set of indices $S(p) = \{j \mid x_j > l_j + y_j, \text{ or } x_{j-1} < l_j - y_j\}$ for a given price vector $p$, if the set $S(p)$ is empty then $p$ is a spanning price policy.

The structure of the worst-case distribution changes according to whether the pricing policy used is spanning or non-spanning in the above sense. The following result, proof of which is relegated to Appendix B, formalizes this claim:

**Lemma 3.1.3** For a given price vector $p$, the maximum regret is

$$r(p) := \begin{cases} 
\Lambda (v - \min_j p_j) & \text{if } S(p) \text{ is empty} \\
\Lambda \max\{(v - \min_j p_j), \max_{j \in S(p)} p_j\} & \text{otherwise}
\end{cases}$$

Furthermore, using this lemma one can easily show the following:

**Lemma 3.1.4** There exists an optimal price vector with equal components.

**Proof:** Consider an optimal solution $p^*$. If the set $S(p^*)$ is empty, the result is obvious in that for the optimal vector $p^*$, the vector $p$ with equal components of $p_j = \min_j p^*_j \ \forall j$ has the same regret by definition in equation (3.2) and by the fact that $S(p)$ is necessarily empty as $p \leq p^*$. Therefore, assume the set $S(p^*)$ is not
empty for the optimal price vector \( p^* \) whose components are not equal. Then we have that \( r(p^*) = \Lambda \max \{ \max_{j \in S(p^*)} p^*_j, (v - \min_j p^*_j) \} \). However, the vector \( p \) with equal components of \( p_j = \min_j p^*_j \ \forall j \) has at most the same regret as shown below, and therefore, is also optimal

\[
r(p^*) = \Lambda \max \{ \max_{j \in S(p^*)} p^*_j, (v - \min_j p^*_j) \} \\
\geq \Lambda \max \{ \min_j p^*_j, (v - \min_j p^*_j) \} \quad \text{as } \min_j p^*_j \leq \max_{j \in S(p^*)} p^*_j \\
\geq r(p) \quad \text{by definition of } r(p) \text{ for } p_j = \min_j p^*_j \ \forall j. \quad \Box
\]

Thus, finding the optimal prices reduces to a single dimensional decision of finding the best uniform price to offer for all versions. Using Lemmas 3.1.3 and 3.1.4, we can now prove our main result for the maximum regret problem:

**Proposition 3.1.4** For fixed number of versions, \( K \), with fixed attributes, \( l \), the optimal regret value is \( r = \Lambda \min \left\{ \frac{\theta \max_j |l_j - l_{j-1}|}{2}, \frac{v}{2} \right\} \), which is achieved by charging equal prices for different versions

\[
p_j = \begin{cases} 
    v - \frac{\theta \max_j |l_j - l_{j-1}|}{2}, & \forall j, \text{ when } v \geq \theta \max_j |l_j - l_{j-1}| \quad \text{(spanning policy)} \\
    v/2, & \forall j, \text{ when } v < \theta \max_j |l_j - l_{j-1}| \quad \text{(non-spanning policy)}
\end{cases}
\]

**Proof:** As a consequence of Lemma 3.1.4, we can restrict our attention to the pricing policies with equal components. This reduces the decision vector to one dimension. Thus, given a decision \( p \in R^+ \), the maximum regret as a function of is:

\[
r(p) = \begin{cases} 
    \Lambda (v - p) & \text{if } S(\{p\}) \text{ is empty} \\
    \Lambda \max \{(v - p), p\} & \text{otherwise}
\end{cases}
\] (3.2)

We prove the result in two parts. First, we show that for \( v \geq \theta \max_j |l_j - l_{j-1}| \), the optimal price vector has \( p_j = v - \frac{\theta \max_j |l_j - l_{j-1}|}{2} \ \forall j \). If we can show that at the
optimal price \( \{p\} \) with equal components of \( p \), the set \( S(\{p\}) \) is empty, then the result follows directly from the analog steps of competitive ratio analysis which shows that minimum value of maximum regret is achieved for \( p = (v - \frac{\theta \max_j |l_j - l_{j-1}|}{2}) \) which yields minimum value of \( \Lambda[v - (v - \frac{\theta \max_j |l_j - l_{j-1}|}{2})] = \Lambda \frac{\theta \max_j |l_j - l_{j-1}|}{2} \). Therefore assume by contradiction that the set \( S(\{p\}) \) is not empty. Then, we have that

\[
r(p) > \Lambda \frac{pj}{\Lambda} > \Lambda (v - \frac{\theta \max_j |l_j - l_{j-1}|}{2}) \geq \Lambda \frac{\theta \max_j |l_j - l_{j-1}|}{2}
\]

where the first inequality follows from definition of \( r(p) \) in equation (3.2), the strict inequality follows from the fact that \( \{p\} \) must violate the inequality \( p_j + p_{j+1} + \theta \max_j |l_j - l_{j-1}| = 2p + \theta \max_j |l_j - l_{j-1}| \leq 2v \) as the set \( S(\{p\}) \) is not empty, and last inequality follows by the specification of \( v \) in this region. Therefore, the non-spanning policy \( \{p\} \) gives strictly more regret, and cannot be optimal in this case.

Second, for \( v < \theta \max_j |l_j - l_{j-1}| \), we show that the optimal price vector has \( p_j = \frac{v}{2} \) \( \forall j \) below. Using the previous case, we know that the firm can guarantee a regret value of \( \Lambda \frac{\theta \max_j |l_j - l_{j-1}|}{2} \) using a price policy low enough to span the whole unit circle. Now, we show that a strictly less regret is possible by charging \( p_j = \frac{v}{2} \) \( \forall j \), which is a non-spanning policy for values of \( v \) in this region, therefore \( S(\{\frac{v}{2}\}) \) is not empty. If the firm uses a non-spanning pricing policy the maximum regret is given by \( \Lambda \max\{p, (v - p)\} \) by equation (3.2). However, this function is minimized when \( p = \frac{v}{2} \), as it is the maximum of two linear functions one increasing the other decreasing in \( p \). As this yields a regret value of \( \Lambda \frac{v}{2} < \Lambda \frac{\theta \max_j |l_j - l_{j-1}|}{2} \), the optimal price of \( p_j = \frac{v}{2} \) \( \forall j \) for this region. Combining the two cases yields the result. \( \square \)

The intuition lying behind this result is as follows: when \( \theta \) is high or \( v \) is low, the firm faces a linear demand curve with downward slope of \( \theta \) for each version,
because cannibalization is not an issue as it is too costly for a customer close to a version to consider some other version. As a result, the firm charges the revenue maximizing price for a monopolist, i.e. $v/2$, which corresponds to a non-spanning pricing policy. On the other hand, when $\theta$ is low or $v$ is high, it is less costly for consumers close to a version to consider other versions; and therefore, cannibalization is an issue. Charging the revenue maximizing price $v/2$ for a version cannibalizes neighboring versions. Therefore, the firm charges a higher price for the version, i.e. $v - \frac{\theta \max_j |l_j - l_{j-1}|}{2} > v/2$.

Note the regret above is minimized irrespective of $v$ when $\max_j |l_j - l_{j-1}|$ is minimized, which again occurs when $|l_j - l_{j-1}| = 1/K$ for all $j$. Thus, we have the same result as in the competitive ratio case:

**Proposition 3.1.5** For fixed $K$, the optimal product line positioning decision is to locate versions at equally spaced intervals along the attribute space and to price them equally according to Proposition 3.1.4, in which case the optimal regret is $r = \Lambda \min \left\{ \frac{\theta}{2K}, \frac{v}{2} \right\}$.

We therefore have shown that for both the competitive ratio and maximum regret criteria, the optimal pricing policy is a function of attribute differences between neighboring versions. This policy reflects the firm’s sensitivity for potential cannibalization among versions (e.g., non-dominating pricing policies). The worst-case performance decreases (i.e. the competitive ratio decreases or the regret increases) as the consumers’ sensitivity for differentiating attribute increases, consumers’ nominal valuations for their ideal product decreases, or the maximum attribute difference among neighboring products increases. The optimal positioning decision is to span the attribute space with equally spaced product versions and to sell them at a uniform
price. In other words, to cover the product space uniformly with a collection of products offered at a single price. Coincidentally, this "equally-space and equally-price" strategy is a celebrated Nash Equilibrium for the classical monopolistic competition model where each version is controlled by a different firm, which is well studied by Salop and others. (See Salop [69].) In the monopolistic competition case, this arises as an equilibrium response among competing firms. In our case, it arises as an optimal hedge against another sort of adversary: "nature" - the unknown preferences of customers.

While stylized, as noted above this uniform positioning and pricing structure is seen commonly in real-world markets. For example, iTunes offers music downloads spanning all genres and artists at a uniform price of $0.99; apparel retailers normally charge the same price for a garment sold in different colors and sizes; ice cream shops carry a range of flavors sold at the same price-per-cone, etc. While a firm could charge different prices if it knew customer preferences certainly, if it makes an error and the actual distribution of preference is different than the firm's estimate, they could do much worse. The uniform positioning and pricing policy serves as a defensive strategy against this risk. Again, there are clearly other considerations that could influence a retailer to price uniformly, such as operational simplicity and customer goodwill, but our results suggest that avoiding risk from lack of information about consumer preferences is another plausible motivation.

Notice also that the optimal prices charged, and thus, revenues earned increases with the number of versions $K$, increasing the competitive ratio and decreasing the regret. This is the typical profit maximization response when there are no economies of scale to production of variety. However, when there are economies of scale, one needs to balance the revenue gain of more variety against the cost of producing many
versions.

As we have noted above we have focused on consumer preference uncertainty; a more complete approach would be to assume uncertainty in other model parameters such as $v$ and $\Lambda$. However, it is not hard to extend our analysis to include the above parameters since we provide closed-form solutions to what would be the inner-optimization problems in such a generalization and the structure of the uniform pricing and differentiation policy is independent of the market size parameter $\Lambda$.

3.2. Vertical product line positioning

We next consider the case of vertical product positioning, in which versions correspond to different quality levels over which consumers share a common ranking (though with heterogenous willingness-to-pay for quality).

Specifically, there are $K$ distinct quality versions of a product that the firm considers potentially offering. The quality level of version $j$ is denoted by $l_j$. The indexing satisfies $l_1 < l_2 < \ldots < l_K$, so that $l_1$ is the lowest quality level, whereas $l_K$ is the highest. In order to reduce the heavy notation induced by the analysis of this setting, we assume there is a continuum of consumers of unit mass. All the results can be scaled to any market size without changing the optimal policy.

Each consumer has a distinct taste for quality, denoted by $v$, which is an independent and identically distributed random variable with unknown distribution $F(\cdot)$ on a known interval $[v, \bar{v}]$. This distribution is the key missing information to the firm in our model.

We adopt the classical "linear utility function of quality" framework of Mussa and Rosen [57]. If a consumer has a quality taste level of $v$ and purchases product version
her net utility is given by \( u(v, j) := v \ l_j - p_j \), where \( p_j \) is the price of product \( j \). Consequently, the higher the taste for quality or the higher the quality level, the more a consumer is willing to pay for a particular version. A consumer prefers version \( j \) over version \( i \) if \( u(v, j) = v \ l_j - p_j > v \ l_i - p_i = u(v, i) \) holds. Therefore, unlike the horizontal differentiation model in the previous section, each consumer enjoys the higher quality versions more. As a tie-breaking assumption, we assume that a consumer prefers the higher quality version when faced with two versions giving the same utility. For a given vector of prices \( p \), a consumer buys the version of the product which yields the highest net utility provided that is nonnegative, i.e. \( u(v) := \max_j \{ v \ l_j - p_j \} \geq 0 \), or will not buy any product otherwise.

The sequence of events is as follows: first, the firm decides on how many versions to offer at what quality levels, and chooses their prices without knowing the distribution of the random taste for quality. Then, we evaluate the decision against the worst-case distribution \( F(\cdot) \) of the taste for quality. Our goal is again to find the optimal product line positioning (this time in terms of quality levels) and pricing policy of the firm. Specifically, we want to answer the questions about how many versions to offer, their quality levels and prices. Also of particular interest is how these decisions change with respect to the uncertainty level of costumer taste for quality. We begin with the competitive ratio case and then analyze the case of maximum regret.

### 3.2.1 Competitive ratio

First, let us assume for the moment that the firm offers all \( K \) quality levels of the product. Similar to the horizontal differentiation setting we define indifference points: \( v_j \) as the quality level where a consumer switches from version \( j \) to version \( (j + 1) \),
then \( v_j \) must satisfy \( v_j \ l_j - p_j = v_j \ l_{j+1} - p_{j+1} \), yielding \( v_j = \frac{p_{j+1} - p_j}{l_{j+1} - l_j} \) for \( j = 1 \ldots K-1 \).

Using the definition of \( u(v, j) \), it is easy to see that version \( j \) is preferred over version \((j + 1)\) by those consumers with \( v < v_j \), and similarly, version \((j + 1)\) is preferred over version \( j \), when \( v \geq v_j \). We say version \( j \) is \textit{strictly dominated} by versions \((j - 1)\) and \((j + 1)\), if version \((j - 1)\) or version \((j + 1)\) yields strictly more utility for every consumers type \( v \in [\underline{v}, \bar{v}] \) than version \( j \). The necessary and sufficient condition for this is \( v_{j-1} > v_j \), which can be easily seen graphically in Figure 3.2.

Note the product line positioning and pricing decisions of the firm can be aggregated into the single decision of offering all \( K \) quality levels of the product and choosing only the prices offered. To see this, assume that some version \( j \) is not offered under some policy. Analytically, this is equivalent to offering that version at some price \( p_j \) that satisfies \( v_{j-1} = v_j \), because at that price level no consumer strictly prefers version \( j \), but a consumer of type \( v = v_{j-1} = v_j \) has the same value of utility for consuming versions \((j - 1), j \) and \((j + 1)\). However, she prefers version \((j + 1)\) over others due to the tie-breaking assumption. Therefore, at price \( p_j \) version \( j \) is effectively a "dummy version" which is not chosen by any consumer, and this case is therefore analytically equivalent to not offering version \( j \).

For the competitive ratio problem, we first prove that there exists an optimal price vector where no product strictly dominates any other product at the optimal solution and further that the consumer with the lowest taste for quality earns zero utility at the optimal solution. These properties are formalized in the following proposition, proof of which is given in Appendix B:

**Proposition 3.2.1** There exists an optimal price vector where no product strictly dominates any other product and which satisfies \( \underline{v} \leq v_1 \leq v_2 \leq \ldots \leq v_{K-1} \leq \bar{v} \).
The next result shows that there exists an optimal policy where only some number of highest quality versions of the product are offered:

**Proposition 3.2.2 (Nested Quality Offers)** There exists an optimal solution where only the \((K - j + 1)\) highest quality products are offered for some \(j = 1 \ldots K\).

**Proof:** Using the previous proposition, the result is trivially true for \(j = 1\) if \(v < v_1\) in the optimal solution. If \(v = v_1 = \ldots = v_{j-1} < v_j\) for some \(j = 2 \ldots K - 1\), we can solve for the optimal price components of the first \(j\) versions as \(p_1 = vl_1, p_2 = vl_2, \ldots, p_j = vl_j\) using the definitions of \(v_i\) for \(i = 1 \ldots j\). Thus, the customer with lowest quality taste chooses version \(j\) by the tie-breaking assumption as \(u(v,1) = u(v,2) = \ldots = u(v,j)\). Furthermore, \(u(v,1) < u(v,2) < \ldots < u(v,j)\) for \(\forall v > v\) as \(l_1 < l_2 < \ldots < l_j\). Consequently, the versions 1 to \(j - 1\) are not bought by any type of customers. Therefore, offering only the \((K - j + 1)\) highest quality version with their optimal prices achieves the same product selection for each type of customer for \(\forall v \in [v, \bar{v}]\) with same prices, resulting in the same revenue collected by the firm and the same response chosen by the adversary, therefore achieving the optimal competitive ratio. \(\square\)

Using Proposition 3.2.1 and our previous observations, we conclude that offering only the \((K - j + 1)\) highest quality products is analytically equivalent to offering all \(K\) quality levels of the product, but fixing the prices of the first \(j\) quality versions to be \(p_1 = vl_1, p_2 = vl_2, \ldots, p_j = vl_j\). This is illustrated in Figure 3.2.

The remainder of the competitive ratio section tries to identify the best competitive ratio that can be achieved by the firm using a policy which offers only the \((K - j + 1)\) highest quality versions, denoted by \(c_j\). Then, the optimal value of the competitive ratio problem satisfies \(c = \max_j \{c_j\}\).
Figure 3.2: An example plotting the utilities by different versions of a feasible policy for $c_3$ when $K = 5$. Slopes of the lines correspond to different quality levels $l_j$ and the intercepts to the corresponding prices $p_j$.

For the problem of finding $c_j$, we show that for a given pricing policy $p$ satisfying above structure, the worst-case distribution allocates the whole unit probability mass arbitrarily close to $\bar{v}$ or to some $v_j$, where a consumer is indifferent between two neighboring quality offers. The resulting value of the competitive ratio for this response is characterized with the following result. The proof is relegated to Appendix B.

**Lemma 3.2.1** When a pricing policy $p$ offering only the $(K - j + 1)$ highest quality products is used, the worst-case distribution yields a competitive ratio arbitrarily close to

$$c_j(p) := \min \left\{ \min_{i=j \ldots K-1} \frac{p_i}{l_K v_i}, \frac{p_K}{l_K \bar{v}} \right\}.$$  

This result helps us write-down the competitive ratio problem when $(K - j + 1)$ highest quality versions are offered as

$$c_j = \max_p \min \left\{ \min_{i=j \ldots K-1} \frac{p_i}{l_K v_i}, \frac{p_K}{l_K \bar{v}} \right\}$$

s.t. $p_1 = y l_1$, $p_2 = y l_2$, $\ldots$, $p_j = y l_j$

$$y < v_j \leq v_{j+1} \leq \ldots \leq v_{K-1} \leq \bar{v}$$
In the proof of the following proposition in Appendix B, we first relax constraint $v < v_j$ above, and define $c_j^r$ as the best ratio that can be achieved by this relaxed formulation, which yields an upper bound on $c_j$. Then, we convert the formulation of $c_j^r$ to an equivalent linear program, and use weak duality to upper bound $c_j^r$, which in turn yields another upper bound on $c_j$. Finally, using strong duality on this dual problem, we show that $c_j$ achieves these upper bounds for all $j$, letting us characterize the optimal policy and the corresponding competitive ratio as follows:

**Proposition 3.2.3** The optimal competitive ratio for the vertical product positioning problem is given by $c = \max_j \{ c_j \}$, where $c_j$ satisfies

$$\frac{y}{v} \prod_{i=j+1}^{K} \frac{(l_i - l_{i-1} + c_j l_K)}{(c_j l_K)^{K+1-i}} = 1 \text{ for } j = 1 \ldots K - 1 \text{ and } c_K = \frac{y}{v}.$$  

If $c = c_j$, only the $(K - j + 1)$ highest quality version are offered with the following optimal prices

$$p_j = y l_j, \text{ and } p_n = y l_j \frac{\prod_{i=j+1}^{n} (l_i - l_{i-1} + c_j l_K)}{(c_j l_K)^{n-j}} \text{ for } n = j + 1 \ldots K.$$  

We see that the optimal policy charges a price premium for each higher quality level as a function of the quality difference offered at each step, $l_i - l_{i-1}$.

Define $\gamma = y/\bar{v}$ and $\gamma_j = \frac{y^{K-j}}{\prod_{i=j+1}^{K} (l_i - l_{i-1} + l_j)}$ for $j = 1 \ldots K - 1$. Observe that $\gamma$ is a measure of potential market heterogeneity for quality: as $\gamma$ decreases, the relative difference between the lowest and the highest taste for quality, $y$ and $\bar{v}$, increases, and vice versa. We further prove the following result tying the heterogeneity level to the optimal portfolio of offered quality levels. We provide the proof of the result in Appendix B.

**Proposition 3.2.4** As the potential market heterogeneity for quality increases (which can also be interpreted as increasing uncertainty in taste for quality for this model),
i.e. as $\gamma$ decreases, it is optimal to offer more quality versions of the product. Specifically, for $\gamma \in [0, \gamma_1)$ the optimal solution offers all the $K$ versions of the product; for $\gamma \in [\gamma_j, \gamma_{j+1})$ the optimal solution offers $(K-j)$-highest quality versions of the product for $j = 2 \ldots K - 2$; and for $\gamma \in [\gamma_{K-1}, 1]$ the optimal solution offers only the highest quality version of the product.

Just as suggested by Bhargava and Choudhary [14] and Lilien et al. [48], this result shows that versioning becomes more attractive when consumers are sufficiently heterogeneous, which is reflected by the range of valuations for quality, $v$ and $\bar{v}$, in this model. When the potential market heterogeneity for quality increases, i.e. $\gamma$ decreases, and the exact valuations of customers for quality are unknown, it is optimal to offer more and more quality levels.

3.2.2 Maximum regret

For the maximum regret problem, it is no longer the case that $u(y) = 0$ and $u(v) > 0$ for $v \in (y, \bar{v}]$ at the optimal solution. The firm can choose a pricing policy in which consumers of type $v \in [y, v_0)$ choose not to buy any version for some $v_0 \in [y, \bar{v})$, i.e. $u(v) < 0$ for $v \in [y, v_0)$. Every feasible price vector $p$ implies a corresponding point $v_0$. Thus, we can divide the decision of the firm into two parts: first selecting the optimal value of $v_0$, i.e. selecting which segment of the market to serve, then selecting optimal price vector which serves that particular segment. That is defining

$$r(v_0) := \min_{p \in P(v_0)} \max_F \left[ R(p^*(F), F) - R(p, F) \right]$$

where $P(v_0) = \{p \mid u(v_0) = 0\}$, we can rewrite maximum regret problem as

$$r^* = \min_{v_0 \in [y, \bar{v}]} \max_p \left[ R(p^*(F), F) - R(p, F) \right] = \min_{v_0 \in [y, \bar{v}]} r(v_0).$$
For fixed $v_0$, there again exists a non-dominating optimal pricing policy, denoted by the vector $p(v_0) \in P(v_0)$, offering only the $(K - j + 1)$ highest quality versions for the problem of identifying $r(v_0)$, as stated by the following results, proofs of which follow identical reasoning with the competitive ratio case and are thus omitted.

**Proposition 3.2.5** There exists an optimal price vector $p(v_0)$ for the problem in (3.3) above, where no product strictly dominates any other product, and which satisfies $v_0 \leq v_1 \leq v_2 \leq \ldots \leq v_{K-1} \leq \bar{v}$.

**Proposition 3.2.6** *(Nested Quality Offers)* There exists an optimal solution $p(v_0)$ for the problem in (3.3) where only the $(K - j + 1)$-highest quality versions are offered for some $j = 1 \ldots K$.

For fixed $v_0$, we again confine our attention to pricing policies offering only the $(K - j + 1)$ highest quality products for $j = 1 \ldots K$, and define the minimum regret that can be achieved using such a pricing policy as $r_j(v_0)$, then by definition the equality $r(v_0) = \min_j \{ r_j(v_0) \}$ holds. The optimal response by the adversary under the problem of identifying $r_j(v_0)$ is to choose a deterministic valuation arbitrarily close to $v_0$, $v_K$ or $v_j$ for some $j = 1 \ldots K - 1$, which is formalized by the following result. The proof again uses identical logic and steps with the corresponding lemma for the competitive ratio case, and therefore is omitted.

**Lemma 3.2.2** For fixed $v_0$, when a pricing policy $p \in P(v_0)$ offering only the $(K - j + 1)$ highest quality products is used, the regret, denoted by $r_j(v_0, p)$, is given
by

\[ r_j(v_0, p) = \max_F \left[ R(p^*(F), F) - R(p, F) \right] \]

\[
= \begin{cases} 
\max \left\{ v_0 l_K, \max_{i=j:K-1} \{ l_K v_i - p_i \}, \ l_K \bar{\nu} - p_K \right\} & \text{if } v_0 > \nu \\
\max \left\{ \max_{i=j:K-1} \{ l_K v_i - p_i \}, \ l_K \bar{\nu} - p_K \right\} & \text{if } v_0 = \nu.
\end{cases}
\]

Then, we have that \( r_j(v_0) = \min_{p \in P_j(v_0)} r_j(v_0, p) \) by definition. However, the exact specification of \( r_j(v_0) \) depends on whether \( v_0 > \nu \) or \( v_0 = \nu \), as shown above.

Using the above results, the optimal regret value the firm can achieve satisfies

\[
r^* = \min_{v_0 \in [\nu, \bar{\nu}]} r(v_0) = \min_{v_0 \in [\nu, \bar{\nu}]} \inf_{v_0 \in [\nu, \bar{\nu}]} r(v_0) = \inf_{v_0 \in [\nu, \bar{\nu}]} \min_{v_0 \in [\nu, \bar{\nu}]} r_j(v_0, p)
\]

\[
= \min_{v_0 \in [\nu, \bar{\nu}]} \min_{v_0 \in [\nu, \bar{\nu}]} r_j(v_0, p) = \inf_{v_0 \in [\nu, \bar{\nu}]} \min_{v_0 \in [\nu, \bar{\nu}]} r_j(v_0, p)
\]

Using Lemma 3.2.2, we can characterize the terms at the right hand side of the last equality as

\[
\inf_{v_0 \in [\nu, \bar{\nu}]} r_j(v_0) = \inf_{v_0 \in [\nu, \bar{\nu}]} \min_{v_0 \in [\nu, \bar{\nu}]} r_j(v_0, p)
\]

\[
= \inf_{v_0 \in [\nu, \bar{\nu}]} \min_{v_0 \in [\nu, \bar{\nu}]} \max \left\{ v_0 l_K, \max_{i=j:K-1} \{ l_K v_i - p_i \}, \ l_K \bar{\nu} - p_K \right\}, \text{ and}
\]

\[
r_j(\nu) = \min_{p \in P_j(\nu)} r_j(\nu, p) = \min_{p \in P_j(\nu)} \max \left\{ \max_{i=j:K-1} \{ l_K v_i - p_i \}, \ l_K \bar{\nu} - p_K \right\}.
\]

These characterizations can be written as equivalent linear programs. For example,
for $r^j_0 := \inf_{v_0 \in (v, \bar{v})} r^j(v_0)$, we have the following formulation:

$$r^j_0 = \min_{r, v_0} r$$

subject to

$$r \geq v_0 l_K$$

$$r \geq v_n l_K - p_n \quad n = j \ldots K - 1$$

$$r \geq \bar{v} l_K - p_K$$

$$v_{n-1} = \frac{p_n - p_{n-1}}{x_n} \leq \frac{p_{n+1} - p_n}{x_{n+1}} = v_n \quad n = j + 1 \ldots K - 1$$

$$v_{K-1} = \frac{p_K - p_{K-1}}{x_K} \leq \bar{v}$$

$$v_j = \frac{p_{j+1} - p_j}{x_{j+1}} \geq v_0$$

$$p_j = v_0 l_j$$

$$v_0 > v$$

where the constraints (3.5), (3.6) and (3.7) reflect the possible choices of the adversary that appear in the characterization of $\inf_{v_0 \in (v, \bar{v})} r^j(v_0)$ above, and the constraints (3.8), (3.9), (3.10) and (3.12) define the feasible region, $v_1 = v_2 = \ldots = v_{j-1} = v_0 < v_j \leq v_{j+1} \leq \ldots \leq v_{K-1} \leq \bar{v}$, for the prices.

Similarly, $r^j(\bar{v})$ can be written as an equivalent linear program that is almost identical to the above LP but without the constraints (3.5) and (3.12) and with the value of the variable $v_0$ fixed at $\bar{v}$. Thus, the feasible region for the prices is given by inequalities $v_1 = v_2 = \ldots = v_{j-1} = \bar{v} < v_j \leq v_{j+1} \leq \ldots \leq v_{K-1} \leq \bar{v}$, in this case, and we denote the optimal value to this LP as $r^j$. This formulation is given in equation (B.17) in Appendix B.

In the proof of Proposition 3.2.7 below, we first relax the strict inequalities ($v_0 > v$ in (3.4) and $v_j > v$ in (B.17)), and solve for the relaxed versions of the LPs in (3.4) and (B.17), establishing lower bounds on $r^j_0$ and $r^j$. Then, we show that the optimal
values, denoted respectively by \( r'^{ij}_0 \) and \( r'^{ij} \), satisfy

\[
\begin{align*}
\frac{v}{l} \frac{l_{K+2-j}}{(l_j + l_K) \prod_{i=j+1}^K (h_i + l_K)}
\end{align*}
\]

and

\[
\begin{align*}
\frac{v}{l} \frac{l_{K+1-j}}{\prod_{i=j+1}^K (h_i + l_K)} - v l_j
\end{align*}
\]

for \( j = 1 \ldots K - 1 \), with \( r'^{lK}_0 := v l_K / 2 \), and \( r'^{lK} := (\bar{v} - v) l_K \).

where \( h_j = l_j - l_{j-1} \) is the quality difference between two neighboring versions. With these values, we are ready to state our main result of this section; the proof is given in Appendix B.

**Proposition 3.2.7** The optimal regret for the vertical product positioning problem satisfies

\[
\begin{align*}
\frac{v}{l} \frac{l_{K+2-j}}{\prod_{i=j+1}^K (h_i + l_K)}
\end{align*}
\]

with the optimal price vector having one of the forms:

- \( p_j = r'^{ij}_0 l_j / l_K \), and \( p_{n+1} = \frac{r'^{ij}_0 h_{n+1} + p_n (l_K + h_{n+1})}{l_K} \) for \( n = j \ldots K - 1 \), if \( r'^* = r'^{ij}_0 \) for some \( j \), or

- \( p_j = v l_j \), and \( p_{n+1} = \frac{r'^{ij} h_{n+1} + p_n (l_K + h_{n+1})}{l_K} \) for \( n = j \ldots K - 1 \), if \( r'^* = r'^{lj} \) for some \( j \).

For example, if \( r'^* = r'^{ij} = r'^{lj} \), the optimal policy is to offer \((K - j + 1)\) versions such that all consumers choose to buy one version, i.e. the whole market is served. The lowest consumer type \( v \) chooses version \( j \) which is priced at \( p_j = v l_j \). The market is segmented with respect to quality types in a monotone manner, and segments with higher quality taste choose higher quality versions and pay higher prices. On the other hand, if \( r'^* = r'^{ij} = r'^{lj} \), the whole market is not served. Consumers of type \( v \in [v, v_0) \) choose not to buy any version, where \( v_0 = r'^{ij}_0 / l_K = \frac{v}{l} \frac{l_{K+1-j}}{(l_j + l_K) \prod_{i=j+1}^K (h_i + l_K)} \), and consumer type \( v_0 \) chooses version \( j \) which is priced at \( p_j = v_0 l_j \).
Below we show that the firm can choose not to serve the whole market only when she offers all $K$ versions, which further reduces the optimal regret formula. This means that some lowest quality types are not served only when the potential market heterogeneity for quality is so large that even offering all $K$ versions is not enough to span the market in an optimal manner.

**Proposition 3.2.8**  The following inequalities hold $\bar{v} \, l_K/2 > r_0^{K-1} > \ldots > r_0^1$, and thus, the optimal regret can be reduced to the following form

$$
r^* = \min \left\{ \frac{\bar{v} l_{K+1}^j}{(l_j + l_K) \prod_{i=2}^{K}(h_i + l_K)}, \frac{\bar{v} l_{K+1-j}^j}{\prod_{i=j+1}^{K}(h_i + l_K)} - v_l_j, \, j = 1 \ldots K-1, \, (\bar{v} - v)l_K \right\}
$$

**Proof:** The first inequality is obtained as follows

$$
\frac{\bar{v} \, l_K^j}{2} = \frac{\bar{v} \, l_K^j}{2l_K^j} > \frac{\bar{v} \, l_K^j}{2l_K + h_K \, l_{K-1}} = \frac{\bar{v} \, l_K^j}{l_K^j + l_K(l_j + h_K) + h_K \, l_{K-1}} = \frac{\bar{v} \, l_K^j}{(l_{K-1} + l_K)(l_K + h_K)} = r_0^{K-1}
$$

Similarly, for the other inequalities, we note that

$$
\frac{1}{l_j + l_K} > \frac{l_K}{l_K^2 + l_K(l_j + h_{j+1}) + h_{j+1} \, l_j} = \frac{l_K}{l_K^2 + l_K(l_j + h_{j+1}) + h_{j+1} \, l_j} = \frac{l_K}{(l_j + l_K)(l_K + h_{j+1})},
$$

multiplying both sides by $\frac{\bar{v} \, l_{K+1-j}^j}{\prod_{i=j+2}^{K}(h_i + l_K)}$ yields $r_0^{j+1} > r_0^j$, which is valid for $j = 1 \ldots K-1$. □

Define again $\gamma = v/\bar{v}$ and $\gamma_j = \frac{l_{K-j+1}^j}{\prod_{i=j}^{K}(h_i + l_K)} = \frac{l_{K-j+1}^j}{\prod_{i=j}^{K}(h_i - n_{i-1} + l_K)}$ for $j = 1 \ldots K$. Remember that $\gamma$ is a measure of the potential market heterogeneity for quality. Similar to the competitive ratio case, we prove the following result tying the degree of heterogeneity to the number of versions offered and their quality levels. The proof is again given in Appendix B.
Proposition 3.2.9  As the potential market heterogeneity for quality increases, i.e. as $\gamma$ decreases, it is optimal to offer more quality versions of the product. Specifically, for $\gamma \in [0, \gamma_1)$ the optimal solution offers all the $K$ versions of the product; and consumers of type $v \in [v, v_0)$ choose not to buy any version, where $v_0 = \frac{e^{1/K}_{K+1-j}}{(1+iK) \prod_{i=1}^K (1+iK)}$, i.e. the whole market is not served. For $\gamma \geq \gamma_1$ the whole market is served. Specifically, for $\gamma \in [\gamma_j, \gamma_{j+1})$ there exists an optimal solution offering $(K+1-j)$-highest quality versions of the product for $j = 1, \ldots, K - 1$; and for $\gamma \in [\gamma_K, 1]$ there exists an optimal solution offering only the highest quality version of the product.

For both the competitive ratio and maximum regret criteria, we have shown that the optimal pricing policy charges a price-premium for each higher quality level as a function of the quality difference between neighboring versions. The optimal pricing and versioning policies are coordinated and take possible cannibalization effects among versions into account (e.g., non-dominating pricing policies again). The optimal positioning policy first decides on how much of the potential market to serve, and segments the market by offering only some number of highest quality versions, i.e. nested quality offers, so that consumers with higher taste for quality choose higher quality versions and pay higher prices. The number of versions offered depends on the market heterogeneity for quality. For both criteria, the number of quality levels offered increases as the uncertainty in the consumers’ preferences increases. This again leads to similar conclusions as the one in the previous section: offering different quality versions can be a response to uncertainty and/or lack of information with respect to consumer preferences as well as serving as a price discrimination mechanism. These results are also consistent with the vertical differentiation strategies for many products we observe in daily life. For products with large and heterogeneous
customer bases, such as micro processors with different speeds and high or low cache memories, professional, standard and student versions of software, and consumer electronics product lines with models having increasing functionality with higher prices, we see many different quality versions offered at different prices.
Chapter 4

Revenue Management Heuristics

Under Limited Market

Information: A Maximum Entropy Approach

In this essay, we consider incorporating censored sales information in a dynamic forecasting and optimization cycle for (airline) capacity allocation problems. Using a two demand class example, Cooper et al. [21] illustrate that most common forecasting methods, when used jointly with optimization routines, produce capacity controls that degenerate and call this phenomenon the “spiral-down effect”. We propose a tractable and intuitive approach based on maximum entropy (ME) distributions to incorporate censored sales information in such dynamic settings. We show that ME algorithm we propose provides capacity controls that avoid the “spiral down” effect and converge to the optimal values, making this approach attractive for problems
of joint forecasting and revenue optimization. In our analysis, we first describe the ME algorithm for the two fare-classes in Section 4.1. In Section 4.2, we prove the convergence of the protection levels produced by the ME algorithm to the optimal values for the underlying unknown demand. Finally, we provide an extension of the algorithm to multifare problems in Section 4.3.

4.1. Single-resource capacity control with two fare-classes

4.1.1 Full information model

We consider the capacity-control problem of an airline with two fare-classes. The demands for the two fare-classes are independent random variables that arrive sequentially in nonoverlapping intervals. The low fare demand, denoted by \( D_2 \), is realized before the high fare demand, denoted by \( D_1 \). The demand for each class is also independent of the capacity controls and remaining capacity at any stage. If there are group bookings, they can be partially accepted. The control variable of the airline is the protection level, denoted by \( L \), which, as mentioned above, is the number of seats to be reserved for the future high fare-class demand. This setup is usually referred to as the Littlewood's setting. The objective of the firm is to maximize its expected revenue, that is

\[
\max_{0 \leq L \leq C} \mathbb{E} \left[ p_1 \min \{ D_1, \max(C - D_2, L) \} + p_2 \min(D_2, C - L) \right],
\]

where \( C \) denotes the total available capacity. The term \( \min(D_2, C - L) \) is the sales for the low fare-class which arrives first; and consequently, the high fare-class sales is
the minimum of demand \( D_1 \) and the remaining number of seats \( C - \min(D_2, C - L) = \max(C - D_2, L) \).

If \( D_1 \) and \( D_2 \) are continuous random variables, the optimal protection-level \( L^* \) is given by the following equality

\[
p_1 \mathbb{P}(D_1 \geq L^*) = p_2 \iff F(L^*) = \gamma := 1 - p_2/p_1,
\]

where \( F(\cdot) \) is the cumulative probability distribution function for \( D_1 \). This equality is commonly referred to as Littlewood’s rule, and is a special case of the more general economical understanding which states “marginal cost = marginal revenue”. The intuitive reasoning behind Littlewood’s rule can be explained as follows: if we protect \( L^* \) seats for the high fare customers, we could sell the last (marginal) seat only if \( D_1 \geq L^* \) to earn \( p_1 \), resulting in the opportunity cost of \( p_1 \mathbb{P}(D_1 \geq L^*) \); whereas we could sell the same seat now to low fare customers for \( p_2 \). For integer values of demand and protection levels, we use the definition

\[
L^* = \min\{L \mid F(L) \geq \gamma\}.
\]

Note that the optimal protection level is independent of the low fare-class demand \( D_2 \).

4.1.2 Unknown demand distribution

If we do not know the distribution of the high fare-class demand in a dynamic setting where the problem repeats at each iteration \( k \), e.g., everyday flights from New York JFK to Atlanta International by a particular airline, we can try to learn the demand using past observations. Common practice involves using passive learning routines where one forecasts the distribution from past sales observations at each iteration \( k \),
and treats this forecast as if it is the real distribution to be used in the optimization of
the protection-level at each step, denoted by $L^k$, through equation (4.2) above. Next,
one can implement the protection-level $L^k$ and observe the new sales, and finally use
them to update the previous forecast. This iterative joint forecast and optimization
loop produces a sequence of protection levels $\{L^k\}$ through time.

Note that the optimal protection level is independent of the low fare-class de-
mand $D_2$ in equation (4.2) above. By exploiting this property and assuming that
the low fare-class demand is abundant, Cooper et. al. [21] show that many common
joint forecast and optimization routines produce protection levels that degenerate and
converge to suboptimal levels, and call this phenomena the spiral-down effect. This
problem arises due to censored sales data; and the assumption that low fare-class de-
mand is abundant corresponds to the worst-case scenario with the maximum amount
of censoring. Under this assumption, sales data at time $k$ is censored if and only if
$D_1^k \geq L^k$. That is, if the realization of the demand for the high fare-class customers at
time $k$, denoted by $D_1^k$, is less than the protection-level $L^k$, we can observe and record
the actual high fare demand level. On the other hand, if sales $S^k := \min(D_1^k, L^k)$
is equal to or larger than $L^k$, we can only deduce that $D_1^k \geq L^k$, that is we have a
censored observation at level $L^k$.

Let $S$ denote the size of the support for discrete demand $D_1^k$. In our setup,
we keep track of censored and uncensored demand realizations as described above.
Specifically, we record vectors $K^k \in \mathbb{R}^S$ and $J^k \in \mathbb{R}^S$ where

$$K_j^k = \# \text{ of uncensored observations at position } j, \text{ and, }$$
$$J_j^k = \# \text{ of censored observations at position } j.$$
As a result, we define the state $\theta^k \in \mathbb{R}^{2S}$ at each iteration $k$ as

$$\theta^k := (\kappa^k, \zeta^k) := (K^k/k, J^k/k), \quad (4.3)$$

where $\kappa^k_j$ denotes the frequency of uncensored observations at position $j$, and $\zeta^k_j$ denotes the frequency of censored observations at $j$. Then, the general problem of interest can be stated as follows:

**The problem:** given a state vector $\theta^k$ of past sales observations, identify a mapping $L(\cdot) : [0, 1]^{2S} \to [0, S]$, such that the sequence of protection levels $\{L(\theta^k)\}$ produced by this mapping converges to the optimal level, denoted by $L^*$, for the underlying actual demand distribution.

In the remainder of the analysis, we assume that the actual demand distribution has some positive probability mass at each level up to the actual protection level, that we use a large enough support for the ME algorithm, and that the low fare demand is ample. More precisely, denoting the discrete probability mass distribution vector for $D_1$ as $\pi$, we assume:

- **Assumption 1**: $\min_j \pi_j > \epsilon$ for some $\epsilon > 0$.
- **Assumption 2**: Support $S$ satisfies $S > 1/\epsilon$.
- **Assumption 3**: $D^k_2 \geq C$ with probability 1 for all $k$.

First two assumptions are not too restrictive for real-life applications. For airlines, it is often the case that each demand level is likely with some positive probability up to the protection level. In other words, one does not expect a demand distribution say with $P(D = x - 1)$ and $P(D = x + 1) > 0$ but $P(D = x) = 0$ for some $x \leq L^*$. The second assumption on the support is a technical one and does not impose a serious constraint on practical applications. Note that $S$ does not need to
correspond to any physical quantity, such as the capacity level or actual support, which might not be known to begin with, and that we are only interested in getting the distribution right up to the correct protection level $L^\star$. The third assumption is imposed to simplify the analysis and to establish a worst-case benchmark. In general, the optimal protection level is independent of the low fare-class demand distribution as also seen in equation (4.2), and assuming ample low fare-class demand corresponds to the worst-case censoring scenario. We establish in the following analysis that the ME algorithm provides protection levels that converge to the optimal value even under this worst-case censoring scenario. Considering that spiral-down is a phenomenon caused by censoring, this would guarantee the convergence of our algorithm in other demand realizations. In Section 4.3, we show that Assumption 3 can be relaxed without any major change in the analysis.

4.1.3 Proposed solution based on the ME distributions

Given the frequency of uncensored and censored observations $\theta^k = (\kappa^k, \zeta^k)$, let us define $\eta_j^k = \kappa_j^k + \zeta_j^k$ as the frequency of total observations at $j$. The key question is now to decide how to “allocate” the censored mass $\zeta^k$. The proposed approach is to
do so in a way that the maximizes the entropy of the resulting distribution as follows:

\[
\min_{p, z} \sum_{j} p_j \ln p_j \tag{4.4}
\]

s.t.

\[
p_j = \kappa_j^k + \sum_{i \leq j} z_{ij}, \quad \forall j \tag{4.5}
\]

\[
\sum_{j \geq i} z_{ij} = \zeta_i^k, \quad \forall i \tag{4.6}
\]

\[
z_{ij} = 0, \quad \forall i < j, \quad z_{ij} \geq 0, \quad \forall i, j \tag{4.7}
\]

\[
\sum_{j} p_j = 1, \quad \tag{4.8}
\]

where \(p \in \mathbb{R}_+^S\), \(z \in \mathbb{R}_+^{S^2}\), and \(z_{ij}\) denotes the probability mass that is allocated to level \(j\), but was observed at level \(i \leq j\) as a censored observation. That is, the vector \(z \in \mathbb{R}_+^{S^2}\) reallocates the censored mass to higher demand levels, and the objective function specifies the criterion according to which the uncensoring is done, which is to maximize the entropy of the forecasted distribution. The above problem has \(S^2\) many \(z_{ij}\) variables, but it can be simplified as follows:

**Proposition 4.1.1** Define the auxiliary vector \(\tilde{\kappa} \in \mathbb{R}_+^S\) as follows: \(\tilde{\kappa}_j^k = \kappa_j^k\) for \(j = 1 \ldots S - 1\), and \(\tilde{\kappa}_S^k = \eta_S\). We can reduce (4.4) to the following formulation where all vectors are in \(\mathbb{R}_+^S\):

\[
\min_{p} \sum_{j} p_j \ln p_j \tag{4.9}
\]

s.t.

\[
p \geq \tilde{\kappa}^k \tag{4.10}
\]

\[
\sum_{i \geq j} p_i \geq \sum_{i \geq j} \eta_i^k \quad \text{if} \quad \zeta_j^k > 0 \tag{4.11}
\]

\[
\sum_{j} p_j = 1 \tag{4.12}
\]
Proof: Denote the feasible set for problem (4.4) defined by constraints (4.5), (4.6), (4.7), and (4.8) as \( P_1 \); and similarly, the feasible region for problem (4.9) defined by constraints (4.10), (4.11), and (4.12) as \( P_2 \). We need to show that i) \( \forall (p, z) \in P_1, p \in P_2; \) and ii) \( \forall p \in P_2, \exists z \in \mathbb{R}_+^{2S} \) such that \( (p, z) \in P_1 \).

i) We first show for each \( (p, z) \in P_1 \), we have \( p \in P_2 \). Given any \( (p, z) \in P_1 \), using (4.5), we get that

\[
p_j = \kappa_j^k + \sum_{i \leq j} z_{ij} \geq \kappa_j^k, \quad j = 1 \ldots S - 1 \quad \text{and} \quad p_S = \kappa_S^k + z_{SS} = \kappa_S^k + \zeta_S^k = \kappa_S^k,
\]

hence, \( p \) satisfies the first set of constraints (4.10) in \( P_2 \).

Also, using constraints (4.5) and (4.6) in \( P_1 \), we have that

\[
\sum_{i \geq j} p_i = \sum_{i \geq j} \kappa_i^k + \sum_{i \geq j} \sum_{m \leq i} z_{mi} = \sum_{i \geq j} \kappa_i^k + \left( \sum_{m < j} \sum_{i \geq j} z_{mi} + \sum_{m \geq j} \sum_{i \geq m} z_{mi} \right) \geq \sum_{i \geq j} \eta_i^k,
\]

which shows that \( p \) satisfies (4.11) in \( P_2 \). As \( \sum_j p_j = 1 \), the last constraint (4.12) also obviously holds, and therefore, we have \( p \in P_2 \).

ii) Next, we show that for all \( p \in P_2 \), there exists a \( z \) such that \( (p, z) \in P_1 \). Given any \( p \in P_2 \), define \( d_j = p_j - \kappa_j^k \) for all \( j \). Observe \( \sum_j d_j = \sum_j p_j - \sum_j \kappa_j^k = 1 - \sum_j \kappa_j^k = \sum_j \zeta_j^k \). Also, note that constraints (4.10) and (4.11) imply \( \sum_{i \geq j} p_i \geq \sum_{i \geq j} \eta_i^k \) for all \( j \). Therefore, we have that \( \sum_{i < j} p_i \leq \sum_{i < j} \eta_i^k \), and hence, \( \sum_{i < j} d_i \leq \sum_{i < j} \zeta_i^k \). Now, let us define a transportation network flow problem as follows: there are \( S \) origin nodes each of which has supply \( \zeta_j^k \) for \( j = 1 \ldots S \), and \( S \) destination nodes each of which has demand \( d_j \) for \( j = 1 \ldots S \). The variables, \( z_{ij} \) denote the flow from origin node \( i \) to destination node \( j \) for all \( i, j \). We impose an upper bound of zero on flows whenever \( i < j \). We minimize the cost \( c z \) where \( c \) is any vector in \( \mathbb{R}_+^{2S} \). That is we
solve the problem
\[
\min_z \left\{ c^T z \mid \sum_{i \leq j} z_{ij} = d_j \forall j, \sum_{j \geq i} z_{ij} = \zeta_k^i \forall i, \ z_{ij} = 0 \forall i < j, \ z_{ij} \geq 0 \right\}. \quad (4.14)
\]

As \( \sum_{i<j} d_i \leq \sum_{i<j} \zeta_i^k \), i.e., the cumulative demand is less than the supply and therefore can be met, and as \( \sum_j d_j = \sum_j \zeta_j^k \), i.e., the transportation problem is balanced, the above problem is feasible and bounded for all \( c \in R^2_{+} \). For any feasible solution \( z \) to the above transportation problem, the corresponding vector \( (p, z) \in P_1 \) by construction. \( \square \)

Consequently, the algorithm we propose can be summarized as follows:

**Maximum entropy capacity allocation algorithm for two fare-classes**

1. At each observation \( k \), update the vector \( \theta^k := (\kappa^k, \zeta^k) \) according to (4.3).

2. Given \( \theta^k \), compute the ME probability mass function \( p_{\theta^k} \) through (4.9)-(4.12); denote the corresponding distribution function as \( F_{\theta^k}(\cdot) \).

3. Set \( L(\theta^k) = \min\{L \mid F_{\theta^k}(L) \geq \gamma\} \).

4. Implement \( L(\theta^k) \) with probability \( 1 - q(\theta^k) \), and \( L(\theta^k) + 1 \) with probability \( q(\theta^k) \), where \( q(\theta^k) \) is the unique solution to
\[
(1 - q(\theta^k)) F_{\theta^k}(L(\theta^k) - 1) + q(\theta^k) F_{\theta^k}(L(\theta^k)) = \gamma. \quad (4.15)
\]

5. Observe new sales in period \( k + 1 \) and go to step 1.

The motivation for this randomized policy is that such a policy with optimal \( L^* \) and \( q^* \) would yield higher expected revenues than implementing a single discrete
protection level with $L^*$ if we knew the actual discrete demand distribution. Although, the mappings $L(\cdot)$ and $q(\cdot)$ are independent of iteration $k$, we use $L^k := L(\theta^k)$ and $q^k := q(\theta^k)$ in the following analysis to reduce the notation.

In our previous language, this algorithm is “passive” in the sense that steps 3 and 4 above treat $F_{\theta^k}(\cdot)$ as if it is the correct demand distribution.

4.2. Convergence analysis for the two fare-class problem

This section proves that the algorithm specified in Section 4.1.3 yields a sequence of protection levels $\{L^k\}$ that converges to the optimal level $L^*$. As we will show, the algorithm, based on the ME distributions, will correctly approximate the entire high fare-class demand distribution up to the critical fractile $\gamma$, and yield the correct protection level $L^*$.

4.2.1 Outline of analysis

Let $\pi \in [0, 1]^S$ represent the probability mass distribution of the actual high-fare-class demand. Through the ME algorithm, the vector $(K^k, J^k)$ evolves recursively as

$$(K^{k+1}, J^{k+1}) = (K^k, J^k) + (W^{k+1}, Q^{k+1})$$  \hspace{1cm} (4.16)
where \( W^{k+1}, Q^{k+1} \in \mathbb{R}^S \) are random vectors satisfying

\[
P(W_j^{k+1} = 1) = \begin{cases} 
\pi_j & \text{for } j < L^k \\
q^k \pi_{L^k} & \text{for } j = L^k, \text{ and} \\
0 & \text{otherwise}
\end{cases}
\]

\[
P(Q_j^{k+1} = 1) = \begin{cases} 
(1 - q^k) \sum_{i \geq L^k} \pi_i & \text{for } j = L^k \\
q^k \sum_{i > L^k} \pi_i & \text{for } j = L^k + 1 \\
0 & \text{otherwise,}
\end{cases}
\]

for \( j = 1 \ldots S \); that is vectors \( W^{k+1} \) and \( Q^{k+1} \) track the realization of uncensored and censored observations at step \( k + 1 \).

Recall that \( \theta^k = (K^k, J^k)/k \) and define \( f(\theta^k) := \left( \mathbb{E}(W^{k+1}), \mathbb{E}(Q^{k+1}) \right) \), where the expectation is taken with respect to the (unknown) true demand distribution for the high fare-class demand so that

\[
\mathbb{E}(W_j^{k+1}) = \begin{cases} 
\pi_j & \text{for } j < L^k \\
q^k \pi_{L^k} & \text{for } j = L^k, \text{ and} \\
0 & \text{otherwise}
\end{cases}
\]

\[
\mathbb{E}(Q_j^{k+1}) = \begin{cases} 
(1 - q^k) \sum_{i \geq L^k} \pi_i & \text{for } j = L^k \\
q^k \sum_{i > L^k} \pi_i & \text{for } j = L^k + 1 \\
0 & \text{otherwise.}
\end{cases}
\]
Dividing both sides of (4.16) by $1/(k + 1)$, we get that

$$\theta^{k+1} = \frac{k \theta^k}{k + 1} + \frac{(W^{k+1}, Q^{k+1})}{k + 1}$$

$$= \frac{k \theta^k}{k + 1} + \frac{(W^{k+1}, Q^{k+1}) - f(\theta^k)}{k + 1} + \frac{f(\theta^k)}{k + 1}$$

$$= \theta^k + \frac{(W^{k+1}, Q^{k+1}) - f(\theta^k)}{k + 1} + \frac{f(\theta^k) - \theta^k}{k + 1}$$

$$= \theta^k + \frac{1}{k + 1} \delta M^k + \frac{1}{k + 1} g(\theta^k)$$

(4.18)

where $\delta M^k := (W^{k+1}, Q^{k+1}) - f(\theta^k)$ are bounded martingale differences for each $k$, and $g(\theta^k) := f(\theta^k) - \theta^k$ is a deterministic function governing the mean drift of the process. The martingale properties of $\delta M^k$ follows from the definition of $f(\theta^k)$, and the fact that the components of both $(W^{k+1}, Q^{k+1})$ and $f(\theta^k)$ are bounded by 1.

Equation (4.18) describes a typical adaptive algorithm, and its asymptotic properties can be analyzed through techniques from the stochastic approximation literature. Below we provide an asymptotic analysis of the process $\{\theta^k\}$ using what is called the "ODE (Ordinary Differential Equations) technique" (see Kushner and Yin [42]), which relies on asymptotic properties of continuous approximations to the process $\{\theta^k\}$. The main intuition behind this technique is that the effect of the random martingale difference terms $\delta M^k$ vanishes as $k$ gets larger, and the process can be approximated accurately by the limit paths of the continuous ODE $\dot{\theta}(t) = g(\theta(t))$. That is intuitively, as $1/(k + 1) \to 0$, $\theta^k$ changes slowly, and, in the absence of the $\delta M^k$ terms, equation (4.18) yields roughly

$$g(\theta^k) = \frac{\theta^{k+1} - \theta^k}{1/(k + 1)} \approx \dot{\theta} |_{\theta^k}.$$
and Yin [42]. For completeness we state a specific version of this theorem, which is adopted for the special case of bounded \( \{\theta^k\} \) process with bounded martingale differences, in Appendix C.1 together with its necessary conditions (A.4.3.1), and (A.5.2.1)-(A.5.2.6). Below we provide the outline of our argument for establishing \( \lim_{k \to \infty} L^k = L^* \):

1. Section 4.2.2 studies the asymptotic behavior of the continuous, deterministic dynamical system governed by the ODE \( \dot{\theta}(t) = g(\theta(t)) \).

2. Section 4.2.3 verifies that the conditions (A.4.3.1), and (A.5.2.1)-(A.5.2.6) needed by Kushner and Yin's theorem are satisfied by the process \( \{\theta^k\} \) and the ODE, and invokes their result to complete our proof.

### 4.2.2 Analysis of the ODE

Denote the domain of problem (4.9) as \( \mathcal{D}(\theta) \in \mathbb{R}^S \) for any given \( \theta = [\kappa, \zeta] \). Let \( p_\theta \in \mathcal{D}(\theta) \) be the optimal solution of problem (4.9) for a given \( \theta \), and let \( [\kappa_\theta, \zeta_\theta] \) denote the corresponding vectors of reallocated uncensored and censored observation frequencies at this solution such that \( \kappa_\theta = \kappa \) and \( p_{\theta,j} = \kappa_{\theta,j} + \zeta_{\theta,j} \) for all \( j \). Hence, \( p_\theta \) is the probability mass function of the ME distribution implied by any given \( \theta \) through problem (4.9), and \( F_\theta(\cdot) \) is the corresponding cumulative distribution function. Hence, the corresponding protection level produced by the ME algorithm is \( L(\theta) := \min\{L \mid F_\theta(L) \geq \gamma\} \), and \( q_\theta \) is the associated randomization probability specified in (4.15).

We start with some preliminary lemmas that establish some of the necessary structural properties of the ODE \( \dot{\theta}(t) = g(\theta(t)) \); their proofs are given in Appendix C.2.
Lemma 4.2.1 \( \mathcal{D}(\theta) \) is a continuous correspondence.

Lemma 4.2.2 The maximum entropy distribution is continuous in \( \theta \), i.e., \( p_\theta \) is continuous in \( \theta \).

Lemma 4.2.3 The function \( g(\theta) = f(\theta) - \theta \) is continuous in \( \theta \).

The next proposition identifies the unique stationary solution of the continuous ODE equation \( \dot{\theta}(t) = g(\theta(t)) \). This solution point, denoted by \( \theta_s \), would serve as the candidate unique limit of the process \( \{\theta^k\} \) if the process converged to a unique equilibrium point. In our case, the process converges to a stable set rather than a unique point as shown further below. When this is the case, all the stationary solutions of the continuous ODE equation must also converge to this stable set. The proof of the proposition is relegated to Appendix C.2.

Proposition 4.2.1 Under assumptions 1, 2, and 3, \( \dot{\theta}(t) = g(\theta(t)) \) has a unique stationary solution point \( \theta_s \) which satisfies:

\[
L(\theta_s) = L^*, \quad \text{and} \quad q(\theta_s) = \frac{(\gamma - \sum_{i=1}^{L^*-1} \pi_i)(\min\{S,S_w\} - L^* + 1)}{1 - \sum_{i=1}^{L^*-1} \pi_i}, \quad \text{where}
\]

\[
S_w = \frac{1 - \sum_{i=1}^{L^*-1} \pi_i}{\sqrt{\pi_{L^*}(\gamma - \sum_{i=1}^{L^*-1} \pi_i)}} + L^* - 1.
\]

We next characterize a candidate limit set of the continuous ODE and the process \( \{\theta^k\} \); and show that it is asymptotically stable for the ODE \( \dot{\theta}(t) = g(\theta(t)) \). This will be needed to prove convergence of \( \{\theta^k\} \) further below. The proof is given in Appendix C.2. We use the candidate Lyapunov function

\[
V(\theta) = \sum_{j < L^*} ||\kappa_j - \pi_j|| + (\gamma - \sum_{j < L^*} \pi_j - \kappa_{L^*})^+,
\] (4.19)
where \(||x|| := \sqrt{x^2}\) is the \(L^2\) norm on \(\mathbb{R}\), and \(x^+ = \max\{x, 0\}\) is the positive part of \(x\). This function is continuous everywhere and continuously differentiable almost everywhere, and the left and right derivatives are strictly negative at breakpoints of \(\dot{V}(\theta)\).

**Proposition 4.2.2** Under assumptions 1, 2, and 3, the set \(\Theta = \{\theta \mid \kappa_j = \pi_j, \forall j < L^*, \gamma - \sum_{j<L^*} \pi_j \leq \kappa_{L^*}\}\) is globally asymptotically stable for the ODE \(\dot{\theta}(t) = g(\theta(t))\).

Observe that, by assumptions 1 and 2 and the definition of \(L^*\), we have \(L_\theta = L^*\) for \(\forall \theta \in \Theta\) as

\[
\sum_{j<L^*} p_{\theta,j} = \sum_{j<L^*} \pi_j < \gamma \quad \text{and} \quad \sum_{j\leq L^*} p_{\theta,j} \geq \sum_{j<L^*} \pi_j + \kappa_{L^*} \geq \gamma, \quad \forall \theta \in \Theta. \tag{4.20}
\]

The proof of the following result is also given in Appendix C.2.

**Proposition 4.2.3** The stationary solution of \(\dot{\theta}(t) = g(\theta(t))\) belongs to the set \(\Theta\), i.e., \(\theta_s \in \Theta\).

### 4.2.3 Proof of convergence

Finally, having established a candidate limit set that is asymptotically stable in which the desired result for the protection level is guaranteed, we show that the process \(\{\theta^k\}\) indeed converges to this set \(\Theta\) by using a theorem provided by Kushner and Yin [42].

**Theorem 4.2.1** Denote the state vector given by the ME algorithm at step \(k\) as \(\theta^k = [\kappa^k, \zeta^k]\), and the protection level implied as \(L^k\). Then, \(\lim_{k \to \infty} L^k = L^*\).

\(^1\)These conditions are sufficient for asymptotic stability.
Figure 4.1: Protection levels produced by the ME algorithm, the empirical distribution, and the uncensored actual demand histogram at each iteration are compared. For the example, $p_2/p_1 = 1 - \gamma = 0.5$, $S = 200$, $D_1 \sim U[50, 80]$. 
Proof: The proof follows from Kushner and Yin’s [42] Theorem 2.1 (Ch. 5, pg. 127) for identifying limits of adaptive algorithms making use of continuous time ordinary differential equations (ODE). This theorem and its conditions, (A.4.3.1) and (A.5.2.1)-(A.5.2.6), are supplied in Appendix C.1.

In our case the process \( \{\theta^k\} \) is bounded in the hyperrectangle \( H := [0, 1]^{2S} \) with probability 1 by construction; hence condition (A.4.3.1) is satisfied. For the ME algorithm, we have that \( \beta^k = 0 \) and \( \epsilon^k = 1/k \), and \( \mathbb{E}[Y^k | \theta^0, Y^i, i < k] = g(\theta^k) = f(\theta^k) - \theta^k \). Consequently, conditions (A.5.2.2), (A.5.2.4) and (A.5.2.5) are trivially satisfied. We also satisfy condition (A.5.2.3) as shown in Lemma 4.2.3 above. Condition (A.5.2.1) is satisfied as \( \sup_k \mathbb{E}|Y^k|^2 \leq S < \infty \) by construction. We have that the ODE \( \dot{\theta}(t) = g(\theta(t)) \) has a unique stationary point \( \theta_s \in \Theta \) as identified by Proposition 4.2.1, and hence, the set \( L^1_H \) of the stationary solutions for the ODE, as defined in condition (A.5.2.6), has a single member \( \theta_s \). Also, we have shown in Proposition 4.2.2 that the set \( A_H = \Theta \) is globally asymptotically stable. Also, we have that \( L^1_H \cup A_H = \Theta \) by Proposition 4.2.3. Consequently, applying Theorem 2.1 of Kushner and Yin, we conclude that limit points of the process \( \{\theta^k\} \) are in \( \Theta \) with probability 1. However, as shown before in equation (4.20) above, we have that \( L_\theta = L^* \) for all \( \theta \in \Theta \).

Figure 4.1 provides an example of the ME algorithm and the spiral-down effect. As illustrated, the protection levels attained by the empirical distribution of observations spiral down as predicted by Cooper et al. [21]. The protection levels \( L^k_H \) provided by the ME algorithm converge to the correct level. Also, observe that the controls obtained by accumulating the true (uncensored) demand observations, which corresponds to the (first) best case for the firm, seem to converge almost at the same rate as those provided by the ME algorithm. The uncensored full information controls
seem to under protect for the high fare-class demand while approaching the optimal level; whereas the protection levels \( \{L^k_d\} \) over protect. In general, in the presence of censored observations, no algorithm can produce protection levels that converge to the correct level by consistently under protecting.

### 4.3. Extension to multifare problem

The corresponding multifare model, where demand for the different classes arrives in nonoverlapping intervals in the order of increasing prices, was studied by Brumelle and McGill [18] using a stochastic dynamic program. Labeling each class such that the fares satisfy \( p_1 > p_2 > \ldots > p_n \), the Bellman equation for each fare-class can be written as (see Talluri and van Ryzin [73]):

\[
V_n(x) = \mathbb{E}\left[ \max_{0 \leq u \leq \min\{D_n,x\}} (p_n u + \delta V_{n-1}(x-u)) \right]
\]

with boundary conditions \( V_0(x) = 0 \), \( \forall x \), where \( x \) denotes the capacity available at stage \( n \), and \( u \) is the optimal sales amount. Brumelle and McGill [18] show that the optimal controls take the form of nested protection levels; that is protection level \( L_n \) denotes the number of seats to be reserved for the \( n \) highest fare-classes \( n, n-1, \ldots, 1 \). For the continuous problem, the optimal levels for the vector \( L \) can be found using the following formulae, which generalize the Littlewood’s rule in (4.1):

\[
p_1 \ P(D_1 > L_1, D_1 + D_2 > L_2, \ldots, \sum_{i=1}^n D_i > L_n) = p_{n+1}, \quad \forall n.
\] (4.21)

A brute force application of the ME approach to the multifare problem would require estimating the joint probability distribution of the demand vector for all classes using a ME formulation analogous to (4.9). Clearly, the number of variables
and problem complexity in such a formulation would increase substantially relatively to that of (4.9).

In the sequel we will focus on a more efficient, albeit a heuristic, approach that exploits the optimality of equations in (4.21). We will describe an algorithm for the general $N$ fare-class problem, but to motivate our discussion, we will first illustrate the main ideas using a three fare-class example below for ease of exposition.

Let $\pi^n \in [0,1]^S$ represent the probability mass distribution of $n^{th}$ fare-class demand $D_n$ for $n = 1, 2, 3$. Assume again the lowest fare-class demand $D_3$ is abundant, and we are interested in finding $L_1$ and $L_2$. For $n = 1$, the equation (4.21) yields the Littlewood's formula above, and the ME algorithm that is applied as above by observing censored and uncensored high-class demand $D_1$ is guaranteed to converge to $L_1$ as demonstrated before. Furthermore, by equation (4.17), the algorithm will also yield the actual demand distribution $\pi^1$ correctly up to $L_1$ in the limit. However, considering the fill event equation for $j = 2$,

$$P(D_1 > L_1, D_1 + D_2 > L_2) = P(D_1 > L_2) + \sum_{j=L_1}^{L_2} P(D_1 = j)P(D_2 > L_2 - j) = \frac{p_3}{p_1},$$

we see that we also need to estimate distribution of $D_1$ between $L_1$ and $L_2$. Obviously, the frequency of observations for $D_1$ between $L_1$ and $L_2$ depend on the distribution of $D_2$ up to $L_2 - L_1$; as $D_2$ is realized before $D_1$, and we can observe a $1^{st}$ fare-class sales in this region only when $D_2$ is small enough.

Assume for the moment that we somehow know both actual $L_1$ and $L_2$, and use them in the above dynamic setting for a long time. In the limit, we would observe uncensored second class demand at level $j$ with frequency $\pi_j^2$ for $j = 1 \ldots L_2 - L_1$ as $D_2$ is realized before $D_1$ at each iteration, and uncensored high-class demand at level $L_1 + j$ with frequency $\pi_{L_1+j}^1 P(D_2 < L_2 - L_1 - j) = \pi_{L_1+j}^1 \sum_{i=1}^{L_2-L_1-j-1} \pi_i^2$. Therefore,
if we used formulation (4.9) to estimate the distribution of $D_2$ using the ME method, we would get it correctly up to $L_2 - L_1$. Then, dividing the uncensored high-class demand observed frequencies by $\sum_{i=1}^{L_2-L_1-j-1} \pi_i^2$, we could find the distribution of $D_1$ between $L_1$ and $L_2$, too. Consequently, assuming again the lowest fare demand $D_N$ is abundant, the heuristic we propose iteratively estimates demand and protection levels for fare-classes in increasing order of fares as follows:

1. At each observation $k$, treat the previous estimates of $L_n^k$ for each class $n$ as the actual protection levels

2. Update the frequency vector of uncensored and censored observations for $D_n$, i.e., $\theta_n^k$, using past sales data.

3. Use $\theta_{N-1}^k$ in ME formulation (4.9) to find the estimate $\hat{\pi}^{N-1}$.

4. For each $n = N - 2, \ldots, 1$ do the following in reverse order of fares:

   • Scale up each component of $\kappa_n^k$, i.e., the vector of uncensored observations for $D_n$, between $L_n$ and $L_{n+1}$ by dividing it by $\sum_{i=1}^{L_{n+1}-L_{n+1}-j-1} \pi_i^{n+1}$ for $j = 1 \ldots L_{n+1} - L_n$. That is assign

   $ \kappa_{1_{L_{n+1}+j}}^k \leftarrow \frac{\kappa_{1_{L_{n+1}+j}}^k}{\sum_{i=1}^{L_{n+1}-L_{n+1}-j-1} \pi_i^{n+1}} \quad j = 1 \ldots L_{n+1} - L_n.$

   • Use the updated $\theta_n^k$ in ME formulation (4.9) to find the estimate $\hat{\pi}^n$. Go back to previous step for $n - 1$.

5. Use $\hat{\pi}^n$ to calculate the next estimate protection level vector $L^{k+1}$ according to fill event equation (4.21) above.

6. Implement the protection levels and observe new sales in period $k + 1$ and go to step 1.
Figure 4.2: An example plotting protection levels for $p_2/p_1 = 0.6$, $p_3/p_1 = 0.4$, $D_1 \sim U[0,40]$, $D_2 \sim U[0,60]$. When the fill equation is not satisfied by any discrete number, the protection levels alternate between closest integer values. This can be thought of randomizing the protection level to meet the fill equation, as described in the previous subsection.

The scaling up operation in step 4 may potentially amplify errors for higher fare-classes as the estimate of each class uses effectively estimates of all lower fare-classes. A numerical example demonstrating this algorithm for a three fare-class problem is provided in Figure 4.2.
Chapter 5

Conclusions

In this thesis, we studied some classical marketing and revenue management models under the assumption that the monopolist seller does not have full information about the market demand, consumer preferences and willingness-to-pay. We demonstrated that this new source of uncertainty may lead to optimal policies that are significantly different from their classical counterparts in the literature when combined with different performance criteria that are suitable for such uncertain settings. Many of our results are consistent with policies observed in practice.

Our first two essays study the pricing and product line positioning problems when the seller has limited information about consumers’ preferences and valuation using the relative performance criteria of competitive ratio and maximum regret. These criteria measure the performance relative to a fully-informed decision maker, and allow us to distinguish between bad market conditions and bad decisions. They produce less conservative results when compared to more common maxmin type performance criteria.

In our first essay, we examine monopoly pricing mechanisms of dynamic pricing,
third-degree price discrimination, and second-price sealed-bid auction under limited WtP information. We assume customers have private WtP drawn from a common distribution that is unknown to the seller. We provide closed-form solutions for the optimal pricing policies and highlight important structural insights for all three pricing schemes. In the dynamic pricing setting, the firm has the ability to change its price over time, and its key decision is to figure out a pricing policy that would perform well when the firm does not know the underlying consumers' WtP distribution. The firm's optimal policy is to adopt a price skimming scheme to minimize and balance the risk of lost sales and foregone revenue under unknown market characteristics. To contrast, if the firm knew the customer WtP distribution, then it would be optimal to charge a static price over the entire sales horizon. Hence, the lack of market information could offer one possible justification for the use of dynamic pricing policies. Analytically, the precise form of the resulting pricing policy ensures that both the firm and the adversary are indifferent with regard to positioning of the representative valuation of the market.

We also analyze the effects of learning in this setting and illustrate how learning and partial information can be incorporated. We consider a version where the firm splits the sales horizon into two periods to use the information extracted in period one to optimize its decisions in period two. When the seller is not restricted in the number of price points that she can experiment with, it is optimal to use a price skimming policy during the learning period and a single price in the remainder of the sales horizon. When there are constraints linking the firm's pricing decision over time, we show that the seller is willing to sacrifice performance to retain pricing flexibility. We also show how to incorporate partial sales information in the form of fractiles of the representative WtP distribution. Through numerical examples, we demonstrate
that even limited learning, e.g., information gained by testing a few price points, leads to significant performance gains with relative performance criteria, and the resulting policies have very good revenue performance across common demand distributions.

The second pricing scheme we study is the third degree price discrimination setting, where the monopolist firm segments the market into subgroups that can be charged different prices. However, in our setting, the firm does not know the representative WtP and relative size of each of these market segments. We show that it is optimal for the firm to set the price for a particular segment equal to its minimum WtP value within the segment. Moreover, if the firm has the ability to choose the market segmentation, it will set equal the (absolute or relative) sizes of the WtP intervals. This strategy is similar to the price skimming policy above in that the firm segments the total market size into smaller subgroups by skimming through the whole valuation range. However, the adversary is strong enough to force the firm to price at the lowest WtP value within each segment. Combined with the segmentation strategy, this results in a piecewise price menu being offered to the market.

The third pricing mechanism we consider is the second price, sealed-bid auction. The choice of this mechanism is motivated by the fact that irrespective of the number of bidders and the underlying WtP distribution, it is still a dominant strategy for each potential buyer to bid her/his true WtP within this mechanism. We show that it is optimal for the seller to set his reservation price so as to balance again the risks of not selling the item with the risk of selling it at a low price. The resulting reservation price has similar properties to the one derived under the classical revenue maximizing setting, but, is slightly lower, reflecting risk aversion of the seller.

In our second essay, we study the joint problem of product line positioning and pricing for a monopolist when consumer preferences and WtP are unknown. We con-
sider a firm that must decide which different versions of a product to offer and how to price each version so as to maximize the performance of the whole product line. Horizontal differentiation is the case where the versions differ in ways that appeal to the heterogeneous tastes of consumers, such as different colors, sizes or flavors. Vertical differentiation describes offering versions that differ in their level of quality and performance. Product line positioning must be coordinated with pricing policies to maximize revenues and prevent cannibalization effect among versions. We extend some classical models of horizontal and vertical differentiation to cover uncertainty in preferences and WtP using the relative performance criteria. Our analysis provides insights into practices observed in many real world markets. For the horizontal differentiation case, we show that the optimal decision for both criteria is to position products at equal intervals in the attribute space and to price them identically. For the vertically differentiated case, we show that the optimal policy consists of offering a number of the highest quality versions, and that the more ambiguity over customers' taste for quality, the more versions the firm should offer. Our results can successfully explain frequently observed versioning practice in information and technology goods by using the classical framework of linear utility functions, as opposed to many earlier work in the marketing literature that failed to do so and concluded that versioning is not optimal for such goods, despite the empirical evidence.

In our first two essays, we study uncertainty in demand information for a single sales horizon only. We change our focus to incorporating continuous stream of partial information in a dynamic capacity allocation setting in our last essay. The inherent complexity of the relative performance criteria often hinder integration of such dynamic information. Therefore, we use a different approach to incorporate partial sales information in such dynamic settings. Revenue management systems consist of joint
forecasting and optimization modules that work in an iterative manner. However, the effect of optimization on future forecasts via policies implemented may result in possible accumulation of errors through time as a result of censored demand observations. Many common uncensoring techniques fail to avoid the producing controls that degenerate, i.e., the "spiral-down" effect, in such settings as illustrated by Weatherford and Polt [78]. We propose a tractable and intuitive approach based on maximum entropy (ME) distributions that easily incorporates partial demand information in the form of censored sales data. Considering an airline capacity control problem where demand for lower fare-classes show up before higher ones in nonoverlapping intervals, we show that the protection levels given by the ME algorithm we propose avoids the "spiral down" effect and converge to the optimal values. We make use of adaptive algorithms and stochastic approximations in our analysis for two fare-classes and provide a heuristic algorithm for the multifare problem. The solution to the ME problem simultaneously applies uncensoring to the raw sales data in an intuitive and statistically sound manner.

Future research opportunities related to our work include establishing possible connections between decision theory, incentive systems and revenue management through relative performance criteria. As previously noted, these criteria successfully explains some of the observed practices in product and revenue management, for which other criteria have failed to do so. It is plausible to hypothesize that a risk averse revenue/product manager could choose to deviate from the firm’s revenue maximization goal in a moral hazard setting to hedge her own performance evaluation risk, which can be successfully modeled using these criteria. Another possible research direction is inclusion of partial demand and preference information of some of the models considered here. We illustrated in the dynamic pricing setting that par-
tial information through limited learning can lead to robust implementable policies with good revenue performance across many demand distributions. We believe similar practical results can be established for other models considered in this thesis. And finally, we believe that the ME approach holds a good potential for future revenue management research. In addition to our theoretical results for the capacity allocation setting in this thesis, our preliminary numerical experiments indicate that this approach can also produce intuitive policies with good performance for monopolistic pricing settings as well.
References


Appendix A

Supplement to Chapter 2: Proofs of Remaining Results

Proof of Proposition 2.2.1: We first establish an upper-bound by interchanging the order of the inner maximization and minimization, and using the minimax inequality. That is, we have that $c \leq \hat{c}$, where

$$
\hat{c} = \max_{t^1} \min_{\bar{f}^1} \min_{\bar{f}^2} \max_{t^2} \frac{\Lambda^1 \sum t_j^1 p_j f_j^1 + \Lambda^2 \sum t_j^2 p_j \bar{f}_j^2}{(\Lambda^1 + \Lambda^2) \max_j p_j \bar{f}_j^2}
$$

subject to

$$
\sum t_j^i = 1, \quad t^i \geq 0 \quad \text{for } i = 1, 2
$$

$$
1 = \bar{f}_1^i \geq \bar{f}_2^i \geq \cdots \geq \bar{f}_k^i \geq 0 \quad \text{for } i = 1, 2
$$

$$
t_j^i(\bar{f}_j^2 - \bar{f}_j^1) = 0, \quad \forall j
$$

However, the information about $\bar{f}_1^i$ and $\bar{f}_2^i$ is revealed successively, and $\bar{f}_1^i$ can be different from $\bar{f}_2^i$ only at points $j$ where $t_j^1 = 0$, i.e. even if $\bar{f}_1^i$ can be different from $\bar{f}_2^i$, this difference does not change the objective function value. Therefore, we can assume $\bar{f}_2^i$ contains the information of $\bar{f}_1^i$. Consequently, letting $\bar{f} := \bar{f}_2 = \bar{f}_1^i$, the
formulation of $\hat{c}$ can be written as follows:

$$\hat{c} = \max_{t^1} \min_{f} \max_{t^2} \frac{\Lambda^1 \sum t^1_j p_j \bar{f}_j + \Lambda^2 \sum t^2_j p_j \bar{f}_j}{(\Lambda^1 + \Lambda^2) \max_j p_j \bar{f}_j}$$

s.t. $\sum t^1_j = 1$, $t^1 \geq 0$, $i = 1, 2$,
$$1 = \bar{f}_1 \geq \cdots \bar{f}_K \geq 0.$$

Now, fix $t^1$ and $\bar{f}$, and consider the inner maximization. The denominator and the first part of the numerator is fixed, and therefore, the inner maximization is simply equivalent to maximizing the second part of the numerator, i.e. maximizing the second period revenue. Therefore, the upper bound problem is equivalent to:

$$\hat{c} = \max_{t^1} \min_{f} \frac{\Lambda^1 \sum t^1_j p_j \bar{f}_j + \max_{t^2} \left\{ \Lambda^2 \sum t^2_j p_j \bar{f}_j : \sum_j t^2_j = 1, \ t^2 \geq 0 \right\}}{(\Lambda^1 + \Lambda^2) \max_j p_j \bar{f}_j}$$

s.t. $\sum_j t^1_j = 1$, $t^1 \geq 0$,
$$1 = \bar{f}_1 \geq \cdots \bar{f}_K \geq 0,$$

As, $\max_{t^2} \left\{ \Lambda^2 \sum t^2_j p_j \bar{f}_j : \sum_j t^2_j = 1, \ t^2 \geq 0 \right\} = \Lambda^2 \max_j \{p_j \bar{f}_j\}$, the upper bound problem reduces to

$$\hat{c} = \max_{t^1} \min_{f} \frac{\Lambda^1 \sum t^1_j p_j \bar{f}_j}{(\Lambda^1 + \Lambda^2) \max_j \{p_j \bar{f}_j\}} + \frac{\Lambda^2}{(\Lambda^1 + \Lambda^2)}$$

s.t. $\sum_j t^1_j = 1$, $t^1 \geq 0$,
$$1 = \bar{f}_1 \geq \cdots \bar{f}_K \geq 0,$$

which is exactly the problem (2.6), whose solution is also the optimal solution of $c$ and satisfies $t^1_\ast > 0$ componentwise as identified by Theorem 2.2.1 above. In other words, the optimal solution of $\hat{c}$ is $(t^1_\ast, t^2_\ast)$ identified by Theorem 2.2.1, at this point the value $c$ achieves its upper-bound $\hat{c}$, i.e. $c = \hat{c}$. Therefore this point is also optimal for $c$ and $t^1_\ast > 0$ componentwise. □
Proof of Lemma 2.2.1: Fix some $t^1$, let the optimal solution to the inner minimization be $\bar{f}^*$, and the resulting optimal ratio be $c(t^1) := n(t^1)/d(t^1)$ where $n(t^1)$ and $d(t^1)$ denote the corresponding values of the numerator and the denominator respectively in (2.10) at the optimal solution.

First observe that $\bar{f}_j = 0$ for $j > j^*$ must hold at the optimal solution for the inner problem. To see this, consider the constraints $1 = \bar{f}_1 \geq \bar{f}_2 \geq \cdots \geq \bar{f}_{j^*} \geq \cdots \bar{f}_K \geq 0$. For any fixed values of $\bar{f}_1, \cdots, \bar{f}_{j^*}$, if some of the variables $\bar{f}_{j^*+1}, \cdots, \bar{f}_K$ have positive values, reducing them does not change the value of $p_{j^*} \bar{f}_{j^*} = \max_j \{p_j \bar{f}_j\}$, and hence the the value of the denominator in (2.10), while strictly reducing the value of the numerator, which would yield a lower competitive ratio than the optimal one, yielding a contradiction.

Now, we show that $\bar{f}_{j^*} = 1$ must hold at the optimal solution of the inner problem, which requires a little more work. Again, assume by contradiction that $\bar{f}_{j^*} < 1$, then increasing $\bar{f}_{j^*}$ by $\epsilon$ increases $n(t^1)$ at most by

$$
\epsilon \left( \Lambda^1 \sum_{i \leq j^*} t^1_i p_i + \Lambda^2 \left[ \sum_{i < j^*} t^1_i p_i + \left( 1 - \sum_{i < j^*} t^1_i \right) p_{j^*} \right] \right),
$$

while increasing the denominator exactly by $\epsilon (\Lambda^1 + \Lambda^2) p_{j^*}$. Therefore, the resulting new ratio, denoted by $c_\epsilon$, satisfies

$$
c_\epsilon \leq \frac{n(t^1) + \epsilon \left( \Lambda^1 \sum_{i \leq j^*} t^1_i p_i + \Lambda^2 \left[ \sum_{i < j^*} t^1_i p_i + \left( 1 - \sum_{i < j^*} t^1_i \right) p_{j^*} \right] \right)}{d(t^1) + \epsilon (\Lambda^1 + \Lambda^2) p_{j^*}} < \frac{n(t^1)}{d(t^1)} = c(t^1),
$$

which results in a contradiction. The strict inequality above holds because, in general
we have,

\[
\frac{\epsilon \left( A^1 \sum_{i<j^*} t^1_i p_i + A^2 \left[ \sum_{i<j} t^1_i p_i + \left(1 - \sum_{i<j} t^1_i\right) p_j^* \right] \right)}{\epsilon (A^1 + A^2)p_j^*} 
\leq \frac{n(t^1)}{d(t^1)} = c(t^1) = \frac{A^1 \sum_{i<j} t^1_i p_i \hat{f}_i + A^2 \left[ \sum_{i<j} t^1_i p_i \hat{f}_i + \left(1 - \sum_{i<j} t^1_i\right) p_j^* \hat{f}_j^* \right]}{(A^1 + A^2)p_j^* \hat{f}_j^*},
\]

as \(1 = \hat{f}_1 \geq \hat{f}_2 \geq \cdots \geq \hat{f}_j^*\) and the inequality holds with equality iff \(1 = \hat{f}_1 = \hat{f}_2 = \cdots = \hat{f}_j^*\) which is not possible by the contradictory assumption. \(\square\)

**Proof of Proposition 2.3.2:** Suppose that for fixed \(n\), the adversary selects \(\bar{f}_n \in (0,1)\). Using the result of Proposition 2.3.1, if \(p_{j_1} < p_{j_2} \bar{f}_n\), then for any vector \(t\), the competitive ratio is of the form

\[
c(n, \bar{f}_n, t) = c_{j_1, j_2} = \frac{\Lambda^1 p_n \bar{f}_n + \Lambda^2 (t_1 p_1 + \sum_{j=2}^{j_2} t_j p_j \bar{f}_n)}{(\Lambda^1 + \Lambda^2)p_{j_2} \bar{f}_n},
\]

for some \(j_2\), \(n \leq j_2 \leq K\). But, for fixed \(n\) and \(\bar{f}_n \in (0,1)\),

\[
c(n, \bar{f}_n, t) = \frac{\Lambda^1 p_n \bar{f}_n + \Lambda^2 (t_1 p_1 + \sum_{j=2}^{j_2} t_j p_j \bar{f}_n)}{(\Lambda^1 + \Lambda^2)p_{j_2} \bar{f}_n} 
= \frac{\Lambda^1 p_n}{(\Lambda^1 + \Lambda^2)p_{j_2}} + \frac{\Lambda^2}{(\Lambda^1 + \Lambda^2)} \frac{t_1 p_1}{p_{j_2} \bar{f}_n} + \frac{\Lambda^2}{(\Lambda^1 + \Lambda^2)} \frac{\sum_{j=2}^{j_2} t_j p_j}{p_{j_2}} 
\geq \frac{\Lambda^1 p_n}{(\Lambda^1 + \Lambda^2)p_{j_2}} + \frac{\Lambda^2}{(\Lambda^1 + \Lambda^2)} \frac{t_1 p_1}{p_{j_2}} + \frac{\Lambda^2}{(\Lambda^1 + \Lambda^2)} \frac{\sum_{j=2}^{j_2} t_j p_j}{p_{j_2}} 
= c(n, 1, t) \quad \forall t, j_2.
\]

That is, whenever it is optimal for the adversary to place \(\bar{f}_n \in (0,1)\) at price \(p_{j_2}\) satisfying \(p_{j_1} < p_{j_2} \bar{f}_n\), it is also optimal to place the whole unit mass at \(p_{j_2}\), considering \(p_{j_1} < p_{j_2}\) for all \(j_1, j_2\) as \(1 \leq j_1 < n \leq j_2 \leq K\).

If \(p_{j_1} \geq p_{j_2} \bar{f}_n\), for any vector \(t\) the competitive ratio is of the form

\[
c(n, \bar{f}_n, t) = c_{j_1, n} = \frac{\Lambda^1 p_n \bar{f}_n + \Lambda^2 (\sum_{j=1}^{j_1} t_j p_j + \sum_{j=j_1+1}^{n} t_j p_j \bar{f}_n)}{(\Lambda^1 + \Lambda^2)p_{j_1}}.
\]
for some \( j_1, 1 \leq j_1 \leq n - 1 \). And, for fixed \( n \) and \( \bar{f}_n \in (0, 1) \),

\[
c(n, \bar{f}_n, t) = \frac{\Lambda_1 p_n \bar{f}_n + \Lambda_2 (\sum_{j=1}^{j_1} t_j p_j + \sum_{j=j_1+1}^{n} t_j p_j \bar{f}_n)}{(\Lambda_1 + \Lambda_2)p_{j_1}} \geq \frac{\Lambda_2 \sum_{j=1}^{j_1} t_j p_j}{(\Lambda_1 + \Lambda_2)p_{j_1}} = c(n, 0, t) \quad \forall t, j_1.
\]  

That is, whenever it is optimal for the adversary to place \( \bar{f}_n \in (0, 1) \) at price \( p_{j_2} \) satisfying \( p_{j_1} \geq p_{j_2} \bar{f}_n \), it is also optimal to place the whole unit mass at \( p_{j_1} \). That is, (A.1) and (A.2) show that irrespective of the value of \( t \) and the indices \( j_1, j_2 \) in the inner problems, there exists a weakly dominant strategy for the adversary that allocates the whole probability mass to a single price. \( \square \)

**Proof of Proposition 2.4.5:** Unlike the competitive ratio case, the firm does not have to serve the whole market. Let us denote the lowest valuation the firm is indifferent between serving and not serving as \( v_d \). If \( y < v_d \), the firm obviously faces at least a regret of \( \Lambda \,(v_d - \epsilon) \) for an arbitrarily small \( \epsilon > 0 \) chosen by the adversary by not serving customer types below \( v_d \), as the adversary can choose the whole market to be at \( v_d - \epsilon \). For notational convenience let \( v_0 := v_d \) and \( v_{K+1} := \bar{v} \). For any given distribution \( F(\cdot) \), define again the probability masses on intervals \([v_j, v_{j+1})\) as \( f_j \) for \( j = 0, \ldots, K - 1 \), the mass on \([v_K, v_{K+1}]\) as \( f_K \), and the mass on \([v, v_d)\) as \( f_d \) whenever \( y < v_d \). Now suppose, \( y < v_d \) holds at optimal segmentation, so that the whole market is not served. Then, using the results in (2.27) and (2.28), the optimal regret satisfies \( r^* = \bar{r} - \epsilon \) for an arbitrarily small \( \epsilon > 0 \) chosen by the adversary where
\( \bar{r} \) is given by

\[
\bar{r} = \min_{v_d,v} \max_f \Lambda \left( \sum_{j=0}^{K} \min\{(v_{j+1} - v_j), \frac{v_{j+1}}{2}\} f_j + v_d f_d \right)
\]

subject to

\[
\sum_j f_j + f_d = 1, \quad f, f_d \geq 0
\]

\[
v_d \leq v_1 \leq \cdots \leq v_K \leq \bar{v}
\]

\[
v_d > v,
\]

The inner maximization is again an LP, and therefore, the optimal value is attained at one of the extreme points of the simplex \( \sum_j f_j = 1, \quad f \geq 0 \). Consequently, as before, the problem can be reduced to

\[
\bar{r} = \min_{v_d,v,r} \Lambda r
\]

subject to

\[
r \geq \Lambda \min\{(v_{j+1} - v_j), \frac{v_{j+1}}{2}\}, \quad j = 0, \ldots, K
\]

\[
r \geq v_d
\]

\[
v_d \leq v_1 \leq \cdots \leq v_K \leq \bar{v}
\]

\[
v_d > v
\]

Now, let \( z_0 := v_d, \quad z_j := \min\{(v_j - v_{j-1}), \frac{v_j}{2}\} \) for \( j = 1, \ldots, K + 1 \) and consider the equivalent problem:

\[
\bar{r} = \min_{v_d,v,z,r} \Lambda r
\]

subject to

\[
r \geq z_j, \quad j = 0, \ldots, K + 1
\]

\[
v_d \leq v_1 \leq \cdots \leq v_K \leq \bar{v}
\]

\[
v_d > v
\]

\[
z_0 = v_d
\]

\[
z_j = \min\{(v_j - v_{j-1}), \frac{v_j}{2}\}, \quad j = 1, \ldots, K + 1,
\]
and its relaxation

\[ \tilde{r} = \min_{z, v, r} \Lambda r \]

s.t. \( r \geq z_j \quad j = 0, \ldots, K + 1 \)

\[ v_d \leq v_1 \leq \cdots \leq v_K \leq \bar{v} \]

\[ z_0 = v_d \]

\[ z_j \leq v_j - v_{j-1}, \quad j = 1, \ldots, K + 1. \]

By construction, \( \tilde{r} \leq \tilde{r} \) clearly. Also, using its dual, it is easy to show that the relaxed formulation has optimal value \( \tilde{r} = \Lambda \frac{\bar{v}}{K+2} \) with the symmetric solution \( v_d = v_j - v_{j-1} = \frac{\bar{v}}{K+2} = z_j \) for all \( j \). Hence, when \( v < \frac{\bar{v}}{K+2} \), our initial assumption that \( v < v_d \) is satisfied, and the optimal solution for the relaxed formulation is also feasible for the problem of \( \tilde{r} \), and consequently, is optimal also for this problem. The case with \( v \geq \frac{\bar{v}}{K+2} \) is analyzed in analogous steps to above, yielding the solution in the statement of the result. \( \square \)

**Proof of Lemma 2.4.1:** Fix some \( v_s \in [\bar{v}, \bar{v}] \). The worst-case distribution selected implies a revenue maximizing reservation price \( v_a \) under full information. We can alternatively look at the problem of the adversary as first selecting a revenue maximizing reservation price \( v_a \) and then selecting the worst-case distribution corresponding to this reservation price. Furthermore, the nature of the analysis is different for \( v_a < v_s \) or \( v_a \geq v_s \). Therefore, defining

\[
c_1(v_s) := \inf_{v_a \in [\bar{v}, v_s]} \min_{F \in F_i(v_a, 1)} \frac{R(v_a, F)}{R(v_a, 1, F)} \quad \text{and} \quad c_2(v_s) := \min_{v_a, 2 \in [v_a, \bar{v}]} \min_{F \in F_2(v_a, 2)} \frac{R(v_a, F)}{R(v_a, 2, F)},
\]

where \( F_i(v_a) = \{ F \mid v_i = \arg\max_{v} R(v, F) \} \) for \( i = 1, 2 \), we have

\[
c(v_s) = \min_{v_a \in [\bar{v}, v_s]} \min_{F \in F(v_a)} \frac{R(v_a, F)}{R(v_a, F)} = \min \{ c_1(v_s), c_2(v_s) \}. \quad (A.3)\]
We first show that \( c_1(v_s) = \frac{v_0}{v_s} \). For this, first fix some \( v_{a,1} < v_s \) and denote this value of \( c_1(v_s) \) as

\[
c_1(v_s, v_{a,1}) := \min_{F \in F_1(v_{a,1})} \frac{R(v_s, F)}{R(v_{a,1}, F)}.
\]

Also, let \( v_1 \) and \( v_2 \) denote the highest and the second highest valuations among \( N \) bidders, and define the probabilities of the following events under which the seller collects different revenues:

\[
p_1 = P(v_1 \geq v_2 \geq v_s), \quad p_2 = P(v_1 \geq v_s > v_2 \geq v_{a,1}), \quad p_3 = P(v_s > v_1 \geq v_2 \geq v_{a,1})
\]

\[
p_4 = P(v_s > v_1 \geq v_{a,1} > v_2), \quad \text{and} \quad p_5 = P(v_{a,1} > v_1, v_2).
\]

Then, conditioning the expected revenue of the auction with respect to these events, the expected revenue of the firm is given by

\[
R(v_s, F) = E(v_2 | v_2 \geq v_s) p_1 + v_s p_2 + v_0 p_3 + v_0 p_4 + v_0 p_5,
\]

the maximum expected revenue under full information is given by

\[
R(v_a, F) = E(v_2 | v_2 \geq v_s) p_1 + E(v_2 | v_1 \geq v_s > v_2 \geq v_{a,1}) p_2
\]

\[
+ E(v_2 | v_s > v_1 \geq v_2 \geq v_{a,1}) p_3 + v_{a,1} p_4 + v_0 p_5,
\]

and consequently, we have

\[
c_1(v_s, v_{a,1}) = \frac{R(v_s, F)}{R(v_{a,1}, F)}
\]

\[
\geq \frac{v_0}{v_s} \quad \forall F \in F_1(v_{a,1}) \text{ and } \forall v_{a,1} < v_s,
\]

where the first inequality follows from the fact that for fixed \( F \in F_1(v_{a,1}) \), the second to fifth terms are all larger than or equal to \( v_0 \) in the numerator \( R(v_s, F) \), and less...
than or equal to $v_s$ in the denominator $R(v_a, F)$ under the respective events. (The inequality is strict when $v_s > v_0$.) For $v_s > v_0$, a distribution that puts all the probability mass at a point arbitrarily close to $v_s$, i.e. at $(v_s - \epsilon)$ for arbitrarily small $\epsilon > 0$, comes arbitrarily close to this lower bound yielding $\frac{v_0}{v_s - \epsilon}$. (Note that for $v_s = v_0$, the distribution which puts all the probability mass at $v_0$ achieves this lower bound exactly, which yields a ratio of $\frac{v_0}{v_s} = 1$. However, under this case the adversary can guarantee a lower ratio by choosing a distribution that yields a revenue maximizing reservation price $v_a$ satisfying $v_a \geq v_s$, as further analyzed below.)

We now identify $c_2(v_s)$ for $v_{a,2} \geq v_s$. Again fix some $v_{a,2}$ and denote this value of $c_2(v_s)$ as $c_2(v_s, v_{a,2})$. Letting $p = P(v \geq v_{a,2})$, define the probabilities of seeing at least two bids above $v_{a,2}$, exactly one bid above $v_{a,2}$, and no bids above $v_{a,2}$ as

\[ p_a = P(v_2 \geq v_{a,2}) = 1 - [Np(1-p)^{N-1} + (1-p)^N] \]
\[ p_b = P(v_1 \geq v_{a,2} > v_2) = Np(1-p)^{N-1}, \quad \text{and} \quad p_c = P(v_1 \leq v_{a,2}) = (1-p)^N. \]

Under the event $v_2 \geq v_{a,2}$, the firm would get $v_2$ using the reservation price of $v_s$ or $v_{a,2}$. Similarly, under the event $v_1 \geq v_{a,2} > v_2$, the firm would earn $\max\{v_2, v_s\}$ using the reservation price of $v_s$, and would earn $v_{a,2}$ using the reservation price of $v_{a,2}$. And finally, under the event $v_1 \leq v_{a,2}$, the firm would get either $\max\{v_2, v_s\}$ (if $v_1 \geq v_s$) or $v_0$ (if $v_1 < v_s$) using the reservation price of $v_0$ (so the firm earns at least $v_0$ under this event) and $v_0$ using $v_{a,2}$. Therefore, we have

\[ c_2(v_s, v_{a,2}) \geq \min_{F \in F_2(v_{a,2})} \frac{E(v_2|v_2 \geq v_{a,2}) p_a + E(\max\{v_2, v_s\}|v_1 \geq v_{a,2} > v_2) p_b + v_0 p_c}{E(v_2|v_2 \geq v_{a,2}) p_a + v_{a,2} p_b + v_0 p_c} \]
\[ := c_2(v_s, v_{a,2}). \]

Now, by contradiction assume that the set of optimal distributions for the adversary that $c_2(v_s, v_{a,2})$ admits does not contain a Bernoulli type valuation distribution. For
any of the optimal distributions, define \( \hat{\rho} = P(v \geq v_{a,2}) \), and also let the corresponding values of the the probabilities \( p_a, p_b, p_c \) be denoted by \( \hat{p}_a, \hat{p}_b, \hat{p}_c \). Observing that \( E(v_2|v_2 \geq v_{a,2}) \geq v_{a,2} \) and \( E(\max\{v_2, v_s\}|v_1 \geq v_{a,2} > v_2) \geq v_s \) holds for any distribution, we have

\[
\zeta_2(v_a, v_{a,2}) = \frac{E(v_2|v_2 \geq v_{a,2}) \hat{p}_a + E(\max\{v_2, v_s\}|v_1 \geq v_{a,2} > v_2) \hat{p}_b + v_0 \hat{p}_c}{E(v_2|v_2 \geq v_{a,2}) \hat{p}_a + v_{a,2} \hat{p}_b + v_0 \hat{p}_c}.
\]

Observe, however, that lower-bound at the right hand side is achieved by the Bernoulli type valuation distribution with \( P(v = v_{a,2}) = \hat{\rho}, \ P(v = v) = 1 - \hat{\rho} \), which leads to a contradiction. Thus, \( \zeta_2(v_s, v_{a,2}) \) achieves its optima at the above Bernoulli type distribution, under which \( c_2(v_s, v_{a,2}) = \zeta_2(v_s, v_{a,2}) \). Therefore, for fixed \( v_{a,2} \geq v_s \) the optimal response of the adversary is to choose Bernoulli type distribution with \( P(v = v_{a,2}) = p, \ P(v = \tilde{v}) = 1 - p \) for some \( p \in [0, 1] \).

Consequently, the problem of identifying \( c_2(v_s) \) reduces to finding the best \( v_{a,2} \) and \( p \), which reduces to

\[
c_2(v_s) = \min_{v_{a,2} \in [0, v_s]} \min_{p \in [0, 1]} \frac{v_{a,2} p_a + v_s p_b + v_0 p_c}{v_{a,2} p_a + v_{a,2} p_b + v_0 p_c}.
\]

For any fixed \( v_{a,2} \), note that the revenue maximizing reserve price is \( v_{a,2} \) when bidders’ valuations have a distribution with \( P(v = v_{a,2}) = p, \ P(v = \tilde{v}) = 1 - p \) for \( \forall \ p \in [0, 1] \), as reflected by the denominator of the ratio above. To see this more clearly, observe that \( v_s \leq v_{a,2} \) needs to hold, and that, for any fixed \( p \in [0, 1] \), the expected revenue for \( v_s \leq v_{a,2} \), i.e. \( v_{a,2} p_a + v_s p_b + v_0 p_c \), is linear and increasing in \( v_s \) and is maximized when \( v_s = v_{a,2} \). That is, for every \( v_{a,2} \), the distribution with \( P(v = v_{a,2}) = p, \ P(v = \tilde{v}) = 1 - p \) is in \( F_2(v_{a,2}) \) for \( \forall \ p \in [0, 1] \), i.e. the selection of \( v_{a,2} \) does not constrain the feasible region of \( p \) in the inner problem above. Therefore,
we can interchange the order of minimization operators above to get

\[ c_2(\upsilon_a) = \min_{p \in [0,1]} \min_{\upsilon_{a,2} \in [\upsilon_a, \bar{\upsilon}]} \frac{\upsilon_{a,2} p_a + \upsilon_a p_b + \upsilon_0 p_c}{\upsilon_{a,2} p_a + \upsilon_a p_b + \upsilon_0 p_c}. \]

For fixed \( p \in [0,1] \), the probabilities \( p_a, p_b, p_c \) are fixed, and therefore, the ratio above is decreasing in \( \upsilon_{a,2} \), letting us to conclude optimal reservation price choice of the adversary satisfies \( \upsilon_{a,2} = \bar{\upsilon} \) independent of chosen \( p \). Therefore, the adversary’s problem reduces to

\[ c_2(\upsilon_a) = \min_{p \in [0,1]} \frac{\bar{\upsilon} p_a + \upsilon_a p_b + \upsilon_0 p_c}{\bar{\upsilon} p_a + \bar{\upsilon} p_b + \upsilon_0 p_c} = \min_{p \in [0,1]} \frac{\bar{\upsilon} \left[ 1 - (Np(1 - p)^{N-1} + (1 - p)^N) \right] + \upsilon_a Np(1 - p)^{N-1} + \upsilon_0 (1 - p)^N}{\bar{\upsilon} (1 - (1 - p)^N) + \upsilon_0 (1 - p)^N}. \]

The ratio within the minimization argument above, call \( c_2(p) \), is continuous and differentiable in \( p \) on the interval \([0,1] \), and attains the value of \( c_2(1) = 1 \) for \( p = 1 \) and \( c_2(0) = 1 \) for \( p = 0 \). Thus, the minimum value of \( c_2(p) \) attained for some \( p \in (0,1) \).

We finally show that \( c_2(p) \) is decreasing on the interval \((0, p^*)\) and increasing on the interval \([p^*, 1)\) for \( p^* = 1 - q \), and therefore minimized at \( p^* \), where \( q \) is the value identified in the statement of the lemma. Differentiating the ratio with respect to \( p \), we get

\[ c_2'(p) = \frac{N(1 - p)^{N-2} (\bar{\upsilon} - \upsilon_a) \left[ (\bar{\upsilon} - \upsilon_0)(1 - p)^N - N\bar{\upsilon}(1 - p) + (N - 1)\bar{\upsilon} \right]}{\left[ \bar{\upsilon} (1 - (1 - p)^N) + \upsilon_0 (1 - p)^N \right]^2}, \]

with \( c_2'(p) = 0 \), if and only if \( (\bar{\upsilon} - \upsilon_0)(1 - p)^N - N\bar{\upsilon}(1 - p) + (N - 1)\bar{\upsilon} := \hat{c}_2(p) = 0 \) on the open interval \((0,1) \). Differentiating the function \( \hat{c}_2(p) \), we get \( \hat{c}_2'(p) = N\bar{\upsilon} - N(1 - p)^{N-1}(\bar{\upsilon} - \upsilon_0) > 0 \), where the strict inequality holds for every \( p \in (0,1) \). As a result, the function \( \hat{c}_2(p) \) is strictly increasing in \( p \in (0,1) \), with values of \( \hat{c}_2(0) = -\upsilon_0 < 0 \) and \( \hat{c}(1) = (N - 1)\bar{\upsilon} > 0 \). Thus, this function crosses 0 only once at the point \( p^* \) satisfying \( \hat{c}_2(p^*) = 0 \), which implies there is only a single local optimum for the
competitive ratio above. \( c'_2(p) \) is negative for \((0, p^*]\), and positive on the interval \([p^*, 1)\), implying the ratio is first decreasing and then increasing, and minimized at the point \( p^* \) satisfying \( \dot{c}_2(p^*) = 0 \).

Obviously, \( p^* \) solving the polynomial \((\bar{v} - v_0)(1 - p)^N - N\bar{v}(1 - p) + (N - 1)\bar{v} = 0\) is independent of \( v_s \) and thus, so are \( p^*_a, p^*_b, p^*_c \). Therefore, the optimal ratio for \( v_{a,2} \geq v_s \) is of the form

\[
c_2(v_s) = \frac{\bar{v} p^*_a + v_s p^*_b + v_0 p^*_c}{\bar{v} p^*_a + \bar{v} p^*_b + v_0 p^*_c},
\]

and consequently, we have

\[
c(v_s) = \min \{ c_1(v_s), c_2(v_s) \} = \min \left\{ \frac{v_0}{v_s - \epsilon}, \frac{\bar{v} p^*_a + v_s p^*_b + v_0 p^*_c}{\bar{v} p^*_a + \bar{v} p^*_b + v_0 p^*_c} \right\}. \quad \square
\]
Appendix B

Supplement to Chapter 3: Proofs of Remaining Results

Proof of Lemma 3.1.3: The part of the result for when $S(p)$ is empty is obvious using the arguments of the previous section as explained above: when $S(p)$ is empty, we have $[z_j, k_j] = [x_j, x_{j+1}]$ for all $j$, and the firm collects $p_j$ from everybody within the interval $[x_j, x_{j+1}]$ for all $j$. Also, the maximum revenue that can be achieved is always less than $\Lambda v$. Thus the adversary restricts her decision to how much probability mass $f_j$ to put on each interval $[x_j, x_{j+1}]$, and allocates each mass $f_j$ on the point $l_j$ for all $j$, achieving the maximum regret of $\Lambda (v - \min_j p_j)$.

For the other part, fix some price $p$ for which $S(p)$ is not empty (i.e. $p$ is a non-spanning price policy), and define the regret as a function of distribution $F$ as $r(p, F) := R(p^*(F), F) - R(p, F)$. For a given $F$, define $o_j = F(z_j) - F(k_j)$ and $r_j = F(k_{j+1}) - F(z_j)$, i.e. $o_j$ is the fraction of customers that are served by price $p_j$, and $r_j$ is the fraction of customers that are not served either by price $p_j$ or $p_{j+1}$, so that $\sum o_j + \sum r_j = 1$. 


Now, we can see the adversary's problem of finding the best distribution to maximize the regret function as a two-step optimization of first choosing $o$ and $r$ vectors, then choosing a distribution function satisfying the requirements of these vectors. That is,

$$\max_F r(p, F) = \max_o \max_r \{ \max_F \{ r(p, F) : o_j = F(z_j) - F(k_j), \ r_j = F(k_{j+1}) - F(z_j), \ \forall j \} \}$$

If we concentrate on the inner maximization for given vectors of $o$ and $r$, we see that $R(p, F) = \Lambda \sum o_j p_j$. As for the function $R(p^*(F), F)$, the revenue generated by the optimal price vector $p^*(F)$ at each region $[k_j, z_j]$ is obviously bounded by $\Lambda o_j v$.

In addition, for the vector $p^*(F)$ to be able to sell to customers with preferences in the region $[z_j, k_{j+1}]$ either $p_j^*(F) < p_j$ or $p_{j+1}^*(F) < p_{j+1}$ must hold. Therefore, the revenue generated by the optimal price vector $p^*(F)$ at each region $[z_j, k_{j+1}]$ is bounded by $\Lambda r_j \max(p_j, p_{j+1})$. Also observe that whenever $r_j > 0$, the prices $p_j$ and $p_{j+1}$ has to satisfy $p_j + p_{j+1} + \theta|l_{j+1} - l_j| > 2v$, which implies $x_j > l_j + y_j$, and $x_j < l_{j+1} - y_{j+1}$, thus both $j, j + 1 \in S(p)$. Combining these observations, we see that the regret function for any given $F$ satisfies:

$$r(p, F) \leq \sum \Lambda o_j v + \sum \Lambda r_j \max(p_j, p_{j+1}) - \sum \Lambda o_j p_j$$

$$\leq \sum \Lambda o_j v + \sum \Lambda r_j \max_{j \in S(p)} \{ p_j \} - \sum \Lambda o_j \min_j \{ p_j \}$$

$$= \Lambda \sum o_j (v - \min_j \{ p_j \}) + \Lambda \sum_{j \in S(p)} r_j \max_{j \in S(p)} \{ p_j \}$$

$$\leq \Lambda \max_j \{(v - \min_j \{ p_j \}) , \ \max_{j \in S(p)} \{ p_j \} \} .$$

Therefore, independent of $F$, the regret function $r(p, F)$ is upper-bounded by $\Lambda \max\{(v - \min_j \{ p_j \}) , \ \max_{j \in S(p)} \{ p_j \} \}$. However, as explained above, the adversary can achieve this bound by putting the whole unit mass to the location of the product with the minimum price achieving a maximum regret of $\Lambda (v - \min_j p_j)$ when $S(p)$ is empty, or
she can select the maximum price whose index-$i$ is in the set $S(p)$, and put the whole mass arbitrarily close to point $l_j + y_j$ (if $x_j > l_j + y_j$) or $l_j - y_j$ (if $x_{j-1} < l_j - y_j$), which ever gives a greater regret value, achieving (or coming arbitrarily close to) a maximum regret of $\Lambda \max\{(v - \min_j p_j), \max_{j \in S(p)} p_j\}$. \hfill\square

**Proof of Proposition 3.2.1:** We prove the result in several steps. As the first step, let us show that for every optimal price vector $p$ where a version is strictly dominated by others, there exists another optimal price vector $\hat{p}$ where no version strictly dominates (i.e., yields strictly more utility for all quality types) any other version. It is sufficient to show the result for any neighboring product pairs. Assume at the optimal solution, some version $j$ is strictly dominated by version $j - 1$ and version $j + 1$. The necessary and sufficient condition for this is easily seen to be $v_{j-1} > v_j$ using the definition of $v_j$. Consider such an optimal solution with $v_{j-1} > v_j$. Decreasing only the price for version $j$, $p_j$, to a level where $\hat{v}_{j-1} = \hat{v}_j$, while keeping the other prices fixed, the revenues earned by the firm does not change regardless of the quality taste distribution selected by the adversary. Decreasing any dominated price in such a fashion yields another optimal price vector, $\hat{p}$ where $\hat{v}_1 < \hat{v}_2 < \cdots < \hat{v}_{K-1}$ holds, i.e., an optimal price vector where no version strictly dominates any other.

Next step is to show that the customer with lowest taste for quality (type) earns zero utility at any optimal price, i.e. $u(v) = 0$, and that there exists an optimal price vector where $u(v) = v l_1 - p_1 = 0$. Notice that $u(v) = \max_j \{v l_j - p_j\} \geq 0$ must hold at the optimal price $p$, as otherwise the adversary would guarantee a competitive ratio of zero by selecting the distribution with $P(v = v) = 1$. Assume $\delta := \max_j \{v l_j - p_j\} > 0$, which would imply that $u(v) > 0$ for $\forall v \in [v, \bar{v}]$, i.e., all consumers have strictly positive utility or surplus, and consider the price vector $\hat{p}$
with \( \hat{p}_j = p_j + \delta \) \( \forall j \). Notice that the resulting indifferent customer types satisfy \( \hat{v}_j = v_j \) \( \forall j \). Thus, \( \hat{p} \) strictly increases the revenues earned by the firm whatever the distribution selected by the adversary is. Consequently, \( \hat{p} \) would yield a strictly better competitive ratio leading to a contradiction of the optimality of \( p \).

As a result, we have \( u(v) = 0 \). Now, using the first step’s conclusion, we have an optimal solution where \( v l_1 - p_1 \leq 0 = \max_j \{ v l_j - p_j \} \), as otherwise \( p_1 \) would be strictly dominated. Assume \( v l_1 - p_1 < 0 \), decreasing only the first price to \( v l_1 \) does not change the revenues earned by the firm under any distribution selected by the adversary, yielding the same optimal competitive ratio, and thus, we always have an optimal price vector with \( v l_1 - p_1 = 0 \). Combining all these results so far, we conclude that there exists an optimal solution where \( v \leq v_1 \leq v_2 \leq \ldots \leq v_{K-1} \) holds. Observe that all customer types choose to buy under such an optimal solution.

The final step is to show that there exists an optimal solution with \( v_{K-1} \leq \bar{v} \). Consider an optimal solution with \( v_{K-1} > \bar{v} \), that is \( p_K > \bar{v} (l_K - l_{K-1}) + p_{K-1} \). The highest quality product is not sold under this optima, and the highest type customer buys version \( K - 1 \). Reducing only the price of the highest quality version to a level \( \hat{p}_K = \bar{v} (l_K - l_{K-1}) + p_{K-1} \), so that \( \hat{v}_{K-1} = \bar{v} \), the firm guarantees to earn at least

![Figure B.1: The policy on the right improves the competitive ratio.](image)

Figure B.1: The policy on the right improves the competitive ratio.
the same revenue regardless of the distribution selected by the adversary. Thus, we achieve another optimal solution satisfying \( v_{K-1} \leq \bar{v} \). Combining all of the arguments above yields the result of the proposition. □

**Proof of Lemma 3.2.1:** For the problem where only the \((K - j + 1)\) highest quality versions are offered, observe that the best competitive ratio satisfies \( C_K = \frac{v_{K-1}}{\bar{v}_{K-1}} = \frac{v}{\bar{v}} \). Let us now consider the problem of finding \( c_j \) for \( j = 1 \ldots K - 1 \). Defining \( f_j = P(v < v_j) \), and \( f_i = P(v \geq v_{i-1}) - P(v \geq v_i) \) for \( i = j + 1 \ldots K - 1 \), and \( f_K = P(v \geq v_{K-1}) \) the competitive ratio achieved by a pricing policy satisfying \( v = \ldots = v_j - 1 < v_j \leq v_{j+1} \leq \ldots \leq v_{K-1} \leq \bar{v} \) is given by:

\[
c_j = \max_p \min_F \frac{\sum_{i=j}^{K-1} f_i p_i}{\int_{v_{j}}^{v_{j+1}} p_F(v) \, dF(v) + \sum_{i=j}^{K-2} \int_{v_i}^{v_{i+1}} p_F(v) \, dF(v) + \int_{v_{K-1}}^{\bar{v}} p_F(v) \, dF(v)}
\]

(B.1)

where \( p_F(v) \) is the price paid by a customer of type \( v \) for her best choice under the revenue maximizing pricing scheme \( p_F(\cdot) \) for a given quality taste distribution \( F(\cdot) \).

Now fix some \( p \) offering only the \((K - j + 1)\) highest quality versions, then the ratio for this fixed price, denoted by \( c_j(p) \), is given by

\[
c_j(p) = \min_F \frac{\sum_{i=j}^{K-1} f_i p_i}{\int_{v_{j}}^{v_{j+1}} p_F(v) \, dF(v) + \sum_{i=j}^{K-2} \int_{v_i}^{v_{i+1}} p_F(v) \, dF(v) + \int_{v_{K-1}}^{\bar{v}} p_F(v) \, dF(v)} \geq \min_F \frac{\sum_{i=j}^{K-1} f_i p_i + f_K p_K}{\sum_{i=j}^{K-1} f_i l_K v_i + f_K l_K \bar{v}} \geq \min_{i=j\ldots K-1} \{p_i/l_K v_i, p_K/l_K \bar{v}\}
\]

where the first inequality follows from the fact that every fixed \( F(\cdot) \), \( p_F(v) \leq l_K v_j \) for \( v \in [v, v_j] \), \( p_F(v) \leq l_K v_{i+1} \) for \( v \in [v_i, v_{i+1}] \), \( i = j \ldots K - 2 \), and \( p_F(v) \leq l_K \bar{v} \) for \( v \in [v_{K-1}, \bar{v}] \). However, selecting a distribution satisfying \( \Pr(v = v_\ast - \epsilon) = 1 \), where \( v_\ast \) is the point where the lower bound above achieves its minima, for some arbitrarily
small $\epsilon \in \mathbb{R}_+$, the adversary can achieve a competitive ratio that is arbitrarily close to the lower bound above. \(\Box\)

**Proof of Proposition 3.2.3:** Using the previous lemma, the problem of finding $c_j$ can be written as:

\[
c_j = \max_p \min \left\{ \min_{i=j...K-1} \left\{ \frac{p_i}{l_i v_i}, \frac{p_K}{l_K \bar{v}} \right\} \right\} \quad (B.2)
\]

\[
s.t. \quad v_{n-1} = \frac{p_n - p_{n-1}}{l_n - l_{n-1}} \leq \frac{p_{n+1} - p_n}{l_{n+1} - l_n} = v_n \quad n = j + 1 \ldots K - 1 \quad (B.3)
\]

\[
v_{K-1} = \frac{p_K - p_{K-1}}{l_K - l_{K-1}} \leq \bar{v} \quad (B.4)
\]

\[
v_j = \frac{p_{j+1} - p_j}{l_{j+1} - l_j} > \underline{v} \quad (B.5)
\]

where the constraints (B.3), (B.4) and (B.5) impose the structure $\underline{v} < v_j \leq v_{j+1} \leq \ldots \leq v_{K-1} \leq \bar{v}$ for the pricing policy.

We will first relax constraint (B.5) above. Define $c'_j$ as the best ratio that can be achieved by relaxing the constraint (B.5), which yields an upper bound on $c_j$. Below we provide an equivalent linear program for the formulation of $c'_j$, then use weak duality to upper bound $c'_j$, which in turn yields another upper bound on $c_j$. Finally, using strong duality on this dual problem, we will show that $c_j$ achieves these upper bounds. We first relax constraint (B.5), and consider the following equivalent
formulation for \( c^*_j \):

\[
c^*_j = \max_{p, c} \quad c \quad \text{s.t.} \quad c \geq \frac{p_n}{l_K v_n} \quad n = j + 1 \ldots K - 1
\]

\[
c \geq \frac{p_K}{l_K \bar{v}}
\]

\[
v_{n-1} = \frac{p_n - p_{n-1}}{l_n - l_{n-1}} \leq \frac{p_{n+1} - p_n}{l_{n+1} - l_n} = v_n \quad n = j + 1 \ldots K - 1
\]

\[
v_{K-1} = \frac{p_K - p_{K-1}}{l_K - l_{K-1}} \leq \bar{v}
\]

\[p_j = v \cdot l_j\]

Renaming the following variables as \( h_j := l_j, \ y_j := p_j = v \cdot l_j \) and \( h_n := l_n - l_{n-1}, \ y_n = p_n - p_{n-1} \quad n = j + 1 \ldots K \), we can reexpress the problem as the following linear program

\[
c^*_j = \max_{c, y} \quad c \quad \text{s.t.} \quad - \sum_{i=j+1}^{K} y_i \leq v \cdot l_j - c \cdot l_K \cdot \bar{v}
\]

\[
c \cdot l_K \cdot y_{n+1} - h_{n+1} \sum_{i=j+1}^{n} y_i \leq v \cdot l_j \cdot h_{n+1} \quad n = j + 1 \ldots K - 1
\]

\[
c \cdot l_K \cdot y_{j+1} \leq v \cdot l_j \cdot h_{j+1}
\]

\[
v_{n-1} = \frac{y_n}{h_n} \leq \frac{y_{n+1}}{h_{n+1}} = v_n \quad n = j + 1 \ldots K - 1
\]

\[
v_{K-1} = \frac{y_K}{h_K} \leq \bar{v}
\]

\[y, c \geq 0\]

For any fixed \( c \), the inner maximization problem above is an LP and is feasible for
some $y$ iff $c \leq c^*_j$. For fixed $c$, consider the dual of this LP:

$$\min_{w,z,t,q} \quad (y_j - c l_K \bar{v}) w + y_j \sum_{i=j}^{K-1} h_{i+1} z_i + \bar{v} q$$

$$\text{s.t.} \quad -w + c l_K z_j - \sum_{i=j+1}^{K-1} h_{i+1} z_i + \frac{t_{j+1}}{h_{j+1}} \geq 0$$

$$-w + c l_K z_{n-1} - \sum_{i=n}^{K-1} h_{i+1} z_i + \frac{t_{n}}{h_{n}} - \frac{t_{n-1}}{h_{n}} \geq 0 \quad n = j + 2 \ldots K - 1$$

$$-w + c l_K z_{K-1} + \frac{q}{h_{K}} - \frac{t_{K-1}}{h_{K}} \geq 0$$

$$w, z, t, q \geq 0$$

Observe that for any fixed $c$, the primal problem is always bounded and the dual problem is always feasible. Therefore, for any given $c$, the primal problem is infeasible, i.e. $c > c^*_j$, if and only if the dual is unbounded.

Consider the dual feasible vectors of the form

$$z_{K-1} = \frac{w}{c l_K} \quad \text{and} \quad z_n = \frac{w \prod_{i=n+2}^{K} (h_i + c l_K)}{(c l_K)^{K-n}} \quad n = j \ldots K - 2$$

that are parameterized over variable $w$ yielding dual objective values

$$w \left[ y_j \frac{\prod_{i=j+1}^{K} (h_i + c l_K)}{(c l_K)^{K-j}} - c l_K \bar{v} \right]$$

Then, let $\gamma = y/\bar{v}$ and define $c^d_j$ such that

$$\gamma l_j \frac{\prod_{i=j+1}^{K} (h_i + c^d_j l_K)}{(c^d_j l_K)^{K+1-j}} = 1$$

The objective function in (B.13) is decreasing in $c$ and is equal to 0 whenever $c = c^d_j$. Thus, above dual solution makes the problem unbounded, whenever $c > c^d_j$ by selecting and infinitely large $w$. Consequently, $c^*_j$ cannot be larger than $c^d_j$. Thus, using the primal-dual feasibility arguments above, we have

$$c_j \leq c^*_j \leq c^d_j, \quad j = 1 \ldots K - 1$$
Now consider the following price vector for the problem in (B.6)

\[ p_n^d = y \cdot h_n, \quad n = 1 \ldots j, \quad \text{and} \]

\[ p_n^d = y \cdot l_j \cdot \frac{\prod_{i=j+1}^{n} (h_i + c_i^d l_K)}{(c_j^d l_K)^{n-j}} = p_{n-1}^d \cdot \frac{(h_n + c_j^d l_K)}{(c_j^d l_K)}, \quad n = j + 1 \ldots K - 1 \]

which yields the objective value of \( c_j^d \). Thus \( p^d \) is optimal for the problem of finding \( c_j^* \) by inequality (B.15) if we can show that it is feasible for formulation (B.6). Therefore, we proceed to show to feasibility of \( p^d \). \( p^d \) satisfies the constraints (B.7) as \( v_{n-1} = \frac{p_n^d - p_{n-1}^d}{h_n} \leq \frac{p_{n+1}^d - p_n^d}{c_j^d l_K} = v_n, \quad n = j + 1 \ldots K - 1, \) and also the constraints (B.8), as \( v_{K-1} = \frac{p_K^d - p_{K-1}^d}{h_K} = \frac{p_{K-1}^d}{c_j^d l_K} \leq \bar{v} \), where the inequality follows from the fact that \( \frac{p_{K-1}^d}{v_{K-1}} \leq \frac{K^d}{v_{K-1}} \leq c_j^d \). As a result, \( p^d \) is feasible and optimal for formulation (B.6) with the optimal objective function value of \( c_j^d \), i.e. \( c_j^* = c_j^d \). Observe that above arguments are valid for every \( j \), and \( c_j \leq c_j^d = c_j^* \quad j = 1 \ldots K - 1. \)

Finally, defining \( c_K^d := \frac{y}{\bar{v}} = c_K := c_K^*, \) and \( c^* = \max_j \{ c_j^* \}, \) \( c^d = \max_j \{ c_j^d \} \) we proceed to show \( c = c^* = c^d \), which will complete the proof.

So far, we have \( c \leq c^* = c^d \). Assume \( c_j \leq c < c^d = c^* = c_j^* \) for some \( j \), which is achieved only if the relaxed constraint is violated, i.e. \( v_j \leq y \) or equivalently \( p_j^* + y \cdot h_{j+1} \). Define \( \delta = p_j^* + y \cdot h_{j+1} - p_{j+1}^* \), and consider \( \hat{p} \) constructed as \( \hat{p}_i = p_i + \delta \) for \( i = j + 1 \ldots K \), and \( \hat{p}_i = p_i \) for \( i \leq j \). Then, \( \hat{v}_1 = \ldots = \hat{v}_j = y \) and \( \hat{v}_i = v_i \) for \( i = j + 1 \ldots K - 1 \), yielding \( p_i/v_i \leq \hat{p}_i/\hat{v}_i \) for \( i = j + 1 \ldots K - 1 \), and \( p_K/\bar{v} \leq \hat{p}_K/\bar{v} \). Consequently, the competitive ratio achieved by \( \hat{p} \) (call \( \hat{c}_j \)) is at least as large as \( c_j^d \). On the other hand, observe that the price vector \( \hat{p} \) is feasible for the primal problem where only the highest \( K - j \) quality products are offered as \( y = \hat{v}_j < \hat{v}_{j+1} \leq \bar{v}_n \quad n = j + 2 \ldots K - 1 \), resulting in \( c_j^* \leq \hat{c}_j \leq c_{j+1} \leq c \) which yields a contradiction.\( \square \)
Figure B.2: The price menu of the policy on the right improves the competitive ratio. The lowest customer type is indifferent between purchasing and not purchasing for multiple versions.

Proof of Proposition 3.2.4: For $\gamma_j = l_j^{K-j} / \prod_{i=j+1}^{K} (h_i + l_j)$, observe that $c_j^d = c_{j+1}^d = l_j/l_K$ by equation (B.14). Also by differentiating equation (B.14), we have

$$\frac{dc_j^d}{d\gamma} = \frac{c_j^d}{\gamma \left( K + 1 - j - c_j^d l_K \sum_{i=j+1}^{K} \frac{1}{h_i + c_j^d l_K} \right)}$$

Combining these,

$$\left. \frac{dc_j^d}{d\gamma} \right|_{\gamma = \gamma_j} - \left. \frac{dc_j^d}{d\gamma} \right|_{\gamma = \gamma_j} = \frac{l_j \left( 1 - l_j \right)}{\gamma_j l_K \left( K + 1 - j - l_j \sum_{i=j+1}^{K} \frac{1}{h_i + l_j} \right) \left( K - j - l_j \sum_{i=j+2}^{K} \frac{1}{h_i + l_j} \right)} > 0$$

which shows that for some $\epsilon_j^+, \epsilon_j^- \in \mathbb{R}_+$, $c_{j+1}^d > c_j^d \forall \gamma \in (\gamma_j, \gamma_j + \epsilon_j^+)$, and $c_{j+1}^d < \ldots$
$c^d_j \quad \forall \gamma \in (\gamma_j - c_j^d, \gamma_j)$. However, whenever $c^d_{j+1} > c^d_j$ for some $\gamma$, we have

$$
\frac{dc^d_{j+1}}{d\gamma} |_{\gamma} - \frac{dc^d_j}{d\gamma} |_{\gamma} =

\frac{c^d_{j+1}l_K \left( K + 1 - j - c^d_jl_K \sum_{i=j+1}^{K} \frac{1}{h_{i+c^d_jl_K}} \right) - c^d_jl_K \left( K - j - c^d_{j+1}l_K \sum_{i=j+2}^{K} \frac{1}{h_{i+c^d_{j+1}l_K}} \right)}{
tl_K \left( K + 1 - j - c^d_jl_K \sum_{i=j+1}^{K} \frac{1}{h_{i+c^d_jl_K}} \right) \left( K - j - c^d_{j+1}l_K \sum_{i=j+2}^{K} \frac{1}{h_{i+c^d_{j+1}l_K}} \right)} \geq

\frac{c^d_{j+1}l_K \left( \left( K + 1 - j - c^d_jl_K \sum_{i=j+1}^{K} \frac{1}{h_{i+c^d_jl_K}} \right) - \left( K - j - c^d_{j+1}l_K \sum_{i=j+2}^{K} \frac{1}{h_{i+c^d_{j+1}l_K}} \right) \right)}{
tl_K \left( K + 1 - j - c^d_jl_K \sum_{i=j+1}^{K} \frac{1}{h_{i+c^d_jl_K}} \right) \left( K - j - c^d_{j+1}l_K \sum_{i=j+2}^{K} \frac{1}{h_{i+c^d_{j+1}l_K}} \right)} \geq

\frac{c^d_{j+1}l_K \left( 1 - \frac{c^d_{j+1}l_K}{h_{j+1+c^d_{j+1}l_K}} \right)}{
tl_K \left( K + 1 - j - c^d_jl_K \sum_{i=j+1}^{K} \frac{1}{h_{i+c^d_jl_K}} \right) \left( K - j - c^d_{j+1}l_K \sum_{i=j+2}^{K} \frac{1}{h_{i+c^d_{j+1}l_K}} \right)} > 0
$$

As a result, we conclude that $c^d_{j+1} > c^d_j$ for $\forall \gamma > \gamma_j$ using the first theorem of calculus. Also similarly we can show that $c^d_{j+1} < c^d_j$ for $\forall \gamma < \gamma_j$. Then, the result follows as the above arguments are valid for all $j = 1 \ldots K - 1$. □

**Proof of Proposition 3.2.7:** The result will be established in several steps. First we will analyze the relaxed versions of the problems $\inf_{v_0 \in [y,y]} r(v_0)$ and $r(y)$, then we will show that these relaxed versions are sufficient to characterize the optima.

In this respect, let us first consider the relaxed version of the LP in (3.4) that ignores (3.12). To find the optimal solution to this relaxed version, we first ignore the constraints (3.8),(3.9) and (3.10); and find $r$ and $p$ that solves for constraints (3.5), (3.6), (3.7) and (3.11) as equalities. This yields the optimal value $r^{ij}_0 = \frac{v_0}{(l_j + l_K) \prod_{i=j+1}^{K} (h_i + l_K)}$, which is attained by the following prices

$$
p_j = v_0 l_j = r^{ij}_0 l_j / l_K \quad \text{and} \quad p_{n+1} = \frac{r^{ij}_0 h_{n+1} + p_n (l_K + h_{n+1})}{l_K} \quad n = j \ldots K - 1 \quad (B.16)
$$

Below we show this solution satisfies constraints (3.8), (3.9) and (3.10), justifying its optimality for the relaxed LP that ignores constraint (3.12). For inequality (3.8),
we use the definition of \( p \) in (B.16):

\[
v_{n-1} = \frac{p_n - p_{n-1}}{h_n} = \frac{r^{i\delta}_0 + p_{n-1}}{l_K} \leq \frac{r^{i\delta}_0 + p_n}{l_K} = \frac{p_{n+1} - p_n}{h_{n+1}} = v_n, \quad n = j + 1 \ldots K - 1
\]

For inequality (3.9), we use constraint (3.7) to get:

\[
v_{K-1} = \frac{p_K - p_{K-1}}{h_K} = \frac{r^{i\delta}_0 + p_{K-1}}{l_K} \leq \frac{r^{i\delta}_0 + p_K}{l_K} = \frac{\bar{v}}{l_K} = \bar{v}
\]

Finally, for inequality (3.10), we first use the definition of \( p \) in (B.16), then constraint (3.5) to get:

\[
v_j = \frac{p_{j+1} - p_j}{h_K} = \frac{r^{i\delta}_0 + p_j}{l_K} = \frac{r^{i\delta}_0 + v_j l_j}{l_K} = \frac{v_0 l_K + v_0 l_j}{l_K} \geq v_0
\]

Similarly, let us now consider finding \( r_j(v) \). We have \( r_j(v) = \min_{p \in \mathcal{P}_j(v)} r_j(v,p) \), that is \( r^j := r_j(v) = \min_{p \in \mathcal{P}_j(v)} \max \{ \max_{i=j}^{j-1} \{ l_K v_i - p_i \}, l_K \bar{v} - p_K \} \), which also can be reexpress with the following LP formulation

\[
r^j = \min_{r, p, r} \quad r \quad \text{s.t.} \quad r \geq v_n l_K - p_n \quad n = j \ldots K - 1 \quad \text{(B.18)}
\]

\[
r \geq \bar{v} l_K - p_K \quad \text{(B.19)}
\]

\[
v_{n-1} = \frac{p_n - p_{n-1}}{h_n} \leq \frac{p_{n+1} - p_n}{h_{n+1}} = v_n \quad n = j + 1 \ldots K - 1 \quad \text{(B.20)}
\]

\[
v_{K-1} = \frac{p_K - p_{K-1}}{h_K} \leq \bar{v} \quad \text{(B.21)}
\]

\[
v_j = \frac{p_{j+1} - p_j}{h_{j+1}} \geq \bar{v} \quad \text{(B.22)}
\]

\[
p_j = \bar{v} l_j \quad \text{(B.23)}
\]

Consider now the relaxed version of this LP which ignores constraint (B.22). To find the optimal solution to this relaxed version, we first ignore the constraints (B.20) and (B.21), and find \( r \) and \( p \) that solves for constraints (B.18), (B.19) and (B.23) as
equalities. This yields, \( r'^j = \frac{\bar{d} l^{K+1-j}}{\prod_{i=j+1}^{K}(l_i + h_i)} - y l_j \) with following optimal prices

\[
p_j = y l_j \text{ and } p_{n+1} = \frac{r'^j h_{n+1} + p_n(l_K + h_{n+1})}{l_K}, \quad n = j \ldots K - 1
\]

(B.24)

Below we show this solution satisfies constraints (B.20) and (B.21), justifying its optimality for the relaxed LP that ignores constraint (B.22). For inequality (B.20), we use the definition of \( p \) in (B.24):

\[
u_{n-1} = \frac{p_n - p_{n-1}}{l_K} = \frac{r'^j + p_{n-1}}{l_K} \leq \frac{r'^j + p_n}{l_K} = \frac{p_{n+1} - p_n}{h_{n+1}} = v_n, \quad n = j + 1 \ldots K - 1
\]

and for inequality (B.21), we use constraint (B.19)

\[
v_{K-1} = \frac{p_K - p_{K-1}}{h_K} = \frac{r'^j + p_{K-1}}{l_K} \leq \frac{r'^j + p_K}{l_K} = \bar{d} l_K = \bar{d}
\]

Finally, we establish that considering these relaxed versions are sufficient to characterize the optima, by showing that

\[
r^* = \min_j \left\{ \min \{ r_j(y), \inf_{v_0 \in [y, \bar{d}]} r_j(v_0) \} \right\} = \min_j \{ r'^j, r'^j_0 \}
\]

\[
= \min_j \{ r'^j, r'^j_0 \}.
\]

We have

\[
\min_j \{ r'^j, r'^j_0 \} \leq \min_j \{ r'^j, r'^j_0 \} = \min_j \{ r_j(y), \inf_{v_0 \in [y, \bar{d}]} r_j(v_0) \}.
\]

Suppose

\[
\min_j \{ r'^j, r'^j_0 \} = \min_j \{ r'^j, r'^j_0 \} \leq \min_j \{ r_j(y), r_j(v_0) \} \}
\]

for some \( j \), which implies \( r'^j < r_j(y) \) or \( r'^j_0 < \inf_{v_0 \in [y, \bar{d}]} r_j(v_0) \) or both.

If \( r'^j < r_j(y) \), we must have \( v_j < y \) at optimal price \( p \) for the relaxed problem of finding \( r'^j \), or equivalently \( p_{j+1} < p_j + y h_{j+1} \). Also, optimality conditions impose that
\[ r^{ij} = v_i l_K - p_i \text{ for } i = j \ldots K - 1. \] Define \( \delta = p_j + y h_{j+1} - p_{j+1} \), and consider some \( \hat{p} \) constructed as \( \hat{p}_i = p_i + \delta \) for \( i = j + 1 \ldots K \), and \( \hat{p}_i = p_i \) for \( i \leq j \). Then, \( \hat{v}_i = \ldots = \hat{v}_j = y \) and \( \hat{v}_i = v_i \) for \( i = j + 1 \ldots K - 1 \), yielding \( \hat{v}_i l_K - \hat{p}_i < v_i l_K - p_i \) for \( i = j + 1 \ldots K - 1 \). Consequently, \( r_{j+1}(y) \leq \max \{ \max_{i=j+1 \ldots K-1} \{ \hat{v}_i l_K - \hat{p}_i \} , \, \bar{v}_K l_K - \bar{p}_K \} < r^{ij} \) as \( \hat{p} \) is feasible for the problem of finding \( r_{j+1}(y) \).

On the other hand, if \( r_0^{ij} < \inf_{v_0 \in [y, v]} r_j(v_0) \), we must have \( v_0 \leq y \) at optimal price \( p \) for the relaxed problem of finding \( r^{ij} \); which implies \( u(y) = y l_{j+1} - p_{j+1} \geq 0 \). Also, optimality conditions impose that \( r_0^{ij} = v_i l_K - p_i \) for \( i = j \ldots K - 1 \). Define \( \delta = u(y) \), and consider, some \( \hat{p} \) constructed as \( \hat{p}_i = p_i + \delta \) for \( i = j + 1 \ldots K \), and \( \hat{p}_i = p_i \) for \( i \leq j \). Then, \( \hat{v}_i = \ldots = \hat{v}_j = y \) and \( \hat{v}_i = v_i \) for \( i = j + 1 \ldots K - 1 \), yielding \( \hat{v}_i l_K - \hat{p}_i \leq v_i l_K - p_i \) for \( i = j + 1 \ldots K - 1 \). Consequently, \( r_{j+1}(y) \leq \max \{ \max_{i=j+1 \ldots K-1} \{ \hat{v}_i l_K - \hat{p}_i \} , \, \bar{v}_K l_K - \bar{p}_K \} \leq r_0^{ij} \) as \( \hat{p} \) is feasible for the problem of finding \( r_{j+1}(y) \). Combining these two cases we have \( \min \{ r^{ij}, r_0^{ij} \} \geq r_{j+1}(y) \geq r^* \) which yields a contradiction. \( \square \)

**Proof of Proposition 3.2.9:** For \( \gamma < \gamma_1 \), we have

\[
\gamma = y - \bar{v} < \frac{l_K^i}{\prod_{i=1}^{K}(h_i + l_K)} \Rightarrow y < \bar{v} \frac{l_K^i}{\prod_{i=1}^{K}(h_i + l_K)} \Rightarrow y l_1 < \bar{v} l_1 \frac{l_K^i}{\prod_{i=1}^{K}(h_i + l_K)} \Rightarrow 0 < \bar{v} l_1 \frac{l_K^i}{h_1 + l_K \prod_{i=2}^{K}(h_i + l_K)} - y l_1 \Rightarrow \bar{v} l_1 \frac{l_K^i}{l_1 + l_K \prod_{i=2}^{K}(h_i + l_K)} < \bar{v} l_1 \frac{l_K^i}{\prod_{i=2}^{K}(h_i + l_K)} - y l_1 \Rightarrow r_0^{ij} < r_1^{ij}.
\]

Similarly, using the reverse arguments, we can show that for \( \gamma \geq \gamma_1 \), we have \( r_0^{ij} \geq r_1^{ij} \).

Now, let us show that for \( \gamma < \gamma_{j+1} \), we have \( r^{ij} < r^{j+1} \); and \( \gamma \geq \gamma_{j+1} \), we have
\( r^{j} \geq r^{j+1} \). For \( \gamma < \gamma_{j+1} \),

\[
\gamma = \frac{y}{\bar{v}} < \frac{l_{K-j}^{K}}{\prod_{i=j+1}^{K}(h_{i} + l_{K})} \\
\Rightarrow \ y \ h_{j+1} < \bar{v} \ h_{j+1} \frac{l_{K-j}^{K}}{\prod_{i=j+1}^{K}(h_{i} + l_{K})} \\
\Rightarrow \ y \ h_{j+1} < \bar{v} \ h_{j+1} \frac{l_{K-j}^{K}}{h_{j+1} \prod_{i=j+2}^{K}(h_{i} + l_{K})} \\
\Rightarrow \ y \ (l_{j+1} - l_{j}) < \bar{v} \left( 1 - \frac{l_{K}}{h_{j+1} + l_{K}} \right) \frac{l_{K-j}^{K}}{\prod_{i=j+2}^{K}(h_{i} + l_{K})} \\
\Rightarrow \ \bar{v} \frac{l_{K-j+1}^{K}}{\prod_{i=j+1}^{K}(h_{i} + l_{K})} - y \ l_{j} < \bar{v} \frac{l_{K-j}^{K}}{\prod_{i=j+2}^{K}(h_{i} + l_{K})} - y \ l_{j+1} \\
\Rightarrow \ r^{j} < r^{j+1}. 
\]

Similarly, using the reverse arguments, we can show that for \( \gamma \geq \gamma_{j+1} \), we have \( r^{j} \geq r^{j+1} \). Then, the result follows as the above arguments are valid for all \( j = 1 \ldots K \). \( \Box \)
Appendix C

Supplement to Chapter 4: Proofs of Remaining Results

C.1. Theorem 2.1 of Kushner and Yin (2003)

Kushner and Yin consider an adaptive process \( \{\theta^k\} \) on some connected compact set \( H \in \mathbb{R}^n \). The adaptive algorithm they analyze is of the form \( \theta^{k+1} = \theta^k + e^k Y^k + e^k Z^k \) where \( e^k \) is the step size, \( e^k Z^k \) are correction terms that take the process back to the nearest point in the set \( H \) when \( \theta^k + e^k Y^k \) is out of this set, and \( Y^k \) are random variables satisfying the below conditions.

The compact set \( H \) and the corresponding correction terms are allowed to take one of several specific forms in their Theorem 2.1; and consequently, the original statement of the theorem is long, accounting for several possible cases. The \( \{\theta^k\} \) process in our problem satisfies a much simpler structure in that it is bounded with probability 1 on a hyperrectangle. Hence, no correction terms \( e^k Z^k \) are necessary, and the simplest specification for set \( H \), which is stated below, is sufficient. Therefore,
we adopt and state the theorem below in a simpler form as it applies to our setting, with only the conditions that are required for our specific structure.

• (A.4.3.1) \( H \) is a hyperrectangle, i.e., \( \exists a_i < b_i \ i = 1, \ldots, n \) such that \( H = \{ \theta : a_i \leq \theta \leq b_i, \ \forall i \} \).

• (A.5.2.1) \( \sup_k \mathbb{E}[|Y^k|^2] < \infty \)

• (A.5.2.2) There is a measurable function \( g(\cdot) \) of \( \theta \) and random variables \( \beta^k \) such that \( \mathbb{E}[Y^k | \theta^0, Y^i, i < k] = g(\theta^k) + \beta^k \).

• (A.5.2.3) \( g(\cdot) \) is continuous.

• (A.5.2.4) \( \sum_i (\epsilon^i)^2 < \infty \)

• (A.5.2.5) \( \sum_i (\epsilon^i) |\beta^i| < \infty \) w.p.1.

**Theorem 2.1 of Kushner and Yin (2003), (Ch. 5, pg. 127)** Let \( L_H \) denote the set of limit points of the mean limit ODE \( \dot{\theta}(t) = g(\theta(t)) \) in the set \( H \) over all initial conditions. Also let \( L_H^1 \subset L_H \) be the set of all stationary points of the ODE, and \( A_H \) be a set that is locally asymptotically stable in the sense of Lyapunov. Assume that

• (A.5.2.6) For any initial condition not in \( L_H^1 \) the trajectories of \( \dot{\theta}(t) = g(\theta(t)) \) goes to \( A_H \).

Then, the limit points of the process \( \{\theta^k\} \) are in \( L_H^1 \cup A_H \) w.p.1.
C.2. Proofs of remaining results

Proof of Lemma 4.2.1: We first show $D(\theta)$ is upper semi-continuous. Denote the universal space of all possible parameters as $\Theta^U$. Consider a generic open set $V$ that has the form $V = \{p \mid p_j > \kappa_j - \epsilon_j, \forall j, \sum_{i \geq j} p_i > \sum_{i \geq j} \kappa_i + \zeta_i - \delta_j \text{ if } \zeta_j > 0, \sum_j p_j = 1\}$, so that $D(\theta) \in V$ for any $\epsilon_j, \delta_j \geq 0$. Now, for any $\epsilon_j, \delta_j \geq 0$ and $V$, define the open set $U = \{p \mid p_j > \kappa_j - \frac{\epsilon_j}{K_1}, \forall j, \sum_{i \geq j} p_i > \sum_{i \geq j} \kappa_i + \zeta_i - \frac{\delta_j}{K_2} \text{ if } \zeta_j > 0, \sum_j p_j = 1\}$, where $K_1, K_2 > 1$ are sufficiently large numbers. Then, if $\theta' = [\kappa', \zeta'] \in U \cap \Theta^U$, we have that $\kappa' > \kappa_j - \epsilon_j$ and $\kappa' + \zeta' > \kappa_i + \zeta_i - \delta_j$, which yields $D(\theta') \in V$. Therefore, $D(\theta)$ is upper semi-continuous at $\forall \theta \in \Theta^U$.

Next we show that $D(\theta)$ is also lower semi-continuous. Fix some $\theta = [\kappa, \zeta] \in \Theta^U$, and let $V$ be an open set satisfying $V \cap D(\theta) \neq \emptyset$, and let $p \in V \cap D(\theta)$. As $V$ is open, there exists some $\delta > 0$, satisfying $p = [\delta, 0, \ldots, 0, -\delta] + p \in V$ as well.

Define the “$\epsilon$-neighborhood” of $\theta$ as $N_\epsilon(\theta) = \{x \mid ||x - \theta|| < \epsilon\}$, where $|| \cdot ||$ is the $L^2$ norm. Now by contradiction suppose that there is no neighborhood of $\theta$ such that $V \cap D(\theta') \neq \emptyset$ for all $\theta'$ in the neighborhood. Let $\{\epsilon_n\} \to 0$ be a sequence of positive reals, and pick some $\theta^n \in N_{\epsilon_n}(\theta)$ such that $V \cap D(\theta_n) = \emptyset$. Note that we can find such $\theta^n$ by the contradictory assumption. Then, using definitions of $V$, $\bar{p}$ and $N_{\epsilon_n}(\theta)$, we have that $\bar{p}_j - \kappa_j^n \to p_j - \kappa_j > 0$ and $\sum_{i \geq j} \bar{p}_i - \sum_{i \geq j} \kappa_i^n + \zeta_i^n \to \sum_{i \geq j} \bar{p}_i - \sum_{i \geq j} \kappa_i + \zeta_i > 0$. Consequently, we have that $\bar{p} \in D(\theta^n)$ for some large $n$, which yields a contradiction as $\bar{p} \in V$ and $V \cap D(\theta^n) = \emptyset$. This completes the proof of the lemma.

Proof of Lemma 4.2.2: The optimal solution in problem (4.9) is $p_\theta \in D(\theta)$ for any given $\theta$. Note that the objective function $\sum_j p_j \ln p_j$ is strictly convex in $p$. As shown in Lemma 4.2.1, $D(\theta)$ is a continuous correspondence, which is also easily seen to be convex and compact valued. Then, the result follows from the “The Maximum
Theorem under Convexity" (see, e.g., Sundaram [72], Theorem 9.17.3), which states that under these conditions $p_{\theta}$ is a continuous function in $\theta$. □

Proof of Lemma 4.2.3: $f(\theta) = (E(W^{k+1}), E(Q^{k+1}))$ satisfies

$$E(W_{j}^{k+1}) = \begin{cases} \pi_j & \text{for } j < L_\theta \\ q_\theta \pi_{L_\theta} & \text{for } j = L_\theta, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

$$E(Q_{j}^{k+1}) = \begin{cases} q_\theta \sum_{i \geq L_\theta} \pi_i & \text{for } j = L_\theta \\ (1 - q_\theta) \sum_{i > L_\theta} \pi_i & \text{for } j = L_\theta + 1 \\ 0 & \text{otherwise}. \end{cases} \quad (C.1)$$

By equation (4.15), the randomization probability $q_\theta$ for the ME algorithm satisfies

$$q_\theta = \frac{\gamma - \sum_{j \leq L_\theta} p_{\theta,j}}{p_{\theta,L_\theta}}. \quad (C.2)$$

As $p_{\theta}$ is continuous in $\theta$, $q_\theta$ and $f(\theta)$ are continuous in $\theta$. □

Proof of Proposition 4.2.1: First, note that any stationary point $\theta$ of the ODE needs to satisfy $\theta(0) = \theta$ and $\dot{\theta}(t) = g(\theta(t)) = 0$, $\forall t$, and therefore, $g(\theta) = f(\theta) - \theta = 0$. Using this, we establish the result in several steps. In step 1, we show that any stationary point $\theta$ of the ODE needs to have $L_\theta \leq L^*$. Then, in step 2, we show that $L_\theta < L^*$ cannot hold by establishing two contradictory inequalities: inequality (C.5) in step 2(a) and inequality (C.7) in step 2(b). Consequently, we conclude that $L_\theta = L^*$ for such a solution point. Finally, in step 3, we establish such a solution must be unique, using the necessary conditions imposed by equations $L_\theta = L^*$ and $g(\theta) = 0$, and identify the solution by solving for it in closed form. The exact form of the solution depends on whether the size of support $S$ is larger than a critical value.
Hence, we characterize the solution by analyzing it in two separate cases in steps 3(a) and 3(b).

Suppose that some point \( \theta \) satisfies \( g(\theta) = f(\theta) - \theta = 0 \), then by definition of \( f(\theta) \) in equation (C.1) above, \( \theta = (\kappa, \zeta) \) satisfies

\[
\kappa_j = \begin{cases} 
\pi_j & \text{for } j < L_\theta \\
q_\theta \pi_{L_\theta} & \text{for } j = L_\theta \\
0 & \text{otherwise}
\end{cases} \quad \text{and} \quad \zeta_j = \begin{cases} 
(1 - q_\theta) \sum_{i \geq L_\theta} \pi_i & \text{for } j = L_\theta \\
q_\theta \sum_{i > L_\theta} \pi_i & \text{for } j = L_\theta + 1
\end{cases} \quad (C.3)
\]

**Step 1:** Above form in equation (C.3) implies that \( \sum_{i < L_\theta} \kappa_i = \sum_{i < L_\theta} \pi_i \). Also, \( \sum_{i < L_\theta} \kappa_i < \gamma \) needs to hold by definition of \( L_\theta \). However, note that the inequality \( \sum_{i < L_\theta} \kappa_i < \gamma \) cannot be satisfied by any \( L_\theta > L^* \), as the definition of the true protection level \( L^* \) requires \( \sum_{i \leq L^*} \pi_i \geq \gamma \). Therefore, we have that \( L_\theta \leq L^* \) for any such solution \( \theta \) to \( g(\theta) = 0 \).

**Step 2:** Now, assume by contradiction that there exists a vector \( \theta \) for which \( g(\theta) = f(\theta) - \theta = 0 \) that has \( L_\theta < L^* \).

**Step 2(a):** By definition of \( L^* \), and the assumption that \( L_\theta < L^* \), we have that

\[
\gamma - \sum_{i < L_\theta} \pi_i > \pi_{L_\theta} . \quad (C.4)
\]

Also, using assumptions 1 and 2, we have that

\[
S > \frac{1}{\epsilon} = \frac{1 - (L_\theta - 1)\epsilon}{\epsilon} + L_\theta - 1 > \frac{1 - \sum_{i < L_\theta} \pi_i}{\pi_{L_\theta}} + L_\theta - 1
\]

where the second inequality follows from Assumption 1; or equivalently, rearranging terms

\[
\pi_{L_\theta} > \frac{1 - \sum_{i < L_\theta} \pi_i}{S - L_\theta + 1} . \quad (C.5)
\]

**Step 2(b):** Now, we show that \( p_{\theta, L_\theta} = \kappa_{\theta, L_\theta} + \zeta_{\theta, L_\theta} \leq \frac{1 - \sum_{i < L_\theta} \pi_i}{S - L_\theta + 1} \) must hold for point \( \theta \) to satisfy \( g(\theta) = f(\theta) - \theta = 0 \) and \( L_\theta < L^* \). This is clearly true when
\( \kappa_{\theta, L_\theta} = \kappa_{L_\theta} < \frac{1 - \sum_{i<L_\theta} \pi_i}{S - L_\theta + 1} \), as the ME algorithm reallocates the mass \( 1 - \sum_{i<L_\theta} \pi_i \) uniformly among \((S - L_\theta + 1)\) points when \( \kappa_{L_\theta} < \frac{1 - \sum_{i<L_\theta} \pi_i}{S - L_\theta + 1} < p_{L_\theta} \) to get \( p_{\theta, L_\theta} = \frac{1 - \sum_{i<L_\theta} \pi_i}{S - L_\theta + 1} \). If, on the other hand, \( \kappa_{\theta, L_\theta} = \kappa_{L_\theta} > \frac{1 - \sum_{i<L_\theta} \pi_i}{S - L_\theta + 1} \), we have that \( \zeta_{\theta, L_\theta} = 0 \) and \( p_{\theta, L_\theta} = \kappa_{\theta, L_\theta} = \kappa_{L_\theta} \) as a result of the ME algorithm. Using the definition of randomization parameter \( q_\theta \) as seen in equation (C.2) above, and the equilibrium condition for the uncensored mass at \( \theta \) in equation (C.3), we can see that \( q_\theta \) has to satisfy the following equations

\[
\kappa_{L_\theta} = q_\theta \pi_{L_\theta} \quad \text{and} \quad q_\theta = \frac{\gamma - \sum_{i<L_\theta} p_{\theta, i}}{p_{\theta, L_\theta}} = \frac{\gamma - \sum_{i<L_\theta} \pi_i}{\kappa_{L_\theta}}
\]

which yields the solution

\[
q_\theta = \sqrt{\frac{\gamma - \sum_{i<L_\theta} \pi_i}{p_{\theta, L_\theta}}} > 1 \quad \text{(C.6)}
\]

where the inequality follows from equation (C.4), yielding a contradiction. Consequently, \( \kappa_{\theta, L_\theta} = \kappa_{L_\theta} > \frac{1 - \sum_{i<L_\theta} \pi_i}{S - L_\theta + 1} \) cannot hold for a point \( \theta \) satisfying \( g(\theta) = f(\theta) - \theta = 0 \) and \( L_\theta < L^* \), hence we have that

\[
p_{\theta, L_\theta} \leq \frac{1 - \sum_{i<L_\theta} \pi_i}{S - L_\theta + 1} \quad \text{(C.7)}
\]

Now, we are ready to derive our contradiction to show \( L_\theta < L^* \) cannot hold at an equilibrium point, and therefore, conclude step 2. For the equilibrium point \( \theta \) with \( L_\theta < L^* \), \( q_\theta \) needs to satisfy

\[
q_\theta = \frac{\gamma - \sum_{i<L_\theta} \pi_i}{p_{\theta, L_\theta}} > \frac{\gamma - \sum_{i<L_\theta} \pi_i}{\pi_{L_\theta}} > 1 \quad \text{(C.8)}
\]

which yields a contradiction. The first inequality follows from inequalities (C.5) and (C.7), as they imply together that \( \pi_{L_\theta} > p_{\theta, L_\theta} \). The second inequality follows from the equation (C.4). As a result, we have established that under assumptions 1 and 2, an equilibrium point \( \theta \) can only have \( L_\theta = L^* \).
Step 3: If there is a solution to \( g(\theta) = f(\theta) - \theta = 0 \) with \( L_\theta = L^* \), it satisfies

\[
\kappa_j = \begin{cases} 
\pi_j & \text{for } j < L^* \\
q_\theta \pi_{L^*} & \text{for } j = L^* \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\zeta_j = \begin{cases} 
q_\theta \sum_{i \geq L^*} \pi_i & \text{for } j = L^* \\
(1 - q_\theta) \sum_{i > L^*} \pi_i & \text{for } j = L^* + 1
\end{cases}
\] (C.9)

where the randomization probability \( q_\theta \) above should satisfy

\[
q_\theta = \frac{\gamma - \sum_{i < L^*} \pi_i}{p_{\theta,L^*}} .
\] (C.10)

Therefore, such an equilibrium point exists and is unique as long as there is a unique solution to equations (C.9) and (C.10). It is easy to establish existence and uniqueness of the solution to the above equations by characterizing it in closed-form. Since the exact form of the solution depends on the value of the support \( S \), we characterize the solution in two cases in steps 3(a) and 3(b) below.

Step 3(a), \( S \geq S_w := \frac{1 - \sum_{i < L^*} \pi_i}{\sqrt{\pi_{L^*} (\gamma - \sum_{i < L^*} \pi_i)}} + L^* - 1 \): If \( \kappa_{L^*} \geq \frac{1 - \sum_{i < L^*} \pi_i}{S - L^* + 1} \), the ME algorithm cannot allocate any censored probability mass at \( L^* \) as discussed above, i.e. \( \zeta_{0,L^*} = 0 \) and \( p_{\theta,L^*} = \kappa_{0,L^*} = \kappa_{L^*} \). In this case solving equations

\[
\kappa_{L^*} = q_\theta \pi_{L^*} \quad \text{and} \quad q_\theta = \gamma - \sum_{i < L^*} \pi_i \cdot \frac{1 - \sum_{i < L^*} \pi_i}{p_{\theta,L^*}} - \gamma - \sum_{i < L^*} \pi_i
\]

to satisfy equations (C.9) and (C.10) yields the unique solution

\[
\kappa_{L^*} = \sqrt{\pi_{L^*} (\gamma - \sum_{i < L^*} \pi_i)} \leq \pi_{L^*} \quad \text{and} \quad q_\theta = \sqrt{\frac{\gamma - \sum_{i < L^*} \pi_i}{\pi_{L^*}}} \leq 1 . \] (C.11)

The inequalities follow from the definition of \( L^* \) in that \( \sum_{i < L^*} \pi_i < \gamma \leq \sum_{i \leq L^*} \pi_i \). Consequently, this solution is a consistent and unique equilibrium point for the ME algorithm if and only if it satisfies our initial assumption that \( \kappa_{L^*} \geq \frac{1 - \sum_{i < L^*} \pi_i}{S - L^* + 1} \).

Substituting formula for \( \kappa_{L^*} \) from equation (C.11), we see that this inequality is satisfied if and only if the support satisfies \( S \geq S_w = \frac{1 - \sum_{i < L^*} \pi_i}{\sqrt{\pi_{L^*} (\gamma - \sum_{i < L^*} \pi_i)}} + L^* - 1 \).
Step 3(b), $S < S_w$: On the other hand, when $S < S_w$, assuming that the total mass at $L^*$ is larger than $\frac{1-\sum_{i<L^*} \pi_i}{S-L^*+1}$, which is to be checked below, the ME algorithm reallocates the distribution mass uniformly such that $p_{0,L^*} = \frac{1-\sum_{i<L^*} \pi_i}{S-L^*+1}$. Consequently in this case, plugging this value to satisfy equations (C.9) and (C.10) yields again the following unique solution

$$
\kappa_{L^*} = q_0 \pi_{L^*} = \frac{(\gamma - \sum_{i<L^*} \pi_i)(S - L^* + 1)}{1 - \sum_{i<L^*} \pi_i} \pi_{L^*} < \pi_{L^*}, \quad \text{and}
$$

$$
q_0 = \frac{(\gamma - \sum_{i<L^*} \pi_i)(S - L^* + 1)}{1 - \sum_{i<L^*} \pi_i} < 1. \quad \text{(C.12)}
$$

The inequalities follow from the definition of $S_w$ and the assumption that $S < S_w$. Observe that the solution is also consistent with the condition of the previous case, where $S \geq S_w$, with respect to the support and the uncensored mass—which says $\kappa_{L^*} \geq \frac{1-\sum_{i<L^*} \pi_i}{S-L^*+1}$ if and only if $S \geq S_w$—as verified by the following inequalities for $S < S_w$:

$$
\kappa_{L^*} = \frac{(\gamma - \sum_{i<L^*} \pi_i)(S - L^* + 1)}{1 - \sum_{i<L^*} \pi_i} \pi_{L^*} < \frac{(\gamma - \sum_{i<L^*} \pi_i)(S_w - L^* + 1)}{1 - \sum_{i<L^*} \pi_i} \pi_{L^*} = \sqrt{\pi_{L^*}} \left( \frac{(\gamma - \sum_{i<L^*} \pi_i)}{S_w - L^* + 1} \right) \frac{1 - \sum_{i<L^*} \pi_i}{S_w - L^* + 1} < \frac{1 - \sum_{i<L^*} \pi_i}{S - L^* + 1}
$$

The last point to check is whether this solution is consistent with our initial assumption that the total mass at $L^*$ is larger than $\frac{1-\sum_{i<L^*} \pi_i}{S-L^*+1}$. This is easily verified
\[ p_{L^*} = \kappa_{L^*} + \zeta_{L^*} = (1 - q_\theta) \sum_{i \geq L^*} \pi_i + q_\theta \pi_{L^*} \]
\[ = (1 - q_\theta) \sum_{i > L^*} \pi_i + \pi_{L^*} \]
\[ > (1 - q_\theta) \sum_{i > L^*} \pi_i + \frac{1 - \sum_{i < L^*} \pi_i}{S - L^* + 1} \]
\[ > \frac{1 - \sum_{i < L^*} \pi_i}{S - L^* + 1} \]

where the first inequality follows from equation (C.5) for \( L_\theta = L^* \) due to Assumption 1.

Observe that at the critical value of the support \( S = S_w \), the two solutions obtained from two cases corresponding to \( S < S_w \) and \( S \geq S_w \) are consistent and continuous:

\[ \kappa_{L^*} = \frac{(\gamma - \sum_{i < L^*} \pi_i)(S_w - L^* + 1)}{1 - \sum_{i < L^*} \pi_i} \pi_{L^*} = \sqrt{\pi_{L^*} (\gamma - \sum_{i < L^*} \pi_i)} \]
\[ q_\theta = \frac{(\gamma - \sum_{i < L^*} \pi_i)(S_w - L^* + 1)}{1 - \sum_{i < L^*} \pi_i} = \frac{\gamma - \sum_{i < L^*} \pi_i}{\pi_{L^*}} \]

Therefore, we can combine the results for two cases under a single formula as follows:

\[ \kappa_{L^*} = \frac{(\gamma - \sum_{i < L^*} \pi_i)(\min\{S, S_w\} - L^* + 1)}{1 - \sum_{i < L^*} \pi_i} \pi_{L^*}, \quad \text{and} \quad (C.13) \]
\[ q_\theta = \frac{(\gamma - \sum_{i < L^*} \pi_i)(\min\{S, S_w\} - L^* + 1)}{1 - \sum_{i < L^*} \pi_i}. \quad \square \]

**Proof of Proposition 4.2.2:** Consider the candidate Lyapunov function

\[ V(\theta) = \sum_{j < L^*} ||\kappa_j - \pi_j|| + (\gamma - \sum_{j < L^*} \pi_j - \kappa_{L^*})^+ \] \quad (C.14)

which is continuous everywhere and continuously differentiable almost everywhere. Observe that \( \Theta = \{ \theta \mid V(\theta) = 0 \} \), and that for all \( \theta \in \Theta \), we have \( L_\theta = L^* \).

Using the above function, we want to show that \( \dot{V}(\theta) < 0 \) for all \( \theta \notin \Theta \) and that \( \dot{V}(\theta) = V(\theta) = 0 \) for all \( \theta \in \Theta \). Let sign(x) = 1 if \( x > 0 \), sign(x) = 0 if
\[ x = 0 \text{ and } \text{sign}(x) = -1 \text{ otherwise. Then, the gradient of the Lyapunov function can be written as follows:} \]

\[ \nabla V(\theta) = (\text{sign}(\kappa_1 - \pi_1), \ldots, \text{sign}(\kappa_{L^* - 1} - \pi_{L^* - 1}), -\text{sign}(\gamma - \sum_{j < L^*} \pi_j - \kappa_{L^*}), 0, \ldots, 0) \]

(0, \ldots, 0) \]

Alternatively, using the fact that the Lyapunov function contains no \( \zeta_j \) term, and hence, ignoring the last \( S \) components with value 0, with a slight abuse of notation, we can rewrite \( \nabla V(\theta) \) and \( g(\theta) \) as

\[ \nabla V(\theta) = \left( \text{sign}(\kappa_1 - \pi_1), \ldots, \text{sign}(\kappa_{L^* - 1} - \pi_{L^* - 1}), -\text{I}(\gamma - \sum_{j < L^*} \pi_j - \kappa_{L^*} > 0), 0, \ldots, 0 \right) \]

\[ g(\theta) = ((\pi_1 - \kappa_1), \ldots, (\pi_{L_0 - 1} - \kappa_{L_0 - 1}), (q_0 \pi_{L_0} - \kappa_{L_0}), -\kappa_{L_0 + 1}, \ldots, -\kappa_S) \]

The analysis of the derivative \( \dot{V}(\theta) \) will be split in three cases.

**Case 1:** \( L_\theta > L^* \). In this case \( \dot{V}(\theta) \) satisfies

\[ \dot{V}(\theta) = -\sum_{j < L^*} ||\kappa_j - \pi_j|| + \text{I}(\gamma - \sum_{j < L^*} \pi_j > \kappa_{L^*})(\kappa_{L^*} - \pi_{L^*}) \]

First, observe that \( \dot{V}(\theta) \) is less than or equal to 0 since clearly \(-\sum_{j < L^*} ||\kappa_j - \pi_j|| \leq 0 \), and and the term \( \text{I}(\gamma - \sum_{j < L^*} \pi_j > \kappa_{L^*})(\kappa_{L^*} - \pi_{L^*}) \) is also non-positive as \( \gamma - \sum_{j < L^*} \pi_j \leq \pi_{L^*} \) by the definition of \( L^* \). Second, note that \( \dot{V}(\theta) = 0 \) would imply \( \sum_{j \leq L^*} p_{\theta, j} = \sum_{j < L^*} \pi_j + p_{\theta, L_\theta} \geq \sum_{j < L^*} \pi_j + \kappa_{L^*} \geq \gamma \), which contradicts the assumption that \( L_\theta > L^* \). Consequently, we have that

\[ \dot{V}(\theta) < 0 \quad \text{for } \theta \text{ with } L_\theta > L^* \quad \text{(C.15)} \]
Case 2: $L_\theta < L^\ast$. In this case $\dot{V}(\theta)$ satisfies

$$
\dot{V}(\theta) = - \sum_{j < L_\theta} ||\kappa_j - \pi_j|| + sign(\kappa_{L_\theta} - \pi_{L_\theta}) (q_\theta \pi_{L_\theta} - \kappa_{L_\theta}) - \sum_{L_\theta < j < L^\ast} \text{sign}(\kappa_j - \pi_j) \kappa_j + \sum_{j < L^\ast} \Pi(\gamma - \sum_{j < L^\ast} \pi_j > \kappa_{L^\ast}) \kappa_{L^\ast}
$$

$$
= \Gamma(\theta) - \sum_{L_\theta < j < L^\ast} \text{sign}(\kappa_j - \pi_j) \kappa_j + \sum_{L_\theta < j < L^\ast} \Pi(\gamma - \sum_{j < L^\ast} \pi_j > \kappa_{L^\ast}) \kappa_{L^\ast}
$$

$$
\leq \Gamma(\theta) + \sum_{L_\theta < j < L^\ast} \pi_j + \sum_{j < L^\ast} \Pi(\gamma - \sum_{j < L^\ast} \pi_j > \kappa_{L^\ast}) \kappa_{L^\ast}
$$

(C.16)

$$
< \Gamma(\theta) + \sum_{L_\theta < j < L^\ast} \pi_j + \sum_{j < L_\theta} \gamma - \sum_{j < L_\theta} \pi_j = \Gamma(\theta) + \gamma - \sum_{j < L_\theta} \pi_j
$$

(C.17)

where the function $\Gamma(\theta)$ is defined as

$$
\Gamma(\theta) := - \sum_{j < L_\theta} ||\kappa_j - \pi_j|| + sign(\kappa_{L_\theta} - \pi_{L_\theta}) (q_\theta \pi_{L_\theta} - \kappa_{L_\theta}).
$$

(C.18)

Define the maximum amount of mass that can be reallocated by the ME algorithm at position $L_\theta$ for any fixed $\theta$ as $p^\mu(\theta) := \min\left\{\frac{1 - \sum_{i < L_\theta} \kappa_i}{S - L_\theta + 1}, \frac{1}{S}\right\}$, which is less than $\pi_{L_\theta}$ by assumption 1. Then, we have that $p_{\theta, L_\theta} = \max\{\kappa_{L_\theta}, p^\mu(\theta)\}$, and we can rewrite $\Gamma(\theta)$ as

$$
\Gamma(\theta) = - \sum_{j < L_\theta} ||\kappa_j - \pi_j|| + sign(\kappa_{L_\theta} - \pi_{L_\theta}) (q_\theta \pi_{L_\theta} - \kappa_{L_\theta})
$$

$$
= \sum_{j < L_\theta} ||\kappa_j - \pi_j|| + sign(\kappa_{L_\theta} - \pi_{L_\theta}) \left(\frac{\gamma - \sum_{i < L_\theta} p_{\theta, i} \pi_{L_\theta} - \kappa_{L_\theta}}{p_{\theta, L_\theta}}\right)
$$

$$
= \sum_{j < L_\theta} ||\kappa_j - \pi_j|| + \Pi(\kappa_{L_\theta} \geq \pi_{L_\theta}) \left(\frac{\gamma - \sum_{i < L_\theta} p_{\theta, i} \pi_{L_\theta} - \kappa_{L_\theta}}{\kappa_{L_\theta}}\right)
$$

$$
+ \Pi(p^\mu(\theta) < \kappa_{L_\theta} < \pi_{L_\theta}) \left(\frac{\gamma - \sum_{i < L_\theta} p_{\theta, i} \pi_{L_\theta}}{p^\mu(\theta) \pi_{L_\theta}}\right)
$$

$$
+ \Pi(\kappa_{L_\theta} \leq p^\mu(\theta)) \left(\frac{\gamma - \sum_{i < L_\theta} p_{\theta, i} \pi_{L_\theta}}{p^\mu(\theta) \pi_{L_\theta}}\right).
$$

Hence, we analyze Case 2 by further conditioning for different values of $\kappa_{L_\theta}$.
Case 2(a): $L_0 < L^*$ and $\kappa_{L_0} \leq p^u(\theta)$. Note that, we have $p_{\theta,L_0} = p^u(\theta)$ in this case, and also, as $\sum_{i \leq L_0} p_{\theta,i} \geq \gamma$, we have $\sum_{i < L_0} p_{\theta,i} \geq \gamma - p^u(\theta) > \gamma - \pi_{L_0} > \sum_{i < L_0} \pi_i$.

Now, fix some constant $M > \sum_{i < L_0} \pi_i$, and consider the following parameterized optimization problem

$$\max_{\kappa_j, j < L_0} \left\{- \sum_{j < L_0} ||\kappa_j - \pi_j|| : \sum_{j < L_0} p_{\theta,j} = M\right\} \quad \text{(C.19)}$$

which has the solution

$$\kappa_j = \pi_j + \frac{M - \sum_{j < L_0} \pi_j}{L_0 - 1}$$

with optimal objective value $\sum_{j < L_0} \pi_j - M = \sum_{j < L_0} \pi_j - \sum_{j < L_0} p_{\theta,j}$.

Therefore, we have for $\kappa_{L_0} \leq p^u(\theta)$,

$$\Gamma(\theta) = - \sum_{j < L_0} ||\kappa_j - \pi_j|| + \left(\kappa_{L_0} - \frac{\gamma - \sum_{j < L_0} p_{\theta,j}}{p^u(\theta)} \pi_{L_0}\right)$$

$$\leq \sum_{j < L_0} \pi_j - \sum_{j < L_0} p_{\theta,j} + \left(\kappa_{L_0} - \frac{\gamma - \sum_{j < L_0} p_{\theta,j}}{p^u(\theta)} \pi_{L_0}\right)$$

$$\leq \sum_{j < L_0} \pi_j - \sum_{j < L_0} p_{\theta,j} + \left(p^u(\theta) - \frac{\gamma - \sum_{j < L_0} p_{\theta,j}}{p^u(\theta)} \pi_{L_0}\right)$$

$$\leq \sum_{j < L_0} \pi_j - \gamma + p^u(\theta)$$

$$< \sum_{j < L_0} \pi_j - \gamma \quad \text{(C.20)}$$

The first inequality follows from the optimization problem (C.19). The second follows from the fact that the term in parenthesis is increasing in $\kappa_{L_0}$ which is less than $p^u(\theta)$ by the case assumption. The third inequality follows from the fact that the right hand side of the inequality is increasing in $\sum_{j < L_0} p_{\theta,j} < \gamma$ as $p^u(\theta) < \pi_{L_0}$. And, the last inequality follows as $p^u(\theta) < \pi_{L_0}$ by assumption 1. As a result, we have that $\hat{\mathcal{V}}(\theta) < 0$ for all $\theta$ with $L_0 < L^*$ and $\kappa_{L_0} \leq p^u(\theta)$ by combining inequalities (C.17) and (C.20) above.
Case 2(b): $L_\theta < L^*$ and $p^u(\theta) < \kappa_{L_\theta} < \pi_{L_\theta}$. Here, we have again $\sum_{i < L_\theta} p_{\theta,i} > \gamma - \pi_{L_\theta} > \sum_{i < L_\theta} \pi_i$. And again, fixing some constant $M > \sum_{i < L_\theta} \pi_i$, and considering the optimization problem (C.19), we have the same optimal objective value of $\sum_{j < L_\theta} \pi_j - \sum_{j < L_\theta} p_{\theta,j}$. Therefore, in a similar reasoning, we have the following inequalities in this case

$$
\Gamma(\theta) = -\sum_{j < L_\theta} ||\kappa_j - \pi_j|| + \left(\kappa_{L_\theta} - \frac{\gamma - \sum_{j < L_\theta} p_{\theta,j}}{\kappa_{L_\theta}} \pi_{L_\theta}\right)
\leq \sum_{j < L_\theta} \pi_j - \sum_{j < L_\theta} p_{\theta,j} + \left(\kappa_{L_\theta} - \frac{\gamma - \sum_{j < L_\theta} p_{\theta,j}}{\kappa_{L_\theta}} \pi_{L_\theta}\right)
< \sum_{j < L_\theta} \pi_j - \sum_{j < L_\theta} p_{\theta,j} + \left(\pi_{L_\theta} - \frac{\gamma - \sum_{j < L_\theta} p_{\theta,j}}{\pi_{L_\theta}} \pi_{L_\theta}\right)
= \sum_{j < L_\theta} \pi_j - \gamma
$$

(C.21)

where the strict inequality follows from the fact that the term in parenthesis is increasing in $\kappa_{L_\theta} < \pi_{L_\theta}$ for every fixed value of $\sum_{j < L_\theta} p_{\theta,j}$. As a result, we have that $V(\theta) < 0$ for all $\theta$ with $L_\theta < L^*$ and $p^u(\theta) < \kappa_{L_\theta} < \pi_{L_\theta}$ by combining inequalities (C.17) and (C.21) above.

Case 2(c): $L_\theta < L^*$ and $\kappa_{L_\theta} \geq \pi_{L_\theta}$. In this case, $\sum_{i < L_\theta} p_{\theta,i} \leq \sum_{i < L_\theta} \pi_i$ is possible. However, the optimal solution to problem (C.19) remains $\kappa_j = \pi_j + \frac{M - \sum_{j < L_\theta} \pi_j}{L_\theta - 1}$, $j < L_\theta$, but the optimal objective value is now $-||\sum_{j < L_\theta} \pi_j - \sum_{j < L_\theta} p_{\theta,j}||$. Therefore, we have that

$$
\Gamma(\theta) = -\sum_{j < L_\theta} ||\kappa_j - \pi_j|| + \left(\frac{\gamma - \sum_{j < L_\theta} p_{\theta,j}}{\kappa_{L_\theta}} \pi_{L_\theta} - \kappa_{L_\theta}\right)
\leq -||\sum_{j < L_\theta} \pi_j - \sum_{j < L_\theta} p_{\theta,j}|| + \left(\frac{\gamma - \sum_{j < L_\theta} p_{\theta,j}}{\kappa_{L_\theta}} \pi_{L_\theta} - \kappa_{L_\theta}\right)
$$

(C.22)
Consequently, if $\kappa_{L_0} \geq \pi_{L_0}$ and $\sum_{i<L_0} p_{\theta,i} < \sum_{i<L_0} \pi_i$ above, we have that

$$\Gamma(\theta) \leq -|| \sum_{j<L_0} \pi_j - \sum_{j<L_0} p_{\theta,j} || + \left( \frac{\gamma - \sum_{j<L_0} p_{\theta,j}}{\kappa_{L_0}} \pi_{L_0} - \kappa_{L_0} \right)$$

$$= \sum_{j<L_0} p_{\theta,j} - \sum_{j<L_0} \pi_j + \left( \frac{\gamma - \sum_{j<L_0} p_{\theta,j}}{\kappa_{L_0}} \pi_{L_0} - \kappa_{L_0} \right)$$

$$\leq \sum_{j<L_0} p_{\theta,j} - \sum_{j<L_0} \pi_j + \left( \frac{\gamma - \sum_{j<L_0} p_{\theta,j}}{\gamma - \sum_{j<L_0} p_{\theta,j}} \pi_{L_0} - (\gamma - \sum_{j<L_0} p_{\theta,j}) \right)$$

$$= 2 \sum_{j<L_0} p_{\theta,j} - \sum_{j<L_0} \pi_j + \pi_{L_0} - \gamma$$

$$< \sum_{j \leq L_0} \pi_j - \gamma \quad \text{(C.23)}$$

where the second inequality follows from the fact that the term in big parenthesis is strictly decreasing in $\kappa_{L_0}$ and $\kappa_{L_0} = p_{\theta,L_0} \geq \gamma - \sum_{j<L_0} p_{\theta,j}$ must hold by definition of $L_0$ in this region. The strict inequality follows from the case assumption $\sum_{i<L_0} p_{\theta,i} < \sum_{i<L_0} \pi_i$ and assumptions 1 and 2. As a result, we have that $\hat{V}(\theta) < 0$ for all $\theta$ with $L_0 < L^*$, $\kappa_{L_0} \geq \pi_{L_0}$ and $\sum_{i<L_0} p_{\theta,i} < \sum_{i<L_0} \pi_i$ by combining inequalities (C.17) and (C.23) above.

On the other hand, if $\kappa_{L_0} \geq \pi_{L_0}$ and $\sum_{i<L_0} \pi_i \leq \sum_{i<L_0} p_{\theta,i} < \gamma - \pi_{L_0}$, we have that

$$\Gamma(\theta) \leq -|| \sum_{j<L_0} \pi_j - \sum_{j<L_0} p_{\theta,j} || + \left( \frac{\gamma - \sum_{j<L_0} p_{\theta,j}}{\kappa_{L_0}} \pi_{L_0} - \kappa_{L_0} \right)$$

$$= \sum_{j<L_0} \pi_j - \sum_{j<L_0} p_{\theta,j} + \left( \frac{\gamma - \sum_{j<L_0} p_{\theta,j}}{\kappa_{L_0}} \pi_{L_0} - \kappa_{L_0} \right)$$

$$\leq \sum_{j<L_0} \pi_j - \sum_{j<L_0} p_{\theta,j} + \left( \frac{\gamma - \sum_{j<L_0} p_{\theta,j}}{\gamma - \sum_{j<L_0} p_{\theta,j}} \pi_{L_0} - (\gamma - \sum_{j<L_0} p_{\theta,j}) \right)$$

$$= \sum_{j \leq L_0} \pi_j - \gamma \quad \text{(C.24)}$$

where the second inequality follows from the fact that the third term in line two is strictly decreasing in $\kappa_{L_0}$, and that $\kappa_{L_0} = p_{\theta,L_0} \geq \gamma - \sum_{j<L_0} p_{\theta,j} > \pi_{L_0}$ must hold.
by definition of $L_\theta$ in this region. As a result, we have that $\dot{V}(\theta) < 0$ for all $\theta$ with $L_\theta < L^*$, $\kappa_{L_\theta} \geq \pi_{L_\theta}$ and $\sum_{i < L_\theta} \pi_i \leq \sum_{i < L_\theta} p_{\theta, i} < \gamma - \pi_{L_\theta}$ by combining inequalities (C.17) and (C.24) above.

Lastly, if $\kappa_{L_\theta} \geq \pi_{L_\theta}$ and $\sum_{i < L_\theta} p_{\theta, i} \geq \gamma - \pi_{L_\theta}$, we have that

$$\Gamma(\theta) \leq -\| \sum_{j < L_\theta} \pi_j - \sum_{j < L_\theta} p_{\theta, j} \| + \left( \frac{\gamma - \sum_{j < L_\theta} p_{\theta, j}}{\kappa_{L_\theta}} \pi_{L_\theta} - \kappa_{L_\theta} \right)$$

$$= \sum_{j < L_\theta} \pi_j - \sum_{j < L_\theta} p_{\theta, j} + \left( \frac{\gamma - \sum_{j < L_\theta} p_{\theta, j}}{\kappa_{L_\theta}} \pi_{L_\theta} - \kappa_{L_\theta} \right)$$

$$\leq \sum_{j < L_\theta} \pi_j - \sum_{j < L_\theta} p_{\theta, j} + \left( \frac{\gamma - \sum_{j < L_\theta} p_{\theta, j}}{\pi_{L_\theta}} \pi_{L_\theta} - \pi_{L_\theta} \right)$$

$$\leq \gamma - \sum_{j < L_\theta} p_{\theta, j}$$

$$\leq \sum_{j < L_\theta} \pi_j - \gamma$$

(C.25)

where the second inequality follows again from the fact that the term in big parenthesis is strictly decreasing in $\kappa_{L_\theta}$ and $\kappa_{L_\theta} = p_{\theta, L_\theta} \geq \gamma - \sum_{j < L_\theta} p_{\theta, j}$ must hold by definition of $L_\theta$ in this region. As a result, we have that $\dot{V}(\theta) < 0$ for all $\theta$ with $L_\theta < L^*$, $\kappa_{L_\theta} \geq \pi_{L_\theta}$ and $\sum_{i < L_\theta} \pi_i \leq \sum_{i < L_\theta} p_{\theta, i} < \gamma - \pi_{L_\theta}$ by combining inequalities (C.17) and (C.25) above.

We have covered all three cases, 2(a), 2(b), and 2(c) for $\theta$ with $L_\theta < L^*$, and showed that $\dot{V}(\theta) < 0$ for such $\theta$. Combining with Case 1 for $L_\theta > L^*$, so far we have shown that $\dot{V}(\theta) < 0$ for $\theta$ with $L_\theta \neq L^*$.

Case 3: $L_\theta = L^*$ and $\theta \notin \Theta$. The derivative $\dot{V}(\theta)$ in this case satisfies

$$\dot{V}(\theta) = -\sum_{j < L^*} ||\kappa_j - \pi_j|| - \mathbb{I}(\gamma - \sum_{j < L^*} \pi_j > \kappa_{L^*})(q_0 \pi_{L^*} - \kappa_{L^*}) .$$

We study two sub-cases depending on the value of the indicator function. Case 3(a): $L_\theta = L^*$, $\theta \notin \Theta$, and $\kappa_{L^*} \geq \gamma - \sum_{j < L^*} \pi_j$. Remember that by definition of $\Theta$ and
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\(V(\cdot)\), we have that \(\Theta = \{\theta \mid V(\theta) = 0\}\). Hence, for \(\theta \notin \Theta\) satisfying \(L_\theta = L^*\) and \(\kappa_{L^*} \geq \gamma - \sum_{j<L^*} \pi_j\), we have that \(V(\theta) = \sum_{j<L^*} ||\kappa_j - \pi_j|| > 0\) by construction of the Lyapunov function. Consequently, in this case, we have \(\dot{V}(\theta) = -\sum_{j<L^*} ||\kappa_j - \pi_j|| = -V(\theta) < 0\).

**Case 3(b):** \(L_\theta = L^*, \theta \notin \Theta,\) and \(\kappa_{L^*} < \gamma - \sum_{j<L^*} \pi_j\). When \(\kappa_{L^*} < \gamma - \sum_{j<L^*} \pi_j\), for \(L_\theta = L^*\) to hold, \(\theta \notin \Theta\) needs to satisfy \(\sum_{j<L^*} \kappa_j = \sum_{j<L^*} \kappa_j - \sum_{j<L^*} \pi_j\) by definition of \(L_\theta\) and \(L^*\). Hence, in this case, we have

\[
\dot{V}(\theta) = -\sum_{j<L^*} ||\kappa_j - \pi_j|| + \kappa_{L^*} - p_{L^*} \pi_{L^*}
\]

\[
\leq -|| \sum_{j<L^*} \kappa_j - \sum_{j<L^*} \pi_j|| + \kappa_{L^*} - q_{L^*} \pi_{L^*}
\]

\[
= -\sum_{j<L^*} \kappa_j + \sum_{j<L^*} \pi_j + \kappa_{L^*} - \frac{\gamma - \sum_{j<L^*} \kappa_j}{P_{L^*}} \pi_{L^*}
\]

\[
\leq -\gamma - \sum_{j<L^*} \pi_j + \kappa_{L^*}
\]

\[
< 0,
\]

where the first inequality follows due to the triangle inequality. Under Assumptions 1, 2, and 3, \(p_{L^*} < \pi_{L^*}\) holds, as previously discussed above, and consequently, the right hand side of the second equality is increasing in \(\sum_{j<L^*} \kappa_j < \gamma\), which yields the second inequality by replacing \(\sum_{j<L^*} \kappa_j\) with \(\gamma\). The strict inequality follows from the case assumption, yielding \(\dot{V}(\theta) < 0\) for this case. As a result, we have that \(\dot{V}(\theta) < 0\) for \(\theta \notin \Theta\) satisfying \(L_\theta = L^*\) and \(\kappa_{L^*} < \gamma - \sum_{j<L^*} \pi_j\).

We have analyzed all cases for \(\theta \notin \Theta\) and shown that \(\dot{V}(\theta) < 0\). Also observing that \(\dot{V}(\theta) = V(\theta) = 0\) if and only if \(\theta \in \Theta\), we conclude that \(\Theta\) is globally asymptotically stable. \(\Box\)

**Proof of Proposition 4.2.3:** As shown in Proposition 4.2.1, \(\theta_s\) satisfies \(L_{\theta_s} = L^*\) and
\( \theta_s = f(\theta_s) \); and hence \( \kappa_j = \pi_j \) for \( j < L^* \) as implied by equation (C.9). Consequently, we need to only show that \( \kappa_{L^*} \geq \gamma - \sum_{i < L^*} \pi_i \) for \( \theta_s \). The component \( \kappa_{L^*} \) satisfies

\[
\kappa_{L^*} = \begin{cases} 
\sqrt{\pi_{L^*} \left( \gamma - \sum_{i < L^*} \pi_i \right)} & \text{for } S \geq S_w \\
\frac{(\gamma - \sum_{i < L^*} \pi_i)(S - L^* + 1)}{1 - \sum_{i < L^*} \pi_i} \pi_{L^*} & \text{for } S < S_w,
\end{cases}
\]

as previously shown in equations (C.11), (C.12) and (C.13) in the proof of Proposition 4.2.1. So, we establish that \( \kappa_{L^*} \geq \gamma - \sum_{i < L^*} \pi_i \) for \( \theta_s \) in two cases.

**Case 1, \( S \geq S_w \):** By definition of \( L^* \), we have that \( \gamma - \sum_{i < L^*} \pi_i \leq \pi_{L^*} \), and the geometric mean of the terms satisfy

\[
\gamma - \sum_{i < L^*} \pi_i \leq \sqrt{\pi_{L^*} \left( \gamma - \sum_{i < L^*} \pi_i \right)} \leq \pi_{L^*}.
\]

Therefore, we have that \( \kappa_{L^*} = \sqrt{\pi_{L^*} \left( \gamma - \sum_{i < L^*} \pi_i \right)} \geq \gamma - \sum_{i < L^*} \pi_i \) for \( S \geq S_w \).

**Case 2, \( S < S_w \):** On the other hand, if \( S < S_w \), using assumptions 1 and 2, we have that

\[
S > \frac{1}{\epsilon} = \frac{1 - (L^* - 1)\epsilon}{\epsilon} + L^* - 1 > \frac{1 - \sum_{i < L^*} \pi_i}{\pi_{L^*}} + L^* - 1
\]

where the second inequality follows from Assumption 1; or equivalently, rearranging terms

\[
\frac{(S - L^* + 1)}{1 - \sum_{i < L^*} \pi_i} \pi_{L^*} > 1, \text{ hence, } \kappa_{L^*} = \frac{(\gamma - \sum_{i < L^*} \pi_i)(S - L^* + 1)}{1 - \sum_{i < L^*} \pi_i} \pi_{L^*} > \gamma - \sum_{i < L^*} \pi_i. \]

\( \square \)