Resource Allocation among Simulation Time Steps

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Resource Allocation among Simulation Time Steps

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Abstract

Motivated by the problem of efficient estimation of expected cumulative rewards or cashflows, this paper proposes and analyzes a variance reduction technique for estimating the expectation of the sum of sequentially simulated random variables. In some applications, simulation effort is of greater value when applied to early time steps rather than shared equally among all time steps; this occurs, for example, when discounting renders immediate rewards or cashflows more important than those in the future. This suggests that deliberately stopping some paths early may improve efficiency. We formulate and solve the problem of optimal allocation of resources to time horizons with the objective of minimizing variance. The solution has a simple characterization in terms of the convex hull of points defined by the covariance matrix of the cashflows. We also develop two ways to enhance variance reduction through early stopping. One takes advantage of the statistical theory of missing data. The other redistributes the cumulative sum to make optimal use of optimal early stopping.

1 Introduction

In a standard finite-horizon simulation, every path reaches every time step. This is an implicit allocation of computational resources among the time steps. This paper proposes and analyzes a variance reduction technique that sets the number of paths reaching each time step optimally according to the solution of a resource allocation problem.

This investigation is motivated by problems of estimating cumulative discounted rewards or cashflows. In various applications, cashflows resulting from business operations or a financial security are potentially generated at each time step; their timing and magnitude are determined by an underlying stochastic model of physical operations or market prices. Mortgage-backed securities provide a specific illustration. The cashflows of these securities result from payments from a pool of mortgages, and valuing such a security entails calculating the expected present values of these payments. Valuing a security backed by 30-year mortgages making monthly payments nominally requires simulating 360 time steps, but most of the value of the security is typically determined over a far shorter horizon. This results from two effects: discounting (which reduces the importance of
future payments) and the right of homeowners to prepay their mortgages (which results in larger payments earlier and smaller payments later). These two features turn out to have opposite effects on the effectiveness of our method.

Put abstractly, our goal is efficient estimation of the expected value of a finite sum of sequentially generated random variables. The summands are correlated—for example, they may be functions of the state of a Markov chain. Simulating each summand entails simulating all previous summands. The length of a path is the number of summands simulated. We plan in advance to simulate a fixed number of paths of each length; by appropriately weighting paths of different lengths we obtain an unbiased estimator. We find the optimal number of paths of each length to simulate by minimizing variance subject to a computational budget constraint and a monotonicity constraint requiring that the number of paths reaching each horizon decreases as the horizon increases. The optimal allocation has a simple characterization in terms of the convex hull of points that reflect the contributions of each step to total work and variance. Section 3 makes this precise.

This method may be interpreted as *stratification in time*. Indeed, our optimal allocation bears a superficial resemblance to the optimal allocation for stratified sampling. However, whereas in stratified sampling different strata can ordinarily be sampled independently, in our setting later time steps can be sampled only if earlier time steps are sampled too.

The idea of varying the time horizon of paths in simulation recalls Fox and Glynn's (1989) investigation of estimation of expected reward over infinite time horizons. However, their work uses random stopping in order to remove bias from truncating an infinite integral, as does Asmussen (1990), while we consider a strategy of deterministic stopping optimized to reduce estimator variance.

The variables not generated when we stop a path early may be viewed as missing data. This interpretation suggests the possibility of improved output analysis using missing data techniques, as in Little and Rubin (1987, § 6.5). With these techniques, Hocking and Smith (1972) find the cost-minimizing experimental design for bivariate data subject to simultaneous constraints on the variance of estimators of six parameters: means, variances, and regression coefficients. In Section 4, we show how the theory of missing data can be applied to reduce variance in our more general setting, and how to solve the resource allocation problem when estimation employs missing data techniques.

Estimators resulting from the application of missing data techniques may be interpreted as redistributing cashflows across time steps. This suggests consideration of other estimators that produce "fictitious cashflows" at intermediate time steps while keeping the sum constant. Section 5 analyzes this case and finds the optimal fictitious cashflows for optimal early stopping.
A discussion of the effectiveness of the techniques developed in this paper appears in Section 6.
Section 7 concludes the paper. All proofs are deferred to Section 8, and Appendix A contains the
algorithm for solving the resource allocation problem.

2 The Simple Early Stopping Problem

We seek to estimate, by simulation, the expectation of the sum $X$ of $m$ discounted cashflows $X_1, \ldots, X_m$ having finite variance. Each $X_k$ is a function of the history $\{S_1, \ldots, S_k\}$, with $S_n$ denoting the state of the simulation at time step $n$. The cost of generating $S_k$ and $X_k$ given the process up to step $k - 1$ is $c_k$, and the total computational budget is $C$. We plan in advance to simulate $X_k$ only on paths $1, \ldots, n_k$, where the $n_k$ satisfy the monotonicity constraint

$$n_k \geq n_{k+1}, \quad k = 1, \ldots, m - 1$$

and the budget constraint

$$\sum_{k=1}^{m} c_k n_k \leq C.$$

Define the feasible set $\mathcal{N}$ as the set of all vectors $(n_1, \ldots, n_m)$ satisfying these two constraints.

There are $n = \max\{n_k\} = n_1$ paths. Let $X_{ik}$ be the value of $X_k$ on path $i$. The simulation generates this value if $i \leq n_k$. This simulation structure also allows us to define the length of the $i$th path $m_i = \max\{k | n_k \geq i\}$, so that $X_{ik}$ is observed (i.e., simulated) if and only if $k \leq m_i$, and again $m = \max\{m_i\} = m_1$. So the simulation produces an $n \times m$ matrix of data $X$ whose $i$th row (the path $i$) is observed up to column $m_i$ and then unobserved, and whose $k$th column (the random variable $X_k$) is observed up to row $n_k$ and then unobserved.

Example. When the vector $n = (4, 2, 1)$ then the vector $m = (3, 2, 1, 1)$ and the matrix $X$ is

$$\begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & \\
X_{31} & \cdot & \\
X_{41} & \cdot & 
\end{bmatrix}$$

This structure for simulation is equivalent to stipulating that if $X_k$ is generated on path $i$, so were $X_1, \ldots, X_{k-1}$ and hence necessarily $S_1, \ldots, S_{k-1}$. Then we say that for $k < \ell$, $X_k$ is more observed than $X_\ell$. This is necessary in simulation when the process is specified in terms of transition probabilities, so that generating $S_k$ requires knowing $S_{k-1}$.

Now define $\bar{X}_{ki} = \frac{1}{n_k} \sum_{i=1}^{n_k} X_{ik}$, the average of $X_k$ on those paths where $X_\ell$ is observed. For this to make sense, we must have $k \leq \ell$ so that $X_k$ is at least as observed as $X_\ell$. The matrix $\bar{X}$ is an upper-triangular $m \times m$ matrix.
Example. Continuing the previous example, the matrix \( \tilde{X} \) is

\[
\begin{bmatrix}
    \frac{(X_{11} + X_{21} + X_{31} + X_{41})}{4} & \frac{(X_{11} + X_{21})}{2} & X_{11} \\
    \cdot & \frac{(X_{12} + X_{22})}{2} & X_{12} \\
    \cdot & \cdot & \cdot 
\end{bmatrix}
\]

In this simple setting, we estimate \( \mu_k \triangleq \mathbb{E}[X_k] \) by \( \bar{\mu}_k = \tilde{X}_{kk} \) and \( \bar{\mu} \triangleq \sum_{k=1}^m \mu_k \) by \( \bar{\mu} = \sum_{k=1}^m \bar{\mu}_k \). Finding the experimental design \( (n_1, \ldots, n_m) \) that minimizes variance given the fixed computational budget is a resource allocation problem with constraints (1) and (2). We ignore the issue that the \( n_i \) must actually be integer; because they will be large in reasonable applications, rounding errors are likely to be insignificant. We suggest rounding up for safety.

To solve the resource allocation problem, we first give an explicit expression for the objective.

**Lemma 1** The variance objective is

\[
\text{Var}[\bar{\mu}] = \sum_{k=1}^m \frac{v_k}{n_k} 
\]

where \( \sigma_{kl} = \text{Cov}[X_k, X_l] \) form the covariance matrix \( \Sigma \) and

\[
v_k = \sigma_{kk} + 2 \sum_{\ell=k+1}^m \sigma_{k\ell}.
\]

The resource allocation problem is then

\[
\min_{n \in \mathbb{N}} \sum_{k=1}^m \frac{v_k}{n_k}
\]

3 Solution of the Resource Allocation Problem

The variance component \( v_k \) defined in equation (4) is that part of the variance of \( \sum_{k=1}^m X_k \) attributable to step \( k \), much as \( c_k \) is the component of a complete path’s cost attributable to simulating step \( k \). Because of the monotonicity constraint (1), doing a unit of work at step \( k \) requires doing a unit of work at all steps \( j \leq k \), so we are also interested in partial sums of the variance components \( v_k \) and cost components \( c_k \). Define the partial sums

\[
V_k \triangleq \sum_{j=1}^k v_j \quad \text{and} \quad C_k \triangleq \sum_{j=1}^k c_j
\]

and the point set

\[
\mathcal{V} \triangleq \{(C_k, V_k) | k = 0, \ldots, m\}.
\]
The tail sum $V_m - V_k = \sum_{i=k+1}^m v_i = \text{Var}[\sum_{i=k+1}^m X_i]$ is the variance of the last $m-k$ discounted cashflows. Likewise, $C_m - C_k$ is the cost of simulating the last $m-k$ steps.

The solution to the resource allocation problem (5) involves the upper convex hull $V^*$ of the graph $\mathcal{V}$. This is defined as that portion of the convex hull above a line connecting the leftmost and rightmost points of the graph, here respectively $(C_0, V_0) = (0,0)$ and $(C_m, V_m)$. Let $V_k^*$ denote $V^*(C_k)$, so that $(C_k, V_k^*)$ is a point on the convex hull, and define $v_k^*$ as the increment $V_k^* - V_{k-1}^*$. For an illustration of taking the cumulative sum of $v$ to get $V$, taking the convex hull of $V$ to get $V^*$, and taking the increments of $V^*$ to get $v^*$, see Figure 1.

Further define the slopes

$$u_k \triangleq v_k / c_k \quad \text{and} \quad u_k^* \triangleq v_k^* / c_k.$$

**Theorem 1** The solution to the resource allocation problem (5) is

$$n_k = \sqrt{\frac{u_k^*}{\nu}}, \quad \nu = \left( \frac{1}{C} \sum_{k=1}^m c_k \sqrt{u_k^*} \right)^2$$

and the ratio of optimal variance to standard variance is

$$R = \frac{\sum_{i=1}^m \sum_{j=1}^m v_i c_j u_j^* / u_i^*}{\sum_{i=1}^m \sum_{j=1}^m v_i c_j}$$

The result parallels the classical result for stratified sampling that the variance of an estimator of a mean is minimized when the number of samples from each stratum is proportional to the square root of the ratio of the variance in that stratum and the cost to sample from that stratum. See, for example, Cochran (1953, Thm. 5.7). To make the analogy, interpret the time steps as strata of equal size. The $v_k^*$ are not variances but they play a similar role.

This resource allocation problem is also nearly a special case of a problem that arises in planning production-distribution systems. Maxwell and Muckstadt (1985, p. 1323) pose “problem RP,” which is to minimize $\sum_k (K_k/T_k + g_k T_k)$, where the sum is over all nodes $k$ in a directed graph, subject to the constraints $T_j \geq T_k$ for each arc $(j,k)$ in the graph. Federgruen and Zheng (1992, Thm. 1) describe the solution of problem RP in greater generality, but the case considered here has special features— in particular, the convex hull characterization—not shared by the general case. Roundy (1986, p. 720) notes connections between the production-distribution planning problem and statistical applications such as isotonic regression and multidimensional scaling, and provides references.
The solution to the resource allocation problem depends on the covariance matrix of the variables \( X_k \), which is presumably unknown in realistic situations. However, the covariance matrix can be estimated in pilot runs, or a good guess may be possible from analysis of a simpler model. In either case, the design optimal for a good estimate of the covariance matrix may be better than either the standard design where all paths reach all time steps or a guess at a design.

The savings due to implementing the scheme should be weighed against the overhead costs of finding the optimal \( n_k \). As expressed in the resource allocation problem, the new scheme is a variance reduction technique, but this is equivalent to a reduction of cost to attain the same variance. If the technique reduces variance to a fraction \( R \) of the original variance, this is equivalent to saving the fraction \( 1 - R \) of the cost, which is \( O(mn) \).

Finding a planar convex hull is ordinarily \( O(m \log m) \), because of the intimate connection to sorting, as remarked by Preparata and Shamos (1985, p.94). In our case, the points of \( \mathcal{V} \) are already sorted by abscissa \( C_i \). Taking advantage of this and our lack of interest in the lower convex hull, we can use a stripped-down Graham scan which is only \( O(m) \). We present the algorithm in Appendix A; for a discussion of the Graham scan, see Preparata and Shamos (1985, §3.3.2). However, simply to compute or guess at the \( m^2 \) elements of the covariance matrix \( \Sigma \) and compute the variance components \( v_k \) from them takes \( O(m^2) \) work. So when \( n \gg m \), it is indeed efficient to use this algorithm.

4 Missing Data Perspective

If some paths stop early, we can use missing data analysis. The resource allocation problem is based on the variance of the estimator used, and should be changed if we use an estimator incorporating missing data analysis. As discussed in Section 2, stopping early on some paths results in a data matrix \( \mathbf{X} \) with missing data, such that for \( k < \ell \), \( X_k \) is more observed than \( X_\ell \). Little and Rubin (1987, Example 6.7) consider data sets with this structure, but with the further assumption that the vector \( (X_1, \ldots, X_m) \) is multivariate normal. Then the maximum likelihood estimate of the mean \( \mu_k \) is

\[
\hat{\mu}_k = \bar{X}_{kk} + \sum_{i=1}^{k-1} \hat{\beta}_ik(\bar{X}_{ik} - \bar{X}_{ik})
\]

(9)

where \( \hat{\beta}_{ik} \) is the estimated coefficient of \( X_k \) on \( X_i \) in the regression on \( X_1, \ldots, X_{k-1} \), using the \( n_k \) observations where all of these variables are observed. The resulting estimator of \( \mu = \sum_{k=1}^{m} \mu_k \) is the sum \( \sum_{k=1}^{m} \hat{\mu}_k \).

In the case of simulation, requiring the random variables \( X_k \) to be multivariate normal would be too restrictive, but their sample averages should be approximately normal. Write \( n = n_qi \), where
$q_i = 1$. As $n \to \infty$, the joint distribution of the appropriately scaled averages $\bar{X}_{ik}$ converges to the multivariate normal, by the central limit theorem. This large-sample result frees us to use finite-variance functions of the data $T_h = t_h(S_1, \ldots, S_h)$ for $h < k$ as predictors of $X_k$, since the vector $(\bar{T}_{1k}, \ldots, \bar{T}_{k-1k}, \bar{X}_{kk})$ of sample averages on the first $n_k$ paths is also approximately multivariate normal, although in general $(T_1, \ldots, T_{k-1}, X_k)$ is not.

Using these $T_1, \ldots, T_{k-1}$ instead of $X_1, \ldots, X_{k-1}$ as predictors for $X_k$, equation (9) becomes

$$\hat{\mu}_k = \bar{X}_{kk} + \sum_{h=1}^{k-1} \hat{\beta}_{hk}(\hat{r}_h - \bar{T}_{hk})$$

(10)

where now $\hat{\beta}_{hk}$ is the estimated multiple regression coefficient of $X_k$ on $T_h$ in the regression on $T_1, \ldots, T_{k-1}$, while $\hat{r}_h$ is a missing-data estimate of the mean $\tau = \mathbb{E}[T_h]$. Expressed with a different subscript, this estimate is

$$\hat{r}_k = \bar{T}_{kk} + \sum_{h=1}^{k-1} \hat{\beta}_{hk}(\hat{r}_h - \bar{T}_{hk})$$

(11)

where $\hat{\beta}_{hk}$ is the estimated coefficient of $T_k$ on $T_h$ in the regression on $T_1, \ldots, T_{k-1}$.

We can express the missing data estimator $\sum_{k=1}^{m} \hat{\mu}_k$ in a cleaner fashion by defining an adjusted data matrix $X' = X + TW$ where $W$ is a matrix of estimated weights. As Lemma 2 states, the missing data estimator takes the form of a sum of averages of random variables $X'_k$. This means that the minimization of this missing data estimator's variance is a resource allocation problem based on $X'$ rather than $X$. Let 1 and 0 be the vectors whose components are respectively 1 and 0. The condition $W1 = 0$ will make the new estimator unbiased, as shown in Theorem 2.

**Lemma 2** Where $\hat{\mu}_k$ is as given in equation (10),

$$\sum_{k=1}^{m} \hat{\mu}_k = \hat{\mu} \triangleq \sum_{k=1}^{m} \frac{1}{n_k} \sum_{i=1}^{n_k} X'_k$$

(12)

with $X' \triangleq X + TW$ for an upper-triangular $m \times m$ matrix $W$ whose elements $w_{hi}$ are given as follows. For $h \leq i \leq k$, define

$$w_{hik} \triangleq \begin{cases} 1 & \text{if } h = i = k \\ -\hat{\beta}_{hi} & \text{if } h < i = k \\ \sum_{j=i+1}^{k-1} \beta_{hj} w_{hij} & \text{otherwise} \end{cases}$$

(13)

$$w_{hik} \triangleq \begin{cases} 0 & \text{if } h = i = k \\ -\hat{\beta}_{ki} & \text{if } h < i = k \\ \sum_{j=i+1}^{k-1} \beta_{kj} w_{hij} & \text{otherwise} \end{cases}$$

(14)

$$w_{hi} \triangleq \sum_{k=i}^{m} w_{hik}$$

(15)

This matrix $W$ satisfies $W1 = 0$. 

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In the definition of the elements of $\mathbf{W}$, the third subscript $k$ indicates that the $\omega_{ik}$ is a weight used in estimating $\hat{v}_k$ and the $w_{ik}$ are weights in $\hat{\mu}_k$; the $w$ with only two subscripts are weights in the estimate \( \hat{\mu} \) of the sum.

Henceforth let $\mathbf{W}$ be any upper-triangular $m \times m$ random matrix that satisfies $\mathbf{W} \mathbf{1} = \mathbf{0}$. Then

$$\mathbf{X}' \mathbf{1} = \mathbf{X} \mathbf{1} + \mathbf{T} \mathbf{W} \mathbf{1} = \mathbf{X} \mathbf{1} + \mathbf{T} \mathbf{0} = \mathbf{X} \mathbf{1}$$

and thus the adjusted $X'_k$ differ from the discounted cashflows $X_k$, but their sums on a complete path are the same.

**Theorem 2** The estimator $\hat{\mu}$ is unbiased for $\mu$.

If $\mathbf{W}$ is constant, the covariance matrix of $(X'_1, \ldots, X'_m)$ is

$$\Sigma' = \Sigma + \mathbf{W}^\top \Xi + \Xi^\top \mathbf{W} + \Theta \mathbf{W}^\top \Theta \mathbf{W}$$

(16)

where $\Sigma$, $\Xi$, and $\Theta$ are respectively the covariance matrices of $(X_1, \ldots, X_m)$, of $(X_1, \ldots, X_m)$ with $(T_1, \ldots, T_m)$, and of $(T_1, \ldots, T_m)$. The following theorem implies that the optimal resource allocation for the estimator $\hat{\mu}$ is given by Theorem 1 with $\Sigma'$ replacing $\Sigma$.

**Theorem 3** If $\mathbf{W}$ is constant, $\hat{\mu}$ has variance $\sigma^2$ given by equations (3) and (4), but with $\Sigma'$ replacing $\Sigma$. As the budget $C$ goes to infinity, the distribution of $(\hat{\mu} - \mu)/\sigma$ converges to standard normal.

What if $\mathbf{W}$ is not constant? It could depend on the simulated data, in which case we think of it as an estimator of some constant matrix $\mathbf{W}^*$.

**Theorem 4** If there is a constant matrix $\mathbf{W}^*$ such that each element $w_{hi}$ of $\mathbf{W}$ is a consistent estimator of $w_{hi}^*$, then as the budget $C$ goes to infinity, the distribution of $(\hat{\mu} - \mu)/\sigma$ converges to standard normal.

This general missing data procedure requires computation of $O(m^3)$ coefficients $w_{hik}$. However, restricted models with fewer coefficients to estimate can save computation, reduce estimation error, and make implementation easier. The key to the effective use of this variance reduction technique is to use knowledge about the model to find predictors which are known as early as possible in the simulation and explain as much as possible of the variance in payoffs. Such knowledge may suggest that it is unnecessary to allow for a completely general relationship between all predictors and payoffs.
For instance, if \( T_k \) is Markov, then \( b_{hk} = 0 \) for \( h < k - 1 \) and should not be estimated. If furthermore \( X_k \) is a function of \( T_k \) only, likewise \( \beta_{hk} = 0 \) for \( h < k - 1 \). In this case, the elements of \( \mathbf{W} \) are

\[
  w_{hi} = \begin{cases} 
    \sum_{k=1}^{m} \beta_{k-1-k} \prod_{j=1}^{k-1} b_{i-j} & \text{if } h = i \\
    -\beta_{hi} - \sum_{k=1}^{m} \beta_{k-1-k} \prod_{j=1}^{k-1} b_{i-j} & \text{if } h < i 
  \end{cases}
\]

To compute these takes only \( O(m^2) \) work because component sums and products can be reused.

This use of missing data techniques to produce improved estimates of the means \( \mu_k \) bears a strong resemblance to control variates. In the above setting, the predictor mean \( \tau_k = \mathbf{E}[T_k] \) is unknown. If each \( T_k \) is a control variate for \( X_k \), \( \tau_k \) is known, and \( X'_k = X_k + \beta_k(\tau_k - T_k) \), where \( \beta_k \) is the estimated simple regression coefficient of \( X_k \) on \( T_k \). Again, the covariance matrix relevant to resource allocation is that of \( (X'_1, \ldots, X'_m) \). A key distinction is that whereas the control variate method relies on \( \mathbf{E}[T_k] - T_k \) having mean zero, the missing data estimator relies on the property (embodied in the condition \( \mathbf{W1} = 0 \)) that each \( T \) gets a net weight of zero when we take the expectation of \( \mathbf{X}' \). This is applicable only because different paths stop at different times.

5 Fictitious Cashflows

The effect of using predictors of future cashflows is to replace the actual step-\( k \) cashflow \( X_k \) with \( X'_k \), which we may interpret as a fictitious cashflow at that time. This suggests consideration of other estimators based on redistributing cumulative cashflows across time steps. More precisely, we consider sequences \( \{X'_1, \ldots, X'_m\} \) having two properties. First, \( X'_k \) is measurable with respect to \( \mathcal{F}_k \), the sigma-algebra generated by the state vectors \( S_1, \ldots, S_k \); that is, it is actually known at step \( k \) of the simulation. Second, the sum on a complete path stays the same: \( \sum_{k=1}^{m} X'_k = \sum_{k=1}^{m} X_k \triangleq X \).

We now have \( \mathbf{X}' = \mathbf{X} + \mathbf{TW} \) after the fashion of Section 4, with \( \mathbf{W} \) equal to the identity matrix and the random variable \( T_k = X'_k - X_k \). Thus Theorem 3 applies. Let \( \mathcal{X} \) be the set of random vectors \( (X'_1, \ldots, X'_m) \) satisfying the two properties of summation to \( X \) and adaptation to the filtration \( \{\mathcal{F}_k\} \). What is the optimal \( (X'_1, \ldots, X'_m) \) in \( \mathcal{X} \) for resource allocation?

The significance of fictitious cashflows is that the covariance matrices \( \Sigma \) of \( (X_1, \ldots, X_m) \) and \( \Sigma' \) of \( (X'_1, \ldots, X'_m) \) generally differ. Let \( v_k \) and \( v'_k \) be respectively the variance components, derived from \( \Sigma' \) according to equation (4), and their partial sums. Because the sums \( \sum_{k=1}^{m} X_k = \sum_{k=1}^{m} X'_k \) are equal, their respective variances \( v_m \) and \( v'_m \) are equal, but the intermediate partial sums \( v_k \) and \( v'_k \) may be unequal. As a consequence, the variance achieved after stopping early may be different. The next result identifies the best set of fictitious cashflows, as measured by the remaining variance after we apply optimal early stopping.
Theorem 5 Given \( X = \sum_{k=1}^{m} X_k \), the optimal sequence of random variables
\[
\arg \min_{\{X'_k\} \in X} \min_{n \in \mathbb{N}} \sum_{k=1}^{m} \frac{\nu_k}{n_k}
\]
is given by \( X'_k = \mathbb{E}[X|\mathcal{F}_k] - \mathbb{E}[X|\mathcal{F}_{k-1}], \ k = 1, \ldots, m \); i.e., \( \sum_{j=1}^{k} X'_j = \mathbb{E}[X|\mathcal{F}_k] \).

The optimal fictitious cashflows are thus the martingale differences associated with the martingale \( \mathbb{E}[X|\mathcal{F}_k], \ k = 1, \ldots, m \). Of course, the conditional expectations \( \mathbb{E}[X|\mathcal{F}_k] \) are presumably unknown. However, the proof of the theorem shows that early stopping is most effective with fictitious cashflows whose partial sums get as close as possible to these conditional expectations, since this reduces the variance \( \text{Var}[X - \sum_{j=1}^{k} X'_j] \) attributable to steps after \( k \). Even if \( \sum_{j=1}^{k} X'_j \) is biased for \( \mathbb{E}[X|\mathcal{F}_k] \), Theorem 2 applies, as remarked earlier, and the estimator under consideration is unbiased for \( \mathbb{E}[X] \triangleq \mu \). We benefit as long as \( \sum_{j=1}^{k} X'_j \) is a better estimator of \( \mathbb{E}[X|\mathcal{F}_k] \) than \( \sum_{j=1}^{k} X_j \) is. Examples of fictitious cashflows designed to resemble unknown conditional expectations appear in Section 6.

We can re-interpret the approach of Section 4 in this light. We want to reduce \( \text{Var}[X - \sum_{j=1}^{k} X'_j] \), but we do not know \( \mathbb{E}[X|\mathcal{F}_k] = \sum_{j=1}^{k} X_j + \mathbb{E}[\sum_{\ell=k+1}^{m} X_{\ell}|\mathcal{F}_k] \). We do know \( \sum_{j=1}^{k} X_j \), and we find the projection of \( \sum_{\ell=k+1}^{m} X_{\ell} \) onto the space of random variables linear in the predictors \( T_1, \ldots, T_k \). The primary attraction of this approach is the possibility of estimating the regression coefficients involved in the linear projection for success it requires only a good choice of linear predictors, not extensive knowledge of the conditional expectations. The other embellishment is the substitution of missing-data estimators \( \tilde{T}_k \) for sample averages \( \bar{T}_k \) in the final estimator of the mean.

6 Effectiveness

Here we will offer some numerical examples which illustrate the dependence of variance reduction on the particular problem. The goal is to give some guidance about when the variance reduction technique is effective and what kind of missing data predictors and fictitious cashflows are practical. First, we discuss more abstractly the characteristics of a problem that make this method effective. Effectiveness consists in achieving a low ratio \( R \) of optimal variance to standard variance as given in equation (8), possibly after introducing fictitious cashflows or missing data techniques.

The infimum of this ratio \( R \) over all problems with \( m \) steps is \( 1/m \), as \( n_1 \to C \) and \( n_k \to 0 \) for \( k > 1 \). This limit would apply if almost all the variance were associated with the first step, in which case almost all resources are optimally allocated to the first step. For instance, if we consider a sequence of problems with identical cashflows and a discount rate going to infinity, steps after the first become negligible compared to the first. With a sufficiently high discount rate, the ratio
$R$ can be made arbitrarily close to $1/m$ for any $X_1, \ldots, X_m$. In the examples we have tried, the variance reduction achieved is modest, but the method is easy to apply.

However, suppose that the number of steps $m$ is not small, and the first step is not of overwhelming importance. Then $m$ is not very important at all. The degree of variance reduction is determined primarily by the shape of the piecewise linear function $V^{(m)}$. A sequence of problems whose size $m$ increases and whose cost-variance shape converges also has convergence of resource allocation shape and variance objective value. For a precise statement and proof of a theorem, see Staum (2001).

In the numerical examples that follow, for the sake of transparency, each step has unit computational cost $c_k = 1$ and the budget is $m$. Then standard simulation has resource allocation $n_k = 1$ for each step, although of course any reasonable budget would be much larger. The results use the optimal $n_k$ computed from the true covariance matrix, ignoring deviation due to pilot estimation of the covariance matrix and rounding the $n_k$.

We also measure costs solely in terms of the numbers of steps $n_k$, disregarding any extra costs incurred in computing predictors or fictitious cashflows from the state vector and true cashflows. Assessment of such costs, while of importance in practice, would be dependent on implementation of numerical algorithms, which is unrelated to the variance reduction method under consideration. For instance, in the example of Section 6.2, simulating the state vector involves generating a normal random variable and taking its exponential, while computing the fictitious cashflows requires two evaluations of the standard normal cumulative distribution function. These costs depend on the desired accuracy of the approximate algorithms.

### 6.1 Mortgage-Backed Security

In this application, the simulation values a mortgage-backed security (MBS), a financial security whose cashflows are the total payments made on a pool containing a large number of mortgages. The difficulty of pricing this security arises from the possibility of prepayment: the mortgages include an option for the homeowner to prepay at any date the balance of the principal and cease making payments thereafter. Prepayments increase when interest rates are low and homeowners have an incentive to refinance their mortgages at more favorable rates.

The prepayment model is based on Richard and Roll (1989), who provide further background on mortgage-backed securities. We assume that the pool is divided into individual mortgages of equal and negligible size. Of mortgages whose rate is $y$, the fraction that prepay in a time interval of length $\Delta t$ when the refinancing rate is $y'$ is illustrated in Figure 2. The instantaneous interest rate $r_t$ obeys the Vasicek model, following the stochastic differential equation $dr_t = \phi(\bar{r} - r_t)dt + \sigma dW_t$
with volatility $\sigma = 1\%$, long-term average interest rate $\bar{r} = 8\%$, and mean reversion strength $\phi = 2\%$. The initial interest rate is $r_0 = 6.5\%$ and the mortgages are 30 years long. The refinancing rate is always 2\% above the yield of a 30-year risk-free bond.

The parameters are such that the expected fraction of mortgages surviving to term is about 0.11 and the price of a 30-year zero-coupon bond is about 17 cents for a dollar of face value. These two quantities reflect the insignificance of the last step relative to the first: at the end of the MBS's life, there are fewer mortgages still in the pool, and payments have a lesser present value.

Figure 3 graphs the variance components $v$ of the MBS's discounted cashflows. As expected, there is very little variance at later steps. The graph also reveals a characteristic feature of MBSs: negative covariance between cashflows at early and late steps. Consider a single mortgage. An unusually large cashflow (the principal balance) occurs when it prepay, and afterwards cashflows are unusually low (zero). This accounts for the presence of negative variance components at steps $k \leq 5$. At these steps, the negative covariance of $X_k$ with the sum of future discounted cashflows outweighs the positive variance of $X_k$ in equation (4).

This feature is unfavorable for variance reduction by resource allocation among time steps. Theorem 5 implies that there is greater variance reduction when early variance components $v_k$ are large, not negative. Figure 4 shows that, in this example, the optimal resource allocation $n$ does not decrease until step 227, or almost 19 years out of 30. With these optimal $n$, variance is reduced to 84.0\% of standard simulation.

It is possible to reduce the impact of negative covariance by using the missing data technique. At step $k$, we have both a discount factor and the number of remaining mortgages at step $k+1$. The discount factor for step $k+1$ is the product of the stochastic discount realized up to step $k$ and a one-period bond price, known in closed form for this model. The number of mortgages left at $k+1$ is the product of the number left at $k$ and the prepayment ratio based on the history up to $k$. We take the predictor $T_k$ to be the product of this discount factor and number remaining at step $k+1$. The intuition is that, ignoring the variability of the nominal cashflow per mortgage, the discounted cashflow is proportional to both the discount factor and the number of mortgages. For this reason, we estimate a reduced model with $\beta_{hk} = 0$ for $h < k-1$.

Applying the missing data technique yields variance components with a smaller initial negative dip. Their partial sums $V'$ are in Figure 5, which shows that $V'$ is larger than $V$ after one year, often much larger. Then Theorem 5 suggests that the variance of the estimator using the missing data technique will be better than that without. Indeed the variance is now only 50.5\% compared to standard estimation. The new optimal resource allocation $n'$ in Figure 4 has more resources devoted to the earliest steps and less at later steps. The optimal allocation stops early more aggressively.
when using the cashflows after adjustment using the missing data technique than for the actual cashflows, because the former bring more of the total variance to early steps in the simulation, which is the recipe for success.

6.2 Asian Option

An Asian derivative is a financial security whose payoff depends on the average of an underlying price over time. There are $m$ averaging dates $t_1, \ldots, t_m$, and the arithmetic average of the underlying price $S$ up to step $k$ is $A_k = \frac{1}{k} \sum_{j=1}^{k} S_j$. This example is an Asian call, with payoff $(A_m - K)^+$. The underlying price obeys the Black-Scholes lognormal model, following the stochastic differential equation $dS_t = S_t \left( r dt + \sigma dW_t \right)$ under a risk-neutral probability measure. The difficulty of pricing the Asian derivative is that while the geometric average of jointly lognormal random variables is lognormal, there is no convenient expression for the distribution of the arithmetic average.

This example has $m = 5$ averaging dates which are the last five days in the option’s one-year life. The constant interest rate $r = 6.5\%$ and the volatility $\sigma = 20\%$. The strike price $K$ and initial underlying price $S_0$ are both 100.

The simulation has $m$ steps because we need to generate a price at each averaging date, but the only cashflow occurs at the terminal date. Therefore resource allocation applied directly produces no benefit. However, it is usable in combination with fictitious cashflows or the missing data approach. The intuition is that early steps in the simulation are more important because early prices both appear directly in the average and influence later prices.

To design fictitious cashflows, we rely on our knowledge about the distribution of a geometric average of lognormal prices. There is a formula $f(k, S_k, G_k)$ which gives the value at time $t_k$ of a call on the geometric average, given the price $S_k$ and geometric average price to date $G_k$ at step $k$. That is, $f(k, S_k, G_k) = e^{-r(t_m-t_k)}E[(G_m - K)^+|S_k, G_k]$, and this price is given by the Black-Scholes call pricing formula, but incorporating the parameters of the distribution not of $S_m$ but of $G_m$, which Curran (1994, §2.2) provides. Our approximation for the step-$k$ value of the arithmetic Asian call is $f(k, S_k, A_k)$. This approximation is exactly correct at step $m$. The fictitious cashflows are $X_k = e^{-r t_k} f(k, S_k, A_k) - e^{-r t_{k-1}} f(k - 1, S_{k-1}, A_{k-1})$.

These produce the cumulative sums $V$ of variance components in Figure 6. This curve coincides with its convex hull. The resulting optimal allocation $n$ in Figure 7 reflects this absence of binding monotonicity constraint: $n$ decreases at each step. However, the variance reduction is to 26.2%, close to the best possible reduction, which is 20%, as discussed at the beginning of this section.

The purpose of the Asian feature in this option might be to smooth the price used in computing
the payoff, diluting the effect of possible large short-term deviations. We also considered an Asian option with \( m = 5 \) averaging dates spaced equally over a year. The averaging feature makes such an option useful, for instance, for limiting the risk of a company which plans to make regular purchases of a commodity whose price underlies the option payoff. For this Asian option with equal spacing, the results are not as good: variance reduction only to 87.6% using fictitious cashflows. The first Asian option was more favorable because almost all the information about the payoff is contained in \( S_1 \), since it has nearly a year of variability in it, while the later steps take only days.

7 Conclusion

We consider multi-step simulations to estimate the mean of some random variable \( X \). The random variables generated at different time steps of a simulation make different contributions to the variance of \( X \). We pose and solve a resource allocation problem that reduces estimator variance by allowing some paths to stop early, thus allocating greater resources to earlier time steps. Further variance reduction can be achieved by decomposing \( X \) into fictitious cashflows; we find the optimal such decomposition.

Examples show that this method is, unsurprisingly, most effective when early steps are intuitively more important than later steps. One way this may happen is when discounting reduces the present value of distant cashflows. Another way is when the near future has more variability than the conditional variability of the distant future given the near future. This resource allocation will yield variance reduction only if the problem has structure of some such sort.

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8 Proofs

**Proof of Lemma 1.**

\[
\text{Var}[\hat{\mu}] = \text{Var} \left[ \sum_{k=1}^{n_k} \frac{X_{ik}}{n_k} \right] = \sum_{i=1}^{m} \text{Var} \left[ \sum_{k=1}^{n_i} \frac{X_{ik}}{n_k} \right] = \sum_{i=1}^{m} \sum_{k=1}^{n_i} \sum_{\ell=1}^{n_i} \frac{\sigma_{k\ell}}{n_k n_{\ell}}
\]

\[
= \sum_{k=1}^{m} \sum_{\ell=1}^{m} \frac{\sigma_{k\ell}}{n_k n_{\ell}} = \sum_{k=1}^{m} \sum_{\ell=1}^{m} \frac{\sigma_{k\ell}}{n_k n_{\ell}} = \sum_{k=1}^{m} \frac{1}{n_k} \left( \sigma_{kk} + \sum_{\ell=k+1}^{m} \sigma_{k\ell} \right)
\]

**Proof of Theorem 1.**

First, see that the stated solution (6) is primal-feasible. It clearly satisfies the budget constraint

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(2). The upper convex hull is a concave function, so the slopes \( u^*_k \) of its segments are nonincreasing. These segments’ endpoints are extreme points of the original graph \( V \), so in particular there is a \( k < m \) such that 
\[
 u^*_m = (V_m - V_k)/(C_m - C_k).
\]
Assuming that costs are positive, \( C_m - C_k > 0 \). Also 
\[
 V_m - V_k = \sum_{t=k+1}^m v_t = \text{Var}[\sum_{t=k+1}^m X_t] > 0.
\]
Therefore \( u^*_m > 0 \), and since these are nonincreasing, all are positive. Consequently, all \( n_k \) are positive and nonincreasing, satisfying the monotonicity constraint (1).

Next, show that the solution is dual-feasible. The Lagrangian is
\[
\mathcal{L} = \sum_{k=1}^{m} \frac{v_k}{n_k} - \nu \left( C - \sum_{k=1}^{m} \alpha_k n_k \right) - \sum_{k=1}^{m-1} \lambda_k (n_k - n_{k+1}).
\]
Differentiating with respect to \( n_j \), the first-order conditions are, for \( j = 1, \ldots, m \),
\[
\lambda_{j-1} = \lambda_j + v_j/n_j^2 - \nu e_j
\]
where \( \lambda_0 \) denotes zero.

Dual-feasibility is the existence of nonnegative Lagrange multipliers \( \lambda_j \) and \( \nu \) which satisfy these first-order equations and the complementary slackness conditions \( \lambda_j (n_j - n_{j+1}) = 0 \). Let \( \nu \) be as given in equation (7), which is positive, and let
\[
\lambda_j = (V_j^* - V_j)/n_j^2
\]
which is nonnegative by definition of the upper convex hull. So \( j \) is the index of an extreme point iff \( \lambda_j = 0 \), and is not iff \( \lambda_j > 0 \). Therefore \( \lambda_j \neq 0 \) implies that \( j \) is not the index of an extreme point, and because \( V^* \) is linear between extreme points, the slopes \( u^*_{j+1} = u^*_j \), hence \( n_j - n_{j+1} = 0 \). Thus complementary slackness is satisfied.

Define \( \ell(j) \) as the index of the first extreme point to the right of \((C_j, V_j)\):
\[
\ell(j) \triangleq \min \{ k > j | V_k^* = V_k \}.
\]
As mentioned above, \( V_m^* = V_m \) so this is well-defined for \( j = 1, \ldots, m - 1 \). Then
\[
\lambda_j n_j^2 = V_j^* - V_j = (V_{\ell(j)} - V_j) - \left(V_{\ell(j)}^* - V_j^*\right) = \sum_{k=j+1}^{\ell(j)} (v_k - v_k^*).
\]
Again, because \( V^* \) is linear between extreme points, \( n_k \) is constant for \( k = j + 1, \ldots, \ell(j) \). Also, from equation (6), \( n_k^2 = v_k^* / \nu \), so
\[
\lambda_j = \sum_{k=j+1}^{\ell(j)} \frac{v_k - v_k^*}{n_k^2} = \sum_{k=j+1}^{\ell(j)} \left(\frac{v_k}{n_k} - \nu \right).$$
Suppose \( j \) is not the index of an extreme point. Then \( \ell(j - 1) = \ell(j) \) and we see directly from this that the first-order condition is satisfied. If \( j \) is the index of an extreme point, then \( \ell(j - 1) = j \) and \( \lambda_j = 0 \), so \( \lambda_{j-1} = 0 + v_j/n_j^2 = \nu_j \) and the first-order condition is satisfied.

Finally, there is the matter of the variance reduction ratio. Under standard simulation,

\[
n_k = \frac{C}{\sum_{i=1}^{m} c_i} \quad \text{and} \quad \text{Var} = \frac{\sum_{k=1}^{m} v_k/n_k = \left( \sum_{i=1}^{m} c_i \right) \left( \sum_{k=1}^{m} v_k \right)}{C} / C.
\]

Using the optimal solution,

\[
n_k = \frac{C \sqrt{u_k}}{\sum_{i=1}^{m} c_i \sqrt{u_i}} \quad \text{and} \quad \text{Var} = \frac{\left( \sum_{j=1}^{m} c_j \sqrt{u_j} \right) \left( \sum_{k=1}^{m} c_k u_k \right) / \left( \sum_{j=1}^{m} c_j \right) \left( \sum_{i=1}^{m} v_i \right)}{C}
\]

so, changing indices and substituting for slopes \( u = v/c \), the ratio of variances is

\[
\frac{\left( \sum_{j=1}^{m} c_j \sqrt{u_j} \right) \left( \sum_{i=1}^{m} v_i / \sqrt{u_i} \right)}{\left( \sum_{j=1}^{m} c_j \right) \left( \sum_{i=1}^{m} v_i \right)}
\]

which equals the formula given in equation (8).

**Proof of Lemma 2.**

First we show that \( \sum_{k=1}^{m} \hat{\mu}_k \) can indeed be written in the form given by equations (12–15). By substituting for \( \hat{\tau}_h \) in the recursive definition (11), we see that \( \hat{\tau}_k \) is a linear combination of averages \( \bar{T} \), where the weights in the linear combination involve estimated regression coefficients \( \hat{b} \). As remarked in Section 2, the average indexed as \( \bar{T}_{hi} \) must have \( h \leq i \) because it is the average of \( T_h \) on paths \( i, \ldots, n_i \) so we must have \( n_h \geq n_i \) for \( T_h \) to be observed on all these paths. Also, if \( \bar{T}_{hi} \) is to feature in \( \hat{\tau}_k \), then we must have \( i \leq k \) so that \( \bar{T}_{hi} \) is based on at least as many paths as \( \bar{T}_{ik} \), the obvious estimate of \( \tau_k \), and thus can be of use in correcting it. Having established that the random variable \( T_{ik} \) does not appear in \( \hat{\tau}_h \) for \( h < k \), by inspection, \( \omega_{kk} = 1 \) is the coefficient of the average \( \bar{T}_{kk} \) in \( \hat{\tau}_k \) and \( \omega_{kk} = \hat{b}_{kk} \) is the coefficient of the average \( \bar{T}_{kk} \) in \( \hat{\tau}_k \). For \( i < k \), the random variable \( T_{hi} \) does not appear directly in \( \hat{\tau}_k \), only through its appearance in \( \hat{\tau}_j \) for \( j = i, \ldots, k - 1 \). Therefore we can write

\[
\hat{\tau}_k = \sum_{i=1}^{k} \sum_{h=1}^{i} \omega_{hik} \bar{T}_{hi}
\]

following the definition of equation (13).

By substituting for \( \hat{\tau}_h \) in definition (10) and repeating the reasoning of the previous paragraph,

\[
\hat{\mu}_k = \hat{X}_{kk} + \sum_{i=1}^{k} \sum_{h=1}^{i} \omega_{hik} \bar{T}_{hi}
\]
following the definition of equation (14). Whereas $\omega_{kk}k = 1$, $w_{kk}k = 0$, because $X_{kk}$ appears in $\mu_k$ where $T_{kk}$ appears in $\hat{\tau}_k$. Then

$$\sum_{k=1}^{m} \mu_k = \sum_{k=1}^{m} \left( \tilde{X}_{kk} + \sum_{i=1}^{k} \sum_{h=1}^{i} w_{hik} \tilde{T}_{hi} \right) = \sum_{k=1}^{m} \tilde{X}_{kk} + \sum_{i=1}^{m} \sum_{h=1}^{i} \left( \tilde{T}_{hi} + \sum_{k=1}^{m} w_{hik} \right)$$

$$= \sum_{k=1}^{m} \tilde{X}_{kk} + \sum_{i=1}^{m} \sum_{h=1}^{i} \tilde{T}_{hi} w_{hi} = \sum_{k=1}^{m} \left( \tilde{X}_{kk} + \sum_{h=1}^{m} \tilde{T}_{hh} w_{hk} \right)$$

using equation (15), which says that $w_{hi} = \sum_{k=i}^{m} w_{hik}$ and is zero if $h > i$. Continuing,

$$\sum_{k=1}^{m} \tilde{\mu}_k = \sum_{k=1}^{m} \frac{1}{n_k} \sum_{i=1}^{n_k} \left( X_{ik} + \sum_{h=1}^{m} T_{ih} w_{hk} \right) = \sum_{k=1}^{m} \frac{1}{n_k} \sum_{i=1}^{n_k} X'_{ik}.$$ 

Next, we show that $W1 = 0$ by virtue of the recursive definitions (13)-(15). Recall that $\omega_{hk}$ and $w_{hik}$ are zero unless $h = i \leq k$. We begin with a proof by induction on $k$ that for any $k > h$, $\sum_{i=1}^{k} \omega_{hi} = 0$.

$$\sum_{i=1}^{k} \omega_{hi} = \omega_{kk} + \sum_{i=h}^{k-1} \sum_{j=i}^{k-1} b_{jk} \omega_{hij} = -b_{hk} + \sum_{j=h}^{k-1} b_{jk} \sum_{i=h}^{j} \omega_{hi}$$

$$= -b_{hk} + b_{hk} \omega_{hh} + \sum_{j=h+1}^{k-1} b_{jk} \sum_{i=h}^{j} \omega_{hi} = -b_{hk} + b_{hk} + \sum_{j=h+1}^{k-1} b_{jk} 0 = 0$$

where $\sum_{i=h}^{j} \omega_{hi} = 0$ is justified by inductive hypothesis, since $h < j < k$.

To complete the proof that $W1 = 0$, we must establish that for any $k \geq h$, $\sum_{i=1}^{k} w_{hik} = 0$. If $k = h$, this is $w_{hh} = 0$. Otherwise,

$$\sum_{i=1}^{k} w_{hik} = w_{hkk} + \sum_{i=h}^{k-1} \sum_{j=i}^{k-1} \beta_{jk} w_{hij} = -\beta_{hk} + \sum_{j=h}^{k-1} \beta_{jk} \sum_{i=h}^{j} \omega_{hi}$$

$$= -\beta_{hk} + \beta_{hk} \omega_{hh} + \sum_{j=h+1}^{k-1} \beta_{jk} \sum_{i=h}^{j} \omega_{hi} = -\beta_{hk} + \beta_{hk} + \sum_{j=h+1}^{k-1} \beta_{jk} 0 = 0.$$

Consequently, for any $h$,

$$\sum_{i=h}^{m} w_{hi} = \sum_{i=h}^{m} \sum_{k=h}^{m} w_{hik} = \sum_{k=h}^{m} \sum_{i=h}^{m} w_{hik} = \sum_{k=h}^{m} 0 = 0.$$

**Proof of Theorem 2.**

The expectation is

$$\mathbb{E} \left[ \sum_{i=1}^{m} \sum_{k=1}^{n_k} \frac{X'_{ik}}{n_k} \right] = \mathbb{E} \left[ \sum_{k=1}^{m} X'_k \right].$$
But
\[
\sum_{k=1}^{m} X'_k = \sum_{k=1}^{m} \left( X_k + \sum_{h=1}^{m} T_h w_{hk} \right) \\
= \sum_{k=1}^{m} X_k + \sum_{h=1}^{m} T_h \sum_{k=1}^{m} w_{hk} \\
= \sum_{k=1}^{m} X_k + \sum_{h=1}^{m} T_h 0 = \sum_{k=1}^{m} X_k
\]
because \( W1 = 0 \) so for each \( h, \sum_{k=1}^{m} w_{hk} = 0 \). By definition, \( \mathbf{E}[\sum_{k=1}^{m} X_k] = \mu \).

Proof of Theorem 3.

We have \( X'_{ik} = X_{ik} + \sum_{h=1}^{m} T_h w_{hk} = X_{ik} + \sum_{h=1}^{m} T_h w_{hk} \). When \( i \) and \( j \) index distinct paths, \( X'_{ik} \) and \( X'_{jk} \) are independent. Also, \( X'_{ik} \) does not involve the decision variables \( n \). Therefore from \( \sum_{k=1}^{m} \sum_{i=1}^{n_k} X'_{ik}/n_k = \sum_{i=1}^{n} \sum_{k=1}^{m} X'_{ik}/n_k \) the proof of Lemma 1 applies, with \( X' \) replacing \( X \).

Similarly, using the \( v'_k \) instead of \( v_k \), Theorem 1 gives the optimal \( n_k \) for this estimator.

Rewrite from equation (12)
\[
\hat{\mu} = \sum_{i=1}^{n_k} \frac{1}{n_k} X'_{ik} = \sum_{k=1}^{m} \hat{v}_k \quad \text{where} \quad \hat{v}_k = \frac{1}{n_k - n_{k+1}} \sum_{i=n_k+1}^{n_k} \sum_{j=1}^{k} X'_{ij} \left( \frac{n_k - n_{k+1}}{n_j} \right)
\]

As the computational budget \( C \) goes to infinity, each \( n_k - n_{k+1} \) goes to infinity, and \( (n_k - n_{k+1})/n_j \) approaches some finite limit \( p_{jk} \) such that \( \sum_{k=1}^{m} p_{jk} = 1 \) for each \( j \). Because \( \sum_{j=1}^{k} p_{jk}X'_j \) has finite variance, the Lindeberg central limit theorem, for which see, e.g., Billingsley (1995, p. 359), implies that for each \( k \), the distribution of
\[
\frac{\hat{v}_k - \mathbf{E} \left[ \sum_{j=1}^{k} p_{jk}X'_j \right]}{\sqrt{\mathbf{Var} \left[ \sum_{j=1}^{k} p_{jk}X'_j \right] / (n_k - n_{k+1})}}
\]
converges to standard normal as \( C \to \infty \). The \( \hat{v}_k \) are independent because they are taken over different sample paths, and they sum to \( \hat{\mu} \). The sum of the expectations in (18) is
\[
\sum_{k=1}^{m} \sum_{j=1}^{k} p_{jk} \mathbf{E}[X'_j] = \sum_{j=1}^{m} \mathbf{E}[X'_j] \sum_{k=j}^{m} p_{jk} = \sum_{j=1}^{m} \mathbf{E}[X'_j] = \mu
\]
and the sum of the variances in (18) is
\[
\sum_{k=1}^{m} \frac{1}{n_k - n_{k+1}} \mathbf{Var} \left[ \sum_{j=1}^{k} p_{jk}X'_j \right] = \sum_{k=1}^{m} \frac{1}{n_k - n_{k+1}} \sum_{i=1}^{k} p_{ik}p_{jk} \sigma'_{ij} \\
= \sum_{i=1}^{m} \sum_{j=1}^{m} \sigma'_{ij} \sum_{k=n_k \{i,j\}}^{m} \left( \frac{p_{ik}p_{jk}}{n_k - n_{k+1}} \right)
\]
From the proof of Lemma 3, with a change of indices, we have
\[
\varsigma^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} \left( \frac{\sigma_{ij}}{\max\{n_i, n_j\}} \right) = \sum_{i=1}^{m} \sum_{j=1}^{m} \sigma_{ij} \sum_{k=\max\{i,j\}}^{m} \left( \frac{n_k - n_{k+1}}{n_i n_j} \right)
\]
The ratios
\[
\left( \frac{p_{ijk} p_{jk}}{n_k - n_{k+1}} \right) / \left( \frac{n_k - n_{k+1}}{n_i n_j} \right)
\]
all converge to 1 as \( C \to \infty \), so the ratio of (19) to \( \varsigma^2 \) also converges to 1. Therefore the distribution of \((\hat{\mu} - \mu)/\varsigma \) converges to standard normal as \( C \to \infty \).

**Proof of Theorem 4.**

While \( \varsigma \) is the standard deviation given by equation (16), let \( \varsigma^2 \) be that produced by substituting \( \mathbf{W}^2 \) in equation (16). The assumption of consistency says that \( w_{hi} - w_{hi}^2 \) converges in probability to 0 for each \( h, i \). Because of this convergence in probability, the random variable \( \varsigma \) converges in probability to \( \varsigma^2 \). Put differently, \( \varsigma^2/\varsigma \) converges in probability to one.

Also let \( \hat{\mu}^2 \) be like \( \hat{\mu} \), but with the constant matrix \( \mathbf{W}^2 \) substituted for the random matrix \( \mathbf{W} \). Then
\[
\hat{\mu} - \hat{\mu}^2 = \sum_{k=1}^{m} \frac{1}{n_k} \sum_{i=1}^{m} \sum_{h=1}^{k} T_{ih}(w_{hk} - w_{hk}^2)
\]
so to prove that \( \hat{\mu} - \hat{\mu}^2 \) converges in probability to zero, it suffices to show that for all \( h \) and \( k \),
\[ T_{ih}(w_{hk} - w_{hk}^2) \]
converges in probability to zero.

We already know that for any positive \( \delta \) and \( \epsilon \), there is some \( C(\delta, \epsilon) \) such that if the budget \( C > C(\delta, \epsilon) \), then \( \mathbf{P}[|w_{hk} - w_{hk}^2| > \epsilon] < \delta \) for each \( h, k \). We must next show this is also true for \( |T_{ih}(w_{hk} - w_{hk}^2)| \). Let \( F \) be the cumulative distribution function of \( \max_h |T_{ih}| \). The event
\[
\{|T_{ih}(w_{hk} - w_{hk}^2)| > \epsilon\} \subset \{|T_{ih}| > F^{-1}(\delta/2)\} \cup \{|w_{hk} - w_{hk}^2| > \epsilon/F^{-1}(\delta/2)\}
\]
Then for \( C > C(\delta/2, \epsilon/F^{-1}(\delta/2)) \),
\[
\mathbf{P}[|T_{ih}(w_{hk} - w_{hk}^2)| > \epsilon] < \mathbf{P}[|T_{ih}| > F^{-1}(\delta/2)] + \mathbf{P}[|w_{hk} - w_{hk}^2| > \epsilon/F^{-1}(\delta/2)]
\]
\[
< \delta/2 + \delta/2 = \delta
\]
i.e. \( |T_{ih}(w_{hk} - w_{hk}^2)| \) converges in probability to zero. This proof works because there are a finite number of \( T_{ih} \), each of which is finite with probability one.

It is established that \( \hat{\mu} - \hat{\mu}^2 \) converges in probability to zero, and since \( \varsigma^2 \) is positive, \( (\hat{\mu} - \mu)/\varsigma \) - \( (\hat{\mu}^2 - \mu)/\varsigma^2 \) also converges in probability to zero. Using the convergence in probability of \( \varsigma/\varsigma^2 \) to one, finally we see \( (\hat{\mu} - \mu)/\varsigma \) - \( (\hat{\mu}^2 - \mu)/\varsigma^2 \) converges in probability to zero. Since \( \mathbf{W}^2 \) is
constant, by Theorem 3, \((\bar{\mu} - \mu)/\kappa\) converges in distribution to standard normal, so 
\((\bar{\mu} - \mu)/\kappa\) does as well.

**Proof of Theorem 5.**

Suppose that sequences \(X'_1, \ldots, X'_m\) and \(X''_1, \ldots, X''_m\) in \(\mathcal{X}\) satisfy \(V'_k \geq V''_k\) for all \(k = 0, \ldots, m\). One is tempted to say that this condition on the partial sums means that \(v'\) majorizes \(v\). However, this is not quite so, for the definition of majorization involves partial sums of terms placed in decreasing order. Majorization is a relation between sets, not vectors; see, for instance, Marshall and Olkin (1979, pp. 7, 12). Here the sequences retain their original order.

Let \(n'\) and \(n''\) be the optimal resource allocations for the problems using respectively \(X'\) and \(X''\). Define the function

\[
f(x_1, \ldots, x_m) = \sum_{k=1}^{m} \frac{x_k}{n_k'}
\]

The derivative of \(f\) with respect to its \(k\)th argument is \(1/n_k''\). This is nondecreasing in \(k\) because the \(n_k''\) are nonincreasing, due to the monotonicity constraint (1). Because we can move from \(v'\) to \(v''\) by subtracting from early components and adding equal amounts to later components, \(f(v'_1, \ldots, v'_m) \leq f(v''_1, \ldots, v''_m)\). This informal argument parallels Theorem A.3 of Marshall and Olkin (1979), but without the restriction that components be decreasing. Then

\[
\sum_{k=1}^{m} \frac{v'_k}{n_k'} \leq \sum_{k=1}^{m} \frac{v''_k}{n_k''} \leq \sum_{k=1}^{m} \frac{v''_k}{n_k''}
\]

where the first inequality is true because \(n'\) minimizes the objective based on \(v'\) and the second inequality is the result about \(f\) just established. Consequently, the sequence \(X'_1, \ldots, X'_m\) produces the resource allocation problem whose optimal objective is least if for any fictitious cashflow \(X''_1, \ldots, X''_m\), \(V'_k \geq V''_k\) for each \(k\).

Next,

\[
V'_k = V'_m - \sum_{\ell=k+1}^{m} v'_\ell = V'_m - \text{Var}\left[\sum_{\ell=k+1}^{m} X'_\ell\right].
\]

Also, since the sequence \(X'_1, \ldots, X'_m\) is in \(\mathcal{X}\), \(\sum_{k=1}^{m} X'_k = X\), so \(\sum_{\ell=k+1}^{m} X'_\ell = X - \sum_{j=1}^{k} X'_j\), and \(\sum_{j=1}^{k} X'_j\) must be \(\mathcal{F}_k\)-measurable. Thus each \(V'_k\) is maximized by minimizing \(\text{Var}[X - Y]\) over \(\mathcal{F}_k\)-measurable random variables \(Y\). As is well known, it is \(E[X|\mathcal{F}_k]\) which minimizes this residual variance; see, for instance, Williams (1991, §9.4).

### A The Algorithm

1. Construct a doubly linked list of points which are candidate extreme points of the upper convex hull in that they are above the line connecting the endpoints:

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(a) Let \( s = V_m/C_m, \ k = m, \) and CARRY=START, a pointer to a newly created blank node.

(b) If \( V_k \geq sC_k, \) create a node with COST=\( C_k, \ \) VALUE=\( V_k, \ \) PREV=CARRY, NEXT=NULL, and let both PREV\( \rightarrow \) NEXT and CARRY be pointers to this node.

(c) Decrease \( k \) by 1. If \( k \geq 0, \) go to step 1b.

2. Scan the list and eliminate points which are not extreme:

(a) Let the pointers K=START\( \rightarrow \) NEXT, J=K\( \rightarrow \) NEXT, and I=J\( \rightarrow \) NEXT.

(b) Compute \( \Delta = V_J(C_K - C_I) - (V_I(C_K - C_J) + V_K(C_J - C_I)). \)

(c) If \( \Delta \geq 0, \) advance the scan: K=J, J=I, I=I\( \rightarrow \) NEXT.

Otherwise, delete node J and back up: J=K, K=K\( \rightarrow \) PREV.

(d) If I\( \neq \)NULL, go to step 2b.

3. Produce \( n \) from the extreme points:

(a) Initialize \( k = m, \ \) K=START\( \rightarrow \) NEXT, J=K\( \rightarrow \) NEXT, and \( u_k = (V_K - V_J)/(C_K - C_J). \)

(b) If \( k \leq J, \) let K=J and J=J\( \rightarrow \) NEXT, and recompute \( u_k. \)

(c) Assign \( v_k = v_k^*c_k \) and decrease \( k \) by 1. If \( k > 0, \) repeat step 3b.

(d) Compute \( n \) from \( v^* \) as in equation (6).

Preparata and Shamos (1985, §3.3.2) provide a proof that the Graham scan does produce the convex hull. Our algorithm departs from the standard Graham scan in three ways:

- It does not include a sort, because the points of the graph \( \mathcal{V} \) are already sorted.

- In step 1b, it discards points below the line connecting \( (0,0) \) and \( (C_m, V_m), \) which can not form part of the upper convex hull. In step 2d, the algorithm stops loop 2 when it reaches \( (0,0) \) and has completed the upper convex hull instead of going on to find the lower convex hull too.

- This means that \( (0,0) \) must always be included in the output of loop 2. We already know \( (0,0) \) and \( (C_m, V_m) \) are the leftmost and rightmost points and must be in the upper convex hull. The algorithm also ensures that \( (C_m, V_m) \) is in the output. In the first iteration of step 1b, the algorithm sets START\( \rightarrow \) PREV=PREV. Then in step 2b if K=START and \( \Delta < 0, \) the result of backing up is to leave K=START and J=START. The next time step 2b executes, \( \Delta = 0 \) and the scan will advance. The result is that the START node representing \( (C_m, V_m) \) can never be deleted.
References


Figure 1: Illustration of Convex Hull Solution
Figure 2: Annual Prepayment Fraction
Figure 3: MBS Example: Variance Components
Figure 4: MBS Example: Resource Allocation
Figure 5: MBS Example: Cumulative Sums
Figure 6: Asian Option Example: Cumulative Sums
Figure 7: Asian Option Example: Resource Allocation