Eliciting Coordination with Rebates

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Abstract. This article considers network routing games, which can readily be used to model competition in telecommunication, traffic, transit or distribution networks. We study a mechanism based on rebates that provides incentives for participants to cooperate. This mechanism is modeled by a Stackelberg game in which the system owner offers rebates, and participants select their routes considering the rebates. The system owner decides how much to offer taking into account both the potential participants’ cost reduction as well as the cost of providing those rebates. Indeed, the rebate budget may come from the savings that arise from the more efficient solution. We characterize the Stackelberg equilibria of the game, and describe a polynomial-time algorithm to compute the optimal rebates. In addition, we provide tight results on their worst-case inefficiency measured by the so-called price of anarchy. Specifically, we describe the tradeoff between the sensitivity of the owner towards rebate costs and the worst-case inefficiency of the system.

Keywords. Network Pricing, Subsidies as Incentives, Wardrop Equilibrium, Stackelberg Games, Price of Anarchy.

1. Introduction

It is well known that because of a misalignment of incentives, an equilibrium solution of a competitive situation does not necessarily maximize social welfare. The most common example is arguably the Prisoner’s Dilemma (Fudenberg and Tirole 1991). In this article, we will concentrate on games in which participants need to select a subset of resources that form a path in a network. The cost function associated to each resource comes from from congestion and cost-demand curves—which model that if many consumers are interested in one resource, demand will drive the price up. We assume that there are a large number of consumers that do not have enough market power to influence prices by themselves (i.e., they are price-taking). This situation was first described by Wardrop (1952) in the context of transportation networks, and the corresponding equilibrium concept has been called a Wardrop equilibrium (under mild conditions this definition coincides with that of a Nash equilibrium).

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Although in some cases a system may be better off without a coordination mechanism because the overall cost of a coordination mechanism may outweigh its benefits, equilibria have been found to be too inefficient in many applications of interest. This makes it necessary to coordinate participants to mitigate the adverse effects of the misalignment of incentives. As imposing decisions to users is not an option in most real-world situations, equilibria can be improved by system (re)design (Roughgarden 2006), by considering routing part of the flow preemptively (Korilis, Lazar, and Orda 1995), or by using pricing mechanisms to create incentives (Bergendorff, Hearn, and Ramana 1997; Labbé, Marcotte, and Savard 1998). This article considers the third approach.

Economists such as Dupuit (1849), Pigou (1920), Knight (1924), and Beckmann, McGuire, and Winsten (1956) proposed long ago to use pricing so participants internalize the externalities, defined as the additional cost they impose to others. If implemented properly, this results in equilibria that are efficient from a social welfare perspective. For a complete treatment of network pricing and many additional references, see, e.g., the book by Yang and Huang (2005). In practice though, the strategy of adding charges—in the form of tolls, prices or taxes—to the most attractive resources may not be acceptable. For example, in the context of transportation this is evidenced by the very few successful implementations of road pricing like those of Singapore and London (Santos 2005; Santos and Fraser 2006). One of the concerns that opponents frequently raise comes from the perspective of social equity. After introducing a pricing mechanism, the better positioned (i.e., rich) participants will be able to use highly demanded resources because they can afford to pay the corresponding charge whereas the not-as-well positioned (i.e., poor) ones will be left with resources that nobody wants but somebody needs to select to be able to achieve a socially efficient solution. In other words, social efficiency is achieved by relegating the poor participants to the unpopular resources. This fact has been thoroughly discussed in previous articles and different measures to alleviate it were suggested (see, e.g., Starkie 1986; Button and Verhoef 1998).

We study the coordinating power that rebates can offer. Rebates are used in logistics, supply chain management, and marketing, with the objective of revenue maximization as well as to create incentives for coordination (Gerstner and Hess 1991; Ali, Jolson, and Darmon 1994; Taylor 2002; Chen, Li, and Simchi-Levi 2005). They are also used in public services, where they are normally referred to by subsidies, to generate incentives for a more efficient utilization of the available resources. The loss of performance arising from the lack of coordination in the system impacts the system owner—a manager, a government, or whoever is in charge of the system—who has a vested interest in the outcome.

For example, in a distribution system of a freight company, although business units could control different markets responding to individual objectives, the company that owns the system cares about the total profit. In relation to urban transportation networks, city authorities care about the population wasting time because of slow traffic or idling in congestion. Not only one could argue that this deteriorates quality of life, but it also represents lost productivity. Both have an impact on the revenue made by the city (e.g., less production implies less taxes paid). The utilization of subsidies to elicit coordination is preferred to pricing in most urban traffic/transit networks in the world as it is better perceived by the public. Indeed, in most cities commuters pay less than the fare associated to the cost of operating their public transport system, and the difference is covered by government
money. This encourages commuters to select public transportation although they may not normally prefer it at a higher price. Users with very high value-of-time are, nevertheless, indifferent to all this because they value the smaller commute time or the comfort of private transportation more than the savings they would get by switching to other modes. Overall, this option improves social welfare as public transportation users incur in less costs because of the subsidy, while the congestion experienced by the rest is reduced by having more users switch to public transportation.

The main question addressed by this article is what rebates should be offered by the system owner. As we argued, the system owner may be leaving money on the table because of not providing the correct incentives. Thus, it may consider using some money of its own to offer rebates for some resources with the hope that the effect outperforms the investment. The amount of money the system owner should invest in rebates depends on the value that is placed on the reduction of the total cost experienced by participants. Indeed, the more value the latter has, the bigger rebates should be. In the extreme case when the participants’ total cost is most important, the system owner can offer rebates that match the cost under a system optimum so participants perceive a cost equal to zero. On the other extreme, when the reduction in total cost is not that valuable to the system owner, there will be no rebates and the outcome will match the Wardrop equilibrium.

We characterize this tradeoff by considering a Stackelberg game in which the system owner is the leader and the participants the followers (von Stackelberg 1934). The leader offers rebates for each resource in a first stage, and participants select resources in a second stage, taking rebates into consideration. We study the Stackelberg game that is parameterized by the relative valuation of the investment in rebates and participants’ total perceived cost, denoted by $\rho$. We first prove that if the system owner values the perceived cost more than rebates ($\rho \leq 1$), then an optimal strategy for the leader is to refund each participant the perceived cost of each resource at a system optimum. On the other hand, when the system owner is more sensitive to the investment in rebates than to the perceived cost ($\rho > 1$), we establish that, for systems with substitutable resources, the proportion of participants that use resources with positive rebates is less than $1/\rho$.

1.1. **The Price of Anarchy.** In many applications, it is not possible to impose a particular equilibrium to participants, and hence it is possible that a bad one arises in practice. This observation led many authors to study the maximum efficiency-loss under an equilibrium, using social welfare as a measure of the quality of different solutions. Koutsoupias and Papadimitriou (1999) defined the *price of anarchy* as the largest possible ratio of the social cost at an equilibrium to the minimum attainable social cost (the term itself was coined by Papadimitriou 2001). Starting from the work of Roughgarden and Tardos (2002), the price of anarchy for network games with price-taking players (the setting suggested by Wardrop 1952) has been characterized by a series of articles that successively considered more general assumptions (Roughgarden 2003; Correa, Schulz, and Stier-Moses 2004; Chau and Sim 2003; Perakis 2004). It turns out that equilibria of these games are reasonably efficient; for example, when the cost of the resources increases linearly with the demand, the extra total cost of an equilibrium is not more than 33% of that of a system optimum. For other typical classes of functions, the inefficiency is somewhat larger but bounded. In practice, these inefficiencies may still be too high and even small improvements can represent a lot in the bottom
line. Hence, some researchers looked for bounds that represent real-world situations better (Friedman 2004; Correa, Schulz, and Stier-Moses 2005; Qiu, Yang, Zhang, and Shenker 2006; Schulz and Stier-Moses 2006), while others proposed ways to deal with the inefficiency. Some references that look at pricing mechanisms from the perspective of the price of anarchy are Koutsoupias (2004), Yang, Xu, and Heydecker (2005), Karakostas and Kolliopoulos (2005), Cole, Dodis, and Roughgarden (2006), Wichiensin, Bell, and Yang (2007) and Xiao, Yang, and Han (2007).

This topic has recently received ample attention from researchers in various communities like Computer Science, Economics, Operations Research, Operations Management, and Transportation. Consequently, there is a growing amount of interdisciplinary literature on the price of anarchy. For example, some additional references in the application domains of telecommunication, transportation, and distribution networks are the articles by Johari and Tsitsiklis (2004), Perakis and Roels (2006), Golany and Rothblum (2006), Weintraub, Johari, and Van Roy (2006), and Acemoglu and Ozdaglar (2007).

1.2. Main Contributions and Structure of the Paper. To our knowledge, this is the first article that formally studies the computation of optimal rebates with the goal of coordinating a congestion game. Our main contribution is a mechanism that provides incentives for coordination and that does not penalize participants, but instead rewards those that were worse-off without such a mechanism by offering them a rebate. In the particular case of affine cost functions and networks with substitutable resources, we characterize the Stackelberg equilibria of the proposed game—which provides us with the optimal rebate vector—and show that it is essentially unique. Moreover, we provide a polynomial-time algorithm that computes those rebates by selecting the arcs in which rebates should be offered. This allows us to then compute the actual value for the rebates.

The characterization of Stackelberg equilibria enables us to explicitly derive the social cost that it achieves, from where we compute the price of anarchy for the rebate mechanism as a function of the sensitivity of the system owner to the cost of offering rebates. From a technical point of view, this is achieved using a graphical decomposition argument that allows us to relate the social costs of a Stackelberg and a Wardrop equilibrium. The main conclusion is that when the sensitivity is low, rebates are cheap and the outcome has very low social cost, and when the sensitivity is high, rebates are expensive and the outcome is close to a Wardrop equilibrium. Our results hold for general networks when the previously mentioned sensitivity is low, and for parallel-link networks (substitutable resources) when the sensitivity is high.

This paper is organized as follows. Section 2 introduces the model and the performance measures we are interested in. The results for general network topologies are given in Section 3, while Section 4 focuses on instances with substitutable resources and characterizes the optimal rebates. In Section 5, we compute the price of anarchy for instances with affine cost functions. Finally, we offer some concluding remarks and open questions in Section 6.

2. Description of the Model

In this section, we introduce the model and its necessary notation. An instance is given by a network, cost functions, a system owner and participants. The most typical example of this model
is given by a urban network in which commuters (the participants) have the choice or driving their cars or taking public transportation (this choice is encoded by the network). Delays, waiting and tolls are encoded in cost functions. In this setting, the transportation authority (the system owner) wants to decide the level at which public transportation must be subsidized to offer an incentive so less commuters drive, hence reducing congestion and improving the welfare of the population.

Each participant is a price-taking individual who controls an infinitesimal amount of flow and whose goal is to select a route from its origin to its destination with minimum cost. To describe an example in the area of logistics, an instance could represent a freight company that manages different business units controlling different markets. The system owner would be the company while the participants would correspond to business units. The company wants to offer incentives to align business units into maximizing the company’s profits.

The network is represented by a directed graph \((V,A)\), where \(A\) is a set of arcs representing the resources and \(V\) is a set of vertices where resources connect to. When the total demand for a resource \(i \in A\) is \(x_i\), then the cost of selecting it is \(c_i(x_i)\). In practice this cost can represent one or more of the actual price paid, the delay incurred, the service level, etc. Functions \(c_i\) are referred to as \textit{cost functions} and are assumed nonnegative, nondecreasing, differentiable and convex (some of these assumptions can be relaxed for some results). Continuing with the example, arcs in the network represent different carriers that transport freight and facilities that process it; namely, sorting facilities, warehouses, flight legs, airports, ship routes, ports, canals, etc. We will assume that these are not controlled by the participants directly and that they are priced according to demand. In other words, resources do not participate in the competition among business units. Their prices are fixed by cost-demand curves, and business units have to pay for their use. These interdependencies create externalities between the participants which is what causes competition among them. If nothing is done, the stable situation would be an equilibrium among the business units, which is generally inefficient in terms of the freight company’s total profit.

As we described before, the system owner offers rebates to elicit coordination. We denote the rebate for resource \(i\) by \(s_i \geq 0\). As participants will not be reimbursed more than their cost, we restrict the actual reimbursement to not exceed \(c_i(x_i)\). Hence, as the rebates are announced before participants make their selections, participants receive a rebate up to the cost of the resource. Indeed, the experienced cost is

\[
c_i^s(x_i) := [c_i(x_i) - s_i]^+ ,
\]

where \([y]^+\) denotes the positive part of \(y\). Equivalently, the actual rebate equals \(\min(s_i, c_i(x_i))\). Collectively, we denote the vector of all rebates with \(s \in \mathbb{R}^A_+\). The outcome of this approach would be that the freight company offers rebates to business units that select the resources that the company wants them to select, therefore achieving a solution maximizing the total profit.

Each participant is associated with a pair of nodes, called an origin-destination pair (OD-pair), and has to select resources corresponding to a path from the origin to the destination.\(^1\) Let us denote the set of OD-pairs by \(K\), the demand corresponding to OD-pair \(k \in K\) by \(r_k\), and the total

\(^1\)Actually, the set of strategies can contain arbitrary set of resources instead of just paths. This competitive situation is called a \textit{nonatomic congestion game}, in which participants are anonymous and the cost of a resource depends only on the number of participants selecting it (Rosenthal 1973). We disregard this to simplify the presentation.
demand $\sum_{k \in K} r_k$ by $r$. In addition, we refer to all the possible paths connecting an OD-pair $k \in K$ by $P_k$ and we let $\mathcal{P} := \bigcup_{k \in K} \mathcal{P}_k$.

We use flows to encode all participants’ decisions, as specific identities are irrelevant. A flow $x$ is feasible if it is nonnegative and it satisfies all demand constraints. Mathematically, this is represented by the set $\{ x \in \mathbb{R}_+^P : \sum_{P \in P_k} x_P = r_k \text{ for all } k \in K \}$. We assume that cost functions are separable, meaning that the only argument of a cost function $c_i(x_i)$ is the congestion along arc $i \in A$, where $x_i := \sum_{P \in \mathcal{P}, P \ni i} x_P$.

Competition leads participants from the same OD-pair to select paths of equal cost because otherwise they would have an incentive to change their selection. This is the basis of the traditional solution concept called Wardrop equilibrium (Wardrop 1952).

**Definition 2.1.** A flow $x^{WE}$ is a Wardrop equilibrium of a network game (without rebates) if it is feasible, and for all $k$ and all $P, Q \in \mathcal{P}_k$ such that $x_P > 0$, $c_P(x) \leq c_Q(x)$, where $c_P(x) := \sum_{i \in P} c_i(x_i)$.

The previous definition provides us with a solution concept that models the behavior of the second stage players:

**Definition 2.2.** If the system owner selects the rebate vector $s$, participants select a solution $x^s$, which is a Wardrop equilibrium with respect to cost functions $[c_i(\cdot) - s_i]^+$.

For a given rebate vector $s$, the corresponding Wardrop equilibrium $x^s$ always exists because the modified cost functions $[c_i(\cdot) - s_i]^+$ are continuous and non-decreasing (Beckmann et al. 1956). In general, the equilibrium $x^s$ need not be unique and any equilibrium can arise in practice; thus, we consider an arbitrary one. Coming back to the example, business units would chose to send their freight using paths of minimum perceived cost (real cost minus rebates). An equilibrium is a selection of resources for all business units so that no unit regrets the choice it has made.

We now focus on the best strategy for the system owner. As it is the leader of the Stackelberg game and fixes the rebates knowing that participants are going to select a Wardrop equilibrium, its optimal strategy is to select the vector $s$ that minimizes the social cost, defined as the sum of the perceived costs to all participants of the system. We do this emulating a standard concept in Economics: the social welfare is defined as the total utility among all participants of the game (Mas-Colell, Whinston, and Green 1995). Indeed, this objective function includes the cost experienced by all the participants and the amount the system owner invests in rebates. As the system owner may be more sensitive to one of the terms than to the other, we consider a parameter $\rho \geq 0$ that transforms the rebate investment into social cost units. This is all captured by the following definition. (Note that we can alternatively define the social cost as the sum of the real costs that participants face by using a modified coefficient as shown in (1b).)

**Definition 2.3.** The strategy $(s, x^s)$ is a Stackelberg equilibrium if the vector of rebates $s$ minimizes the social cost, defined as

$$C_\rho(s) := \sum_{i \in A} x_i^s [c_i(x_i^s) - s_i]^+ + \rho \sum_{i \in A} x_i^s \min(c_i(x_i^s), s_i),$$

where the sum is over participants’ perceived cost and the cost of rebates.
which can also be expressed as
\[
\sum_{i \in A} x_i^s c_i(x_i^s) + (\rho - 1) \sum_{i \in A} x_i^s \min\{c_i(x_i^s), s_i\}.
\]  
\hspace{1cm} (1b)

The parameter \(\rho\) allows the system owner to control the tradeoff between the social cost of the solution and its investment. Alternatively, it can be viewed as the Lagrangian multiplier of the system owner’s rebate budget constraint. In fact, \(1/\rho\) represents the investment the system owner is willing to commit to make the participants’ perceived cost decrease by one unit:

- \(\rho = 1\) corresponds to the situation in which the system owner is only interested in minimizing the participants’ real cost \(\sum_{i \in A} x_i c_i(x_i)\), regardless of the rebate cost (see (1b)).
- \(\rho = +\infty\) corresponds to the situation in which the system owner does not want to spend any money on rebates. Here, the outcome will be a Wardrop equilibrium, as without rebates.
- Values of \(\rho < 1\) correspond to the case where the network planner values the participants’ perceived cost more than its own investments.

Structurally, the Stackelberg equilibrium problem is a mathematical program with equilibrium constraints (MPEC). If the leader wants to compute optimal rebates for a particular instance, there are relatively standard optimization techniques to solve this problem, even if more constraints are added to the problem (e.g., one can only offer rebates in a subset of the resources). For a background on MPECs and solution methods, we refer the interested reader to the book by Luo, Pang, and Ralph (1996). Instead, we will work with the optimality conditions of this problem to explicitly characterize the Stackelberg equilibrium. This will allow us to design an efficient algorithm and to find its worst-case inefficiency.

Not only do we want to compare the social cost of different solutions with rebates, but we also want to compare using rebates to not using them. Therefore, another measure of interest is the participants’ real cost, represented by the objective function
\[
C(x) := \sum_{i \in A} x_i c_i(x_i).
\]

The following definition captures the situation when the system owner controls the whole system.

**Definition 2.4.** A flow \(x^{SO}\) is a system optimum if it is feasible and minimizes \(C(\cdot)\).

The corresponding analogy for the example of a freight company would be the situation without business units and without competition. Instead, the (whole) company would be making all decisions based on the goal of minimizing the total cost.

The following proposition draws on the first-order optimality conditions to the mathematical program that defines a system optimum.

**Proposition 2.5** (Beckmann et al. 1956). For instances with derivable and convex cost functions, a flow \(x^{SO}\) is a system optimum if and only if it is a Wardrop equilibrium with respect to the modified cost functions \(c'_i(x_i) := c_i(x_i) + x_i c'_i(x_i)\).

Notice that if \(\rho \geq 1\), the social cost of a Stackelberg equilibrium \((s, x^*)\) satisfies
\[
C(x^{SO}) \leq C_\rho(s) \leq C(x^{WE}).
\]  
\hspace{1cm} (2)
Figure 1. Pigou’s example. Arcs are labeled with their cost functions.

The lower bound follows from (1b) because its second term is non-negative, and the upper bound comes from the feasibility of \( s = 0 \) because \( C(x^\text{WE}) = C_\rho(0) \).

2.1. Examples. In this section we introduce two concrete instances that will be the running examples for the rest of the article. These instances will be used to illustrate the different concepts and calculations along the way.

Instance 1 (Roughgarden and Tardos 2002). The first instance represents a competitive situation first described by Pigou (1920). As illustrated in Figure 1, participants must select one of two available resources: the first is expensive but its cost is not influenced by demand, while the second one is cheap under low demand but becomes expensive if it attracts many participants. This instance models a decision that commuters make daily in many cities. A person can use mass transit and experience an almost constant but large commute time, or can drive to (hopefully) experience a short commute while being exposed to the possibility of congestion.

The total demand in this instance is equal to 1, composed of an infinite number of price-taking users. The Wardrop equilibrium routes all flow in the lower arc (because in that case all participants take lowest-cost routes). Under this solution, \( C(x^\text{WE}) = 1 \). It is not hard to see that the system optimum is the flow that assigns half of the participants to each resource, implying that \( C(x^\text{SO}) = 3/4 \).

If \( \rho \leq 1 \), the system owner will propose rebates equal to the costs under the system optimum. Specifically, the Stackelberg equilibrium consists of the rebate vector \((1, 1/2)\) for which the corresponding equilibrium matches the system optimum solution. Actually, for arbitrary instances when \( \rho \leq 1 \), the system optimum is at equilibrium for optimal rebates because the experienced costs are zero. For details see Section 3.1 below, where we analyze this case in detail.

Let us now consider the case \( \rho > 1 \). It does not make sense to offer a rebate for the two resources because subtracting a constant everywhere will not change the equilibrium. Therefore, the system owner should only consider giving a rebate in the upper resource (the lower one is always cheaper so it should not be subsidized). Let the rebate be equal to \( s \in [0, 1] \) and, thus, the perceived cost on this resource equals \( 1 - s \). Therefore, the corresponding Wardrop equilibrium \( x^s \) is the flow that routes \( s \) units in the upper arc. After some algebra, \( C_\rho(s) = 1 - s + \rho s^2 \). The minimum, which provides the Stackelberg equilibrium, is \( s = 1/(2\rho) \) and achieves a social cost of \( 1 - 1/(4\rho) \).

Instance 2. The second network is similar to Pigou’s but contains an extra resource. As depicted in Figure 2, the three resources, numbered from 1 to 3 for simplicity, have cost functions equal
Our running example. Arcs are labeled with their cost functions.

to \( c_i(x_i) := (i - 1) + x_i \). At the Wardrop equilibrium, all participants select the first resource, and therefore \( C(x_{WE}) = 1 \). The system optimum is given by the flow \((3/4, 1/4, 0)\), with total cost \( C(x_{SO}) = 7/8 \). Finally, an optimal rebate vector for \( \rho > 1 \) is \( s = (0, 1/(2\rho), y) \), with \( 0 \leq y \leq 1 + 1/(4\rho) \). The corresponding Wardrop equilibrium \( x^s \) is \((1 - 1/(4\rho), 1/(4\rho), 0)\), and its social cost equals \( C_{\rho}(s) = 1 - 1/(8\rho) \).

2.2. Coordination Mechanisms based on Transfer Payments. In this section, we introduce some measures derived from the price of anarchy that will be useful to quantify the quality of equilibria resulting from a coordination mechanism. As we said in Section 1.1, Roughgarden and Tardos (2002) were the first to measure the price of anarchy in the network competition model introduced by Wardrop (1952). They defined the coordination ratio of an instance as

\[
\frac{C(x_{WE})}{C(x_{SO})}
\]

and the price of anarchy as the supremum of (3) among all possible instances, meaning all possible networks, demands and allowed cost functions. Note that this value is at least 1 and it can be interpreted as follows: if it is low, then there is not much improvement to be expected from the introduction of a coordination mechanism in the game that was considered. On the other hand, a large price of anarchy suggests that there is a potentially large benefit to be made.

For example, going back to Pigou’s instance (Instance 1), its coordination ratio is \( 4/3 \), and it turns out that this ratio is the largest possible, as the following result establishes.

**Proposition 2.6** (Roughgarden and Tardos 2002). The price of anarchy for instances with affine cost functions is \( 4/3 \).

For quadratic, cubic and quartic cost functions, the price of anarchy is 1.626, 1.896, and 2.151, respectively (Roughgarden 2003; Correa et al. 2004). For a simple proof of these results we direct the reader to Correa et al. (2005).

Recognizing the suboptimality of equilibria, classical studies have looked at ways to improve coordination by using taxes (Pigou 1920). The traditional mechanism to produce a socially efficient outcome has been to charge users the externalities they produce. Indeed, the marginal cost taxation mechanism charges users \( c^*(x) - c(x) = xc'(x) \) on each resource, which leads to \( x_{SO} \) (see Proposition 2.5). Typically, efficiency has been defined in terms of the total cost \( C(\cdot) \) because taxes are transfer payments that stay inside the system, or alternatively by assuming that these payments can
be redistributed back to the users. More recently, some articles looked at social cost functions that included a term corresponding to taxation. Under these more general social cost functions, a system owner may take a more holistic view, and care not only about outcomes, but also about investments. Cole et al. (2006) considered the problem of finding the taxes $\tau$ that minimize $\sum_{i \in A} x_i (c_i(x_i) + \tau_i)$, where $x$ is a Wardrop equilibrium with respect to modified cost functions $c(\cdot) + \tau$. In networks with linear cost functions, although marginal cost taxes can only increase the social cost, an optimal mechanism can do better by highly taxing some resources to effectively delete them from the network. Unfortunately, finding this optimal mechanism is NP-hard for arbitrary instances. Although they did not explicitly specialize their results to networks with substitutable resources, a generalization of the results of Section 4 can be used to compute optimal payments in polynomial time (still considering a general conversion factor $\rho$ like in (1a)). Karakostas and Kolliopoulos (2005) extended the previous analysis and found bounds for the social cost achieved by an extension of the marginal taxation mechanism to heterogeneous values-of-time. Under this setting, the ratio of the social cost of an equilibrium to the solution of minimum social cost with respect to the optimal taxes is bounded with a smaller constant than that of Proposition 2.6 and its generalizations. In addition, the social cost is not too large compared to the minimum possible total cost (without taxes).

One can employ different variations of the concept of the price of anarchy to quantify the power of a coordination mechanism. We consider the two definitions that are most interesting in our opinion. Both consist of ratios of the same cost function under two different solutions, thereby not falling into the situation of comparing apples and oranges. In addition, both compare the outcome provided by the coordination mechanism to an upper or lower bound, depending on the circumstances.

The first measure we consider is a straightforward extension of (3). Indeed, to quantify the loss of efficiency due to the limited coordinating power of the system owner, we consider the ratio

$$\frac{C(x^s)}{C(x^{SO})}. \tag{4}$$

For example, looking at Instance 2, this ratio equals 1 for $\rho \leq 1$ and $(8 - 2/\rho + 1/\rho^2)/7$ for $\rho > 1$. Note that although the previous ratio measures the quality of a given solution $(s, x^s)$ for a fixed instance, our main interest is on the supremum of the coordination ratio of an arbitrary Stackelberg equilibrium over all possible instances, as it is done for the price of anarchy. Another option would have been to define the price of anarchy as in (4) but using perceived costs, as Cole et al. (2006) proposed for their study of taxes in networks. Remark 5.5 shows that the bound that can be obtained is the same as that for (4).

Previous research has determined that the price of anarchy is sometimes a pessimistic measure, as can be expected from a general worst-case bound. In practice, instances are not arbitrary but have a certain structure. As a way to tighten the bound and reflect better what happens in practice, Correa et al. (2005) proposed to parametrize the price of anarchy based on the congestion level of the network. They found that equilibria are more inefficient for medium values of congestion because under low congestion externalities are low so an equilibrium does well, and under high congestion the bottleneck comes from the lack of capacity so all solutions are bad. Another aspect of the previous definitions is that they do not consider that in certain settings a system optimum is unrealistic and cannot be implemented. For example, Schulz and Stier-Moses (2006) proposed to
quantify the performance of a route guidance system for vehicular traffic by comparing the solutions with and without guidance instead of using a social optimum.

To get a measure that is both less pessimistic and more realistic, we consider that the best possible outcome is what the system owner can enforce by setting rebates correctly. Hence, we consider the ratio of the social cost of a Wardrop equilibrium to that of a Stackelberg equilibrium. Letting \( s \) be the optimal rebate vector, this ratio is expressed as

\[
\frac{C_\rho(0)}{C_\rho(s)} = \frac{C(x^{WE})}{C_\rho(s)}. \tag{5}
\]

When \( \rho \geq 1 \), the lower bound in (2) implies that this ratio is less pessimistic (smaller) than (3).

For the examples provided before, we get that for Instance 1, this ratio equals \( 4\rho/(4\rho - 1) \), while the coordination ratio displayed in (3) equals \( 4/3 \). The corresponding values for Instance 2 are \( 8\rho/(8\rho - 1) \) and \( 8/7 \), respectively.

3. General Network Topologies

We start our study of the structural characteristics of Stackelberg equilibria. In this section, we study general network topologies, with possibly several OD-pairs. We start by considering the case of the system owner assigning more value to the participants’ perceived cost than to its own rebate investment, and characterize the optimal strategy when setting the rebates. Later, we turn into the opposite case and provide some properties that will be used later on.

3.1. Small \( \rho \). In this section we consider that \( \rho \leq 1 \). We shall prove that it is optimal to propose rebates equal to the cost under a system optimum. When cost functions are strictly increasing, it turns out that there is a unique allocation at a Stackelberg equilibrium, which matches the system optimum. If we only consider weakly increasing functions, then a system optimum is always a Stackelberg equilibrium (but there may be other equilibria).

**Proposition 3.1.** For any instance, there exists a rebate vector whose corresponding Wardrop equilibrium matches the system optimum. Moreover, if cost functions are strictly increasing then this Wardrop equilibrium is unique.

**Proof.** Consider the rebate vector given by \( s_i = c_i(x_i^{SO}) \) for all \( i \in A \). The system optimum \( x_i^{SO} \) is at equilibrium since all participants experience a zero cost (which is the absolute minimum because of the non-negativity of the modified cost functions).

We now prove the uniqueness of equilibria in the case of strictly increasing cost functions. Using the variational inequality characterization of Wardrop equilibria (Smith 1979),

\[
\sum_{i \in A} (x_i^{SO} - x_i^s) \left[ c_i(x_i^s) - c_i(x_i^{SO}) \right]^+ \geq 0
\]

holds for an arbitrary Wardrop equilibrium \( x^s \) and the unique system optimum \( x^{SO} \). The summands vanish on arcs \( i \) such that \( x_i^s \leq x_i^{SO} \), and are strictly negative on arcs \( i \) for which \( x_i^s > x_i^{SO} \). Consequently, \( x_i^s \leq x_i^{SO} \) for all \( i \in A \), resulting in \( x^s = x^{SO} \) because \( x^{SO} \) minimizes the participants’ real cost. \( \square \)
Although we consider rebates, payments (also called tolls or taxes in the literature) can also be used to elicit coordination. Proposition 2.5 proves that payments equal to marginal costs at the system optimum also lead to a fully efficient coordinated solution. Moreover, any convex combination of optimal transfers (payments or rebates) is also optimal (Bergendorff, Hearn, and Ramana 1997), which implies that the set of transfers that lead to system optimality is a polyhedron.

**Remark 3.2.** A system optimum is at equilibrium for a rebate vector of the form \( s = [(1 - \kappa)c_i(x^{SO}) - \kappa(x^{SO}_i c_i'(x^{SO}))]_{i \in A} \) with \( 0 \leq \kappa \leq 1 \), where positive values represent rebates and negative values represent payments. Moreover, if cost functions are strictly increasing the Wardrop equilibrium \( x^s \) is unique.

The next proposition shows that the rebates previously mentioned are optimal for the system owner for instances with strictly increasing cost functions. Indeed, Proposition 3.1 guarantees that a system optimum is at equilibrium for a rebate vector of the form \( \sum_{i \in A} \rho \} ≥ \rho C(x^{SO}) \). Bounding each of the terms in (1b) separately, \( C_\rho(s) ≥ C(x^{SO}) + (\rho - 1)C(x^{SO}) = \rho C(x^{SO}) \), where we used that \( x^{SO} \) minimizes \( C(\cdot) \) and that \( \rho ≤ 1 \). Hence, \( \rho C(x^{SO}) \) is a lower bound for the optimal social objective that is attained at \( s \), which establishes the proposition.

Proposition 3.3 establishes what we claimed in Section 2.1 regarding the optimality of offering rebates equal to the cost under a system optimum for the case \( \rho ≤ 1 \).

### 2.2. Large \( \rho \)

In this section we consider that \( \rho > 1 \). We will prove that under an optimal rebate vector there is always at least one used resource with positive experienced cost, and one used resource for which there is no rebate offered. Both properties will be useful later to find the optimal rebates.

We define the set \( I \subseteq A \) to contain the used resources at equilibrium, and we partition it into the sets \( I_s \) for which rebates are positive, and \( I_0 \) for which there are no rebates.

**Definition 3.4.** For a given rebate vector \( s \), define \( I := \{i \in A \mid x^s_i > 0\} = I_s \cup I_0 \), where \( I_s := \{i \in I \mid s_i > 0\} \) and \( I_0 := \{i \in I \mid s_i = 0\} \).

Without loss of generality, we will sometimes assume that rebates for resources in \( A \setminus I \) are zero. Indeed, if a resource is unused with a positive rebate, it will still be unused without the
rebate. Consequently, the corresponding Wardrop equilibrium and all the aggregate measures we considered do not change when the rebate is removed. For example, for Instance 2, we have that \( I_0 = \{1\} \) and \( I_s = \{2\} \). The third arc is unused by the Stackelberg equilibrium so it does not belong to any of those sets.

**Lemma 3.5.** Assume that \( \rho > 1 \) and that all cost functions are strictly increasing. If \((s, x^s)\) is a Stackelberg equilibrium, then there exists a resource \( i \in I \) such that \( s_i < c_i(x_i^s) \).

**Proof.** Assume that all perceived costs are zero, i.e., \( s_i \geq c_i(x_i^s) \) for all \( i \in I \). Without loss of generality, it is enough that \( s_i = c_i(x_i^s) \) for all those resources. Then the social cost equals \( C_\rho(s) = \rho C(x^s) \geq \rho C(x^{SO}) \). Since the social cost \( \rho C(x^{SO}) \) can be attained with rebates \((c_i(x_i^{SO}))_{i \in A}\) as stated in Proposition 3.1 and \( s \) was assumed to be optimal, \( C(x^s) = C(x^{SO}) \), and thus \( x^s \) is a system optimum. Because of Proposition 2.5, \( x^s \) is at equilibrium with respect to modified costs \( c_i(x_i^s) + x_i^s c_i'(x_i^s) \).

As perceived costs are zero and cost functions are strictly increasing, \( s_i > 0 \) for all \( i \in I \), or equivalently \( I_0 = \emptyset \). Hence, there exists a small enough \( \epsilon > 0 \) such that \( \bar{s} \geq 0 \), where

\[
\tilde{s}_i := \begin{cases} 
  s_i - \epsilon (c_i(x_i^s) + x_i^s c_i'(x_i^s)) & \text{if } i \in I, \\
  0 & \text{if } i \in A \setminus I.
\end{cases}
\]

Under rebates \( \tilde{s} \) and flow \( x^s \), the perceived cost on each resource is \([c_i(x_i^s) - \tilde{s}_i]^+ = \epsilon (c_i(x_i^s) + x_i^s c_i'(x_i^s))\) for \( i \in I \). Similarly, \([c_i(x_i^s) - \tilde{s}_i]^+ = \epsilon c_i(x_i^s) = \epsilon (c_i(x_i^s) + x_i^s c_i'(x_i^s))\) for \( i \in A \setminus I \). The last two equations imply that \( x^s \) is at equilibrium under rebates \( \tilde{s} \), and the perceived cost on each used resource is strictly positive. Finally, \( x^s \) is the unique equilibrium under \( \tilde{s} \) since the potential function \( F(x) := \sum_{i \in A} \int_0^{x_i} [c_i(z_i) - \tilde{s}_i]^+ dz_i \) is convex in general, strictly convex in a vicinity of \( x^s \) as the cost functions are strictly increasing, and achieves a minimum at \( x^s \). (We refer the reader to Beckmann et al. (1956) for details on the characterization of Wardrop equilibria with this type of potential function.) Consequently,

\[
C_\rho(\tilde{s}) = \sum_{i \in A} (x_i^s c_i(x_i^s) + (\rho - 1) x_i^s \tilde{s}_i) = C_\rho(s) - \epsilon (\rho - 1) \sum_{i \in I} x_i^s \left( c_i(x_i^s) + x_i^s c_i'(x_i^s) \right) > C_\rho(s),
\]

which is a contradiction to the optimality of \( s \). □

When we presented the examples in Section 2.1, we mentioned that it cannot be optimal to offer rebates in all resources. The next lemma generalizes this observation to any network topology. It shows that, if all resources are used, then \( I_0 \) is necessarily nonempty. In Section 4, we will generalize it to instances in which not all resources are used.

**Lemma 3.6.** Assume that \( \rho > 1 \) and that all cost functions are strictly increasing. If \((s, x^s)\) is a Stackelberg equilibrium and all resources are used, then there exists a resource \( i \in I \) such that \( s_i = 0 \).

**Proof.** With the purpose of deriving a contradiction, let us assume that \( s \) is an optimal vector of rebates such that \( s_i > 0 \) for all \( i \in I \). Let us consider new rebates \( \tilde{s} = [c_i(x_i^s) - \eta(c_i(x_i^s) - s_i)^+]_{i \in A} \), where

\[
\eta := \min_{i \in I} \frac{c_i(x_i^s)}{(c_i(x_i^s) - s_i)^+}.
\]
The definition implies that \( \bar{s} \geq 0 \) and Lemma 3.5 implies that \( \eta < \infty \), so the new rebates are well-defined. The perceived cost for resource \( i \) under the new rebates equals \( (c_i(x^*_i) - \bar{s}_i) = \eta(c_i(x^*_i) - s_i) \), meaning that \( x^* \) is also at equilibrium under \( \bar{s} \). Furthermore, as \( s > 0 \), we have that \( \eta > 1 \) and \( \bar{s} \leq s \). Hence, looking at (1b), the participants’ real cost is unchanged, whereas the cost of rebates strictly decreases because \( \bar{s}_i = 0 \) for the argument \( i \) achieving the minimum. This contradicts the optimality of \( s \).

\[ \Box \]

4. Substitutable Resources

Equipped with the structural results of the previous section, we now embark in the design of an efficient algorithm for computing Stackelberg equilibria. The outline of procedure described in this section is as follows. First, we will separate resources into those in which rebates must be offered, those in which no rebates must be offered and those that are not used in an equilibrium. With this partition, we will be able to compute the actual rebates for the corresponding resources.

We focus on networks in which participants have to select exactly one out of many possible resources. This models applications in which resources are substitutes. The network topology that corresponds to this situation comprises two nodes joined by several arcs representing resources (see Figure 3). Networks with substitutable resources extend the classic two-route network introduced by Pigou (1920). They have been widely used because of its relevance in practical applications—such as transportation, telecommunication, scheduling and resource allocation problems—and because of its tractability (see, e.g., Korilis et al. 1995; Koutsoupias and Papadimitriou 1999; Roughgarden 2004; Weintraub et al. 2006; Acemoglu and Ozdaglar 2007; Wichiensin et al. 2007; Xiao et al. 2007). Note that the restriction to simple topologies seems necessary if we hope to find the optimal rebates in polynomial time because Cole et al. (2006) proved that finding optimal taxes in general networks with affine cost functions is hard. Finally, we only consider the case of \( \rho > 1 \), since the optimal rebates for \( \rho \leq 1 \) were already found in Section 3.1. We can assume without loss of generality that \( s_i \leq c_i(x^*_i) \), as it is never beneficial to offer more.

\[ \text{Figure 3. A network of substitutable resources} \]

\footnote{Cole et al. (2006, Theorem 6.2) prove that an approximation algorithm with guarantee better than \( 4/3 - \epsilon \) cannot exist unless P=NP. Although their reduction does not work for our problem, we conjecture that finding the optimal rebates in a general network with affine cost functions is also NP-hard because of the similarity between their social cost function and (1b) (see Section 2.2).}
Consider a Stackelberg equilibrium \((s, x^s)\) of an instance in which cost functions are strictly increasing. The equilibrium conditions imply that there is a constant \(L_\rho \geq 0\) such that

\[
L_\rho = c_i(x^s_i) - s_i \quad \forall \ i \in I \tag{6a}
\]

\[
L_\rho \leq c_i(0) - s_i \quad \forall \ i \in A \setminus I. \tag{6b}
\]

Moreover, Lemma 3.5 implies that \(L_\rho\) has to be strictly positive which implies that \(c_i(x^s_i) > s_i\) for all \(i \in A\). For networks with substitutable resources, then, we do not need to enforce the constraint that the system owner cannot offer rebates that are larger than the cost of resources. In this case (1) simplifies to \(C_\rho(s) = \sum_{i \in A} x^+_i[c_i(x^+_i) + (\rho - 1)s_i]\).

**Remark 4.1.** The positivity of \(L_\rho\) also implies that when cost functions are strictly increasing there is a unique Wardrop equilibrium corresponding to the optimal \(s\) since the potential function \(F(x) = \sum_{i \in A} \int_0^{x_i} [c_i(z) - s_i]^{+} dz\) is strictly convex in a vicinity of \(x^s\). Later, we shall prove that in this case the optimal \(s\) is also unique.

Going back to the examples in Section 2.1, it is not hard to check that \(L_\rho\) for Instances 1 and 2 equals \(1 - 1/(2\rho)\) and \(1 - 1/(4\rho)\), respectively.

### 4.1. General Cost Functions

We start with general cost functions and then, in the next section, switch to the particular case of affine cost functions. This section proves a result that will allow us to decide for which resources we must offer positive rebates. To get there, we first have to present a series of lemmas. The first one establishes that a rebate vector that is optimal for a given network is also optimal when some unused resources are taken out. In other words, removing \(i \in A \setminus I\) does not affect the optimality of \(s\).

**Lemma 4.2.** Consider a parallel-link network and a corresponding optimal rebate vector \(s\). If \(l\) is a resource in \(A \setminus I\), then the vector \(s'\) with the entry corresponding to \(l\) removed is an optimal rebate vector for a similar instance with resource \(l\) removed.

**Proof.** Let \(\tilde{A} := A \setminus \{l\}\) and \(\tilde{s}\) be the restriction of \(s\) to \(\tilde{A}\). Assume that \(\tilde{s}\) is not optimal for \(\tilde{A}\), and let \(\tilde{s}^*\) be an optimal rebate vector for that network. Then, \(C_\rho(\tilde{s}^*) < C_\rho(\tilde{s}) = C_\rho(\tilde{s})\), where the superscript represents the instance and the equality holds because if no participant selects the resource, it makes no difference whether the resource exists or not. Now, we take the optimal rebate vector \(\tilde{s}^*\) and extend it to the original network by setting \(s^*_i := 0\) and \(s^*_i := s^*_i\) for \(i \in \tilde{A}\). Comparing the two equilibria \(x^{\tilde{s}^*}\) and \(x^{s^*}\), (re)introducing resource \(l\) can only reduce the participants’ real cost at equilibrium as a situation like Braess’ paradox (1968) cannot occur because of the network structure. Together with the fact that resource \(l\) is not subsidized, we have that \(C_\rho(\tilde{s}^*) \leq C_\rho(\tilde{s}^*),\) which contradicts the optimality of \(s\) in the original instance. \(\square\)

Notice that the previous lemma generalizes Lemma 3.6 to an arbitrary instance with substitutable resources. Indeed, Lemma 4.2 implies that an optimal rebate vector \(s\) is still optimal for the network consisting only of resources in \(I\). Because that instance makes use of all resources, it must contain at least one resource without rebate.

In the following propositions, we derive necessary conditions for a rebate vector \(s\) to be optimal. These conditions are implied by the first-order optimality conditions of the optimization problem.

that finds the optimal rebate. The next proposition shows that the optimal rebates satisfy the following equilibrium conditions: rebates are offered only in resources for which the expression $c_i^*()$ is minimal. Contrast this to Proposition 2.5 that states that in a system optimum, participants are assigned only to resources for which the expression $c_i^*()$ is minimal.

**Proposition 4.3.** Consider a parallel-link network with strictly increasing and differentiable cost functions, and a Stackelberg equilibrium $(s, x^*)$. There exists $V_\rho > 0$ such that

\[
V_\rho = c_i(x_i^*) + x_i^*c_i'(x_i^*) \quad \forall i \in I_s
\]

\[
V_\rho \leq c_i(x_i^*) + x_i^*c_i'(x_i^*) \quad \forall i \in A \setminus I_s.
\]

**Proof.** Without loss of generality assume that $s_i = c_i(0) - L_\rho$ for all $i \in A \setminus I$. Consider two fixed resources $i \in I_s$ and $j \in A$. Since $s_i$ is strictly positive, it is possible to simultaneously reduce $s_i$ by a positive infinitesimal $ds_i$ and increase $s_j$ so that the only effect is that some participants switch from resource $i$ to $j$. In other words, we have that $dx_j = -dx_i$, where we denote an infinitesimal variation of a quantity $w$ by $dw$. By design, the perceived cost $L_\rho$ at equilibrium remains the same.

The local effect at the resources in question is $d(c_i(x_i^*) - s_i) = 0$ and $d(c_j(x_j^*) - s_j) = 0$. Because $s$ was optimal, this modification cannot decrease the total rebate cost $\sum_{i \in A} x_i^*s_i$, as it does not modify the total participants’ perceived cost. This implies that $d(x_i^*s_i + x_j^*s_j) \geq 0$. Putting all together,

\[
dx_i^* (x_i^*c_i'(x_i^*) + s_i - x_j^*c_j'(x_j^*) - s_j) \geq 0.
\]

As $dx_i^* < 0$, we must have $x_i^*c_i'(x_i^*) + s_i \leq x_j^*c_j'(x_j^*) + s_j$, and adding $c_i(x_i^*) - s_i = c_j(x_j^*) - s_j$, we finally obtain that $c_i(x_i^*) + x_i^*c_i'(x_i^*) \leq c_j(x_j^*) + x_j^*c_j'(x_j^*)$. We get the claim by letting $i$ and $j$ vary.

From (6a) and (7), we get that there exists a constant $D_\rho := 2L_\rho - V_\rho$ such that

\[
D_\rho = c_i(x_i^*) - x_i^*c_i'(x_i^*) - 2s_i \quad \forall i \in I_s
\]

\[
D_\rho \geq c_i(x_i^*) - x_i^*c_i'(x_i^*) - 2s_i \quad \forall i \in I_0.
\]

The common perceived cost at equilibrium therefore equals $L_\rho = (V_\rho + D_\rho)/2$. Comparing the expressions, it is clear that $D_\rho < L_\rho < V_\rho$. For example, looking at the Stackelberg equilibrium of Instance 2, the constants are $V_\rho = 1 + 1/(2\rho)$ and $D_\rho = 1 - 1/\rho$.

In the sequel, we will make extensive use of the following definition to characterize and to compute optimal rebates:

**Definition 4.4.** For $X \subseteq A$, let $K(X) := \sum_{i \in X} c_i(x_i^*)^{-1}$. For the special case of an empty set, it is assumed that $K(\emptyset) := 0$.

The following technical lemma provides a formula that will be useful later. Its proof considers another feasible direction from the optimal rebate vector.

**Lemma 4.5.** Consider a parallel-link network with strictly increasing and differentiable cost functions, and a Stackelberg equilibrium $(s, x^*)$. Then,

\[
\sum_{i \in I_s} \left( x_i^*K(I) + \frac{s_i}{c_i'(x_i^*)}K(I_0) \right) = \frac{r}{\rho}K(I_s).
\]


Proof. To ensure that the modification to the rebates we are going to make does not change the sets \( I_s \) and \( I_0 \), we first remove all unused resources. Indeed, Lemma 4.2 proves that if \( s \) is optimal for the original network, it is also optimal for the instance containing the resources in \( I \) only. The proposition is obvious for \( I_s = \emptyset \), so let us assume the opposite. We consider adding or subtracting a common infinitesimal \( ds \) to all rebates that are strictly positive. After modifying \( s \) the outcome is still at equilibrium and all resources are still used; hence, differentials of perceived costs are equal for all resources in \( I \). For a fixed \( i_s \in I_s \) and a fixed \( i_0 \in I_0 \neq \emptyset \), we have that 

\[
\begin{align*}
    dC_{i_s} &= c'_{i_s}(x^s) dx_s - ds = c'_{i_0}(x^s) dx_{i_0} \\
    &
\end{align*}
\]

As the total demand does not change, we must have that \( 0 = \sum_{i \in I} dx^s_i = K(I_s)c'_{i_s}(x^s_i) dx_{i_s} + K(I_0)c'_{i_0}(x^s_i) dx_{i_0} \). After some algebra, \( c'_{i_s}(x^s_i) dx_{i_s} = ds K(I_0) / K(I) \). Finally, let us consider how the social cost changes.

\[
\begin{align*}
    dC_{\rho}(s) &= d\left( r(c_{i_s}(x_{i_s}) - s_{i_s}) + \rho \sum_{i \in I} x_i s_i \right) \\
    &= r(c'_{i_s}(x^s_i) dx_{i_s} - ds) + \rho \sum_{i \in I_s} \left( c'_{i_s}(x^s_i) dx_{i_s} s_i / c'_{i_s}(x^s_i) + x^s_i ds \right) \\
    &= c'_{i_s}(x^s_i) dx_{i_s} \left( r + \rho \sum_{i \in I_s} s_i / c'_{i_s}(x^s_i) \right) + ds \left( \rho \sum_{i \in I_s} x^s_i - r \right) \\
    &= ds \left( \frac{K(I_0)}{K(I)} - 1 \right) r + \rho \frac{K(I_0)}{K(I)} \sum_{i \in I_s} s_i / c'_{i_s}(x^s_i) + \rho \sum_{i \in I_s} x^s_i \\
    &= ds \frac{\rho}{K(I)} \left( -\frac{r}{\rho} K(I_0) + K(I_0) \sum_{i \in I_s} s_i / c'_{i_s}(x^s_i) + K(I) \sum_{i \in I_s} x^s_i \right).
\end{align*}
\]

The claim follows because the optimality of \( s \) implies that \( dC_{\rho}(s) \geq 0 \) for feasible directions \( ds > 0 \) and \( ds < 0 \). \( \square \)

Using the previous results, we can characterize the sets \( I_0 \) and \( I_s \), which will allow us to compute the optimal rebates.

Proposition 4.6. Consider a parallel-link network with strictly increasing and differentiable cost functions, and a Stackelberg equilibrium \((s, x^s)\). Then, for all \( i \in A \),

\[
\begin{align*}
    i \in I_0 & \iff D_{\rho} \geq c_i(x^s_i) - x^s_i c'_i(x^s_i) \quad (10a) \\
    i \in A \setminus I & \iff V_{\rho} \leq c_i(0). \quad (10b)
\end{align*}
\]

Proof. We start with (10a). The forward implication is (8b). Conversely, consider \( i \in A \), and assume that \( D_{\rho} \geq c_i(x^s_i) - x^s_i c'_i(x^s_i) \). If \( i \in A \setminus I \), then \( x^s_i = 0 \) and \( c_i(0) \leq D_{\rho} < L_{\rho} \), contradicting the Wardrop equilibrium condition. If \( i \in I \), then (8a) implies that \( c_i(x^s_i) - x^s_i c'_i(x^s_i) = D_{\rho} + 2s_i > D_{\rho} \), yielding a contradiction again.
The forward implication of (10b) follows from (7b). Conversely, consider an \( i \in A \), and assume that \( V_\rho \leq c_i(0) \). If \( i \in I_0 \), then \( c_i(0) < L_\rho < V_\rho \), which yields a contradiction. If \( i \in I_s \) then (7a) implies that \( c_i(0) < c_i(x^s_i) + x^s_i c'_i(x^s_i) = V_\rho \), which is again a contradiction. \( \square \\

In other words, we have the following partition of the resources according to the expression \( c_i(x^s_i) - x^s_i c'_i(x^s_i) \): considering \( i_0 \in I_0 \), \( i_s \in I_s \), and \( j \in A \setminus I \), we have

\[
    c_{i_0}(x_{i_0}^s) - x_{i_0}^s c'_{i_0}(x_{i_0}^s) \leq D_\rho < c_{i_s}(x_{i_s}^s) - x_{i_s}^s c'_{i_s}(x_{i_s}^s) < V_\rho \leq c_j(0).
\]

(11)

This characterizes which resources are used naturally because they are cheap, which resources are used because of the rebates offered, and which resources are not used, even having the possibility of offering rebates, because they are too expensive. Of course, to use this result constructively one would first need to know the Stackelberg equilibrium. In the next section, we will see how to work around that problem for affine cost functions. Going back to Instance 2, one can see that \( c_1(x^s_1) - x^s_1 c'_1(x^s_1) = 0 \leq D_\rho = 1 - 1/\rho < c_2(x^s_2) - x^s_2 c'_2(x^s_2) = 1 < V_\rho = 1 + 1/(2\rho) \leq c_3(0) = 2 \).

The following proposition determines how many of the participants are benefited from having rebates in the network.

**Proposition 4.7.** Consider a parallel-link network with strictly increasing and differentiable cost functions, and a Stackelberg equilibrium \((s,x^s)\). The proportion of participants that receive a rebate is strictly lower than \(1/\rho\).

**Proof.** Assume that \( I_s \neq \emptyset \) because otherwise the claim is obvious. Dividing (9) by \( K(I_s) \),

\[
    \frac{r}{\rho} = \frac{K(I_0)}{K(I_s)} \sum_{i \in I_s} \left( x^s_i + \frac{s_i}{c'_i(x^s_i)} \right) + \sum_{i \in I_s} x^s_i = K(I_0)(V_\rho - L_\rho) + \sum_{i \in I_s} x^s_i.
\]

Therefore, \( \sum_{i \in I_s} x^s_i/r = 1/\rho - K(I_0)(V_\rho - L_\rho)/r < 1/\rho \), as we wanted to show. \( \square \)

The previous bound turns out to be tight as demonstrated by the following instance.

**Instance 3.** Consider a network similar to that depicted in Figure 1 but with cost functions \( c_1(x) = 1 - (1 - \epsilon)/\rho + \alpha x \) and \( c_2(x) = x \), where \( 0 < \epsilon < 1 \) and \( \alpha > 0 \). Using results we will develop in Section 4.2, we must have that \( I_0 = \{2\} \) and \( I_s = \{1\} \) (because \( b_2 < L_\infty(1 - 1/\rho) = 1 - 1/\rho < b_1 < L_\infty(1 + 1/\rho) \); see the next section for the notation). Hence, the rebate \( s = (\epsilon/(2\rho),0) \) is optimal and the corresponding equilibrium is given by

\[
    x^s = \left( \frac{2 - \epsilon}{2(1 + \alpha)\rho}, \frac{1}{2(1 + \alpha)\rho} \right).
\]

The proportion of participants that receive positive rebates is \( x^s_i \), which tends to \( 1/\rho \) as \( \epsilon \) and \( \alpha \) tend to 0.

4.2. **Affine Cost Functions.** Having derived properties for general cost functions, this section considers instances with affine cost functions and explicitly provides expressions for the optimal rebates.Instances with affine cost functions are rich enough for many congestion phenomena to appear. Even for applications in which cost functions are more complex, an affine approximation can already show evidence of first-order effects (Weintrab et al. 2006; Acemoglu and Ozdaglar 2007). We denote the cost function on resource \( i \in A \) by \( c_i(x) = a_i x + b_i \), with \( a_i > 0 \) and \( b_i \geq 0 \).
Without loss of generality, we consider that resources are sorted according to \( b_i \), so we have that \( b_1 \leq b_2 \leq \ldots \leq b_{|A|} \). For ease of notation, we let \([i] := \{1, \ldots, i\}\), and \( b_{|A|+1} = +\infty \).

In the case of affine functions, we can simplify some of the formulas we provided in previous sections. For example, Definition 4.4 becomes \( K(X) = \sum_{i \in X} 1/a_i \) for \( X \subset A \). Notice also that a consequence of (8) is that \( D_\rho \geq 0 \) and \( s_i \leq b_i/2 \) for all \( i \in I_\rho \). In the case of affine cost functions, it is easy to partition the resources into the sets \( I_0, I_s \) and \( A \setminus I \). Indeed, rewriting (11), we get the following proposition:

**Proposition 4.8.** Consider a parallel-link network with affine cost functions. If we consider \( i_0 \in I_0 \), \( i_s \in I_s \), and \( j \in A \setminus I \), then \( b_{i_0} \leq D_\rho < b_{i_s} < V_\rho \leq b_j \).

The following lemma and theorem show that if we know how the resources are partitioned, we can compute the optimal rebate values for all resources.

**Lemma 4.9.** Consider a parallel-link network with affine cost functions, and a Stackelberg equilibrium \((s, x^\ast)\). If rebates are beneficial (i.e., if \( I_s \neq \emptyset \)), then

\[
D_\rho = \frac{1}{K(I_0)} \left( r^\frac{\rho - 1}{\rho} + \sum_{i \in I_0} \frac{b_i}{a_i} \right), \\
V_\rho = \frac{1}{K(I)} \left( r^\frac{\rho + 1}{\rho} + \sum_{i \in I} \frac{b_i}{a_i} \right).
\]

**Proof.** On the one hand, (6a) and (7a), respectively, imply that

\[
x_i^s = \begin{cases} 
\frac{V_\rho + D_\rho - 2b_i}{2a_i} & \text{if } i \in I_0 \\
\frac{V_\rho - b_i}{2a_i} & \text{if } i \in I_s.
\end{cases}
\]

Since \( \sum_{i \in I} x_i^s = r \), we have

\[
\frac{V_\rho}{2} K(I_0) = r - \frac{D_\rho}{2} K(I_0) + \sum_{i \in I_0} \frac{b_i}{2a_i} + \sum_{i \in I} \frac{b_i}{a_i}.
\]

On the other hand, (7a), (8a) and Lemma 4.5 imply that

\[
\frac{r}{\rho} K(I_s) = \frac{V_\rho}{2} K(I_s) K(I) + (K(I_0) - K(I)) \sum_{i \in I_s} \frac{b_i}{2a_i} - \frac{D_\rho}{2} K(I_0) K(I_s)
\]

and since \( I_s \neq \emptyset \),

\[
\frac{V_\rho}{2} K(I) = \frac{r}{\rho} + \sum_{i \in I_s} \frac{b_i}{2a_i} + \frac{D_\rho}{2} K(I_0).
\]

Adding and subtracting (12) and (13) yield the claim. \( \square \)

If \( I_0 \) is known, making use of the previous lemma, we can compute the optimal rebates using the relations that we developed in the previous section. This result implies that, essentially, there is a unique optimal rebate vector.
**Theorem 4.10.** Consider a parallel-link network with affine cost functions, and let \((s, x^*)\) be an arbitrary Stackelberg equilibrium. Then, rebates must satisfy that
\[
s_i = \left[ \frac{b_i - D^2}{2} \right]^+
\]
for all \(i \in \mathcal{I}\). Moreover, if this formula is used for all resources, the corresponding solution is a Stackelberg equilibrium.

*Proof.* Consider a resource \(i \in \mathcal{I}\). If \(i \in \mathcal{I}_0\), then \(s_i = 0\) by definition and this agrees with the proposed formula because of Proposition 4.8. If \(i \in \mathcal{I}_s\), then solving for \(s_i\) in \((8a)\) also gives the proposed formula.

Now consider using the proposed formula for all \(i \in A\). We must prove that each resource \(j \in A \setminus \mathcal{I}\) is not used under the corresponding Wardrop equilibrium. Proposition 4.8 implies that \(b_j > V_\rho\). Therefore, the rebate computed by the theorem is positive and \(b_j - s_j = (b_j + D_\rho)/2\). We conclude that the experienced cost under no flow is \(b_j - s_j \geq L_\rho\), which means that \(x_i^* = 0\). \(\square\)

Evidently, plugging the values into the expression of the previous theorem for Examples 1 and 2 gives us the rebates that we indicated in Section 2.1. What remains to be done to finish the characterization of optimal rebates is to find \(\mathcal{I}_0\) in order to determine the value of \(D_\rho\). The following result provides a characterization of the common cost experienced by participants under a Stackelberg equilibrium. It will allow us to compute the values of \(D_\rho\) and \(V_\rho\).

**Proposition 4.11.** Consider a parallel-link network with affine cost functions and total demand \(r > 0\). For \(j \in A\), define \(\gamma(j, r) := \left( r + \sum_{i=1}^{[j]} \frac{b_i}{a_i} \right) / K([j])\). There exist unique resources \(i_0, i_1 \in A\) such that
\[
\begin{align*}
&b_{i_0} \leq \gamma(i_0, r) < b_{i_0+1} \\
&b_{i_1} < \gamma(i_1, r) \leq b_{i_1+1}. 
\end{align*}
\]
Moreover, \(\gamma(i_0, r) = \gamma(i_1, r) = L_\infty\), where \(L_\infty\) is the common cost experienced by participants under a Wardrop equilibrium (without rebates).

*Proof.* Let us define \(i_0 := \max\{ i \in A : b_i \leq L_\infty \}\), and let \(x\) be the Wardrop equilibrium. From the definition, \(i_0\) satisfies that \(b_{i_0} \leq L_\infty < b_{i_0+1}\). The equilibrium condition implies that \(x_i = (L_\infty - b_i)/a_i\) for all \(i \leq i_0\). Summing over that range we get that \(L_\infty = \gamma(i_0, r)\). What is left to prove is that there is no other \(i_0\) that satisfies \((14a)\). Hence, assume that there is another index \(\tilde{i}_0\), and define \(\tilde{x}\) equal to \((\gamma(i_0, r) - b_i)/a_i\) for \(i \leq \tilde{i}_0\) and 0 otherwise. This flow is feasible as it is nonnegative and its total demand equals \(r\). Furthermore, it satisfies the Wardrop equilibrium conditions with cost equal to \(\gamma(\tilde{i}_0, r)\) for all participants. Recall that since cost functions are strictly increasing, there exists a unique Wardrop equilibrium. As \(x\) and \(\tilde{x}\) are both at equilibrium, they must be equal. This implies that \(\gamma(i_0, r) = \gamma(\tilde{i}_0, r)\), from where \(\tilde{i}_0 = i_0\) because of \((14a)\).

A similar argument proves the existence of a unique index \(i_1 := \max\{ i \in A : b_i < L_\infty \}\) that verifies \((14b)\). \(\square\)
Computing $\gamma(i, r)$ for the different resources in Instance 2, we get that $\gamma(1, r) = r$, $\gamma(2, r) = (r + 1)/2$ and $\gamma(3, r) = (r + 3)/3$. Then $i_0 = 1$ when $0 \leq r < 1$, $i_0 = 2$ when $1 \leq r < 3$, and $i_0 = 3$ when $r \geq 3$. Similarly, $i_1 = 1$ when $0 \leq r \leq 1$, $i_1 = 2$ when $1 < r \leq 3$, and $i_1 = 3$ when $r > 3$.

In the sequel, we consider the function $L_\infty(z)$ which represents the perceived cost under a Wardrop equilibrium (without rebates) when the total demand is $z$. When we do not explicitly display the demand, we mean the regular demand of $r$. It is well known that the function $L_\infty(z)$ is non-decreasing and continuous (Hall 1978). In addition, Proposition 4.11 implies that it is piecewise-affine with slope $1/K[i]$ when its value is between $b_i$ and $b_{i+1}$. Therefore, it is a concave function. Under our assumptions, $L_\infty(\cdot)$ is easy to compute using an incremental loading algorithm. The bottleneck of this procedure is sorting the resources with respect to the fixed costs $b_i$.

Using the previous result, we can now express the perceived cost of participants at the Stackelberg equilibrium. In addition, the next proposition will clearly identify the sets $I_0$ and $I_s$. First, $I_0 = [i_0]$, where $i_0$ corresponds to the index introduced in Proposition 4.11, and the resources without rebates that are used in a Stackelberg equilibrium coincide with those that are used under a Wardrop equilibrium (without rebates) with a total demand of $r(1 - 1/\rho) + \epsilon$, for a sufficiently small $\epsilon > 0$. Likewise, $I = [i_1]$, where $i_1$ is the index introduced in Proposition 4.11, and the used resources under a Stackelberg equilibrium coincide with those that are used under a Wardrop equilibrium with a total demand of $r(1 + 1/\rho)$.

**Proposition 4.12.** Consider a parallel-link network with affine cost functions, and a Stackelberg equilibrium $(s, x^*)$. If rebates are beneficial (i.e., if $I_s \neq \emptyset$), then

$$D_\rho = L_\infty \left( r \frac{\rho - 1}{\rho} \right) \quad \text{and} \quad V_\rho = L_\infty \left( r \frac{\rho + 1}{\rho} \right),$$

and the perceived cost of each participant under $x^*$ is

$$L_\rho = \frac{1}{2} \left( L_\infty \left( r \frac{\rho - 1}{\rho} \right) + L_\infty \left( r \frac{\rho + 1}{\rho} \right) \right).$$

**Proof.** From Proposition 4.8 and Lemma 4.9, we know that there exist $i_0, i_1 \in A$ such that $I_0 = [i_0]$, $I = [i_1]$, $b_{i_0} \leq \gamma \left( i_0, r \frac{\rho - 1}{\rho} \right) < b_{i_0+1}$, and $b_{i_1} < \gamma \left( i_1, r \frac{\rho + 1}{\rho} \right) \leq b_{i_1+1}$.

Hence, Proposition 4.11 implies the first two claims. The third follows simply from the relation displayed right after (8). \qed

Using the values of $i_0$ that we previously computed for our running example (Instance 2), it is easy to see that $L_\infty(r) = r$ when $0 \leq r < 1$, $L_\infty(r) = (r + 1)/2$ when $1 \leq r < 3$, and $L_\infty(r) = (r + 3)/3$ when $r \geq 3$. Using this, $D_\rho = 1 - 1/\rho$, $V_\rho = 1 + 1/(2\rho)$, and $L_\rho = 1 - 1/(4\rho)$ as expected.

Notice that Proposition 4.12 provides an explicit way to compute $D_\rho$. Hence, this value is unique and relying on Proposition 4.11, the vector of optimal rebates is unique as well (disregarding that a rebate for a resource $l \in A \setminus I$ can take any value between 0 and $c_l(0) - L_\rho$, which does not count as multiple equilibria because $l$ is unused). As for any given rebate vector such that $L_\rho > 0$ there is a unique Wardrop equilibrium, the Stackelberg game has a unique solution.

The following proposition provides an easily verifiable condition to check whether rebates can help lower the social cost in a specific instance or not. Note that when the inequality does not hold, the formula must hold with equality because of the concavity of $L_\infty(r)$.
Proposition 4.13. Consider a parallel-link network with affine cost functions. Rebates are beneficial (i.e., $\mathcal{I}_s \neq \emptyset$) if and only if
$$\frac{1}{2} \left( L_\infty \left( r \frac{\rho-1}{\rho} \right) + L_\infty \left( r \frac{\rho+1}{\rho} \right) \right) < L_\infty .$$

Proof. Rebates are beneficial only if the social cost of a Stackelberg equilibrium is lower than that of the Wardrop equilibrium. In that case, $L_\rho < L_\infty$ and, hence, the strict inequality of the claim holds because of Proposition 4.12.

We now focus on the reverse implication. Assuming that the inequality in the hypothesis holds, there exists an $i \in A$ such that
$$L_\infty \left( r \frac{\rho-1}{\rho} \right) < b_i < L_\infty \left( r \frac{\rho+1}{\rho} \right) .$$
Proposition 4.8 implies that if $i \in A \setminus \mathcal{I}$ then $b_i \geq V_\rho$, and if $i \in \mathcal{I}_0$ then $b_i \leq D_\rho$. Therefore, $i \in \mathcal{I}_s$, which is consequently nonempty. □

The results we have presented imply that optimal rebates can be computed in polynomial time. To summarize, the system owner can follow the following algorithm to determine them. First, Proposition 4.13 indicates whether rebates need to be used or not. (Remember that $L_\infty(z)$ can be computed efficiently for networks with substitutable resources.) If rebates are beneficial, Proposition 4.12 provides the value for $D_\rho$ that Theorem 4.10 needs to compute the rebates.

5. Computing the Price of Anarchy

Now that we have already characterized the optimal rebates for a particular instance of the problem, we are ready to analyze the performance of this coordination mechanism. We continue to work with networks with substitutable resources and affine cost functions.

We start by providing a bound between the uncoordinated solution (no rebates) and the Stackelberg equilibrium. The case of $\rho \leq 1$ follows from Proposition 3.3. Indeed, using Proposition 2.6, we have that $C_\rho(0)/C_\rho(s) = C(x^{WE})/(\rho C(x^{SO})) \leq 4/(3\rho)$. This means that the price of anarchy arising from (5) is $4/(3\rho)$ for an arbitrary network with affine cost functions. The case of $\rho > 1$ is more involved. We start by computing the social cost of the Stackelberg equilibrium making use of the relations developed in the previous section.

Lemma 5.1. Consider a parallel-link network with affine cost functions. For $\rho > 1$, the optimal social cost equals $\left( \rho/2 \right) \int_{r(1-1/\rho)}^{r(1+1/\rho)} L_\infty(z)dz$.

Proof. We rewrite the expression $2r(V_\rho - D_\rho)/\rho$ using the graphical decomposition shown in Figure 4. Indeed, the area of the rectangle equals
$$K(\mathcal{I}_0) \frac{(V_\rho - D_\rho)^2}{2} + \sum_{i \in \mathcal{I}_s} \frac{(V_\rho - b_i)^2}{2a_i} + \int_{r(1-1/\rho)}^{r(1+1/\rho)} (L_\infty(z) - D_\rho)dz =$$
$$\frac{V_\rho - D_\rho}{2} \left( V_\rho - D_\rho K(\mathcal{I}_0) + \sum_{i \in \mathcal{I}_s} \frac{V_\rho - b_i}{a_i} \right) - 2 \sum_{i \in \mathcal{I}_s} x_i^s s_i + \int_{r(1-1/\rho)}^{r(1+1/\rho)} (L_\infty(z) - D_\rho)dz ,$$
where we used the expression for $x_i^s$ in the proof of Lemma 4.9, the expression for $s_i$ in Theorem 4.10, and that $(V_\rho - b_i)^2 = (V_\rho - b_i)(V_\rho - D_\rho + D_\rho - b_i)$. The term with the brace equals $2r/\rho$ because of (13). After some algebra,

$$
\sum_{i \in I_s} x_i^s s_i = \frac{1}{2} \int_{r(1-1/\rho)}^{r(1+1/\rho)} (L_\infty(z) - D_\rho) dz - \frac{r}{2\rho} (V_\rho - D_\rho) = \frac{1}{2} \int_{r(1-1/\rho)}^{r(1+1/\rho)} L_\infty(z) dz - \frac{r}{\rho} L_\rho. \quad (15)
$$

Consequently, the optimal social cost is

$$
C_\rho(s) = rL_\rho + \rho \sum_{i \in I_s} x_i^s s_i = (\rho/2) \int_{r(1-1/\rho)}^{r(1+1/\rho)} L_\infty(z) dz. \quad (16)
$$

Theorem 5.2. Consider a parallel-link network with affine cost functions. For $\rho > 1$, the unique Stackelberg equilibrium $(s, x^s)$ satisfies that

$$
\frac{C_\rho(0)}{C_\rho(s)} \leq \frac{4\rho}{4\rho - 1}.
$$
Proof. Let us assume that $I_s \neq \emptyset$ because otherwise the result is trivial. We need to compare the cost $C_\rho(s)$ computed in the previous lemma to $rL_\infty(r)$. Since $L_\infty(z)$ is a positive and concave function, $L_\infty(z)/z$ is a non-increasing function. Bounding the integral from below as Figure 5 illustrates, we get that

$$C_\rho(s) \geq \frac{\rho r L_\infty(r)}{2\rho} \left( 2 - \frac{1}{2\rho} \right) = C_\rho(0) \left( 1 - \frac{1}{4\rho} \right)$$

as claimed.

The previous result characterizes the tradeoff between willingness to offer rebates and coordination power of the mechanism. The corresponding bound is tight, as Instance 1 demonstrates. (Note that the top-most arc has a constant cost, but one can take that cost equal to $ax + 1$ for an arbitrarily small $a$ and nothing changes.) When the system owner’s willingness to offer rebates is high ($\rho$ is not much larger than 1), the optimal social cost is approximately equal to the total cost under a system optimum; hence, the previous theorem provides a bound that is close to $4/3$. Here, recall that $4/3$ is the price of anarchy when the coordination mechanism can achieve a socially optimal solution (Proposition 2.6). Not surprisingly, when the willingness to offer rebates decreases (big $\rho$), the previous theorem gives a bound that is close to 1 because the system owner cannot do much better than in a Wardrop equilibrium.

Finally, we compute the worst-case ratio between the participants’ real cost under a Stackelberg equilibrium and under a system optimum, as we proposed in (4). In the case of $\rho \leq 1$, the flow $x^{SO}$ is at equilibrium (and it is the unique one for strictly increasing cost functions, see Proposition 3.1), which implies that for an arbitrary network with affine cost functions the mechanism coordinates the network. The following results provide the bound corresponding to the case of $\rho > 1$. First, we express the cost of the system optimum as a function of $L_\infty$ to be able to relate it to the Stackelberg equilibrium.
Lemma 5.3. For parallel-link networks with affine cost functions, the minimal value of the participants’ real cost is

\[ C(x^{SO}) = \frac{1}{2} \int_{z=0}^{2r} L_\infty(z)dz. \]

Proof. Proposition 2.5 implies that there exists a constant \( L^{SO} > 0 \) such that

\[ L^{SO} = 2a_i x_{i}^{SO} + b_i \quad \forall \ i \ \text{s.t.} \ x_{i}^{SO} > 0 \]

\[ L^{SO} \leq b_i \quad \forall \ i \ \text{s.t.} \ x_{i}^{SO} = 0. \]

Proceeding as in Stackelberg equilibrium case, \( x_{i}^{SO} = [L^{SO} - b_i]/(2a_i) \). If we let \( i^{SO} := \max \{ i \in A : b_i \leq L^{SO} \} \), we have that \( r = \sum_{j=1}^{i^{SO}} (L^{SO} - b_j)/(2a_j) \). Hence,

\[ L^{SO} = \frac{1}{K([i^{SO}])} \left( 2r + \sum_{j=1}^{i^{SO}} \frac{b_j}{a_j} \right). \]

Since \( b_i^{SO} \leq L^{SO} < b_i^{SO+1} \), Proposition 4.11 implies that \( L^{SO} = L_\infty(2r) \). Then

\[ C(x^{SO}) = \sum_{j=1}^{i^{SO}} x_{j}^{SO} \left( L_\infty(2r) - \frac{L_\infty(2r) - b_j}{2} \right) \]

\[ = r L_\infty(2r) - \frac{1}{2} \sum_{j=1}^{i^{SO}} \frac{(L_\infty(2r) - b_j)^2}{2a_j}. \]

Finally, using a similar decomposition as in Figure 4, it can be shown that

\[ 2r L_\infty(2r) = \int_{z=0}^{2r} L_\infty(z)dz + \sum_{i \in A, b_i \leq L_\infty(2r)} \frac{(L_\infty(2r) - b_i)^2}{2a_i}. \]

The claim follows from the last two equations. \( \square \)

Theorem 5.4. Consider a parallel-link network with affine cost functions. The Stackelberg equilibrium \((s, x^s)\) described in the previous section satisfies that

\[ \frac{C(x^s)}{C(x^{SO})} \leq \frac{4\rho}{3\rho + 1}. \] (17)

Proof. Equations (1b), (15) and (16) imply that

\[ \sum_{i \in A} x_i^s c_i(x_i^s) = r^{\rho-1} D_{\rho} + \frac{1}{2} \int_{r(1+1/\rho)}^{r(1-1/\rho)} L_\infty(z)dz. \] (18)

From Lemma 5.3, the concavity of \( L_\infty \), and decomposing the area under the curve as Figure 6 illustrates, we have that

\[ 2C(x^{SO}) \geq r^{\rho-1} \frac{D_{\rho}}{2} + r^{\rho-1} V_{\rho} + \int_{r(1-1/\rho)}^{r(1+1/\rho)} L_\infty(z)dz \]

\[ = 2 \sum_{i \in A} x_i^s c_i(x_i^s) - r^{\rho-1} \frac{D_{\rho}}{2}. \]
\[
\geq 2 \sum_{i \in A} x_i^c(x_i^s) - r \frac{\rho - 1}{\rho} L_\rho \frac{L_\rho}{2}
\]
\[
\geq C(x^s) \left(2 - \frac{\rho - 1}{2\rho}\right),
\]
where the second, third and fourth lines hold because of (18), \(D_\rho \leq L_\rho\), and \(rL_\rho \leq \sum_i x_i^c(x_i^s)\), respectively.

Figure 6. Illustration of the bound for \(C(x^{SO})\).

This bound is close to 1 for \(\rho \approx 1\) because in that case a Stackelberg equilibrium is similar to a system optimum, and close to \(4/3\) when \(\rho\) is large because in that case it is similar to a Wardrop equilibrium. As for the previous bound, Theorem 5.4 provides the curve that characterizes the tradeoff between willingness to offer rebates and coordinating power. We highlight that this bound is tight, which can be observed by taking \(\epsilon = 0\) and letting \(\alpha\) tend to 0 in Instance 3.

Remark 5.5. The bound provided by Theorem 5.4 is also valid if one takes the ratio of the participants’ perceived cost in the Stackelberg equilibrium to that in the system optimum. This holds because \(\sum_i x_i^c(x_i^s) - s_i^+ \leq C(x^s)\). Moreover, the same instance as before shows that this bound is tight.

6. Conclusions

We have studied the possible improvement that can stem from the use of rebates to coordinate a decentralized system. As the price of anarchy is a pessimistic measure of the loss of efficiency due to the lack of coordination, we have suggested to take into account the limited power available to a system owner to coordinate participants. Indeed, in practice one attractive mechanism is to offer participants the correct incentives. The system owner’s goal is to minimize social cost, defined as the sum of the participants’ perceived cost and the cost of offering rebates. For parallel-link networks,
which model that resources are substitutes, we have established that the coordination mechanism
directly affects a limited proportion of the demand that is routed optimally among the resources
for which a rebate is offered. We have computed the worst-case inefficiency of this mechanism
for affine cost functions, and have found that the worst-case is achieved with only two resources.
These bounds are parametrized by the system owner’s sensitivity to the cost of offering rebates, and
illustrate the fact that the coordinating power of a rebate scheme increases as the owner’s sensitivity
to the rebate cost decreases.

Several questions remain open. Obtaining efficient algorithms and bounds for the ratio of the
social cost with and without rebates for general cost functions would provide additional insights
about the benefits of using such mechanisms. In addition, it is not immediate how to extend our
results to arbitrary network topologies. Similarly, it is an interesting open problem to determine
the complexity of finding optimal rebates in general. For quadratic cost functions for example, the
optimal rebates can be irrational numbers.\textsuperscript{3} Hence, an optimal rebate vector cannot be computed
in polynomial time but it would be interesting to find a way to approximate it. Finding an efficient
algorithm would allow the system owner to compute its optimal strategy. Proving its hardness
would shed light into this problem and would imply the need to look for good heuristics. We have
provided a partial answer to this question by proving that a solution can be found efficiently for
parallel-link networks with affine cost functions with a polynomial time algorithm. Finally, another
interesting open question is whether optimal rebates must be unique or not. Again, we have shown
that the affirmative is true for parallel-link networks with affine cost functions.

In the future, we plan to incorporate the possibility that the system owner both taxes some
resources and offers rebates in others. Such extended model will be useful to model systems in
which both incentive mechanisms can co-exist (for example London uses congestion pricing and
subsidizes public transportation).

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\textsuperscript{3}For example, considering the instance shown in Figure 1 with costs functions $1$ and $x^2$, and $\rho = 2$, it is optimal to
offer a rebate of $(11 - \sqrt{13})/18$ for the resource with constant cost.
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