Efficient Cost Allocation

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Abstract

Firms routinely allocate the costs of common corporate resources down to divisions. The main insight of this paper is that any efficient allocation rule must reflect the firm's underlying cost structure. We propose a new allocation rule (the polynomial rule), which achieves efficiency and approximate budget balance. We also examine conditions under which simple allocation rules induce efficiency. Finally, we show that welfare losses due to linear allocation rules increase with firm size. Thus polynomial allocation rules should be preferred to linear rules for larger firms.

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1 Introduction

The multiple divisions within a firm often share a variety of common resources, such as information technology, legal services, human resource management, executive time, etc. Managerial accounting textbooks (Horngren et al. (2005), Zimmerman (2006)) and surveys of company practice (Fremgen and Liao (1981), Atkinson (1987), Ramadan (1989), Dean et al. (1991)) document the widespread practice of common cost allocation to induce appropriate consumption of corporate resources. For example, if divisions were not allocated any corporate costs, they may have adverse incentives to overconsume such common resources. The objective of this paper is to examine cost allocation rules that solve this free-rider problem, i.e. induce efficient resource use by divisions acting simultaneously and independently.

We demonstrate that the main feature of any efficient allocation is that it must reflect the firm's underlying costs. While this point may seem obvious, the linear rules used in practice make allocations without regard to the shape of the firm's cost function, and this keeps such rules from achieving efficiency. The reason for this failure is straightforward: charging for each unit of common resource used at the same constant rate (whether an actual average cost or a budgeted per-unit overhead rate) ignores the fact that the actual marginal cost of each unit of resource used may depend on the total amount of resources used (for example, if the firm's cost function is highly convex, it is much more costly for the firm to acquire an additional unit of the resource if it already has a hundred than if it only had one). Consequently, under such cost allocation schemes, the price that a division pays for an additional unit of resource (the private cost to the division) differs from the actual marginal cost to the firm, which causes inefficient resource consumption decisions by the division.

The analysis here operates in environments that more closely resemble real-world settings, with the aim of recommending cost allocations that will be practically useful to managers. First, we depart from formal mechanism design theory (such as Green and Laffont (1979)) in that we assume that the private information of the divisional managers is too complex to be embedded within the firm's contracts. Therefore, the firm cannot perfectly obtain the manager's entire private information through a complex reporting game and through contracts that depend on announcements of private information. While formal mechanism design has spawned an enormous literature on incentives, it requires a large amount of information (knowledge of all production and utility functions, rich contract spaces, etc.). Thus it has been unable to provide concrete advice to actual managers, since it relies on menus of contracts that enjoy nice theoretical properties but are rarely adopted in practice (see Rogerson (2003) and Wilson (1987) for a fully articulated critique). Instead, we propose a world where private information is sufficiently complex, communication is sufficiently costly, and contracts are sufficiently incomplete that the Revelation Principle no longer applies. Divisional managers have private information on their divisional production functions, but cannot perfectly communicate this information to the central office via a complex menu of contracts.

The class of efficient cost allocations turns out to be large. However, the class of efficient cost allocations to be used in practice can be narrowed down by imposing additional desirable properties on these allocations. In line with our main goal of capturing a more realistic firm environment, we require cost allocation rules to satisfy certain properties of actual allocation methods used in practice. Like the early cooperative cost allocation literature, we impose certain axioms on allocation rules and explore when some or all of these axioms can be satisfied. In particular, an allocation is *budget balancing* if the sum of the allocated costs equals total cost; an allocation is *fair* if a division pays nothing if it consumes none of the resource; and an allocation is *simple* if it can be written as a ratio. Linear allocation rules commonly used in practice satisfy all three properties, though they are not efficient. Requiring these properties constrains the set of possible efficient allocation rules. For example, the firm could easily charge every division the full corporate cost. While this would achieve efficiency for each division, it would grossly break the budget. The question we set out to answer is whether it is possible to construct efficient cost allocation rules that possess any of these additional desirable properties.

We show that it is possible to construct allocations that are efficient and *approximately* budget balancing. This allocation rule (called the polynomial allocation) induces efficient resource levels, but may exhibit a small budget imbalance. For firms with more divisions, this budget imbalance shrinks, eventually vanishing altogether. Numerical simulations show that for a firm with as few as four divisions, these imbalances are a small fraction of total cost. We give an explicit algorithm for calculating the polynomial allocation from the firm's cost function: first, fit a polynomial to the firm's cost function, and then use the coefficients of that polynomial to construct the allocation rule (specifically, use the coefficients to determine the transfers to different divisions). This illustrates the main message that an efficient allocation must reflect the firm's cost function. In fact, the firm can use this explicit algorithm even if it does not know its cost function exactly, but must estimate its cost function from internal cost data. This makes the polynomial allocation useful in practice, as it reduces the informational requirements of the allocation.

In addition to the result discussed above, we also show that it is possible to construct allocations that are efficient, budget balancing, and simple if one of two special cases holds: either all divisions have the same production function (i.e., are equally productive), or the firm knows the relative efficient resource use levels (i.e., for each pair of divisions, i and j, the firm knows that in an equilibrium division i will consume α times as much of the common resource as division j will). However, it is generally not possible to construct cost allocations that, in addition to efficiency and budget balance, also satisfy fairness.

Even though the linear rules used in practice are in general not efficient, they are widely used in practice. Therefore, we conclude our analysis by exploring the welfare losses of linear rules. In particular, we show that these welfare losses increase with the number of divisions. Intuitively, linear rules are inefficient because they do not reflect the firm's underlying costs, and therefore do not adjust to changes in the firm's cost function. The linear rule is a blunt instrument to control managerial behavior compared to the efficient rule, which varies with the firm's underlying costs. An increase in the number of divisions aggravates the free-rider problem, and linear rules are less capable of resolving this problem compared to efficient rules.

The existing literature on cost allocation spans both accounting and economics, and relies on both cooperative and non-cooperative games. The older cooperative game theory approach began with Shubik (1962), who suggested that the Shapley value of a game can be used to allocate accounting costs. Subsequent papers have expanded on the Shapley allocation by incorporating notions of equity (Hughes and Scheiner (1980)), bargaining between agents (Roth and Verrecchia (1979)), and even variations on the Shapley allocation closer to actual practice (Moriarity (1975), Louderback (1976), Gangolly (1981), Balachandran and Ramakrishnan (1981)). The main problem with the cooperative approach is not only an inability to consider agent-level incentives, but also the severe informational requirements of the cost function. In cooperative models the cost function is defined on all subsets of agents, and this information is necessary when forming the allocation. For these reasons, recent analytical work in cost allocation has shifted into the non-cooperative realm.

Agency models of cost allocation take place in single-agent and multiple agent settings. Single agent settings consider a principal who must compensate and possibly allocate costs to an agent. For example, Baiman and Noel (1985) show that allocating costs can assist in dynamic performance measurement. Magee (1988) shows that the agent's optimal contract can include a cost component based on activity levels to better control his unobservable effort levels. Demski (1981) also takes a performance measurement approach and argues that cost allocation is valuable if it provides additional information for contracting purposes. Because these papers involve only a single agent, they do not consider issues of common cost allocation, i.e., cost allocation across multiple divisions.

Some papers consider multiple agents. Suh (1987) shows that the principal may want to include non-controllable costs in order to discourage collusion between the agents. Yet that model does not explicitly speak to the form of the allocation rule, and instead asks whether the compensation should or should not include non-controllable costs. Therefore Suh (1987) is more a paper on *whether* you should allocate costs instead of *how* you should allocate costs. Rajan (1992) shows that cost allocation schemes can serve a coordination purpose when multiple agents have correlated private information. Baldenius et al. (2006) find that a cost allocation based on hurdle rates of divisional reports to a central office is an optimal mechanism in a multiple division, multiperiod setting. These last two papers both allow communication between the agents and the principal and assume the principal can commit to a menu of contracts. We do not make these assumptions on communication and commitment here.

There has been a recent surge of interest in simple and robust mechanism design. All such papers begin with the observation that real-life mechanisms are much simpler than the complex mechanisms articulated in theory. A handful of papers seek to calculate the welfare losses from simple, common mechanisms used in practice. For example, Rogerson (2003) examines fixed-price cost reimbursement contracts in the defense industry, McAfee (2002) considers matching and rationing problems using only two priority classes, and Satterthwaite and Williams (2002) explore the double auction as a simple trading mechanism. All three papers show that simple mechanisms fare quite well, despite small efficiency losses. Hansen and Magee (2003) show that linear allocation rules are robust in a model of a single decision-maker who must allocate capacity to multiple products. In particular, as the number of products grows, linear allocation rules become optimal because the expected benefit per unit of capacity is set equal across project types. Another cluster of papers responds to the Wilson critique of mechanism design (Wilson (1987)). Wilson argues that the main problem with complex mechanisms is that they assume the mechanism designer knows the game that is played. Bergemann and Morris (2005) and Arya et al. (2005) consider mechanisms that are robust to small perturbations in the environment.

Like much of the prior cost allocation literature, this paper has several normative messages. First and foremost, we show that, in order to achieve efficiency, the cost allocation rule must reflect the firm's underlying cost structure. Second, we take as given certain properties of allocation rules which may reflect exogenous constraints, such as ease of accounting (budget balance), bounded rationality (simplicity), and equity (fairness). We then explore when it is possible to achieve efficiency subject to these constraints. The polynomial and simple allocation rules proposed in this paper are intended for actual implementation in real-world environments, the former when the firm has a large number of diverse divisions having private information about their production functions, the latter when the firm is small and simplicity is paramount. Both these recommended allocations share the feature that they reflect the firm's underlying costs, and show that embedding such costs into the allocation itself will bring the divisional resources closer to efficient levels.

The paper is organized as follows. Section 2 provides a motivating example; Section 3 presents the main model and explores efficient allocation rules; Section 4 investigates budget balance and constructs the efficient polynomial allocation rule that (approximately) implements it; Section 5 addresses the additional requirement of fairness; Section 6 examines simple allocation rules; Section 7 investigates the welfare losses from linear allocation rules, and Section 8 concludes.

2 A Motivating Example

To fix ideas, consider the following simple example of corporate cost allocation, as taught in textbooks, classrooms, and as practiced in corporations. Imagine a firm that consists of two divisions, labeled 1 and 2. Each division selects a resource consumption level for its own plant. Each plant requires information technology (IT) support that the firm provides to all divisions. The corporate IT department has both variable costs (number of computers per plant, number of IT support engineers dedicated to each division) as well as fixed costs (overhead for the IT division, salary of the IT department manager, general administrative costs of running the department). The firm allocates these IT costs to each division to induce the efficient use of corporate IT resources. Assume that plant resource level drives IT costs: as plant resource level increases, so do corporate IT costs. Suppose division 1 consumes \$100m worth of the resource and division 2 consumes \$200m worth. This generates \$9m of corporate IT costs for the firm. Since plant resource consumption is the cost driver for IT costs, the firm will allocate IT costs to each division based on its own resource level relative to total resource consumption. Thus the allocated costs to each division are (all figures in millions):

$$\left(\frac{\$100}{\$100+\$200}\right)\$9=\$3 \quad \text{for division 1,}$$
$$\left(\frac{\$200}{\$100+\$200}\right)\$9=\$6 \quad \text{for division 2.}$$

More generally, let x be division 1's resource level, y be division 2's resource level, and let $S_i(x,y)$ be the share of corporate IT costs allocated to division i, for $i=1,2$. Then, allocating costs according to relative resource levels is an allocation rule of

$$S_1(x,y)=\frac{x}{x+y} \quad \text{and} \quad S_2(x,y)=\frac{y}{x+y}.$$

Call this allocation rule the *linear* rule. Observe that the linear rule satisfies budget balance, or $S_1(x,y)+S_2(x,y)=1$ for all x,y. For any set of resource levels, this rule always allocates all costs down to the divisions, presumably to ease accounting calculations. Second, observe that each division pays nothing if it consumes nothing, or $S_1(0,y)=0$ for all y and $S_2(x,0)=0$ for all x. This seems to satisfy some notion of equity or fairness, since a division is not charged if it produces nothing. Finally, observe that the linear rule is simple in the sense that it can be written as a ratio of each division's activity level (resource level).

We take as given that these three properties of the linear allocation rule are a reduced-form expression for the underlying economic environment of the firm. For example, perhaps making accounting calculations is a time consuming activity and therefore budget balanced allocation rules reduce these computing costs, or perhaps managers can more easily understand allocation rules that are written as a simple ratio, or perhaps interdivisional conflict is a serious drain on productivity and fair allocation rules reduce such conflict. Whatever the reasons, these three properties represent the environment of the firm, and therefore set the stage for the forthcoming analysis. The linear allocation rule certainly satisfies all three properties but it is not efficient, as shown later. But is it possible to find an efficient allocation rule that also satisfies any or all of the three properties? To get traction on this question, it is necessary to articulate the model more precisely and define efficiency accordingly.

3 The Model

Consider a firm with n divisions and a central office. The firm has a decentralized structure: each division acts as a profit center and therefore each divisional manager's goal is to maximize the profit of his or her division. Each division simultaneously selects a resource level k_i. These resources are assets such as plants, machines, human capital, etc. The production of the firm is given by Φ, which is a function of all divisions' resource levels, k_1 through k_n. We assume that $\Phi(0,\ldots,0)=0$ and that the marginal productivity of division i takes the form

$$\frac{\partial\Phi}{\partial k_i}(k_1,\ldots,k_n)=z(n)\phi_i(k_i)$$

for all i, where ϕ_i are positive-valued, strictly decreasing functions and z is a positive-valued, strictly increasing function. In words, we assume that the marginal productivity of division i decreases in its resource use, but increases in the size of the firm. This allows us to model synergies in production, at the same time keeping the production function separable in the production levels of individual divisions. We believe that this is a simple, reduced-form way to model synergies between multiple divisions. As the number of divisions grows, these synergies increase, and therefore each division's marginal productivity increases. We now denote division i's production function by $f_i(k_i) \equiv \int_0^{k_i} z(n)\phi_i(x)dx$. Note that our assumptions on ϕ_i imply that f_i is strictly increasing and strictly concave for any i. Also notice that $f_i(k_i)/z(n)$ would give the production of a firm consisting of just division i. Lemma 1 in the Appendix shows that $\Phi(k_1,\ldots,k_n)=\sum_{i=1}^n f_i(k_i)$, i.e., the production function is separable in individual divisions' productions, as claimed above.

All divisions of the firm make use of a common, firm-wide resource, such as information technology, corporate human resources, executive time, etc. The cost to the firm of the use of this common resource given each division's resource level is $C(k_1 + \cdots + k_n)$, where C is strictly increasing, weakly convex, continuous, and twice continuously differentiable.[1] Let $k = (k_1, \ldots, k_n)$ be the resource vector, and let $k_{-i} = (k_1, \ldots, k_{i-1}, k_{i+1}, \ldots k_n)$ be the resource vector for all divisions other than i. Furthermore, let $K = k_1 + \ldots + k_n$ be the total resource level and $K_{-i} = \sum_{j \neq i} k_j$, the total resource level for all divisions except i. We assume that the feasible resource level set for each division i is bounded above by $\bar{k}_i$, so that $k \in \prod_{i=1}^n [0, \bar{k}_i]$ and $K \in [0, \bar{K}]$, where $\bar{K} = \sum_{i=1}^n \bar{k}_i$.

The firm's total profit is

$$\sum_{i=1}^{n} f_i\left(k_i\right) - C\left(K\right).$$

Let $k^* \equiv \{k_i^*\}_{i=1}^n$ denote the first-best efficient resource levels, i.e., the resource levels that maximize the firm's total profit. The first-order conditions for a maximum require that (for all i) $f_i'(k_i^*) = C'\left(K^*\right)$, where $K^* = \sum k_j^*$ is the efficient total resource level.[2] The production functions f_i are private information of the respective divisions, but resource level decisions k_i, current production levels $f_i(k_i)$ and costs $C(\cdot)$ are common knowledge.[3] This stands in contrast to many agency models where effort is unobservable but utility functions are common knowledge. While it is plausible that effort is legitimately unobservable, it is hard to believe that resource consumption is unobservable. Internal accounting records document resource levels, as the firm must use such numbers for budgeting and compensation purposes.

Contracts within the firm are incomplete, and so the firm cannot perfectly obtain the divisional manager's private information through a complex menu of contracts and incentive constraints. In this setting, the Revelation Principle does not apply. Contractual incompleteness reflects the high costs of including complex private information within the contracts of divisional managers. Indeed, while menus of contracts exhibit nice theoretical properties, they are rarely observed in practice, and certainly not to the extent and complexity as predicted by theory. One reason for this is that either the agents or the mechanism designer does not perfectly know the game that is being played, and therefore simple and incomplete contracts

[1] The resource level k_i is measured in units (plants, machines, factories) while the cost of these resources is measured in dollars.

[2] The assumptions on C and f_i guarantee that the second-order conditions for a maximum are met, and Lemma 2 in the Appendix shows that the solution k^* is unique.

[3] That is, each division (and the central office of the firm) observes the current production of the other divisions, but it does not observe the other divisions' full production *functions*: everybody sees what each division produces at its current resource level, but only the division itself knows what it would produce if its resource level were to change.

are robust to this uncertainty, as modeled in Bergemann and Morris (2005) and Arya et al. (2005).

Even though resource levels are observable, the private information of the divisions prevents the firm from implementing first-best resource levels through a forcing contract, i.e., a contract that pays each division a positive amount if it selects the first-best resource level, and zero otherwise. A forcing contract is impossible because the firm does not even know the first best resource levels. The firm can, however, induce first-best resource levels through an appropriate cost allocation rule. Suppose that the firm charges $A_i(k_1 \dots k_n)$ to division i, based on the resource levels of all divisions.[4] Let S_i be the *proportion* of common costs charged to division i, so $S_i = A_i/C$. Each division then maximizes

$$\Pi_i = f_i(k_i) - S_i(k_i, k_{-i}) \cdot C(k_1 + \cdots + k_n).$$

Thus the agency problem here is the classic free-rider problem. Each division's resource consumption generates common costs for the firm, and thus imposes negative externalities on other divisions. The objective of the firm is to choose the allocation rule to induce the selection of efficient resource levels. For example, if the firm does not allocate any of these common costs ($S_i = 0$), then each division will select a resource level that maximizes its own private return without considering its effect on other divisions. This will lead other divisions to over-consume; in other words, they will select a privately optimal resource level that exceeds the socially optimal (efficient) level.

Casting the common cost allocation problem in terms of implementing efficient resource levels gives guidance on what the "right" allocation rule is. The incentive effects of cost allocations have been known in the accounting literature at least dating back to Zimmerman (1979) and are now acknowledged by most modern accounting textbooks (such as Horngren et al. (2005) or Garrison et al. (2004)); see in particular the discussion of cost allocations as a system for taxing excessive consumption in Zimmerman (2006, Chapter 7C). Nonetheless, the exact form of incentive-optimal cost allocations has not been studied extensively, particularly in an environment with incomplete contracts. In this paper, we seek to fill this gap.

3.1 Efficient Allocation Rules

Each division chooses its resource level simultaneously; therefore it is necessary to solve for the Nash equilibrium of the resource level selection game. Let $\tilde{k}_i$ denote the equilibrium resource

[4]This includes charging each division a capital charge rate for its resource level, in which case $A_i(k) = \mu_i k_i$ for some $\mu_i > 0$. Of course, A_i can be much more general than this.

level actually chosen by division i. These actual resource levels will be determined by the system of n first-order conditions from the individual divisions' optimization problems:

$$f_i'(\tilde{k}_i) = S_i(\tilde{k}_i, \tilde{k}_{-i})C'(\tilde{K}) + C(\tilde{K})\frac{\partial S_i(\tilde{k}_i, \tilde{k}_{-i})}{\partial k_i},$$

where $\tilde{k}_{-i}$ is the equilibrium resource levels of all divisions other than i, and $\tilde{K}$ is the equilibrium total resource level. Thus in equilibrium, the marginal return to additional resource consumption equals the marginal cost. Observe that there are in fact two marginal costs of resource consumption. For every dollar's worth of resources, the division bears not only the direct marginal cost from use of the common resource, but also the marginal change in the allocation rule; these are the first and second terms on the right-hand side in the equation above. This shows that cost allocations indeed have incentive effects. If the firm allocates costs according to certain activity levels (such as resource levels), then the manager will select his activity level depending on the actual allocation rule. The firm can therefore control the actual resource levels by choosing the appropriate cost allocation rules.[5]

We take the position that the single most important goal the firm must consider in designing cost allocation rules is efficiency. That is, the rules should work as a tool for aligning the interests of divisional decision makers with those of the firm as a whole, inducing the individual managers to use resources in a way that maximizes overall firm profit. We now formalize this notion. Let $S \equiv \{S_i\}_{i=1}^n$ be a set of cost allocation rules.

Definition 1 *S is **efficient** if, for any set of production functions, $\tilde{k}_i = k_i^*$ for all i.*

In other words, a set of cost allocation rules S is efficient if each allocation rule S_i induces efficient resource levels for every division. Let S_i^* denote an efficient allocation rule and S^* the corresponding set of efficient allocation rules. Since the firm does not know the individual production functions, it can only ensure efficiency if it induces $\tilde{k}_i = k_i^*$ for *all* possible production functions. The differential equations given by the first-order conditions for the first best and for the individual divisions' problems immediately yield a straightforward characterization of efficient allocation rules (all proofs are in the Appendix):

Proposition 1 *S is efficient if and only if there exist transfers $r_i : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, for all i and all $(k_1, \ldots k_n)$,*

$$S_i^*(k_i, k_{-i}) = 1 - \frac{r_i(k_{-i})}{C(K)}. \tag{1}$$

[5] Zimmerman (1979) first articulated the incentive effects of cost allocations. In particular, he argued that the firm can use cost allocations to tax undesirable or excessive investment, thus controlling divisional managers' behavior. This model formalizes Zimmerman's early insight.

Therefore the firm can implement efficiency (i.e., induce first-best resource levels) by setting an allocation rule with an appropriate transfer scheme $r_i(k_{-i})$, which constitutes a payment between division i and the central office. The intuition behind this result becomes apparent if we rewrite (1) by multiplying through by $C(K)$: $A(k_i, k_{-i}) = C(K) - r_i(k_{-i})$. We can thus imagine the cost allocation as a two-step process. Each division first pays the full cost of the firm ($C(K)$) and then receives a refund in the form of a transfer that depends only on the *other* divisions' resource choices ($r_i(k_{-i})$). The first step (paying the full common cost) makes each division's perceived cost move one-to-one with the firm's common cost (i.e., it equates each division's individual marginal cost of resource consumption to that of the firm), thus inducing the division to select the optimal resource level. The second step (the transfer) allows the firm to actually charge each division less than the total common cost without distorting the incentives of the division. This is because the transfer to each division does not depend on that division's decisions: the division cannot affect its own transfer by manipulating its resource level.

To see the logic in the proposition above, note that, under efficiency, the allocation to division i ($A_i(k_i, k_{-i})$) as a function of k_i must be a parallel shift of the total cost ($C(K) = C(k_i + k_{-i})$). This is because the division equates it marginal benefit $f_i'(k_i)$ to its *private* marginal cost $\frac{\partial}{\partial k_i} A_i(k_i, k_{-i})$, whereas efficiency requires that the same marginal benefit be equated to the firm's *overall* marginal cost $\frac{\partial}{\partial k_i} C(k_i + k_{-i}) = C'(K)$. Thus, if the division's decision is to coincide with the efficient decision, its private marginal cost must equal the overall marginal cost, i.e., $\frac{\partial}{\partial k_i} A_i(k_i, k_{-i}) = C'(K)$. Put differently, the functions $A_i(k_i, k_{-i})$ and $C(k_i, k_{-i})$ must have the same slope at every value of k_i, which means that one must be a parallel shift of the other: the two functions can differ only by a term independent of k_i. This term is the transfer $r_i(k_{-i})$ in the expression above. Note that, as far as efficiency is concerned, the transfer can be *any* function of k_{-i}: after receiving the payment $C(K)$ from each division, the firm can pay back as much or as little of it as it pleases, as long as the transfer given back to each division is independent of that division's own resource use. The transfers therefore act as an instrument that the firm can use to adjust its budget balance without creating incorrect incentives.

That the transfer for division i depends only on k_{-i} bears similarity to the Groves scheme in direct revelation mechanisms (Groves (1973)); hence also the term "transfer." However, the efficient rule S_i^* in Proposition 1 is not a Groves mechanism, since the game played here is not a direct revelation game, i.e., the transfers do not depend on announcements of the private information of the divisions. Nonetheless, the essential logic of the Groves scheme applies here. Division i's transfer, being independent of division i's actions, allows the mechanism designer, in this case, the firm, to adjust the total payment by division i without negatively affecting the division's incentives.

The class of efficient rules is quite large: any allocation is efficient, as long as it satisfies (1) and the transfer to each division does not depend on that division's resource level. Observe that the proposition does not require these transfers to take any specific form, only that they do not depend on the target division's resource level. Nonetheless, the efficient rule in (1) does include the common cost function, and therefore any allocation rule that does not include the common cost function cannot be efficient.

An allocation rule commonly used in practice is the linear rule $S_i^L(k_1, \ldots, k_n) = k_i/K$, where each division is allocated costs based on its relative resource level. The linear rule does not include the common cost function, and therefore it is not efficient (for more discussion on linear rules, see Section 7). Nonetheless, the linear rule satisfies some convenient and intuitive properties. For example, the shares in the linear rule all sum to one, and if any division selects zero resources, it bears none of the common cost. Therefore, we now consider more general allocation rules that also satisfy these properties. Take these constraints on the class of allocation rules as reduced-form expressions for complexity of the environment or a desire for equity among different parties within the firm. The question we ask is whether these additional properties are compatible with our key criterion of efficiency. In the next section, we explore the notion of budget balance, while Sections 5 and 6 address the concepts of fairness and simplicity, respectively.

4 Budget Balance

In this section, we explore the implications of the additional requirement of budget balance, namely, the idea that the cost shares allocated should sum up to one. We begin by defining this notion precisely.

Definition 2 *S is* ***budget balancing (BB)*** *if, for all* $(k_1, \ldots k_n)$,

$$\sum_{i=1}^{n} S_i(k_i, k_{-i}) = 1.$$

Budget balance simply requires the allocations to sum to one, or the sum of the allocated costs to exactly equal total costs.[6] The equality must hold at all values of $(k_1, \ldots k_n)$ (not just at the equilibrium), because the firm does not know the production functions and therefore does not know the equilibrium nor efficient resource levels.

[6]Demski (1981) called allocations that sum to one "tidy." We use the term "budget balance," following the extensive literature on public decisions and cost-sharing (Groves (1973), Green and Laffont (1979), Moulin and Shenker (1992), Moulin and Sprumont (2005)).

In practice, firms do not always allocate all of their common costs. For example, firms may not allocate corporate legal expenses to individual divisions, as it is difficult to determine which individual divisions generate firm-wide legal costs. However, firms do desire budget balance within those common costs the firms do decide to allocate. Thus, a firm that decides to allocate its information technology costs to various divisions often desires budget balance among all IT costs. In other words, the common costs considered here are those costs that the firm does decide to allocate.

With this qualification, the requirement of budget balance is an intuitive and natural one, and typically satisfied by actual cost allocation rules used in practice. In particular, budget balance is satisfied by the linear rule. Textbook examples of cost allocations (such as Zimmerman, 2006, Chapter 7) are also budget balancing: the identified common costs are fully distributed among cost objects (such as divisions of a firm), based on some allocation base (such as hours of resource use). Even when allocations are made *prospectively*, based on budgeted, rather than actual, numbers (as in Zimmerman, 2006, Chapter 9C), a form of budget balance is used: namely, the budgeted allocations equal the budgeted common costs. Furthermore, budget balance also has normative appeal: it simplifies accounting and allows the firm to cover the full costs incurred without putting undue stress on the individual divisions' budgets (which would not be the case if the allocations exceeded the total cost). In addition, if budget balance were not satisfied, divisional performance measurement in a decentralized organization could become meaningless, since the sum of individual divisional profits could be far from the total firm profits.

In general, budget balance constrains the set of efficient allocation rules. This section investigates conditions under which efficient and budget balancing allocation rules exist. While Section 4.1 gives the somewhat discouraging result that exact budget balance is compatible with efficiency only for a particular class of cost functions, we go on in Section 4.2 to construct an efficient rule that is *approximately* budget balancing for any cost function. While this constructed allocation rule cannot be expressed as a ratio and therefore is not a simple rule (in the sense defined precisely in Section 6), it is not difficult to imagine a firm constructing this allocation rule from its cost data. In fact, it can be constructed *without knowledge of the cost function*, as Section 4.3 shows.

4.1 Exact Budget Balance

When do efficient and budget balancing allocation rules exist in general? The following example shows that the search is not futile, even with strictly convex costs and zero fixed costs.

Example. Let $n = 3$ and let $C(K) = K^2$. Our goal is to create an efficient and budget balancing cost allocation.

Recall from Proposition 1 that efficiency requires the allocations to take the form

$$A_i(k_i, k_{-i}) = C(K) - r_i(k_{-i}).$$

Therefore all three allocations together sum to

$$A_1(k_1, k_{-1}) + A_2(k_2, k_{-2}) + A_3(k_3, k_{-3}) = 3C(K) - r_1(k_{-1}) - r_2(k_{-2}) - r_3(k_{-3}).$$

Budget balance requires that this total allocated cost be equal to the total common cost, i.e., $3C(K) - r_1(k_{-1}) - r_2(k_{-2}) - r_3(k_{-3}) = C(K)$, or

$$r_1(k_{-1}) + r_2(k_{-2}) + r_3(k_{-3}) = 2C(K).$$

Expanding

$$C(K) = K^2 = (k_1 + k_2 + k_3)^2 = k_1^2 + k_2^2 + k_3^2 + 2k_1k_2 + 2k_1k_3 + 2k_2k_3$$

and plugging into the expression for the sum of the transfers above yields

$$r_1(k_{-1}) + r_2(k_{-2}) + r_3(k_{-3}) = 2k_1^2 + 2k_2^2 + 2k_3^2 + 4k_1k_2 + 4k_1k_3 + 4k_2k_3.$$

To obtain the individual transfers, we now just have to regroup the terms in the sum above, making sure that r_i does not contain any terms containing k_i for any i. One (symmetric) way to do this is to write

$$\begin{aligned} r_1(k_{-1}) &= k_2^2 + k_3^2 + 4k_2k_3; \\ r_2(k_{-2}) &= k_1^2 + k_3^2 + 4k_1k_3; \\ r_3(k_{-3}) &= k_1^2 + k_2^2 + 4k_1k_2. \end{aligned}$$

Letting $A_i(k_i, k_{-i}) = C(K) - r_i(k_{-i})$ for all i now yields our desired efficient, budget balancing solution. ■

Notice that in the three-division example above the third derivative of the cost function was zero. The following proposition shows that this is no coincidence:

Proposition 2 *An efficient and budget balancing allocation rule exists if and only if the n^{th} derivative of C is identically 0.*

This completely characterizes the set of efficient and budget balancing allocation rules. The main insight is that every efficient rule must satisfy (1) from Proposition 1, and the allocations must sum to one. This reduces to the expression:

$$\frac{1}{n-1} \sum_{i=1}^{n} r_i(k_{-i}) = C(K).$$

In words, the average transfer must equal the total cost.

Differentiating both sides of the equation above n times with respect to $k_1, \ldots, k_n$ shows that the n^{th} derivative of C is 0. Thus a necessary condition for an efficient and budget balancing allocation is that the n-th derivative of the cost function be zero. Moreover, any cost function whose n^{th} derivative is 0 must be a polynomial of degree less than or equal to $n-1$. The proof of Proposition 2 shows that it is possible to construct a set of transfers based on the coefficients of that polynomial such that the equation above holds. Therefore, $C^{(n)} \equiv 0$ is also a sufficient condition for an efficient and budget balancing allocation. The algorithm for the construction of this rule, given a cost function with $C^{(n)} \equiv 0$, is a simple algebraic exercise based on the multinomial expansion theorem. This is unsurprising, since the only role of the transfer scheme here is to mechanically ensure budget balance; the scheme has no incentive effects. For details of the algorithm, refer to the proof in the Appendix. The construction there might appear quite complex at first glance. However, the algorithm can easily be implemented on a computer and it is not important that the users of the algorithm understand its every detail.

To illustrate the algorithm, consider the special case when $n = 3$ (this is a generalization of the motivating example at the beginning of this section). We know from Proposition 2 that when $n = 3$, an efficient and budget balancing allocation rule exists if and only if $C^{(3)} \equiv 0$, that is, if and only if the cost function can be written as

$$C(K) = a_0 + a_1 K + a_2 K^2$$

for some constants a_0, a_1, and a_2. By the argument above, the efficient and budget balancing rule satisfies $A_i(k_1, k_2, k_3) = C(K) - r_i(k_{-i})$, where $\frac{1}{n-1} \sum_{i=1}^{n} r_i(k_{-i}) = C(K)$.

The construction of the actual transfers proceeds similarly to the construction in the example at the beginning of this section. Recall that in that example we expanded out the expression $2C(K) = 2(k_1 + k_2 + k_3)^2$ and then regrouped the terms in the expression so that r_i did not depend on k_i. The construction in the general case follows the same lines. We expand all the powers of $K = k_1 + \ldots + k_n$ according to the multinomial expansion theorem and then group them so that r_i is independent of k_i for all i.

We begin by constructing the sets P_i^j from the proof of the proposition. For the $n = 3$ case, each of these sets consists of vectors with three elements, where the first corresponds to a power of k_1, the second to a power of k_2 and the third to a power of k_3. For example, $(1, 2, 0)$ would correspond to $k_1 k_2^2 k_3^0 = k_1 k_2^2$. The set P_i^j simply lists all the terms in the expansion of $(k_1 + k_2 + k_3)^j$ that do not contain k_i. For example, $(0, 2, 0)$, which corresponds to k_2^2, is in P_1^2, but neither in P_2^2 (because it contains k_2) nor in P_1^1 (because the expansion of K^1 does not contain any square terms). For our $n = 3$ case, all of the sets are as follows:

$$\begin{aligned}
P_1^1 &= \{(0,1,0),(0,0,1)\}; & P_1^2 &= \{(0,2,0),(0,0,2),(0,1,1)\};\\
P_2^1 &= \{(1,0,0),(0,0,1)\}; & P_2^2 &= \{(2,0,0),(0,0,2),(1,0,1)\};\\
P_3^1 &= \{(1,0,0),(0,1,0)\}; & P_3^2 &= \{(2,0,0),(0,2,0),(1,1,0)\}.
\end{aligned}$$

The first step above accomplished two tasks: it enumerated all the terms in the expansions of all the relevant powers of K and it grouped together the terms that can be used for r_i (i.e., terms that do not contain k_i). All that remains to be done to complete the calculation of the r_is is to multiply each of the terms from each P_i^j by an appropriate constant (coming from the multinomial expansion theorem), so that all the terms from P_1^j, P_2^j and P_3^j sum to $2K^j$. We also need to multiply all these terms by a_j (since K^j is multiplied by that coefficient in $C(K)$). The resulting expressions are the β_i^j from the proof of the theorem (β_i^j is the contribution to r_i that comes from the expansion of K^j). In our case, they are as follows (shown here only for $i = 1$; the other cases can easily be obtained by symmetry):

$$\beta_1^0 = \frac{2}{3}a_0; \quad \beta_1^1 = a_1\frac{2}{2}\cdot\frac{1}{1\cdot 1\cdot 1}(k_2+k_3); \quad \beta_1^2 = a_2\left(\frac{2}{2}\cdot\frac{2}{2\cdot 1\cdot 1}(k_2^2+k_3^2) + \frac{2}{1}\cdot\frac{2}{1\cdot 1\cdot 1}k_2k_3\right).$$

Finally, we obtain the transfers $r_i = \beta_i^0 + \beta_i^1 + \beta_i^2$:

$$\begin{aligned}
r_1 &= \frac{2}{3}a_0 + a_1(k_2+k_3) + a_2(k_2^2+k_3^2+4k_2k_3);\\
r_2 &= \frac{2}{3}a_0 + a_1(k_1+k_3) + a_2(k_1^2+k_3^2+4k_1k_3);\\
r_3 &= \frac{2}{3}a_0 + a_1(k_1+k_2) + a_2(k_1^2+k_2^2+4k_1k_2). \qquad (2)
\end{aligned}$$

Observe that the transfer to each division i (r_i) depends only on the resource levels of the other divisions (k_j for $j \neq i$). Once we have chosen the transfers, we are effectively done. Set the final allocations $A_i(k_1,k_2,k_3) = C(K) - r_i(k_{-i})$ for each division. Note that this rule is efficient by Proposition 1 and budget balancing, because $\frac{1}{2}(r_1+r_2+r_3) = a_0 + a_1K + a_2K^2 = C(K)$, as is easy to check.

The result of Proposition 2 makes a theoretical link between the number of divisions of the firm and its cost function. As long as there are more divisions in the firm than the degree of the cost function, then there will exist an efficient and budget balancing allocation rule. For example, suppose the cost function is the quadratic cost function $C(K) = \gamma K^2$. The third derivative of this function is 0, so any firm with three or more divisions can use the allocation rule constructed in the proof of Proposition 2. This allocation rule is efficient and budget balancing.

One implication is that it is easier to achieve efficiency and budget balance in firms with many divisions. These firms allow for cost functions with large degrees, as the high number of divisions permits a class of increasingly fine polynomials. And even though the allocation in Proposition 2 cannot be written as a ratio, it is easy to calculate using modern computing technology. If the firm knows its cost function, as assumed at the outset, and this cost function is a polynomial with degree less than the number of divisions in the firm, then Proposition 2 provides an algorithm for constructing an efficient and budget balancing allocation rule.

4.2 Approximate Budget Balance with Known Cost Function

Of course, Proposition 2 begs the question of what happens if the cost function is not a polynomial, as some cost functions do not satisfy this property for any n (consider, for example, $C(K) = e^K$). However, recalling the basic result from real analysis that every measurable function can be arbitrarily well approximated by polynomials (which do satisfy the property above for a finite n), it is possible to construct efficient rules that are *approximately* budget balancing, as long as the number of divisions is sufficiently large. Furthermore, the rules get closer and closer to budget balancing as the number of divisions increases, eventually achieving exact budget balance. To make this notion more precise, consider the following definition.

Definition 3 *Let* $\left\{\{S_i^n\}_{i=1}^n\right\}_{n=1}^{\infty}$ *be a sequence of sets of allocation rules. The sequence* ***converges to budget balance in*** n *if*

$$\lim_{n\to\infty} \max_k \left| \sum_{i=1}^{n} S_i^n(k_i, k_{-i}) - 1 \right| = 0.$$

Thus, a sequence of sets of allocation rules converges to budget balance in n if, as the number of divisions increases, the maximum possible deviation from budget balance (over all possible equilibrium values of k) gets arbitrarily small. The reason we focus on the maximum possible deviation over all k is that firm does not known the production functions of the individual divisions in advance and hence also does not know the equilibrium values of k. Since any k is a possible equilibrium (as shown in Lemma 3 in the Appendix), the firm must design the cost allocation rule so as to guarantee that the deviation from budget balance will be small, no matter what the equilibrium k turns out to be.

The idea that the firm aims to minimize the maximum possible deviation is akin to the concept of *minimax* in game theory. Now, bounding this maximum possible deviation becomes easier as the firm becomes larger, because the set of cost functions that allow the exact implementation of efficient and budget balancing allocation rules becomes larger (as per Proposition 2), and consequently better and better approximations of arbitrary cost functions can be found in this set. Convergence to budget balance simply requires that the worst-case deviation from budget balance decreases with the number of divisions so much that this deviation becomes negligible when the number of divisions becomes sufficiently large. Thus, if an allocation rule converges to budget balance and the firm that uses this rule has many divisions, we can rest assured that, no matter what the true production functions of the divisions are and no matter what resource levels are selected at the equilibrium, the deviation from budget balance will be close to zero. This is indeed the case for the efficient allocation rules constructed in the proof of Proposition 2:

Proposition 3 *There always exists a sequence of efficient allocation rules that converges to budget balance in n.*

Call the allocation rule constructed in Proposition 3 the "polynomial allocation," as the allocation itself is built from a Chebyshev polynomial approximation of the cost function (other approximation methods might be used; the Chebyshev approximation was chosen because of its superior convergence properties). The idea behind the polynomial allocation is simple: we know from Proposition 2 how to construct an efficient and budget balancing rule when the cost function is a polynomial. When the cost function is not a polynomial, we can *approximate* the function by a polynomial and construct the allocation rule from this approximated function, instead of the true one. By construction, the rule will still be efficient. Furthermore, the allocations will sum to the approximated cost function. This will result in a budget imbalance. However, as the quality of the approximation improves, the approximated function (and hence also the sum of the allocations) will be closer and closer to the true cost function. Now, as the number of divisions grows, higher and higher order polynomials can be used in the approximation (recall that we can use polynomials of order at most $n-1$). Consequently, the quality of the approximation improves, and the budget imbalance gradually vanishes.

As an example, consider the case $C(K) = e^K$ and let $\bar{K} = 5$ (so that K is restricted to the interval $(0, 5]$). To construct the polynomial allocation, an n-division firm begins by approximating the cost function using Chebyshev least squares approximation of degree $n-1$ over the interval $(0, \bar{K}]$ (this purely numerical exercise can easily be completed by a computer program). For example, if the firm has three divisions, it will use the degree-2 approximation, which in this case turns out to be $\hat{C}(K) = 9.86 - 25.23K + 9.95K^2$. If the firm has five divisions, it will use the degree-4 approximation; in the $C(K) = e^K$ case, $\hat{C}(K) = 1.66 - 5.14K + 9.28K^2 - 3.93K^3 + 0.69K^4$. The first five polynomial approximations are shown in Figure 1. We see that the approximated cost functions are quite close to the true C starting already at the square approximation, which corresponds to $n = 3$, and the fit improves as the order of the polynomial increases. Once the cost function has been approximated by a polynomial, the firm uses the coefficients of the polynomial obtained to calculate the cost allocations according to the algorithm in the proof of Proposition 2. For example, if the firm has three divisions, it uses the degree-2 approximation $\hat{C}(K) = a_0 + a_1K + a_2K$; in the $C(K) = e^K$ case, $a_0 = 9.86$,

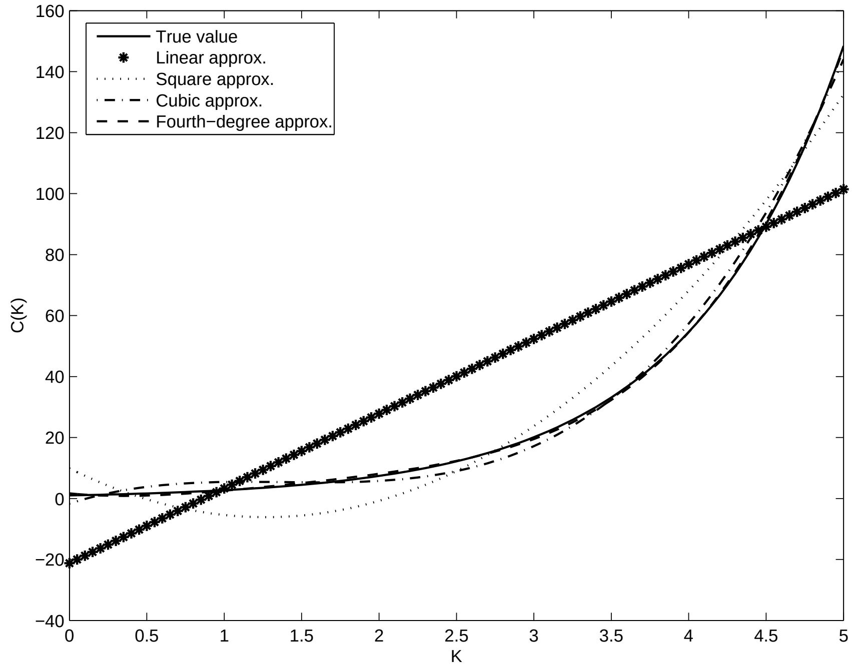

Figure 1: The quality of fit of polynomial approximations

$a_1 = -25.23$ and $a_2 = 9.95$ (as shown above). Finally, the transfers r_i for this three-division case are found by substituting the values of a_0, a_1, and a_2 into equation (2), and the allocation is given by $A_i = C(K) - r_i$.

We know that the allocation thus constructed is efficient. But how does it do in terms of budget balance? Since, by construction, the allocated costs sum to the value of the estimated cost function, the budget imbalance at a given point will equal the difference between the true and estimated cost functions at that point. Figure 2 uses this fact to show deviations from budget balance ($|\sum_{i=1}^{n} S_i^n(k_i, k_{-i}) - 1|$) as a function of K for the first few polynomial approximations. The deviations are high for low degree approximations when K is low, but quickly go to zero as either K or n increases. This already suggests that the convergence to budget balance is quite fast. Figure 3 confirms this. It shows the maximum deviation $\max_k |\sum_{i=1}^{n} S_i^n(k_i, k_{-i}) - 1|$ over the entire interval $[0, 5]$ as a function of the degree of the polynomial approximation. The convergence is very fast (note that the y-axis is on a logarithmic scale). Linear approximations allow for budget imbalances more than 20 times the total cost (however, as we saw before, these occur only when the equilibrium K is very low; the "typical" budget imbalance is much lower than this upper bound), while high-order approximations have essentially negligible imbalances. In particular, with ten divisions (ninth-degree

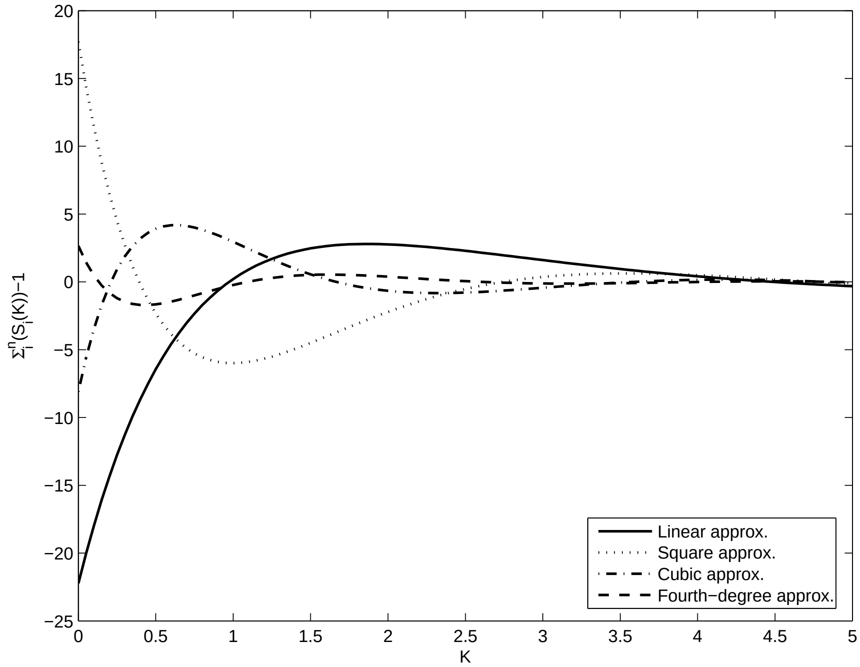

Figure 2: Deviations from budget balance for polynomial rules

approximation) the budget imbalance is no more than 5.8×10^{-4} times the total cost, and with twenty divisions it is at most 8.4×10^{-13} times the total cost—that is, budget balance is achieved almost exactly.

The polynomial allocation constructed above is efficient for any size firm, and converges to budget balance as the number of divisions increase. Therefore, firms with more divisions that make use of this allocation rule will eventually achieve budget balance. This feature of the polynomial allocation makes it superior to other efficient rules. For example, the efficient and fair allocation of Section 5 (Equation 3) falls wildly short of achieving budget balance, even though it induces first-best resource levels. The allocation rule constructed here, on the other hand, is not only efficient, but also achieves approximate budget balance, with the deviation from budget balance vanishing as the number of divisions increase. Furthermore, because the coefficients of the polynomial fitted to the cost function form the basis for the allocation, Proposition 3 further illustrates the message that the "right" allocation must reflect the firm's costs.

This analysis shows that budget balance and efficiency are not as incompatible as previously thought, especially in light of the impossibility results of the formal mechanism design literature (Green and Laffont (1979)). We have always known that it is possible to achieve efficiency by

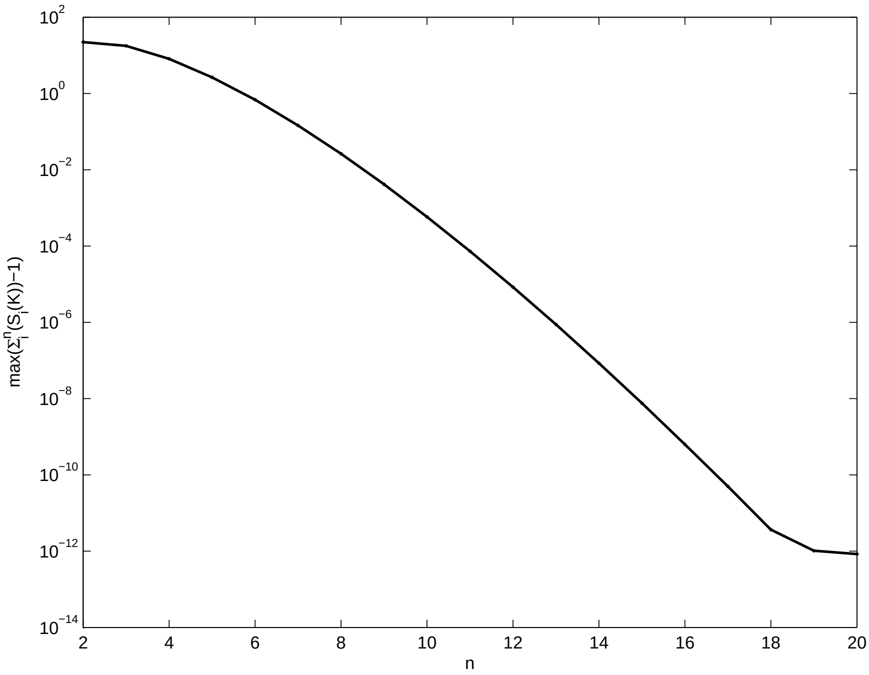

Figure 3: Maximum deviation from budget balance as a function of n

breaking the budget, and that it is impossible to simultaneously achieve efficiency and budget balance for general cost functions. This section shows that the impossibility is not as severe as it seems. In particular, it is possible to construct allocation rules that are simultaneously efficient and converge to budget balance. The main intuition relies on the simple fact that polynomials approximate cost functions arbitrarily well, and the coefficients from these polynomials can be used to construct divisional transfers to induce efficiency. Thus, if the firm is willing to relax its need for exact budget balance, and is willing to replace it with approximate budget balance, then the polynomial allocation can do this and achieve efficiency as well. In some sense, this shows that the impossibility result is "discontinuous," as it is possible to approximate the cost function arbitrarily closely and still achieve efficiency and approximate budget balance. Finally, another remarkable property of the polynomial allocation is that it continues to enjoy nice properties even when the firm does not know its cost function, as shown below.

4.3 Approximate Budget Balance with Estimated Cost Function

In the previous subsection, we assumed that the firm knows the entire cost function. However, this may be unlikely in practice. It is much more likely that the firm has observed the values of the cost function at a finite set of points and has to extrapolate the function elsewhere.

For example, the firm has chosen certain resource levels in the past and this has generated certain common costs. As a result, the firm has a finite amount of observations from its prior activities. It knows the values of its cost function at the points that correspond to its prior choices and must use this information to estimate the costs that it would incur if it were to make a resource choice that is different from those of the past.

In this case, the firm can obtain a polynomial approximation of the cost function by running an OLS regression of observed cost function values on observed resource levels. The firm can then use the polynomial allocation rule with polynomial coefficients from the OLS regression instead of those from the Chebyshev approximation (the latter is unavailable for an unknown cost function). However, it is not immediately obvious that the resulting rule will approach budget balance, even when both the number of divisions and the number of sample points is high; for each n, the OLS regression used suffers from omitted variable bias (due to truncating the series of powers of K at $n-1$), which usually does not vanish as sample size goes to infinity. Fortunately, the special structure of the regressor matrix, along with the convergence result from the previous subsection, guarantees that in this particular case the bias does disappear and the rule does become budget balancing as *both* the number of divisions and the sample size for each number of divisions increase. We now turn to stating this result more formally.

Let an n-division firm have data on cost function values at m points. The data consist of a vector $K^{n,m}$ of m observed total resource levels and a vector $y^{n,m}$ of the corresponding cost levels:

$$\tilde{K}^{n,m} = (\tilde{K}_1^{n,m}, \tilde{K}_2^{n,m}, \ldots, \tilde{K}_m^{n,m})' \text{ and}$$
$$\tilde{y}^{n,m} = (C(\tilde{K}_1^{n,m}), C(\tilde{K}_2^{n,m}), \ldots, C(\tilde{K}_m^{n,m}))'.$$

The firm constructs an allocation rule in two steps as follows:

1. Run an OLS regression of $\tilde{y}^{n,m}$ on the first $n-1$ powers of $\tilde{K}^{n,m}$ and a constant to estimate the cost function by $\tilde{C}$:

$$\tilde{C}(K) = \tilde{c}_0^{n,m} + \tilde{c}_1^{n,m} K + \cdots + \tilde{c}_{n-1}^{n,m} K^{n-1},$$

   where $\tilde{c}^{n,m} = (\tilde{c}_0^{n,m}, \tilde{c}_1^{n,m}, \ldots \tilde{c}_{n-1}^{n,m})'$ is the vector of estimated coefficients from the OLS regression.

2. Construct the cost allocation rules $\{S_i^{n,m}\}_{i=1}^n$ as outlined in the proof of Proposition 3, with $\tilde{c}_i^{n,m}$ in place of $\hat{c}_i$.

Call this the *estimated* polynomial allocation rule. Before stating the approximate budget balance result for this case, it is necessary to expand the definition of budget balance to the case of estimated cost functions:

Definition 4 *Let* $\left\{\{S_i^{n,m}\}_{i=1}^n\right\}_{n=1}^{\infty}$ *be a sequence of sets of allocation rules. The sequence* ***converges to budget balance in*** n ***and*** m *if*

$$\lim_{n\to\infty}\lim_{m\to\infty}\max_k\left|\sum_{i=1}^n S_i^{n,m}(k_i,k_{-i})-1\right|=0.$$

That is, the sharing rules converge to budget balance in n and m if the deviation from budget balance becomes negligible when the firm estimates its cost functions from large samples and the number of divisions becomes large. The intuition for this definition is essentially the same as that for convergence to budget balance in n alone (from the case of known cost functions). The only difference is that in the case of estimated cost functions, one additional variable influences the quality of the approximation and hence also the maximum deviation from budget balance. This variable is sample size: the more observations of the cost function the firm has made in the past, the better it is able to estimate the cost function. An allocation rule converges to budget balance if the maximum possible deviation from budget balance goes to zero as the number of divisions and the number of cost function sample points grow.

Intuitively, this happens for two reasons. First, for any firm with a given number of divisions, the estimated cost function becomes closer and closer to the true cost function as the number of cost function observations (m) grows (this is the inner limit in the definition above). Second, just as in Section 4.2, the allocation rule guarantees that the maximum possible deviation from budget balance diminishes as the number of divisions (n) grows (this is the outer limit in the definition). Consequently, if a large firm with a large number of prior observations of the cost function implements cost allocation rules that converge to budget balance, it can be certain that the deviation from budget balance will be small, regardless of the shapes of the divisions' production functions.

It turns out that, as long as the firms' samples are sufficiently well dispersed over the range of feasible resource levels, the estimated polynomial allocation rule does indeed converge to budget balance:

Proposition 4 *The estimated polynomial allocation rule converges to budget balance in* n *and* m*, if for each* n *and* i*,* $\lim_{m\to\infty}\left(\tilde{K}_i^{m,n}-\frac{\bar{K}}{m}i\right)=0$.

Before we move to a specific numerical example, let us take a moment to summarize the algorithm that the firm uses to construct the estimated polynomial allocation.

1. The firm has observed its total resource use, K, and its corresponding total cost $C(K)$ on m previous occasions. The observed values are stored in a dataset containing two variables/ columns, K and C, and m observations/ rows.

2. Auxiliary variables $K_2, K_3, \ldots, K_{n-1}$ are computed, where K_i corresponds to the i^{th} power of K.

3. Ordinary least squares (OLS) regression is run with C as the dependent variable and K through K_{n-1} as the regressors. The coefficient estimates are a_0 through a_{n-1}, where a_0 is the constant; a_1 is the coefficient on K, and a_2 through a_{n-1} are the coefficients on K_2 through K_{n-1}.

4. Cost allocations are calculated by a computer program using the algorithm in the proof of Proposition 2.

Thus, the algorithm for the estimated polynomial rule differs from that for the simple polynomial rule described in the previous section only in the way the cost function is approximated by a polynomial: the simple polynomial rule uses the Chebyshev least squares approximation, while the estimated polynomial rule relies on OLS regression on a set of previously observed values. The simple polynomial rule requires exact knowledge of the cost function, whereas the estimated rule uses random samples of values of the function.

Let us return to our earlier example $C(K) = e^K$ for $K \in [0, 5]$. Suppose the firm has a sample of m observations of the values of the cost function. As described above, the firm runs OLS regression on these observations to obtain a polynomial approximation of the cost function and then proceeds to construct the cost allocation, as described in the examples in the previous two sections. Figure 4 illustrates the speed of convergence to budget balance. Each line corresponds to a sample size. We see that for each sample size the budget imbalance decreases as the number of divisions grows and that the budget imbalance for any given number of divisions is smaller, the larger the sample size. We see that with a sample size of 100 the deviation from budget balance is less than ten percent (10^{-1}) for firms with more than seven divisions, less than one percent (10^{-2}) for firms with more than eight divisions and less than one-hundredth of one percent (10^{-4}) for firms with twelve or more divisions.

4.4 Prospective Cost Allocation

Note that the cost allocation rules defined above are based on *actual* resource levels consumed by all divisions; that is, these allocations are retrospective or *ex post*. In practice, most firms define their cost allocation rules *prospectively*, on the basis of a combination of actual and budgeted numbers. Typically, an overhead rate is computed at the beginning of a year based on budgeted resource use, and during the year each division is charged this budgeted rate times that division's actual resource use (Zimmerman, 2006, Chapter 9C). In addition, when a clear dichotomy between fixed and variable costs exists, a two-step scheme is sometimes used,

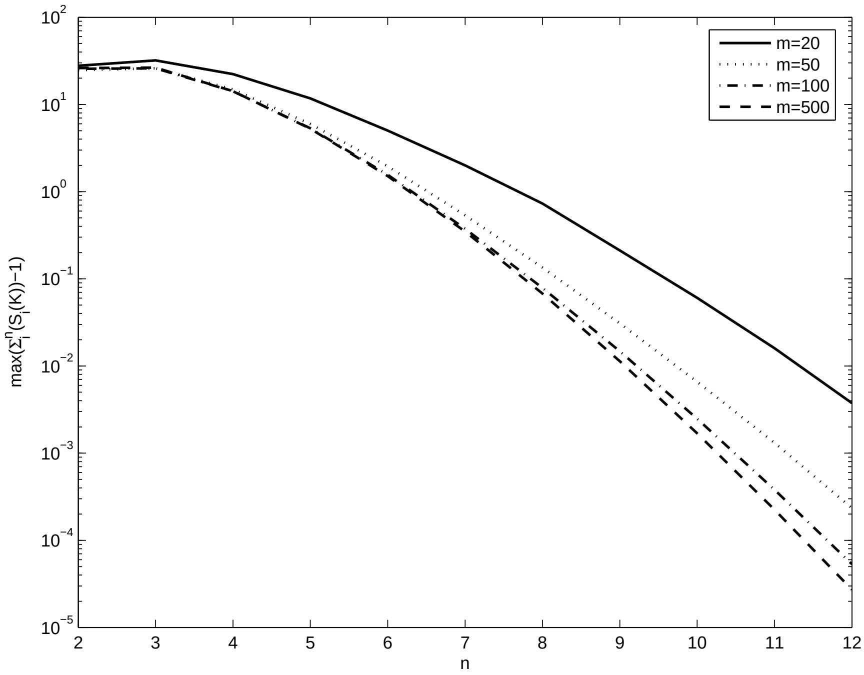

Figure 4: Maximum deviation from budget balance as a function of n and m

whereby fixed costs are allocated based entirely on budgeted numbers and each division's peak demand (Garrison et al., 2004, Chapter 15). What are the implications of these practices in light of the findings delineated above?

The key insight is once again that any efficient allocation must reflect the firm's cost structure, based on the actual resource use. If the allocation does not take the form in Proposition 1, it is not efficient. In particular, simple prospective allocations based on constant overhead rates are not efficient for the same reason that general linear allocations are not efficient: they neglect the fact that the true marginal cost of an additional unit of resource depends on the total level of resource used. Furthermore, prospective allocation schemes will result in deviations from efficiency even if they do take into account the shape of the cost function, as long as the budgeted resource levels differ from the true levels (since efficiency requires that (1) in Proposition 1 hold at the actual resource levels). Therefore, the use of prospective allocations results not only in possible under- or over- absorption of costs, as in Zimmerman (2006, Chapter 9C2), but also in possible deviations from efficiency.

This result, however, does not mean that prospective cost allocations should not be used at all. What it does show is that firms that use prospective cost allocations should fully recalculate cost allocations at the end of the year, after actual numbers become available, and make the necessary adjustments from the prospective allocation. If the divisional managers know that such recalculation will be made, they will make their resource choices based on the expected final, not the tentative budgeted allocations. Consequently, as long as the recalculated allocations are in the form of (1) in Proposition 1, the induced resource use will be efficient. In addition, if the recalculation follows the polynomial scheme discussed above, the allocations will be approximately budget balancing.

Thus, our results suggest that whenever prospective allocations are used, a two-step process should be followed: firms should be charged according to the prospectively determined rate during the course of the year, but an adjustment should be made at the end of the year. We recommend that the polynomial rule be used in the calculation of the adjusted allocation, resulting in efficiency and approximate budget balance.

5 Fair Allocation Rules

In the previous section, we analyzed conditions under which cost allocation rules are budget balancing. In this section, we look at *fairness*: an additional property that, like budget balance, is also satisfied by our motivating example, the linear rule. We begin by defining fairness and finding conditions under which efficient cost allocation rules are fair and then proceed to ask whether fairness is compatible with budget balance (under efficiency).

Definition 5 *S is **fair** if $S_i(0, k_{-i}) = 0$ for all i and all k_{-i}.*

Fair allocation rules neither reward nor penalize divisions for producing zero output.[7] Just like budget balance, we take fairness as an exogenous constraint on the class of feasible allocation rules. Combining efficiency (Proposition 1) and fairness yields the following (unique) allocation rule:

$$S_i(k_i, k_{-i}) = 1 - \frac{C(K_{-i})}{C(K)}. \tag{3}$$

Fairness implies that $r_i(k_{-i}) = C(K_{-i})$: each division pays the additional costs that it incurs. In particular, the efficient, fair allocation above allocates to each division its relative incremental contribution to total cost. Specifically, this is its incremental contribution to total cost, $C(K) - C(K_{-i})$, divided by the total cost $C(K)$. Charging each division its relative incremental contribution to total cost induces each division to consume resources at the efficient level.

[7]This "fairness" property of allocation rules has appeared elsewhere. For example, Baldenius et al. (2006) call this the "no play no pay" condition.

Recall that any efficient rule *must* include the common cost function of the firm. This is essential to obtain efficiency and is a key distinction between efficient allocation rules and linear allocation rules. To illustrate the differences between linear and efficient allocation rules, let us consider a particular example. Let $C(K) = F + K^m$, pick one division (labeled i), and fix the total resource use of all other divisions at $K_{-i} = 1$. We want to compare the share of total costs allocated to division i (as function of k_i) under the two allocation rules: the linear allocation, $S_i^L = \frac{k_i}{K}$, and the efficient and fair allocation, given in equation (3). Figure 5 graphs the share allocated to division i according to the linear rule and according to the efficient and fair rule under two different scenarios: highly convex C ($F = 0$, $m = 2.5$) and high fixed costs with linear variable costs ($F = 1$, $m = 1$).

Suppose that the cost function C is highly convex. Therefore, additional resources for any division are highly costly for the firm. The firm would like to discourage such resource use, and can do this by increasing the share of allocated costs. In particular, for any given resource level k_i, the firm will allocate more of the common cost under the fair and efficient rule than under the linear rule. Essentially, the firm adjusts its allocation to respond to its highly convex cost function. Figure 5 shows that for the cost function $C(K) = F + K^m$, the fair and efficient rule lies above the linear rule if costs are sufficiently convex (if m is sufficiently high).[8] The efficient rule essentially *accelerates* the cost allocation for any given resource level.

Similarly, suppose that the firm's cost function has a high fixed component but a low variable component. In this case, the marginal effect of additional resource use by any division on the total resource level will be small. The firm would like to encourage additional resource use, and can do so by reducing the share of allocated cost for any given resource level. Therefore, the efficient and fair allocation will lie below the linear allocation rule, as shown in Figure 5. The efficient and fair rule essentially *decelerates* the cost allocation for any given resource level, compared to the linear rule. Unlike the linear rule, the efficient and fair rule varies as the firm's cost function varies. Efficiency forces the allocation to reflect the underlying costs; linear rules do not.

A natural question to ask is whether efficient, budget balancing, and fair allocation rules even exist. For example, linear rules are budget balancing and fair, but not necessarily efficient. The polynomial rule constructed in the previous section is efficient and (approximately) budget balancing, but not necessarily fair. The efficient, fair allocation rule in (3) does not always balance the budget. Unfortunately, these examples are not a coincidence: for virtually any commonly used cost function (except for constant marginal cost with zero fixed cost), allocation rules that satisfy all three of the criteria above (efficiency, budget balance, and fairness) do not exist, as the following proposition shows:

[8]See Corollary 1 in the Appendix for a derivation of this result.

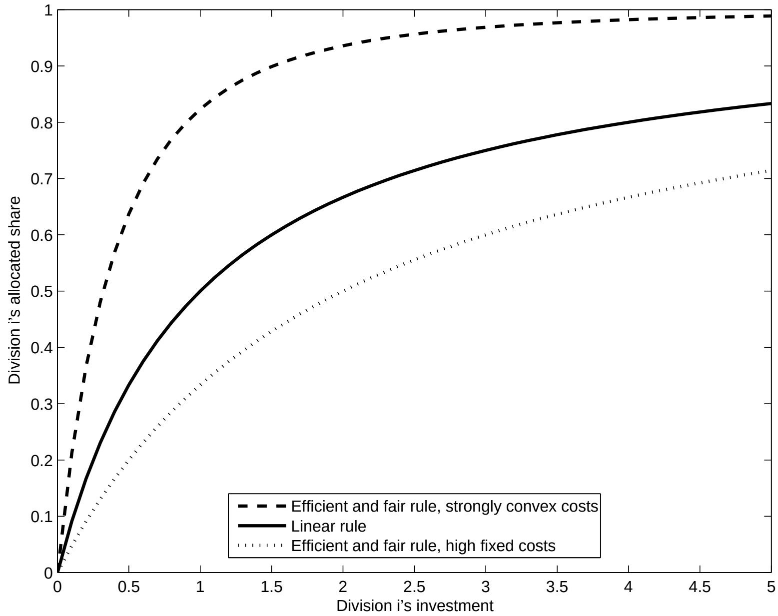

Figure 5: Efficient versus linear allocation rules

Proposition 5 *An efficient, budget balancing, and fair allocation rule exists if and only if $C(K) \equiv \alpha K$, for some $\alpha \in \mathbb{R}_+$.*

Any efficient and fair allocation rule must satisfy (3), or in words, must allocate to each division its relative incremental contribution to total cost. But budget balance constrains these allocations to one. Said differently, budget balance requires that the sum of each division's incremental contribution to total cost exactly equals total cost:

$$\sum_{i=1}^{n} (C(K) - C(K_{-i})) = C(K) = C\left(\sum_{i=1}^{n}(K - K_{-i})\right).$$

The only cost function that satisfies this condition for all $(k_1 \ldots k_n)$ is the constant-marginal-cost function with no fixed cost ($C(K) = \alpha K$). Intuitively, the linearity of the variable costs (constant marginal cost), along with the absence of fixed costs, allows us to exchange the C and $\sum$ in the equation above. Moreover, no other cost function permits this.

This proposition shows that satisfying all three of the target conditions for cost allocation rules is in general impossible. Once again, there is a connection with the literature on public decisions. Green and Laffont (1979) show that it is impossible to find mechanisms that satisfy ex post efficiency, budget balance, and implementation in dominant strategies. Of course, their setting is different from here because they invoke the Revelation Principle, allow contracts and transfers to depend on arbitrarily complex private information, and work within the stronger equilibrium of dominant strategies. Our setting here uses a weaker equilibrium notion (Nash instead of dominant-strategy), a less complete class of contracts (those that do not vary with private information), a more restrictive set of mechanisms (allocation rules instead of general mechanisms), and an additional requirement that the allocations be fair. Nonetheless, the general flavor of the public decisions literature continues to operate here: efficiency, budget balance, and fairness are generally incompatible (except in the knife-edge case when costs are linear and with zero fixed costs).

Therefore, it is necessary to dispose of at least one of the three requirements. A natural candidate is fairness: while it would be hard to defend sharing rules that do not lead to efficient outcomes or fail to balance the budget, it can reasonably be argued that divisions may be punished ($S_i(0, k_{-i}) > 0$) or rewarded ($S_i(0, k_{-i}) < 0$) for zero resource use. For example, a division may receive a bonus if it creates no common costs, since it is not imposing any negative externalities on other divisions.

6 Simple Allocation Rules

This section explores allocation rules that are both efficient and budget balancing, combined with the requirement that they are *simple*. An allocation is simple if it can be written as a ratio. For example, the linear allocation rule, while not efficient, does satisfy budget balance and is expressed as a ratio. The simplicity of the linear rule may contribute to its wide use in practice. Presumably this simplicity eases accounting calculations and is easy to understand and implement for divisional managers. In this section we look for conditions under which simplicity is compatible with efficiency and budget balance.

Call a cost allocation rule S *simple* if it takes the form

$$S_i(k_1, \ldots, k_n) = \frac{g_i(k_i)}{g_1(k_1) + \cdots + g_n(k_n)}$$

for functions $g_i : [0, \bar{k}_i] \to \mathbb{R}_+$. If S is simple and there is some function g such that for each $i, g_i = g$, call S *very simple*. If for each $i, g_i(k_i) = k_i$, call the rule *linear*. Observe that all such rules satisfy budget balance by definition, and if $g_i(0) = 0$ for all i, then they also satisfy fairness.

If there are no fixed costs and the variable costs are linear, i.e., $C(K) = \alpha K$, it is easy to see that the linear rule will achieve first best. In fact, this rule will be the efficient, budget balancing, and fair rule whose existence is guaranteed by Proposition 5 (this is also apparent from Lemma 4). Note, however, that the linear allocation rule will not lead to first best with general affine cost functions incorporating non-zero fixed cost. This is also apparent from

Proposition 5; since the linear rule is budget balancing and fair, it cannot be efficient when $C'' = 0$ and $C(0) \neq 0$.

In the case of symmetric production functions, first best can in fact be achieved with a rule in the very simple form.

Proposition 6 *Let $f_i = f$ for each i. There exists a very simple allocation rule that implements efficient resource levels.*

In this symmetric equilibrium, each division selects the efficient resource level k^* and pays $\frac{1}{n}$ of the common costs. The very simple allocation rule that implements efficiency is given by $g(k) = C(nk)$. Essentially the allocation rule must incorporate the common cost function C, much like the efficient allocation rule does. And, of course, very simple allocation rules balance the budget by definition. Thus, if the divisions are identical, Proposition 6 shows that it is possible to implement efficiency with an allocation rule that is much simpler and more intuitive than the efficient allocation rule. This is good news for a firm that is willing to sacrifice some notion of equity in favor of efficiency. The linear rule is budget balancing and fair, but the very simple rule in Proposition 6 is budget balancing and efficient.

In the case of asymmetric production functions where f_i is not necessarily equal to f_j, it is possible to construct cost allocation rules that lead to efficient equilibria if the relative *ratios* of efficient levels are known in advance, that is, if the central office of the firm knows the ratios $\alpha_i = k_i^*/K^*$ for each i. In other words, the firm may still not know the exact equilibrium levels of resource use, but it knows, for example, that division 1 will use twice as much of the resource as division 2 will. In an example below, we show that such a situation can arise naturally if the firm knows the relative productivities of divisions.

Proposition 7 *Suppose the firm knows $\alpha_i = k_i^*/K^*$. Then there exists a simple allocation rule that implements efficient resource levels.*

The simple rule S_i depends only on $\{\alpha_i\}$, not on K^*. The firm only needs to know the relative resource levels in advance. In the proof of Proposition 7, the allocation rule that implements efficiency is given by $g_i(k_i) = \alpha_i C\left(\frac{1}{\alpha_i} k_i\right)$. In this equilibrium, each division selects its efficient resource level k_i^*, and pays α_i of the shared common costs.

When might a firm know the relative resource levels α_i in advance, outside of the symmetric case? The following example illustrates the applicability. Suppose that $f_i(\beta_i k_i) = \beta_i f(k_i)$ for each i, where $\{\beta_i\}$ is common knowledge and f may be unknown but satisfies $f'' < 0$. So the firm may not know the absolute production functions, but knows the relative production functions, e.g., the firm knows that division 1 is twice as productive as division 2.

The optimal resource level will satisfy $f_i'(k_i^*) = C'(K^*)$ for each i, i.e., marginal private benefit equals marginal social cost. By construction,

$$f_i'(k_i^*) = f_i'\left(\beta_i \frac{k_i^*}{\beta_i}\right) = f'\left(\frac{k_i^*}{\beta_i}\right).$$

So $f'\left(\frac{k_i^*}{\beta_i}\right)$ is constant across i, and the concavity of the production function implies that $\frac{k_i^*}{\beta_i}$ must also be constant across i. Let $\frac{k_i^*}{\beta_i} = D$. Then $k_i^* = D\beta_i$, $K^* = D\sum\beta_j$, and $k_i^*/K^* = \beta_i/\sum\beta_j$. So Proposition 7 applies, with $\alpha_i = \beta_i/\sum\beta_j$. Then, under optimal resource levels, division i will produce $k_i^* = (\beta_i/\sum\beta_j)K^*$. Thus if the firm knows the relative productivity of the different divisions, it will still be able to implement efficient resource levels with a simple allocation rule.

7 Inefficiency of Linear Rules

In practice, linear rules are often used even though they may not be efficient. How inefficient are linear allocation rules? This section considers the relationship between linear and efficient allocation rules. This will give insight into the welfare losses from using linear rules compared to the various efficient rules in this paper, such as the simple, very simple, and polynomial rules. In general, the use of linear rules leads to over-consumption of resources, relative to first-best. The superscript 'L' will designate the linear rule and the superscript '$*$' will designate the efficient rule.

Proposition 8 *Let $C(0) = 0$. If $f_i = f$ for all i, then $k_i^L > k_i^*$.*

In the case of symmetric production, at the symmetric equilibrium, the resource levels under a linear allocation rule will be larger for each division than the efficient symmetric value.

Consider a symmetric case with

$$f_i(k_i) = A(k_i)^p \quad \text{for each} \quad i \quad \text{and} \quad C(K) = BK^q$$

where $0 < p < 1 < q < \infty$. Let $\Delta \equiv 1 - \frac{\pi^L}{\pi^*}$. Take this as the measure of welfare loss; a larger Δ corresponds to a greater welfare loss.

Proposition 9 *The welfare loss Δ increases in n. Δ also increases in q for sufficiently high values of n or p and for sufficiently low values of q, but it decreases in q for sufficiently high values of q.*

Most importantly, the welfare increases in the number of divisions. This shows that larger firms with more divisions suffer more from using linear rules than smaller firms with fewer divisions. Combined with Proposition 3, this gives a general prediction on efficient cost allocation and the size of firms. If n is small, then Proposition 9 shows that the welfare loss from using the linear rule is small, and so small firms can use linear rules with few efficiency losses. But if n is big, not only do the welfare losses from the linear rules increase, but the benefits of the polynomial allocation also increase, since the polynomial allocation converges to budget balance for large n. Therefore, this paper gives an explicit recommendation on allocation practices depending on the number of divisions in the firm: small firms can use linear rules, while large firms should use non-linear rules (such as the polynomial allocation). This recommendation from the model is consistent with the normative flavor of much of the cost allocation literature.

8 Conclusion

Cost allocations have incentive effects. They determine how a division is charged for use of a common resource, and thus become an instrument for the firm to indirectly control the behavior of its decentralized divisions. For example, suppose a firm allocates its information technology (IT) costs based on IT headcount, i.e., the number of corporate IT employees assigned to different divisions. On the margin, this gives incentives for the divisions to reduce resource levels that require a high volume of IT support. Knowing that divisions respond in such a manner, the firm can choose an allocation rule to induce divisional resource levels that are optimal for the firm as a whole.

The cost allocation method that dominates time in cost accounting classrooms, space in cost accounting textbooks, and the energy of corporate financial officers is the linear allocation rule. This rule satisfies certain key features that reflect the underlying economic and managerial environment of the firm, such as budget balance, fairness, and simplicity. That it satisfies these three properties possibly explains its wide use in teaching and practice. However, the linear rule is not efficient in that it does not solve the firm's free-rider problem of multiple divisions simultaneously and independently making resource selection decisions. When is it possible to satisfy the three properties of budget balance, simplicity, and fairness, as well as efficiency? Like much of the prior literature in cost allocation, the aim here has been normative: to recommend allocation rules that bring the firm closer to efficiency but also are feasible for the firm, by virtue of satisfying some or all of the three aforementioned properties. We propose a new allocation—the polynomial allocation—that achieves efficiency and converges to budget balance, as the number of divisions in the firm increases. We also show that if the firm knows the relative efficient investment levels (and, in particular, if all divisions have the same production function), a simple, budget balancing allocation rule can achieve efficiency.

The main message is that any efficient allocation will reflect the firm's underlying costs. As the cost of a common resource increases, the firm would like to discourage use of the common resource (to mitigate the negative externalities of resource consumption) and therefore will accelerate the allocation, relative to the linear rule. Similarly, if the cost of a common resource decreases, the firm can afford to encourage more use of the resource, and therefore will decelerate the allocation, relative to the linear rule. The efficient rule will thus vary with the firm's cost structure. This drives home the broader point that an efficient accounting system will reflect the firm's underlying economic environment.

Collecting all the results in the paper, another message from the analysis is that linear rules "should" be used for small, not large, firms. First, the welfare loss of linear rules increases as the number of divisions grows, as this exacerbates the free-rider problem. And second, the polynomial allocation converges to budget balance as the number of divisions grows. For firms with sufficiently many divisions, the polynomial allocation is efficient and exactly budget balancing. Large, multi-divisional firms may experience significant negative externalities from incorrect cost allocation; therefore getting the allocation "right" in terms of achieving efficiency is perhaps more important for large, rather than small, firms.

Future research in this area will proceed in similar manner by examining what firms actually do and making explicit, concrete recommendations for changes to practice. While it is true that the contracts and mechanisms observed in practice are much simpler than those articulated in the formal mechanism design theory, it is still possible to strike a balance halfway between theory and practice. This requires more careful observation of real life mechanisms, a better understanding of what constraints those mechanisms satisfy, and analyzing optimal mechanisms subject to those constraints.

9 Appendix

Lemma 1 $\Phi(k_1, \ldots, k_n) = \sum_{i=1}^{n} f_i(k_i)$.

Proof of Lemma 1

$$\begin{aligned}\Phi(k_1, \ldots, k_n) = \Phi(k_1, \ldots, k_n) - \Phi(0, \ldots, 0) = \\ [\Phi(k_1, \ldots, k_n) - \Phi(k_1, \ldots, k_{n-1}, 0)] + [\Phi(k_1, \ldots, k_{n-1}, 0) - \Phi(k_1, \ldots, k_{n-2}, 0, 0)] \\ + \cdots + [\Phi(k_1, 0, \ldots, 0) - \Phi(0, \ldots, 0)]\end{aligned}$$

$$= \int_0^{k_n} \frac{\partial \Phi}{\partial k_n}(k_1, k_2, \ldots, k_{n-1}, x)dx + \int_0^{k_{n-1}} \frac{\partial \Phi}{\partial k_{n-1}}(k_1, \ldots, k_{n-2}, x, 0)dx$$

$$+ \cdots + \int_0^{k_1} \frac{\partial \Phi}{\partial k_1}(x, 0, \ldots, 0)dx$$

$$= \sum_{i=1}^{n} f_i(k_i).$$

■

Lemma 2 *Given C and the set of production functions $(f_1, \ldots, f_n)$, there is a single vector of resource levels $(k_1^*, \ldots, k_n^*)$ satisfying the first-best conditions $f_i'(k_i^*) = C'(\sum_i k_i^*) = 0$ for all i.*

Proof of Lemma 2 Suppose that $k^a = (k_1^a, ..., k_n^a)$ and $k^b = (k_1^b, ..., k_n^b)$ both satisfy $f_i'(k_i^j) = C'(K^j)$ for $j \in \{a, b\}$, where $K^j \equiv \sum_{i=1}^n k_i^j$. First, suppose $K^a \neq K^b$. Without loss of generality, $K^a < K^b$. Then

$$f_i'(k_i^a) = C'(K^a) < C'(K^b) = f_i'(k_i^b).$$

This implies that $k_i^a > k_i^b$ for each i, since $f_i'' < 0$. So $K^a = \sum k_i^a > \sum k_i^b = K^b$. Contradiction.

Therefore, $K^a = K^b = K^*$. And because $f_i'(k_i^j) = C'(K^*)$, it must also hold that $k_i^a = k_i^b = k_i^*$ for each i. ■

Lemma 3 *For any vector $(\hat{k}_1, \ldots, \hat{k}_n)$ there is a vector of production functions $(\hat{f}_1, \ldots, \hat{f}_n)$ that leads to optimal resource levels $k_i^* = \hat{k}_i$ for each i.*

Proof of Lemma 3 Let the function $f^\alpha(x) \equiv \alpha x^{\frac{1}{2}}$. Then $f^{\alpha\prime}(x) = \dfrac{\alpha}{2x^{\frac{1}{2}}}$. So for each i, let

$$\hat{f}_i = f^{\alpha_i} \quad \text{for} \quad \alpha_i = 2\hat{k}_i^{\frac{1}{2}} C'(\sum_{i=1}^{n} \hat{k}_i).$$

These production functions generate our desired optimal resource levels. ■

Proof of Proposition 1 We will prove the equivalent statement that a set of cost allocations induces $k_i^* = \tilde{k}_i$ for all i and all $\{f_i\}_{i=1}^n$ if and only if there exist $r_i : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, for all i and all $(k_1, \ldots k_n)$, $A_i(k_i, k_{-i}) = C(\sum_{i=1}^n k_i) - r_i(k_{-i})$.

Recall that, for a given set of production functions, k_i^* and $\tilde{k}_i$ are defined by the first-order conditions for the firm's and the individual divisions' optimization problems, respectively:

$$f_i'(k_i^*) = C'(\sum_{i=1}^{n} k_i^*) \quad \text{and} \quad f_i'(\tilde{k}_i) = \frac{\partial A_i}{\partial k_i}(\tilde{k}_i, \tilde{k}_{-i}).$$

Therefore, $k_i^* = \tilde{k}_i$ for all i if and only if

$$C'(\sum_{i=1}^{n} k_i^*) = \frac{\partial A_i}{\partial k_i}(k_i^*, k_{-i}^*).$$

This completes the proof of the "if" part of Proposition 1: if $A_i(k_i, k_{-i}) = C(\sum_{i=1}^{n} k_i) - r_i(k_{-i})$, then $C'(\sum_{i=1}^{n} k_i) = \frac{\partial A_i}{\partial k_i}(k_i, k_{-i})$ at all $(k_1, \ldots, k_n)$ and thus also at $(k_1^*, \ldots k_n^*)$.

The "only if" part requires some more work, because, for a given set of production functions, the relationship $C' = \frac{\partial A_i}{\partial k_i}$ must hold only at one point, namely, at the corresponding first-best resource vector. This precludes us from determining a global relationship between A and C directly. However, the fact that we must ensure $k_i^* = \tilde{k}_i$ for *arbitrary* production functions guarantees that the relationship will hold at every point $(k_1, \ldots, k_n)$, as Lemma 3 shows.

Since every vector $(k_1, \ldots, k_n)$ is first-best for some set of production functions, the relationship

$$C'(\sum_{i=1}^{n} k_i) = \frac{\partial A_i}{\partial k_i}(k_i, k_{-i})$$

must hold at all $(k_1, \ldots, k_n)$. Holding k_{-i} fixed and integrating with respect to k_i, we readily obtain

$$A_i(k_i, k_{-i}) = C(\sum_{i=1}^{n} k_i) - r_i(k_{-i}).$$

■

Corollary 1 *Let $C(K) = F + K^m$. Let S_i^L and S_i^* denote the "linear" and "fair and efficient" allocation rules, respectively. For any i and k_{-i},*

1. *$S_i^*(k_i, k_{-i}) > S_i^L(k_i, k_{-i})$ for sufficiently high m.*
2. *$S_i^*(k_i, k_{-i}) < S_i^L(k_i, k_{-i})$ for sufficiently high F.*

Proof of Corollary 1 Observe $S_i^L(k_i, k_{-i}) = k_i/K$ and $S_i^*(k_i, k_{-i}) = 1 - \frac{C(K_{-i})}{C(K)}$. So $S_i^*(k_i, k_{-i}) > S_i^L(k_i, k_{-i})$ if and only if

$$\frac{K_{-i}}{K} > \frac{C(K_{-i})}{C(K)} = \frac{F + (K_{-i})^m}{F + K^m} \equiv D.$$

Observe that

$$D = \frac{1}{1 + \frac{K^m}{F}} + \frac{\left(\frac{K_{-i}}{K}\right)^m}{\frac{F}{K^m} + 1} \to 0 \text{ as } m \to \infty$$

.

Both terms vanish as $m \to \infty$ since $K_{-i} < K$. This proves part one.

Observe that

$$D = \frac{1 + \frac{K_{-i}^m}{F}}{1 + \frac{K^m}{F}} \to 1 \text{ as } F \to \infty,$$

since $K_{-i}/K < 1$. This proves part two. ■

Proof of Proposition 2

$\Longrightarrow$: Suppose an efficient and budget balancing allocation rule does exist. By efficiency, there are functions $(r_1, \ldots, r_n)$ satisfying

$$S_i(k_i, k_{-i}) = 1 - \frac{r_i(k_{-i})}{C\left(\sum_{i=1}^n k_i\right)}.$$

Summing over all divisions and applying the (BB) condition yields

$$\sum_{i=1}^n r_i(k_{-i}) = (n-1)C\left(\sum_{i=1}^n k_i\right).$$

Apply $\dfrac{\partial^n}{\partial k_1 \cdots \partial k_n}$ to both sides to get

$$0 = (n-1)C^{(n)}\left(\sum_{i=1}^n k_i\right).$$

$\Longleftarrow$: If $C^{(n)}$ is identically 0, then C must be a polynomial of degree less than or equal to $n-1$:

$$C(K) = a_{n-1}K^{n-1} + \cdots + a_1 K + a_0.$$

We will now define some helpful terminology. First define the sets

$$P^j \equiv \left\{p = (p_1, ..., p_n) \;\middle|\; p_l \text{ a nonnegative integer}, (\sum_{l=1}^n p_l) = j\right\}$$

$$P_i^j \equiv \left\{p \in P^j \middle|\; p_i = 0\right\}$$

for $j = 1, ..., n-1$; $i = 1, ..., n$. Next, for $p \in P^j$, let $G(p)$ be the number of nonzero coordinates of p: $G(p) \equiv \left|\left\{l \mid p_l \neq 0\right\}\right|$. Note that $G(p)$ is at most j. Finally, define $\begin{pmatrix} j \\ p \end{pmatrix} \equiv \frac{j!}{p_1! \cdots p_n!}$.

By the multinomial expansion theorem, it holds that

$$(k_1 + \cdots + k_n)^j = \sum_{p \in P^j} \begin{pmatrix} j \\ p \end{pmatrix} k_1^{p_1} \cdots k_n^{p_n}.$$

Now we will define a series of β_i^j for $j = 0, ..., n-1;\ i = 1, ..., n$:

$$\beta_i^0 = a_0 \frac{n-1}{n}$$

$$\beta_i^j = a_j \sum_{p \in P_i^j} \frac{n-1}{n - G(p)} \begin{pmatrix} j \\ p \end{pmatrix} k_1^{p_1} \cdots k_n^{p_n} \text{ for } j = 1, ..., n-1.$$

Observe that for a given vector $p \in P^j$, p is in P_i^j for $n - G(p)$ values of i. Therefore

$$\sum_{i=1}^{n} \beta_i^j = a_j (n-1)(k_1 + \cdots + k_n)^j = a_j (n-1) K^j.$$

By construction, β_i^j is independent of k_i for each j. So now let

$$r_i(k_{-i}) = \sum_{j=0}^{n-1} \beta_i^j$$

and define

$$S_i(k_i, k_{-i}) = 1 - \frac{r_i(k_{-i})}{C(K)}.$$

By Proposition 1, this rule is efficient. Furthermore, it satisfies budget balance, because

$$\sum_{i=1}^{n} r_i(k_{-i}) = \sum_{i=1}^{n} \sum_{j=0}^{n-1} \beta_i^j = \sum_{j=0}^{n-1} \sum_{i=1}^{n} \beta_i^j = \sum_{j=0}^{n-1} a_j (n-1) K^j = (n-1) C(K).$$

■

Proof of Proposition 3 For each $n \geq 2$, let $\hat{C}_{n-1}$ be the Chebyshev least squares approximation of C over the interval $(0, \bar{K}]$. Note that, by definition, $\hat{C}_{n-1}$ is a polynomial of degree $n-1$,

$$\hat{C}_{n-1}(K) = \hat{c}_{n-1} K^{n-1} + \cdots + \hat{c}_1 K + \hat{c}_0.$$

Construct the sets of allocation rules as in the proof of Proposition 2. That is, let

$$\beta_i^0 = \hat{c}_0 \frac{n-1}{n}$$

$$\beta_i^j = \hat{c}_j \sum_{p \in P_i^j} \frac{n-1}{n - G(p)} \begin{pmatrix} j \\ p \end{pmatrix} k_1^{p_1} \cdots k_n^{p_n} \text{ for } j = 1, ..., n-1,$$

where P_i^j and $G(p)$ are as defined in the proof of Proposition 2, and let

$$S_i^n(k_i, k_{-i}) = 1 - \frac{r_i^n(k_{-i})}{C(K)},$$

where

$$r_i^n(k_{-i}) = \sum_{j=0}^{n-1} \beta_i^j.$$

As before, the rule is efficient by Proposition 1, and

$$\sum_{i=1}^{n} r_i^n(k_{-i}) = \sum_{i=1}^{n}\sum_{j=0}^{n-1} \beta_i^j = \sum_{j=0}^{n-1}\sum_{i=1}^{n} \beta_i^j = \sum_{j=0}^{n-1} \hat{c}_j (n-1)K^j = (n-1)\hat{C}_{n-1}(K).$$

Therefore, for a given $k \in \prod_{i=1}^{n}[0, \bar{k}_i]$,

$$\left|\sum_{i=1}^{n} S_i^n(k_i, k_{-i}) - 1\right| = \frac{n-1}{C(K)}|C(K) - \hat{C}_{n-1}(K)|.$$

Since C is C^2, the standard results on the convergence of Chebyshev approximations apply (see, e.g., Judd (1998), pages 210–215). In particular, there exists a $B < \infty$ such that $|C(K) - \hat{C}_{n-1}(K)| \leq B\frac{\ln(n)}{n^2}$ for all $K \in (0, \bar{K}]$. Therefore, for each K,

$$\left|\sum_{i=1}^{n} S_i^n(k_i, k_{-i}) - 1\right| \leq \frac{B}{C(K)}\frac{\ln(n)}{n} \to 0,$$

as $n \to \infty$, because $C(K) > 0$ for $K > 0$. ■

Proof of Proposition 4 In what follows, we will omit the superscript m, n for visual clarity. Thus, we will use $\tilde{K}$, $\tilde{y}$, and $\tilde{c}$ to denote $\tilde{K}^{m,n}$, $\tilde{y}^{m,n}$, and $\tilde{c}^{m,n}$, respectively.

Let

$$\tilde{\mathbf{K}} = \begin{pmatrix} 1 & \tilde{K}_1 & (\tilde{K}_1)^2 & \dots & (\tilde{K}_1)^{n-1} \\ 1 & \tilde{K}_2 & (\tilde{K}_2)^2 & \dots & (\tilde{K}_2)^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \tilde{K}_m & (\tilde{K}_m)^2 & \dots & (\tilde{K}_m)^{n-1} \end{pmatrix}.$$

Estimating the polynomial coefficients by OLS, we obtain

$$\tilde{c} = (\tilde{\mathbf{K}}'\tilde{\mathbf{K}})^{-1}\tilde{\mathbf{K}}'\tilde{y}.$$

Recall that we construct the estimated polynomial sharing rule by using $\tilde{c}$ instead of $\hat{c}$ in the algorithm outlined in the proof of Proposition 3. That proof then shows that the budget imbalance arising from using this rule when total resource level is K is

$$\frac{n-1}{C(K)}\left|C(K) - \tilde{C}(K)\right|,$$

where $\tilde{C}(K) = \tilde{c}_0 + \tilde{c}_1 K + \cdots + \tilde{c}_{n-1}K^{n-1}$. Our goal is therefore to show that (for any K)

$$\lim_{n\to\infty}\lim_{m\to\infty} \frac{n-1}{C(K)}\left|C(K) - \tilde{C}(K)\right| = 0.$$

Now, let $\hat{C}$ be the $(n-1)^{st}$-degree Chebyshev least squares approximation of C, and let δ be the residual, $\delta(K) = C(K) - \hat{C}(K)$. Recall that $\delta(K) = O(\frac{\ln n}{n^2})$ (where $O(\cdot)$ denotes the asymptotic operator "of order no more than"). We can now write

$$\tilde{y}_i = C(\tilde{K}_i) = \hat{C}(\tilde{K}_i) + \delta(\tilde{K}_i) = \begin{pmatrix} 1 & \tilde{K}_i & (\tilde{K}_i)^2 & \dots & (\tilde{K}_i)^{n-1} \end{pmatrix} \hat{c} + \delta(\tilde{K}_i),$$

so that $\tilde{y} = \tilde{\mathbf{K}}\hat{c} + \Delta$, where $\Delta = \begin{pmatrix} \delta(\tilde{K}_1) & \delta(\tilde{K}_2) & \dots & \delta(\tilde{K}_m) \end{pmatrix}'$. Now,

$$\begin{aligned} \tilde{C}(K) &= \begin{pmatrix} 1 & K & K^2 & \dots & K^{n-1} \end{pmatrix} \tilde{c} \\ &= \begin{pmatrix} 1 & K & K^2 & \dots & K^{n-1} \end{pmatrix} (\tilde{\mathbf{K}}'\tilde{\mathbf{K}})^{-1}\tilde{\mathbf{K}}' \left(\tilde{\mathbf{K}}\hat{c} + \Delta\right) \\ &= \hat{C}(K) + \begin{pmatrix} 1 & K & K^2 & \dots & K^{n-1} \end{pmatrix} \left((\tilde{\mathbf{K}}'\tilde{\mathbf{K}})^{-1}\tilde{\mathbf{K}}'\Delta\right). \end{aligned}$$

Now, given our assumption that $\lim_{m\to\infty}\left(K_i - \frac{\bar{K}}{m}i\right) = 0$, we know that

$$\lim_{m\to\infty} \tilde{\mathbf{K}} = \lim_{m\to\infty} \begin{pmatrix} 1 & \frac{\bar{K}}{m} & \left(\frac{\bar{K}}{m}\right)^2 & \dots & \left(\frac{\bar{K}}{m}\right)^{n-1} \\ 1 & 2\frac{\bar{K}}{m} & \left(2\frac{\bar{K}}{m}\right)^2 & \dots & \left(2\frac{\bar{K}}{m}\right)^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \bar{K} & \left(\bar{K}\right)^2 & \dots & \left(\bar{K}\right)^{n-1} \end{pmatrix} = \lim_{m\to\infty} VD,$$

where V is the rectangular (n,m) Vandermonde matrix on integer nodes ($V_{ij} = i^{j-1}$) and $D = \text{diag}\begin{pmatrix} 1 & \frac{\bar{K}}{m} & \dots & \left(\frac{\bar{K}}{m}\right)^{n-1} \end{pmatrix}$. Therefore, for each n,

$$\lim_{m\to\infty} \begin{pmatrix} 1 & K & K^2 & \dots & K^{n-1} \end{pmatrix} \left((\tilde{\mathbf{K}}'\tilde{\mathbf{K}})^{-1}\tilde{\mathbf{K}}'\Delta\right) = \lim_{m\to\infty} \begin{pmatrix} 1 & K & K^2 & \dots & K^{n-1} \end{pmatrix} D^{-1}V^{+}\Delta,$$

where V^+ is the Moore-Penrose pseudo-inverse of V, $V^+ = (V'V)^{-1}V'$.

Using the factorization in Eisinberg et al. (2001), we can write $V^+ = \bar{S}\mathcal{M}$, where $\bar{S}$ is an $n \times n$ upper-triangular matrix whose entries do not depend on m and $\mathcal{M}$ is given by

$$\mathcal{M}_{i,j} = \sum_{t=1}^{n}\sum_{k=1}^{n} (-1)^t \binom{j-1}{t-1} \frac{\binom{i+k-2}{i-1}\binom{m-i}{m-k}\binom{t+k-2}{t-1}\binom{m-t}{m-k}}{\binom{2k-2}{k-1}\binom{m+k-1}{2k-1}}.$$

Now, $\binom{j-1}{t-1} \le \binom{m-1}{t-1} = \Theta(m^{t-1})$, where Θ denotes the asymptotic operator "of the same order as". Similarly, $\binom{m-i}{m-k} = \Theta(m^{k-i})$, $\binom{m-t}{m-k} = \Theta(m^{k-t})$, and $\binom{2k-2}{k-1}\binom{m+k-1}{2k-1} = \Theta(m^{2k-1})$. Consequently, each term in the finite summation for $\mathcal{M}_{ij}$ (and therefore $\mathcal{M}_{ij}$ itself) is

$$O(m^{t-1+k-i+k-t-(2k-1)}) = O(m^{-i}),$$

where O denotes the asymptotic operator "of order no more than." Finally, the (i,j) element of $D^{-1}V^+ = D^{-1}\bar{S}\mathcal{M}$ is

$$\left(D^{-1}V^+\right)_{i,j} = \sum_{l=1}^{n} \left(\frac{m}{\bar{K}}\right)^{i-1} \bar{S}_{i,l}\mathcal{M}_{l,j}.$$

Since $\mathcal{M}_{l,j} = O(m^{-l})$, all terms are $O(m^{i-l-1})$. Noting that $\bar{S}_{i,l} = 0$ when $i > l$ we see that $i - l - 1 \leq -1$ for all non-zero terms, so that each element of $\left(D^{-1}V^{+}\right)_{i,j}$ is decreasing at rate at least m^{-1}. Hence, for any n,

$$\lim_{m\to\infty} \begin{pmatrix} 1 & K & K^2 & \dots & K^{n-1} \end{pmatrix} D^{-1}V^{+}\Delta = \mathbf{0},$$

so that

$$\lim_{m\to\infty} \tilde{C}(K) = \hat{C}(K).$$

Now,

$$\lim_{m\to\infty} \frac{n-1}{C(K)} \left| C(K) - \tilde{C}(K) \right| = \frac{n-1}{C(K)} \left| C(K) - \hat{C}(K) \right| = \frac{n-1}{C(K)} \left| \delta(K) \right| = O\left(\frac{\ln(n)}{n} \right),$$

so that

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{n-1}{C(K)} \left| C(K) - \tilde{C}(K) \right| = 0.$$

■

Proof of Proposition 5

1. A set of allocation rules is efficient, budget balancing, and fair if and only if the following three conditions hold for all i and all $(k_1, \dots, k_n)$:

$$S_i(k_i, k_{-i}) = 1 - \frac{r_i(k_{-i})}{C\left(\sum_{i=1}^n k_i\right)};$$

$$\sum_{i=1}^n S_i(k_i, k_{-i}) = 1;$$

$$S_i(0, k_{-i}) = 0.$$

The first and the third condition (efficiency and fairness) together are equivalent to

$$S_i(k_i, k_{-i}) = 1 - \frac{C\left(\sum_{j\neq i} k_j\right)}{C\left(\sum_{i=1}^n k_i\right)}.$$

Combining this with budget balance yields

$$n - \frac{\sum_{i=1}^n C\left(\sum_{j\neq i} k_j\right)}{C\left(\sum_{i=1}^n k_i\right)} = 1,$$

that is,

$$C\left(\sum_{i=1}^n k_i\right) = \frac{1}{n-1} \sum_{i=1}^n C\left(\sum_{j\neq i} k_j\right). \tag{4}$$

We therefore see that a set of efficient, budget balancing, and fair allocation rules exists if and only if (4) holds for all $k \in \prod_{i=1}^n [0, \bar{k}_i]$.

2. Now, pick any $k_1 \in \mathbb{R}$ and take $k = (k_1, 0, \ldots, 0)$. By (4), if an efficient, budget balancing and fair allocation rule exists, then $C(k_1) = \frac{1}{n-1}(C(0) + (n-1)C(k_1))$, that is, $C(0) = 0$.

3. Next, suppose that C is strictly convex and an efficient, budget balancing and fair allocation rule exists. Let $K \equiv \sum_{i=1}^{n} k_i$ and let $K_{-i} \equiv \sum_{j \neq i} k_j$.

   Now, observe that

$$C(K_{-i}) = C\left(\frac{K_{-i}}{K} \cdot K + \frac{k_i}{K} \cdot 0\right) < \frac{K_{-i}}{K} C(K) + \frac{k_i}{K} C(0) = \frac{K_{-i}}{K} C(K) \tag{5}$$

   for all i, where the inequality follows by the convexity of C and $C(0) = 0$ as shown above. Therefore,

$$\begin{aligned} \sum_{i=1}^{n} C(K_{-i}) < \sum_{i=1}^{n} \frac{K_{-i}}{K} C(K) &= \frac{C(K)}{K} \sum_{i=1}^{n} K_{-i} \\ &= \frac{C(K) \cdot (n-1)K}{K} = (n-1)C(K), \end{aligned}$$

   which contradicts (4).

4. Therefore, we see that if an efficient, budget balancing and fair allocation rule exists, C must not be strictly convex and must have $C(0) = 0$. That is, the only candidate functions are the linear cost functions $C(K) = \alpha K$. It is also easy to see that this family of functions satisfies (4):

$$\frac{1}{n-1} \sum_{i=1}^{n} C(K_{-i}) = \frac{1}{n-1} \left(\alpha \sum_{i=1}^{n} \sum_{j \neq i} k_j\right) = \alpha K = C(K),$$

   which completes the proof.

■

Lemma 4 *Take $S = (S_1, \ldots, S_n)$ in the simple form $S_i(k_1, \ldots, k_n) = \frac{g_i(k_i)}{g_1(k_1) + \cdots + g_n(k_n)}$. Then S will imply an efficient equilibrium if and only if (for each i)*

$$\frac{C(K^*)}{C'(K^*)} = \frac{\sum_j g_j(k_j^*)}{g_i'(k_i^*)}.$$

Proof of Lemma 4 From the firm's objective function, we see that k_i^* is defined by the FOC

$$f_i'(k_i^*) - C'(K^*) = 0 \tag{6}$$

Division i maximizes $f_i(k_i) - S_i(k_1, \ldots, k_n) \cdot C(K)$. Let $\partial_i S_i$ denote the derivative of $S_i(k_1, \ldots, k_n)$ with respect to k_i. With an arbitrary allocation rule S, we get the first order condition for division i's problem,

$$f_i'(\tilde{k}_i) - \partial_i S_i(\tilde{k}_1, ..., \tilde{k}_n) \cdot C(\tilde{K}) - S_i(\tilde{k}_1, ..., \tilde{k}_n) \cdot C'(\tilde{K}) = 0. \tag{7}$$

Combining (6) and (7), we see that $k_i^* = \tilde{k}_i$ for each i if and only if

$$C'(K^*) = \partial_i S_i(k_1^*, ..., k_n^*) \cdot C(K^*) + S_i(k_1^*, ..., k_n^*) \cdot C'(K^*). \tag{8}$$

By assumption, $S_i(k_1, \ldots, k_n) = \frac{g_i(k_i)}{\sum_j g_j(k_j)}$, so

$$\begin{aligned}\partial_i S_i &= \frac{(\sum_j g_j(k_j)) \cdot g_i'(k_i) - g_i(k_i) \cdot g_i'(k_i)}{(\sum_j g_j(k_j))^2} \\ &= \frac{g_i'(k_i) \cdot \sum_{j \neq i} g_j(k_j)}{(\sum_j g_j(k_j))^2}.\end{aligned}$$

Plugging this and S_i into (8) and simplifying gives us the desired result,

$$\frac{C(K^*)}{C'(K^*)} = \frac{\sum_j g_j(k_j^*)}{g_i'(k_i^*)}. \tag{9}$$

■

Proof of Proposition 6

By Lemma 4 it suffices to check that (9) holds. k_i^* is determined by (7) and all values of k_i^* are equal to some k^*. So $K = nk^*$. Thus,

$$\frac{\sum_j g_j(k_j^*)}{g_i'(k_i^*)} = \frac{ng(k^*)}{g'(k^*)} = \frac{nC(nk^*)}{nC'(nk^*)} = \frac{C(K^*)}{C'(K^*)}.$$

■

Proof of Proposition 7

To show that an equilibrium is generated, by Lemma 4 it suffices to check that (9) holds:

$$\frac{\sum_j g_j(k_j^*)}{g_i'(k_i^*)} = \frac{\sum_j \alpha_j C(\frac{1}{\alpha_j}\alpha_j K^*)}{\frac{1}{\alpha_i}\alpha_i C'(\frac{1}{\alpha_i}\alpha_i K^*)} = \frac{C(K^*)}{C'(K^*)}.$$

So optimal resource levels are an equilibrium. And in this equilibrium, division i pays

$$S_i = \frac{\alpha_i C(K^*)}{\sum_j \alpha_j C(K^*)} = \alpha_i.$$

■

Proof of Proposition 8. Fix K_{-i}. Let k_i^{Lc} be the resource levels induced by the linear cost allocation rule for division i, conditional on K_{-i}, and let k_i^{*c} be the conditional efficient resource level.

Given K_{-i}, we see from (6) that the conditional efficient k_i^{*c} will satisfy

$$f_i'(k_i^{*c}) = C'(K_{-i} + k_i^{*c}).$$

The linear rule will have

$$S_i^L(k) = \frac{k}{K_{-i} + k}, \quad S_i^{L\prime}(k) = \frac{K_{-i}}{(K_{-i} + k)^2},$$

and so from (7), the linear k_i^{Lc} will satisfy

$$f_i'(k_i^{Lc}) = \frac{K_{-i}}{(K_{-i} + k_i^{Lc})^2} \cdot C(K_{-i} + k_i^{Lc}) + \frac{k_i^{Lc}}{K_{-i} + k_i^{Lc}} \cdot C'(K_{-i} + k_i^{Lc}).$$

And, because C is convex,

$$C(x) < x \cdot C'(x) + C(0). \tag{10}$$

We have assumed $C(0) = 0$, so plug this back in with $x = K_{-i} + k_i^{Lc}$:

$$f_i'(k_i^{Lc}) < \frac{K_{-i}(K_{-i} + k_i^{Lc})}{(K_{-i} + k_i^{Lc})^2} \cdot C'(K_{-i} + k_i^{Lc}) + \frac{k_i^{Lc}}{K_{-i} + k_i^{Lc}} \cdot C'(K_{-i} + k_i^{Lc}) = C'(K_{-i} + k_i^{Lc}).$$

Because $f_i'(k)$ is decreasing in k and $C'(K_{-i} + k)$ is increasing in k, we have that

$$\begin{aligned} f_i'(k_i^{Lc}) &< C'(K_{-i} + k_i^{Lc}) \\ \text{and} \quad f_i'(k_i^{*c}) &= C'(K_{-i} + k_i^{*c}) \end{aligned}$$

iff $k_i^{Lc} > k_i^{*c}$. If production is symmetric, $k_i^* = k^*$ for each i. Let $k_i^{*c}(K_{-i})$ be the optimal conditional resource levels for a given K_{-i}. $k_i^{*c}(K_{-i})$ satisfies $f'(k_i^{*c}(K_{-i})) - C'(k_i^{*c}(K_{-i}) + K_{-i}) = 0$, so we can implicitly differentiate and see that

$$\frac{dk_i^{*c}(K_{-i})}{dK_{-i}} = \frac{C''}{f'' - C''} < 0.$$

At any symmetric equilibrium, $K_{-i} = (n-1)k$ and $K = nk$. Now $k_i^{Lc}(K_{-i}) > k_i^{*c}(K_{-i})$ for any K_{-i}. Also, for $K_{-i} \leq K_{-i}^*$, the above differentiation shows that $k_i^{*c}(K_{-i}) \geq k_i^{*c}(K_{-i}^*) = k^*$. So assume that $K^L \leq K^*$. Then, $K_{-i}^L \leq K_{-i}^*$ for all i, and therefore

$$k_i^L = k_i^{Lc}(K_{-i}^L) > k_i^{*c}(K_{-i}^L) \geq k^*.$$

But this implies that $K^L > K^*$. Contradiction. So $K^L > K^*$ and $k^L > k^*$. ■

Lemma 5 *If $f_i(k_i) = A(k_i)^p$ for each i and $C(K) = BK^q$ for $0 < p < 1 < q < \infty$ then*

$$k^* = \left(\frac{Ap}{Bq}\,n^{1-q}\right)^{\frac{1}{q-p}} \qquad\qquad k^L = \left(\frac{Apn^{2-q}}{B(n-1+q)}\right)^{\frac{1}{q-p}}$$

$$\frac{\pi^L}{\pi^*} = n^{\frac{p}{q-p}}\left(\frac{q}{n-1+q}\right)^{\frac{p}{q-p}}\left(\frac{1-\frac{np}{n-1+q}}{1-\frac{p}{q}}\right).$$

Proof: In the respective symmetric equilibria, $k_i^* = k_j^*$ and $k_i^L = k_j^L$, so we will drop the subscripts from these quantities in the following proof. The optimal production k^* solves (6), so

$$\begin{aligned} Ap(k^*)^{p-1} &= q\,B(nk^*)^{q-1} \\ (k^*)^{q-p} &= \frac{Ap}{Bq}\,n^{1-q} \\ k^* &= \left(\frac{Ap}{Bq}\,n^{1-q}\right)^{\frac{1}{q-p}}. \end{aligned}$$

And k^L solves (7), so

$$\begin{aligned} Ap(k^L)^{p-1} - \frac{(n-1)k^L}{n^2(k^L)^2}B(nk^L)^q - \frac{1}{n}Bq(nk^L)^{q-1} &= 0 \\ Ap(k^L)^{p-1} &= n^{q-2}(k^L)^{q-1}[B(n-1+q)] \\ k^L &= \left(\frac{Apn^{2-q}}{B(n-1+q)}\right)^{\frac{1}{q-p}}. \end{aligned}$$

Now, $\pi(k) = nAk^p - B(nk)^q$ for a symmetric output level k, so

$$\begin{aligned} \pi^* &= An\left(\frac{Ap}{Bq}n^{1-q}\right)^{\frac{p}{q-p}} - B\left(n^{q-p}\,\frac{Ap}{Bq}n^{1-q}\right)^{\frac{q}{q-p}} \\ &= n^{\frac{q-pq}{q-p}}\left[A\left(\frac{Ap}{Bq}\right)^{\frac{p}{q-p}} - B\left(\frac{Ap}{Bq}\right)^{\frac{q}{q-p}}\right] \\ &= An^{\frac{q-pq}{q-p}}\left(\frac{Ap}{Bq}\right)^{\frac{p}{q-p}}\left(1-\frac{p}{q}\right). \end{aligned}$$

And

$$\begin{aligned} \pi^L &= An\left(\frac{Ap}{B(n-1+q)}n^{2-q}\right)^{\frac{p}{q-p}} - B\left(n^{q-p}\frac{Ap}{B(n-1+q)}n^{2-q}\right)^{\frac{q}{q-p}} \\ &= An^{\frac{q+p-qp}{q-p}}\left(\frac{Ap}{B(n-1+q)}\right)^{\frac{p}{q-p}}\left(1-\frac{np}{n-1+q}\right). \end{aligned}$$

So

$$\frac{\pi^L}{\pi^*} = \frac{An^{\frac{q+p-qp}{q-p}}\left(\frac{Ap}{B(n-1+q)}\right)^{\frac{p}{q-p}}\left(1-\frac{np}{n-1+q}\right)}{An^{\frac{q-pq}{q-p}}\left(\frac{Ap}{Bq}\right)^{\frac{p}{q-p}}\left(1-\frac{p}{q}\right)}$$
$$= n^{\frac{p}{q-p}}\left(\frac{q}{n-1+q}\right)^{\frac{p}{q-p}}\left(\frac{1-\frac{np}{n-1+q}}{1-\frac{p}{q}}\right).$$

■

Proof of Proposition 9

Note $\Delta = 1-\pi^L/\pi^*$, where π^L/π^* is calculated in Lemma 5.

Without calculations, it is clear that $\dfrac{\partial \Delta}{\partial A} = \dfrac{\partial \Delta}{\partial B} = 0$. Now

$$\frac{\partial \Delta}{\partial n} = \frac{(n-1)n^{\frac{p}{q-p}-1}p(q-1)^2 q\left(\frac{q}{n+q-1}\right)^{\frac{p}{q-p}}}{(q-p)^2(n+q-1)^2} > 0.$$

$$\frac{\partial \Delta}{\partial p} = \frac{1}{(p-q)^3}\Bigg(n^{\frac{p}{q-p}}\Big(\frac{q}{n-1+q}\Big)^{\frac{q}{q-p}}\cdot$$
$$\Big((n-1)(p-q)(q-1)+q(n-np+q-1)\log(\frac{nq}{n-1+q})\Big)\Bigg).$$

So $\frac{\partial \Delta}{\partial p}$ has the opposite sign of $(n-1)(p-q)(q-1)+q(n-np+q-1)\log(\frac{nq}{n-1+q})$. Plugging in the inequality $\log(x) \le x-1$,

$$(n-1)(p-q)(q-1)+q(n-np+q-1)\log(\frac{nq}{n-1+q})$$
$$\le (n-1)(p-q)(q-1)+q(n-np+q-1)\left(\frac{nq}{n-1+q}-1\right)$$
$$= -\frac{(n-1)^2p(q-1)^2}{n-1+q}$$
$$< 0.$$

Thus, $\frac{\partial \Delta}{\partial p} > 0$. Now

$$\frac{\partial \Delta}{\partial q} = \frac{1}{(p-q)^3(n-1+q)^2}\Bigg(n^{\frac{p}{q-p}}pq\Big(\frac{q}{n-1+q}\Big)^{\frac{p}{q-p}}\cdot$$
$$\Big((n-1)(q-p)(q-1)-(q-1+n-np)(n-1+q)\log(\frac{nq}{n-1+q})\Big)\Bigg)$$

So $\frac{\partial\Delta}{\partial q}$ has the opposite sign of
$(n-1)(q-p)(q-1)-(q-1+n-np)(n-1+q)\log(\frac{nq}{n-1+q})$.
In other words, $\frac{\partial\Delta}{\partial q}$ has the sign of

$$\log(\frac{nq}{n-1+q})-\frac{(n-1)(q-p)(q-1)}{(n-1+q)(n-np+q-1)}.$$

As $n\longrightarrow\infty$, this goes to $\log(q)-0=\log(q)>0$. So for n large enough holding other parameters fixed, $\frac{\partial\Delta}{\partial q}>0$.

As $q\longrightarrow\infty$, this goes to $\log(n)-(n-1)$ which is strictly less than 0 for $n\geq 2$. So for q large enough, $\frac{\partial\Delta}{\partial q}<0$.

As $q\longrightarrow 1$, we can find the sign by dividing through by the fraction and applying L'Hôpital's rule to the limit. This has the sign of $n-1>0$, so for q small enough, $\frac{\partial\Delta}{\partial q}>0$.

As $p\longrightarrow 1$, the expression goes to $\log(\frac{nq}{n-1+q})-\frac{(n-1)(q-1)}{n-1+q}$. Applying the inequality $\log(x)\leq x-1$, we get $\log(\frac{nq}{n-1+q})\leq\frac{(n-1)(q-1)}{n-1+q}$. So $\frac{\partial\Delta}{\partial p}<0$ for p large enough, when other parameters are held constant. ■

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