Abstract. Using the copula approach to modeling consumer preferences, this paper establishes several new results on the profitability and effects of product bundling. A multiproduct firm achieves higher profit from bundling than from separate sales if consumer values for two products are negatively dependent, independent, or have limited positive dependence. This profitability condition of monopoly bundling extends to competitive markets with a multiproduct firm and a single-product entrant or with symmetric multiproduct firms. Under a similar sufficient condition, a multiproduct firm charges higher standalone prices for each product under mixed bundling than under separate sales, and bundling sometimes raises prices for all consumers. Mixed bundling can reduce the profit of a single-product entrant, although its entry deterrence effect may be weaker than a commitment to pure bundling.
1. INTRODUCTION

Bundling has been an important topic in industrial organization. Stigler (1963) showed with a simple example that bundling can be profitable even without demand complementarity or economies of scope. Adams and Yellen (1976) expanded on this view, showing mostly with examples that mixed bundling can be a profitable way to segment markets. Schmalensse (1984) studied profitability and prices under pure and mixed bundled when consumer values for two goods have a bivariate normal distribution, and found sufficient conditions on the marginal distributions for pure bundling to dominate separate sales for any correlation coefficient. McAfee, McMillan, and Whinston (1989; henceforth MMW) proved that (mixed) bundling dominates separate sales when values are distributed independently. Nalebuff (2004) examined examples for which pure bundling does or does not deter single-product entry depending on whether the incumbent can or cannot commit to prices.

Much of the attention in the economic analysis of bundling has been placed on how the correlation of values between products matters for the profitability of bundling. However, analytical results under general distributions of values have been lacking in the literature (other than the result in MMW under independent values). In this paper, we utilize the new approach of representing the joint distributions of consumer values by copulas. A copula is a function that couples distributions of random variables to form a joint distribution, making it straightforward to vary dependence while holding marginal distributions constant. We are then able to develop new analytical results under general distributions of values. In particular, advancing the analysis of MMW, we show that (mixed) bundling dominates separate sales if values for two products are negatively dependent, independent, or have limited positive dependence.

The bundle is sold under a discount relative to the separately priced goods under mixed bundling. Intuitively, one suspects that the discount on the bundle might be based on inflated prices on individual goods, but, to the best of our knowledge, the literature has not addressed this possibility in a setting with general distributions of consumer values. Under a sufficient condition similar to that for the profitability of bundling, we show that a multiproduct firm will charge higher standalone prices for each product under mixed bundling than under separate sales, so that mixed bundling leads to higher prices for at least some consumers. Furthermore, even the price of
the bundle is sometimes higher than the sum of prices under separate sales, in which case bundling raises prices for all consumers.

Bundling could also increase a firm’s profit through its effects on deterring entry or excluding competition. While Nalebuff (2004) argues that pure bundling is highly effective as an entry barrier against a single-product entrant, the effects of mixed bundling on entry has not been well understood. Our analysis finds that mixed bundling also can reduce the profit of a single-product entrant, although its entry deterrence effect may be weaker than the effect under a commitment to pure bundling.

The rest of the paper is organized as follows. Section 2 sets up the multiproduct monopoly model. Section 3 establishes the main profitability result, and Section 4 analyzes the price and consumer welfare effects of bundling. Sections 5 and 6 consider extensions to competitive market situations with asymmetric or symmetric firms. Section 7 concludes.

2. THE MULTIPRODUCT MONOPOLY MODEL

There are two goods, X and Y. The size of consumer population is normalized to 1. Each consumer demands at most one unit of each good, and her consumption of one good does not affect her decision to purchase the other good. A consumer’s value for X is $u$ and for Y is $v$, which has marginal distributions $F(u)$ and $G(v)$, with corresponding strictly positive density functions $f(u)$ and $g(v)$ on their respective supports. The constant marginal costs for X and Y are $m_x$ and $m_y$, respectively. The value of two goods together is $u + v$, with marginal cost $m_x + m_y$. To avoid trivial situations, we assume that there are positive measures of $u$ and $v$ for which $u > m_x$ and $v > m_y$. As in Adams and Yellen (1976) and in MMW, this framework rules out product complementarity or economies of scale as explanations for bundling.

By Sklar’s Theorem (Nelsen, 2006), it is without loss of generality to represent the joint distribution of consumers’ values for the two products by a copula and the marginal distributions. A copula is a bivariate uniform distribution. Let $x = F(u)$, $y = G(v)$, and the copula associated with the joint distribution of $(u, v)$ be $C(x, y)$. Then the joint distribution of $(u, v)$ is $C(F(u), G(v))$, with $u(x) \equiv F^{-1}(x)$ and $v(y) \equiv G^{-1}(y)$ for $(x, y) \in [0, 1]^2$. Uniform margins for $x$ and $y$ mean $C(x, 1) = x$ and $C(1, y) = y$. A copula additionally satisfies $C(x, 0) = 0 = C(0, y)$ and lies
between the Fréchet-Hoeffding lower and upper bounds: \( \max\{0, x+y-1\} \leq C(x, y) \leq \min\{x, y\} \). We assume that \( C(x, y) \) is a twice-differentiable function that admits a density on the full support of \([0, 1]^2\).

The copula describes the statistical dependence of consumer values for the two products. For example, Kendall’s tau, \( \tau_C \equiv 4E\{C(x, y)\} - 1 \in [-1, 1] \), which is a function of the copula, measures the concordance of consumer values. The independent copula is \( C(x, y) = xy \), for which \( \tau_C = 0 \). Positive quadrant dependence means \( C(x, y) \geq xy \), for which obviously \( \tau_C \geq 0 \), and negative quadrant dependence means just the opposite. The copula also indicates stochastic dependence. In particular, \( C_1(x, y) \equiv \frac{\partial C(x, y)}{\partial x} \) is the conditional distribution of \( y \) given \( x \). The value \( v \) is stochastically increasing in \( u \) if \( C_{11}(x, y) \equiv \frac{\partial^2 C(x, y)}{\partial x \partial y} \leq 0 \); conversely, \( v \) is stochastically decreasing in \( u \) if \( C_{11}(x, y) \equiv \frac{\partial^2 C(x, y)}{\partial x^2} \geq 0 \). Positive (negative) stochastic dependence implies positive (negative) quadrant dependence (Nelsen 2006, 195-197).

As a benchmark, if the two goods are sold separately at prices \( p \) and \( q \), consumers will purchase \( X \) if \( u(x) \geq p \), or if \( x \geq F(p) \), and will purchase \( Y \) if \( v(y) \geq q \), or if \( y \geq G(q) \). The optimal prices for \( X \) and \( Y \) under separate sales, \( p^s \) and \( q^s \), satisfy first-order conditions

\[
1 - F(p^s) - (p^s - m_x) f(p^s) = 0, \tag{1}
\]

\[
1 - G(q^s) - (q^s - m_y) g(q^s) = 0. \tag{2}
\]

We assume that the supports of \( u \) and \( v \) are not too much above the marginal costs so that \( p^s \) and \( q^s \) are indeed interior values satisfying (1) and (2). Since \( F(\cdot) \), \( G(\cdot) \), and costs are all given, \( p^s \) and \( q^s \) are given for our analysis that follows.

Without loss of generality, we formulate bundling as mixed bundling, which allows pure bundling as a special case. Suppose \( X \) and \( Y \) are offered separately at prices \( p \) and \( q \), respectively, and the XY bundle is offered at price \( r \leq p + q \). The value of the outside option is normalized to zero. Consumers are willing to purchase the bundle if \( u(x) + v(y) - r \geq \max\{0, u(x) - p, v(y) - q\} \), \( X \) alone if \( u(x) - p \geq 0 \) and \( v(y) \leq r - p \), and \( Y \) alone if \( v(y) - q \geq 0 \) and \( u(x) \leq r - q \). Consequently, demands
for each good and the bundle are, respectively,

\[ Q_x(p, r) \equiv G(r - p) - C(F(p), G(r - p)), \]
\[ Q_y(q, r) \equiv F(r - q) - C(F(r - q), G(q)), \]
\[ Q_{xy}(p, q, r) \equiv \int_{F(r - q)}^{F(p)} [1 - C_1(x, G(r - u(x))] \, dx + [1 - F(p)] - Q_x(p, r). \]  

(3) \hspace{1cm} (4) \hspace{1cm} (5)

Profit function is

\[ \pi (p, q, r) = (p - m_x) Q_x (p, b) + (q - m_y) Q_y (q, b) + (b - m_x - m_y) Q_B (p, q, r). \]  

(6)

The monopolist chooses \((p, q, r)\) to maximize profits subject to \(r \leq p + q\). Bundling is profitable if \(r < p + q\) at the solution; pure bundling is optimal if \(p\) and \(q\) are such that separate sales for X and Y are zero, which is equivalent to not offering the goods separately; otherwise mixed bundling is optimal.

### 3. PROFITABILITY OF BUNDLING

In this section, we show that the multiproduct monopolist achieves higher profit from bundling than from separate sales if values for two products are not too positively dependent. This new condition substantially extends the sufficient condition in MMW that bundling dominates separate sales under independent values.

**Proposition 1** There exists some \(\bar{\delta} > 0\) such that bundling has higher profits than separate sales if \(C_{11} \geq -\bar{\delta}\).

**Proof.** Consider

\[ \psi (\varepsilon) = \pi (p^*, q^*, p^* + q^* - \varepsilon) \text{ for } \varepsilon \geq 0, \]

where recall \(p^*\) and \(q^*\) are the optimal prices when the two goods are sold separately. Bundling is more profitable than separate sales if \(\psi' (0) > 0\). We have

\[ \psi' (0) = - (p^* - m_x) \frac{\partial Q_x (p^*, p^* + q^*)}{\partial b} - (q^* - m_y) \frac{\partial Q_y (q^*, p^* + q^*)}{\partial b} \]
\[ - (p^* + q^* - m_x - m_y) \frac{\partial Q_b (p^* + q^*, p^*, q^*)}{\partial b} - Q_b (p^* + q^*, p^*, q^*). \]  

(7)
From (3), (4), and (5), simple differentiation and substitution yield: 

$$
\psi' (0) = 
\begin{align*}
- (p^s - m_x) g (q^s) [1 - C_2 (F (p^s), G (q^s))] - (q^s - m_y) f (p^s) [1 - C_1 (F (p^s), G (q^s))] \\
+ (p^s + q^s - m_x - m_y) \{[1 - C_1 (F (p^s), G (q^s))] f (p^s) + [1 - C_2 (F (p^s), G (q^s))] g (q^s)\} \\
- [1 - F (p^s)] - [G (p^s) - C (F (p^s), G (q^s))] \\
\end{align*}
$$

$$
= (p^s - m_x) f (p^s) [1 - C_1 (F (p^s), G (q^s))] + (q^s - m_y) g (p^s) [1 - C_2 (F (p^s), G (q^s))] \\
- [1 - F (p^s)] + [G (q^s) - C (F (p^s), G (q^s))].
$$

Using the first-order conditions for $p^s$ and $q^s$, (1) and (2), $\psi' (0)$ becomes

$$
\psi' (0) = [1 - F (p^s)] [1 - C_1 (F (p^s), G (q^s))] + [1 - G (q^s)] [1 - C_2 (F (p^s), G (q^s))] \\
- [1 - F (p^s)] + [G (q^s) - C (F (p^s), G (q^s))] \\
= [1 - G (q^s)] [1 - C_2 (F (p^s), G (q^s))] + \int_{F(p^s)}^{1} [C_1 (x, G (q^s)) - C_1 (F (p^s), G (q^s))] \, dx \\
= [1 - G (q^s)] [1 - C_2 (F (p^s), G (q^s))] + \int_{F(p^s)}^{1} \int_{F(p^s)}^{z} C_{11} (z, G (q^s)) \, dz \, dx.
$$

Thus, since $C_2 (F (p^s), G (q^s)) = \Pr (X \leq F (p^s)|G (q^s)) < 1$, $\psi' (0) > 0$ if $C_{11} \geq 0$. Now, for $\delta \geq 0$, define function

$$
\phi (\delta) \equiv \frac{[1 - G (q^s)] [1 - \sup \{C_2 (F (p^s), G (q^s)) : -\delta \leq C_{11} \leq 0\}]}{\frac{1}{2} [1 - F (p^s)]^2}.
$$

Then, $\phi (\delta)$ is non-increasing. Since $C$ has full support on $[0, 1]^2$, any $C_2 (F (p^s), G (q^s))$ is bound away from 1. Thus $\phi (\delta) > \delta$ for $\delta$ sufficiently close to zero. Therefore, there exists some $\hat{\delta} > 0$ such that $\delta \geq \phi (\delta)$ first occurs at $\hat{\delta}$ (i.e., either $\hat{\delta} = \phi (\hat{\delta})$ or $\phi (\delta)$ jumps down to cause $\delta > \phi (\delta)$ at $\hat{\delta}$). Let $\tilde{\delta} = \hat{\delta}/2 > 0$, then $\tilde{\delta} < \phi (\hat{\delta})$. It follows that
if $C_{11} \geq -\delta$, 

$$
\psi'(0) \geq [1 - G(q^*)] [1 - C_2(F(p^*) , G(q^*))] - \delta \frac{1}{2} [1 - F(p^*)]^2 \\
> [1 - G(q^*)] [1 - C_2(F(p^*) , G(q^*))] - \phi(\delta) \frac{1}{2} [1 - F(p^*)]^2 \\
= [1 - G(q^*)] [1 - C_2(F(p^*) , G(q^*))] \\
- [1 - G(q^*)] [1 - \sup \{ C_2(F(p^*) , G(q^*)) : -\delta \leq C_{11} \leq 0 \}] \\
\geq 0.
$$

Adams and Yellen (1976) argued with diagrammatic examples that mixed bundling is more profitable than separate pricing if values are not too positively dependent, and Schmalensee (1984) proved that negative correlation is sufficient condition for the superiority of mixed bundling in the bivariate normal special case. The above proposition establishes this argument much more generally. Note that if $C$ is a symmetric copula, then $C_{11} = C_{22}$. Note also that dependence is entirely a property of the copula, even though the positive dependence allowed in the sufficient condition generally depends on both marginal distributions and marginal costs that jointly determine the prices under separate sales.

4. PRICE AND WELFARE EFFECTS OF BUNDLING

This section studies the price and welfare effects of bundling. We first show that, under a sufficient condition similar to that for the profitability of bundling, the stand-alone prices for each good are higher under mixed bundling than under separate sales.\footnote{The stand-alone prices for each good are trivially higher under pure bundling than under separate sales.} We then further study two special cases, the FGM-uniform Case and the Hotelling Case.

Analytical Result

Denote the optimal prices under bundling by $p^*$, $q^*$ and $b^*$. Recall that under mixed bundling $p^*$ and $q^*$ are interior values. The next result utilizes assumption A1 below,
where (i) is satisfied if, for instance, \( m_x = m_y = 0 \), and (ii) is the familiar monotonic hazard rate condition:

**A1.** (i) \( r^* - p^* > m_y \) and \( r^* - q^* > m_x \); (ii) \( d \left[ \frac{f(u)}{1-F(u)} \right] \geq 0 \), and \( d \left[ \frac{g(v)}{1-G(v)} \right] \geq 0 \).

**Proposition 2** Suppose that A1 holds. Then, under mixed bundling \( p^* > p^* \) and \( q^* > q^* \) if \( \min \{C_{11}, C_{22}\} > -\delta \) for sufficiently small \( \delta > 0 \).

**Proof.** Under mixed bundling, \( p^* \) satisfies

\[
\frac{\partial \pi(p^*, q^*, r^*)}{\partial p} = Q_x(p^*, r^*) + (p^* - m_x) \frac{\partial Q_x(p^*, r^*)}{\partial p} + (r^* - m_x - m_y) \frac{\partial Q_b(p^*, q^*, r^*)}{\partial p} = 0.
\]

From (3), (4), and (5), simple differentiation and substitution yield: 0 =

\[
G(r^* - p^*) - C(F(p^*), G(r^* - p^*)) + (p^* - m_x) [-g(r^* - p^*)] [1 - C_2(F(p^*), G(r^* - p^*))] - C_1(F(p^*), G(r^* - p^*)) f(p^*) + (r^* - m_x - m_y) [g(r^* - p^*)] [1 - C_2(F(p^*), G(r^* - p^*))] = G(r^* - p^*) - C(F(p^*), G(r^* - p^*)) - (p^* - m_x) f(p^*) C_1(F(p^*), G(r^* - p^*)) + (r^* - p^* - m_y) g(r^* - p^*) [1 - C_2(F(p^*), G(r^* - p^*))].
\]

Now, suppose that, to the contrary, \( p^* \leq p^* \). Then, from part (ii) of A1, \( (p^* - m_x) f(p^*) \leq 1 - F(p^*) \) and 0 = \( \frac{\partial \pi(p^*, q^*, r^*)}{\partial p} \)

\[
\geq G(r^* - p^*) - C(F(p^*), G(r^* - p^*)) - [1 - F(p^*)] C_1(F(p^*), G(r^* - p^*)) + (r^* - p^* - m_x) g(r^* - p^*) [1 - C_2(F(p^*), G(r^* - p^*))]
\]

\[
= \int_{F(p^*)}^{1} \left[ C_1(x, G(r^* - p^*)) - C_1(F(p^*), G(r^* - p^*)) \right] dx + (r^* - p^* - m_y) g(r^* - p^*) [1 - C_2(F(p^*), G(r^* - p^*))]
\]

\[
= \int_{F(b^*)}^{1} \int_{F(p^*)}^{1} C_{11}(z, G(b^* - p^*)) dx + (r^* - p^* - m_y) g(b^* - p^*) [1 - C_2(F(p^*), G(b^* - p^*))] > 0
\]

if \( C_{11} > -\delta \) for \( \delta > 0 \) sufficiently small. This is a contradiction.
Similarly, \( q^* > q^s \) if \( C_{22} > -\delta \) for sufficiently small \( \delta > 0 \). □

Thus, although under mixed bundling the bundle is sold at a discount relative to individually-priced goods, the standalone prices for X and Y are higher under mixed bundling than under separate sales. The sufficient condition for this result is similar to that for the profitability of bundling. Since the optimal prices \( p^* \) and \( q^* \) depend on the copula, the condition is stated as \( C_{22} > -\bar{\delta} \) for sufficiently small \( \bar{\delta} > 0 \), instead of for some fixed \( \bar{\delta} \). As we shall see shortly in a numerical example, the \( \bar{\delta} \) allowed need not be close to zero.

An immediate implication of Proposition 2 is that mixed bundling will reduce the welfare of at least some consumers. A different issue is whether the bundle is offered at a discount relative to \( (p^s + q^s) \), and how does aggregate consumer welfare under bundling \( (W^*) \) compare with that under separate sales \( (W^s) \). We address this issue in two special cases.

**FGM-Uniform Case and Hotelling Case**

We first conduct numerical analysis of the FGM-Uniform Case, for which the copula is the FGM copula family and the marginal distribution is uniform. The FGM copula family is given by

\[
C(u, v) = uv + \theta uv (1 - u)(1 - v),
\]

with \( \theta \in [-1, 1] \) and density

\[
c(u, v) = \frac{\partial^2 (uv + \theta uv (1 - u)(1 - v))}{\partial u \partial v} = 1 + \theta (2v - 1)(2u - 1).
\]

The FGM family is useful for modelling a limited range of positive and negative dependence, including independence \( (\theta = 0) \). Kendall’s tau for the FGM family is \( \tau(\theta) = \frac{2\theta}{9} \), with \( \theta < 0 \) and \( \theta > 0 \) indicating negative and positive dependence, respectively. For ease of computations, we further assume \( m_x = m_y = m \) and the marginal distributions satisfy \( F(\cdot) = G(\cdot) \), with

\[
F(u) = \frac{u - 4}{5}, \quad 4 < w < 9.
\]
Tables 1-4 in the Appendix details results with $m = 0$, $m = 2$, $m = 4$, and $m = 4$. The specific uniform distribution and values of $m$ are chosen to encompass situations where either all consumers or only some consumers value the products more than their costs, which allows us to consider how marginal costs and preference dependence interact to determine the profitability and effects of bundling. The following is a summary of our findings:

First, for the parameter values we consider, bundling always increases profit. Pure bundling is optimal if $m$ is low relative to valuations ($m \leq 2$), whereas mixed bundling is optimal when $m$ is relatively high ($m \geq 4$; in which case not every consumer’s product valuation is strictly above $m$). An advantage of mixed bundling over pure bundling is that it limits inefficient production for consumers who would purchase the bundle but have a low value for one of the goods. This explains why mixed bundling dominates pure bundling when $m$ is high enough. The result also suggests that the limited positive dependence allowed in Proposition 1 can be fairly large; in the case here, bundling is profitable even when $\theta = 1$.

Second, the standalone prices under mixed bundling are always higher than the corresponding prices under separate selling. Again, this result does not require $C_{11}$ to approach zero as in the sufficient condition of Proposition 2. The price for the bundle can be higher than the sum of prices under separate sales, but it appears to occur only under pure bundling (when $m = 0$ or when $m = 2$ and $\theta$ is relatively low). In this sense, pure bundling is more likely to lower consumer welfare. Notice, however, that when $m = 2$ and $\theta = 1$, pure bundling both maximizes profit and increases consumer welfare.

Third, the effects of bundling on prices and consumer welfare depend on the interactions between marginal costs and preference dependence. When $m = 0$, or when $m = 2$ and $\theta$ is relatively low ($\theta \leq -0.5$), bundling raises prices and (weakly) lowers welfare for all consumers. When $m \geq 4$, bundling lowers prices for consumers purchasing both products but raises prices for other consumers: with $m = 4$, (aggregate) consumer surplus is higher under bundling if $\theta \leq 0.5$ but lower if $\theta = 1$, while with $m = 6$ consumer surplus is always lower under bundling.

Fourth, as $\theta$ increases, the bundle price decreases under pure bundling (which occurs when $m$ is relatively low) but increases under mixed bundling. The standalone
prices under mixed bundling decrease in $\theta$ when $m = 4$ but increase in $\theta$ when $m = 6$. Consumer surplus increases with $\theta$ under pure bundling but varies non-monotonically with $\theta$ under mixed bundling. Moreover, as $\theta$ increases, profit under bundling always decreases.

Schmalensee (1984) points out that pure bundling operates by reducing the effective dispersion of consumer values. As $\theta$ increases, this mechanism is less useful, which might explain why price under pure bundling goes down; and since consumers are more likely to have similar values for both products with higher $\theta$, their value for the bundle tends to increase with higher $\theta$. The price and value effects together appear to explain why consumer surplus increases with $\theta$ under pure bundling. Under mixed bundling, the firm may still sell separately priced goods to consumers who do not buy the bundle, so its incentive to price the bundle is more complicated; but it is intriguing that the bundle price increases with $\theta$. As $\theta$ increases, consumer preferences for the two products become more similar, which reduces the value of sorting out consumers through bundling. This could explain why profit decreases with $\theta$ (under either pure or mixed bundling).

We further analyze the price and welfare effects of bundling for the "Hotelling case" in which symmetric marginals are uniform and the copula exhibits perfect negative dependence. In this case, $u(x) = a + bx$ for $b > \max \{a, -a\}$ and $C(x, y) = \max \{1, x + y - 1\}$ is the Fréchet-Hoeffding lower bound, which implies that, with probability 1, $v(y) = a + by = a + b(1 - x)$. The marginal distribution of consumer values for each good is $F(u) = \frac{u - a}{b}$. Marginal costs are set to zero. The symmetric price when the goods are sold separately is $p^* = (a + b) / 2$; the parameter insures an interior solution.

The support of the copula in this case is degenerated—it is the secondary diagonal on $[0, 1]^2$ and hence does not satisfy the full support assumption of our model. We thus cannot apply Propositions 1 and 2 to the Hotelling case. Nevertheless, Proposition 3 below shows that the conclusions there extend to the Hotelling case, provided that there is a positive measure of $u$ and $v$ for which $u(x) + u(y) > m_x + m_y$.2 Notice that with perfectly negative dependence, $u(x) + u(y) = 2a + b$ for all $(x, y)$, and by assumption $m_x + m_y = 0$.

2 Otherwise, the value of the bundle is (weakly) lower than its cost for all consumers, in which case trivially bundling cannot be part of an optimal selling scheme.
Proposition 3 In the Hotelling case, with $2a + b > 0$, bundling is more profitable than separate pricing. If $a \geq 0$, then pure bundling is optimal and $r^* = 2a + b \geq 2p^*$; if $a < 0$, then mixed bundling is optimal with $r^* = 2a + b < 2p^*$ and $p^* = q^* = \frac{3a + 2b}{2} > p^*$. Furthermore, bundling (weakly) reduces the welfare of all consumers.

Thus, in case of perfect negative dependence and uniform margins, bundling is profitable as long as production cost for the bundle is not weakly higher than its value for all consumers. Furthermore, price under pure bundling is higher than under separate sales for all consumers. Under mixed bundling, the price of the bundle extracts all surplus from all consumers purchasing the bundle, whereas the standalone prices are higher than those under separate sales. Thus, compared to separate sales, all consumers are (weakly) worse off under both pure and mixed bundling.

This Hotelling case, which features perfectly negative dependence, is closely related to the example in Stigler (1963): If Customer A is willing to pay 80 for X and 25 for Y, and customer B is willing to pay 70 for X and 30 for Y, then the multiproduct monopolist can charge 100 to each for the XY bundle, and earn revenue of 200, while under separate pricing charges 70 for X and 25 for Y and earns only 190 in revenue. If the example were changed slightly so that A was only willing to pay 75 for X, then nothing changes in the analysis except that both consumers have the same value for the bundle and the monopolist fully extracts consumer surplus.

Stigler (1963)’s example suggests that negative dependence is important for profitability of pure bundling. In the Hotelling case, however, mixed bundling is sometimes more profitable than pure bundling. As discussed by Adams and Yellen (1976), the reason for the superiority of mixed bundling is that it economizes on costs for consumers who have a low value of one of the goods.

5. COMPETITIVE ASYMMETRIC FIRMS

Our analysis under multiproduct monopoly can be extended to competitive markets. We first investigate the incentives and effects of bundling in markets where a multiproduct firm, $A$, competes with a single-produce firm, $B$, whom we interpret as a (potential) entrant. Firm $A$ offers both X and $Y_A$, whereas Firm $B$ offers product $Y_B$. A consumer’s value for X is $u(x)$, and for $Y_i$ is $v(y_i)$ with $(x, y_A, y_B) \in I^3$. Therefore, the marginal distribution of consumer values for X is $F(u)$, and the sym-
metric distribution for each variety of product Y is \( G(v) \). The copula \( C(x, y_A, y_B) \), with \( y_A \) and \( y_B \) exchangeable, describes the population of consumers. We assume that \( C(x, y_A, y_B) \) is twice-differentiable and admits a density on the full support of \( I^3 \).

Since \( x, y_A, \) and \( y_B \) all have uniform margins, \( C_1(x, y_A, y_B) \) is the joint distribution of \((y_A, y_B)\) conditional on \( X = x \), and the values of product Y are positively dependent, independent, or negatively dependent in the values of X (in the sense of stochastically increasing or stochastically decreasing) if

\[
C_1(x, y_A, y_B) = \Pr(Y_A \leq y_A, Y_B \leq y_B|X = x)
\]

is decreasing, independent, or increasing in \( x \), or \( C_{11}(x, y_A, y_B) \) is decreasing, independent, or increasing in \( y_A \), or \( C_{12}(x, y_A, y_B, y_B) \) is decreasing, independent, or increasing in \( y_B \), respectively. Thus the dependence concept under bivariate distribution extends naturally to the multivariate distribution. Similarly, \( C_2(x, y_A, y_B) \) is the joint distribution of \((x, y_B)\) conditional on \( Y_A = y_A \), and \( C_3(x, y_A, y_B) \) is the joint distribution of \((x, y_A)\) conditional on \( Y_B = y_B \).

Under mixed bundling, let \( p \) denote the standalone price of \( X \), \( q_i \) the standalone price \( y_i \), and \( r \leq p + q_A \) the price of Firm A’s bundle. Consumers will purchase the bundle if

\[
\begin{align*}
  u(x) + v(y_A) - r & \geq \max \{0, v(y_B) - q_B\}, \\
  u(x) + v(y_A) - r & \geq u(x) - p + \max \{0, v(y_B) - q_B\}, \\
  u(x) + v(y_A) - r & \geq v(y_A) - q_A,
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
  y_A & \geq G(r - u(x) + \max \{0, v(y_B) - q_B\}), \\
  y_A & \geq G(r - p + \max \{0, v(y_B) - q_B\}), \\
  x & \geq F(r - q_A).
\end{align*}
\]

Consumers will purchase \( X \) as a standalone product, rather than as part of the bundle,
if

\[ x \geq F(p), \]
\[ y_A \leq G(r - p + \max\{0, v(y_B) - q\}); \]

and \( Y_A \) as a standalone product if

\[ x < F(r - q_A), \]
\[ y_A \geq G(q_A + \max\{0, v(y_B) - q\}). \]

The standalone demands for \( X \) and \( Y_A \) are

\[
Q_X(p, q_B, r) = C(1, G(r - p), G(q_B)) - C(F(p), G(r - p), G(q_B)) \\
+ \int_{G(q_B)}^{1} [C_3(1, G(r - p + v(y) - q_B), y) - C_3(F(p), G(r - p + v(y) - q_B), y)] \, dy,
\]

\[
Q_{Y_A}(q_A, q_B, r) = C(F(r - q_A), 1, G(q_B)) - C(F(r - q_A), G(q_A), G(q_B)) \\
+ \int_{G(q_B)}^{1} [C_3(F(r - q_A), 1, y) - C_3(F(r - q_A), G(q_A + v(y) - q_B), y)] \, dy,
\]

and the demand for the bundle is

\[
Q_{XY_A}(p, q_A, q_B, r) = 1 - F(p) - Q_x(p, q_B, r) \\
+ \int_{F(r-q_A)}^{F(p)} [C_1(x, 1, G(q_B)) - C_1(x, G(r - u(x)), G(q_B))] \, dx \\
+ \int_{F(r-q_A)}^{F(p)} \int_{G(q_B)}^{1} [C_{13}(x, 1, y) - C_{13}(x, G(r - u(x) + v(y) - q_B), y)] \, dy \, dx.
\]

The demand for \( Y_B \) is analogous.
The profit of Firm A is
\[
\pi_A (p, q_A, q_B, r) = (p - m_x) Q_X (q_A, q_B, r) + (q_A - m_y) Q_{Y_A} (q_A, q_B, r) + (r - m_x - m_y) Q_{XY_A} (p, q_A, q_B, r).
\]

**Separate Pricing**

Separate pricing is equivalent to \( r = p + q_A \), in which case the demands for X and \( Y_A \) are respectively
\[
Q^*_x (p) = 1 - F (p),
Q^*_Y (q_A, q_B) = 1 - C_1 (1, G (q_A), G (q_B)) - \int_{G(q_B)}^1 C_2 (1, G (q_A + v(y) - q_B), y) dy.
\]

Equilibrium separate prices \((p^*, q^*)\) satisfy
\[
1 - F (p^*) = (p^* - m_x) f (p^*),
\]
\[
\frac{1}{2} [1 - C_1 (1, G (q^*), G (q^*))] = (q^* - m_y) \left[ C_2 (1, G (q^*), G (q^*)) g(q^*) + \int_{G(q^*)}^1 C_{23} (1, y, y) g(v(y)) dy \right].
\]

An interior equilibrium satisfying these conditions is assumed to exist. See Chen and Riordan (2009) for details on symmetric equilibrium in the \( Y \) market.

**Profitability of Bundling**

MMW observed that, if \((y_A, y_B)\) is independent of \( x \), then
\[
\tilde{v} = v (y_A) - \max \{0, v(y_B) - q_B\}
\] (10)
is independent of \( u \), and the best response problem for Firm A is equivalent to a multiproduct monopoly problem with a population of consumers with values \((u, \tilde{v})\). Consequently in the independence case, it is straightforward that mixed bundling dominates separate sales. The issue, however, is more complicated for more general dependence relationships. We next show that, similarly as in the multiproduct monopoly model, with competition by asymmetric firms bundling also dominates separate sales for Firm A if values for X and Y are not too positively dependent. The result does not depend on the dependence relationship between \( Y_A \) and \( Y_B \).
Proposition 4 With competing asymmetric firms, there exists some $\bar{\delta}_1 > 0$ such that bundling is profitable for the multiproduct firm if $C_{11} \geq -\bar{\delta}_1$.

Proof. See the appendix. ■

This substantially extends Schmalensee (1982)'s result that negative correlation and perfect competition in the Y market is sufficient for mixed bundling to be profitable for a single-product monopolist.

Pricing

Our discussion of how mixed bundled affects pricing is organized around an example in which marginal distributions are uniform on $[0, 1]$, and the copula is the independence copula, i.e. $F(u) = G(u) = u$ for $u \in [0, 1]$, and $C(x, y_A, y_B) = xy_Ay_B$ for $(x, y_A, y_B) \in I^3$. We consider a mixed bundling (Nash) equilibrium in which Firm A (the incumbent) chooses $(p, q_A, r)$ and Firm B (the entrant) chooses $q_B$.

Numerical analysis of the independent-uniform example calculates candidate equilibrium prices $p = 0.604$, $q_A = 0.602$, $q_B = 0.419$, and $r = 0.792$. Corresponding market shares are $Q_X = 0.141$, $Q_{Y_A} = 0.047$, $Q_{Y_B} = 0.369$, and $Q_{XY_A} = 0.377$. The profit of the incumbent is $\Pi_A = 0.459$ and of the entrant is $\Pi_B = 0.155$.

The separate pricing equilibrium imposes the restriction $r = p + q_A$. The prices in this equilibrium are $p = 0.50$, $q_A = 0.414$, $q_B = 0.414$, and $r = 0.914$, and profits are $\Pi_A = 0.422$ and $\Pi_B = 0.172$. Thus all prices except the price of the bundle $(r)$ are higher in the mixed bundling equilibrium. Clearly mixed bundling is profitable for the incumbent, as expected from Proposition 4. Furthermore, the entrant’s profit is lower in the mixed bundling equilibrium. Thus, for a range of fixed cost, mixed bundling deters entry relative to separate pricing.

This example differs from the one in Nalebuff (2004) which assumes that the two Y goods are perfect substitutes and that the incumbent follows a pure bundling strategy. Otherwise, the examples are comparable for the case in which the incumbent cannot precommit to a price strategy; Nalebuff (2004) assumes standard uniform margins and independence for X and Y goods. These different assumptions lead to the contrary conclusion in Nalebuff (2004) that in Nash equilibrium bundling increases post entry profits, and, therefore, is entry accommodating rather than entry deterring. The reason for the different conclusion is familiar from Carbajo, De Meza,
and Seidman (1990) and Chen (1997)\(^3\); pure bundling vertically differentiates the incumbent’s product from the entrant’s product, thus relaxing competition in the Y market. Nalebuff (1984) also studied a subgame perfect equilibrium of a Stackelberg price game, in which the incumbent moved first set the price of the bundle before the entrant followed by setting the price of good Y to reach the opposite conclusion that bundling is entry deterring.

Nalebuff (2004) argues that pure bundling is a more effective entry deterrent than mixed bundling because it prevents consumers from mixing and matching, i.e. purchasing products X and Y\(_B\).\(^4\) Restricting the incumbent to a pure bundling strategy, equilibrium prices are \(r = 0.727\) and \(q_B = 0.404\), and profits are \(\Pi_A = 0.423\) and \(\Pi_B = 0.126\). Thus the proposition that pure bundling is better entry deterrent than mixed bundling is born out in our example, and as Nalebuff (2004) suggests, without very much compromising the incumbent’s profit.

6. SYMMETRIC MULTIPRODUCT FIRMS

The condition for bundling to increase profit under multiproduct monopoly can also be extended to competing symmetric multiproduct firms, under additional restrictions. Suppose that firm A offers X\(_A\) and Y\(_A\) while firm B offers X\(_B\) and Y\(_B\). Consumer values for X\(_i\) and Y\(_i\) are \(u_i\) and \(v_i\), which have marginal distributions \(F(\cdot)\) and \(G(\cdot)\). The joint distribution of \((u_A, u_B, v_A, v_B)\) is associated with copula \(C(x_A, x_B, y_A, y_B)\) such that \(x_i = F(u_i)\) and \(y_i = G(v_i)\), or \(u_i = u(x_i) = F^{-1}(x_i)\) and \(v_i = v(y_i) = G^{-1}(y_i)\), and \(C(x_A, x_B, y_A, y_B) = C(x_B, x_A, y_B, y_A)\).

If values for all products are independent, then \(C(x_A, x_B, y_A, y_B) = x_A x_B y_A y_B\). The general dependence relations between products are more complicated to describe now. For negative dependence, one condition is that \(X_2, Y_1, Y_2\) are all stochastically decreasing in \(X_1\), or \(C_{11}(x_A, x_B, y_A, y_B) \geq 0\). One further condition is that \(C_{11}(x_A, x_B, y_A, y_B) \geq C_{11}(x_A, x_B, 1, 1)\) and \(C_{112}(x_A, x_B, y_A, y_B) \geq C_{112}(x_A, x_B, 1, 1)\), so that the stochastic effect of a higher \(x_A\) is (weakly) more profound when the joint distribution of \((x_A, x_B, y_A, y_B)\) is considered than when only the joint distribution of

\(^3\)In Chen (1997)’s model, a monopoly market structure for X is endogenous.

\(^4\)Nalebuff (2004) makes this argument informally in the context of the Stackelberg model. The argument appears to depend on the assumption that the Y products are perfect substitutes.
(x_A, x_B) is considered. For positive dependence, these inequalities are reversed.

For any given price \( p_2 \) by firm 2, define

\[
\tilde{u} = u_1 - \max \{0, u_2 - p_2\}.
\]

Let the marginal distribution for \( \tilde{u} \) be \( \tilde{F}(\tilde{u}) = x \), and the joint distribution of \( (\tilde{u}, v_1, v_2) \) be associated with copula \( \tilde{C}(x_1, y_1, y_2) \). Then:

**Lemma 1**

\[
\tilde{F}(\tilde{u}) = C(F(\tilde{u}), F(p_2), 1, 1) + \int_{F(p_2)}^1 C_2(F(\tilde{u} + u(x_2) - p_2), x_2, 1, 1) \, dx_2,
\]

\[
\tilde{f}(u) = C_1(F(\tilde{u}), F(p_2), 1, 1) \tilde{f}(\tilde{u}) + \int_{F(p_2)}^1 C_{12}(F(\tilde{u} + u(x_2) - p_2), x_2, 1, 1) \, dx_2,
\]

\[
\tilde{C}(x_1, y_1, y_2) = C(F(\tilde{u}), F(p_2), y_1, y_2) + \int_{F(p_2)}^1 C_2(F(\tilde{u} + u(x_2) - p_2), x_2, y_1, y_2) \, dx_2.
\]

**(11)**

**Proof.** See appendix. ■

Given any prices of firm B for \( X_B \) and \( Y_B \) under separate sales, \( p_B \) and \( q_B \), firm A’s demand functions for \( X_A, Y_A \) and the bundle are given by (7), (8), and (9), with \( \tilde{F}(\cdot) \) and \( \tilde{C}(\cdot) \) respectively replacing \( F(\cdot) \) and \( C(\cdot) \). Therefore, if independence, negative dependence, or a sufficiently limited positive dependence under \( C(\cdot) \) leads to the same dependence properties under \( \tilde{C}(\cdot) \), then Proposition 4 would imply that under these dependence properties firm A’s profit is higher with bundling than without. The next Proposition makes this argument, which uses the definition below:

\[
\alpha(x_A, x_B, y_A, y_B) \equiv \min \{C_{11}(x_A, x_B, y_A, y_B), C_{11}(x_A, x_B, y_A, y_B) - C_{11}(x_A, x_B, 1, 1)\},
\]

\[
\beta(x_A, x_B, y_A, y_B) \equiv \min \{C_{112}(x_A, x_B, y_A, y_B), C_{112}(x_A, x_B, y_A, y_B) - C_{112}(x_A, x_B, 1, 1)\}.
\]

**Proposition 5** If valuations for all products are independent, or if \( \min \{\alpha(\cdot), \beta(\cdot)\} \geq -\delta \) for sufficiently small \( \delta > 0 \) and \( f(\cdot) \) is loglinear (i.e., \( d \ln f(u)/du^2 = 0 \)), then in equilibrium at least one firm chooses bundling.

**Proof.** From Proposition 4, it suffices to show that under the stated conditions
\[ \tilde{C}_{11} (x, y_1, y_2) \geq -\delta_1 \] for some given \( \delta_1 > 0 \).

First, from (11), the conditional distribution of \((y_1, y_2)\) given \(x\) is

\[ \tilde{C}_1 (x, y_1, y_2) = C_1 (F (\tilde{u}), F (p_2), y_1, y_2) \frac{f (\tilde{u})}{f (u)} \]

\[ + \int_{x_2 = F (p_2)}^{1} C_{12} (F (\tilde{u} + u (x_2) - p_2), x_2, y_1, y_2) \frac{f (\tilde{u} + u (x_2) - p_2)}{f (u)} dx_2 \]

\[ = \frac{C_1 (F (\tilde{u}), F (p_2), y_1, y_2) f (u) + \int_{F (p_2)}^{1} C_{12} (F (\cdot), x_2, y_1, y_2) f (\tilde{u} + u (x_2) - p_2) dx_2}{C_1 (F (\tilde{u}), F (p_2), 1, 1) f (\tilde{u}) + \int_{F (p_2)}^{1} C_{12} (F (\tilde{u}), x_2, 1, 1) f (\tilde{u} + u (x_2) - p_2) dx_2} \]

\[ = \frac{C_1 (x, F (p_2), y_1, y_2) + \int_{F (p_2)}^{1} C_{12} (F (\tilde{u} + u (x_2) - p_2), x_2, y_1, y_2) \frac{f (\tilde{u} + u (x_2) - p_2)}{f (\tilde{u})} dx_2}{C_1 (x, F (p_2), 1, 1) + \int_{F (p_2)}^{1} C_{12} (F (\tilde{u} + u (x_2) - p_2), x_2, 1, 1) \frac{f (\tilde{u} + u (x_2) - p_2)}{f (\tilde{u})} dx_2} \]

\[ \equiv \frac{\Phi (x, y_1, y_2)}{\Phi (x, 1, 1)} \in [0, 1], \]

which implies \( \Phi (x, 1, 1) \geq \Phi (x, y_1, y_2) \). If valuations for all products are independent, we have \( C (x_1, x_2, y_1, y_2) = x_1 x_2 y_1 y_2 \) and

\[ \tilde{C}_1 (x, y_1, y_2) = y_1 y_2 \frac{F (p_2)}{F (p_2)} + \int_{F (p_2)}^{1} \frac{f (\tilde{u} (x) + u (x_2) - p_2)}{f (\tilde{u} (x))} dx_2 \]

\[ = y_1 y_2, \]

implying \( \tilde{C}_{11} (x, y_1, y_2) = 0 \geq -\delta \) for any \( \delta > 0 \).

Next, \( d \left[ f (\tilde{u} (x) + u (x_2) - p_2) / f (\tilde{u} (x)) \right] / dx = 0 \) since \( f (\cdot) \) is loglinear, and thus

\[ \Phi_1 (x, y_1, y_2) \]

\[ = C_{11} (x, F (p_2), y_1, y_2) \]

\[ + \int_{F (p_2)}^{1} \left[ C_{112} (F (\tilde{u} (x) + u (x_2) - p_2), x_2, y_1, y_2) \frac{f (\tilde{u} + u (x_2) - p_2)^2}{f (\tilde{u}) f (\tilde{u})} \right] dx_2. \]
If $\min \{ \alpha (\cdot), \beta (\cdot) \} \geq 0$, then $\Phi_1 (x, y_1, y_2) \geq 0$ and

$$\Phi_1 (x, y_1, y_2) - \Phi_1 (x, 1, 1) = C_{11} (x, F (p_2), y_1, y_2) - C_{11} (x, F (p_2), 1, 1) +$$

$$+ \int_{x_2=F(p_2)}^1 [C_{112} (F (\cdot), x_2, y_1, y_2) - C_{112} (F (\cdot), x_2, 1, 1)] \frac{[f (\bar{u} (x) + u (x_2) - p_2)]^2}{f (\bar{u} (x)) f (\bar{u})} \, dx_2 \geq 0.$$ 

But

$$\Phi_1 (x, y_1, y_2) - \Phi_1 (x, 1, 1) \geq 0$$

$$\implies \frac{\Phi_1 (x, y_1, y_2) \Phi (x, 1, 1) - \Phi_1 (x, 1, 1) \Phi (x, y_1, y_2)}{\Phi^2 (x, 1, 1)} = \tilde{C}_{11} (x, y_1, y_2) \geq 0.$$

By continuity, for given $\tilde{\delta}_1 > 0$, if $\min \{ \alpha (\cdot), \beta (\cdot) \} \geq -\delta$ for sufficiently small $\delta > 0$, $\tilde{C}_{11} (x, y_1, y_2) \geq -\tilde{\delta}_1$. ■

As we discussed earlier, $\min \{ \alpha (\cdot), \beta (\cdot) \} \geq -\delta$ for sufficiently small $\delta > 0$ can be considered the condition that products X and Y are not too positively dependent. Under this sufficient condition, bundling is profitable for at least one multiproduct firm. This sufficient condition relies on the additional restriction that $f (\cdot)$ is loglinear, which is satisfied if $F (\cdot)$ is uniform or exponential. Proposition 5 extends our result under monopoly. It also extends MMW by going beyond independent preferences.

7. CONCLUSION

To be added.

APPENDIX

Proof of Proposition 3.—

Let $r \leq 2a + b$ be the price of the bundle and $p$ the standalone price for each of the goods. Then a type $x$ consumer will prefer to purchase good X instead of the XY bundle if $r - p \geq a + b (1 - x)$. Therefore, the fraction of consumers purchasing
X on a standalone basis is $0 < (r - p - a)/b < \frac{1}{2}$, and similarly for Y. Remaining consumers purchase the bundle, so profits are

$$2p\frac{r - p - a}{b} + r \left[ 1 - \frac{2(r - p - a)}{b} \right] = r - \frac{2(r - p)(r - p - a)}{b} = r - \frac{2\delta(\delta - a)}{b}$$

where $\delta \equiv r - p$. Given $r$, this profit is maximized at $\delta = a/2$, in which case $0 < (r - p - a)/b < \frac{1}{2}$ implies $-b < a < 0$. If $a \geq 0$, then pure bundling is the solution, in which case $r = 2a + b$. Given $-b < a < 0$ and $\delta = a/2$, the maximum price of the bundle $r = 2a + b$ is optimal, in which case $p = \frac{3a + 2b}{2}$. If follows that profit under optimal mixed bundling is

$$r - \frac{2\delta(\delta - a)}{b} = 2a + b + \frac{a^2 + 4ab + 2b^2}{2b} = \frac{(a + b)^2}{2b} + \frac{2ab + b^2}{2b} = \frac{(a + b)^2}{2b} + \frac{2a + b}{2}$$

In contrast, the profit under separate pricing is $\frac{(a + b)^2}{2b}$. Therefore, bundling is more profitable than separate pricing if and only if $2a + b > 0$. The consumer welfare result is obvious.

**Proof of Proposition 4.**—

Let $(p^s, q^s)$ denote the prices of X and Y products in a separate pricing equilibrium. Then

$$\psi(\varepsilon) \equiv \pi_A \left( p^s + \epsilon, q^s, q^s, p^s + q^s \right)$$

is Firm A’s profit from increasing the standalone price of X, while holding constant the standalone prices of the Y product and the price of the bundle at $r = p^s + q^s$. Bundling must be profitable in equilibrium if $\psi'(0) > 0$. Noticing that $\partial Q_{Y A}(q_A, q_B, r)/\partial p = 0$, we have

$$\psi'(0) = \frac{\partial \pi_A \left( p^s, q^s, q^s, p^s + q^s \right)}{\partial p} = Q_X \left( p^s, q^s, p^s + q^s \right) + (p^s - m_x) \frac{\partial Q_X \left( p^s, q^s, p^s + q^s \right)}{\partial p} + (p^s + q^s - m_x - m_y) \frac{\partial Q_{XY A} \left( p^s, q^s, q^s, p^s + q^s \right)}{\partial p}$$
with

$$Q_x (p^s, q^s, p^s + q^s) = C (1, G (q^s), G (q^s)) - C (F (p^s), G (q^s), G (q^s))$$
$$+ \int_{G(q^s)}^1 \left[ C_3 (1, y, y) - C_3 (F (p^s), y, y) \right] dy$$
$$= \frac{1}{2} \left[ 1 - F (p^s) \right] + \frac{1}{2} \left[ C (1, G (q^s), G (q^s)) - C (F (p^s), G (q^s), G (q^s)) \right];$$

$$\frac{\partial Q_x (p^s, q^s, p^s + q^s)}{\partial p} = - [C_2 (1, G (q^s), G (q^s)) - C_2 (F (p^s), G (q^s), G (q^s))] g (q^s)$$
$$- C_1 (F (p^s), G (q^s), G (q^s)) f (p^s)$$
$$- \int_{G(q^s)}^1 \left[ C_{23} (1, y, y) - C_{23} (F (p^s), y, y) \right] g (v (y)) dy$$
$$- \int_{G(q^s)}^1 C_{13} (F (p^s), y, y) dy f (p^s),$$

$$\frac{\partial Q_{xy} (p^s, q^s, p^s + q^s)}{\partial p} = - f (p^s) - \frac{\partial Q_x (p^s, q^s, p^s + q^s)}{\partial p}$$
$$+ \left[ C_1 (F (p^s), 1, G (q^s)) - C_1 (F (p^s), G (q^s), G (q^s)) \right] f (p^s)$$
$$+ \int_{G(q^s)}^1 \left[ C_{13} (F (p^s), 1, y) - C_{13} (F (p^s), y, y) \right] dy f (p^s)$$

$$= - f (p^s) + [C_2 (1, G (q^s), G (q^s)) - C_2 (F (p^s), G (q^s), G (q^s))] g (q^s)$$
$$+ \int_{G(q^s)}^1 \left[ C_{23} (1, y, y) - C_{23} (F (p^s), y, y) \right] g (v (y)) dy + C_1 (F (p), 1, 1) f (p).$$

Therefore,

$$\psi' (0) = Q_x (\cdot) + (p^s - m_x) \frac{\partial Q_x (\cdot)}{\partial p} +$$

$$(p^s - m_x) \left\{ - f (p^s) - \frac{\partial Q_x (\cdot)}{\partial p} + \left[ C_1 (F (p^s), 1, G (q^s)) - C_1 (F (p^s), G (q^s), G (q^s)) \right] f (p^s) \right.$$
$$+ \int_{G(q^s)}^1 \left[ C_{13} (F (p), 1, y_2) - C_{13} (F (p), y_2, y_2) \right] f (p) dy_2 \right\}$$

$$+ (q^s - m_y) \left\{ - f (p^s) + \left[ C_2 (1, G (q^s), G (q^s)) - C_2 (F (p^s), G (q^s), G (q^s)) \right] g (q^s) \right.$$
$$+ \int_{G(q^s)}^1 \left[ C_{23} (1, y, y) - C_{23} (F (p^s), y, y) \right] g (v (y)) dy + C_1 (F (p), 1, 1) f (p) \right\}$$

After cancelling out two terms of $(p^s - m) \frac{\partial Q_x}{\partial p}$, and noticing $C_1 (F (p^s), 1, 1) = 1,$
\[ C_{13} (1, y_2, y_2) = \frac{1}{2} \frac{dC_1 (1, y_2, y_2)}{dy_2}, \] we have \( \psi' (0) = Q_x (\cdot) + \)

\[(p^s - m_x) \left\{ -f (p^s) + [C_1 (F (p^s), 1, G (q^s)) - C_1 (F (p^s), G (q^s), G (q^s))] f (p^s) \right. \]
\[\left. + f (p^s) - C_1 (F (p), 1, G (q^s)) f (p^s) - \int_{G(q^s)}^{1} \frac{1}{2} dC_1 (F (p) , y_2, y_2) f (p) \right\} \]
\[+(q^s - m_y) \left\{ [C_2 (1, G (q^s), G (q^s)) - C_2 (F (p^s), G (q^s), G (q^s))] g (q^s) \right. \]
\[\left. + \int_{G(q^s)}^{1} [C_{23} (1, y, y) - C_{23} (F (p^s), y, y)] g (v (y)) dy \right\} . \]

Substituting \( Q_x (\cdot) \), and using \( (p^s - m) f (p^s) = 1 - F (p^s) \),

\[ \psi' (0) = \frac{1}{2} [1 - F (p^s)] + \frac{1}{2} \int_{F(q^s)}^{1} C_1 (x, G (q^s), G (q^s)) dx - \frac{1}{2} [1 - F (p^s)] \]
\[-\frac{1}{2} C_1 (F (p^s), G (q^s), G (q^s)) [1 - F (p^s)] \]
\[+(q^s - m_y) \left\{ [C_2 (1, G (q^s), G (q^s)) - C_2 (F (p^s), G (q^s), G (q^s))] g (q^s) \right. \]
\[\left. + \int_{G(q^s)}^{1} [C_{23} (1, y, y) - C_{23} (F (p^s), y, y)] g (v (y)) dy \right\} \]
\[= \frac{1}{2} \int_{F(q^s)}^{1} \int_{F(q^s)}^{x} C_{11} (z, G (q^s), G (q^s)) dz dx + \varphi (C_2 (\cdot), C_{23} (\cdot)) , \]

where

\[ \varphi (C_2 (\cdot), C_{23} (\cdot)) \equiv (q^s - m_y) \left\{ [C_2 (1, G (q^s), G (q^s)) - C_2 (F (p^s), G (q^s), G (q^s))] g (q^s) \right. \]
\[\left. + \int_{G(q^s)}^{1} [C_{23} (1, y, y) - C_{23} (F (p^s), y, y)] g (v (y)) dy \right\} > 0 \]
since
\[ C_2(1, G(q^*), G(q^*)) - C_2(F(p^*), G(q^*), G(q^*)) > 0, \]
\[ C_{23}(1, y, y) - C_{23}(F(p^*), y, y) > 0. \]

Thus, \( \psi'(0) > 0 \) if \( C_{11} \geq 0 \). Now, for \( \delta \geq 0 \), consider function
\[ \Gamma(\delta) \equiv \inf \{ \varphi(C_2(\cdot), C_{23}(\cdot)) : -\delta \leq C_{11} \leq 0 \}. \]

Then, \( \Gamma(\delta) \) is non-increasing. Since \( C(\cdot) \) has full support on \( I^3 \), \( \Gamma(\delta) \) is bound away from zero, and hence \( \Gamma(\delta) > \delta \) for \( \delta \) sufficiently close to zero. Therefore, there exists some \( \hat{\delta}_1 > 0 \) such that \( \delta \geq \Gamma(\delta) \) first occurs at \( \hat{\delta}_1 \). Let \( \tilde{\delta}_1 = \hat{\delta}_1/2 > 0 \), then \( \tilde{\delta}_1 < \Gamma(\hat{\delta}_1) \).

It follows that if \( C_{11} \geq -\hat{\delta}_1 \),
\[ \psi'(0) \geq -\delta_1 \left[ 1 - F(p^*) \right]^2 + \varphi(C_2(\cdot), C_{23}(\cdot)) \]
\[ > -\Gamma(\tilde{\delta}_1) \left[ 1 - F(p^*) \right]^2 + \varphi(C_2(\cdot), C_{23}(\cdot)) \]
\[ = -\inf \{ \varphi(C_2(\cdot), C_{23}(\cdot)) : -\delta_1 \leq C_{11} \leq 0 \} + \varphi(C_2(\cdot), C_{23}(\cdot)) \]
\[ \geq 0. \]

Proof of Lemma 1.—

\[ \tilde{F}(u) = \Pr(U < u) \]
\[ = \Pr(u_1 - \max \{0, u_2 - p_2\} < u) = \Pr(F^{-1}(x_1) - \max \{0, F^{-1}(x_2) - p_2\} < u) \]
\[ = \int_0^{F(u)} \int_0^{F(p_2)} C_{12}(x_1, x_2, 1, 1) \, dx_2 \, dx_1 + \int_0^{F(u)+F^{-1}(x_2)-p_2} \int_{F(p_2)}^1 C_{12}(x_1, x_2, 1, 1) \, dx_2 \, dx_1 \]
\[ = C(F(u), F(p_2), 1, 1) + \int_{F(p_2)}^1 C_2(F(u + F^{-1}(x_2) - p_2), x_2, 1, 1) \, dx_2. \]
\[ f(u) = C_1(F(u), F(p_2), 1, 1) f(u) \]
\[ + \int_{F(p_2)}^1 C_{12}(F(u + F^{-1}(x_2) - p_2), x_2, 1, 1) f(u + F^{-1}(x_2) - p_2) \, dx_2. \]

\[ C(x, y_1, y_2) = \Pr(X < x, Y_1 < y_1, Y_2 < y_2) \]
\[ = \Pr(\tilde{F}(U) < \tilde{F}(u), Y_1 < y_1, Y_2 < y_2) = \Pr(U < u, Y_1 < y_1, Y_2 < y_2) \]
\[ = \Pr(U_1 - \max\{0, U_2 - p_2\} < u, Y_1 < y_1, Y_2 < y_2) \]
\[ = \Pr(F^{-1}(x_1) - \max\{0, F^{-1}(x_2) - p_2\} < u, Y_1 < y_1, Y_2 < y_2) \]
\[ = \int_{x_2=0}^{F(p_2)} \int_{x_1=0}^{F(u)} \int_{y_1=0}^{y_1} \int_{y_2=0}^{y_2} c(x_1, x_2, y_1, y_2) \, dx_1 \, dx_2 \, dy_1 \, dy_2 \]
\[ + \int_{x_2=F(p_2)}^1 \int_{x_1=0}^{F(u)} \int_{y_1=0}^{y_1} \int_{y_2=0}^{y_2} c(x_1, x_2, y_1, y_2) \, dx_1 \, dx_2 \, dy_1 \, dy_2 \]
\[ = C(F(u), F(p_2), y_1, y_2) + \int_{x_2=F(p_2)}^1 C_2(F(u + F^{-1}(x_2) - p_2), x_2, y_1, y_2) \, dx_2. \]

**Tables for the FGM-Uniform Case.**

When \( m = 0, p^* = 4.5, \pi^* = 8.1, \) and \( W^* = 4.05. \) Pure bundling is optimal, and \( r^* > 2p^* = 9. \) Bundling makes all consumers worse off. Both the bundle price and profit decrease as \( \theta \) increase, whereas consumer surplus increases as \( \theta \) increases.

**Table 1. \( m = 0 \)**

<table>
<thead>
<tr>
<th>( m = 0 )</th>
<th>( p^* )</th>
<th>( r^* )</th>
<th>( \pi^* )</th>
<th>( \Delta \pi )</th>
<th>( W^* )</th>
<th>( \Delta W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta = -1 )</td>
<td>( \geq 9 )</td>
<td>10.564</td>
<td>9.7361</td>
<td>1.6361</td>
<td>2.4877</td>
<td>-1.5623</td>
</tr>
<tr>
<td>( \theta = -0.5 )</td>
<td>( \geq 9 )</td>
<td>10.412</td>
<td>9.4636</td>
<td>1.3636</td>
<td>2.6552</td>
<td>-1.3948</td>
</tr>
<tr>
<td>( \theta = 0 )</td>
<td>( \geq 9 )</td>
<td>10.210</td>
<td>9.2127</td>
<td>1.1127</td>
<td>2.8620</td>
<td>-1.188</td>
</tr>
<tr>
<td>( \theta = 0.5 )</td>
<td>( \geq 9 )</td>
<td>9.9556</td>
<td>8.9925</td>
<td>0.8925</td>
<td>3.1102</td>
<td>-0.9398</td>
</tr>
<tr>
<td>( \theta = 1 )</td>
<td>( \geq 9 )</td>
<td>9.6752</td>
<td>8.8114</td>
<td>0.7114</td>
<td>3.3777</td>
<td>-0.6723</td>
</tr>
</tbody>
</table>

When \( m = 2, p^* = 5.5, \pi^* = 4.9, \) and \( W^* = 2.45. \) Pure bundling is optimal, and \( r^* > 2p^* = 11 \) when \( \theta \leq -0.5 \) but \( r^* < 2p^* \) when \( \theta \geq 0. \) Both the bundle price and profit decrease as \( \theta \) increase, whereas consumer surplus increases as \( \theta \) increases. (Aggregate) consumer surplus is lower under bundling when \( \theta \leq 0.5 \) but higher when \( \theta = 1. \)
Table 2. $m = 2$

<table>
<thead>
<tr>
<th>$m = 2$</th>
<th>$p^*$</th>
<th>$r^*$</th>
<th>$\pi^*$</th>
<th>$\Delta \pi$</th>
<th>$W^*$</th>
<th>$\Delta W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = -1$</td>
<td>$\geq 9$</td>
<td>11.100</td>
<td>6.1468</td>
<td>1.2468</td>
<td>2.0079</td>
<td>$-0.4421$</td>
</tr>
<tr>
<td>$\theta = -0.5$</td>
<td>$\geq 9$</td>
<td>11.042</td>
<td>5.9421</td>
<td>1.0421</td>
<td>2.1020</td>
<td>$-0.348$</td>
</tr>
<tr>
<td>$\theta = 0$</td>
<td>$\geq 9$</td>
<td>10.961</td>
<td>5.7404</td>
<td>0.8404</td>
<td>2.2121</td>
<td>$-0.2379$</td>
</tr>
<tr>
<td>$\theta = 0.5$</td>
<td>$\geq 9$</td>
<td>10.846</td>
<td>5.5434</td>
<td>0.6434</td>
<td>2.3458</td>
<td>$-0.1042$</td>
</tr>
<tr>
<td>$\theta = 1$</td>
<td>$\geq 9$</td>
<td>10.680</td>
<td>5.3547</td>
<td>0.4547</td>
<td>2.5152</td>
<td>$0.0652$</td>
</tr>
</tbody>
</table>

When $m = 4$, $p^* = 6.5$, $\pi^* = 2.5$, and $W^* = 1.25$. Mixed bundling is optimal, $p^* > p^*$, and $r^* < 2p^* = 13$. As $\theta$ increases, the standalone price and profit decrease, the bundle price increases, whereas consumer surplus first increases then decreases. Consumer surplus is higher under bundling when $\theta \leq 0.5$ but lower when $\theta = 1$.

Table 3. $m = 4$

<table>
<thead>
<tr>
<th>$m = 4$</th>
<th>$p^*$</th>
<th>$r^*$</th>
<th>$\pi^*$</th>
<th>$\Delta \pi$</th>
<th>$W^*$</th>
<th>$\Delta W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = -1$</td>
<td>7.4999</td>
<td>12.072</td>
<td>2.9174</td>
<td>0.4174</td>
<td>1.2636</td>
<td>0.0136</td>
</tr>
<tr>
<td>$\theta = -0.5$</td>
<td>7.4311</td>
<td>12.170</td>
<td>2.8286</td>
<td>0.3286</td>
<td>1.278</td>
<td>0.028</td>
</tr>
<tr>
<td>$\theta = 0$</td>
<td>7.3333</td>
<td>12.310</td>
<td>2.746</td>
<td>0.246</td>
<td>1.2749</td>
<td>0.0249</td>
</tr>
<tr>
<td>$\theta = 0.5$</td>
<td>7.2111</td>
<td>12.496</td>
<td>2.6741</td>
<td>0.1741</td>
<td>1.2530</td>
<td>0.003</td>
</tr>
<tr>
<td>$\theta = 1$</td>
<td>7.0917</td>
<td>12.700</td>
<td>2.6171</td>
<td>0.1171</td>
<td>1.2209</td>
<td>$-0.0291$</td>
</tr>
</tbody>
</table>

When $m = 6$, $p^* = 7.5$, $\pi^* = 0.9$, and $W^* = 0.45$. Mixed bundling is optimal, $p^* > p^*$, and $r^* < 2p^* = 15$. As $\theta$ increases, the standalone price and bundle price increase, profit decreases, and consumer surplus varies non-monotonically. Consumer surplus is always lower under bundling.

Table 4. $m = 6$

<table>
<thead>
<tr>
<th>$m = 6$</th>
<th>$p^*$</th>
<th>$r^*$</th>
<th>$\pi^*$</th>
<th>$\Delta \pi$</th>
<th>$W^*$</th>
<th>$\Delta W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = -1$</td>
<td>7.682</td>
<td>14.226</td>
<td>0.95609</td>
<td>0.05609</td>
<td>0.44433</td>
<td>$-0.00567$</td>
</tr>
<tr>
<td>$\theta = -0.5$</td>
<td>7.6846</td>
<td>14.381</td>
<td>0.94548</td>
<td>0.04548</td>
<td>0.44677</td>
<td>$-0.00323$</td>
</tr>
<tr>
<td>$\theta = 0$</td>
<td>7.6916</td>
<td>14.52</td>
<td>0.93861</td>
<td>0.03861</td>
<td>0.44478</td>
<td>$-0.00522$</td>
</tr>
<tr>
<td>$\theta = 0.5$</td>
<td>7.7046</td>
<td>14.636</td>
<td>0.93751</td>
<td>0.03751</td>
<td>0.42157</td>
<td>$-0.02843$</td>
</tr>
<tr>
<td>$\theta = 1$</td>
<td>7.7244</td>
<td>14.730</td>
<td>0.93241</td>
<td>0.03241</td>
<td>0.43518</td>
<td>$-0.01482$</td>
</tr>
</tbody>
</table>
REFERENCES


