Valuing private equity*

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Abstract

To evaluate the performance of private equity (PE) investments, we solve and calibrate a simple portfolio-choice model for a risk-averse institutional investor. In addition to investing in traditional stocks and bonds, this investor is a limited partner (LP) in a PE fund. The fund is illiquid and holds a PE asset, managed by a general partner (GP) who generates alpha and charges management and performance fees. Under incomplete markets, we derive tractable expressions for the LP’s portfolio weights, consumption rule, and certainty-equivalent valuation of the PE investment. We find that the typically observed 2-20 compensation contract requires a substantial alpha for the LP to break even. Leverage reduces this break-even alpha, because the GP can generate alpha on a larger asset base. Evaluating empirical PE performance measures under our model, we find that their usual interpretation may be optimistic. On average, LPs may just break even.

Keywords: Private equity, LP portfolio choice, certainty-equivalent valuation, incomplete markets, illiquidity, non-diversifiable risk, alpha, GP compensation, management fees, carried interest.

JEL Classification: G11, G23, G24.

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A substantial fraction of institutional investors’ assets is allocated to alternative investments. Yale University’s endowment targets a 63% allocation (34% to private equity (PE), 20% to real estate, and 9% to natural resources). The California Public Employees’ Retirement System (CalPERS) allocates 14% to PE and 10% to real assets.\footnote{See http://news.yale.edu/2011/09/28/investment-return-219-brings-yale-endowment-value-194-billion, and http://www.calpers.ca.gov/index.jsp?bc=/investments/assets/assetallocation.xml, accessed on March 23rd., 2012.} More generally, allocations by public pension funds range from 0% for Georgia’s Municipal Retirement System, which is prohibited by law from making alternative investments, to 46% for the Pennsylvania State Employees’ Retirement System.\footnote{NY Times, April 1, 2012, “After Riskier Bets, Pension Funds Struggle to Keep Up” by Julie Creswell.} Given the magnitude and diversity of these allocations, it is important to understand the economic performance of alternative investments compared to traditional traded stocks and bonds. This study focuses on PE investments, specifically investments by a limited partner (LP) in a buyout (BO) fund, managed by a general partner (GP), but similar issues arise for venture capital, real estate, natural resources, and other alternative assets.

PE investments have four distinguishing features. First, they are illiquid. PE funds have ten-year maturities, and the secondary market for PE positions is opaque, making it difficult for LPs to rebalance PE investments and common to hold them to maturity. Moreover, PE funds hold illiquid privately-held companies, making the funds difficult to value during the life of the PE investment. Second, the management of a PE fund is delegated to a GP, raising concerns about incentives, aligning GP and LP interests, and screening for high-quality GPs. Third, the compensation schedule is non-linear and performance based, giving the GP both ongoing management fees, resembling a fixed-income stream, and a performance fee (“carried interest”), resembling a call option, both of which affect the LP’s net-of-fees risk and return (Metrick and Yasuda (2010)). Fourth, PE assets are distinct from publicly traded assets. The returns to these asset classes are not perfectly correlated, and publicly traded stocks and bonds cannot fully hedge PE returns. The resulting incomplete markets and non-diversifiable risks lead to potential diversification benefits and volatility costs for a risk-averse LP.

To evaluate PE performance given these features, we model and calibrate the LP’s portfolio-choice problem. We introduce a PE asset in the standard Merton (1971) model...
where an investor continuously rebalances a portfolio with publicly traded equity and a risk free asset. Specifically, the investor (LP) also invests in a PE fund, managed by the GP. This PE investment must be held to maturity (ten years). During the life of the fund, the GP receives ongoing management fees (typically 1.5%–2% of committed capital). At maturity, the fund is liquidated, and the LP and GP share the proceeds according to a given schedule (“waterfall”), typically compensating the GP with a 20% performance fee (“carried interest”) after the LP achieves a specified preferred return (“hurdle rate”). The return to the PE investment is risky. Only part of this risk is spanned by publicly-traded assets and can be hedged. Hence, in addition to the standard risk premium for systematic risk, the LP requires a premium for holding illiquid non-diversifiable risk. To compensate the LP, the GP must generate sufficient excess return (alpha) by effectively managing the fund’s assets.

The model delivers a tractable solution and intuitive expression for the LP’s certainty-equivalent valuation of the PE investment. The certainty-equivalent solves a non-linear differential equation, which extends the standard Black-Scholes formula with four new terms, accounting for alpha, management fees, carried interest, and the non-linear pricing of illiquid non-diversifiable risk. We obtain closed-form solutions for the optimal hedging portfolio and consumption rules. Finally, we solve for the alpha that the GP must generate for the LP to break even in certainty-equivalence terms. Ignoring leverage, our baseline calibration requires a break-even alpha of 2.6–3.1%, annually.

Surprisingly, leverage reduces the breaks-even alpha. Axelson, Jenkinson, Stromberg, and Weisbach (2011) report a historical average D/E ratio of 3.0 for buyout transactions. In our baseline calibration, when the D/E ratio increases from zero to three, the break-even alpha declines from 3.1% to 2.1%. The benefit of leverage is that it increases the amount of assets for which the GP generates alpha, yet leaves fees unchanged. The cost of leverage is that it increases risk and volatility. In our calibrations, the first effect dominates. This may provide an answer to the “PE leverage puzzle” introduced by Axelson, Jenkinson, Stromberg, and Weisbach (2011). They find that the credit market is the primary predictor of leverage used in PE transactions, and that PE funds appear to use as much leverage as tolerated by the market.\(^3\) This behavior is inconsistent with standard theories of capital structure (see

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\(^3\)In their conclusion, Axelson, Jenkinson, Stromberg, and Weisbach (2011) state that “the factors that predict capital structure in public companies have no explanatory power for buyouts. Instead, the main factors that do affect the capital structure of buyouts are the price and availability of debt; when credit is
also Axelson, Stromberg, and Weisbach (2009)). In our model, it is optimal.

Finally, our model produces tractable expressions for the performance measures used in practice. Given the difficulties of estimating traditional risk and return measures, such as CAPM alphas and betas, several alternative measures have been adopted, including the Internal Rate of Return (IRR), Total Value to Paid-In capital multiple (TVPI), and Public Market Equivalent (PME). While these alternative measures are easier to compute, they are more difficult to interpret. For example, Harris, Jenkinson, and Kaplan (2011) report a value-weighted average PME of 1.27 and conclude that “it seems very likely that buyout funds have outperformed public markets in the 1980s, 1990s, and 2000s.” Whether or not this outperformance is sufficient, however, to compensate LPs for the risks, illiquidity, fees, and other features of PE investing can be evaluated within our model. Given the break-even alpha, we calculate the corresponding break-even values of the IRR, TVPI, and PME measures. We find that these break-even values are close to their empirical counterparts. Our baseline calibration gives a break-even PME of 1.30, suggesting that the reported outperformance may be just sufficient for LPs to break even, on average. While the exact break-even values depend on the specific calibration, the more general message is that the traditional interpretation of these performance measures may be misleading.


abundant and cheap, buyouts become more leveraged [...] Private equity practitioners often state that they use as much leverage as they can to maximize the expected returns on each deal. The main constraint they face, of course, is the capital market, which limits at any particular time how much private equity sponsors can borrow for any particular deal.”

4As explained below, the PME is calculated by dividing the present value (PV) of cash flows distributed to the LP by the PV of cash flows paid by the LP, where the discounting is performed using realized market returns. A PME exceeding one is interpreted as outperformance relative to the market. Specifically, a PME of 1.27 is interpreted as outperforming the market by 27% over the life of the investment.

5Kaplan and Schoar (2005) find substantial persistence in the performance of subsequent PE funds managed by the same PE firm, indicating that PE firms differ in their quality and ability to generate returns. Lerner, Schoar, and Wongsuwai (2007) find systematic variation in PE performance across LP types, suggesting that LPs differ in their ability to identify and access high-quality PE firms. Hence, some specific LPs may well be able to consistently outperform (or underperform) the average PE performance. Our analysis considers only average performance.
and Yasuda (2010) calculate present values of different features of the GP’s compensation, including management fees, carried interest, and the hurdle rate, assuming a risk-neutral investor. Franzoni, Nowak, and Phalippou (2012) estimate a factor model with a liquidity factor for buyout investments. Robinson and Sensoy (2012) investigate the correlation between PE investments and the public market, and the effect of this correlation on the PME measure. None of these studies, however, evaluate PE performance in the context of the LP’s portfolio-choice decision.

Our analysis also relates to the extensive literature about valuation and portfolio choice with illiquid assets, such as restricted stocks, executive compensation, non-traded labor income, illiquid entrepreneurial businesses, and other illiquidity frictions. For example, Kahl, Liu, and Longstaff (2003) analyze a continuous-time portfolio choice model with restricted stocks. Ang, Papanikolaou, and Westerfield (2012) analyze a model with an illiquid asset that can be traded and rebalanced at Poisson arrival times. Despite the substantial literature, we are unaware of any existing model that fully captures the illiquidity, non-diversifiable risk, and compensation features of PE investments. Capturing these institutional features in a model that is sufficiently tractable to evaluate actual PE performance is a main contribution of this study.

The paper is structured as follows. Section 1 presents the model. The model is solved, under complete and incomplete markets, in Sections 2 and 3. The calibration and comparison to existing empirical performance measures are in Section 4. Section 5 evaluates the effects of leverage, and Section 6 evaluates management fees and carried interest. Section 7 concludes. Technical derivations are in the Appendix.

1 Model

An institutional investor (LP) invests in public and private equity, in addition to a risk-free asset. Both the public equity, represented by a public market portfolio, and the risk-free asset are continuously traded and rebalanced. The risk-free asset pays a constant rate of interest, \( r \). The public market portfolio’s return is independently and identically distributed, given by

\[
\frac{dS_t}{S_t} = \mu_S dt + \sigma_S dB_t^S, \quad (1)
\]
where $B_t^S$ is a standard Brownian motion, $\mu_S$ and $\sigma_S$ are constant drift and volatility parameters, and the Sharpe ratio is

$$
\eta = \frac{\mu_S - r}{\sigma_S}.
$$

(2)

1.1 PE return

At time 0, the LP makes an initial investment of $I_0 > 0$ into a PE fund. The fund is managed by a general partner (GP), who uses this initial investment to acquire $A_0$ worth of corporate assets, consisting of a portfolio of privately-held companies fully owned by the PE fund. Without leverage, $A_0 = I_0$. With leverage, $A_0 = I_0(1 + l)$, where $l$ is the D/E ratio. The return to the PE asset follows the geometric Brownian motion

$$
\frac{dA_t}{A_t} = \mu_Ad t + \sigma_A dB^A_t,
$$

(3)

where $B^A_t$ is a standard Brownian motion, $\mu_A$ is the drift, and $\sigma_A$ is the volatility. The PE investment is held to maturity, $T$, at which point the PE asset is liquidated for total proceeds of $A_T$. These proceeds are divided between the LP and GP, as specified below. The PE investments is a one-shot investment. After time $T$, the LP is restricted to investing in the risky market portfolio and the risk-free asset, and the problem reduces to the standard Merton (1971) portfolio problem.

The PE asset is correlated with the public market portfolio. Let $\rho$ be the correlation coefficient between $B_t^S$ and $B^A_t$. When $|\rho| \neq 1$, markets are incomplete, and the LP cannot fully hedge the PE investment by dynamically trading the public market portfolio and the risk free asset. We decompose the volatility of the PE return, $\sigma_A$, into the part spanned by the public market portfolio, $\rho \sigma_A$, and the remaining non-diversifiable part, given by

$$
\epsilon = \sqrt{\sigma_A^2 - \rho^2 \sigma_S^2}.
$$

(4)

Define beta as the unlevered beta relative to the public market portfolio,

$$
\beta = \frac{\rho \sigma_A}{\sigma_S},
$$

(5)

and restate the non-diversifiable volatility as

$$
\epsilon = \sqrt{\sigma_A^2 - \beta^2 \sigma_S^2}.
$$

(6)
Similarly, we define the excess return (alpha) of the PE asset with respect to the public market portfolio as
\[ \alpha = \mu_A - r - \beta(\mu_S - r). \] (7)

Informally, we interpret alpha as a measure of the GP’s managerial skill and ability to outperform the market. With appropriate data, the alpha and beta can be estimated by regressing the excess returns to the PE asset on the corresponding excess returns to the public market portfolio. Note, we define the alpha and beta of the PE asset relative to the public market portfolio, not relative to the LP’s entire portfolio, which also includes the PE investment. Existing empirical studies of PE performance also measure PE risks relative to the public market portfolio. Adopting this definition allows us to use existing estimates in our calibration. Defining alphas and betas relative to the total portfolio, comprising both public and private equity, is impractical because the total portfolio of PE assets is not observed.

When markets are complete and PE risks are spanned by traded assets, no arbitrage implies that there can be no risk-adjusted excess returns, hence \( \alpha = 0 \).\(^6\) In this case, the illiquidity of the PE investment does not require a risk premium. When PE risks are not fully spanned, however, markets are incomplete, and the un-spanned volatility, \( \epsilon \), introduces an additional non-diversifiable risks into the LP’s overall portfolio. Importantly, the spanned component of the volatility, \( \rho \sigma_A \), and the non-diversifiable part, \( \epsilon \), play distinct roles in the LP’s certainty equivalent valuation. The LP requires a premium for bearing these risks (in addition to the alpha required to justify the GP’s compensation). Below, we calculate the break-even alpha (\( \alpha > 0 \)), defined as the minimum alpha required for the LP to break even in certainty equivalence terms.

### 1.2 GP compensation

The GP receives ongoing management fees and performance-based carried interest. To define management fees, let \( X_0 \) denote the LP’s total committed capital, given as the sum of the initial investment, \( I_0 \), and the total management fees paid over the life of the fund. The

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\(^6\)However, given management and incentive fees, the initial valuation of the PE asset when the manager acquires from a third party for example can still be below the public valuation, the gap of which justifies the management compensation. One interpretation is that the GP can find an identical deal available from the private market and take the deal to investors, but the GP finds it at a discount due to the GP’s expertise. The valuation discount justifies management compensation in this example.
annual management fees are specified as a fraction $m$ (typically 1.5%-2%) of the committed capital, hence

$$I_0 + mT X_0 = X_0,$$

implying $X_0 = I_0/(1 - mT)$ and $I_0 = X_0(1 - mT)$. To illustrate, total committed capital of $X_0 = $12.5, with a 2% annual management fee paid over ten years, leaves $I_0 = $10 for the initial investment.

At maturity, $T$, the PE assets are liquidated with total proceeds of $A_T$, leaving $A_T - X_0$ of profits. The proceeds are divided between the LP and GP according to the schedule ("waterfall") given below. The GP’s share of the liquidating proceeds is the carried interest.\(^7\) The waterfall has three distinct regions, depending on $A_T$. The boundaries, denoted $F$ and $Z$, are defined as follows.

**Region 1: Preferred return ($A_T \leq F$).** First, GP receives no carried interest, and the LP receives the entire proceeds, until the LP achieves a given preferred return ("hurdle"), $h$ (typically 8%). The boundary, $F$, where the LP’s IRR equals the hurdle return is defined as

$$F = I_0 e^{hT} + \int_0^T mX_0 e^{hs} ds = I_0 e^{hT} + \frac{mX_0}{h}(e^{hT} - 1).$$

Without a hurdle, $h = 0$ and $F = X_0$. With a hurdle, $h > 0$ and $F > X_0$.

**Region 2: Catch-up ($F \leq A_T \leq Z$).** With a positive preferred return ("hurdle"), the initial profits are all paid to the LP. Hence, for the GP to catch up to the given share of profits, $k$ (typically, 20%), this region gives the GP a substantial fraction, $n$ (typically 100%), of the subsequent proceeds as carried interest. This region lasts until the GP’s carried interest equals the given share of profits. Hence, the boundary, $Z$, solves

$$k(Z - X_0) = n(Z - F).$$

The left-hand side is the GP’s share of total profits. The right-hand side is the GP’s carried interest received in the catch-up region.

\(^7\)Note that “clawbacks,” where a GP must return excess carried interest to the LP, cannot arise in our model, because the carried interest is only paid when the fund matures.
Region 3: Profit sharing \((A_T > Z)\). After the GP catches up with the LP, the GP’s total carried interest is simply a fraction, \(k\) (typically 20\%), of total profits. The LP receives the remaining proceeds, \(X_0 + (1 - k)(A_T - X_0)\).

1.3 LP’s problem

Objective. The LP has standard time-additive separable utility, given by

\[
E \left[ \int_0^\infty e^{-\zeta t} U(C_t) \, dt \right],
\]

where \(\zeta > 0\) is the subjective discount rate and \(U(C)\) is a concave function. For tractability, we choose \(U(C) = -e^{-\gamma C}/\gamma\), where \(\gamma > 0\) is the coefficient of absolute risk aversion (CARA).

Wealth dynamics. Let \(W_t\) denote the LP’s liquid wealth process, including the public market portfolio and risk-free asset, but excluding the PE investment. The LP allocates the amount \(\Pi_t\) to the public market portfolio and the remaining \(W_t - \Pi_t\) to the risk-free asset, resulting in

\[
dW_t = (rW_t - mX_0 - C_t) \, dt + \Pi_t \left( (\mu_S - r) \, dt + \sigma_S dB^S_t \right).
\]

The first term is the wealth accumulation when the LP is fully invested in the risk-free asset, net of management fees, \(mX_0\), and the consumption/expenditure \(C_t\). The second term is the excess return from investing in the public market portfolio.

Voluntary participation. With illiquidity and incomplete markets, standard no-arbitrage pricing does not apply. Instead, we value the PE investment in terms of its certainty equivalent. Let \(V(A_0, 0)\) be the certainty equivalent at time 0, taking into account the illiquidity, risk, and fees. The LP’s voluntary participation constraint is then

\[
V(A_0, 0) \geq I_0.
\]

Voluntary participation requires that the certainty-equivalent valuation, \(V(A_0, 0)\), exceeds the initial investment, \(I_0\), not the committed capital, \(X_0\), because management fees and carried interest are included in \(V(A_0, 0)\). With leverage, the cost of debt is also included in \(V(A_0, 0)\), and the voluntary participation constraint (13) remains unchanged.
2 Complete markets solution

With complete markets (CM), the LP’s problem can be decomposed into two independent optimization problems, namely total wealth maximization and complete-markets utility maximization. The PE investment can be perfectly replicated by dynamic trading of the publicly traded assets, and thus the LP requires no additional illiquidity premium. With a constant interest rate, \( r \), and a single risk factor (market portfolio) with Sharpe ratio \( \eta = (\mu_S - r)/\sigma_S \), the CAPM holds.\(^8\) The equilibrium expected rate of return, \( \mu_A \), for the PE asset is

\[
\mu_A = r + \beta(\mu_S - r),
\]

where \( \beta = \rho \sigma_A / \sigma_S \) as defined in (5).

Using the standard dynamic replicating portfolio argument, as in the Black-Scholes setting, we obtain a PDE for the LP’s value of the PE investment, \( V(A,t) \), as

\[
rV(A,t) = -mX_0 + V_t(A,t) + rAV_A(A,t) + \frac{1}{2} \sigma^2_V A^2 V_{AA}(A,t).
\]

(15)

The term \(-mX_0\) captures management fees. The other terms are standard.

**Boundary Conditions.** The LP’s payoff can be viewed as the sum of three distinct claims or tranches, corresponding to the three regions in the waterfall: (1) the LP’s senior preferred return claim; (2) the GP’s catch-up region, corresponding to a mezzanine claim for the LP; and (3) the profit-sharing region, corresponding to a junior equity claim. Adding these three claims gives the first boundary condition. At maturity \( T \), the LP’s total payoff is

\[
V(A_T, T) = LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T),
\]

(16)

where \( LP_1(A_T, T) \), \( LP_2(A_T, T) \), and \( LP_3(A_T, T) \) are the incremental payoffs in the three regions of the waterfall, as described next.

In the preferred return region, the LP’s payoff is

\[
LP_1(A_T, T) = \min \{ A_T, F \},
\]

(17)

\(^8\)First, with CM, the unique stochastic discount factor (SDF) \( M \) satisfies \( dM_t = -r M_t dt - \eta M_t dB^S_t \). We may verify this by using the fact that \( \{ M_t S_t \} \) is a martingale (i.e. the drift of \( MS \) is zero). Second, the process \( \{ M_t A_t \} \) is also a martingale under CM. Using Ito’s formula \( d(M_t A_t) = M_t dA_t + A_t dM_t + dM_t dA_t \) and the CM no-arbitrage martingale pricing result, the drift of \( MA \) satisfies \( \mu_A - r - \rho \eta \sigma_A = 0 \). Since \( \eta = (\mu_S - r)/\sigma_S \), we may rewrite the zero drift condition for the \( MA \) process as (14), a CAPM formula with the standard definition of \( \beta \) given in (5).
Here, $F$ is given in (9) and can be interpreted as the principal of a senior debt claim. In the catch-up region, the incremental payoff resembles a $(1 - n)$ fraction of mezzanine debt,\(^9\) given as
\[
LP_2(A_t, T) = (1 - n) \left( \max \{ A_T - F, 0 \} - \max \{ A_T - Z, 0 \} \right).  
\]
where $Z$ is given in (10) and can be interpreted as the sum of the principals of the senior and mezzanine debt claims. Finally, in the profit-sharing region, the incremental payoff is a junior claim, resembling a $(1 - k)$ equity share, given by
\[
LP_3(A_t, T) = (1 - k) \max \{ A_T - Z, 0 \}.  
\]

The LP must honor any remaining management fees regardless of the fund’s performance. The resulting liability is effectively a fixed-income claim, and the second boundary condition equates the certainty-equivalent value to the (negative) present value (PV) of the remaining fees, as
\[
V(0, t) = \int_t^T e^{-r(T-s)} (-mX_0) ds = -\frac{mX_0}{r} (1 - e^{-r(T-t)}) < 0.  
\]

**Solution.** For notation, let $BS(A_t, t; K)$ denote the Black-Scholes call option pricing formula, given in (A.4) of the appendix. Under complete markets, we value each tranche separately, as
\[
V^*(A_t, t) = LP^*_1(A_t, t; F) + LP^*_2(A_t, t; F) + LP^*_3(A_t, t; F) - \frac{mX_0}{r} (1 - e^{-r(T-t)}) ,  
\]
where
\[
LP^*_1(A_t, t) = A_t - BS(A_t, t; F) ,  
\]
\[
LP^*_2(A_t, t) = (1 - n)(BS(A_t, t; F) - BS(A_t, t; Z)) ,  
\]
\[
LP^*_3(A_t, t) = (1 - k)BS(A_t, t; Z) .  
\]

where $LP^*_1$, $LP^*_2$, and $LP^*_3$ are the LP’s values of the incremental payoffs in the three tranches or regions. The last term in (22) is the cost of management fees. Under complete markets there is no risk-adjusted excess return, $\alpha = 0$. Hence, $V^*(A_t, t) < A_t$, and a rational LP will never voluntarily allocate capital to the PE investment.

\(^9\)Note that PE usually have catch-up rates of $n = 100\%$, leaving $1 - n = 0\%$. For generality, we keep this tranche in the analysis, even if it does not exist for PE. Real estate partnerships commonly use catch-up rates of $n = 80\%$. 

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3 Incomplete markets solution

Consider the valuation of a PE investment under incomplete markets, without leverage. Leverage is introduced in Section 5. After the PE investment matures, at time $T$, the LP investor is left investing in the risk-free asset and the risky market portfolio, reducing the problem to the standard Merton (1969, 1971) consumption/portfolio allocation problem. The solution is summarized in Proposition 1. The value function $J^*(W)$ from this solution enters the pre-exit problem.

**Proposition 1** The LP’s post-exit value function takes the form:

$$J^*(W) = -\frac{1}{\gamma r} e^{-\gamma r(W+b)},$$  \hfill (25)

where $b$ is a constant,

$$b = \frac{\eta^2}{2\gamma r^2} + \frac{\zeta - r}{\gamma r^2}. \hfill (26)$$

Optimal consumption, $C$, is

$$C = r(W + b), \hfill (27)$$

and the optimal allocation to the risky market portfolio, $\Pi$, is

$$\Pi = \frac{\eta}{\gamma r \sigma_S}. \hfill (28)$$

Certainty equivalent valuation. Let $J(W,A,t)$ denote the LP’s value function for $t \leq T$. The LP’s consumption $C$ and allocation to the risky market portfolio $\Pi$ solve the Hamilton-Jacobi-Bellman (HJB) equation,

$$\zeta J(W,A,t) = \max_{\Pi, C} U(C) + J_t + (rW + \Pi(\mu_S - r) - mX_0 - C)J_W$$

$$+ \frac{1}{2} \Pi^2 \sigma_S^2 J_{WW} + \mu_A A J_A + \frac{1}{2} \sigma_A^2 A^2 J_{AA} + \rho \sigma_S \sigma_A \Pi A J_{WA}. \hfill (29)$$

In the appendix, we first show that the solution takes the exponential form,

$$J(W,A,t) = -\frac{1}{\gamma r} \exp \left[ -\gamma r (W + b + V(A,t)) \right], \hfill (30)$$

where $b$ is given in (26) and $V(A,t)$ is the LP’s private certainty-equivalent valuation for the PE investment.
Next, in the appendix, we show that \( V(A,t) \) solves the nonlinear partial differential equation (PDE)

\[
    rV(A,t) = -mX_0 + V_t + (r + \alpha) AV_A + \frac{1}{2}\sigma^2 A^2 V_{AA} - \frac{\gamma^T}{2} \epsilon^2 A^2 V_{\epsilon}^2,
\]

(31)

where \( \alpha \) is given by (7) and \( \epsilon \) is the volatility of the PE investment’s non-diversifiable risk, given in (6). The nonlinear PDE (31) is solved subject to the same boundary conditions as in the complete-markets case, given in equations (16) and (20).

Unlike the complete-markets case, the PDE in equation (31) is non-linear, because the last term involves \( V_{A}^2 \), invalidating the standard law-of-one-price valuation. An LP whose certainty equivalent of two individual PE investments are \( V_1 \) and \( V_2 \), when valued in isolation, no longer values the portfolio containing both of these PE investments as \( V_1 + V_2 \).

**Consumption and portfolio rules.** The solution implies that the LP’s optimal consumption \( C \) is given by

\[
    C(W,A,t) = r (W + b + V(A,t)) ,
\]

(32)

which is a version of the permanent-income/precautionary-saving models.\(^{10}\) Comparing this expression to (27), we see that the LP’s total certainty-equivalent wealth is the sum of the liquid wealth \( W \) and the certainty equivalent of the PE investment \( V(A,t) \).

Further, the LP’s optimal allocation to the public market portfolio is given as

\[
    \Pi(A,t) = \frac{\eta}{\gamma r \sigma_S} - \beta AV_A(A,t).
\]

(33)

The first term is the standard mean-variance term. The second term is the intertemporal hedging demand, with the unlevered \( \beta \) of the PE asset given by (5). Following the option pricing terminology, we interpret \( V_A(A,t) \) as the delta of the LP’s valuation with respect to \( A \), the value of the underlying asset. Greater values of \( \beta \) and \( V_A(A,t) \) result in a larger hedging demand.

4 Performance calibration

Given a set of parameter values, the PDE for the LP’s valuation of the PE investment from the previous section is straightforward to solve. When possible, we use parameter values from Metrick and Yasuda (2010) (henceforth MY) as a baseline. All parameters are annualized when applicable. MY use a volatility of 60% per individual buyout investment, with a pairwise correlation of 20% between any two buyout investments. MY report that the average buyout fund invests in around 15 buyouts (with a median of 12). From these figures, we calculate a volatility of 25% for the total PE asset.

We use a risk-free rate of 5%, as MY. For parameters not in MY, we set the volatility of the market portfolio to $\sigma_S = 20\%$, with an expected return of $\mu_S = 11\%$, implying a risk premium of $\mu_S - r = 6\%$ and a Sharpe ratio of $\eta = 30\%$. The unlevered $\beta$ of the PE asset is set to 0.5, implying a correlation between the PE asset and the market portfolio of $\rho = \beta \sigma_S / \sigma_A = 0.4$, but we consider a range of other values of $\beta$ as well.

Finally, to find reasonable values of the risk aversion, $\gamma$, and the initial investment, $I_0$, we derive a new invariance result, given in proposition 2.

**Proposition 2** Define $\gamma = \gamma I_0$, $a = A / I_0$, $x_0 = X_0 / I_0$, $z = Z / I_0$ and $f = F / I_0$. It is straightforward to verify that $V(A,t) = v(a,t) \times I_0$, where $v(a,t)$ solves the following ODE,

$$rv(a,t) = -mx_0 + v_t + (r + \alpha) av_a(a,t) + \frac{1}{2} \sigma_A^2 a^2 v_{aa}(a,t) - \frac{\eta}{2} \epsilon^2 a^2 v_a(a,t)^2, \quad (34)$$

subject to the boundary conditions,

$$v(a,T) = a - n (\max\{a - f, 0\} - \max\{a - z, 0\}) - k \max\{a - z, 0\}, \quad (35)$$

$$v(0,t) = -\frac{mx_0}{r} \left(1 - e^{-(r-t)}\right). \quad (36)$$

Proposition 2 shows that $v(a,t)$ depends on the product $\gamma = \gamma I_0$ but not on $\gamma$ and $I_0$ individually. The LP’s certainty equivalent valuation $V(A,t)$ is proportional to the invested capital $I_0$, holding $\gamma$ constant.\(^{11}\) An LP larger PE investment and smaller risk aversion will have the same $v(a,t)$ valuation as another LP with a proportionally smaller PE investment but a proportionally larger risk aversion. To illustrate, when $LP_1$ values a (scalable) PE

\(^{11}\)This does not imply the law of one price, which does not hold for this model, as discussed earlier.
investment with invested capital $I_0 = $100 at $V_0 = $120, then $LP_2$ values a similar investment with $I_0 = $200 at $V_0 = $240, holding $\gamma$ constant. Holding $\gamma$ constant while doubling $I_0$ implies $\gamma_2 = \gamma_1/2$.

To determine a reasonable value of $\gamma$, we use the following approximation. Note that, by definition, the relative risk aversion equals $\gamma_R = \gamma C$. Substituting expression (32) for $C(W, A, t)$ and expression (26) for $b$, and assuming that the LP’s time preference equals the risk-free rate ($\zeta = r$), we get

$$\gamma_R = \gamma r(W + b + V(A, t))$$  
$$= \gamma I_0 r\frac{W + V(A, t)}{I_0} + \gamma r \frac{\eta^2}{2\gamma r^2}. \tag{38}$$

Solving for $\gamma$ gives

$$\gamma = \gamma I_0 = \frac{\gamma R - \frac{\eta^2}{2r}}{r} \left(\frac{I_0}{W + V(A, 0)}\right). \tag{39}$$

Hence, $\gamma$ can be determined from the fraction of the LP’s portfolio allocated to the PE investment (the term in parenthesis) and the LP’s relative risk aversion. With $\eta = 30\%$, $r = 5\%$, and assuming a PE allocation of $I_0/(W + V(A, 0)) = 25\%$, the expression reduces to $\gamma = 5 \times (\gamma_R - 0.9)$. Hence, a relative risk aversion of $\gamma_R = 2$ implies $\gamma = 5.5$. An LP with a smaller risk aversion or a smaller PE allocation has a smaller $\gamma$. Our baseline calibration uses $\gamma = 2$, but we consider values ranging from $\gamma = 5$ to $\gamma \to 0$ for a risk-neutral LP.

### 4.1 Cost of unskilled manager

Applying the invariance result, we focus on the properties of $v(a, t)$ and its delta, $v_a(a, t)$ (note $V_A(A, t) = v_a(a, t)$). Figure 1 illustrates the case where the GP is unskilled ($\alpha = 0$). With CM, the non-diversifiably illiquidity risk premium for the LP vanishes, and the valuation coincides with the Black-Scholes pricing formula (22).\footnote{The limit case when $\gamma \to 0$ has the same certainty equivalent valuation as the complete-markets case does. See Chen, Miao and Wang (2010) for a similar observation.} With no excess returns ($\alpha = 0$) but the costs of management fees and carried interest, $v(1, 0) < 1$, meaning that a rational LP would not voluntarily invest in PE. In our baseline calibration, $v(1, 0) = 0.746$, stating that the LP loses 25.4\% of the invested capital $I_0$ when the GP has no skill. Of these 25.4\%, the 19.7\% are due to management fees, and the remaining 5.7\% are due to incentive fees.
In other words, without skill, an LP’s initial investment of $I_0 = $100 (equivalent to committed capital of $X_0 = $125 with $25 of management fees paid over a ten-year period) has an economic value of $74.6 (net of fees and carried interest) to the LP. Management fees and carried interest imply that an LP investing with unskilled GP loses 25 cents for each dollar invested in the underlying PE investment, $I_0$. As discussed below, to break even ($v(1, 0) = 1$), the LP requires $\alpha = 2.61\%$.

A risk-averse LP demands an additional risk premium as compensation for the illiquidity and non-diversifiable risk. For $\gamma = 2$, $v(1, 0) = 0.715$. This extra 3.1% discount reflects the non-diversifiable illiquidity risk premium. Hence, management fees, carried interest and non-diversifiable illiquidity risk premium jointly imply that a risk-averse LP with $\gamma = 2$ investing with unskilled GP loses 29 cents for each dollar invested in the underlying PE investment, $I_0$. In this case, the break-even alpha is 3.08%.

Panel B in Figure 1 shows that a more risk averse LP has a lower delta $v_a(a, t)$, which implies that the LP’s certainty equivalent is less sensitive to changes in the value of the underlying asset. Intuitively, the more risk averse LP values the PE asset less and has a flatter valuation $v(a, t)$. Hence, we have the seemingly counter-intuitive result that a more risk averse LP has a lower hedging demand.
4.2 Performance measures

Because of management fees, carried interest, and non-diversifiable illiquidity risks, the LP will only invest with skilled managers. We define the break-even alpha as the minimal level of alpha that the GP must generate in order for the LP to participate. This break-even alpha solves $I_0 = V(A_0, 0)$, which is equivalent to $v(1, 0) = 1$. Given $\gamma$, the break-even alpha is independent of the amount of invested capital $I_0$. In the baseline calibration, the break-even alphas are 2.61% for $\gamma = 0$ and 3.08% for $\gamma = 2$.

The alpha generated by a GP is difficult to observed directly, and more readily observable performance measures are used in practice. Harris, Jenkinson, and Kaplan (2011) summarize studies with empirical estimates of the three most common performance measures: the internal rate of return (IRR), the total-value-to-paid-in-capital multiple (TVPI), and the public market equivalent (PME).

To define these performance measures divide the cash flows between the LP and GP into capital calls and distributions. Capital calls, $Call_t$, are cash flows from the LP to the GP, and distributions, $Dist_t$, are those from the GP to the LP. The performance measures are then defined as follows: IRR solves $1 = \frac{\sum Dist_t}{\sum Call_t}$, TVPI = $\frac{\sum Dist_t}{\sum Call_t}$, and PME = $\frac{\sum Dist_t}{\sum Call_t}$, where $r_t$ is the cumulative realized return on the market portfolio up to time $t$. PME is the value of distributed capital relative to called capital, discounted by the realized market return. Assuming $\beta = 1$, the empirical studies typically interpret $PME > 1$ as the PE investment outperforming the market.14

Analytical performance measures. In our model, we can solve for the analytical counterparts to these performance measures. The LP’s required IRR for the PE asset, denoted $\phi$, solves,

$$I_0 + \int_0^T mX_0 e^{-\phi t} dt = e^{-\phi T} \mathbb{E} [LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T)] ,$$

13These studies include Driessen, Lin, Phalippou (2011), Jegadeesh et al. (2009), Kaplan and Schoar (2005), Korteweg and Sorensen (2010), Ljungqvist and Richardson (2003), Metrick and Yasuda (2010), Phalippou and Gottschalg (2009), Robinson and Sensoy (2011), and Stucke (2011).

14Robinson and Sensoy (2012) present a new beta-adjusted PME calculation. As with the traditional PME, one limitation of this adjusted PME is that it is unclear under which conditions a value greater than 1.0 implies economic outperformance.
which simplifies to
\[
I_0 + \frac{mX_0}{\phi} (1 - e^{-\phi T}) = e^{-\phi T}[A_0 - n(\text{EC}(A_0; F) - \text{EC}(A_0; Z)) - k\text{EC}(A_0; Z)].
\]

Here, \(\text{EC}(A; K)\) is the expected payoff of a call option with strike price \(K\) under the physical measure, as given in (A.14) in the Appendix. The expression for \(\text{EC}(A; K)\) looks similar to the Black-Scholes formula, but it calculates the expected payoff of a call option under the physical measure, not the risk-neutral one.

The ex-ante expected TVPI is defined as
\[
\mathbb{E}[\text{TVPI}] = \frac{\mathbb{E}[LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T)]}{X_0},
\]
where the numerator is the LP’s expected payoff net of carried interest, and the denominator is the total committed capital. The solution is
\[
\mathbb{E}[\text{TVPI}] = \frac{e^{\mu_A T}[A_0 - n(\text{EC}(A_0; F) - \text{EC}(A_0; Z)) - k\text{EC}(A_0; Z)]}{X_0}.
\]

In our model, the PME is defined as
\[
\text{PME} = \frac{\mathbb{E}[e^{-\mu_S T}(LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T))]}{I_0 + \mathbb{E}\left[\int_0^T e^{-\mu_S s}mX_0 ds\right]} = \frac{e^{(\mu_A - \mu_S) T}[A_0 - n(\text{EC}(A_0; F) - \text{EC}(A_0; Z)) - k\text{EC}(A_0; Z)]}{I_0 + \frac{mA_0}{\mu_S} (1 - e^{-\mu_S T})}.
\]

There are three concerns with the standard PME measure. First, the denominator combines two types of cash flows, the investment \(I_0\) and the management fees. Management fees are effectively a risk-free claim and should be discounted at a rate close to the risk-free rate. Second, the numerator contains the total proceeds net of carried interest. The carried interest is effectively a call option, making the LP’s total payoff at maturity less risky than the underlying PE investment. Hence, it should be discounted at a lower rate than the underlying PE investment. Finally, the beta of the PE investment may not equal one.

To address these concerns, we define the adjusted PME as follows,
\[
\text{Adj. PME} = \frac{\widetilde{\mathbb{E}}[e^{-\tau T}(LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T))]}{I_0 + \widetilde{\mathbb{E}}\left[\int_0^T e^{-\tau s}mX_0 ds\right]}.
\]
Here, $\bar{E}\{\cdot\}$ denotes the expectation under the risk-adjusted measure, analogous to the Black-Scholes option-pricing methodology.\footnote{Under the risk-adjusted measure $\bar{E}\{\cdot\}$, the calculation is analogous to the expectation in Black-Scholes, but it allows for positive alpha. With CM and no alpha, this pricing formula is the Black-Scholes formula.} We treat management fees as a risk-free claim, discounted at the risk-free rate. We discount the carried interest and the underlying PE investment, taking into account their different risks. Finally, we allow the underlying PE investment to have arbitrary beta. Our adjusted PME captures the systematic risks of these different cash-flow components. It does not capture the effects of non-diversifiable illiquidity risk. More precisely, when an LP’s risk aversion approaches zero, an adjusted PME exceeding one is equivalent to positive performance. With risk aversion, an adjusted PME exceeding one is a necessary, but not sufficient, condition for positive performance.

In our model, the adjusted PME is

\[
\text{Adj. PME} = \frac{e^{\alpha T} [A_0 - n(RC(A_0; F) - RC(A_0; Z)) - kRC(A_0; Z)]}{I_0 + \frac{m\lambda_0}{r} (1 - e^{-rT})},
\]

where $RC(A; K)$ is the expected payoff of a call option under the risk-adjusted measure, given in (A.18) of the Appendix.

### 4.3 Break-even performance

Table 1 reports break-even values of the various performance measures. The break-even alpha solves $v(1, 0) = 1$ for various levels of risk aversion and beta. The break-even values of IRR, TVPI, PME, and adjusted PME are implied by the corresponding break-even alpha.

Insert Table 1 here.

**Effectively risk-neutral LP.** For an effectively risk-neutral LP ($\gamma \to 0$), the break-even alpha is 2.61%. The GP must generate this excess return to compensate the LP for management fees and carried interest. This break-even alpha is independent of $\beta$, because alpha is the risk-adjusted excess return and there is no premium for non-diversifiable illiquidity risk. The break-even IRR, however, increases with beta, because IRR does not account for risk. Quantitatively, the break-even IRR increases in lockstep with $\beta$. For example, as $\beta$ increases from 0.5 to 1, the systematic risk premium increases by $0.5 \times 6\% = 3\%$, and the IRR also
increases by this amount. Similarly, the break-even TVPI and PME also increase with $\beta$, reflecting the increasing expected return.

Empirically, a PME exceeding one is typically interpreted as earning positive economic value for the LPs. Table 1 shows that the conventional interpretation of this measure is misleading, when $\beta$ differs from one. For example, with $\beta = 0$, the PME only needs to exceed 0.57 for the LPs to earn positive economic value.

In contrast, the break-even value of the adjusted PME, our newly constructed measure, equals one at all levels of $\beta$, because this measure appropriately accounts for the systematic risks of different tranches (e.g. management fees, carried interest) of the cash flows.

Hence, an effectively risk-neutral LP participates in the PE investment as long as the investment’s alpha exceeds the break-even alpha, 2.61%, or the adjusted PME exceeds one, regardless of the systematic risk exposure. In contrast, the interpretations of IRR, TVPI, and conventional PME measures depend on the investment’s beta.

Risk-averse LP. A risk-averse LP demands additional premium for bearing non-diversifiable illiquidity risks. As the LP’s risk aversion increases, the required compensation increases, and the break-even alpha increases. For example, with $\beta = 0.5$, the break-even alpha increases by 47 basis points, from 2.61% to 3.08%, as $\gamma$ increases from 0 to 2. Correspondingly, the break-even IRR increases by almost the same amount. The break-even values of the remaining measures also increase with risk aversion. For the adjusted PME, the break-even value now exceeds one due to the non-diversifiable illiquidity-risk premium. To illustrate, as $\gamma$ increases from 0 to 2, the break-even adjusted PME increases from 1 to 1.04.

In Table 1, the total volatility is constant at $\sigma_A = 25\%$. Recall that the systematic component of the volatility is $\rho \sigma_A = \beta \sigma_S$ and thus the non-diversifiable illiquidity volatility is $\epsilon = \sqrt{\sigma_A^2 - \beta^2 \sigma_S^2}$. As beta increases, the non-diversifiable illiquidity volatility $\epsilon$ decreases, which causes the break-even alpha to decline. To illustrate, for $\gamma = 2$, the break-even alpha decreases by 56 basis points from 3.17% to 2.61% as beta increases from 0 to 1.25.

In our calibration, $\beta = 1.25$ is the case with no non-diversifiable illiquidity risk, $\epsilon = 0$. With no non-diversifiable illiquidity risk premium, the break-even alpha equals 2.61%, regardless of risk aversion $\gamma$. The other performance measures, imputed from this break-even alpha, also remain unchanged.

Absent leverage, the primary determinants of the break-even alpha appear to be the
management fees and carried interest. The non-diversifiable illiquidity risk has a smaller effect.

5 Leverage

Motivating Leverage The use of leverage in PE has presented a puzzle. Axelson, et al. (2011) show that the patterns of leverage in PE deals is largely independent of the patterns of publicly traded companies. PE firms appear to be primarily driven by credit markets, and employ as much leverage as creditors permit. Under the standard Modigliani-Miller argument, greater leverage allows shareholders to earn a higher expected return but at a higher risk, these effects exactly offset, leaving the shareholders indifferent. To overcome this, Axelson, Stromberg, and Weisbach (2009) present a model of ex-ante and ex-post discipline to explain the use of leverage. Our results provide a simpler explanation. With incomplete markets and GP ability, the benefit of leverage is that it allows the GP to apply their value-adding ability across a wider asset base, and the cost of leverage is that it exposes to LP to greater non-diversifiable illiquidity and systematic risks. But there is no reason to expect these two effects to cancel. Our findings suggest that the first effect dominates, and a simple explanation for the use of leverage is that it permits GPs to apply their value adding ability across a wider asset base.\(^{16}\) The economic cost of the corresponding increases in risk appears comparatively modest.

5.1 Pricing with leverage

At time 0, let \( l \) denote the debt-equity ratio. For a given amount of invested capital \( I_0 \) (equity), the value of debt equals \( D_0 = lI_0 \), and the total PE asset equals \( A_0 = (l + 1)I_0 \). We next price debt from the perspective of well diversified risk-averse investors.

Debt pricing. Our model applies to general forms of debt, but for simplicity we consider balloon debt with no intermediate payments and where the principal and all accrued interest are due at maturity \( T \). Let \( y \) denote the yield for the debt, which we derive below. The

\(^{16}\)Our model assumes constant returns to scale of the GP’s value-adding ability. This assumption is not essential for the argument, and the same logic applies with decreasing returns.
payment at maturity $T$ is the sum of principal and compounded interest,

$$D(A_T, T) = \min \{ A_T, D_0e^{yT} \} . \quad (47)$$

Intuitively, the expected rate of return on debt is higher than the risk-free rate because it is a risky claim on the underlying PE asset. Additionally, the PE asset has alpha, which implies that this debt has a higher value than the one implied by the standard Black-Scholes pricing formula. In the appendix, we show that debt is priced by

$$rD(A, t) = D_t(A, t) + (r + \alpha)AD_A(A, t) + \frac{1}{2}\sigma_A^2A^2D_{AA}(A, t) , \quad (48)$$

subject to the boundary condition (47). Despite the resemblance to the Black-Scholes debt pricing formula, our debt pricing formula is fundamentally different. Unlike Black-Scholes, our model allows for positive $\alpha$, and hence the risk-adjusted drift is $r + \alpha$. A positive alpha is critical, because it allows the LP to break even given fees, illiquidity, and non-diversifiable illiquidity risk. Like Black-Scholes, the risk-adjusted expected return on the risky debt equals the risk-free rate. The market value of debt at time 0 is then given by

$$D_0(A) = e^{\alpha(T-t)} \left[ A - RC(A; D_0e^{yT}) \right] , \quad (49)$$

where $RC(A; \cdot)$ is the expected payoff of a call option under the risk-adjusted measure, given in (A.18) in the Appendix.

We solve for the equilibrium yield $y(A_0, T)$ as follows. We calculate the candidate final payoff in (47) using the original debt $D_0$ and a candidate $y$. The resulting candidate value of debt at time zero $\hat{D}_0$ then follows from the pricing formula (49), and the equilibrium yield is determined when $\hat{D}_0 = D_0$.

**Boundary conditions with leverage.** With leverage, the LP’s claim is junior to the debt, but it can still be valued as three tranches over three regions. For the preferred-return region, due to the seniority of debt, the LP’s payoff equals the difference between the payoffs of two call options,

$$LP_0(A_T, T) = \max \{ A_T - D_0e^{yT} , 0 \} - \max \{ A_T - (D_0e^{yT} + F) , 0 \} , \quad (50)$$

where $F$ is given in (9).
In the catch-up region, the LP’s incremental payoff equals the fraction \((1 - n)\) of the difference between the payoffs of two other call options,

\[
LP_1(A_T, T) = (1 - n) \left[ \max \{A_T - D_0e^{yT} - F, 0\} - \max \{A_T - Z_{Lev}, 0\} \right],
\]

where \(Z_{Lev}\) solves

\[
k \left( Z_{Lev} - D_0e^{yT} - X_0 \right) = n \left( Z_{Lev} - D_0e^{yT} - F \right).
\]

Here, \(Z_{Lev}\) is the level of \(A_T\) such that the GP catches up.

Finally, in the profit region, the LP’s incremental payoff equals the fraction \((1 - k)\) of the payoff of the option with strike price \(Z_{Lev},\)

\[
LP_2(A_T, T) = (1 - k) \max \{0, A_T - Z_{Lev}\}.
\]

As in the case without leverage, the LP’s total payoff equals \(V(A_T, T) = LP_0(A_T, T) + LP_1(A_T, T) + LP_2(A_T, T)\).

In the appendix, we derive expressions for the performance measures, IRR, TVPI, PME, and adjusted PME with leverage. These are analogous to the expressions without leverage, albeit more complex. We use these formulas in the following analysis.

### 5.2 Risk and Risk Aversion

Tables 2 and 3 report break-even values of the performance measures for various degrees of leverage. Because of leverage, the break-even alpha now solves \(v(1 + l, 0) = 1\), and the break-even IRR, TVPI, PME, and adjusted PME are implied by this break-even alpha.

Effectively risk-neutral LP. We first consider the effects of systematic risk on the performance measures. For an effectively risk-neutral LP, the non-diversifiable illiquidity risk is not priced. Tables 2 and 3 report break-even performance measures when initial debt-equity ratio of 1 and 3 \((l = 1, 3)\), respectively. The patterns are largely similar. Quantitatively, the effects are stronger with higher leverage. Unlike the case without leverage, we now have the credit spread for debt.
The first column in Table 2, with $l = 1$, shows that the break-even alpha is 1.68% compared to 2.61% without leverage, a reduction of 93 basis points. This break-even alpha is independent of beta, because alpha measures the excess return adjusted for systematic risk. The credit spread $y - r$ for debt is 1.05%. Importantly, this spread is also independent of beta because debt is priced as a derivative on the underlying PE asset with alpha.

As the unlevered $\beta$ increases from 0.5 to 1, the PE asset’s systematic risk premium increases by $0.5 \times 6\% = 3\%$, and consequently the break-even IRR for the LP increases by 4.2%, which is higher than 3% due to leverage. The break-even TVPI and PME are more sensitive to changes in $\beta$ with leverage than without. For example, as we increase beta from 0.5 to 1, without leverage, the TVPI increases from 2.07 to 2.73. With leverage, however, the TVPI increases from 2.43 to 3.60, which is much more significant. Moreover, the PME increases from 0.88 to 1.30, and the usual interpretation of a PME exceeding one as equivalent to outperformance may be more misleading with leverage. For the adjusted PME, the break-even value equals one at all levels of $\beta$, because this measure appropriately accounts for systematic risk and leverage.

**Risk-averse LP.** As risk aversion increases, the LP demands additional non-diversifiable illiquidity risk premium, and the break-even alpha increases. In Table 2, with $\beta = 0.5$, the break-even alpha increases by 78 basis points, from 1.68% to 2.46%, as $\gamma$ increases from 0 to 2. Correspondingly, the break-even IRR increases by 1.2%, and the break-even values of the remaining measures also increase. In contrast, the credit spread declines, because a higher alpha increases the debt value, implying a lower spread. In the table, total volatility $\sigma_A$ is constant, hence an increase in beta reduces non-diversifiable illiquidity volatility, which also reduces the break-even alpha. The effect of beta on the break-even alpha is also significant. For example, as we increase the unlevered beta from 0.5 to 1, the break-even alpha decreases from 2.46% to 2.03% for $\gamma = 2$. Finally, as the case without leverage, $\beta = 1.25$ implies no non-diversifiable illiquidity risk, and in this case all performance measures remain independent of risk aversion. Table 3 shows that all of these effects become more pronounced as leverage increases.

In sum, with leverage, the management fees and carried interest are no longer the primary determinants of the break-even alpha. The break-even values now also heavily depend on the magnitudes of risk aversion and beta.
5.3 Effect of Leverage

Insert Tables 4 and 5 here.

Tables 4 demonstrates the effects of leverage for an effectively risk-neutral LP ($\gamma \to 0$). We consider various leverage levels of $l$ from 0 to 9. The effect of leverage on the break-even alpha is substantial. The break-even alpha decreases from 2.61% when $l = 0$ to 1.00% when $l = 3$, and this annual alpha compounds over the life of the fund. Note that the decline on the break-even alpha is independent of the unlevered $\beta$. The economic intuition for this decline is as follows. With greater leverage, the GP can apply the alpha across more assets, and hence a lower alpha is required to generate sufficient returns to compensate the LPs. The classical Modigliani-Miller argument would say that the LPs should be indifferent to leverage, but this argument does not hold with positive alpha and fees. Increasing leverage allows the GP to generate alpha for a larger amount of assets and charge lower fees per dollar of assets under management. To illustrate, by simply leveraging three times, the unlevered return for the asset value $A_T$ that the GP needs to generate in order for the LP to break even is reduced by 15%.

The adjusted PME equals one for all cases, because this measure appropriately accounts for the systematic risk and the effectively risk neutral LP demands no non-diversifiable illiquidity risk premium. The credit spread increases with the amount of leverage. The leverage effect on credit spread is significant. For example, the credit spread is 3.48% when $l = 3$.

For beta greater than zero, the IRR, TVPI, and PME increase with leverage. With $l = 3$ and the unlevered $\beta = 0.5$, the LP’s IRR equals 11.2% increasing from 7.9% by 3.3%. The TVPI increases to 2.81 from 2.07 when $l = 0$. The PME increases to 1.02 from 0.75 when $l = 0$. For the special case with $\beta = 0$, levered beta equals zero and hence IRR, TVPI and PME do not change with leverage (see Panel A).

Importantly, as the break-even alpha, credit spread is independent of the unlevered beta because debt is effectively priced as a derivative on the underlying PE asset. However, unlike the standard derivatives analysis, our underlying PE asset is not tradable and hence admits a positive alpha. Other performance measures including IRR, TVPI and PME all increase with the unlevered beta.
5.4 Comparison to empirical findings

We now compare the break-even values of the performance measures to the actual performance of PE investors. Axelson, Jenkinson, Stromberg, and Weisbach (2007) consider 153 buyouts during 1985–2006, and report that, on average, equity accounted for only 25% of the purchase price, corresponding to \( l = 3 \) in our model. Turning to the unlevered beta, a number of studies report levered betas ranging from a low of 0.7–1.0 (Jegadeesh, Kraussl, and Pollet (2010)) to a high of 1.3 (Driessen, Lin, and Phalippou (2011)). For a levered beta of one and assuming a debt beta of zero for simplicity, we obtain an implied unlevered beta of 0.25. This estimate seems unreasonably low because typical PE funds target firms whose risks are comparable to publicly traded ones. Using an average debt-equity ratio of 0.5 for a representative firm with levered beta of one, we obtain an approximate implied unlevered firm beta of 2/3. Hence, we use an unlevered beta of 0.5, which seems a reasonable starting point.

We view a sensible range of LP’s risk aversion is \( 0 \leq \gamma \leq 5 \), with a preferable value around 1-2. These choices of leverage, unlevered beta, and risk aversion point to the second and third columns of Panel B in Table 3. The implied break-even values for the IRR are 12.7–13.8%, and for the TVPI are 3.24–3.61. Harris, Jenkinson, and Kaplan (2011) report separate performance figures for various datasets, with average value-weighted IRRs of 12.3–16.9%, which compare favorably to our break-even IRR of 12.7% and 13.8% for \( \gamma = 1, 2 \). Their value-weighted TVPI multiple, however, ranges from 1.76 to 2.30, which is somewhat lower than our break-even figure. The empirical and theoretical IRR and TVPI measures are difficult to compare, however, since they are absolute performance measures, and hence they are sensitive to the realized market performance over the life of the fund. In our calibration, we assume an excess market return of 6%, but actual market returns have varied substantially over the past decades.

The PME is closer to a relative performance measure. Harris, Jenkinson, and Kaplan (2011) report average PMEs of 1.20–1.27. These figures are very close to our break-even values of 1.17–1.30. While an average PME greater than one is often interpreted as evidence that LPs have outperformed the market, this out-performance appears to be almost exactly equal to the amount required to compensate a risk-averse LPs for illiquidity and non-diversifiable illiquidity risk in our model.
6 Management fees and carried interest

We first quantify the effects of changing management fees and carried interest on the break-even alpha for the complete-markets case. Then, we turn to the incomplete-markets setting.

6.1 Complete markets

Table 6 presents changes in the break-even alpha as management and incentive fees change, focusing on the complete-markets case (or equivalently, the case where the LP is close to being risk neutral). Panel A shows the impact of management fees for different levels of the debt-equity ratio $l$. Without leverage ($l = 0$), the break-even alpha moves almost in lockstep with $m$. As $m$ increases from 1.5% to 2.0% and again from 2.0% to 2.5%, in 50 basis-point steps, the break-even alpha increases from 2.1% to 2.6% and again from 2.6% to 3.1%, in 50 basis-point steps. With a debt-equity ratio of three ($l = 3$), the break-even alpha becomes less sensitive to $m$. The break-even alpha only increases by about 15-16 basis points from 0.85% to 1.00% and again from 1.00% to 1.16%, as $m$ increases from 1.5% to 2.0% and again from 2.0% to 2.5%.

Panel B illustrates the impact of the carried interest, holding management fees fixed at $m = 2\%$. Without leverage ($l = 0$), the break-even alpha is less sensitive to $k$. As $k$ increases from 20% to 25% and again from 25% to 30%, the break-even alpha increases from 2.61% to 2.83% and again to 3.07%, i.e., by 22-24 basis-point increments. The effect of a 5% increase of the incentive fee (carry) $k$ has roughly half the effect on the break-even alpha as a 50 basis-point increase in the management fee $m$.

With a debt-equity ratio of three ($l = 3$), the break-even alpha becomes less sensitive to changes in $k$. In our example, the break-even alpha increases by about 16-18 basis points from 1% to 1.16% and further to 1.34%, as $k$ increases from 20% to 25% and again to 30%. Thus, with a debt-equity ratio of three, the effect of a 5% increase in the incentive fee (carry) $k$ has roughly the same effect on the break-even alpha as a 50 basis-point increase in the management fee $m$. Note that the break-even alpha is independent of beta under complete markets, although this independence no longer holds under incomplete markets.

These results provide an alternative comparison for different combinations of management versus incentive fees (carry). The results summarize various compensation structures in terms of the required managerial skills, as measured by the break-even alpha. With a
leverage of three, starting from a typical 2-20 compensation contract, an increase in the management fees $m$ from 2.0% to 2.5% is roughly comparable to an increase in the incentive fees (carry) $k$ from 20% to 25%.

### 6.2 Incomplete markets

Table 7 shows how the break-even alpha relates to management and incentive fees for a risk-averse LP with a relative risk aversion around two. With incomplete markets, the break-even alpha depends on the beta for the underlying PE asset, and we focus on the case of $\beta = 0.5$. Panel B shows the impact of management fees for different levels of the debt-equity ratio $l$. Without leverage ($l = 0$), the break-even alpha moves almost in lockstep with $m$. As $m$ increases from 1.5% to 2.0% and again from 2.0% to 2.5%, in 50 basis-point steps, the break-even alpha increases from 2.56% to 3.08% and again from 3.08% to 3.65%, in 52-57 basis points increments. With a debt-equity ratio of three ($l = 3$), the break-even alpha is less sensitive to $m$. The break-even alpha now only increases by about 18-20 basis points, from 1.87% to 2.05% and again from 2.05% to 2.25%, as $m$ increases from 1.5% to 2.0% and again from 2.0% to 2.5%.

Panel B in Table 8 illustrates the impact of increasing carried interest, fixing management fees at $m = 2\%$ and the unlevered beta at $\beta = 0.5$. Without leverage ($l = 0$), the break-even alpha is less sensitive to $k$ than to $m$. As $k$ increases from 20% to 25% and again from 25% to 30%, the break-even alpha increases from 3.08% to 3.29% and again from 3.29% to 3.50%, i.e., in 21 basis-point increments. The effect of a 5% increase of the incentive fee (carry) $k$ has roughly 40% of the effect on the break-even alpha as a 50 basis-point increase in the management fee $m$.

Leverage makes the break-even alpha less sensitive to changes in $k$. In our example, with a debt-equity ratio of three ($l = 3$), the break-even alpha increases by about 13-14 basis points from 2.05% to 2.18% and again from 2.18% to 2.32%, as $k$ increases from 20% to 25% and again from 25% to 30%. With a debt-equity ratio of three, the effect of a 5% increase in the incentive fee (carry) $k$ has roughly comparable, but slightly smaller effects on the break-even alpha as a 50 basis-point increase in the management fee $m$ (about 14 basis points versus 18-20 basis points).

To a first-order approximation, our incomplete-markets results corroborate the complete-
markets results. Specifically, starting from a typical 2-20 compensation contract, an increase
in the management fees $m$ from 2.0% to 2.5% is roughly comparable to an increase in the
incentive fees (carry) $k$ from 20% to 25%. Both changes require an additional 20-25 basis
points of alpha generated by the GP on the PE asset for the LP to remain indifferent.

7 Conclusion

We develop a model of the asset allocation problem facing an institutional investor (LP),
who invests in traditional liquid assets as well as an illiquid long-term private equity (PE)
investment. The model captures the primary features of PE investing, including illiquidity,
non-diversifiable risk, and the management compensation structure. The compensation
structure distinguishes management fees from carried interest, and includes preferred re-
turns, hurdle rate, and catch-up provisions. In addition to fees, the LP is also exposed to
non-diversifiable illiquidity risk of the PE investment as the PE asset is illiquid and markets
are incomplete. In order for the GP to justify fees and for LPs to break even in utility terms,
the GP needs to generate a sufficiently large alpha by effectively managing the PE assets
consisting of the underlying portfolio companies. We calculate the alpha required for the LP
to break even in certainty equivalent terms.

Despite the fictions arising from these institutional features, we derive tractable expres-
sions for the LP’s asset allocation rule and provide an analytical characterization of the
certainty equivalent valuation of the PE investment. Intuitively, we interpret the GP’s man-
age ment fees and carried interest as a fixed-income claim and a call-option-type contract on
the underlying PE investment, respectively. Using this analogy, we build on the standard
Black-Scholes option pricing formula to price the various “tranches” of the cash flows that
are distributed to the GP and the LP based on the contractual agreement between the two.

Importantly, we extend the standard Black-Scholes-based valuation to capture the insti-
tutional features of PE investments: First, the GP must generate alpha, which the standard
Black-Scholes formula rules out by assumption. Second, the LP demands a non-diversifiable
illiquidity risk premium, which is also ruled out by the assumption of complete markets in
the Black-Scholes framework. Third, management compensation (both management fees
and carried interest) shall be included into the incomplete-markets valuation framework.

Our formula for the LP’s certainty equivalent valuation of the PE asset captures all three
features. Implied by the LP’s optimal asset allocation rule, our valuation model generalizes the Black-Scholes option pricing formula with four new terms: (1) the alpha term, which augments the risk-adjusted drift of the PE asset from the risk-free rate \( r \) to \( r + \alpha \); (2) the non-diversifiable illiquidity risk premium term, which depends on the LP’s risk aversion and the part of the PE asset volatility that is orthogonal to the volatility of the publicly traded equity; (3) management fees \( mX_0 \) and (4) the waterfall compensation structure that appear in the boundary conditions. Because of incomplete markets and illiquidity, the law of one prices no longer holds in our model. Given the prevalent usage of leverage in the PE industry, we further extend our model to allow for external debt and price the debt accordingly.

Quantitatively, we find that common variations of the management fees and carried interest lead to significant changes in the break-even alpha, representing the required value added by the GP for LPs to break even. In our calibration, for an un-levered PE investment, this break-even alpha equals 2.6% annually. Importantly, leveraging the PE investment, substantially reduces this break-even value. With an debt-equity ratio of three, the GP only needs to generate an alpha of 1.0% for LP to break even. Intuitively, leverage allows the to GP manage more assets and generate alpha on a greater asset base. Since management fees are calculated from the LP’s committed capital, leverage reduces the effective management fee per dollar of managed PE assets. The decline in the break-even alpha with increasing leverage provides a new justification for the observed use of debt in PE transactions (given the existing structure of management contracts). Additionally, we show that both risk aversion and leverage have significant effects on the break-even alpha in our incomplete-markets valuation framework.

Moreover, we quantify the effects of changing management compensation structure on the break-even alpha for the manager. We find that for commonly adopted leverage (with a debt-equity ratio around three for example), starting from a typical 2-20 compensation contract, an increase in the management fees from 2.0% to 2.5% is roughly comparable to an increase in the performance fee (“carried interest”) from 20% to 25%.

Three new insights emerge from our analysis. First, we present a new risk-adjusted PME measure and shows that this measure can diverge substantially from the standard PME measures, raising concerns about the usual interpretation of a PME exceeding one as indicating outperforming the market. Even when the beta of the overall cash flows equals
one (as typically assumed), the risks of the calls and distributions are substantially different, leading to a biased measure. Our results indicate that this bias may be substantial, and our new adjusted PME measure may better reflect the economic value of PE investments.

Second, the existing literature has largely evaluated the compensation structure by calculating the PV of the different parts and comparing changes in these PVs as the term changes, for example by increasing the carried interest from 20% to 30%. These PVs and their changes are difficult to interpret, however, and we recast this analysis in terms of a more meaningful trade off. Specifically, our model naturally frames the question in terms of how much greater value a GP must generate, in terms of alpha, for an LP to be indifferent to a change in the terms.

Third, our results provide a new explanation for the leverage puzzle in PE. The existing literature has struggled with explaining the leverage patterns in PE. Axelson et al. (2011) report that leverage in PE appears largely unrelated to leverage patterns for publicly traded companies. To explain this apparent puzzle, Axelson, Stromberg, and Weisbach (2009) propose a model of ex-ante and ex-post monitoring. Our results, however, provide a different explanation. Our analysis indicates that a substantial benefit of leverage is that it allows GPs to apply their value-adding ability across a greater amount of assets, and hence effectively lowers the management fees per unit of assets under management.

The model assumes that all capital is invested initially and all exits are realized at maturity. The analysis does not capture the dynamics of investment decisions over the life of a fund as well as issues surrounding the valuation of unfounded liabilities (in our model, management fees are simply valued as a risk-free annuity). While it is straightforward to allow for deterministic investment dynamics, as in Metrick and Yasuda (2010), modeling stochastic investment and exit processes, possibly depending on the market return, introduces substantial additional complexity.
Appendices

A Technical details

We first derive the complete-markets benchmark solution and then sketch out the derivation for the incomplete-market solution.

A.1 For complete-markets benchmark

\[
V^*(A_t, t) = \tilde{E}_t \left[ \int_t^T e^{-r(s-t)} (-mX_0) ds + e^{-r(T-t)} V^*(A_T, T) \right]
\]

\[
= \frac{mX_0}{r} \left( e^{-r(T-t)} - 1 \right) + e^{-r(T-t)} \tilde{E}_t \left[ \min \{ A_T, F \} + (1 - k) \max \{ 0, A_T - Z \} \right] \\
+ e^{-r(T-t)} \tilde{E}_t \left[ (1 - n) \max \{ A_T - F, 0 \} - \max \{ A_T - Z, 0 \} \right]
\] (A.2)

Simplifying, we have

\[
V^*(A_t, t) = -\frac{mX_0}{r} \left( 1 - e^{-r(T-t)} \right) + A_t - n \left[ BS(A_t, t; F) - BS(A_t, t; Z) \right] - kBS(A_t, t; Z),
\] (A.3)

where \( BS(A_t, t; K) \) is the Black-Scholes call option pricing formula,

\[
BS(A_t, t; K) = A_t N(d_1(t; K)) - Ke^{-r(T-t)} N(d_2(t; K)),
\] (A.4)

with

\[
d_1(t; K) = d_2(t; K) + \frac{\sigma_A \sqrt{T-t}}{\sigma_A \sqrt{T-t}},
\] (A.5)

\[
d_2(t; K) = \frac{\ln \left( \frac{A_t}{K} \right) + \left( r - \frac{\sigma^2_A}{2} \right) (T-t)}{\frac{\sigma_A \sqrt{T-t}}{\sigma_A \sqrt{T-t}}}.
\] (A.6)

A.2 Incomplete-markets solution

Solution after maturity \( T \). After exiting from holding the illiquid asset, investors solve a classic Merton-type consumption and portfolio allocation problem by investing in the risk-free asset and the risky market portfolio. The wealth dynamics is given by

\[
dW_t = (rW_t - C_t) dt + \Pi_t \left( (\mu_S - r) dt + \sigma_S dB_t^S \right), \quad t \geq T.
\] (A.7)
Let $J^*(W)$ denote investors’ value function after time $T$, i.e.

$$J^*(W) = \max_{\Pi, C} \mathbb{E} \left[ \int_T^\infty e^{-\zeta(s-T)} U(C_s) \, ds \right]. \quad (A.8)$$

The following HJB equation holds

$$\zeta J^*(W) = \max_{\Pi, C} U(C) + (rW + \Pi(\mu_S - r) - C)J^*(W)' + \frac{1}{2}\Pi^2\sigma_S^2 J^*(W)''. \quad (A.9)$$

The FOCs for $\Pi$ and $C$ are

$$U_C(C) = J^*(W)', \quad (A.10)$$

$$\Pi = -\frac{(\mu_S - r)J^*(W)'}{\sigma_S^2 J^*(W)''}. \quad (A.11)$$

We conjecture that $J^*(W)$ is given by (25). Using the FOCs (A.10) and (A.11) for $C$ and $\Pi$, we obtain the optimal consumption and portfolio allocation given in Proposition 1.

Solution before maturity $T$. Substituting (30) into the HJB equation (29), we obtain

$$-\frac{\zeta}{\gamma r} = \max_{\Pi, C} \frac{-e^{-\gamma(C-r(W+b))}}{\gamma} + V + rW + \Pi(\mu_S - r) - mX_0 - C$$

$$+\mu_A V_A + \frac{1}{2}\sigma_A^2 V_{AA} - \frac{\gamma r}{2} \left( \Pi^2\sigma_S^2 + 2\rho\sigma_S\sigma_A\Pi\sigma_A + \sigma_A^2\right). \quad (A.12)$$

Using the FOCs for $C$ and $\Pi$, we have the optimal consumption and portfolio rules given in (32) and (33), respectively. After some algebras, we have ODE (31).

Derivation for Proposition 2. Substituting $V(A,t) = v(a,t) \times I_0$ into (31) and using $\gamma = \gamma I_0$, $a = A/I_0$, $x_0 = X_0/I_0$, we obtain (34). Using (9), (10), and (16), we have (35). Finally, substituting $x_0 = X_0/I_0$ into (20), we obtain (36).

A.3 Technical details for various performance measures

For results in Section 4.2. Let $EC(A; K)$ denote the expected payoff of a call option with strike price $K$ under the physical measure,

$$EC(A; K) = \mathbb{E}_0 \left[ e^{-\mu_A T} \max \{A_T - K, 0\} \right], \quad (A.13)$$

$$= AN(p_1(K)) - K e^{-\mu_A T} N(p_2(K)). \quad (A.14)$$
where \( p_1(K) \) and \( p_2(K) \) are given by
\[
\begin{align*}
    p_1(K) &= p_2(K) + \sigma_A \sqrt{T}, \quad \text{(A.15)} \\
p_2(K) &= \frac{\ln\left(\frac{A}{K}\right) + \left(\mu_A - \frac{\sigma_A^2}{2}\right) T}{\sigma_A \sqrt{T}}, \quad \text{(A.16)}
\end{align*}
\]

Denote \( RC(A; K) \) as the expected payoff of a call option with strike price \( K \) under the risk-adjusted measure defined in the text,
\[
\begin{align*}
    RC(A; K) &= \tilde{\mathbb{E}}_0 \left[ e^{-rT} \max\{A_T - K, 0\} \right], \quad \text{(A.17)} \\
    &= e^{\alpha T} (AN(q_1(K)) - K e^{-(r+\alpha)T} N(q_2(K))), \quad \text{(A.18)}
\end{align*}
\]

where
\[
\begin{align*}
    q_1(K) &= q_2(K) + \sigma_A \sqrt{T}, \quad \text{(A.19)} \\
    q_2(K) &= \frac{\ln\left(\frac{A}{K}\right) + \left(r + \alpha - \frac{\sigma_A^2}{2}\right) T}{\sigma_A \sqrt{T}}. \quad \text{(A.20)}
\end{align*}
\]

For Section 5 with leverage. The market value of debt at time \( t \) is
\[
\begin{align*}
    D(A, t) &= e^{\alpha(T-t)} A - \left[ Ae^{\alpha(T-t)} N(\tilde{d}_1) - D_0 e^{yT} e^{-r(T-t)} N(\tilde{d}_2) \right], \quad \text{(A.21)}
\end{align*}
\]

where
\[
\begin{align*}
    \tilde{d}_1 &= \tilde{d}_2 + \sigma_A \sqrt{T} - t, \quad \text{(A.22)} \\
    \tilde{d}_2 &= \frac{\ln\left(\frac{A}{D_0 e^{yT}}\right) + \left(r + \alpha - \frac{\sigma_A^2}{2}\right) (T-t)}{\sigma_A \sqrt{T} - t} \quad \text{(A.23)}
\end{align*}
\]

Incorporating the risky debt, we define the ex-ante TVPI as
\[
\begin{align*}
    \mathbb{E}[\text{TVPI}] &= \frac{\mathbb{E}\left[ LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T) \right]}{X_0} \quad \text{(A.24)} \\
    &= \frac{e^{\mu_A T} \left[ EC(A_0; D_0 e^{yT}) - n \left(EC(A_0; F + D_0 e^{yT}) - EC(A_0; Z_{Lev})\right) - k EC(A_0; Z_{Lev}) \right]}{X_0}.
\end{align*}
\]

Similarly, the LP’s ex-ante IRR solves the equation
\[
I_0 + \int_0^T m X_0 e^{-\phi t} dt = e^{-\phi T} \mathbb{E}\left[ LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T) \right], \quad \text{(A.25)}
\]
which simplifies to

\[ I_0 + \frac{mX_0}{\phi} (1 - e^{-\phi T}) \]  

(A.26)

\[ = e^{(\mu_A - \phi)T} \left[ EC(A_0; D_0e^{yT}) - n \left( EC(A_0; F + D_0e^{yT}) - EC(A_0; Z_{Lev}) \right) - kEC(A_0; Z_{Lev}) \right] . \]

The PME is then given by

\[ \text{PME} = \frac{\mathbb{E} \left[ e^{-\mu_S T} \left[ LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T) \right] \right]}{\mathbb{E} \left[ \int_0^T e^{-\mu_S s} mX_0 ds + I_0 \right]} \]  

(A.27)

\[ = \frac{e^{(\mu_A - \mu_S)T} \left[ EC(A_0; D_0e^{yT}) - n \left( EC(A_0; F + D_0e^{yT}) - EC(A_0; Z_{Lev}) \right) - kEC(A_0; Z_{Lev}) \right]}{I_0 + \frac{mX_0}{\mu_S} (1 - e^{-\mu_S T})} . \]

The adjusted ex-ante PME is given by

\[ \text{Adjusted PME} = \frac{\mathbb{E} \left[ e^{-r T} \left[ LP_1(A_T, T) + LP_2(A_T, T) + LP_3(A_T, T) \right] \right]}{\mathbb{E} \left[ \int_0^T e^{-rs} mX_0 ds + I_0 \right]} \]  

(A.28)

\[ = \frac{e^{\alpha T} \left[ RC(A_0; D_0e^{yT}) - n \left( RC(A_0; F + D_0e^{yT}) - RC(A_0; Z_{Lev}) \right) - kRC(A_0; Z_{Lev}) \right]}{I_0 + \frac{mX_0}{r} (1 - e^{-r T})} . \]
Table 1: Effects of risk aversion on performance measures: The case with no leverage

<table>
<thead>
<tr>
<th>Panel A.</th>
<th>( \beta = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk Aversion (( \gamma ))</td>
<td>0_+</td>
</tr>
<tr>
<td>Alpha (( \alpha ))</td>
<td>2.61%</td>
</tr>
<tr>
<td>IRR (( \phi ))</td>
<td>5.0%</td>
</tr>
<tr>
<td>( \text{E}[\text{TVPI}] )</td>
<td>1.58</td>
</tr>
<tr>
<td>PME</td>
<td>0.57</td>
</tr>
<tr>
<td>Adj. PME</td>
<td>1.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B.</th>
<th>( \beta = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk Aversion (( \gamma ))</td>
<td>0_+</td>
</tr>
<tr>
<td>Alpha (( \alpha ))</td>
<td>2.61%</td>
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<tr>
<td>IRR (( \phi ))</td>
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</tr>
<tr>
<td>( \text{E}[\text{TVPI}] )</td>
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</tr>
<tr>
<td>PME</td>
<td>0.75</td>
</tr>
<tr>
<td>Adj. PME</td>
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</table>

<table>
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<th>Panel C.</th>
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</tr>
</thead>
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<td>Risk Aversion (( \gamma ))</td>
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</tr>
<tr>
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<td>2.61%</td>
</tr>
<tr>
<td>IRR (( \phi ))</td>
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</tr>
<tr>
<td>( \text{E}[\text{TVPI}] )</td>
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</tr>
<tr>
<td>PME</td>
<td>0.99</td>
</tr>
<tr>
<td>Adj. PME</td>
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<table>
<thead>
<tr>
<th>Panel D.</th>
<th>( \beta = 1.25 )</th>
</tr>
</thead>
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<td>Risk Aversion (( \gamma ))</td>
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</tr>
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<tr>
<td>IRR (( \phi ))</td>
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</tr>
<tr>
<td>( \text{E}[\text{TVPI}] )</td>
<td>3.13</td>
</tr>
<tr>
<td>PME</td>
<td>1.13</td>
</tr>
<tr>
<td>Adj. PME</td>
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</tr>
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</table>
Table 2: Effects of risk aversion on performance measures: The case with $l = 1$

<table>
<thead>
<tr>
<th>Panel</th>
<th>$\beta$ = 0</th>
<th>$\beta$ = 0.5</th>
<th>$\beta$ = 1</th>
<th>$\beta$ = 1.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A.</td>
<td>$\gamma = 0$+</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Alpha ($\alpha$)</td>
<td>1.68%</td>
<td>2.16%</td>
<td>2.60%</td>
<td>3.79%</td>
</tr>
<tr>
<td>IRR ($\phi$)</td>
<td>5.0%</td>
<td>5.9%</td>
<td>6.6%</td>
<td>8.5%</td>
</tr>
<tr>
<td>Credit spread ($y - r$)</td>
<td>1.05%</td>
<td>0.93%</td>
<td>0.83%</td>
<td>0.61%</td>
</tr>
<tr>
<td>E[TVPI]</td>
<td>1.58</td>
<td>1.71</td>
<td>1.83</td>
<td>2.19</td>
</tr>
<tr>
<td>PME</td>
<td>0.57</td>
<td>0.62</td>
<td>0.66</td>
<td>0.79</td>
</tr>
<tr>
<td>Adj. PME</td>
<td>1.00</td>
<td>1.08</td>
<td>1.16</td>
<td>1.39</td>
</tr>
</tbody>
</table>

| Panel B. | $\gamma = 0$+ | 1 | 2 | 5 |
| Alpha ($\alpha$) | 1.68% | 2.08% | 2.46% | 3.49% |
| IRR ($\phi$) | 9.6% | 10.2% | 10.8% | 12.3% |
| Credit spread ($y - r$) | 1.05% | 0.95% | 0.86% | 0.66% |
| E[TVPI] | 2.43 | 2.57 | 2.72 | 3.13 |
| PME | 0.88 | 0.93 | 0.98 | 1.13 |
| Adj. PME | 1.00 | 1.07 | 1.13 | 1.33 |

| Panel C. | $\gamma = 0$+ | 1 | 2 | 5 |
| Alpha ($\alpha$) | 1.68% | 1.86% | 2.03% | 2.51% |
| IRR ($\phi$) | 13.8% | 14.0% | 14.3% | 14.9% |
| Credit spread ($y - r$) | 1.05% | 1.00% | 0.96% | 0.85% |
| E[TVPI] | 3.60 | 3.68 | 3.77 | 4.01 |
| PME | 1.30 | 1.33 | 1.36 | 1.45 |
| Adj. PME | 1.00 | 1.03 | 1.06 | 1.14 |

| Panel D. | $\gamma = 0$+ | 1 | 2 | 5 |
| Alpha ($\alpha$) | 1.68% | 1.68% | 1.68% | 1.68% |
| IRR ($\phi$) | 15.7% | 15.7% | 15.7% | 15.7% |
| Credit spread ($y - r$) | 1.05% | 1.05% | 1.05% | 1.05% |
| E[TVPI] | 4.33 | 4.33 | 4.33 | 4.33 |
| PME | 1.56 | 1.56 | 1.56 | 1.56 |
| Adj. PME | 1.00 | 1.00 | 1.00 | 1.00 |
Table 3: Effects of risk aversion on performance measures: The case with $l = 3$

<table>
<thead>
<tr>
<th>Panel A. $\beta = 0$</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk Aversion ($\gamma$)</td>
<td>$0_+$</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Alpha ($\alpha$)</td>
<td>1.00%</td>
<td>1.67%</td>
<td>2.22%</td>
<td>3.71%</td>
</tr>
<tr>
<td>IRR ($\phi$)</td>
<td>5.0%</td>
<td>7.1%</td>
<td>8.6%</td>
<td>12.1%</td>
</tr>
<tr>
<td>Credit spread ($y - r$)</td>
<td>3.48%</td>
<td>2.91%</td>
<td>2.53%</td>
<td>1.74%</td>
</tr>
<tr>
<td>$E[TVPI]$</td>
<td>1.57</td>
<td>1.91</td>
<td>2.20</td>
<td>3.07</td>
</tr>
<tr>
<td>PME</td>
<td>0.57</td>
<td>0.69</td>
<td>0.79</td>
<td>1.11</td>
</tr>
<tr>
<td>Adj. PME</td>
<td>1.00</td>
<td>1.21</td>
<td>1.39</td>
<td>1.94</td>
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</table>

<table>
<thead>
<tr>
<th>Panel B. $\beta = 0.5$</th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk Aversion ($\gamma$)</td>
<td>$0_+$</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Alpha ($\alpha$)</td>
<td>1.00%</td>
<td>1.58%</td>
<td>2.05%</td>
<td>3.33%</td>
</tr>
<tr>
<td>IRR ($\phi$)</td>
<td>11.2%</td>
<td>12.7%</td>
<td>13.8%</td>
<td>16.5%</td>
</tr>
<tr>
<td>Credit spread ($y - r$)</td>
<td>3.48%</td>
<td>2.98%</td>
<td>2.64%</td>
<td>1.91%</td>
</tr>
<tr>
<td>$E[TVPI]$</td>
<td>2.81</td>
<td>3.24</td>
<td>3.61</td>
<td>4.66</td>
</tr>
<tr>
<td>PME</td>
<td>1.02</td>
<td>1.17</td>
<td>1.30</td>
<td>1.68</td>
</tr>
<tr>
<td>Adj. PME</td>
<td>1.00</td>
<td>1.18</td>
<td>1.34</td>
<td>1.80</td>
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</table>

<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk Aversion ($\gamma$)</td>
<td>$0_+$</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Alpha ($\alpha$)</td>
<td>1.00%</td>
<td>1.28%</td>
<td>1.50%</td>
<td>2.11%</td>
</tr>
<tr>
<td>IRR ($\phi$)</td>
<td>16.6%</td>
<td>17.2%</td>
<td>17.7%</td>
<td>18.9%</td>
</tr>
<tr>
<td>Credit spread ($y - r$)</td>
<td>3.48%</td>
<td>3.23%</td>
<td>3.04%</td>
<td>2.60%</td>
</tr>
<tr>
<td>$E[TVPI]$</td>
<td>4.70</td>
<td>5.00</td>
<td>5.24</td>
<td>5.89</td>
</tr>
<tr>
<td>PME</td>
<td>1.70</td>
<td>1.81</td>
<td>1.89</td>
<td>2.13</td>
</tr>
<tr>
<td>Adj. PME</td>
<td>1.00</td>
<td>1.08</td>
<td>1.16</td>
<td>1.36</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel D. $\beta = 1.25$</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk Aversion ($\gamma$)</td>
<td>$0_+$</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Alpha ($\alpha$)</td>
<td>1.00%</td>
<td>1.00%</td>
<td>1.00%</td>
<td>1.00%</td>
</tr>
<tr>
<td>IRR ($\phi$)</td>
<td>19.0%</td>
<td>19.0%</td>
<td>19.0%</td>
<td>19.0%</td>
</tr>
<tr>
<td>Credit spread ($y - r$)</td>
<td>3.48%</td>
<td>3.48%</td>
<td>3.48%</td>
<td>3.48%</td>
</tr>
<tr>
<td>$E[TVPI]$</td>
<td>5.96</td>
<td>5.96</td>
<td>5.96</td>
<td>5.96</td>
</tr>
<tr>
<td>PME</td>
<td>2.15</td>
<td>2.15</td>
<td>2.15</td>
<td>2.15</td>
</tr>
<tr>
<td>Adj. PME</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
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</table>

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Table 4: Effects of leverage: The case with risk aversion $\overline{\gamma} = 0.4$

<table>
<thead>
<tr>
<th>Leverage $(l)$</th>
<th>Panel A. $\beta = 0$</th>
<th>Panel B. $\beta = 0.5$</th>
<th>Panel C. $\beta = 1$</th>
<th>Panel D. $\beta = 1.25$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$0$</td>
<td>$1$</td>
<td>$3$</td>
<td>$6$</td>
</tr>
<tr>
<td>Alpha $(\alpha)$</td>
<td>2.61%</td>
<td>1.68%</td>
<td>1.00%</td>
<td>0.63%</td>
</tr>
<tr>
<td>IRR $(\phi)$</td>
<td>5.0%</td>
<td>5.0%</td>
<td>5.0%</td>
<td>5.0%</td>
</tr>
<tr>
<td>Credit spread $(y - r)$</td>
<td>N/A</td>
<td>1.05%</td>
<td>3.48%</td>
<td>5.69%</td>
</tr>
<tr>
<td>$\mathbb{E}[\text{TVPI}]$</td>
<td>1.58</td>
<td>1.58</td>
<td>1.58</td>
<td>1.58</td>
</tr>
<tr>
<td>PME</td>
<td>0.57</td>
<td>0.57</td>
<td>0.57</td>
<td>0.57</td>
</tr>
<tr>
<td>Adj. PME</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$\mathbb{E}[\text{TVPI}]$</td>
<td>1.58</td>
<td>1.58</td>
<td>1.58</td>
<td>1.58</td>
</tr>
<tr>
<td>PME</td>
<td>0.57</td>
<td>0.57</td>
<td>0.57</td>
<td>0.57</td>
</tr>
<tr>
<td>Adj. PME</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

38
Table 5: Effects of leverage: The case with risk aversion $\gamma = 2$

<table>
<thead>
<tr>
<th>Panel A.</th>
<th>$\beta = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leverage ($l$)</td>
<td>0</td>
</tr>
<tr>
<td>Alpha ($\alpha$)</td>
<td>3.17%</td>
</tr>
<tr>
<td>IRR ($\phi$)</td>
<td>5.6%</td>
</tr>
<tr>
<td>Credit spread ($y - r$)</td>
<td>N/A</td>
</tr>
<tr>
<td>$E[TVPI]$</td>
<td>1.66</td>
</tr>
<tr>
<td>PME</td>
<td>0.60</td>
</tr>
<tr>
<td>Adj. PME</td>
<td>1.05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B.</th>
<th>$\beta = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leverage ($l$)</td>
<td>0</td>
</tr>
<tr>
<td>Alpha ($\alpha$)</td>
<td>3.08%</td>
</tr>
<tr>
<td>IRR ($\phi$)</td>
<td>8.4%</td>
</tr>
<tr>
<td>Credit spread ($y - r$)</td>
<td>N/A</td>
</tr>
<tr>
<td>$E[TVPI]$</td>
<td>2.16</td>
</tr>
<tr>
<td>PME</td>
<td>0.78</td>
</tr>
<tr>
<td>Adj. PME</td>
<td>1.04</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C.</th>
<th>$\beta = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leverage ($l$)</td>
<td>0</td>
</tr>
<tr>
<td>Alpha ($\alpha$)</td>
<td>2.82%</td>
</tr>
<tr>
<td>IRR ($\phi$)</td>
<td>11.1%</td>
</tr>
<tr>
<td>Credit spread ($y - r$)</td>
<td>N/A</td>
</tr>
<tr>
<td>$E[TVPI]$</td>
<td>2.78</td>
</tr>
<tr>
<td>PME</td>
<td>1.00</td>
</tr>
<tr>
<td>Adj. PME</td>
<td>1.02</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel D.</th>
<th>$\beta = 1.25$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leverage ($l$)</td>
<td>0</td>
</tr>
<tr>
<td>Alpha ($\alpha$)</td>
<td>2.61%</td>
</tr>
<tr>
<td>IRR ($\phi$)</td>
<td>12.3%</td>
</tr>
<tr>
<td>Credit spread ($y - r$)</td>
<td>N/A</td>
</tr>
<tr>
<td>$E[TVPI]$</td>
<td>3.13</td>
</tr>
<tr>
<td>PME</td>
<td>1.13</td>
</tr>
<tr>
<td>Adj. PME</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Table 6: Effects of management and incentive fees on break-even alpha: The case with risk neutrality $\gamma = 0_{+}$

<table>
<thead>
<tr>
<th>Panel A. $k = 0.2$</th>
<th>$l$</th>
<th>$m = 1.5%$</th>
<th>$m = 2%$</th>
<th>$m = 2.5%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>2.11%</td>
<td>2.61%</td>
<td>3.14%</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1.41%</td>
<td>1.68%</td>
<td>1.99%</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0.85%</td>
<td>1.00%</td>
<td>1.16%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B. $m = 2%$</th>
<th>$l$</th>
<th>$k = 0.2$</th>
<th>$k = 0.25$</th>
<th>$k = 0.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>2.61%</td>
<td>2.83%</td>
<td>3.07%</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1.68%</td>
<td>1.90%</td>
<td>2.13%</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>1.00%</td>
<td>1.16%</td>
<td>1.34%</td>
</tr>
</tbody>
</table>

Table 7: Effects of management fees on break-even alpha: The case with risk aversion $\gamma = 2$

<table>
<thead>
<tr>
<th>Panel A. $\beta = 0$</th>
<th>$l$</th>
<th>$m = 1.5%$</th>
<th>$m = 2%$</th>
<th>$m = 2.5%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>2.65%</td>
<td>3.17%</td>
<td>3.74%</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>2.29%</td>
<td>2.60%</td>
<td>2.93%</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>2.04%</td>
<td>2.22%</td>
<td>2.43%</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B. $\beta = 0.5$</th>
<th>$l$</th>
<th>$m = 1.5%$</th>
<th>$m = 2%$</th>
<th>$m = 2.5%$</th>
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</thead>
<tbody>
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<td>3.08%</td>
<td>3.65%</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>2.16%</td>
<td>2.46%</td>
<td>2.79%</td>
</tr>
<tr>
<td>3</td>
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<td>1.87%</td>
<td>2.05%</td>
<td>2.25%</td>
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</table>

<table>
<thead>
<tr>
<th>Panel C. $\beta = 1$</th>
<th>$l$</th>
<th>$m = 1.5%$</th>
<th>$m = 2%$</th>
<th>$m = 2.5%$</th>
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</thead>
<tbody>
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<td></td>
<td>1.74%</td>
<td>2.03%</td>
<td>2.35%</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>1.34%</td>
<td>1.50%</td>
<td>1.68%</td>
</tr>
</tbody>
</table>
Table 8: Effects of incentive fees on break-even alpha: The case with risk aversion $\gamma = 2$

<table>
<thead>
<tr>
<th>Panel A. $\beta = 0$</th>
<th>$l$</th>
<th>$k = 0.2$</th>
<th>$k = 0.25$</th>
<th>$k = 0.3$</th>
</tr>
</thead>
<tbody>
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<td>3.17%</td>
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<td>3.59%</td>
<td></td>
</tr>
<tr>
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<td>2.78%</td>
<td>2.98%</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.22%</td>
<td>2.35%</td>
<td>2.49%</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B. $\beta = 0.5$</th>
<th>$l$</th>
<th>$k = 0.2$</th>
<th>$k = 0.25$</th>
<th>$k = 0.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.08%</td>
<td>3.29%</td>
<td>3.50%</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.46%</td>
<td>2.65%</td>
<td>2.85%</td>
<td></td>
</tr>
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<td>3</td>
<td>2.05%</td>
<td>2.18%</td>
<td>2.32%</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel C. $\beta = 1$</th>
<th>$l$</th>
<th>$k = 0.2$</th>
<th>$k = 0.25$</th>
<th>$k = 0.3$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
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<td>3.03%</td>
<td>3.26%</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.03%</td>
<td>2.23%</td>
<td>2.45%</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.50%</td>
<td>1.65%</td>
<td>1.81%</td>
<td></td>
</tr>
</tbody>
</table>
Table 9: Summary of Key Variables and Parameters

This table summarizes the symbols for the key variables in the model and baseline parameter values.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP’s Consumption or expenditure</td>
<td>$C$</td>
<td>Risk-free rate</td>
<td>$r$</td>
<td>5%</td>
</tr>
<tr>
<td>LP’s Value Function</td>
<td>$J$</td>
<td>Expected return of market portfolio</td>
<td>$\mu_S$</td>
<td>11%</td>
</tr>
<tr>
<td>LP’s Value Function after exiting illiquid asset</td>
<td>$J^*$</td>
<td>Expected return of PE asset</td>
<td>$\mu_A$</td>
<td></td>
</tr>
<tr>
<td>LP’s Certainty Equivalent</td>
<td>$V$</td>
<td>Volatility of market portfolio</td>
<td>$\sigma_S$</td>
<td>20%</td>
</tr>
<tr>
<td>Future value of investment and fees</td>
<td>$F$</td>
<td>Volatility of PE asset</td>
<td>$\sigma_A$</td>
<td>25%</td>
</tr>
<tr>
<td>Debt</td>
<td>$D$</td>
<td>Aggregate equity risk premium</td>
<td>$\mu_S - r$</td>
<td>6%</td>
</tr>
<tr>
<td>Wealth</td>
<td>$W$</td>
<td>Market Sharpe ratio</td>
<td>$\eta$</td>
<td>30%</td>
</tr>
<tr>
<td>Assets</td>
<td>$A$</td>
<td>Hurdle rate</td>
<td>$h$</td>
<td>8%</td>
</tr>
<tr>
<td>Brownian Motion for Market Return</td>
<td>$B^R$</td>
<td>Carried interest</td>
<td>$k$</td>
<td>20%</td>
</tr>
<tr>
<td>Brownian Motion for PE Return</td>
<td>$B^A$</td>
<td>Management fee</td>
<td>$m$</td>
<td>2%</td>
</tr>
<tr>
<td>Committed Capital</td>
<td>$X_0$</td>
<td>Catch-up rate</td>
<td>$n$</td>
<td>100%</td>
</tr>
<tr>
<td>Invested Capital</td>
<td>$I_0$</td>
<td>Life of PE investment</td>
<td>$T$</td>
<td>10</td>
</tr>
<tr>
<td>Market portfolio allocation</td>
<td>$\Pi$</td>
<td>non-diversifiable illiquidity risk premium</td>
<td>$\alpha$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Correlation between market and PE asset</td>
<td>$\rho$</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Systematic Risk</td>
<td>$\beta$</td>
<td>0.5</td>
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References


